A Timescale Model for Predicting Stream-Flow Permanence in a Confined Watershed

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Abstract

Stream discharge in a confined watershed can be examined as a simple model of conservation. Precipitation is the system's input and stream flow the system's output. By monitoring the water entering a measurable watershed, one can predict the discharge of the watershed's stream.

Using equations to model groundwater storage and flux through the stream-wetted perimeter, we create a predictive model for stream discharge through time. Our equation demonstrates that in response to a precipitation event, stream discharge spikes and then undergoes natural decay as the water table drains. The e-folding time interval of the decay is proportional to a drainage timescale constant, τ_D, which is composed of measurable parameters of confined watersheds. Our predictive model uses τ_D as a measure of how quickly stream flow decays after a precipitation event and the precipitation recurrence timescale, τ_R, as a measure of how frequently precipitation events occur. We use real data to calibrate our discharge model and then test the influence of various parameters on stream-flow permanence.

We explore if our discharge equation accurately predicts stream flow, if τ_d and τ_R can be used to characterize a stream as perennial or intermittent, and if the magnitude and/or frequency of precipitation events is important in determining stream-flow permanence. Through our experiments, we conclude that the drainage timescale is the most important factor in determining stream-flow permanence.

1 Developing the Model

1.1 Basic Assumptions

For the length of this report, we will consider a single watershed as a closed system and liquid water as the only significant fluid entering and exiting the system. The regional water balance equation has inputs of precipitation and groundwater flow and outputs of groundwater flow, evaporation, and stream flow discharge [9]. In our model, we assume that all precipitation entering a watershed will exit and that the amount of water evaporating out of a watershed is negligible.

1.2 Mass Balance Equation

We have thus simplified the water balance equation to take the form

$$P = Q \tag{1}$$

where P is net precipitation [L^3] and Q is net discharge [L^3]. Equation 1 demonstrates that all precipitation inputs are conserved as discharge outputs and over infinite time, all precipitation that enters a watershed will exit via stream flow.

Examining the water entering and exiting a watershed over an infinite amount of time is neither meaningful nor useful. To describe stream flow as a function of precipitation, discharge and precipitation must be functions of time, $Q(t)$ and $P(t)$, respectively. $Q(t)$ is discharge measured in units [L^3T^{-1}] and $P(t)$ is precipitation rate measured in units [LT^{-1}].

To find a complete equation for $Q(t)$ we define discharge in two ways; as a change in groundwater storage and as flux through the stream-wetted perimeter of the watershed's stream. Both of these water "routes" consider only water moving through the subsurface. We ignore surface flow in creating our model because in most humid regions, rainfall intensity will not exceed the infiltration capacity of the ground [8]. Surface flow occurs mostly in areas without vegetation [8], so we will only apply our model to humid areas such as the portion of United States east of the Mississippi River.

1.3 Groundwater Storage and Flux Through the Stream-Wetted Perimeter

Let us consider the groundwater storage of a watershed, GW_S.

$$GW_S = A\phi h \tag{2}$$

where A is the water table (or watershed) surface area, ϕ is the porosity of the subsurface material, and h is the height of the water table. Water table height is continuously fluctuating in response to inputs and outputs [7]. Change in GW_S can be defined in terms of the change in the water table height through time, $\frac{dh}{dt}$, such that

$$\Delta GW_S = A\phi \frac{dh}{dt}. \tag{3}$$

By conservation of mass, ΔGW_S is equal to the rate at which water enters groundwater storage minus the rate at which water exits groundwater storage. We define the entering rate as the precipitation rate over the watershed surface area (we assume precipitation rates are uniform over the watershed) and we define the exiting rate as stream discharge such that

$$\Delta GW_S = AP(t) - Q(t) \qquad \text{and} \qquad A\phi\frac{dh}{dt} = AP(t) - Q(t). \tag{4}$$

Now we define stream discharge, $Q(t)$, in terms of the water entering a stream through the subsurface. Discharge is equal to fluid flux, or specific discharge, over area. Our flux is water moving through the water table and our area the total surface area through which water can enter the stream. Thus,

$$Q(t) = va. \tag{5}$$

where v is the specific discharge and a is stream-wetted perimeter, the surface area of the stream in contact with the ground. Specific discharge is the product of hydraulic conductivity, κ, (a characteristic of porous media) and hydraulic gradient, i. i is itself a function of the height of the water table, h, and the distance of a particle at that height from the stream, X.

$$v = i\kappa \quad \text{and} \quad i = \frac{h}{X} \qquad \therefore \qquad Q(t) = a\kappa\frac{h}{X}. \tag{6}$$

