Flow Over a Wavy Bedform

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1 Equations of Motion and Continuity Equation

We begin by considering flow in a stream with a wavy, sinusoidal bedform of infinitesimal amplitude. The channel is straight and of uniform width such that we can neglect flow near the banks. Water flows in the positive x direction, the z direction is with water depth, and the y direction is across the stream from bank to bank. Neglecting flow near the banks, we need only consider two of the Navier-Stokes equations and a simplified continuity equation,

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial x} - g\frac{\partial z_0}{\partial x} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial z}\tau_{zx}\Big) \tag{1}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial z} - g\frac{\partial z_0}{\partial z} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial z}\tau_{zz}\Big) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

where z_0 is a vertical coordinate in the earth's reference frame.

2 Simplifying the Advective Terms

Note that, by the chain rule,

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + u\frac{\partial w}{\partial z} \tag{4}$$

and, rearranging,

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = u\Big(\frac{\partial u}{\partial x} + u\frac{\partial w}{\partial z}\Big) + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}. \tag{5}$$

Noting that the parenthetical term is our continuity equation 3 and is equal to 0,

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}, \tag{6}$$

and we can replace the advective terms in equation 1 with the left side of equation 6.

We can repeat a similar process,

$$\frac{\partial}{\partial x}(uw)+\frac{\partial}{\partial z}(w^2) = u\frac{\partial w}{\partial x}+w\frac{\partial u}{\partial x}+w\frac{\partial w}{\partial z}+w\frac{\partial w}{\partial z} = w\Big(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\Big)+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} = u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}. \tag{7}$$

Using equation 6 and 7 we rewrite equations 1 and 2,

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = \frac{-1}{\rho}\frac{\partial p}{\partial x} - g\frac{\partial z_0}{\partial x} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial z}\tau_{zx}\Big) \tag{8}$$

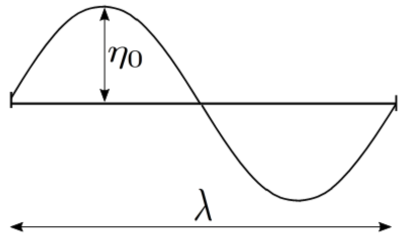

Figure 1: Sinusoidal bedform with amplitude η_0 and wavelength λ.

$$\frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial z}(w^2) = \frac{-1}{\rho}\frac{\partial p}{\partial z} - g\frac{\partial z_0}{\partial z} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial z}\tau_{zz}\Big) \tag{9}$$

We now want to see if there quantities in our equations that might be insignificant in comparison to others. Let U_* be a characteristic streamwise velocity (parallel to the x axis) and W_* be a characteristic velocity normal to the stream bed (parallel to the z axis). We wish to nondimensionalize the continuity equation using U_*, W_*, and additional characteristic quantities. Note that water along the bottom of the stream, when travelling from the bottom to the top of a sinusoidal wave form must travel a distance $\lambda/2$ streamwise, where λ is half the wavelength, and a distance $2\eta_0$, where η is the amplitude. The continuity equation can thus be nondimensionalized

$$\frac{2U_*}{\lambda}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{W_*}{2\eta_0}\frac{\partial \hat{w}}{\partial \hat{z}}. \tag{10}$$

Recognizing that the two partial derivative terms are non-dimensional and thus vary between 0 and 1, we know that W_* is of the order $\frac{4\eta_0 U_*}{\lambda}$. For a bedform of very small or infinitesimal amplitude, $\eta_0 << \lambda$ and thus, from the dimensional argument above, we expect that $W_* << U_*$. Comparing the advective terms in equations 8 and 9, we also conclude that $\frac{\partial}{\partial x}(uw) << \frac{\partial}{\partial x}(u^2)$ and $\frac{\partial}{\partial z} << \frac{\partial}{\partial x}(u^2)$. Thus the advective terms from equation 9 can be removed and our equations of motion are now

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = \frac{-1}{\rho}\frac{\partial p}{\partial x} - g\frac{\partial z_0}{\partial x} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial z}\tau_{zx}\Big) \tag{11}$$

$$0 = \frac{-1}{\rho}\frac{\partial p}{\partial z} - g\frac{\partial z_0}{\partial z} + \frac{1}{\rho}\Big(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial z}\tau_{zz}\Big) \tag{12}$$

We can assume that at a chosen position z, $\tau_{xx} = -\rho\overline{u'^2}$, $\tau_{zx} = -\rho\overline{u'w'}$, and τ_{xx} and τ_{zx} are of the same order. Noting that with the wavy bedform of infinitesimal amplitude, $\eta_0 << \lambda$, both stress derivative terms, $\frac{\partial \tau_{xx}}{\partial x}$ and $\frac{\partial \tau_{xz}}{\partial x}$ are insignificant and the equations of motion can be approximated

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial z}(uw) = \frac{-1}{\rho}\frac{\partial p}{\partial x} - g\frac{\partial z_0}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{13}$$

$$0 = \frac{-1}{\rho}\frac{\partial p}{\partial z} - g\frac{\partial z_0}{\partial z} + \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z} \tag{14}$$

3 Depth-Integrated Continuity Equation

We begin by noting that water depth is a function of (x, y), $h = h(x, y)$. We can rewrite the continuity equation 3

$$\frac{\partial}{\partial x}u(x, y, z) + \frac{\partial}{\partial z}w(x, y, z) = 0. \tag{15}$$

We now integrate over the depth of flow, from the bed, $z = \eta$, to the water surface, $z = \zeta$, where $h = \zeta - \eta$,

$$\int_\eta^\zeta \frac{\partial}{\partial x} u(x, y, z) dz + \int_\eta^\zeta \frac{\partial}{\partial z} w(x, y, z) dz = 0. \tag{16}$$

Expanding the second term,

$$\int_\eta^\zeta \frac{\partial}{\partial z} w(x, y, z) dz = w(x, y, z)\Big|_\eta^\zeta = w(x, y, \zeta) - w(x, y, \eta), \tag{17}$$

Consider $w(x, y, \zeta)$, which represents the vertical component of velocity at the water surface. Thinking in a Lagrangian manner, we can envision the motion of a particle at the water surface with position (x, y, ζ). The water surface is a steady function of (x, y), $\zeta(x, y)$. The particle motion can be described $x = x(t)$ and $y = y(t)$, thus $\zeta = \zeta(x(t), y(t))$. The vertical component of the particle velocity is $\frac{\partial \zeta}{\partial t}$ and it must equal the Eulerian velocity component, $w(x, y, \zeta)$, thus

$$w(x, y, \zeta) = \frac{\mathrm{d}}{\mathrm{d}t} \zeta(x(t), y(t)) = \frac{\partial \zeta}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \zeta}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t}. \tag{18}$$

Note that $\frac{\mathrm{d}x}{\mathrm{d}t} = u(x, y, \zeta)$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = v(x, y, \zeta)$ so we can rewrite $w(x, y, \zeta)$

$$w(x, y, \zeta) = \frac{\partial \zeta}{\partial x} u(x, y, \zeta) + \frac{\partial \zeta}{\partial y} v(x, y, \zeta). \tag{19}$$

Because we are ignoring flow in the cross-stream direction (parallel to they y axis), $v(x, y, \zeta) = 0$, and we are left with

$$w(x, y, \zeta) = \frac{\partial \zeta}{\partial x} u(x, y, \zeta) \tag{20}$$

which can be rewritten

$$w_\zeta = u_\zeta \frac{\partial \zeta}{\partial x} \tag{21}$$

Rewriting equation 16 with equations 17 and 21,

$$\int_\eta^\zeta \frac{\partial}{\partial x} u(x, y, z) dz + w_\zeta - w(x, y, \eta) = 0. \tag{22}$$

Because of the no slip condition, $w(x, y, \eta) = 0$, and we are left with

$$\int_\eta^\zeta \frac{\partial}{\partial x} u(x, y, z) dz + w_\zeta = 0. \tag{23}$$

