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Mathematical formulae for calculating the base temperature for growing degree days

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Abstract

Much research has been done on finding methods to determine the base temperature, a very important variable in computation of growing degree days (GDD). Four common methods have been reported often in the literature: (1) the least standard deviation in growing degree days; (2) the least standard deviation in days; (3) coefficient of variation; (4) regression coefficient. The procedures to calculate the base temperature associated with these methods are tedious and lack a theoretical basis in mathematics. The objective of this research was to find simple and mathematically sound formulae to calculate the base temperature for GDD. Mathematical formulae are proposed, proved and tested using temperature data for snap bean, sweet corn, and cowpea. Compared with previous procedures, these proposed mathematical formulae can produce the base temperature easily and accurately. These formulae are applicable to calculating the base temperature for GDD of any developmental stage for any crop.

1. Introduction and literature review

In modeling crop growth, accurate prediction of crop development is necessary. Much research has been done in this area (Anderson et al., 1978; Angus et al., 1981; Bewick et al., 1988). Researchers have found that air temperature is a dominant factor controlling crop development. To predict crop development with air temperature, growing degree days (GDD) or a similar linear unit system is widely used (Madariaga and Knott, 1951; Hoover, 1955; Gilmore and Rogers, 1958; Hortic and Arnold, 1965). The selection of an appropriate base temperature is critical to the GDD or any heat unit model.

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The concept of base temperature can be described either physiologically or statistically. Physiologically, it is assumed that below a certain temperature level, crop growth and development will cease. However, it is difficult to determine the physiological base temperature, and each developmental phase may have a different base temperature. In physiology, the base temperature should be similar for a given crop developmental stage in any growing season. However, in practice, the base temperature selected may vary among years or growing seasons. For example, Arnold (1959) reported that the base value for corn was 6°C in 1954 and 4.3°C in 1955. Statistically, the base temperature is that which results in the lowest variation in GDD accumulations. In most cases, the base temperature is determined statistically rather than physiologically, owing to criteria of selecting the base temperature (Arnold, 1959; Goyne et al., 1977; Fernandez and Chen, 1989). Even if the base temperature is said to be determined biologically, the final criterion to accept or reject it is the lowest standard deviation in GDD accumulations or in days. However, the base temperature may sometimes be calculated to be below zero, which is difficult to explain in biology.

Many methods to determine the base temperature have been reported. Some of the most important are: (1) the least standard deviation in GDD (Magoon and Culpepper, 1932; Stier, 1939); (2) the least standard deviation in days (Arnold, 1959); (3) the coefficient of variation in days (Nuttonson, 1958); (4) the regression coefficient (Hoover, 1955).

1.1. The least standard deviation in GDD method

In this method the base temperature is selected so that the resultant variation in GDD using a series of plantings is minimized. The least standard deviation in GDD is defined as

$$SD_{\mathrm{gdd}} = \left[\frac{\sum_{i=1}^{n}(GDD_i - MGDD)^2}{n-1}\right]^{\frac{1}{2}} \tag{1}$$

where SD_{gdd} is the least standard deviation in GDD, GDD_i is growing degree days of the ith planting, MGDD is the overall mean GDD of all plantings, and n is the number of plantings.

In this method, GDD is calculated using a series of candidate base temperatures, each one resulting in a set of GDDs and a standard deviations. The temperature that generates GDDs with the smallest standard deviation is selected as the base temperature. Goyne et al. (1977) used temperatures from −6°C to 9°C to calculate the base temperature for sunflowers (*Helianthus annuus* L.). Perry et al. (1986) selected the base temperature from five temperatures (0°C, 10°C, 13°C, 18°C and 25.5°C) in the prediction of cucumber (*Cucumis sativus* L.) harvest date.