Solving for h as a function of time, we find

$$h(t) = \frac{X}{a\kappa}Q(t). \tag{7}$$

Taking the derivate of h with respect to time t, we find

$$\frac{dh}{dt} = \frac{X}{a\kappa}\frac{dQ}{dt}. \tag{8}$$

1.4 Discharge as a Continuous Convolution Integral

Equation 8 can be combined with equation 4 to create the first order differential equation,

$$Q(t) = AP(t) - \tau_D\frac{dQ}{dt} \quad \text{where} \quad \tau_D = \frac{A\phi X}{a\kappa} \tag{9}$$

and τ_D is called the time-constant of the system or the drainage timescale. It is possible to show that the solution of a differential equation that takes the form of equation 9 is the convolution integral [6]

$$Q(t) = \int_0^\infty \upsilon(u)P(t-u)du \tag{10}$$

where $v(u)$ is the impulse response or "weighting" function, defined

$$v(u) = \frac{A}{\tau_D} e^{\frac{-u}{\tau_D}}. \tag{11}$$

Therefore, $Q(t)$ is calculated from $P(t)$ as a continuously weighted aggregate [6]. In simpler terms, discharge at a single point in time is a function of all past precipitation events, each past event weighted by the exponential decay function $v(u)$.

To apply the model in equation 10 to a real system, we must adjust the limits of integration because we cannot measure precipitation inputs indefinitely into the future. A perfect model would have a lower bound of $-\infty$ and an upper bound of the present moment, accounting for all precipitation events in the past. Obviously we could not use real data for such a model, so we adjust the limits of integration,

$$Q(t) = \int_0^t v(t-u)P(u)du \tag{12}$$

where t is the present time and 0 is the earliest time from which we will account for precipitation.

1.5 The Discrete Model

Equation 12 treats the function $P(u)$ as continuous between the bounds of integration, when precipitation data is invariably recorded as a step function, where the length of the step is the time interval between precipitation measurements. To utilize this model with real data, we must estimate the integral and consider $P(u)$ a step function or time series. In doing this, the function $P(u)$ changes from a precipitation rate (measured in units [LT^{-1}]) to a precipitation measurement in units [L]. Including the weighting function from equation 11, the integral in equation 12 takes the form

$$Q(t) = \frac{A}{\tau_D} \sum_{u=0}^{t} e^{\frac{-(t-u)}{\tau_D}} P(u) \tag{13}$$

where [u,t] is the time interval over which discharge is measured discretely and u iterates through each time step such that precipitation events early in the time series are weighted more heavily than events closer to time t. Let's apply equation 13 to a simple data set to understand the equation more fully. Let us set $t = 4$ days such that we are calculating the discharge based on precipitation events from 5 days; where the first day is represented by $t = 0$ and today is represented by $t = 4$.

$$Q(4) = v(4)P(0) + v(3)P(1) + v(2)P(2) + v(1)P(3) + v(0)P(4) \tag{14}$$

Examining the weighting function (equation 11), we see that $v(4) < v(3) < v(2)$... such that precipitation events further in the past have a lesser influence on present discharge.

2 Characteristic Drainage Timescale

To utilize the model in equation 13, we must contend with all of it's parameters. We need values for A, the watershed surface area, and τ_D, what Box and Jenkins [6] call the time-constant of the weighting function and what we call the characteristic drainage timescale. Let's examine τ_D in detail.

$$\tau_D = \frac{A\phi X}{a\kappa} \tag{15}$$

where ϕ is the porosity of the subsurface material, X is the distance from the stream that water in the water table lies, a is the stream-wetted perimeter, and κ is the hydraulic conductivity of the subsurface material. All of these variables are specific to a particular watershed and thus τ_D is a characteristic of a watershed and different watersheds have different values of τ_D.

A, ϕ, and κ are generally easy to find for a watershed and a particular subsurface material. a and X, which we call the characteristic width, must be approximated. If we assume a stream lays in the center of a watershed, then the average distance of water in the water table to the stream is 1/4 of the width of the watershed, half the distance from the stream to the edge of the watershed. If we assume an approximately square watershed, then $X = \frac{1}{4}\sqrt{A}$.

Stream-wetted perimeter is the length of a stream multiplied by the stream's cross-sectional perimeter in contact with the ground (the sides and bottom). Assuming again that a watershed is approximately square, the length of the stream will be the length of one side, $\sqrt{A}$. If we are dealing with relatively small watersheds, we assume that the bottom of the stream is approximately 2 meters, and each side 1 meter, such that $a = 4\sqrt{A}$. Regardless of the accuracy of these approximations, real values would vary at most by 1 order of magnitude for a watershed of relatively small size and therefore we move forward with our model.

