Report Electromagnetism I

FDTD Simulation of Lossless Transmission Lines

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1 Introduction

The aim of this project is to simulate the behavior of voltages and currents in a lossless transmission line in function of time and space. To accomplish this objective, the Finite-Difference Time-Domain method (FDTD) was used. This method makes use of a leapfrog scheme in both time and space and turns a continuous problem in discrete equations. The transmission line studied in this project is depicted in Figure 1 and its discretized version in Figure 2.

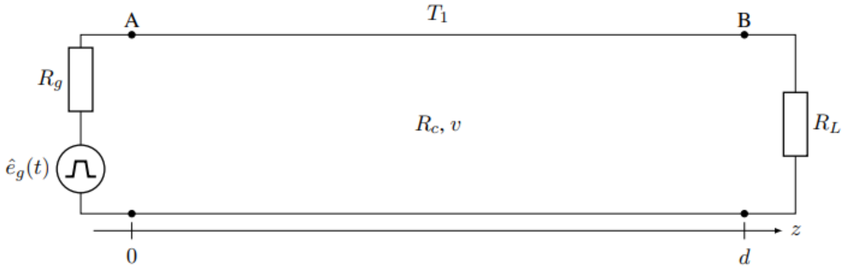


Figure 1: Transmission line

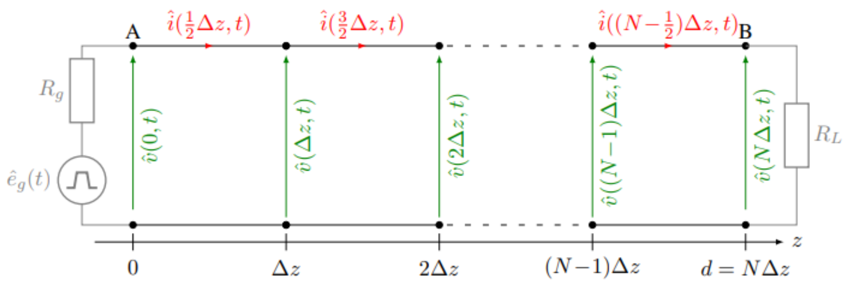


Figure 2: Discretization of the transmission line

In the first section, the correct update equations and constants are derived. Using this information, the behavior of a bit traveling through the transmission line is studied, both with default values and with varying parameters. This is all considered with just a resistor as load. Eventually a capacitor is added in parallel to the load. This changes the bit behavior substantially. To conclude the report, an application of transmission line theory inside the field of biomedical engineering is discussed.

2 A bit traveling through a lossless transmission line

2.1 Numerical implementation

In the assignment paper the update functions are given as

$$\tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}} = \tilde{I}_{n+\frac{1}{2}}^{m-\frac{1}{2}} + \alpha\left(V_n^m - V_{n+1}^m\right), \qquad V_n^{m+1} = V_n^m + \alpha\left(\tilde{I}_{n-\frac{1}{2}}^{m+\frac{1}{2}} - \tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}}\right), \tag{1}$$

with

$$\alpha \triangleq \frac{v\Delta t}{\Delta z} \tag{2}$$

the dimensionless Courant factor and

$$\tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}} = I_{n+\frac{1}{2}}^{m+\frac{1}{2}} R_c \tag{3}$$

the rescaled current.
The boundary condition at the generator ($z = 0$) is briefly discussed in the assignement but is not finalized. It's final form at $z = 0$ is

$$V_0^{m+1} = K_1 V_0^m + 2\kappa_1\left(E_g^{m+\frac{1}{2}}\frac{R_c}{R_g} - \tilde{I}_{\frac{1}{2}}^{m+\frac{1}{2}}\right) \tag{4}$$

where

$$K_1 = \frac{R_g - \alpha R_c}{R_g + \alpha R_c}, \tag{5}$$

$$\kappa_1 = \frac{\alpha R_c}{R_g + \alpha R_c}, \tag{6}$$

are two dimensionless constants which will be discussed in a following section.

At the load ($z = d$) a similir situation occurs as at the generator ($z = 0$), namely there is a resistance. This is depicted in figure 3.