1.2. The least standard deviation in days method

Arnold (1959) suggested that 'those who use heat units as a tool are not interested in the error in heat units as such but rather the error in days which the heat unit error represents'. He defined the least standard deviation in days as

$$SD_{\text{day}} = \frac{SD_{\text{gdd}}}{x_t - t_b} \tag{2}$$

where SD_{day} is the standard deviation in days, SD_{gdd} is the standard deviation in GDDs, X_t is the overall mean temperature of all plantings, and t_b is the base temperature.

As in Method 1, this procedure also requires the same selection of a base temperature from a series of candidate temperatures. There is no improvement in this method in terms of calculation procedures.

1.3. Coefficient of variation in days method

Methods 1 and 2 both give absolute magnitudes of the variation in GDD and in days, but fail to mention relative magnitudes of the variation. Nuttonson (1958) defined coefficient of variation with the equation

$$CV_{\text{day}} = \frac{SD_{\text{day}}}{x_d} \times 100\% \tag{3}$$

where CV_{day} is coefficient of variation in days, SD_{day} is the standard deviation in days, and X_d is the mean of number of days required to reach a given developmental stage. Because X_d is a constant for all plantings, independent of the base temperature selected, there is no difference between this method and Methods 1 and 2 in terms of the base temperature selected and calculation procedures.

It is obvious that the three methods discussed above to calculate the base temperature are empirical. To obtain a base temperature which generates the least variation in GDD or days, a range of candidate temperatures must be selected to calculate GDD or days and their associated standard deviations in GDD or days. If the base temperature that generates GDD with the least variation is much below zero, it is very possible to miss it in the process of selecting candidate base temperatures. Because the calculation procedure is empirical in all three methods, it is difficult to select the correct temperature that generates GDD or days with the least variation.

1.4. Regression coefficient method

To overcome shortcomings of the three methods discussed above, Hoover (1955) developed a regression coefficient method to estimate the base temperature, t_b, with the equation

$$Y_i = a + bT_i, \qquad Y_i = (T_i - t_b)d_i \tag{4}$$

where a and b are constants, T_i is the mean of temperature for the ith planting, d_i is the number of days required for a developmental stage for the ith planting, and Y_i is GDD for the ith planting.

He studied the relationship between the mean temperature and GDD with a linear regression model. In this linear regression model, the mean temperature was the independent variable and GDD was the dependent variable. Theoretically, if the selected temperature was too high, the regression coefficient in the model was positive; otherwise it was negative. When the regression coefficient was zero, the selected temperature was considered to be the base temperature. This method is based on the assumption that GDD is constant and independent of mean temperature. Hoover located points in the positive and negative range and found the zero point by graphic interpolation.

In statistics, the regression method is not much different from the least standard deviation method, and shows no improvement in GDD calculation procedures (Arnold, 1959). As with Methods 1, 2 and 3 it also requires the calculation of GDD using a series of values to find the base temperature. In addition, this graphic interpolation method cannot give an accurate base temperature because it is difficult to determine exactly a number from a graph.

Another widely used method to find the base temperature similar to the regression method is the x-intercept or development rate method (Arnold, 1959). The general formula for this method is $1/d = a + Bt$, where d is the number of days between developmental stages, $1/d$ is developmental rate, a is intercept, b is regression coefficient, and T is mean temperature. When developmental rate $1/d = 0$, the base temperature is $-a/b$.

This method is simple and analytic, but has a serious limitation that precluded its use in this research. In statistics, it is not appropriate to extrapolate in regression models. In the developmental rate method, setting the developmental rate ($1/d$) to zero to find the base temperature is extrapolation and therefore not acceptable. In contrast, the regression coefficient (b) in the regression method is set to zero and no extrapolation is involved.

As mentioned above, procedures associated with these four popular methods to calculate the base temperature for GDD have shortcomings. The objective of this paper is to provide mathematical formulae to facilitate the calculation of the four discussed methods and to test the formulae using field data from snap bean (*Phaseolus vulgaris* L.), cowpea (*Vigna unguiculata* L.), and sweet corn (*Zea mays* L.) phenology studies.