2.1 Theoretically and Empirically Calculating the Drainage Timescale

The weighting function in equation 11 allows to test the accuracy of our theoretical drainage timescale equation by using real hydrographs to empirically calculate τ_D for a watershed. $\upsilon(u)$ is an exponential decay function and contains the drainage timescale, τ_D, in the exponent. Taking the natural logarithm of both sides of equation 11, we find that

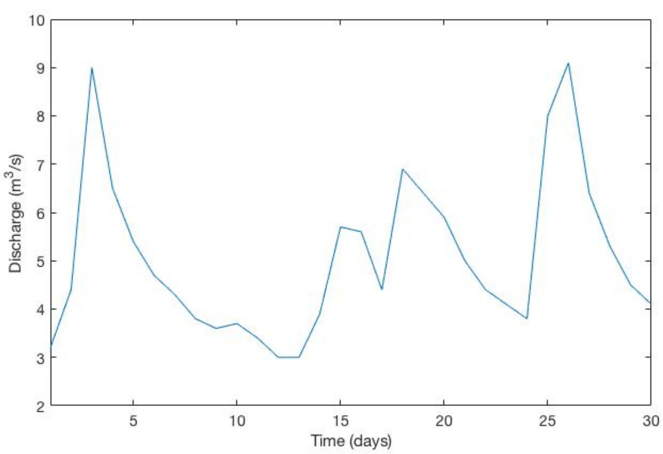
$$\ln(\upsilon(u)) = \ln(\frac{A}{\tau_D}) + \frac{-1}{\tau_D}u, \tag{16}$$

which, as we would expect from a natural decay function, produces a linear equation. We see that the slope m of the line is $\frac{-1}{\tau_D}$ such that

$$\tau_D = \frac{-1}{m}. \tag{17}$$

Examining real hydrographic data [2], we can see the natural decay of discharge occur after sharp rises in response to precipitation events. Figure 1 has particularly clear natural decay after the first and last discharge peaks.

Figure 1: 30 Day Hydrograph, Manokin Branch in Princess Anne, MD

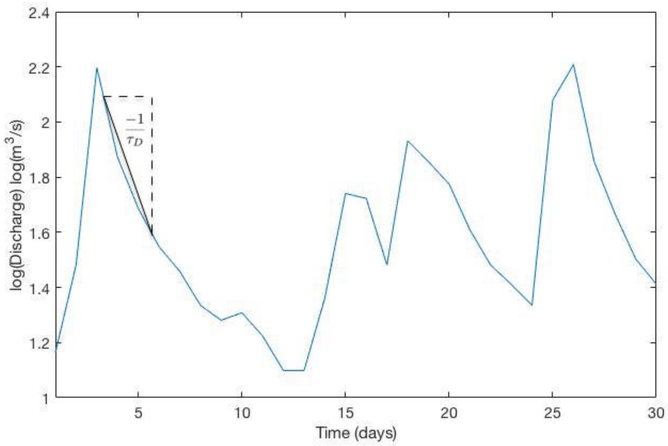


Plotting the natural logarithm of discharge data, the natural decay should become linear. If we find the linear rate of change during a period of decay, we can find τ_D with equation 17. Figure 2 shows the same hydrograph as in figure 1 but with the natural logarithm of the discharge plotted with respect to time.

It is evident that stream discharge in our altered hydrograph is not undergoing perfect natural decay; if it did, then figure 2 would have straight lines descending from discharge peaks. Despite this, we can still estimate τ_D by estimating the slope of the function. To do this, we select 30 time intervals from a year-long data set that contain discharge decay. After altering the data sets by taking the natural logarithm of Q, we perform linear regressions on each data set to find the line of best fit. We then use equation 17 to find the empirical drainage timescale, $\tau_{D,e}$, for the Manokin Branch watershed and we found $\tau_{D,e} = 4.47$.

Now that we have the empirical value of τ_D for the Manokin Branch watershed, we can test the accuracy of our drainage timescale equation (equation 15). To fill out the equation, we must find parameters for

Figure 2: 30 Day Hydrograph, Manokin Branch in Princess Anne, MD



our watershed. The watershed surface area is 12431942m^2 [2], the porosity of the local silt loam soil [4] is approximately 0.425 [3], and the hydraulic conductivity is approximately 1 ms^{-1} [5]. Our equation yields a drainage timescale of 3.30×10^5. The difference between the empirically and theoretically calculated drainage timescales is massive; the two differ by 5 orders of magnitude. For the purposes of moving this model forward, we will add a constant, c, to τ_D such that

$$c\tau_D = \tau_{D,e}. \tag{18}$$

Solving for c, we find $c = 1.3536 \times 10^{-5}$. Our discrete discharge equation includes c such that

$$Q(t) = \frac{A}{c\tau_D}\sum_{u=0}^{t} e^{\frac{-(t-u)}{c\tau_D}} P(u). \tag{19}$$

As we move forward, we must keep in mind that the current value of c is specific to the Manokin Branch watershed. It is possible that the calibration coefficient would produce accurate results for other watersheds, but we will leave that discovery to future work.

