CONTROL AND ACTUATING DEVICES FOR MECHANICAL SYSTEMS

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ASSIGNMENT 2: SPEED CONTROL OF MOTOR SHAFT

The system, represented in figure 1, consists of an electric motor whose shaft is connected to an inertial load J and is subject to a constant resistance torque Cr. The flexibility of the motor shaft is taken into account through a torsional spring of constant kT. The energy dissipation in the system is modelled considering a viscous type of dissipation, proportional to the square of the speed of rotation of the shaft through a torsional damping coefficient c. The parameters of the system are listed in table 1, where A and B represent the derivatives of the motor torque Cm with respect to the speed of rotation of the motor shaft and to the control variable respectively, both computed about the steady-state condition.

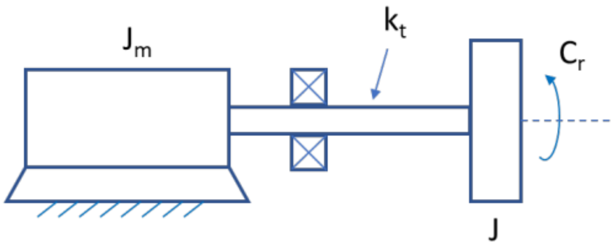


Figure 1 – Scheme of the system

System data			
Motor mass moment of inertia	Jm	[kgm^2]	0.01
Motor mass moment of inertia	J	[kgm^2]	0.15
Equivalent damping coefficient	c	[Nms/rad]c	0.01
Derivative of characteristic curve with respect to the speed of motor shaft at s.s.	A	[Nm/(rad/s)]	-0.986
Derivative of characteristic curve with respect to the control variable at s.s.	B	[Nm/V]	1.1
Equivalent torsional spring	kt	[Nm]	1000

Table 1 – System data

$$C_m = C_m(\omega, y) \;\Rightarrow\; A = \left.\frac{\partial C_m}{\partial \omega}\right|_{\bar{\omega},\bar{y}} \quad , \qquad B = \left.\frac{\partial C_m}{\partial y}\right|_{\bar{\omega},\bar{y}}$$

QUESTIONS: Define a **proportional control on the speed** $\boldsymbol{\omega}$ of the motor shaft and **study the stability** and the **performances** (draw Bode Plots, Nyquist and Root locus, compare the open loop and closed loop transfer functions) of the controlled system in the following cases:

CASE A: Neglecting the flexibility of the shaft (**kt**→∞), therefore considering the system as a 1 dof system.

CASE B*:* Representing the flexibility of the shaft trough the **torsional spring kt.**
Consider for this second case:

1) A colocated control
2) A non-colocated contro

CASE A: kt→∞, 1 dof system.

First order nonlinear equation due to the torque $C_m(\omega, y)$ where y is the regulator parameter of Cm (it is like voltage for the electric motors) and ω is the speed of the motor:

$$(J_m + J)\dot{\omega} + c\ \omega = C_m(\omega, y) - C_r$$

Linearised first order differential equation of small perturbations around the steady state condition $\omega = \bar{\omega}$, $y = \bar{y}$:

$$(J_m + J)\delta\dot{\omega} + (c - A)\delta\omega = B\ \delta y$$

Proportional control:

$$\delta y = kp\ (\delta\omega_{ref} - \delta\omega)$$

In time domain:

<table>
<tr><th>Without control</th><th colspan="2">With control</th></tr>
<tr><td rowspan="2">$\lambda = -\frac{c-A}{J_m+J} = -6.23\ [s^{-1}]$</td><td colspan="2">$\lambda_c = -\frac{c-A+B*kp}{J_m+J}$</td></tr>
<tr><td>kp = 10
$\lambda_c = -74.98\ [s^{-1}]$</td><td>kp = 100
$\lambda_c = -693.72\ [s^{-1}]$</td></tr>
<tr><td>If A < 0: asymptotically stable system
If A > 0: we must check the condition c-A
Typically for DC electric motors A < 0 (our case)</td><td colspan="2">Increasing kp:
- we stabilise the system in the case of c-A < 0
- when c -A > 0, we make the system faster as $\tau = -\frac{1}{\lambda} \downarrow$ (our case)</td></tr>
<tr><td colspan="3">Steady state error considering a unitary step as input:
$\delta\omega_\infty = \frac{Bk_p}{c-A+Bk_p} < 1 \rightarrow e_\infty = 1 - \delta\omega_\infty \neq 0$</td></tr>
</table>

Tab. 2 – Time domain analysis of the 1 DOF system

Obviously the first order system response doesn’t oscillate, so we do not have overshoot problems.

We are going to consider the case with **c-A > 0**, stable uncontrolled system (A has a negative sign from the data of the problem - *Table 1*). Introducing a proportional controller, we can assess that it is not possible to make the system unstable.

ROOT LOCUS

The root locus of the 1 DOF system has a real negative pole, therefore the controlled system is asymptotically stable whatever kp value. As kp increase it results in a more stable and faster system since the pole moves toward more negative real values on the left, having asymptote $\theta_a = \pm\pi\ [n = 0, m = 1, k = 1, q = 1]$.