2. Materials and methods

Eqs. (1)–(4) in Section 1 were used as the basis to develop mathematical formulae to calculate the base temperature. The base temperatures of snap bean, cowpea, and sweet corn from planting to harvest were calculated with newly developed mathematical formulae. Snap bean was studied in a series of ten planting dates in 1989 and 1990 at Knoxville, Tennessee, and a series of four plantings in 1991 at Knoxville and Greenville, Tennessee. Cowpea was studied in 1991 and 1992 at Knoxville and Greenville, with four plantings at each location. Sweet corn was studied in 1992 at Crossville, with eight plantings. Ambient air temperature (1.5 m above sod) was measured with Standard National Weather Service maximum and minimum thermometers.

3. Results and discussion

3.1. Mathematical formulae

3.1.1. The least standard deviation in GDD method

Let $f_i(x)$ be GDD of the ith planting, the function of the base temperature (x) selected. Then $f_i(x)$ is expressed as

$$f_i(x) = (T_i - x)d_i \tag{5}$$

where T_i is the overall mean temperature of the ith planting, and d_i is the number of days of the ith planting to reach a given developmental stage under study. Let

$$f(x) = \frac{\sum_{i=1}^{n} f_i(x)}{n} \tag{6}$$

be the mean of GDD accumulations for the all plantings, where n is the number of plantings. Then the standard deviation in GDD accumulations for all plantings is defined as

$$SD_{\text{gdd}} = \left\{ \frac{\sum_{i=1}^{n} [f_i(x) - f(x)]^2}{n-1} \right\}^{\frac{1}{2}} \tag{7}$$

where SD_{gdd} is the abbreviation of the standard deviation of GDD accumulations.

By taking the derivative of SD_{gdd}, the equation

$$\frac{\mathrm{d}SD_{\text{gdd}}}{\mathrm{d}x} = \left(\left\{ \frac{\sum_{i=1}^{n} [f_i(x) - f(x)]^2}{n-1} \right\}^{\frac{1}{2}} \right)' \tag{8}$$

is obtained.

Let Eq. (8) equal zero and the base temperature (x) can be calculated with equation

$$x = \frac{\sum_{i=1}^{n} T_i d_i \sum_{i=1}^{n} d_i - n \sum_{i=1}^{n} d_i^{\,2} T_i}{(\sum_{i=1}^{n} d_i)^2 - n \sum_{i=1}^{n} d_i^2} \tag{9}$$

where T_i is the overall mean temperature of the ith planting and d_i is the number of days of the ith planting to reach a developmental stage under study.

3.1.2. The least standard deviation in days method

Standard deviation in days is defined as

$$SD_{\text{day}} = \frac{SD_{\text{gdd}}}{T - X} \tag{10}$$

where SD_{day} is standard deviation in days, SD_{gdd} is standard deviation in GDD, T is the overall mean temperature of all plantings, and x is the base temperature.

By taking the derivative of SD_{day}, the equation

$$\frac{\mathrm{d}SD_{\text{day}}}{\mathrm{d}x} = \left(\left\{\frac{\sum_{i=1}^{n}[f_i(x) - f(x)]^2}{(n-1)(T-x)^2}\right\}^{\frac{1}{2}}\right)' \tag{11}$$

is obtained, where T is the overall mean of temperature of all plantings and n is the number of plantings.

Let Eq. (11) be zero and the base temperature can be calculated with the equation

$$x = T - \frac{(\sum_{i=1}^{n} t_i d_i)^2 - n\sum_{i=1}^{n} t_i^2 d_i^2}{n\sum_{i=1}^{n} d_i^2 t_i - n\sum_{i=1}^{n} t_i d_i \sum_{i=1}^{n} d_i} \tag{12}$$

where d_i is the number of days required to reach a developmental stage for the ith planting and t_i is the difference of the overall mean of temperature in all plantings and the mean temperature of the ith plantings.