The first term contains two independent variables, x and z, and y is treated as a constant. Because the order of integration/differentiation does not matter, we rewrite the first term

$$\frac{\partial}{\partial x} \int_\eta^\zeta u(x, y, z) dz \tag{24}$$

and the Leibniz rule tells us

$$\frac{\partial}{\partial x} \int_{\eta(x,y)}^{\zeta(x,y)} u(x, y, z) dz = \int_{\eta(x,y)}^{\zeta(x,y)} \frac{\partial u}{\partial x} dz + u(x, y, \zeta) \frac{\mathrm{d}\zeta}{\mathrm{d}x} - u(x, y, \eta) \frac{\mathrm{d}\eta}{\mathrm{d}x}. \tag{25}$$

The no-slip condition tells us that $u(x, y, \eta) = 0$. Rearranging equation 25 we get

$$\int_{\eta(x,y)}^{\zeta(x,y)} \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_{\eta(x,y)}^{\zeta(x,y)} u(x, y, z) dz - u(x, y, \zeta) \frac{\mathrm{d}\zeta}{\mathrm{d}x}. \tag{26}$$

Recognize that the second term on the right side is equal to equation 21. Plugging equation 26 into equation 23, the two w_ζ terms cancel out and we are left with

$$\frac{\partial}{\partial x} \int_{\eta(x,y)}^{\zeta(x,y)} u(x, y, z) dz = 0. \tag{27}$$

Regardless of how ζ and η vary with (x, y), the integral quantity may be repalce with the product of $h = \zeta - \eta$ and the vertically integrated streamwise velocity, u_0, thus

$$\frac{\partial}{\partial x}(hu_0) = 0. \tag{28}$$

For simplicity, from this point forward we allow u to represent the depth-averaged streamwise velocity, u_0. We are left with two main take-aways from our depth averaging the continuity equation:

$$\frac{\partial}{\partial x}(hu) = 0 \quad \text{and} \quad w_\zeta = u_\zeta \frac{\partial \zeta}{\partial x} \tag{29}$$

4 Depth-Integrated Advective Terms

We depth-integrate the first term on the right side of equation 13 and apply the Leibniz rule,

$$\int_\eta^\zeta \frac{\partial}{\partial x}(u^2) dz = \frac{\partial}{\partial x} \int_\eta^\zeta u^2 dz - u_\zeta^2 \frac{\partial \zeta}{\partial x} + u_\eta^2 \frac{\partial \eta}{\partial x}. \tag{30}$$

and depth-integrate the second term on the right side of equation 13,

$$\int_\eta^\zeta \frac{\partial}{\partial z}(uw) dz = uw\Big|_\eta^\zeta = u_\zeta w_\zeta - u_\eta w_\eta. \tag{31}$$

The first two terms in equation 13 can be replaced with equations 30 and 31 and become

$$\frac{\partial}{\partial x} \int_\eta^\zeta u^2 dz - u_\zeta^2 \frac{\partial \zeta}{\partial x} + u_\eta^2 \frac{\partial \eta}{\partial x} + u_\zeta w_\zeta - u_\eta w_\eta. \tag{32}$$

which, because the no-slip condition means $u_\eta = 0$ and $w_\eta = 0$, simplifies to

$$\frac{\partial}{\partial x} \int_\eta^\zeta u^2 dz - u_\zeta^2 \frac{\partial \zeta}{\partial x} + u_\zeta w_\zeta \tag{33}$$

or

$$\frac{\partial}{\partial x} \int_\eta^\zeta u^2 dz - u_\zeta \Big(u_\zeta \frac{\partial \zeta}{\partial x} - w_\zeta \Big). \tag{34}$$

Plugging in w_ζ from equation 29 removes the parenthetical term and using the mean value theorem,

$$\frac{\partial}{\partial x} \int_\eta^\zeta u^2 dz = \frac{\partial}{\partial x}(h\langle u^2 \rangle) \tag{35}$$

where $\langle u^2 \rangle$ is the depth averaged square of the streamwise velocity. If we assume that $\langle u^2 \rangle = \langle u \rangle^2$ we rewrite 35

$$\frac{\partial}{\partial x}(hu^2) \tag{36}$$

where u now represents the depth averaged streamwise velocity. Using the chain rule we find

$$\frac{\partial}{\partial x}(hu^2) = \frac{\partial u}{\partial x}(hu) + u\frac{\partial}{\partial x}(hu), \tag{37}$$

and with equation 29 we can remove the second term on the right hand side such that

$$\frac{\partial}{\partial x}(hu^2) = \frac{\partial u}{\partial x}(hu). \tag{38}$$

Thus the two non-linear advective terms on the left side of equation 13 can be approximated

$$hu\frac{\partial u}{\partial x} \tag{39}$$

5 Depth-Integrated Stress Terms

We depth integrate the stress term in equation 13

$$\int_{\eta}^{\zeta} \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} dz = \frac{1}{\rho}\tau_{zx}\Big|_{\eta}^{\zeta} = \frac{1}{\rho}(\tau_{\zeta x} - \tau_{\eta x}). \tag{40}$$

If we treat the water surface as a boundary across which no momentum can be diffused, then the water surface is a zero-stress boundary, and the stress term in equation 13 simplifies further to

$$-\frac{\tau_{\eta x}}{\rho} \tag{41}$$

where $\tau_{\eta x}$ represents stress at the stream bed.

Depth integrating the stress term in equation 14,

$$\int_{\eta}^{\zeta} \frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z} dz = \frac{1}{\rho}\tau_{zz}\Big|_{\eta}^{\zeta} = \frac{1}{\rho}(\tau_{\zeta z} - \tau_{\eta z}). \tag{42}$$

Given that $\tau_{zz} = -\rho\overline{w'^2}$ and $\overline{w'^2}$ must vanish and the water surface and stream bed, both $\tau_{\zeta z} = 0$ and $\tau_{\eta z} = 0$, and the stress term in equation 14 goes to 0.

6 Depth-Integrated Gravitational Term

Note that $\frac{\partial z_0}{\partial x} = sin\theta$ where θ is the angle of incline of the streambed, the angle at which the x axis is inclined from earth's horizontal plane. Integrating the gravitational term in equation 13,

$$\int_{\eta}^{\zeta} -g\frac{\partial z_0}{\partial x} dz = -g\frac{\partial z_0}{\partial x}\int_{\eta}^{\zeta} dz = -g\frac{\partial z_0}{\partial x} z\Big|_{\eta}^{\zeta} = -gh\frac{\partial z_0}{\partial x} = -ghsin\theta. \tag{43}$$

7 Depth-Integrated Pressure Term

Note that equation 14, ignoring the stress term which, as we determined in section 5, goes to 0, is simply the hydrostatic equation altered by a factor $\frac{\partial z_0}{\partial z} = cos\theta$, where θ is negative,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - gcos\theta \tag{44}$$

or

$$\frac{\partial p}{\partial z} = -\rho gcos\theta. \tag{45}$$

Integrating with respect to z,

$$p = -\rho gcos\theta z + C_1 \tag{46}$$

applying the boundary condition that at the water surface, $z = \zeta$, $p = p_0$, where p_0 is atmospheric pressure,

$$p_0 = -\rho gcos\theta\zeta + C_1 \tag{47}$$

$$C_1 = p_0 + \rho gcos\theta\zeta \tag{48}$$

$$p = p_0 + \rho gcos\theta(\zeta - z) \tag{49}$$

We then differentiate with respect to x, noting that z is independent of x and ζ is dependent on x,

$$\frac{\partial p}{\partial x} = \frac{\partial p_0}{\partial x} + \rho gcos\theta(\frac{\partial \zeta}{\partial x} - \frac{\partial z}{\partial x}) \tag{50}$$

$$\frac{\partial p}{\partial x} = \rho gcos\theta\frac{\partial \zeta}{\partial x}. \tag{51}$$