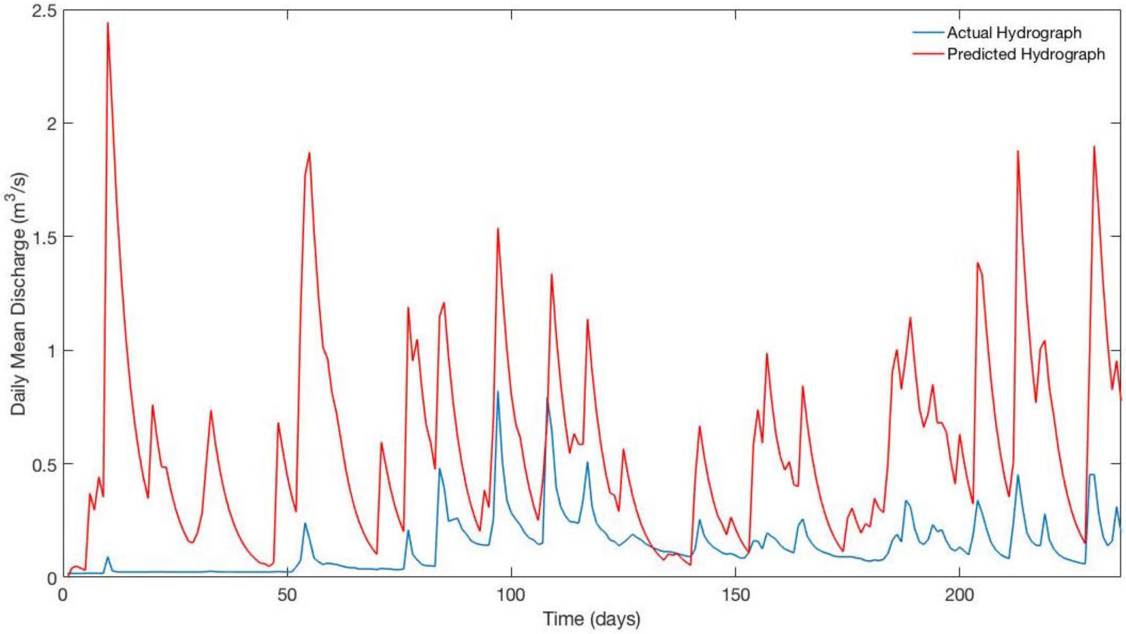
3 Checking Our Model with Precipitation and Discharge Data

Now that we have calibrated our drainage timescale, we should expect the model to create a reasonably accurate hydrograph using real precipitation data. To test the model, we use daily rain records [1] for the location of the Manokin Branch as the time series for equation 19 to operate on. We then compare the hydrograph predicted by our model to a real Manokin Branch hydrograph from the same time interval [2]. Figure 3 contains our results. Our predicted discharge is of a greater magnitude than the actual discharge, but our model's response to precipitation events is similar to the actual hydrograph. Most of the peaks in the predicted hydrograph are matched by peaks in the actual hydrograph. [1]

Now that we have found that our model accurately predicts discharge to within one order of magnitude and that our impulse response function is quite accurate in predicting the response of discharge to precipitation events, we can turn to testing the effect of various parameters on our model.

[1]The blatant difference in the first 50 days seems to be an error in the stream-flow measurement device. See the real hydrograph with daily precipitation data in figure 4. We can see that actual discharge consistently responded to precipitation events of magnitude equivalent to the precipitation events in the first 50 days. The flat lining hydrograph of the first 50 days is unlike any other part of the hydrograph so we think that these days contain inaccurate discharge measurements.

Figure 3: Actual and Predicted Hydrographs, Manokin Branch, Princess Anne, MD



4 Predicting Stream-Flow Permanence

Assuming that our model is a reasonable estimate of discharge response to precipitation events, tweaking the parameters of our model will produce changes in discharge that would be mimicked in a real system.

Our main goal is to discover what parameters control stream-flow permanence. A stream can be characterized as either perennial, intermittent, or ephemeral. Ignoring the details distinguishing intermittent and ephemeral streams, we define an intermittent stream as a stream that, at some points in time, reaches negligible discharge, and we define a perennial stream as one that maintains significant discharge.