3.1.3. Coefficient of variation in GDD method

Coefficient of variation in GDD is defined as

$$CV_{\text{gdd}} = \frac{SD_{\text{gdd}}}{f(x)} \times \frac{100}{100} \tag{13}$$

where CV_{gdd} is the coefficient of variation in GDD, SD_{gdd} is the standard deviation of GDD accumulations in n plantings, and $f(x)$ is the mean of GDD accumulations of n plantings.

By taking the derivative of CV_{gdd}, the equation

$$\frac{\mathrm{d}CV_{\text{gdd}}}{\mathrm{d}x} = \left(\left\{\frac{\sum_{i=1}^{n}[f_i(x) - f(x)]^2}{(n-1)f(x)}\right\}^{\frac{1}{2}}\right)' \tag{14}$$

is obtained.

Let Eq. (14) be zero and the base temperature (x) can be expressed as

$$x = \frac{\sum_{i=1}^{n} T_i d_i^2 \sum_{i=1}^{n} T_i d_i - \sum_{i=1}^{n} d_i \sum_{i=1}^{n} T_i^2 d_i^2}{\sum_{i=1}^{n} d_i^2 \sum_{i=1}^{n} T_i d_i - \sum_{i=1}^{n} d_i \sum_{i=1}^{n} T_i d_i^2} \tag{15}$$

where T_i is the overall mean of temperature of the ith planting, d_i is the number of days of the ith planting, and n is the number of all plantings.

3.1.4. Regression coefficient method

The equation $Y_i = a + Bt_i$ was discussed in the literature review. By studying this simple linear regression model, it was found that there was an analytical way to calculate the base temperature with the regression coefficient method. The basic idea was that if the true base temperature is used in the calculation of GDD_i of the ith planting, GDD_i should be independent of T_i, the average temperature of the ith planting, that is, the regression coefficient of T_i, b, should be equal to zero. In this

Table 1
The base temperatures (T), standard deviation in growing degree day accumulation (SD_{gdd}), standard deviation in days (SD_{day}), coefficients of variation in growing degree day accumulations (CV_{gdd}), and coefficients of variation in days (CV_{day}) calculated for snap bean, snow pea, and sweet corn with the standard deviation in GDD (ST), the standard deviation in days (SD), coefficient of variation (CV), and regression method (RE)

Crop	Method	T (°C)	SD_{gdd} (GDD)	SD_{day} (day)	CV_{gdd} (%)	CV_{day} (%)
Snap bean	SD	6.1	50.8	1.70	3.47	3.45
	ST	8.3	47.2	1.90	3.74	3.71
	CV	6.1	50.9	1.70	3.47	3.45
	RE	5.6	52.4	1.70	3.47	3.46
Cowpea	SD	11.0	138.0	6.50	7.92	7.83
	ST	15.0	111.9	8.10	9.82	9.66
	CV	10.9	138.9	6.50	7.92	7.83
	RE	11.0	137.8	6.50	7.92	7.83
Sweet corn	SD	6.4	57.1	2.30	2.66	2.66
	ST	10.4	48.0	2.70	3.16	3.16
	CV	6.4	57.1	2.30	2.66	2.66
	RE	6.3	57.6	2.30	2.66	2.66

regression model, the regression coefficient of T_i can be expressed as

$$b = \frac{n\sum_{i=1}^{n} \mathrm{GDD}_i T_i - \sum_{i=1}^{n} T_i \sum_{i=1}^{n} \mathrm{GDD}_i}{n\sum_{i=1}^{n} T_i^2 - (\sum_{i=1}^{n} T_i)^2} \tag{16}$$

and then the base temperature (x) can be calculated by the equation

$$x = \frac{\sum_{i=1}^{n} T_i \sum_{i=1}^{n} d_i T_i - n\sum_{i=1}^{n} d_i T_i^2}{\sum_{i=1}^{n} d_i \sum_{i=1}^{n} T_i - n\sum_{i=1}^{n} d_i T_i} \tag{17}$$

where T_i is the overall mean of temperature of the ith planting, d_i is the number of days of the ith planting, and n is the number of all plantings.