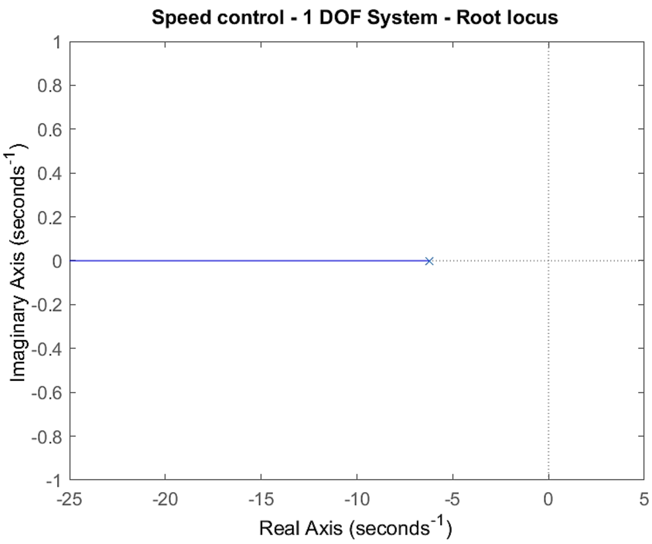
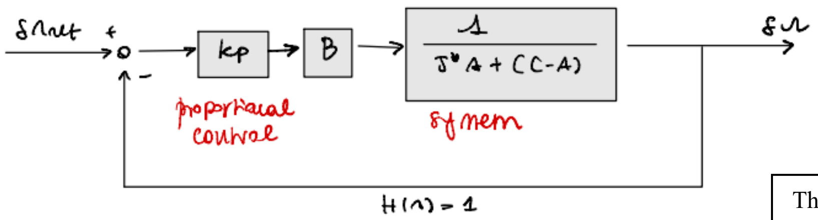


Fig.2 – Root locus 1 DOF system

In Laplace domain: $\delta\Omega = \mathcal{L}\{\delta\omega\}$

$$(J^* s + c - A + Bk_p)\ \delta\Omega = Bk_p\ \ \delta\Omega_{ref}$$



The pole p1 is the eigenvalue of the controlled system

$$GH = \frac{Bk_p}{J^* s + c - A} \qquad L = \frac{Bk_p}{J^* s + c - A + Bk_p}$$

Considering a stable uncontrolled system $Re\{p_1\} < 0$, we can apply the undirect Bode Criterion in order to define the stability conditions.

BODE DIAGRAM

kp = 10		kp = 100	
$GH = \dfrac{11}{0.16s + 0.996}$	$L = \dfrac{11}{0.16\ s + 12}$	$GH = \dfrac{110}{0.16s + 0.996}$	$L = \dfrac{110}{0.16\ s + 111}$
Concerning GH: $\mu = \frac{Bk_p}{c-A} > 0$ and $p_1 = -\frac{c-A}{J^*} < 0$			

Tab.3 – Open loop and closed loop transfer function 1 DOF system varying control gain from 10 to 100.

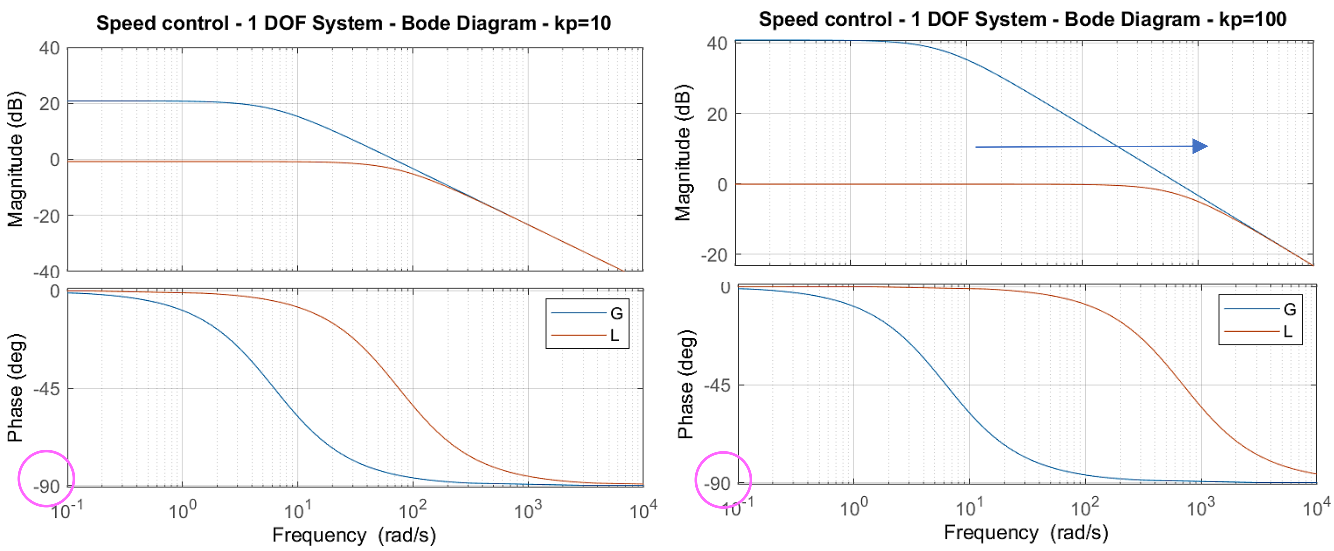
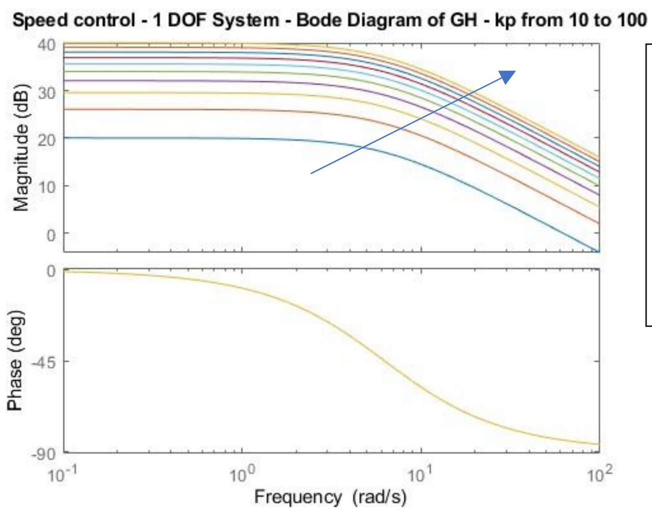


Fig.3 – Bode diagram of GH and L for the 1 DOF system having kp = 10 and kp = 100.



