Notes from TGRA lab #5

Introduction

This document contains solutions to the problems solved in lab class 17th September 2013 (Lab 5). Note that this document is created from notes from TA and might contain misspelled words or minor calculation errors. If you find such errors please send us email with your findings or create a pull request for this document on Github (`https://github.com/hlysig/tgra-2013`). The current version of this document is 1[1].

[1] (22. September 2013)

Problem 1

i) Define a 2D transformation matrix A for translating by $(3, 5)$.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate the point $p = (1, 2)$ with A.

$$p_A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 2+5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

ii) Make a 2D transformation matrix B for double scaling.

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transform the point $p = (5, 4)$ with B.

$$p_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 1 \end{bmatrix}$$

iii) Make a 2D transformation matrix C for $36°$ rotation.

$$C = \begin{bmatrix} cos36° & -sin36° & 0 \\ sin36° & cos36° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Transform the point $p = (3, -1)$ with C.

$$p_C = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4+0.6 \\ 1.8-0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

iv) Make a 2D transformation matrix that applies a combined transformation. The combined transformation is as so: translate a coordinate frame first as in A, then as in B (relative to the previous coordinate frame) and finally as in C (again relative to the previous coordinate frame)

$$
\begin{aligned}
ABC &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
ABC &= \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
ABC &= \begin{bmatrix} 1.6 & -1.2 & 3 \\ 1.2 & 1.6 & 5 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Transform point $p = (-2, 3)$.

$$
\begin{aligned}
P_{ABC} &= ABCp \\
P_{ABC} &= \begin{bmatrix} 1.6 & -1.2 & 3 \\ 1.2 & 1.6 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3.8 \\ 7.4 \\ 1 \end{bmatrix}
\end{aligned}
$$

Problem 2

Values are set in the modelview matrix transforming coordinates in relation to the direction and orientation of the camera as in (2).

$$\begin{bmatrix} u_x & u_y & u_z & -eye \cdot u \\ v_x & v_y & v_z & -eye \cdot v \\ n_x & n_y & n_z & -eye \cdot n \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

a) How will the matrix look if the following code will be run:

1. `glLoadIdentity();`
2. `gluLookat(5,7,3,4,7,1,0,1,0)`

After line 1: the modelview matrix is

$$vm = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

After line 2: From the parameters in gluLookat we have the following variables.

$$eye = (5,7,3), \quad center = (4,7,1), \quad \vec{up} = (0,1,0) \tag{4}$$

`gluLookat` builds matrix that converts world coordinates to eye coordinates. We then must create a coordinate frame for the camera coordinates from the variables eye, $center$ and $\vec{up}$. The camera coordinate axes are defined as follows.

1. $\vec{n} = eye - center$
2. $\vec{u} = \vec{up} \times \vec{n}$
3. $\vec{v} = \vec{n} \times \vec{u}$

Note that these vectors are not normalized. Before we can use them we must normalize them as we do later in our solution. The vector $\vec{n}$ is our local z-axis so we will be looking along the negative $\vec{n}$. The vector $\vec{u}$ should point straight to the right and since we're looking along negative $\vec{n}$ and the up vector points in a general up direction, $\vec{u}$ should be perpendicular to both of them. Finally $\vec{v}$ should point straight up and be orthogonal to both $\vec{u}$ and $\vec{n}$.

Let us calculate the abovementioned vectors.

$$\begin{aligned} \vec{n} &= (5,7,3)-(4,7,1)=(1,0,2) && (5)\\ \vec{u} &= (0,1,0)\times(1,0,2)=(2,0,-1) && (6)\\ \vec{v} &= (1,0,2)\times(2,0,-1)=(0,5,0) && (7) \end{aligned}$$

Next we normalize $\vec{n}$, $\vec{u}$ and $\vec{v}$.

$$\begin{aligned} \hat{n} &= \left(\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\right) && (8)\\ \hat{u} &= \left(\frac{2}{\sqrt{5}},0,\frac{-1}{\sqrt{5}}\right) && (9)\\ \hat{v} &= (0,1,0) && (10) \end{aligned}$$

Now we got u,v, n for the matrix in (2). Now we only need to find the values for $-eye\cdot\hat{u}$, $-eye\cdot\hat{v}$, $-eye\cdot\hat{n}$.

$$\begin{aligned} -eye\cdot\hat{u} &= (-5,-7,-3)\cdot\left(\frac{2}{\sqrt{5}},0,\frac{-1}{\sqrt{5}}\right) && (11)\\ &= \frac{-10}{\sqrt{5}}+\frac{3}{\sqrt{5}}=\frac{-7}{\sqrt{5}} && (12) \end{aligned}$$

$$\begin{aligned} -eye\cdot\hat{v} &= (-5,-7,-3)\cdot(0,1,0) && (13)\\ &= -7 && (14) \end{aligned}$$

$$\begin{aligned} -eye\cdot\hat{n} &= (-5,-7,-3)\cdot\left(\frac{1}{\sqrt{5}},0,\frac{2}{\sqrt{5}}\right) && (15)\\ &= \frac{-5}{\sqrt{5}}+\frac{-6}{\sqrt{5}}=\frac{-11}{\sqrt{5}} && (16) \end{aligned}$$

After the second line has been executed, that is the `gluLookat` line, the modelview matrix becomes:

$$vm = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} & \frac{-7}{\sqrt{5}}\\ 0 & 1 & 0 & -7\\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} & \frac{-11}{\sqrt{5}}\\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (17)$$

b) Show the values in the modelview matrix if the following lines of code are executed after the code in a)