3.2. Base temperature

The base temperature of snap bean, cowpea, and sweet corn from planting to harvest were calculated easily and accurately with four new mathematical formulae and without cumbersome calculation processes. Results (Table 1) showed that base temperatures calculated from the standard deviation in days (SD), coefficient variation (CV), and regression coefficient (RE) methods were very similar, but departed considerably from the standard deviation in GDD method (ST) because the method of choosing the base temperature with the ST method was different those with the other methods. Coefficients of variation in both GDD and days were slightly larger from the ST method than those from the SD, CV, and RE methods, although standard deviation in GDD from ST method was the smallest. Therefore, the SD, CV, and RE methods are superior to the ST method and should be used to calculate the base temperature.

One additional advantage of the mathematical formula approach to the determination of the base temperature for GDD is the detection of inconsistent crop phenological data, owing to either observer error or widely fluctuating environmental (other than temperature) conditions. The resultant base temperature looks unrealistic if crop phenological data are inconsistent. For example, when suspicious harvest data from snap bean harvests at Knoxville in 1990 and 1991 were pooled with the good data, unrealistic base temperatures of −126.1°C, 21.7°C, −128.9°C, and −106.7°C were calculated with the ST, SD, CV, and RE methods, respectively. The good data alone produced equivalent base temperatures of 6.7°C, 8.8°C, 6.8°C, and 6.2°C; the bad data alone produced extremely unrealistic results. If calculation procedure prior to this study had been used, unrealistic base temperature such as −126.1° or 21.7°C, which, in fact might produce the lowest variation in both GDD and days, would never have been chosen as candidate base temperatures. Instead, more realistic temperature such as 8°C or 10°C would have been chosen, even though, in reality, they would not result in the lowest variation. With the calculation procedures presented in this study, unrealistic base temperatures are detected quickly and the suspicious harvest data (in this case, data collected by an untrained observer) can be eliminated.

4. Conclusion

Four mathematical formulae for calculating the base temperature in heat unit systems using the least standard deviation in GDD and in days, coefficients of variation in GDD, and regression coefficient methods were proposed and proved. With these mathematical formulae, an accurate base temperature can be easily obtained without cumbersome calculation processes. Also, inconsistent observation of crop phenology or wide environmental fluctuations can be detected with these formulae. These mathematical formulae are new in terms of calculation procedures, but derived from the existing, widely used methods. Therefore, as long as researchers can determine the base temperature with the methods discussed in Section 1, these new mathematical formulae are applicable to calculate the base temperature for any crop and any developmental stage of the crop.

Appendix: Mathematical proofs of four proposed formulae in calculation of the base temperature

The least standard deviation in GDD method

Let SD_{gdd} represent the standard deviation in GDD; then

$$SD_{\mathrm{gdd}} = \left\{ \frac{\sum [f_i(x) - f(x)]^2}{n-1} \right\}^{\frac{1}{2}}$$

the derivative of SD_{gdd} is

$$\frac{\mathrm{d}SD_{\mathrm{gdd}}}{\mathrm{d}x} = \frac{1}{2}\left\{\frac{\sum[f_i(x) - f(x)]^2}{n-1}\right\}^{-\frac{1}{2}} \times \frac{2\sum[f_i(x) - f(x)][f_i'(x) - f'(x)]}{n-1}$$

Let

$$\frac{\mathrm{d}SD_{\mathrm{gdd}}}{\mathrm{d}x} = 0$$

then

$$\sum[f_i(x) - f(x)][f_i'(x) - f'(x)] = 0$$

Therefore

$$\begin{aligned}
\sum[f_i(x) - f(x)][f_i'(x) - f'(x)] &= \sum\left[(T_i - x)d_i - \frac{\sum(T_i - x)d_i}{n}\right]\left(-d_i + \frac{\sum d_i}{n}\right) \\
&= \sum\left[\frac{n(T_i - x)d_i - \sum(T_i - x)d_i}{n}\right]\left(\frac{\sum d_i - nd_i}{n}\right) \\
&= \frac{1}{n}\sum\left(nT_id_i - nxd_i - \sum T_id_i + x\sum d_i\right) \\
&\quad \times \left(\sum d_i - nd_i\right)
\end{aligned}$$