To predict stream-flow permanence we return to our original mass balance equation and consider the drainage of groundwater storage and the occurrence of precipitation events which increase groundwater storage. If groundwater storage drains completely before the recurrence of a precipitation event, then the watershed's stream flows intermittently. If a watershed does not drain before precipitation recurrence, then stored groundwater will continue to drain into the stream and the stream will flow perennially.

We already have a measure of how rapidly a watershed drains, τ_D, and we will soon introduce a second time constant, the characteristic precipitation recurrence timescale, to compare to τ_D. We will then explore if the ratio between the drainage timescale and the precipitation recurrence timescale can determine stream-flow permanence by keeping precipitation frequency constant and tweaking τ_D. Then, we will compare the importance of the magnitude of precipitation events with the frequency of precipitation events by keeping τ_D constant and tweaking precipitation inputs.

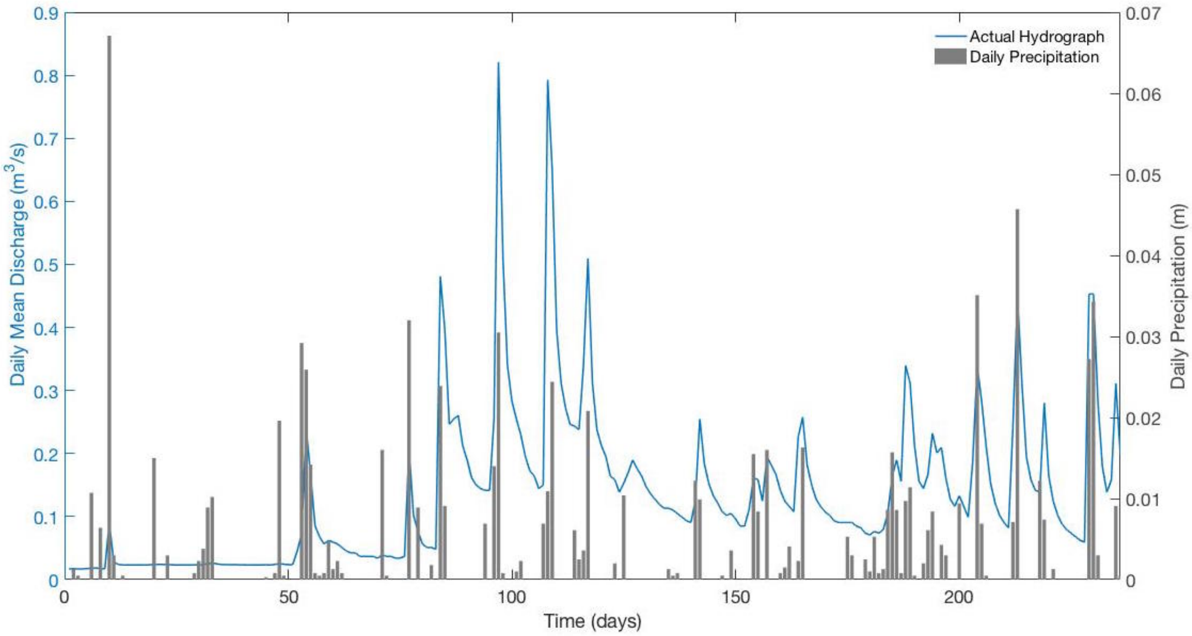
4.1 Characteristic Precipitation Recurrence Timescale

The precipitation recurrence timescale, τ_R, is determined by the frequency of rainy days. If α is the proportion of rainy days to total days in a watershed, then

$$\tau_R = \frac{1}{\alpha} \tag{20}$$

For example, in a watershed where it rains every 3 days out of 10, $\alpha = 0.3$, $\tau_d = 3.33$, and it will rain approximately every 3.33 days. The Manokin Branch watershed is a particularly rainy place. Using historical precipitation data [1], we find that $\alpha = 0.415$ and $\tau_R = 2.41$ for our time interval of the Manokin Branch watershed.

Figure 4: Actual Hydrograph and Daily Precipitation, Manokin Branch, Princess Anne, MD



4.2 The Drainage Timescale to Precipitation Recurrence Timescale Ratio

Before we explore the $\tau_D : \tau_R$ ratio, we must remind ourselves that our drainage timescale is not a measurement of how long it takes for a watershed to drain, but is a measurement of the rate at which discharge decays after a precipitation event causes a discharge spike. We should also keep in mind that our model will never allow discharge to truly reach zero because once a precipitation event occurs, discharge undergoes natural decay and becomes infinitesimally small (if no precipitation events follow) [2].

To create hydrographs with varying $\tau_D : \tau_R$ ratios, we leave τ_R constant and tweak τ_D [3]. We will use our Manokin Branch watershed precipitation data and will begin with the true drainage timescale value, 4.47. With $\tau_D = 4.47$ and $\tau_R = 2.41$, our ratio $\tau_D : \tau_R = 1.85$. The predicted hydrograph in figure 3 has this ratio.