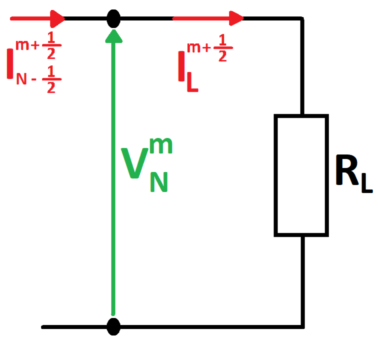


Figure 3: Boundary situation at the $z = d$

The voltage update equation becomes:

$$V_N^{m+1} = V_N^m + \frac{2\Delta t}{C\Delta z}\left(I_{N-\frac{1}{2}}^{m+\frac{1}{2}} - I_L^{m+\frac{1}{2}}\right) \tag{7}$$

Kirchoff's voltage law in discretized form states that

$$\begin{aligned} I_L^{m+\frac{1}{2}} &= \frac{V_N^{m+\frac{1}{2}}}{R_L} \\ &= \frac{V_N^m + V_N^{m+1}}{2R_L} \end{aligned} \tag{8}$$

Subsituting (8) in (7) and using the same relations as for $z = 0$ yields, after some rearrangements:

$$V_N^{m+1} = K_2 V_N^m + 2\kappa_2 \tilde{I}_{N-\frac{1}{2}}^{m+\frac{1}{2}}, \tag{9}$$

where

$$K_2 = \frac{R_L - \alpha R_c}{R_L + \alpha R_c}, \tag{10}$$

$$\kappa_2 = \frac{\alpha R_c}{R_L + \alpha R_c}, \tag{11}$$

are two dimensionless constants which also will be discussed in a following section.

2.2 General behavior of a bit traveling through a transmission line

This section aims to predict, and thus validate, the behavior of the numerical simulation of the voltage in a LTL. A rough analytical approach is used, since it gives more insight in the physical nature of the problem.

Rearranging the telegrapher's equations for a LTL yields

$$\frac{\partial^2 \hat{v}(z,t)}{\partial z^2} - k^2 \frac{\partial^2 \hat{v}(z,t)}{\partial t^2} = 0, \quad k = \frac{1}{c} = \sqrt{LC} = \text{cte}. \tag{12}$$

This means that $\hat{v}$ satisfies the wave equation and thus can be written as a superposition of two voltage waves traveling in opposite directions with constant speed c. That is,

$$\hat{v}(z,t) = \underbrace{\hat{v}^{+}(z - ct)}_{\text{forward wave}} + \underbrace{\hat{v}^{-}(z + ct)}_{\text{backward wave}}. \tag{13}$$

Realising that the voltage and current are related;

$$\hat{i}(z,t) = \frac{1}{R_c}(\hat{v}^{+}(z,t) - \hat{v}^{-}(z,t)), \tag{14}$$

with $R_c = \sqrt{\frac{L_{TL}}{C_{TL}}}$ and L_{TL}, C_{TL} p.u.l. inductance resp. capacitance of the TL, a constant that depends on the characteristics of the TL, hence the name characteristic impedance. Although it has the units of resistance (Ω) it is not reactive; there is no energy dissipation over the TL. The characteristic impedance can be interpreted as a scale for (1) the current to voltage amplitude when there is only a forward or backward wave present in the TL and (2) the dissipation of energy to the load or generator (1). Furthermore, the CI is the consequence of the model we provided for the TL; it counts for the p.u.l capacitance and inductance (depend on type of TL). In further sections, R_c is assumed to be finite and constant.

2.3 Reflections and amplitude damping

In section 2.2 the wave behavior of the voltage and current is discovered. To this end, the analysis of voltage waves at the boundaries for specific values of the generator resistance R_g and load resistance R_L is discussed. We highly recommend watching the simulation video's[1].

Our first question concerns the entrance of the bit to the transmission line (TL): the simulation starts at $t = 0$ and the generator $\hat{e}_g$ produces a bit with amplitude V_0. Before the bit enters the TL, it meets the generator resistance R_g and thus energy will be dissipated i.e. the voltage amplitude is dampened. This can be clarified by expressing Kirchoff's Voltage Law at the origin ($z = 0$) of the TL;

$$\hat{e}_g(t) = (R_g + R_c)\hat{i}(0,t) \tag{15}$$

$$\Rightarrow \hat{i}(0,t) = \frac{1}{R_g + R_c}\hat{e}_g(t) = \frac{1}{R_c}(\hat{v}^{+}(0,t) - \hat{v}^{-}(0,t)) \tag{16}$$

$$\Rightarrow \hat{v}(0,t) = \hat{v}^{+}(0,t) = \kappa \hat{e}_g(0,t). \tag{17}$$