1. `glRotated(30,1,0,0);`
2. `glTranslate(0,10,0);`

We define transformation matrices for 1) and 2). We name them m_1 and m_2 respectively.

$$m_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos30^\circ & -sin30^\circ & 0 \\ 0 & sin30^\circ & cos30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

$$m_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

Next we perform the calculation $(vm_1)m_2$

$$vm_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} & \frac{-7}{\sqrt{5}} \\ 0 & 1 & 0 & -7 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} & \frac{-11}{\sqrt{5}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos30^\circ & -sin30^\circ & 0 \\ 0 & sin30^\circ & cos30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-0.5}{\sqrt{5}} & \frac{-0.866}{\sqrt{5}} & \frac{-7}{\sqrt{5}} \\ 0 & 0.866 & -0.5 & -7 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1.722}{\sqrt{5}} & \frac{-11}{\sqrt{5}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

Therefore, our modelview matrix becomes the matrix in (21) after executing the first line of the code.

Now we know the matrix vm_1. Let us calculate $(vm_1)m_2$ for the second line of the abovementioned code.

$$(vm_1)m_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-0.5}{\sqrt{5}} & \frac{-0.866}{\sqrt{5}} & \frac{-7}{\sqrt{5}} \\ 0 & 0.866 & -0.5 & -7 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1.722}{\sqrt{5}} & \frac{-11}{\sqrt{5}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{0.5}{\sqrt{5}} & -0.866 & \frac{-12}{\sqrt{5}} \\ 0 & 0.866 & -0.5 & 1.66 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1.722}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Therefore, our modelview matrix becomes the matrix in (23) after executing the second line of the code.

Problem 3

An equation of plane can be given on a *Point-normal form (PNF)*. To fulfill this form we need a normal of the plane, $\hat{n}$ and a point, B that lies on the plane. Then we can find any point R on the plane by solving the equation

$$(R - B) \cdot \hat{n} = 0 \tag{24}$$

where the vector $R - B$ must be orthogonal with the normal $\hat{n}$ using the orthogonal property of the dot product.

There are many ways to give a formula for a line (or a ray). One is the *Parametric form.* Given one point p on the line (or an initial point of a ray) and the directional vector $\vec{c}$ of the line. We can find any point p' on the line by scaling the vector $\vec{c}$ with some parameter t and translate the point p with that vector.

$$p' = p + t\vec{c} \tag{25}$$

In the following problem we wish to find where a given line hits a plane, that is the point of hit. We will call that point p_{hit}. We are also interested in the time of hit, that we call t_{hit}.

With our hypothetical line (that is hitting the plane) and from (25) we know that

$$p_{hit} = A + ct_{hit} \tag{26}$$

and from (24) we know that

$$(p_{hit} - B) \cdot \hat{n} = 0 \tag{27}$$

where the point p_{hit} lies on the plane (where the line hits the plane) and the vector $(p_{hit} - R)$ is therefore orthogonal to the normal of the plane.

By combining (26) and (27) we get

$$(A + ct_{hit} - B) \cdot \hat{n} = 0 \tag{28}$$

We can now rewrite (28) as follows (the aim is to isolate t_{hit}):

$$\begin{aligned}
((A - B) + ct_{hit}) \cdot \hat{n} &= 0 \\
(A - B) \cdot \hat{n} + ct_{hit} \cdot \hat{n} &= 0 \\
ct_{hit} \cdot \hat{n} &= (B - A) \cdot \hat{n} \\
t_{hit}(c \cdot \hat{n}) &= (B - A) \cdot n \\
t_{hit} &= \frac{(B - A) \cdot \hat{n}}{c \cdot \hat{n}}
\end{aligned}$$

From this we can fill in (26):

$$p_{hit} = A + c\left(\frac{(B-A)\cdot\hat{n}}{c\cdot\hat{n}}\right) \tag{29}$$

and we can use this formula to find where our line hits a plane.

Solving the problem using t_{hit} and p_{hit}

We are given three points on a plane:

$$p_1 = (3,2,4), \quad p_2 = (4,3,0), \quad p_3 = (5,2,3)$$

We then have a ray, starting at point $A = (-1,-2,3)$ which has the direction $\vec{c} = (1,5,3)$. Let us find p_{hit} and t_{hit} where the line hits the plane.

First we need the normal vector of the plane

$$\begin{aligned}
\hat{n} &= (p_2 - p_1) \times (p_3 - p_1) \\
\hat{n} &= (1,1,-4) \times (2,0,-1) \\
\hat{n} &= (-1,-7,-2)
\end{aligned}$$

We can now use the t_{hit} formula the we derived above. We select $B = p_1$.

$$\begin{aligned}
t_{hit} &= \frac{(B-A)\cdot\hat{n}}{c\cdot\hat{n}} \\
t_{hit} &= \frac{(4,4,1)\cdot(-1,-7,-2)}{(1,5,3)\cdot(-1,-7,-2)} \\
t_{hit} &= \frac{34}{42}
\end{aligned}$$

We now know t_{hit} and we can find the point where the ray hits the plane with p_{hit}.

$$\begin{aligned}
p_{hit} &= A + ct_{hit} \\
p_{hit} &= (-1,-2,3) + \left(\frac{34}{42}(1,5,3)\right) \\
p_{hit} &= \left(-1 + \frac{34}{42}, -2\frac{170}{42}, \frac{102}{42}\right)
\end{aligned}$$