We can substitute equation 51 into the pressure term in equation 13. Depth-integrating it,

$$\int_{\eta}^{\zeta} -\frac{1}{\rho}\frac{\partial p}{\partial x}dz = \int_{\eta}^{\zeta} -\frac{1}{\rho}\rho gcos\theta\frac{\partial \zeta}{\partial x}dz = -gcos\theta\frac{\partial \zeta}{\partial x}\int_{\eta}^{\zeta} dz = -gcos\theta h\frac{\partial \zeta}{\partial x}. \tag{52}$$

8 New Equations of Motion

Combining the work of previous sections, we can rewrite the equation of motion 13. The depth integrated stress terms on the left side are simplified to equation 39, the depth integrated pressure term is simplified to equation 52, the depth integrated gravitational term is simplified to equation 43, and the depth integrated stress term is simplified to equation 41, and our depth integrated equation of motion for the x direction is

$$hu\frac{\partial u}{\partial x} = -ghcos\theta\frac{\partial \zeta}{\partial x} - ghsin\theta - \frac{\tau_{\eta x}}{\rho} \tag{53}$$

and dividing through by h,

$$u\frac{\partial u}{\partial x} = -gcos\theta\frac{\partial \zeta}{\partial x} - gsin\theta - \frac{\tau_{\eta x}}{h\rho}. \tag{54}$$

In the engineering literature, it is common to dscribe bed stress in terms of velocity. The stress term $\tau_{\eta x}$ can be represented

$$\tau_{\eta x} = \rho\Gamma u^2 \tag{55}$$

where Γ is a friction coefficient similar to the Darcy-Weisbach coefficient. Thus our final depth-integrated equation of motion for the x direction is

$$u\frac{\partial u}{\partial x} = -gcos\theta\frac{\partial \zeta}{\partial x} - gsin\theta - \frac{\Gamma u^2}{h}. \tag{56}$$

This equation and the depth-integrated continuity equation,

$$\frac{\partial}{\partial x}(hu) = 0, \tag{57}$$

will allow us to solve the remainder of the problem.

9 Linearization

Equations 56 and 57 approximate the system that we want to solve for but are unfortunately nonlinear. We now linearize the equations. We perform Reynold's averaging on the variables u, h, ζ, and η where the averaging is over a many bedform wavelengths, λ. Reynolds averaging introduces reach-averaged values U, H, Z, and H and perturbations u', h', ζ', and η' such that

$$u = U + u', \quad h = H + h', \quad \zeta = Z + \zeta', \quad \eta = H + \eta'. \tag{58}$$

The averaged values are *zeroth-order* and the perturbed values are first-order values. This allows us to make a dimensional argument with our equations. Say that H and U are on O(1). If we assume that perturbations over the stream reach are on O(0.1), we will later be able to approximate to the zeroth-order momentum balance (equations of motion) by removing terms involving first-order terms. This would involve an error of at most 10%.

9.1 Linearization of the Equation of Motion (56)

We proceed by plugging the expressions 58 into equation of motion 56 and noting that $h' = \zeta' - \eta'$

$$(U + u')\frac{\partial}{\partial x}(U + u') + gcos\theta\frac{\partial}{\partial x}(Z + \zeta') + gsin\theta + \frac{\Gamma(U + u')^2}{H + \zeta' - \eta'} = 0. \tag{59}$$

Multiplying some things out...

$$(U + u')\Big(\frac{\partial U}{\partial x} + \frac{\partial u'}{\partial x}\Big) + gcos\theta\Big(\frac{\partial Z}{\partial x} + \frac{\partial \zeta'}{\partial x}\Big) + gsin\theta + \frac{\Gamma}{H + \zeta' - \eta'}\Big(U^2 + 2Uu' + u'^2\Big) = 0. \tag{60}$$

Our first step at simplifying this mess is recognizing that derivatives of zeroth-order terms are equal to zero. This leaves us with the (marginally) simpler

$$(U + u')\Big(\frac{\partial u'}{\partial x}\Big) + gcos\theta\Big(\frac{\partial \zeta'}{\partial x}\Big) + gsin\theta + \frac{\Gamma}{H + \zeta' - \eta'}\Big(U^2 + 2Uu' + u'^2\Big) = 0. \tag{61}$$

Multiplying out...

$$U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} + gcos\theta\frac{\partial \zeta'}{\partial x} + gsin\theta + \frac{\Gamma}{H + \zeta' - \eta'}\Big(U^2 + 2Uu' + u'^2\Big) = 0. \tag{62}$$

We can simplify again by recognizing that if first-order terms are on O(0.1) then terms involving the product of two first-order terms, which may be called second-order terms, are on O(0.01). Thus an acceptable approximation can ignore second-order terms. We remove these and find

$$U\frac{\partial u'}{\partial x} + gcos\theta\frac{\partial \zeta'}{\partial x} + gsin\theta + \frac{\Gamma}{H+\zeta'-\eta'}\Big(U^2 + 2Uu'\Big) = 0. \tag{63}$$

Now we multiply through by the denominator in the fourth term (on the left hand side),

$$U\frac{\partial u'}{\partial x}(H+\zeta'-\eta') + gcos\theta\frac{\partial \zeta'}{\partial x}(H+\zeta'-\eta') + gsin\theta(H+\zeta'-\eta') + \Gamma\Big(U^2 + 2Uu'\Big) = 0 \tag{64}$$

multiply out again...

$$\begin{aligned} HU\frac{\partial u'}{\partial x} + \zeta' U\frac{\partial u'}{\partial x} - \eta' U\frac{\partial u'}{\partial x} + Hgcos\theta\frac{\partial \zeta'}{\partial x} + \zeta' gcos\theta\frac{\partial \zeta'}{\partial x} - \eta' gcos\theta\frac{\partial \zeta'}{\partial x} + \\ Hgsin\theta + \zeta' gsin\theta - \eta' gsin\theta + \Gamma U^2 + 2\Gamma Uu' = 0. \end{aligned} \tag{65}$$

Again, we can remove second-order terms, and we are left with

$$HU\frac{\partial u'}{\partial x} + Hgcos\theta\frac{\partial \zeta'}{\partial x} + Hgsin\theta + \zeta' gsin\theta - \eta' gsin\theta + \Gamma U^2 + 2\Gamma Uu' = 0. \tag{66}$$

A zeroth-order momentum balance, which describes the average momentum balance, involves only the two zeroth-order terms in the equation above,

$$Hgsin\theta + \Gamma U^2 = 0. \tag{67}$$

Using this, we can reduce equation 66 to a first-order momentum balance,

$$HU\frac{\partial u'}{\partial x} + Hgcos\theta\frac{\partial \zeta'}{\partial x} + \zeta' gsin\theta - \eta' gsin\theta + 2\Gamma Uu' = 0. \tag{68}$$

9.2 Linearization of the Continuity Equation (57)

We proceed by plugging the expressions 58 into the depth-averaged continuity equation 57,

$$\frac{\partial}{\partial x}\Bigg((H+h')(U+u')\Bigg) = 0, \tag{69}$$

$$\frac{\partial}{\partial x}\Bigg(HU + Hu' + Uh' + h'u'\Bigg) = 0, \tag{70}$$

$$\frac{\partial}{\partial x}(HU) + \frac{\partial}{\partial x}(Hu') + \frac{\partial}{\partial x}(Uh') + \frac{\partial}{\partial x}(h'u') = 0, \tag{71}$$

$$H\frac{\partial U}{\partial x} + U\frac{\partial H}{\partial x} + H\frac{\partial u'}{\partial x} + u'\frac{\partial H}{\partial x} + U\frac{\partial h'}{\partial x} + h'\frac{\partial U}{\partial x} + h'\frac{\partial u'}{\partial x} + u'\frac{\partial h'}{\partial x} = 0, \tag{72}$$

Removing derivatives of zeroth-order terms (which equal 0),

$$H\frac{\partial u'}{\partial x} + U\frac{\partial h'}{\partial x} + h'\frac{\partial u'}{\partial x} + u'\frac{\partial h'}{\partial x} = 0. \tag{73}$$

The third and fourth terms on the left hand side are second-order, so we remove them, and we also plug in $h' = \zeta' - \eta'$,

$$H\frac{\partial u'}{\partial x} + U\frac{\partial}{\partial x}(\zeta' - \eta') = 0 \tag{74}$$

and we are left with

$$H\frac{\partial u'}{\partial x} + U\frac{\partial \zeta'}{\partial x} - U\frac{\partial \eta'}{\partial x} = 0, \tag{75}$$

a first-order expression of continuity. Note that there is no zeroth-order expression of continuity.

