EECS 769 Homework 5

Elise McEllhiney

December 7, 2016

0.1 Uniformly distributed noise

Let the input random variable X to a channel be uniformly distributed over the interval $-1/2 \leq x \leq +1/2$. Let the output of the channel be $Y = X + Z$, where the noise random variable is uniformly distributed over the interval $-a/2 \leq z \leq +a/2$.

The density of Y is as follows:
For $a < 1$:

$$p_Y(y) = \begin{cases} (\frac{1}{2a})(y + \frac{1+a}{2}) & -\frac{1+a}{2} \leq y \leq -\frac{1-a}{2} \\ 1 & -\frac{1-a}{2} \leq y \leq +\frac{1-a}{2} \\ (\frac{1}{2a})(-y - \frac{1+a}{2}) & +\frac{1-a}{2} \leq y \leq +\frac{1+a}{2} \end{cases}$$

For $a = 1$: Y is triangular over [-1, +1]
For $a > 1$:

$$p_Y(y) = \begin{cases} y + \frac{1+a}{2} & -\frac{1+a}{2} \leq y \leq -\frac{1-a}{2} \\ \frac{1}{a} & -\frac{1-a}{2} \leq y \leq +\frac{1-a}{2} \\ -y - \frac{1+a}{2} & +\frac{1-a}{2} \leq y \leq +\frac{1+a}{2} \end{cases}$$

(a) Find $I(X;Y)$ as a function of a.

If $a < 1$ the output Y will be conditionally uniformly distributed with probability $1 - a$ over the interval $[-\frac{1-a}{2}, +\frac{1-a}{2}]$. Y will have a triangular density with the base of the triangle having width a with probability a.

$$h(Y) = H(a) + (1-a)\ln(1-a) + a(\frac{1}{2} + \ln a)$$

$$h(Y) = -a \ln a - (1-a)\ln(1-a) + (1-a)\ln(1-a) + \frac{a}{2} + a \ln a$$

$$h(Y) = \frac{a}{2} nats$$

If $a > 1$ Y's trapezoidal density be scaled by a factor a:

$$h(Y) = \ln a + \frac{1}{2a}$$

Therefore:

$$I(X;Y) = \begin{cases} \frac{a}{2} - \ln a & a \leq 1 \\ \frac{1}{2a} & a \geq 0 \end{cases}$$

(b) For $a = 1$ find the capacity of the channel when the input X is peak-limited; that is, the range of X is limited to $-1/2 \leq x \leq +1/2$. What probability distribution on X maximizes the mutual information $I(X;Y)$?

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z)$$

Both X and Y are restricted to the interval $[-\frac{1}{2}, +\frac{1}{2}]$. This means that since $Y = X + Z$, it is limited to the interval $[-1, +1]$. The differential entropy of Y is $\leq$ the entropy of a uniformly distributed random variable on the interval $[-1, +1]$. This means, $h(Y) \leq 1$. This equation for maximum entropy achieves equality iff $x = \pm 1$ with probabilities of $\frac{1}{2}$ each.

$$I(X;Y) = h(Y) - h(Z) = 1 - 0 = 1$$

$$\hat{X} = \begin{cases} -\frac{1}{2} & y < 0 \\ +\frac{1}{2} & y \geq 0 \end{cases}$$

(c) (Optional) Find the capacity of the channel for all values of a, again assuming that the range of X is limited to $-\frac{1}{2} \le x \le \frac{1}{2}$

0.2 Channel with uniformly distributed noise:

Consider a additive channel whose input alphabet $\mathcal{X} = \{0, \pm 1, \pm 2\}$, and whose output $Y = X + Z$, where Z is uniformly distributed over the interval $[-1, 1]$. Thus the input of the channel is a discrete random variable, while the output is continuous. Calculate the capacity $C = \max_{p(x)} I(X;Y)$ of this channel

