Formularium - Electromagnetism I

1 Introduction

$$AB + BA = \heartsuit \tag{1}$$

2 Maxwell's Equations

3 Electrostatics

4 Magnetostatics

5 Plane Waves

5.1 Plane waves in a lossless dielectric

Assumptions Source free, homogenous, lossless and isotrpoic dielectric characterized by $\epsilon = \epsilon_0 \epsilon_r$ and $\mu = \mu_0 \mu_r$.
Plane wave propagation to an arbitrary direction $\vec{u}$:

$$\vec{e}(\vec{r}) = \vec{A} e^{-jk\vec{u}\cdot\vec{r}}, \tag{2}$$

$$\vec{h}(\vec{r}) = \frac{1}{Z_c} \vec{u} \times \vec{e}(r), \tag{3}$$

with $\vec{A} \cdot \vec{u} = 0$, $Z_c = \frac{\mu}{\epsilon}$, $k^2 = \omega^2 \epsilon \mu$ and Poynting's vector:

$$\vec{p}(\vec{r}) = \frac{|\vec{A}|^2}{2Z_c} \vec{u}. \tag{4}$$

Plane waves in a lossy dielectric

Equations 2 and 3 with $\vec{A} \cdot u = 0$ remain unchanged but k and Z_c are now complex valued (in VI). Denote $k = \beta - j\alpha$, the new Poynting vector becomes

$$\vec{p}(\vec{r}) = \frac{|\vec{A}|^2}{2Z_c} e^{-2\alpha\vec{u}\cdot\vec{r}} \vec{u}. \tag{5}$$

This shows that when a wave propagates over a distance d in a lossy medium, its power descreases by a factor $e^{-2\alpha d}$. Expressed in decibel (dB) gives

$$-10 \log_{10}(e^{-2\alpha d}) \approx 8.686 \alpha d, \tag{6}$$

with $L = 8.686\alpha$ the relative power loss in (dB/m).

Considering a material with conduction losses ($\epsilon + \frac{\sigma}{j\omega}$):

$$k = \omega \sqrt{\epsilon \mu} \sqrt{1 + \frac{\sigma}{j\omega}}, \tag{7}$$

$$Z_c = \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\sqrt{1 + \frac{\sigma}{j\omega}}}, \tag{8}$$

gives rise to two interesting situations:

1. Low-loss dielectric ($\sigma \ll \omega\epsilon$):

$$k = \beta - j\alpha \approx \omega\sqrt{\epsilon\mu} - j\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}, \tag{9}$$

$$Z_c \approx \sqrt{\frac{\mu}{\epsilon}}, \tag{10}$$

2. Good conductor ($\sigma \gg \omega\epsilon$):

$$k = \beta - j\alpha \approx \frac{1-j}{\delta}, \tag{11}$$

$$Z_c \approx \frac{1+j}{\sigma\delta}, \tag{12}$$

with $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ known as the *skin depth*.

5.2 Reflection and transmission at a plane interface

Given the incident fields in medium 1:

$$\vec{e_i} = \vec{A}e^{-jk_1\vec{u_i}\cdot\vec{r}}, \tag{13}$$

$$\vec{h_i} = \frac{1}{Z_1}\vec{u_i} \times \vec{e_i}, \tag{14}$$

with $u_i = \cos(\theta_i)\vec{u}_z + \sin(\theta_i)\vec{u}_x$ and $\vec{A}\cdot\vec{u}_i = 0$. Then:

1. The reflected wave becomes

$$\vec{e_r} = \vec{B}e^{-jk_1\vec{u_r}\cdot\vec{r}}, \tag{15}$$

$$\vec{h_r} = \frac{1}{Z_1}\vec{u_r} \times \vec{e_r}, \tag{16}$$

with $u_r = -\cos(\theta_i)\vec{u}_z + \sin(\theta_i)\vec{u}_x$ and $\vec{B}\cdot\vec{u}_r = 0$.