↑ kp:

- ↑ the magnitude diagram
- The phase doesn’t change
- ↑ ω_{BW}

Fig.4 – Bode diagram of GH for the 1 DOF system increasing progressively kp from 10 to 100.

Concerning the GH function, the Bode criterion says that:

$$\begin{cases} p_m = \measuredangle GH\left(\Omega = \Omega_{gc}\right) + \pi > 0 \\ |G_m|_{dB} \to \infty \end{cases} \quad \forall kp \rightarrow \text{stable}$$

The L function for low value of Ω is *almost* equal to 0 dB and it has almost 0 phase. This happens because $L(s=0) = \delta\omega_\infty \neq 0$, therefore, we have a steady state error; it is small, but it is not 0 as shown below:

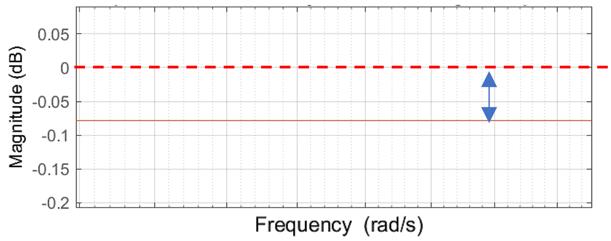
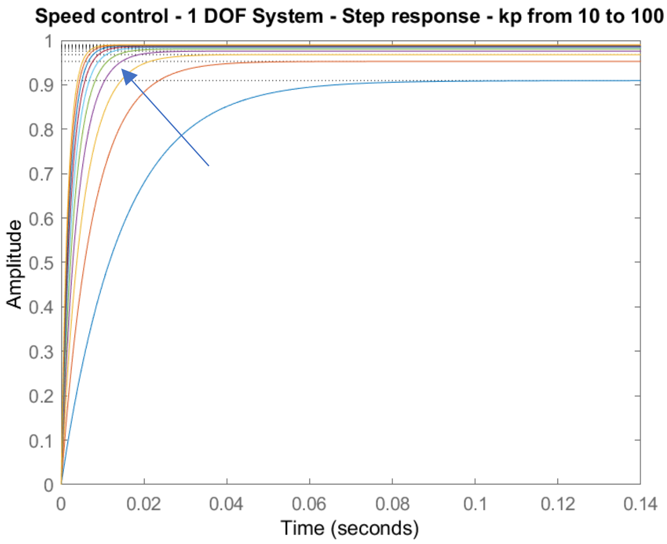


Fig.5 – Steady state error of the L function with kp = 100: the closed loop transfer function doesn’t have magnitude exactly equal to 1 in the bandwidth region.

From the step response of the controlled system, it can be seen that ↑ kp, ↓e_∞, ↓tr

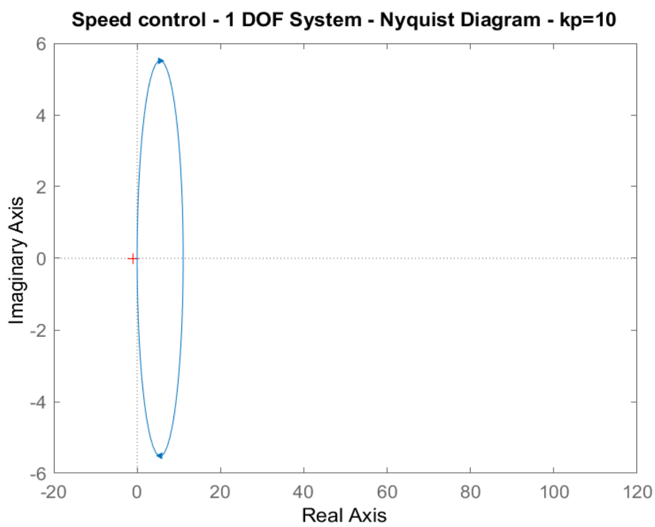


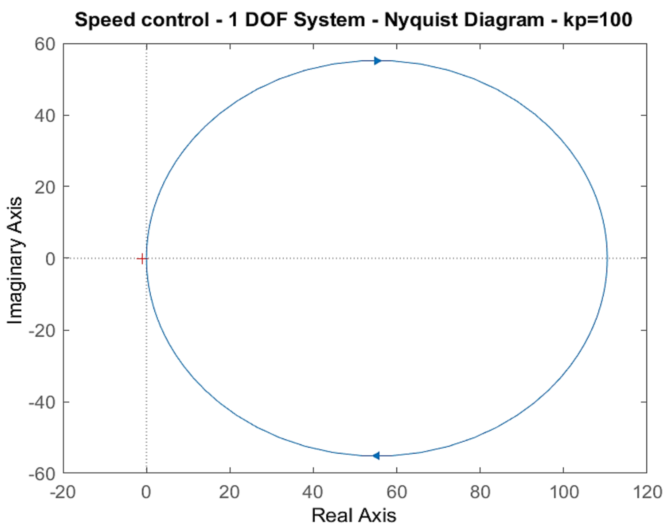
$$\boxed{\begin{aligned} \delta\omega(t) &= -\frac{Bk_p}{c - A + Bk_p} e^{-\lambda_c t} + \frac{Bk_p}{c - A + Bk_p} \\ &= \delta\omega_\infty\left(1 - e^{-\lambda_c t}\right), t \geq 0 \\ \delta\omega(0) &= 0 \end{aligned}}$$