10 Final Equations

Rearranging equation 67

$$gsin\theta = -\frac{\Gamma U^2}{H}, \tag{76}$$

we can replace the $gsin\theta$ terms in equation 68 and we are left with

$$HU\frac{\partial u'}{\partial x} + Hgcos\theta\frac{\partial \zeta'}{\partial x} - \zeta'\frac{\Gamma U^2}{H} + \eta'\frac{\Gamma U^2}{H} + 2\Gamma U u' = 0. \tag{77}$$

Recognizing that for small values of θ, $cos\theta = 1$, dividing through by HU^2, and moving the term containing η' to the right side, we are left with a final momentum balance equation that we will solve,

$$\boxed{\frac{1}{U}\frac{\partial u'}{\partial x} + \frac{g}{U^2}\frac{\partial \zeta'}{\partial x} - \frac{\Gamma}{H^2}\zeta' + \frac{2\Gamma}{HU}u' = -\frac{\Gamma}{H^2}\eta'.} \tag{78}$$

Dividing our first-order expression of continuity (equation 75) through by HU and moving the term involving η' to the right side, we are left with a final continuity equation that we will solve,

$$\boxed{\frac{1}{U}\frac{\partial u'}{\partial x} + \frac{1}{H}\frac{\partial \zeta'}{\partial x} = \frac{1}{H}\frac{\partial \eta'}{\partial x}.} \tag{79}$$

11 Solutions using Euler's Identity

Recall that our goal in this problem is to understand the dynamics of flow over a wavy bedform. To solve equations 78 and 79 we specify the bedform shape, η', and then solve for the fluid velocity and the height of the flow above the bedform, u' and ζ'. We want to decribe the bedform as a sinusoid and we can do this in two ways. The first involves a simple sinuisoidal function (we use cosine here although using sine would work just as well),

$$\eta' = \eta_0 cos(\omega x). \tag{80}$$

The second way involves using Euler's identity, which indicates that $e^{i\omega x} = cos(\omega x) + isin(\omega x)$.

$$\eta' = \eta_0 e^{i\omega x}. \tag{81}$$

Thus the two forms above are equivalent apart from a complex term, which, as we will see, can factor out of all equations. We use the second form because the use of Euler's identity actually makes solving for u' and ζ' easier. Familiarity with linear differential equations (such as 78 and 79) indicates that the solutions of ζ' and u' will also be sinusoids and so we assume solutions similar to the form that we used to describe bedform,

$$\zeta' = \zeta_0 e^{i\omega x} \quad \text{and} \quad u' = u_0 e^{i\omega x} \tag{82}$$

Note that the terms η_0, ζ_0, and u_0 are amplitudes and ω is a wavenumber and $\omega = 2\pi/\lambda$ where, as before, λ is the wavelength of the sinusoidal bedform.

We will later see that our solutions take the form

$$\eta' = \eta_0 cos(\omega x), \quad \zeta' = \zeta_0 cos(\omega x + \phi_\zeta) \quad \text{and} \quad u' = u_0 cos(\omega x + \phi_u) \tag{83}$$

where the leading coefficients are amplitudes and ϕ_ζ and ϕ_u are phase angles of the sinusoid represented by the $e^{i\omega x}$ term via Euler's rule. First we will find the amplitudes for the above equations and then we will find the phase angles.

11.1 Solving for the Amplitudes

We solve for the amplitudes by assuming solutions of the form shown in equations 81 and 82 and replacing all terms in equations 78 and 79 with our assumed solutions and the spatial derivatives of those equations,

$$\frac{\partial \eta'}{\partial x} = \eta_0 i\omega e^{i\omega x}, \quad \frac{\partial \zeta'}{\partial x} = \zeta_0 i\omega e^{i\omega x}, \quad \text{and} \quad \frac{\partial u'}{\partial x} = u_0 i\omega e^{i\omega x}. \tag{84}$$

Replacing all terms in equations 78 and 79 with the above we find

$$\frac{1}{U}u_0 i\omega e^{i\omega x} + \frac{g}{U^2}\zeta_0 i\omega e^{i\omega x} - \frac{\Gamma}{H^2}\zeta_0 e^{i\omega x} + \frac{2\Gamma}{HU}u_0 e^{i\omega x} = -\frac{\Gamma}{H^2}\eta_0 e^{i\omega x} \tag{85}$$

and

$$\frac{1}{U}u_0 i\omega e^{i\omega x} + \frac{1}{H}\zeta_0 i\omega e^{i\omega x} = \frac{1}{H}\eta_0 i\omega e^{i\omega x}. \tag{86}$$

We can factor out all exponential terms from equation 85 and all exponential and imaginary terms from equation 86,

$$\frac{u_0}{U}i\omega + \frac{\zeta_0 g}{U^2}i\omega - \frac{\zeta_0\Gamma}{H^2} + \frac{2u_0\Gamma}{HU} = -\frac{\eta_0\Gamma}{H^2} \tag{87}$$

$$\frac{u_0}{U}\omega + \frac{\zeta_0}{H}\omega = \frac{\eta_0}{H}\omega. \tag{88}$$

The two equations above could be solved directly as a system of equations, as each term in the two equations have the same units. However, for simplicity, we make some further replacements involving the coefficients,

$$U_0 = \frac{u_0}{U}, \quad Z_0 = \frac{\zeta_0}{H}, \quad H_0 = \frac{\eta_0}{H}, \quad \alpha = \frac{\Gamma}{H}, \quad \text{and} \quad \beta = \frac{gH}{U^2}. \tag{89}$$

Combining these relationships with equations 87 and 88 we are left with two simpler equations,

$$U_0 i\omega + 2\alpha U_0 + \beta Z_0 i\omega - \alpha Z_0 = -H_0\alpha \tag{90}$$

$$U_0\omega + Z_0\omega = H_0\omega \tag{91}$$

which can be rewritten as a system of equations in matrix form,

$$\begin{bmatrix} (i\omega + 2\alpha) & (\beta i\omega - \alpha) \\ \omega & \omega \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} -H_0\alpha \\ H_0\omega \end{bmatrix} \tag{92}$$

which represent two equations and two unknowns and we later refer to these matrices with the matrix variables

$$Ax = B. \tag{93}$$

11.1.1 Using Cramer's Rule

Cramer's rule allows us to easily solve for U_0 and Z_0. For a matrix equation $Ax = B$ where A is a nxn matrix with a non-zero determinant and vector x is a column matrix of unknown variables,the system has a unique solution given by

$$x_i = \frac{det(A_i)}{det(A)} \tag{94}$$

where A_i is a matrix formed by replacing the i^{th} column of A by the column vector B. First we find $det(A)$ which will be used in solving both U_0 and Z_0.

$$det(A) = \omega(i\omega + 2\alpha) - \omega(\beta i\omega - \alpha) \tag{95}$$
$$det(A) = (i\omega^2 + 2\alpha\omega) - (\beta i\omega - \alpha\omega) \tag{96}$$
$$det(A) = i\omega^2(1 - \beta) + 3\alpha\omega. \tag{97}$$

We can call the variable U_0 x_1 because it is the first variable in the x matrix. To solve for U_0 we need the matrix A_1, which is formed as described above, and is defined

$$A_1 = \begin{bmatrix} -H_0\alpha & \beta i\omega - \alpha \\ H_0\omega & \omega \end{bmatrix}. \tag{98}$$

We can then find the determinant of this matrix,

$$det(A_1) = \omega(-H_0\alpha) - H_0\omega(\beta i\omega - \alpha) \tag{99}$$
$$det(A_1) = -H_0\alpha\omega - H_0\beta i\omega^2 + H_0\alpha\omega) \tag{100}$$
$$det(A_1) = -H_0\beta i\omega^2. \tag{101}$$