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z)$$

$$h(Z) = \log 2 \qquad \text{because Z is uniform over the interval [-1,1]}$$

If the probabilities of X are $p_{-2}, p_{-1}, ..., p_2$, then Y has a uniform output distribution with weight $\frac{p_{-2}}{2}$ for $-3 \le Y \le -2$, weight $\frac{(p_{-2} - p_{-1})}{2}$ for $-2 \le Y \le -1$,... Since Y is within the range of $[-3, 3]$, the maximum entropy possible occurs for a uniform distribution over the range. This is achieved if X has the distribution $(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$. This means $h(Y) = \log 6$ and the capacity is: $C = \log 6 - \log 2 = \log 3$

0.3 The two-look Gaussian channel.

Consider the ordinary Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$

$$Y_2 = X + Z_2$$

with a power constraint P on X, and $(Z_1, Z_2)\ \mathcal{N}_2(0, K)$, where

$$K = \begin{bmatrix} N & N_\rho \\ N_\rho & N \end{bmatrix}$$

$$\begin{aligned} C_2 &= \max I(X; Y_1, Y_2) \\ &= h(Y_1, Y_2) - h(Y_1, Y_2|X) \\ &= h(Y_1, Y_2) - h(Z_1, Z_2|X) \\ &= h(Y_1, Y_2) - h(Z_1, Z_2) \end{aligned} \tag{1}$$

$$(Z_1, Z_2) \sim \mathcal{N}(0, \begin{bmatrix} N & N_\rho \\ N_\rho & N \end{bmatrix})$$

$$h(Z_1, Z_2) = \frac{1}{2} \log{(2\pi e)^2 |K_Z|} = \frac{1}{2} \log{(2\pi e)^2 N^2 (1 - \rho^2)}$$

Because,

$$Y_1 = X + Z_1$$

$$Y_2 = X + Z_2$$

We know

$$(Y_1, Y_2) \sim \mathcal{N}(0, \begin{bmatrix} P + N & P + \rho N \\ P + \rho N & P + N \end{bmatrix})$$

$$
\begin{aligned}
h(Y_1, Y_2) = \frac{1}{2}\log{(2\pi e)^2(N^2(1-\rho^2) + 2PN(1-\rho))} \\
C_2 = h(Y_1, Y_2) - h(Z_1, Z_2) \\
= \frac{1}{2}\log{(1 + \frac{2P}{N(1+\rho})}
\end{aligned} \tag{2}
$$

Find the capacity for:

(a) $\rho = 1$

$$C = \frac{1}{2}\log{(1 + \frac{P}{N})}$$

This is the same as the capacity of a single look channel

(b) $\rho = 0$

$$C = \frac{1}{2}\log{(1 + \frac{2P}{N})}$$

(c) $\rho = -1$

$$C = \infty$$

If we add Y_1 and Y_2 we can recover X

0.4 Fading Channel

Consider and additive noise fading channel where Z is additive noise, V is a random variable representing fading, and Z and V are independent of each other and of X. Argue that the knowledge of the fading factor V improves capacity by showing

$$I(X;Y|V) \geq I(X;Y)$$

$$
\begin{aligned}
I(X;Y,V) &= I(X;V) + I(X;Y|V) \\
&= I(X;Y) + I(X;V|Y)
\end{aligned} \tag{3}
$$

$$I(X;V) + I(X;Y|V) = I(X;Y) + I(X;V|Y)$$

$$I(X;Y|V) = I(X;Y) + I(X;V|Y) \qquad \text{Since X and V are independent}$$

$$I(X;Y|V) \geq I(X;Y) \qquad \text{Since} I(X;V|Y) \geq 0$$

0.5 Multipath Gaussian Channel

Consider a Gaussian noise channel of power constraint P, where the signal takes two different paths and the received noisy signals are added together at the antenna.