Fig. 6 – Step response of the controlled 1 DOF system

NYQUIST DIAGRAM

$$\begin{cases} \tilde{N} = 0 \\ P = 0 \end{cases} \quad \forall kp \rightarrow \text{stable}$$





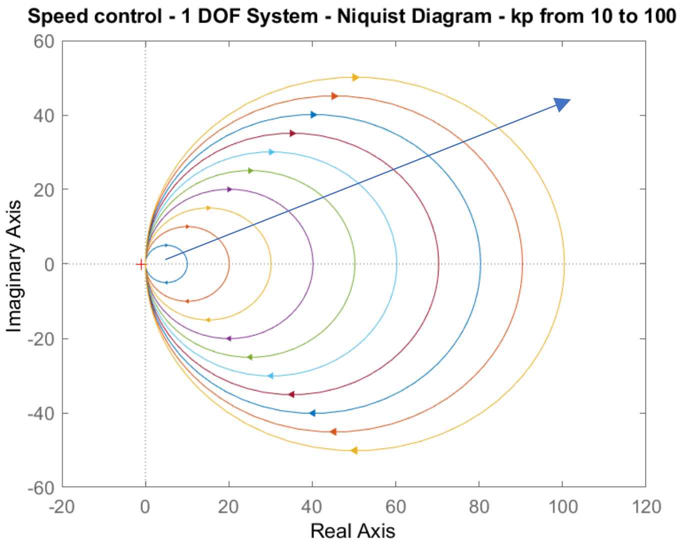
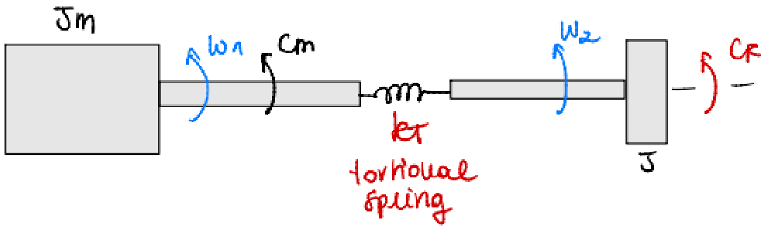


Fig.7 – Nyquist diagram 1 DOF system.

Whatever kp, the control system is asymptotically stable as a matter of fact the closed-loop system is always stable according to Nyquist criterion.

CASE B: Representing the flexibility of the shaft trough the torsional spring kt.

For high frequencies the flexibility of the shaft must be taken into account:



$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \delta\ddot{\vartheta}_1 \\ \delta\ddot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} c-A & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \delta\dot{\vartheta}_1 \\ \delta\dot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} k_T & -k_T \\ -k_T & k_T \end{bmatrix}\begin{bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}\delta y$$

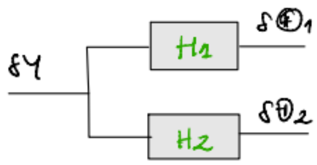
[M] is symmetric and definite positive

[C] is symmetric and since c-A has been defined from the data > 0, it is also definite positive

[K] is symmetric but not definite positive, since det([K]) = 0, therefore it is semi-definite positive (one pole will be in the origin) → the steady state condition is characterised by neutral equilibrium (labile system)

The proportional speed control will introduce an equivalent damping coefficient in the [C] matrix.

Passing in Laplace domain, we can derive the block diagram associated with the system, in green are highlighted the 2 important transfer functions H1 and H2:



$H1 = \dfrac{0.165\ s^2 + 0.011\ s + 1100}{0.0015\ s^4 + 0.1495\ s^3 + 160\ s^2 + 1006\ s}$	$H2 = \dfrac{1100}{0.0015\ s^4 + 0.1495\ s^3 + 160\ s^2 + 1006\ s}$
We have a: - pole in the origin $p_1 = 0$ - then a second real pole $p_2 = -\alpha_1$, - after we see the anti-resonance $\boldsymbol{z_{1,2} = -\alpha_2 \pm i\omega_{d2}}$ - Lastly, we have the second resonance, associated with the 2° mode of vibration $p_{3,4} = -\alpha_3 \pm i\ \omega_{d3}$.	We also have a pole in the origin followed by a real pole and the second resonance; but we <u>do not have the counter resonance</u> as in H1. - $p_1 = 0$ - $p_2 = -\alpha_1$ - $p_{3,4} = -\alpha_3 \pm i\ \omega_{d3}$.