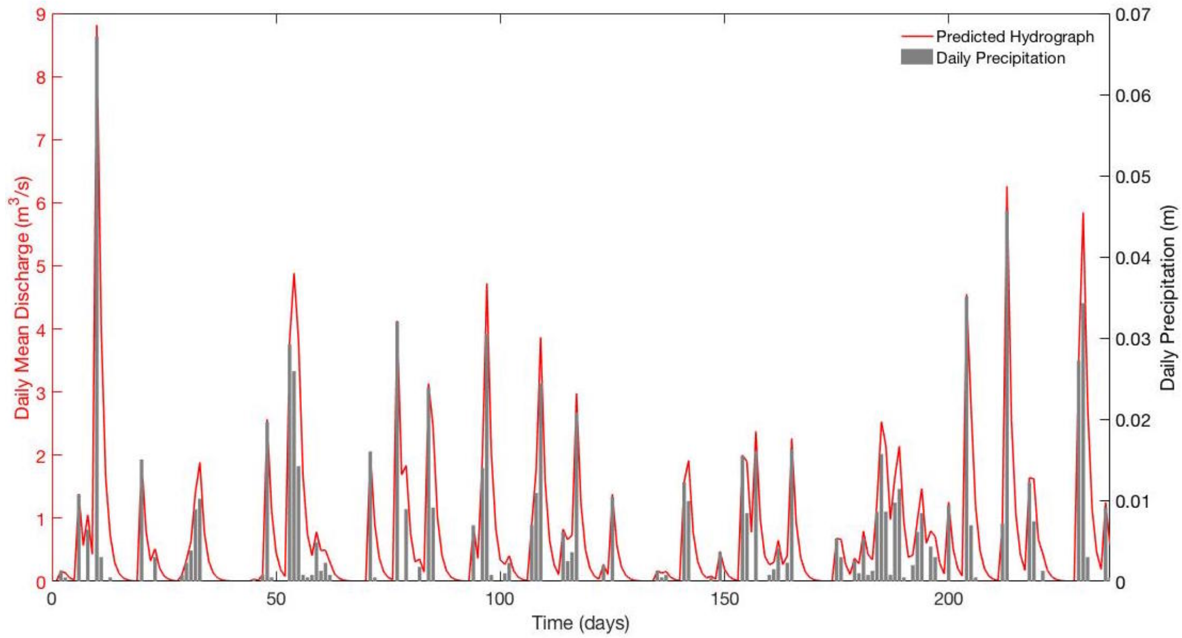
The minimum discharge value of the predicted hydrograph is $0.0317\text{m}^3/\text{s}$ and the average discharge $0.576\text{m}^3/\text{s}$. The minimum discharge value of the actual hydrograph is $0.0170\text{m}^3/\text{s}$ and the average discharge $0.1372^3/\text{s}$. The minimum discharge of the predicted hydrograph is 5.5% of the average discharge and for the actual hydrograph the minimum is 12% of the average. This difference is relatively small and the actual discharge hydrograph never reaches zero, so we can say that our model did not reach zero discharge in this case. This makes sense if our watershed takes longer to drain that it does for rain to return, and, even though our drainage timescale is not the amount of time it takes for a watershed to drain, our $\tau_D : \tau_R$ ratio is greater than 1 and so we observe that this ratio indicates perennial flow.

In the next scenario, we use a $\frac{1}{4}\tau_D = 1.12$ and keep $\tau_R = 2.41$ such that our ratio $\tau_D : \tau_R = 0.465$. Such a change could be the result of a different subsurface porosity; changing porosity from 0.425 to 0.1 would change τ_D approximately the same amount. Our ratio is now less than 1, so we predict that the stream will flow intermittently. Figure 5 shows the predicted hydrograph with precipitation. It is obvious from the figure that scaling the drainage timescale by $\frac{1}{4}$ had a significant enough effect on stream discharge decay to force discharge to approach zero multiple times. The minimum discharge value of this hydrograph is $1.0017\times10^{-4}\text{m}^3/\text{s}$ (the discharge reaches this small order of magnitude multiples times) and the average discharge is $0.7973\text{m}^3/\text{s}$. The minimum discharge being 0.012% of the average discharge, it seems safe to

[2] The exception to this rule is that our predicted hydrographs always have zero discharge at the beginning of the time series because before any precipitation events, our model has not "revved up". This is due to the fact that our model does not include every precipitation event before the present moment.

[3] This allows us to use 1 precipitation data time series for all of our models.