The channel simplifies to an additive noise channel with $2X$ as input, $Z_1 + Z_2$ as additive noise, and Y as output. The power constraint on 2X is 4P. Z_1 and Z_2 are zero mean. This means that

$Z_1 + Z_2$ is also zero mean.

$$\begin{aligned} Var(Z_1+Z_2) &= E[(Z_1+Z_2)^2] \\ &= E[Z_1^2 + Z_2^2 + 2Z_1Z_2] \\ &= 2\sigma^2 + 2\rho\sigma^2 \end{aligned} \tag{4}$$

(a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix:

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

$$\begin{aligned} C &= \frac{1}{2}\log(1 + \frac{4P}{2\sigma^2(1+\rho)}) \\ &= \frac{1}{2}\log(1 + \frac{2P}{\sigma^2(1+\rho)}) \end{aligned} \tag{5}$$

(b) What is the capacity for $\rho = 0$, $\rho = 1$, $\rho = -1$?

For $\rho = 0$

$$C = \frac{1}{2}\log(1 + \frac{2P}{\sigma^2})$$

For $\rho = 1$

$$C = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

For $\rho = -1$

$$C = \infty$$

0.6 Time varying channel

A train pulls out of the station at constant velocity. The received signal energy thus falls of with time as $\frac{1}{i^2}$. The total received signal at time i. is:

$$Y_i = (\frac{1}{i})X_i + Z_i$$

where $Z_1, Z_2, \ldots$ are i.i.d. $N(0, N)$. The transmitter constraint for block length n is

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2(w) \leq P, \quad w \in 1, 2, \ldots, 2^{nR}$$

Using Fano's inequality, show that the capacity C is equal to zero for this channel.

$$\begin{aligned}
nR &= H(W) \\
&= I(W;\hat{W}) + H(W|\hat{W}) \\
&\leq I(W:\hat{W}) + n\epsilon_n \\
&\leq I(X^n;Y^n) + n\epsilon_n \\
&= h(Y^n) - h(Y^n|X^n) + n\epsilon_n \\
&= h(Y^n) - h(Z^n) + n\epsilon_n \\
&\leq \sum_{i=1}^{n} h(Y_i) - \sum_{i=1}^{n} h(Z_i) + n\epsilon_n \\
&= \sum_{i=1}^{n} I(X_i;Y_i) + n\epsilon_n
\end{aligned} \tag{6}$$

Let P_i be the average power of the ith column of the codebook

$$P_i = \frac{1}{2^{nR}} \sum_w x_i^2(w)$$

Because:

$$Y_i = \frac{1}{i} X_i + Z_i$$

And X_i and Z_i are independent,
The average power of Y_i is $\frac{1}{i^2} P_i + N$
And since entropy is maximized by the normal distribution, we have:

$$h(Y_i) \leq \frac{1}{2} \log 2\pi e(\frac{1}{i^2} P_i + N)$$

Conversely,

$$\begin{aligned}
nR &\leq \sum (h(Y_i) - h(Z_i)) + n\epsilon_n \\
&\leq \sum (\frac{1}{2} \log (2\pi e(\frac{1}{i^2} P_i + N)) - \frac{1}{2} \log 2\pi e N) + n\epsilon_n \\
&= \sum (\frac{1}{2} \log (1 + \frac{P_i}{i^2 N})) + n\epsilon_n
\end{aligned} \tag{7}$$

The average satisfies the power constraint since each of the codewords satisfies the power constraint

$$\frac{1}{n} \sum_i P_i \leq P$$

If we use water-filling, the optimal solution is to use our limited power on the channels with the least noise first. The noise in each channel i is:

$$N_i = i^2 N$$

Thus we put power only into channels where:

$$P_i + N_i \leq \lambda$$

The height of the water $\leq N + nP$
For all the channels where we put power:

$$i^2 N < nP + N$$

The average rate is $< \frac{1}{n}\sqrt{n}\frac{1}{2}\log(1 + \frac{nP}{N})$
The capacity per transmission goes to 0. Therefore, the channel capacity goes to 0.