Tab. 4 –Transfer functions associated with the 2 DOFs system

The bode diagrams associated are shown below:

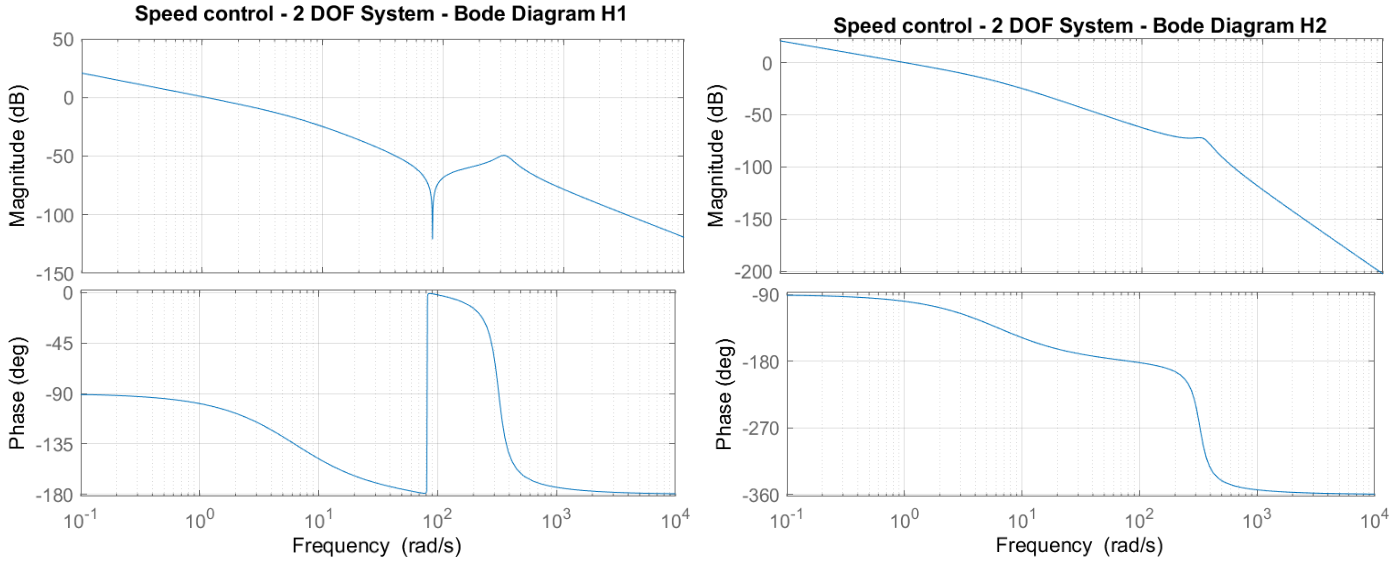


Fig. 8 –bode diagram associated with the transfer functions H1 and H2 of the 2 DOFs system

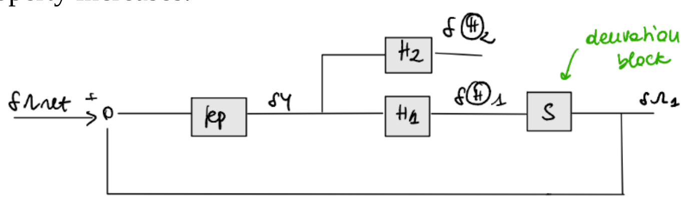
1) Colocated control

We apply the motor torque to the shaft whose variable is $\delta\vartheta_1$ and we control $\delta\vartheta_1$. The term introduced by the proportional control on $\delta\dot{\vartheta}_1$ is added on the C_{11} terms of the damping matrix (principal diagonal).

$$[C_T] = \begin{bmatrix} c - A + Bkp & 0 \\ 0 & c \end{bmatrix}$$

$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \delta\ddot{\vartheta}_1 \\ \delta\ddot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} c - A + \boldsymbol{Bkp} & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \delta\dot{\vartheta}_1 \\ \delta\dot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} k_T & -k_T \\ -k_T & k_T \end{bmatrix}\begin{bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}\delta y$$

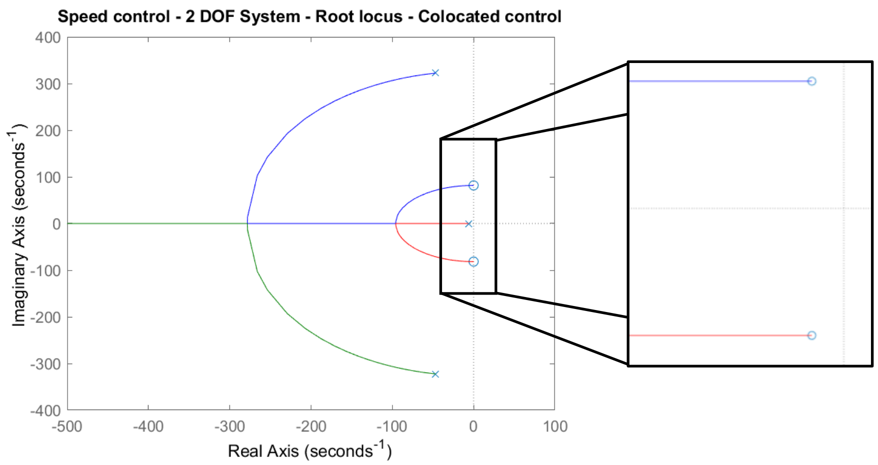
Increasing k_p, stability property increases.



$GH_1 = \dfrac{k_p B(J^* s^2 + cs + k_T)\cancel{s}}{\cancel{s}\ (a_4 s^4 + a_3 s^3 + a_2 s + a_1)}$	$L_1 = \dfrac{k_p B(J^* s^2 + cs + k_T)\cancel{s}}{\cancel{s}\ (a_4 s^4 + a_3 s^3 + a_2 s + a_1) + k_p B(J^* s^2 + cs + k_T)\cancel{s}}$
$GH_{1\mid kp=100} = \dfrac{16.5\ s^2 + 1.1\ s + 1.1\cdot 10^5}{0.0015\ s^3 + 0.1495\ s^2 + 160\ s + 1006}$	$L_{1\mid kp=100} = \dfrac{16.5\ s^2 + 1.1\ s + 1.1\cdot 10^5}{0.0015\ s^3 + 16.65\ s^2 + 161.1\ s + 1.11\cdot 10^5}$

Tab. 5 – Open and closed loop transfer function of the collocated control of the 2 DOFs system for kp = 100

ROOT LOCUS



$n = 3, m = 2, q = 1, k = 1$

$\rightarrow \theta_a = \pm\pi$

↑ kp we always get a stable system.