$$\sum d_i = D \text{ and } \sum T_id_i = A$$

then

$$\begin{aligned}
&\sum[f_i(x) - f(x)][f_i'(x) - f'(x)] = \tfrac{1}{n}\sum(nT_id_i - nxd_i - A + xd)(D - nd_i) \\
&= \frac{1}{n}\left(\sum nDT_id_i - nxd_iD - AD + XD^2 - n^2T_iD_i^2 + n^2xd_i^2 + nAd_i - xDnd_i\right) \\
&= \frac{1}{n}\left(\sum nAD - nxD^2 - nAD + nxD^2 - n^2\sum T_id_i^2 + n^2x\sum d_i^2\right. \\
&\quad \left. + nAD - nxD^2\right) \\
&= -n\sum T_id_i^2 + nx\sum d_i^2 + AD - xD^2 = 0
\end{aligned}$$

Therefore

$$x = \frac{AD - n\sum T_id_i^2}{D^2 - n\sum d_i^2} = \frac{\sum T_id_i \sum d_i - n\sum T_id_i^2}{(\sum d_i)^2 - n\sum d_i^2}$$

The least standard deviation in days method

Let SD_{day} represent the standard deviation in days, then

$$SD_{\text{day}} = \frac{SD_{\text{gdd}}}{T-x} = \left\{\frac{\sum[f_i(x)-f(x)]^2}{(n-1)(T-x)^2}\right\}^{\frac{1}{2}} = \frac{1}{(n-1)^{\frac{1}{2}}}\left\{\frac{\sum[f_i(x)-f(x)]^2}{(T-x)^2}\right\}^{\frac{1}{2}}$$

Let $y = T - x$, then $x = T - y$. Therefore $f_i(x) = f_i(T-y) = (T_i - T + y)d_i$ $(t_i + y)d_i = f_i(y)$. Therefore

$$SD_{\text{day}} = \frac{1}{(n-1)^{\frac{1}{2}}}\left\{\frac{\sum[f_i(y)-f(y)]^2}{y^2}\right\}^{\frac{1}{2}}$$

The derivative of SD_{day} is

$$\frac{\mathrm{d}SD_{\text{day}}}{\mathrm{d}y} = \frac{1}{2\times(n-1)^{\frac{1}{2}}}\left\{\frac{\sum[f_i(y)-f(y)]}{y^2}\right\}^{-\frac{1}{2}} \times \left\{\frac{2\sum[f_i(y)-f(y)][f_i'(y)-f'(y)]y^2 - 2y\sum[f_i(y)-f(y)]^2}{y^4}\right\}$$

Let

$$\frac{\mathrm{d}SD_{\text{day}}}{\mathrm{d}y} = 0$$

then

$$\left\{\frac{2\sum[f_i(y)-f(y)][f_i'(y)-f'(y)]y^2 - 2y\sum[f_i(y)-f(y)]^2}{y^4}\right\} = 0$$

Therefore

$$\sum[f_i(y)-f(y)][f_i'(y)-f'(y)]y - \sum[f_i(y)-f(y)]^2 = 0$$

Because

$$f_i'(y) = d_i, \qquad f'(y) = \frac{\sum d_i}{n}$$

therefore

$$\begin{aligned}
&\sum[f_i(y)-f(y)][f_i'(y)-f'(y)] - \sum[f_i(y)-f(y)]^2 \\
&= \sum[f_i(y)-f(y)][d_i - \frac{\sum d_i}{n}]y - \sum[f_i(y)-f(y)]^2 \\
&= \sum\left\{[f_i(y)-f(y)]\left[(t_i+y)d_i - t_i d_i - \frac{\sum(t_i+y)d_i}{n} + \frac{\sum t_i d_i}{n}\right]\right\} \\
&\quad - \sum[f_i(y)-f(y)]^2
\end{aligned}$$