We can now define U_0

$$U_0 = \frac{det(A_1)}{det(A)} = \frac{-H_0\beta i\omega^2}{i\omega^2(1 - \beta) + 3\alpha\omega}. \tag{102}$$
$$U_0 = \frac{-H_0\beta i\omega}{i\omega(1 - \beta) + 3\alpha}. \tag{103}$$

To solve for Z_0 we recognize that it can be called x_2 because it is the second variable in the x matrix. We need matrix A_2,

$$A_2 = \begin{bmatrix} i\omega + 2\alpha & -H_0\alpha \\ \omega & H_0\omega \end{bmatrix}. \tag{104}$$

Finding the determinant of A_2,

$$det(A_2) = H_0\omega(i\omega + 2\alpha) - (-H_0\alpha)\omega \tag{105}$$
$$det(A_2) = H_0 i\omega^2 + 2H_0\alpha + H_0\alpha\omega \tag{106}$$
$$det(A_2) = H_0 i\omega^2 + 3H_0\alpha\omega. \tag{107}$$

We can now define Z_0

$$Z_0 = \frac{det(A_2)}{det(A)} = \frac{H_0 i\omega^2 + 3H_0\alpha\omega}{i\omega^2(1 - \beta) + 3\alpha\omega}. \tag{108}$$
$$Z_0 = \frac{H_0 i\omega + 3H_0\alpha}{i\omega(1 - \beta) + 3\alpha}. \tag{109}$$

11.1.2 Finding the Magnitudes of Complex Numbers

We have solved U_0 and Z_0 but they still contain imaginary numbers. To remove them, we begin by defining the ratios

$$\frac{U_0}{H_0} = \frac{-\beta i\omega}{i\omega(1 - \beta) + 3\alpha}. \tag{110}$$

and

$$\frac{Z_0}{H_0} = \frac{i\omega + 3\alpha}{i\omega(1-\beta) + 3\alpha}. \tag{111}$$

Note that, from the relations 89 that $U_0/H_0 = u_0/\eta_0$ and $Z_0/H_0 = \zeta_0/\eta_0$ and so if we solve for the two ratios defined above, we can simply move η_0, which we define explicitly, to the right side, and we will have solutions for both u_0 and ζ_0.

To solve the ratios, we first find the magnitude of the complex numbers 110 and 110. Finding the magnitude of the ratio of two complex numbers A and B can be done

$$\left|\frac{A}{B}\right| = \frac{|A|}{|B|} = \frac{\sqrt{A_{real}^2 + A_{imag}^2}}{\sqrt{B_{real}^2 + B_{imag}^2}} \tag{112}$$

where the subscripts *real* and *imag* refer to the real part and the imaginary part of the complex number, respectively.

We find the magnitude of the ratio U_0/H_0 following this process,

$$\left|\frac{U_0}{H_0}\right| = \frac{|U_0|}{|H_0|} = \frac{\sqrt{0^2 + (-\beta\omega)^2}}{\sqrt{(3\alpha)^2 + (\omega(1-\beta))^2}} = \frac{\sqrt{\beta^2\omega^2}}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}} = \frac{\beta\omega}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}. \tag{113}$$

We find the magnitude of the ration Z_0/H_0 similarly,

$$\left|\frac{Z_0}{H_0}\right| = \frac{|Z_0|}{|H_0|} = \frac{\sqrt{(3\alpha)^2 + \omega^2}}{\sqrt{(3\alpha)^2 + (\omega(1-\beta))^2}} = \frac{\sqrt{9\alpha^2 + \omega^2}}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}. \tag{114}$$

11.1.3 Velocity and Flow Height Amplitudes

With the relations 89, we know that

$$\frac{U_0}{H_0} = \frac{u_0}{U}\frac{H}{\eta_0} \quad \text{and} \quad \frac{Z_0}{H_0} = \frac{\zeta_0}{H}\frac{H}{\eta_0} \tag{115}$$

and thus the solutions to u_0 and ζ_0 are

$$\boxed{u_0 = \frac{U}{H}\frac{\beta\omega}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}\eta_0} \tag{116}$$

and

$$\boxed{\zeta_0 = \frac{\sqrt{9\alpha^2 + \omega^2}}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}\eta_0.} \tag{117}$$

11.2 Velocity and Flow Height Phase Angles

We now know the amplitude of the sinusoids that describe the velocity and the water surface height. We can also solve for the phase angles, which will provide us with all the information we need to plot velocity and water surface height with a given bedform sinusoid. We want to find the phase angles of the complex numbers 110 and 110.

The angle, an operation that can be represented with the symbol $ang()$, is equal to the arctan of the ratio of the imaginary part to the real part of a number,

$$ang(A) = tan^{-1}\left(\frac{A_{imag}}{A_{real}}\right). \tag{118}$$

Furthermore, the angle of a ratio A/B can be found $ang(A/B) = ang(A) - ang(B)$. Thus,

$$\phi_u = ang(U_0/H_0) \tag{119}$$
$$\phi_u = ang(U_0) - ang(H_0) \tag{120}$$
$$\phi_u = tan^{-1}\Big(\frac{-\beta\omega}{0}\Big) - tan^{-1}\Big(\frac{\omega(1-\beta)}{3\alpha}\Big). \tag{121}$$

Recognizing that as the argument of *arctan* approaches negative infinity, the angle approaches radians $-\pi/2$,

$$\phi_u = -\frac{\pi}{2} - tan^{-1}\Big(\frac{\omega(1-\beta)}{3\alpha}\Big). \tag{122}$$

Furthermore,

$$\phi_\zeta = ang(Z_0/H_0) \tag{123}$$
$$\phi_\zeta = ang(Z_0) - ang(H_0) \tag{124}$$
$$\tag{125}$$

$$\phi_\zeta = tan^{-1}\Big(\frac{\omega}{3\alpha}\Big) - tan^{-1}\Big(\frac{\omega(1-\beta)}{3\alpha}\Big). \tag{126}$$

12 Plotting the Solutions

Here we gather all various the solutions and parameters that we have defined along the way that will allow us to plot our final results. Our wavy bedform, fluid surface height, and velocity function are described by the equations

$$\eta' = \eta_0 cos(\omega x), \quad \zeta' = \zeta_0 cos(\omega x + \phi_\zeta) \quad \text{and} \quad u' = u_0 cos(\omega x + \phi_u) \tag{127}$$

respectively. We have defined the amplitudes ζ_0 and u_0 in terms of a chosen value for η_0,

$$u_0 = \frac{U}{H}\frac{\beta\omega}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}\eta_0 \qquad \text{and} \qquad \zeta_0 = \frac{\sqrt{9\alpha^2+\omega^2}}{\sqrt{9\alpha^2 + \omega^2(1-\beta)^2}}\eta_0, \tag{128}$$

and we have defined the phase angles ϕ_ζ and ϕ_u

$$\phi_u = -\frac{\pi}{2} - tan^{-1}\Big(\frac{\omega(1-\beta)}{3\alpha}\Big) \qquad \text{and} \qquad \phi_\zeta = tan^{-1}\Big(\frac{\omega}{3\alpha}\Big) - tan^{-1}\Big(\frac{\omega(1-\beta)}{3\alpha}\Big). \tag{129}$$

Remembering that

$$\alpha = \frac{\Gamma}{H}, \quad \beta = \frac{gH}{U^2}, \quad \text{and} \quad \omega = \frac{2\pi}{\lambda}, \tag{130}$$

we see that we can plot all the equations 127 over a chosen x domain by defining only the parameters Γ, H, g, U, λ, and η_0, which represent, respectively, a friction coefficient, the reach-averaged fluid depth, earth's gravitational constant, the reach-averaged streamwise velocity, the bedform wavelength, and the bedform wave amplitude. Except where a selected variable is varied intentionally, we choose the following parameter values:

12.1 Interpretations

If we replace α, β, and ω in the amplitude and phase angle equations above with the actual parameters, it will be easier to learn what our solutions imply about the physics of flow over a wavy bedform.