We would have had 2 possible root loci, and both would satisfy the rules we know. In both cases the system will be always stable.

Fig.9 – Root locus 2 DOF system collocated control

BODE DIAGRAM

- $\boldsymbol{\mu > 0}$
- $\boldsymbol{p_1} \in \mathbb{R}$: first mode of vibration with the mass in phase (rigid movement)
  - we had 2 real poles but one of them has been simplified (Pole in the origin)
- $\boldsymbol{p_{2,3}} \in \mathbb{C}$ conjugates: second mode of vibration mass in counter phase
- $\boldsymbol{z_{1,2}} \in \mathbb{C}$ conjugates: they represent the anti-resonance we have in 2 DOF systems

Generally: $|p_1| < |z_{1,2}| < |p_{2,3}|$

Since the uncontrolled system is stable, we can use Bode criterion to assess stability:

$$\begin{cases} p_m > 0 \\ |G_m|_{dB} \to \infty \end{cases} \quad \forall kp \rightarrow \text{stable}$$

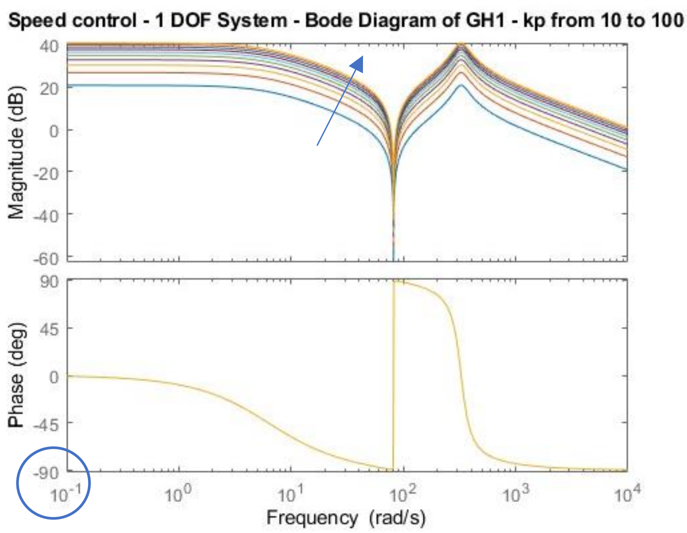


Fig. 10 – Bode diagram of the 2 DOFs system with a colocated control

NYQUIST DIAGRAM

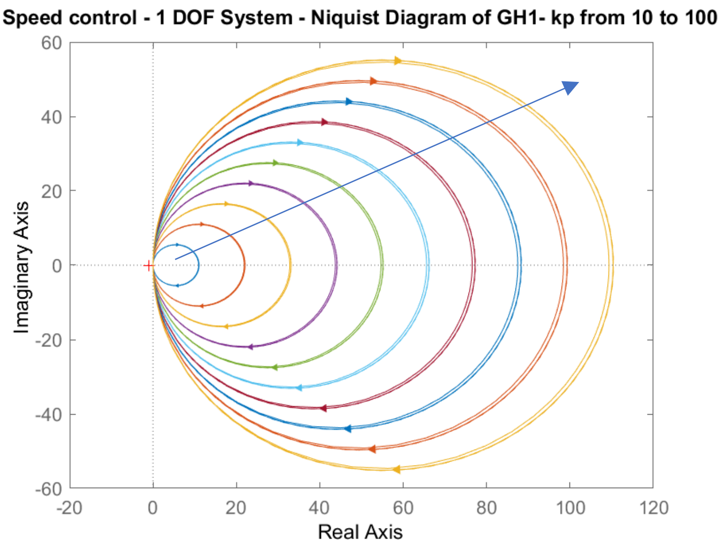


Fig. 11 – Nyquist diagram 2 DOF system with a colocated control

$$\begin{cases} \tilde{N} = 0 \\ P = 0 \end{cases} \quad \forall kp \rightarrow \text{stable}$$

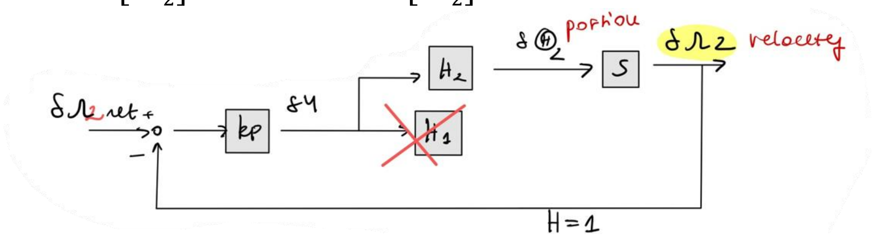
Whatever kp, the control system is asymptotically stable, as a matter of fact the closed-loop system is always stable according to Nyquist criterion. The diagram enlarges remaining in the right part of the diagram.

