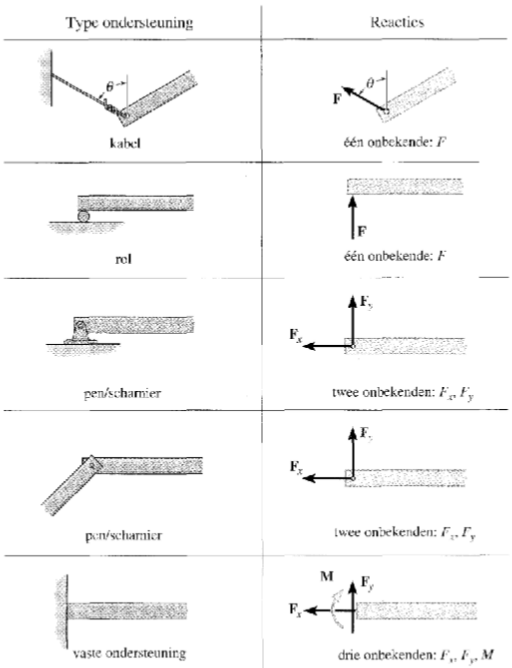
Inhoudsopgave

Hoofdstuk 1

Krachten, momenten, spanningne en rekken

1.1 STATICA EN EVENWICHT VAN CONSTRUCTIES

1.1.1 Types ondersteuningen



1.1.2 Evenwicht van een constructie

$$\sum \mathbf{F} = 0$$
$$\sum \mathbf{M}_O = 0$$

1.2 INTUÏtIEF BEGRIP VAN SPANNINGEN EN REKKEN

$$\sigma = \frac{F}{A_0}$$
$$\varepsilon = \frac{\Delta L}{L_0}$$
$$\sigma = E \cdot \varepsilon$$

1.3 SPANNINGEN

1.3.1 Definitie

De spanningsvector

$$\vec{\phi}^{(n)} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1.1}$$

De normaalspanning

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \tag{1.2}$$

De schuifspanning

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A} \tag{1.3}$$

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.4}$$

1.3.2 Verband tussen spanningsvector $\vec{\phi}^{(n)}$ en spanningstensor $[\sigma]$

Het verband tussen de spanningsvector en spanningstensor

$$\sigma_{ij} \cdot n_i = \phi_j^{(n)} \quad (i, j = x, y, z) \tag{1.5}$$

1.3.3 Vergelijkingen van het evenwicht

De vergelijkingen van het evenwicht

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0 \end{aligned} \tag{1.6}$$

Wet van de wederkerigheid der schuifspanningen

$$\begin{aligned} \tau_{xy} = \tau_{yx} \\ \tau_{xz} = \tau_{zx} \\ \tau_{yz} = \tau_{zy} \end{aligned} \tag{1.7}$$

1.3.4 Transformatie van coördinaten en hoofdrichtingen

Transformatieregels

$$[\sigma'] = [a] \cdot [\sigma] \cdot [a]^\top \tag{1.8}$$

met

$$a_{rk} = \vec{e}_r' \cdot \vec{e}_k \quad r, k = x, y, z \tag{1.9}$$

Diagonaalsom van $[\sigma]$

$$\Sigma_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{1.10}$$

Som van de hoofdminoren van $[\sigma]$

$$\Sigma_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \tag{1.11}$$

Determinant van $[\sigma]$

$$\Sigma_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{xz}\tau_{yz} \tag{1.12}$$

1.3.5 Kromlijnige coördinaten

Cilindercoördinaten

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \tau_{x\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} \tag{1.13}$$

De evenwichtsvergelijkingen

$$\begin{aligned} \frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial z} + F_r = 0 \\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + F_\theta = 0 \\ \frac{\partial\tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + F_z = 0 \end{aligned} \tag{1.14}$$

Bolcoördinaten

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \tau_{x\theta} & \tau_{r\phi} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{r\phi} & \tau_{\theta\phi} & \sigma_{\phi\phi} \end{bmatrix} \tag{1.15}$$