$$= \sum [f_i(y) - f(y)]\left\{[f_i(y) - f(y)] - \left(t_i d_i - \frac{\sum t_i d_i}{n}\right)\right\} - \sum [f_i(y) - f(y)]^2$$

$$= -\sum [f_i(y) - f(y)]\left[t_i d_i - \frac{\sum t_i d_i}{n}\right]$$

$$= \sum [f_i(y) - f(y)]\left[t_i d_i - \frac{\sum t_i d_i}{n}\right] = 0$$

Therefore

$$\sum [f_i(y) - f(y)]\left[t_i d_i - \frac{\sum t_i d_i}{n}\right]$$

$$= \sum \left[t_i d_i + d_i y - \frac{\sum t_i d_i + y \sum d_i}{n}\right]\left[t_i d_i - \frac{\sum t_i d_i}{n}\right]$$

$$= \sum \left[t_i d_i - \frac{t_i d_i \sum t_i d_i}{n} + t_i d_i^2 y - d_i y \sum t_i d_i - \frac{t_i d_i \sum t_i d_i}{n} - \frac{t_i d_i y \sum t_i d_i}{n} + \frac{(\sum t_i d_i)^2}{n^2} + \frac{y \sum t_i d_i \sum d_i}{n^2}\right]$$

$$= \sum t_i^2 d_i^2 - \frac{(\sum t_i d_i)^2}{n} + y \sum t_i d_i^2 - \frac{y \sum t_i d_i \sum d_i}{n} = 0$$

Therefore

$$y = \frac{(\sum t_i d_i)^2 - n \sum t_i^2 d_i^2}{n \sum t_i d_i^2 - \sum t_i d_i \sum d_i}$$

Therefore

$$x = \frac{\sum T_i}{n} - \frac{(\sum t_i d_i)^2 - n \sum t_i^2 d_i^2}{n \sum t_i d_i^2 - \sum t_i d_i \sum d_i}$$

$$= \frac{\sum T_i}{n} - \frac{[\sum (T - T_i) d_i]^2 - n \sum (T - T_i)^2 d_i^2}{n \sum (T - T_i) d_i^2 - \sum (T - T_i) d_i \sum d_i}$$

Coefficient of variation in GDD method

Let CV_{gdd} represent the coefficient of variation in GDD; then

$$CV_{\mathrm{gdd}} = \frac{SD_{\mathrm{gdd}}}{f(x)} = \left\{\frac{\sum [f_i(x) - f(x)]^2}{(n-1) f^2(x)}\right\}^{\frac{1}{2}}$$

The derivative of CV_{gdd} is

$$\frac{\mathrm{d}CV_{\mathrm{gdd}}}{\mathrm{d}x} = \frac{1}{2}\left\{\frac{\sum [f_i(x) - f(x)]^2}{(n-1) f^2(x)}\right\}^{-\frac{1}{2}} \left\{\frac{\sum [f_i(x) - f(x)]^2}{(n-1) f^2(x)}\right\}'$$

Because

$$\sum[f_i(x) - f(x)]^2 = \sum f_i^2(x) + nf^2(x) - 2f(x)\sum f_i(x)$$

$$nf^2(x) - 2f(x)\sum f_i(x) = f(x)[\frac{n\sum f_i(x)}{n} - 2\sum f_i(x)] = -f(x)\sum f_i(x)$$

therefore

$$\sum[f_i(x) - f(x)]^2 = \sum f_i^2(x) - f(x)\sum f_i(x)$$

Therefore

$$\left\{\frac{\sum[f_i(x) - f(x)]^2}{(n-1)f^2(x)}\right\} = \frac{1}{(n-1)}\left[\frac{\sum f_i^2(x)}{f^2(x)} - \frac{\sum f_i(x)}{f(x)}\right]$$