$$u_0 = \frac{2\pi g}{U\lambda\sqrt{9\Gamma^2/H^2 + (4\pi^2/\lambda^2)(1-gH/U^2)^2}}\eta_0 \qquad \text{and} \qquad \zeta_0 = \frac{\sqrt{9\Gamma^2/H^2 + 4\pi^2/\lambda^2}}{\sqrt{9\Gamma^2/H^2 + (4\pi^2/\lambda^2)(1-gH/U^2)^2}}\eta_0, \tag{131}$$

$$\phi_u = -\frac{\pi}{2} - tan^{-1}\left(\frac{2\pi}{3}\frac{H(1-gH/U^2)}{\lambda\Gamma}\right) \qquad \text{and} \qquad \phi_\zeta = tan^{-1}\left(\frac{2\pi}{3}\frac{H}{\lambda\Gamma}\right) - tan^{-1}\left(\frac{2\pi}{3}\frac{H(1-gH/U^2)}{\lambda\Gamma}\right). \tag{132}$$

12.2 Friction Coefficient, Γ

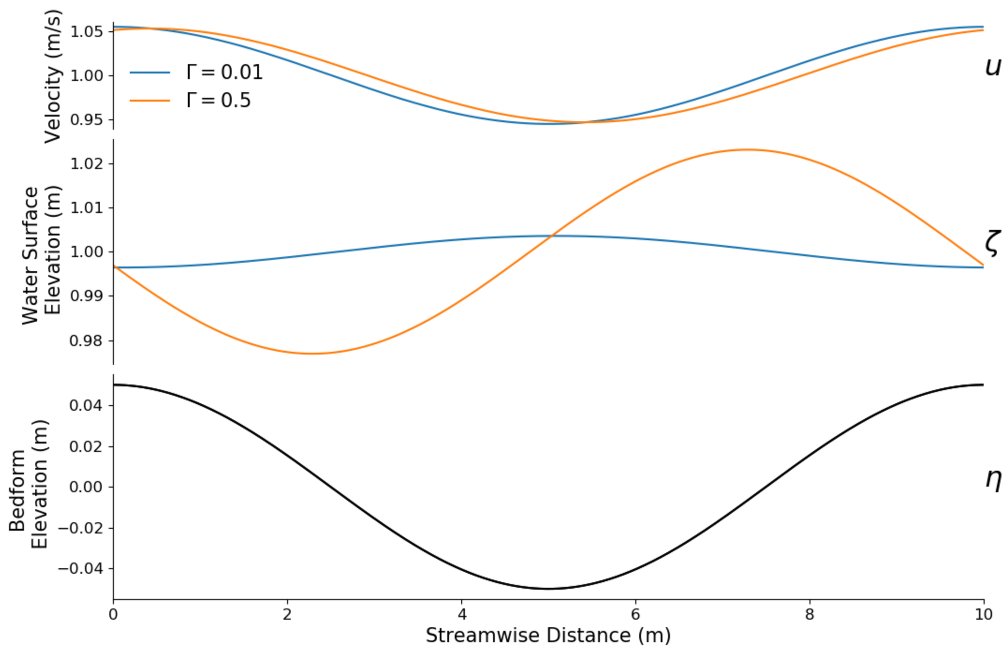

Figure 2: Varying the friction coefficient, Γ. $\eta_0 = 0.05$, $\lambda = 10$, $U = 1.0$, $g = 9.81$, and $H = 1.0$.

We can see from the above equations that as Γ increases, u_0 approaches 0, ζ_0 approaches some constant, ϕ_u approaches $-\pi/2$, and ϕ_ζ approaches 0. As the friction of the stream bed increases, reach-wise velocity perturbations are minimized, maximum flow velocity moves over the lee-side of the bedform bumps, and the sinuisoidal form of the fluid surface is aligned with the bedform such that the highest points on the fluid surface are above the highest part of bedform bumps. As Γ decreases, u_0 and ζ_0 approach some constants, ϕ_u approaches 0 (as long as $(1 - gH/U^2)$ remains negative, which, at all reasonable values, it does), and ϕ_ζ approaches π. As streambed friction decreases, maximum streamwise velocity is over the bedform humps and the water surface is highest above bedform troughs.

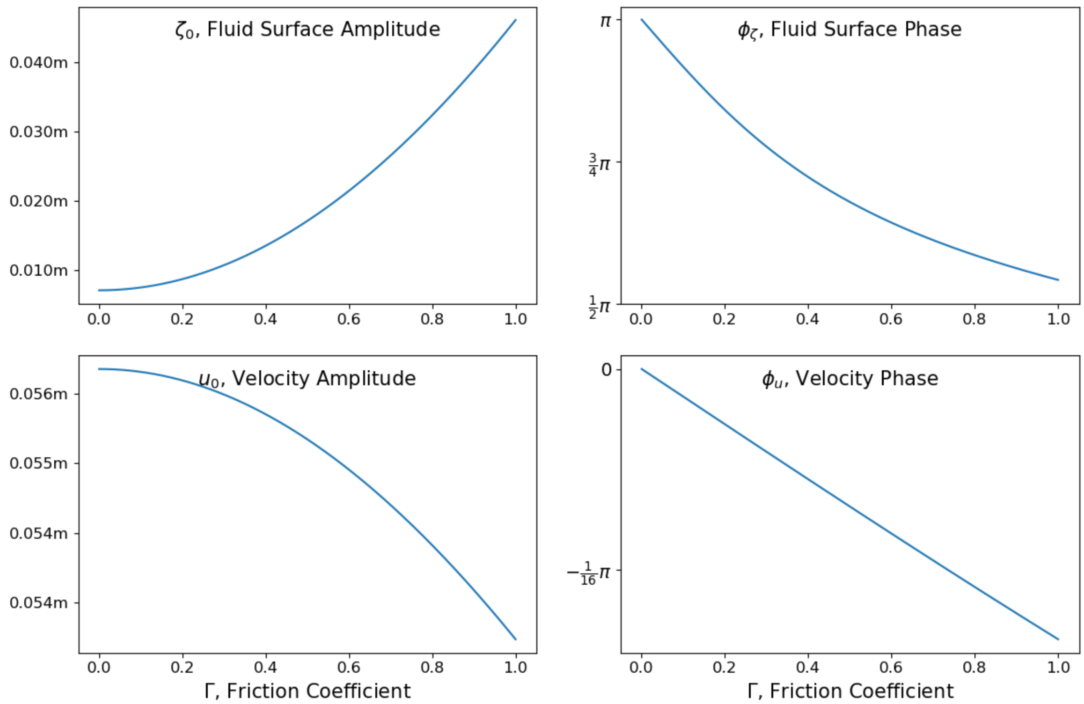

Figure 3: How the amplitudes and phases vary with the bed friction coefficient, Γ. Amplitudes and phases of the sinusoids representing water surface elevation and depth-averaged streamwise velocity. $\eta_0 = 0.05$, $\lambda = 10$, $U = 1.0$, $g = 9.81$, and $H = 1.0$.

It is worth noting that estimates for Γ in natural streams are on the order of $10^{-2} - 10^{-3}$. Thus under realistic friction conditions, the fluid surface waveform will almost always be out of phase by approximately π and the velocity waveform almost always be approximately in phase with the stream bed waveform. Furthermore, fluid surface and velocity waveform amplitudes will not vary much with realistic variations in Γ.

12.3 Reach-Averaged Fluid Depth, H

As fluid depth increases, ζ_0 and u_0 approach 0, ϕ_u approaches 0, and ϕ_ζ approaches π. This is rather intuitive; as fluid depth increases, velocity perturbations become negligible and variations in water surface height disappear. We would expect that as water depth increases, the effect of the bedform on the fluid surface is minimized, and that interpretation is supported by the mathematics. The phase angles are less interesting in the case of increasing fluid depth because both amplitudes decrease and so the location of the crest/trough of the waves mean little. Plotting the amplitudes against a range of H values (Figure 4) demonstrates that velocity amplitude approaches a maximum value at low fluid depths before again approaching 0. It is interesting, and perhaps physically unrealistic, that velocity waveform amplitude approaches 0 and fluid surface amplitude approaches very high values as H approaches 0. At realistic ranges of H, let's say 0.5 – 5 meters, the fluid surface will always be out of phase with the bed waveform by approximately π the velocity waveform approximately in phase with the bed waveform, and both the fluid surface and velocity waveform amplitudes are very small.