De evenwichtsvergelijkingen

$$\begin{aligned} \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta} + \frac{1}{r}\left(2\sigma_{rr} + \tau_{r\theta}\cot\theta - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + F_r = 0 \\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\phi} + \frac{1}{r}\left(3\tau_{r\theta} + \sigma_{\theta\theta}\cot\theta - \sigma_{\phi\phi}\cot\theta\right) + F_\theta = 0 \\ \frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\phi} + \frac{1}{r}\left(3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta\right) + F_\phi = 0 \end{aligned} \tag{1.16}$$

1.4 REKKEN

1.4.1 Veralgemeende vervormingstoestand

Verband tussen rekken en verplaatsing

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned} \tag{1.17}$$

1.4.2 Transformatie van coördinaten en hoofdrichtingen

Transformatieregels

$$[\varepsilon'] = [a] \cdot [\varepsilon] \cdot [a]^\top \quad \text{met } a_{rk} = \vec{e}'_r \cdot \vec{e}_k \tag{1.18}$$

De rekstensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{1.19}$$

Diagonaalsom van $[\varepsilon]$

$$i_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{1.20}$$

Som van de hoofdminoren van $[\varepsilon]$

$$I_2 = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xy}^2 - \varepsilon_{xz}^2 - \varepsilon_{yz}^2 \tag{1.21}$$

Determinant van $[\varepsilon]$

$$I_3 = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xx}\varepsilon_{yz}^2 - \varepsilon_{yy}\varepsilon_{xz}^2 - \varepsilon_{zz}\varepsilon_{xy}^2 + 2\varepsilon_{xy}\varepsilon_{xz}\varepsilon_{yz} \tag{1.22}$$

1.4.3 Compatibiliteitsvoorwaarden

$$\begin{aligned}
\frac{\partial^2\varepsilon_{xx}}{\partial y\partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial\varepsilon_{xz}}{\partial y} + \frac{\partial\varepsilon_{xy}}{\partial z} - \frac{\partial\varepsilon_{yz}}{\partial x}\right) \\
\frac{\partial^2\varepsilon_{yy}}{\partial x\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial\varepsilon_{xy}}{\partial z} + \frac{\partial\varepsilon_{yz}}{\partial x} - \frac{\partial\varepsilon_{xz}}{\partial y}\right) \\
\frac{\partial^2\varepsilon_{zz}}{\partial x\partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial\varepsilon_{yz}}{\partial x} + \frac{\partial\varepsilon_{xz}}{\partial y} - \frac{\partial\varepsilon_{xy}}{\partial z}\right) \\
\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} &= \frac{1}{2}\left(\frac{\partial\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2}\right) \\
\frac{\partial^2\varepsilon_{xz}}{\partial x\partial z} &= \frac{1}{2}\left(\frac{\partial\varepsilon_{zz}}{\partial x^2} + \frac{\partial^2\varepsilon_{xx}}{\partial z^2}\right) \\
\frac{\partial^2\varepsilon_{yz}}{\partial y\partial z} &= \frac{1}{2}\left(\frac{\partial\varepsilon_{yy}}{\partial z^2} + \frac{\partial^2\varepsilon_{zz}}{\partial y^2}\right)
\end{aligned} \tag{1.23}$$

1.4.4 Kromlijnige coördinaten

Cilindercoördinaten

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta z} \\ \frac{1}{2}\gamma_{rz} & \frac{1}{2}\gamma_{\theta z} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \frac{1}{2}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial r}\right) \\ \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right) \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{1.24}$$

Bolcoördinaten

$$\begin{aligned}
[\varepsilon] &= \begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta\phi} \\ \frac{1}{2}\gamma_{r\phi} & \frac{1}{2}\gamma_{\theta\phi} & \varepsilon_{\phi\phi} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\ \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) & \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi}{r}\cot\theta\right) \\ \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi}{r}\cot\theta\right) & \frac{1}{r\sin\theta} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta \end{bmatrix}
\end{aligned}$$

1.5 LINEAIR ELASTISCH MATERIAALGEDRAG

1.5.1 Wet van Hooke

Wet van Hooke

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \frac{\tau_{xz}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G}
\end{aligned} \tag{1.25}$$