Figure 5: Predicted Hydrograph and Daily Precipitation, $\tau_D : \tau_R = 0.464$



say that this hydrograph represents a scenario in which actual discharge would reach zero. The increased τ_D value also has a significant effect on the magnitude of discharge. The hydrograph with a timescale ratio of less than 1 (in figure 5) reaches a maximum discharge value that more than triples the maximum discharge value of the hydrograph with a ratio of greater than 1 (predicted hydrograph in figure 3). This makes sense because a watershed that drains quickly will funnel more water to the stream flow at one time than a watershed that drains slowly.

Although we cannot make any definitive statements about a specific $\tau_D : \tau_R$ ratio that determines stream-flow permanence, we can say with confidence that seemingly small changes to the drainage timescale, and thus the timescale ratio, can change a stream from perennial to intermittent and vice versa.

4.3 Comparing the Importance of Precipitation Magnitude and Frequency to Stream-Flow Permanence

An interesting question that our model raises is whether the magnitude of precipitation events or the frequency with which precipitation occurs is more significant in determining stream-flow permanence. To explore this question, we model two additional hydrographs. In both models, we will use our minimized τ_D value ($\tau_D = 1.12$) and attempt to avoid zero discharge states by first increasing the magnitude of precipitation events and second increasing the frequency of precipitation events.

Figure 6 shows two hydrographs, the discharge response to our actual precipitation data and the discharge response to the precipitation data scaled by a factor of 2. We can see that scaling the precipitation events by a factor of 2, a huge change in climate, has almost no effect on the stream-flow permanence. The discharge responses to the scaled precipitation events is significantly larger, as we would expect, but the discharge decays rapidly, quickly aligning with the discharge response to the normal precipitation events. At every time where the discharge response to the normal precipitation approaches zero, the discharge response to the increased precipitation approaches zero.

For our second model, we create a false precipitation time series to increase the frequency of precipitation events and decrease τ_R. We randomly add precipitation values to existing "dry days" such that $\alpha = 0.65$ and $\tau_R = 1.54$. Our $\tau_D : \tau_R$ ratio in this case is equal to 0.726, so we predict that the hydrograph will still reach a zero discharge state, although this should happen with less frequency than in our previous model. Our results are in figure 7. We see that the increased frequency of rainy days creates a much more jagged and variable hydrograph. There are far fewer points in time than in figure 5 or figure 6 where discharge