2) Non-colocated control

The term introduced by the proportional control on $\delta\dot{\vartheta}_2$ is added on the extra diagonal term C_{12} :

$$[C_T] = \begin{bmatrix} c - A & B * kp \\ 0 & c \end{bmatrix}$$

$$\begin{bmatrix} J_m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \delta\ddot{\vartheta}_1 \\ \delta\ddot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} c - A & \boldsymbol{Bkp} \\ 0 & c \end{bmatrix}\begin{bmatrix} \delta\dot{\vartheta}_1 \\ \delta\dot{\vartheta}_2 \end{bmatrix} + \begin{bmatrix} k_T & -k_T \\ -k_T & k_T \end{bmatrix}\begin{bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}\delta y$$



It is no more symmetric, therefor the system can be no more dissipative.

$GH_2 = \dfrac{k_p B\, k_T\, \cancel{s}}{\cancel{s}\,(a_4 s^4 + a_3 s^3 + a_2 s + a_1)}$	$L_2 = \dfrac{k_p B\, k_T\, \cancel{s}}{\cancel{s}\,(a_4 s^4 + a_3 s^3 + a_2 s + a_1) + k_p B\, k_T\, \cancel{s}}$
$GH_{2\|\,kp=100} = \dfrac{1.1\cdot 10^5}{0.0015\, s^3 + 0.1495\, s^2 + 160\, s + 1006}$	$L_{2\|\,kp=100} = \dfrac{1.11\cdot 10^5}{0.0015\, s^3 + 0.1495\, s^2 + 160\, s + 1.11\cdot 10^5}$

Tab. 6 – Open and closed loop transfer function of the non colocated control of the 2 DOFs system for kp = 100

ROOT LOCUS

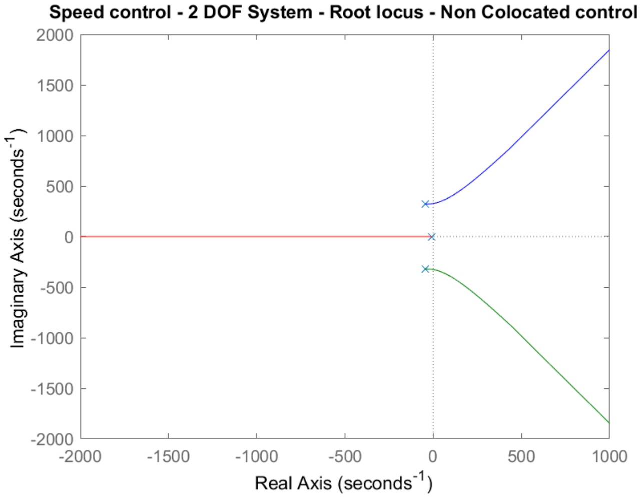


Fig.12 – Root locus 2 DOF system non collocated control

$n = 3, m = 0, q = 3, k = 1, 3$

$\rightarrow \vartheta_a = \pm\pi \; and \pm\frac{\pi}{3}$

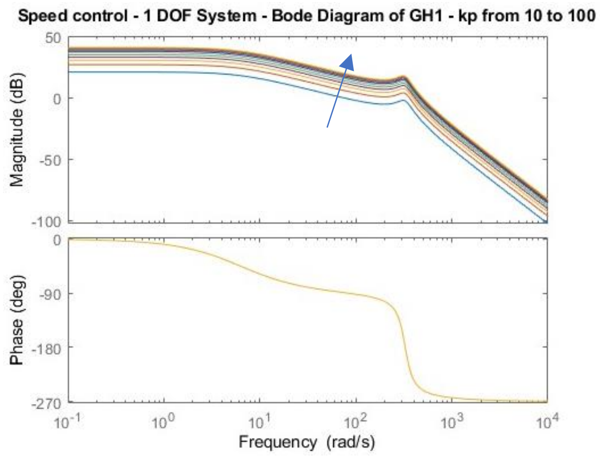
- $\downarrow kp$ → stable
- $\uparrow kp$ → unstable (dynamic instability → CC poles with Re > 0)

BODE DIAGRAM

- $\boldsymbol{\mu > 0}$
- $\boldsymbol{p_1} \in \mathbb{R}$
- $\boldsymbol{p_{2,3}} \in \mathbb{C}$
  - For the same system, the transfer functions have the same poles
- $\boldsymbol{z}$: NO, we do not have the anti-resonance in H2

$$\begin{cases} p_m > 0 \\ |G_m|_{dB} = -|GH(\Omega_{pc})| > 0 \end{cases} \quad \downarrow kp \rightarrow \text{stable}$$

$$\begin{cases} p_m < 0 \\ |G_m|_{dB} = -|GH(\Omega_{pc})| < 0 \end{cases} \quad \uparrow kp \rightarrow \text{unstable}$$