2. The transmitted wave becomes

$$\vec{e_t} = \vec{C}e^{-jk_2\vec{u_t}\cdot\vec{r}}, \tag{17}$$

$$\vec{h_t} = \frac{1}{Z_2}\vec{u_t} \times \vec{e_t}, \tag{18}$$

with $u_t = \cos(\theta_t)\vec{u}_z + \sin(\theta_t)\vec{u}_x$, $\vec{C}\cdot\vec{u}_t = 0$ and where θ_t is defined as

$$k_2 \sin\theta_t = k_1 \sin\theta_i. \tag{19}$$

equation 19 gives rise to some problems. To better understand these problems, consider the case of lossless materials. Two cases must be distinguished:

i. Medium 2 is more dense than medium 1 ($k_2 > k_1$ or $N_2 > N_1$ with $N_i = \sqrt{\epsilon_{ri}\mu_{r_i}}$). Then

$$\vec{u_t} = \vec{u}_z\sqrt{1 - \frac{k_1^2}{k_2^2}\sin^2(\theta_i)} + \sin(\theta_t)\vec{u}_x, \tag{20}$$

and equation 19 can be rewritten as, what is known as *Snell's law*,

$$\sin(\theta_t) = \frac{N_1}{N_2}\sin(\theta_i). \tag{21}$$

ii. Medium 2 is less dense than medium 1 ($k_2 < k_1$ or $N_2 < N_1$). If

$$\sin(\theta_i) < \frac{N_2}{N_1} = \sin(\theta_c), \tag{22}$$

with θ_c the critical angle, then equation 20 still holds. When θ_i exceeds θ_c, equation 20 becomes

$$\vec{u_t} = -j\vec{u}_z\sqrt{\frac{k_1^2}{k_2^2}\sin^2(\theta_i) - 1} + \sin(\theta_t)\vec{u}_x, \tag{23}$$

and the transmitted wave decays exponentially when propagating in medium 2 with skin depth

$$\delta = \frac{1}{\sqrt{k_1^2 \sin^2(\theta_i) - k_2^2}}, \quad \theta_i \in [\theta_c, \frac{\pi}{2}] \tag{24}$$

Solving for $\vec{B}$ and $\vec{C}$: an incident wave with arbitrary elliptical polarization can be written as the superposition of a TE and a TM polarized contribution.

1. **TE polarization**

$$\begin{aligned}
\vec{e}_i(\vec{r}) &= Ae^{-jk_1\vec{u}_i\cdot\vec{r}}\vec{u}_y, \\
\vec{h}_i(\vec{r}) &= \frac{1}{Z_1}\vec{u}_i \times \vec{e}_i, \\
\vec{e}_r(\vec{r}) &= R_{TE}Ae^{-jk_1\vec{u}_r\cdot\vec{r}}\vec{u}_y, \\
\vec{h}_r(\vec{r}) &= \frac{1}{Z_1}\vec{u}_r \times \vec{e}_r, \\
\vec{e}_t(\vec{r}) &= T_{TE}Ae^{-jk_2\vec{u}_t\cdot\vec{r}}\vec{u}_y, \\
\vec{h}_t(\vec{r}) &= \frac{1}{Z_2}\vec{u}_t \times \vec{e}_t,
\end{aligned} \tag{25}$$

with

$$R_{TE} = \frac{Z_2\cos(\theta_i) - Z_1\cos(\theta_t)}{Z_2\cos(\theta_i) + Z_1\cos(\theta_t)}, \tag{26}$$

$$T_{TE} = \frac{2Z_2\cos(\theta_i)}{Z_2\cos(\theta_i) + Z_1\cos(\theta_t)}, \tag{27}$$

2. **TM polarization**

$$\begin{aligned}
\vec{h}_i(\vec{r}) &= \frac{1}{Z_1}Ae^{-jk_1\vec{u}_i\cdot\vec{r}}\vec{u}_y, \\
\vec{e}_i(\vec{r}) &= -Z_1\vec{u}_i \times \vec{h}_i, \\
\vec{h}_r(\vec{r}) &= \frac{1}{Z_1}R_{TM}Ae^{-jk_1\vec{u}_r\cdot\vec{r}}\vec{u}_y, \\
\vec{e}_r(\vec{r}) &= -Z_1\vec{u}_r \times \vec{h}_r, \\
\vec{h}_t(\vec{r}) &= \frac{1}{Z_2}T_{TM}Ae^{-jk_2\vec{u}_t\cdot\vec{r}}\vec{u}_y, \\
\vec{e}_t(\vec{r}) &= -Z_2\vec{u}_t \times \vec{h}_t,
\end{aligned} \tag{28}$$

with

$$R_{TM} = \frac{Z_1\cos(\theta_i) - Z_2\cos(\theta_t)}{Z_1\cos(\theta_i) + Z_2\cos(\theta_t)}, \tag{29}$$

$$T_{TM} = \frac{2Z_2\cos(\theta_i)}{Z_1\cos(\theta_i) + Z_2\cos(\theta_t)}, \tag{30}$$

Total transmission: for TM polarization R_{TM} becomes zero provided the angle of incidence is equal to the *Brewster agnle* defined by

$$\tan(\theta_B) = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{N_2}{N_1} \tag{31}$$