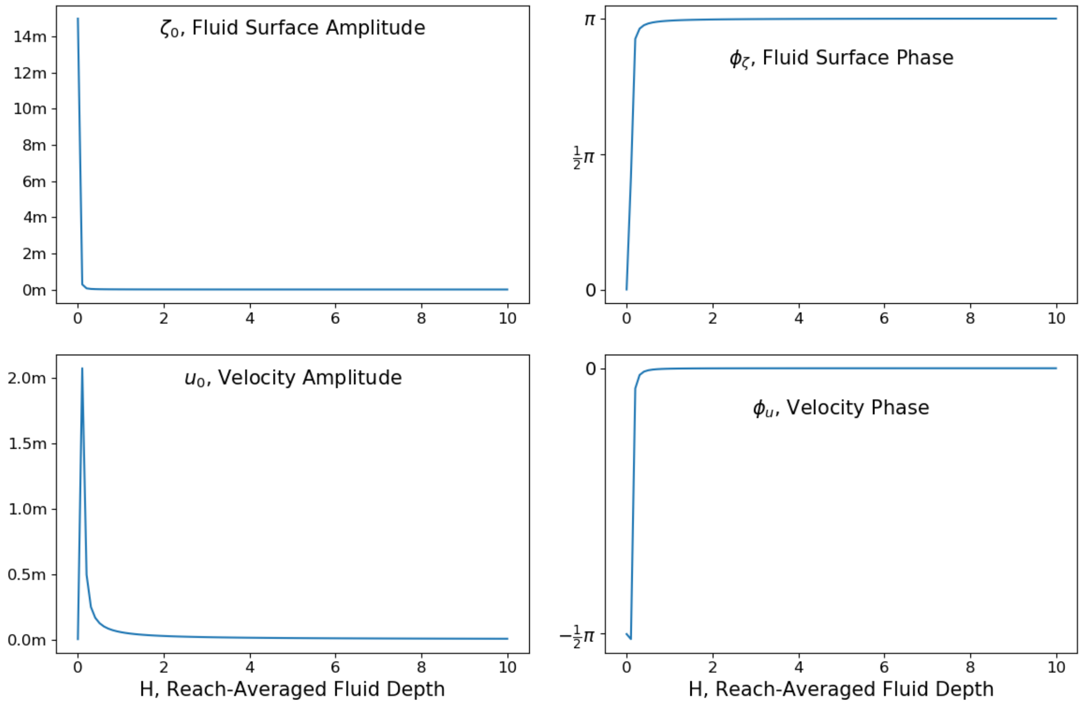

Figure 4: How the amplitudes and phases vary with average water depth, H. Amplitudes and phases of the sinusoids representing water surface elevation and depth-averaged streamwise velocity. $\eta_0 = 0.05$, $\lambda = 10$, $U = 1.0$, $g = 9.81$, and $\Gamma = 0.01$

12.4 Reach-Averaged Streamwise Velocity, U

Fluid flow over a wavy bedform responds in a surprisingly varied manner to changes in streamwise velocity. Plotting amplitudes and phases against a realistic range of streamwise velocities, let's say $0.5 - 10$ m s^{-1}, demonstrates that amplitudes reach a maximum value at a streamwise velocity about 3.5 m s^{-1}. The maximum amplitudes here, however, may by unrealistic because, for the parameters chosen, the fluid surface and velocity amplitude profiles will reach below 0.

The fluid surface and velocity phase angles show interesting behavior, with the fluid surface phase quickly changing from π to 0 around $U = 3$ and the velocity phase quickly changing from 0 to $-\pi$ around $U = 3$. We can observe these changes in Figure 6, which demonstrates that a fluid velocity of 5 m s^{-1} pushes the fluid surface waveform into phase with the bed waveform. When $U = 3$, the fluid surface is out of phase by approximately π. We can also see how between streamwise velocities of 1 and 5 m s^{-1}, fluid surface waveform amplitude increases and then decreases.

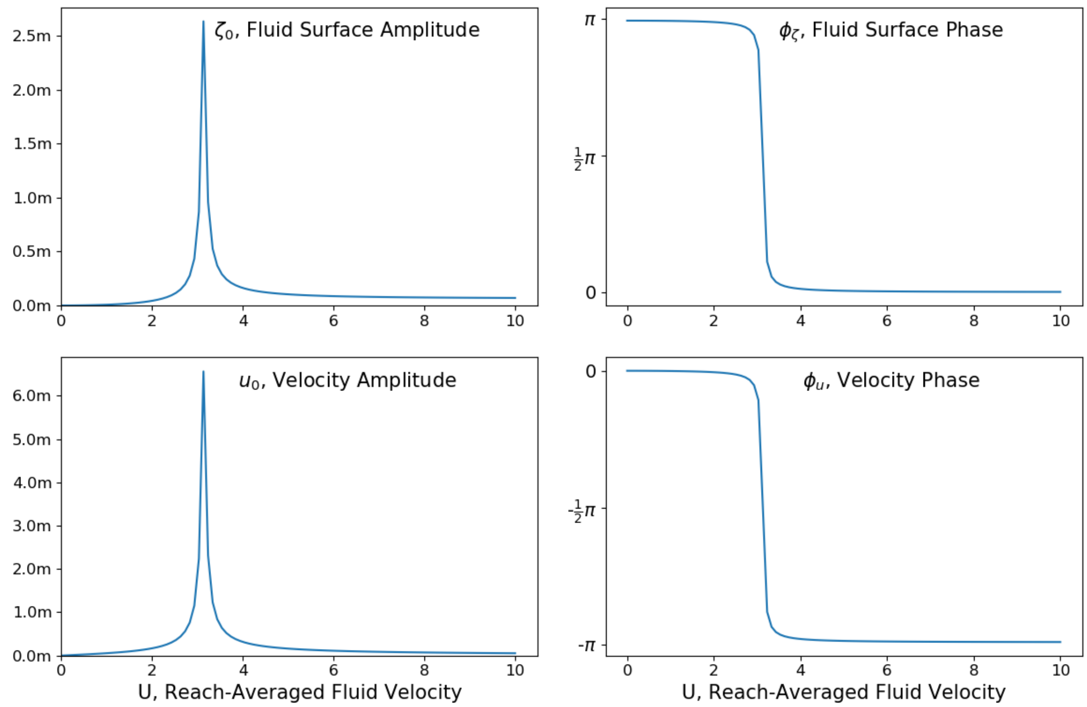

Figure 5: How the amplitudes and phases vary with averages streamwise velocity, U. Amplitudes and phases of the sinusoids representing water surface elevation and depth-averaged streamwise velocity. $\eta_0 = 0.05$, $\lambda = 5$, $g = 9.81$, $H = 1.0$, and $\Gamma = 0.01$