$$= \frac{1}{(n-1)}\left[\frac{\sum f_i^2(x)}{f^2(x)} - \frac{n\sum f_i(x)}{\sum f_i(x)}\right]$$

$$= \frac{1}{(n-1)}\left[\frac{\sum f_i^2(x)}{f^2(x)} - n\right]$$

Let

$$\frac{\mathrm{d}CV_{\mathrm{gdd}}}{\mathrm{d}x} = 0$$

Then

$$\left\{\frac{\sum[f_i(x) - f(x)]^2}{(n-1)f^2(x)}\right\}'$$

$$= \frac{1}{(n-1)}\left[\frac{\sum f_i^2(x)}{f^2(x)} - n\right]'$$

$$= \left[\frac{1}{n-1}\right]\frac{2\sum f_i(x)f_i'(x)f^2(x) - 2f(x)f'(x)\sum f_i^2(x)}{f^4(x)}$$

$$= \left[\frac{1}{n-1}\right]\frac{2\sum f_i(x)f_i'(x)f(x) - 2f'(x)\sum f_i^2(x)}{f^3(x)} = 0$$

Therefore

$$\sum f_i(x)f_i'(x)f(x) = f'(x)\sum f_i^2(x)$$

Therefore

$$\sum(T_i - x)d_i(-d_i)\frac{\sum(T_i - x)d_i}{n} = -\frac{\sum d_i}{n}\sum(T_i d_i - xd_i)^2$$

Let $\sum d_i = D$ and $\sum T_i d_i = A$; then

$$\sum (T_i - x)d_i(-d_i)\frac{\sum(T_i - x)d_i}{n} = \frac{[\sum(-T_i d_i^2 + x d_i^2)](A - xD)}{n}$$

$$- \frac{\sum d_i}{n}\sum(T_i d_i - x d_i)^2$$

$$= -\frac{D}{n}\sum(T_i^2 d_i^2 + x^2 d_i^2 - 2x T_i d_i^2)$$

Therefore

$$-A\sum T_i d_i^2 + Ax\sum d_i^2 + Dx\sum T_i d_i^2 - Dx^2\sum d_i^2$$
$$= D\left(-\sum T_i^2 d_i^2 - x^2\sum d_i^2 + 2x\sum T_i d_i^2\right) - A\sum T_i d_i^2 + Ax\sum d_i^2$$
$$= -D\sum T_i^2 d_i^2 + x\sum T_i d_i^2$$

Therefore

$$x = \frac{A\sum T_i d_i^2 - D\sum T_i^2 d_i^2}{A\sum d_i^2 - D\sum T_i d_i^2} = \frac{\sum T_i d_i \sum T_i d_i^2 - \sum d_i \sum T_i^2 d_i^2}{\sum T_i d_i \sum d_i^2 - \sum d_i \sum T_i d_i^2}$$

Regression coefficient method

Growing degree days can be expressed as $f_i(x) = a + Bt_i$, where x is the base temperature selected and T_i is the mean of average daily temperature of the ith planting. Because GDD is a constant for crop development, $f_i(x)$ does not depend on T_i. Therefore, the regression coefficient of T_i will be zero.

Because

$$b = \frac{n\sum T_i f_i(x) - \sum T_i \sum f_i(x)}{n\sum T_i^2 - (\sum T_i)^2} = 0$$

therefore

$$n\sum T_i f_i(x) - \sum T_i \sum f_i(x) = 0, \qquad n\sum T_i f_i(x) = \sum T_i \sum f_i(x)$$

Therefore

$$n\sum T_i(T_i - x)d_i = \sum T_i \sum (T_i - x)d_i$$

Therefore

$$n\sum T_i^2 d_i - nx\sum T_i d_i = \sum T_i \sum T_i d_i - x\sum T_i \sum d_i$$

Therefore

$$x = \frac{\sum T_i \sum T_i d_i - n\sum T_i^2 \sum d_i}{\sum T_i \sum d_i - n\sum T_i \sum d_i}$$

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