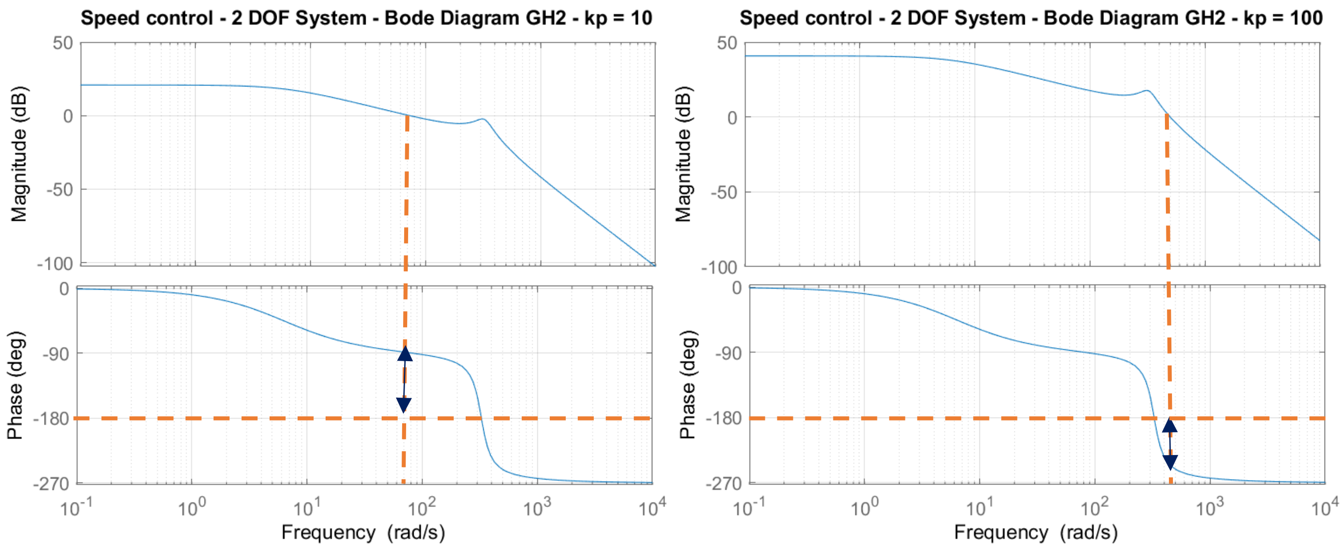
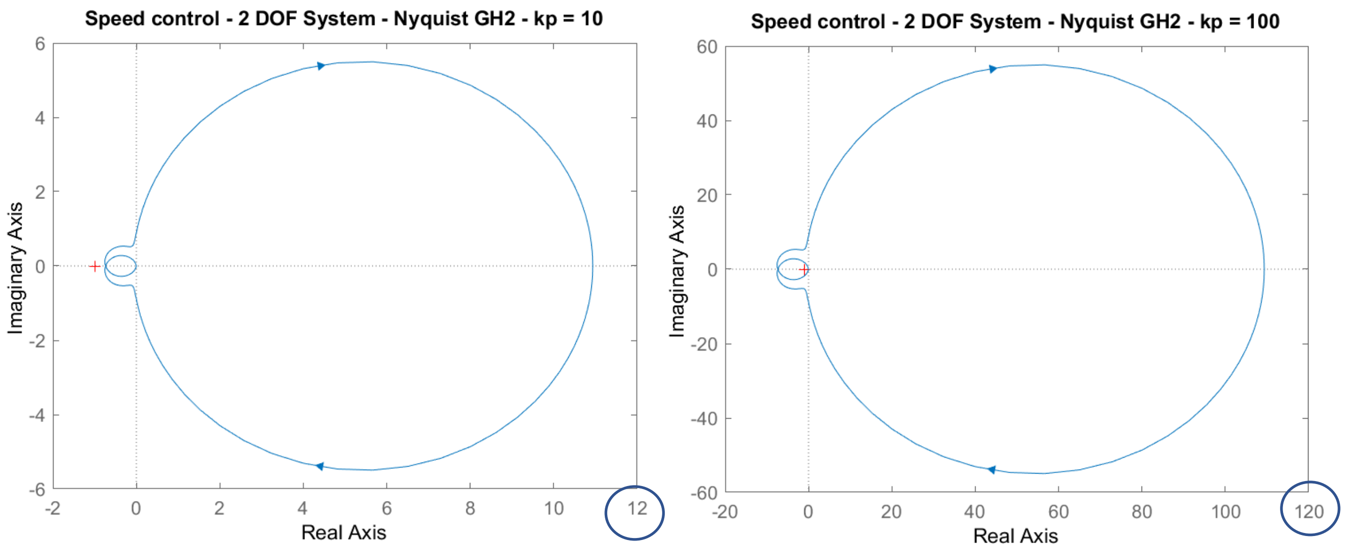
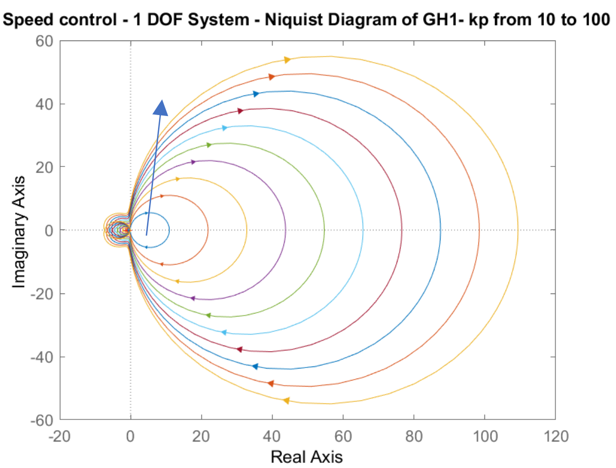


Fig. 13 – Bode diagram of GH2 in the 2 DOFs system with a non-colocated control

Stability is conditioned by the value that we choose for k_p. We have a limit in the pass band of the system, otherwise it becomes unstable. The limit value of kp for which the system become unstable can be detected from the root locus.

NYQUIST DIAGRAM





$$\begin{cases} \tilde{N} = 0 \\ P = 0 \end{cases} \quad \downarrow kp \rightarrow \text{stable}$$

$$\begin{cases} \tilde{N} = -2 \\ P = 0 \end{cases} \quad \uparrow kp \rightarrow \text{unstable}$$

Fig.13 – Nyquist diagram 2 DOF system with a not colocated control

$\uparrow kp$ the Nyquist diagram encircles 2 times the point (-1, 0) in clockwise direction leading to instability since GH_2 do not have any unstable poles.