Figure 6: Predicted Hydrograph with Scaled Precipitation

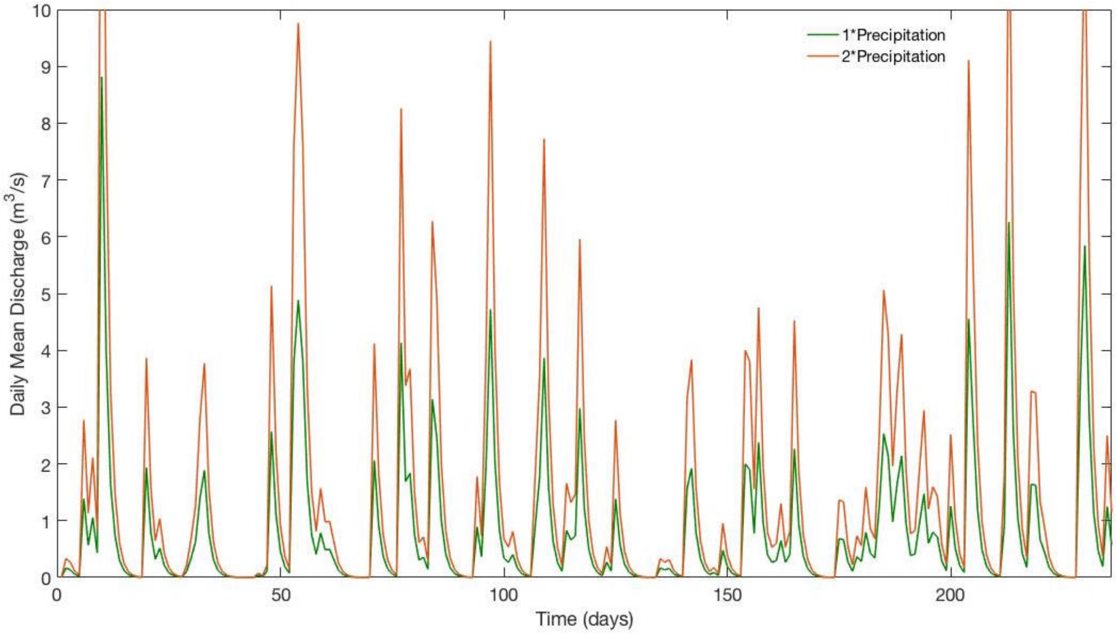
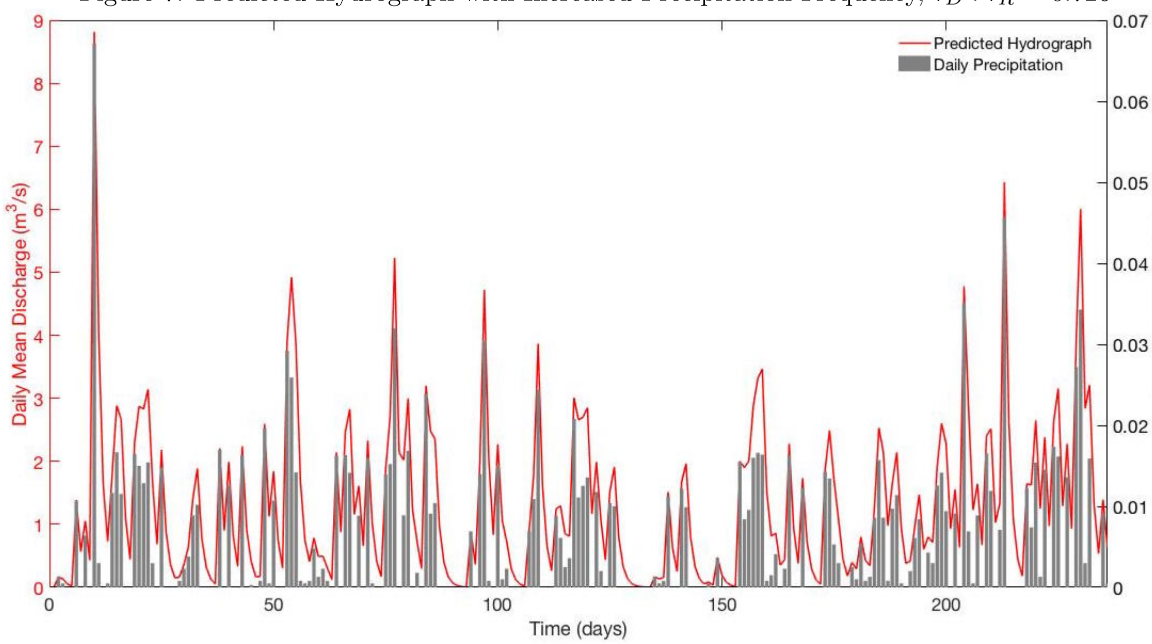


Figure 7: Predicted Hydrograph with Increased Precipitation Frequency, $\tau_D : \tau_R = 0.726$



approaches zero. There are a few points, just before day 100 and just before day 150, where discharge seems to approach a zero state. Even though we increase the probability of rain by a scalar of about 1.5, a huge climate change, the hydrograph still approaches zero discharge multiple times. This leads us to the conclusion that precipitation parameters - both frequency and magnitude - matter far less in determining stream flow permanence than the rate at which discharge decays.

5 Conclusion

For a theoretical model, the predictive power of equation 19 is surprisingly accurate. The magnitude of discharge values were consistently predicted to within one order of magnitude and the timing response of discharge to precipitation events was accurate. In simplifying the watershed system and allowing our model to only account for precipitation inputs and stream-flow outputs, we assume that all rain that enters a watershed exits the watershed via stream-flow. This is not the case in reality. Our model ignores all water that leaves the watershed in the subsurface and for a stream to exist (in most cases), water *must* be in the subsurface. We also ignore the influence of evapotranspiration. Future work could attempt to account for both subsurface outflow and evapotranspiration. More work should also go towards finding the drainage timescale both empirically and theoretically for watersheds of varying size and with varying subsurface materials to see if there are particular watersheds for which this model works particularly well.

A future model could also seek to perfect the concept of a timescale ratio. Our ratio treats τ_D as if it was a measurement of the time it takes for a watershed to drain. This is impossible because our model, as we discussed, never actually reaches zero discharge. The current ratio does offer some insight into stream-flow permanence. As a ratio less than 1 becomes smaller, the modeled hydrographs more frequently exhibit zero discharge states. Future work could to set a minimum discharge threshold to decide when the discharge is essentially zero. Using the log of the discharge function (equation 16), you could find the true time at which discharge reaches the threshold. From this, a true timescale ratio could be created.

Our model did reveal some important insights. It is clear that the rate at which a watershed drains is the most important factor in determining stream-flow permanence. Increasing the magnitude of precipitation events, even to an unrealistically large degree, does not affect stream-flow permanence. Increasing the frequency of precipitation does change the likelihood of stream-flow permanence, but the affect of the drainage timescale still reigns supreme.

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