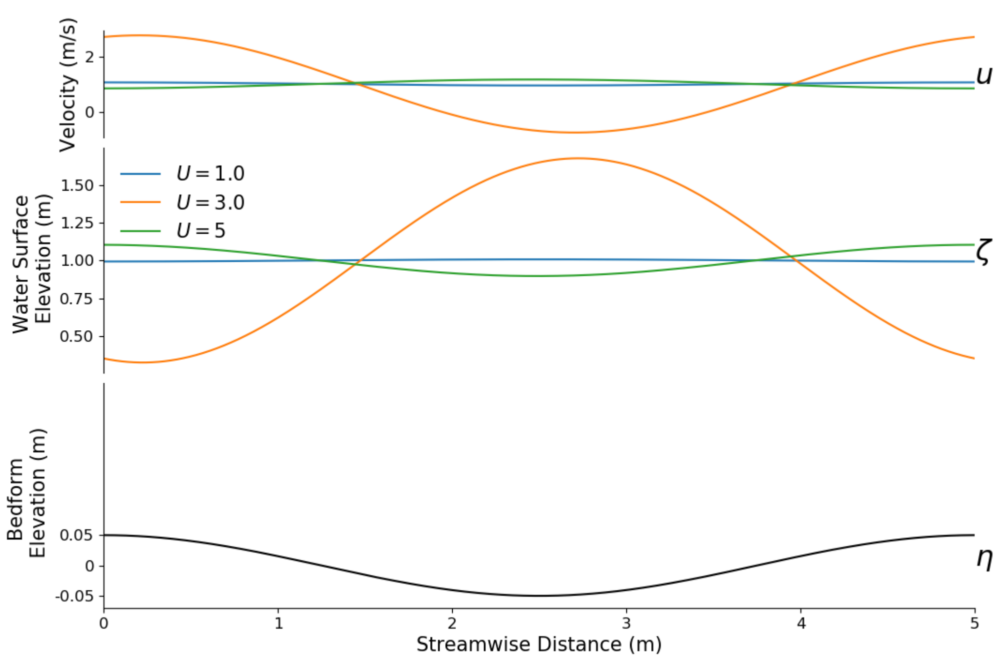

Figure 6: Varying streamwise velocity, U. $\eta_0 = 0.05$, $\lambda = 5$, $g = 9.81$, $H = 1.0$, and $\Gamma = 0.01$.

12.5 Bedform Wavelength, λ

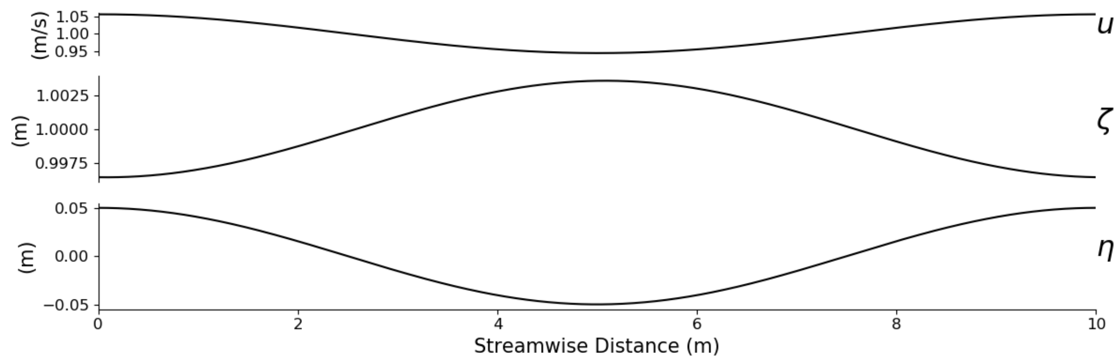

Figure 7: Waveforms with a bedform wavelength of 10 m. $\eta_0 = 0.05$, $\lambda = 10$, $U = 1.0$, $g = 9.81$, $H = 1.0$, and $\Gamma = 0.01$

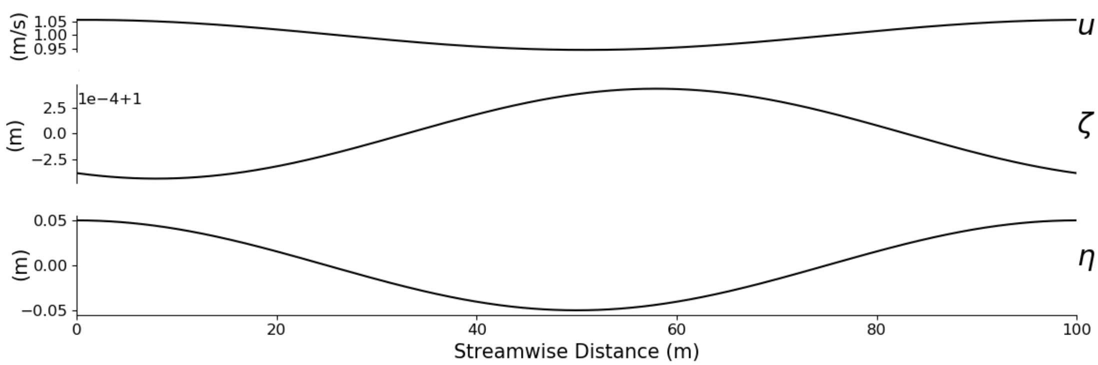

Figure 8: Waveforms with a bedform wavelength of 100 m. $\eta_0 = 0.05$, $\lambda = 100$, $U = 1.0$, $g = 9.81$, $H = 1.0$, and $\Gamma = 0.01$

Changes in bedform wavelength have significant effects on fluid surface amplitude, which drops from 0.03 m to close to 0 over the range $\lambda = [0, 100]$, and fluid surface phase, which decreases slowly from π at $\lambda = 0$. Velocity ampltiude and phase does not change much over the same range of λ values. We can observe the shifting of the fluid surface waveform in Figures 7 and 8.

12.6 Bedform Amplitude, η_0

We provide no figures for changes in bedform amplitude because their only effect is to scale amplitudes of ζ_0 and u_0 proportionately.

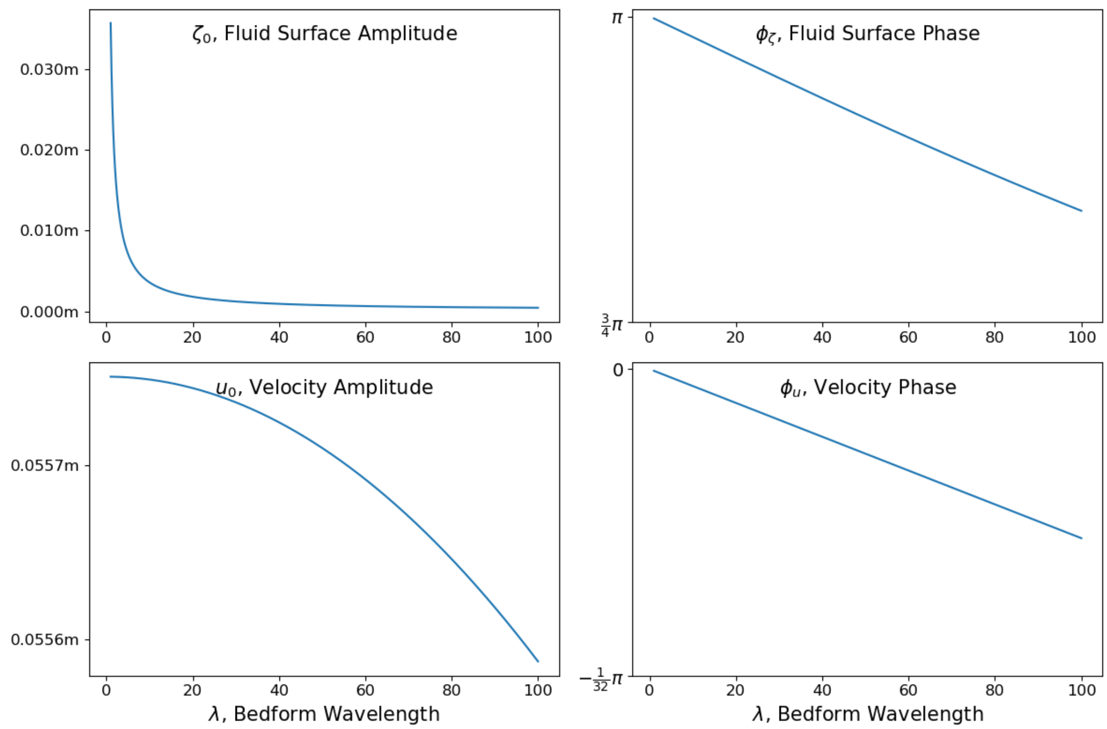

Figure 9: Amplitudes and phases varying with bedform wavelength, λ. $\eta_0 = 0.05$, $U = 1.0$, $g = 9.81$, $H = 1.0$, and $\Gamma = 0.01$.