Geïnverteerde wet van Hooke

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] \\
\tau_{xy} &= G\gamma_{xy} \\
\tau_{xz} &= G\gamma_{xz} \\
\tau_{yz} &= G\gamma_{yz}
\end{aligned} \tag{1.26}$$

1.5.2 Bijzondere belastingsgevallen

Zuivere rek

(x-as volgens trekrichting)

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.27}$$

Zuivere afschuiving

(x-as volgens trekrichting)

$$[\sigma] = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \tag{1.28}$$

Hydrostatische belasting

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -|p| & 0 & 0 \\ 0 & -|p| & 0 \\ 0 & 0 & -|p| \end{bmatrix} \tag{1.29}$$

Torsie of wringing

(x-as in de langsrichting)

$$[\sigma] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \tag{1.30}$$

in polaire coördinaten

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\theta z} \\ 0 & \tau_{\theta z} & 0 \end{bmatrix} \tag{1.31}$$

1.5.3 Relaties tussen de elastische constanten

Verband tussen E, ν en G

$$G = \frac{E}{2(1+\nu)} \tag{1.32}$$

Volumeverandering en compressiemodulus

Volumerek of dilatatie

$$\varepsilon_{\text{vol}} = \frac{\Delta V}{dV} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{1.33}$$

Compressiemodulus of volume-elasticiteitsmodulus

$$K = \frac{-|p|}{\frac{\Delta V}{dv}} = \frac{E}{3(1-2\nu)}, \quad \nu\frac{1}{2} \tag{1.34}$$

1.5.4 Kromlijnige coördinaten

Cilindercoördinaten

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right] \\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\ \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{\theta\theta}\right)\right] \\ \gamma_{x\theta} &= \frac{\tau_{x\theta}}{G} \\ \gamma_{xz} &= \frac{\tau_{rz}}{G} \\ \gamma_{\theta z} &= \frac{\tau_{\theta z}}{G} \end{aligned} \tag{1.35}$$

1.6 OPLOSSING VAN HET LINEAIR ELASTISCH PROBLEEM

Gebruik *vergelijkingen van het evenwicht*, *vergelijkingen tussen rek en verplaatsing* en de wet van Hooke samen met randvoorwaarden om het linear statisch probleem op te lossen.

1.7 THERMISCHE SPANNINGEN

1.7.1 Vergelijkingen

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] + \alpha T \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] + \alpha T \\ \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] + \alpha T \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \frac{\tau_{xz}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \end{aligned} \tag{1.36}$$

Geïnverteerd

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx}+\nu\left(\varepsilon_{yy}+\varepsilon_{zz}\right)\right]-\frac{E}{1-2\nu}\alpha T\\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy}+\nu\left(\varepsilon_{xx}+\varepsilon_{zz}\right)\right]-\frac{E}{1-2\nu}\alpha T\\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz}+\nu\left(\varepsilon_{xx}+\varepsilon_{yy}\right)\right]-\frac{E}{1-2\nu}\alpha T\\
\tau_{xy} &= G\gamma_{xy}\\
\tau_{xz} &= G\gamma_{xz}\\
\tau_{yz} &= G\gamma_{yz}
\end{aligned} \tag{1.37}$$

1.8 ARBEID EN ELASTISCHE ENERGIE

1.8.1 Arbeid van een kracht

Verichte arbeid

$$dU_{\text{uitw}} = F \cdot dx \tag{1.38}$$

Totale arbeid over afstand x

$$U_{\text{uitw}} = \int_0^x F(x) \cdot dx \tag{1.39}$$

1.8.2 Arbeid van een moment

Verichte arbeid

$$dU_{\text{uitw}} = M \cdot d\theta \tag{1.40}$$

Totale arbeid over hoekverdraaiing

$$U_{\text{uitw}} = \int_0^\theta M(\theta) \cdot \theta \tag{1.41}$$

1.8.3 Wet van behoud van mechanishce energie

$$U_{\text{uitw}} = U_{\text{inw}} \tag{1.42}$$

Hoofdstuk 2

Tweedimensionel problemen