The amplitude V_0 of the generated bit is reduced by a factor $\kappa = \frac{R_c}{R_g + R_c}$. Here, the role of the characteristic impedance becomes clear; at the beginning of the TL, the bit 'sees' it as input impedance, but does not dissipate energy to it; the bit amplitude remains constant during its passage through the TL. Indeed, the TL is lossless and only contains p.u.l capacitances and inductances (passive). One can observe two interesting cases:

[1] Simulation details and visualizations are stored in a OneDrive. Only accessible for UGhent members.

- $R_g = 0 \Rightarrow \kappa = 1$
  The generated bit freely enters the TL since it does not meet resistance.

- $R_g = \infty \Rightarrow \kappa = 0$
  The circuit is open. The bit can not enter the TL.

Now, assume the bit is able to enter the transmission line ($R_g \neq \infty$) and propagates lossless towards the load with, obviously, constant speed c. Again, at the load ($z = d$), Kirchoff's Voltage Law must be satisfied. Consequently, a reflected wave with lower or equal amplitude will be originated (Figure 5);

$$\hat{v}(d,t) = \hat{v}^+(d,t) + \hat{v}^-(d,t) = R_L \hat{i}(d,t) \tag{18}$$

$$\Leftrightarrow \hat{v}^+(d,t) + \hat{v}^-(d,t) = \frac{R_L}{R_c}(\hat{v}^+(d,t) - \hat{v}^-(d,t)) \tag{19}$$

$$\Leftrightarrow (1 + K_L)\hat{v}^+(d,t) = \frac{R_L}{R_c}(1 - K_L)\hat{v}^+(d,t) \tag{20}$$

$$\Rightarrow K_L = \frac{R_L/R_c - 1}{R_L/R_c + 1} = \frac{R_L - R_c}{R_c + R_L}. \tag{21}$$

K_L is also know as the reflection coefficient at the load. This constant determines how much of the amplitude is reflected back:

1. $R_L > R_c \Rightarrow K_L > 0$ (Figure 4 A)
   The voltage wave arriving at the load does not have enough power. Since the load impedance is not matched with the characteristic impedance, the voltage at the load must take the load impedance into account (KCL); the net voltage $\hat{v}$ is compensated such that its amplitude at the load is higher than the amplitude of the wave arriving at R_L; a backward wave is originated that satisfies $\hat{v}^- = K_L \hat{v}^+$, because than

   $$\hat{v}(d,t) = (1 + K_L)\hat{v}^+(d,t).$$

   This might be counterintuitive, since one might think it violates the law of conservation of energy. Luckily, this is not true; comparing the input power to the power delivered to the load (first reflection) gives

   $$P_{\text{in}} > P_L \tag{22}$$

   $$\Leftrightarrow \frac{\kappa^2 V_0^2}{R_c} > \frac{(1 + K_L)^2 \kappa^2 V_0^2}{R_L} \tag{23}$$

   $$\Leftrightarrow (R_L - R_c)^2 > 0. \tag{24}$$

   Equation (24) is always true and the law of conservation of energy still holds. The reflected wave still contains $P_{\text{in}} - P_L$ power.

2. $R_L = R_c \Rightarrow K_L = 0$: matching (Figure 4 B, video matching)
   From an engineering point of view, this is the most wanted; the energy of the forward wave is fully dissipated to the load. Maximum power is reached and there is no reflection back.

3. $R_L < R_c \Rightarrow K_L < 0$
   The same principle holds as in case 1, with the difference that the reflected wave amplitude has an opposite sign compared to the incident wave; that is, because the voltage wave arriving at the load contains too much energy.

4. $R_L \longrightarrow \infty \Rightarrow K_L \longrightarrow 1$
   The electric circuit is open. When the voltage wave arrives at the end of the TL, it is fully reflected to the load and no energy is dissipated.

Obviously, when the reflected wave arrives back at the generator it might be reflected back to the load, depending on the value of the generator impedance. In summary, the generated bit enters the TL weakened (17) and starts to move back and forth between the load impedance and generator impedance, whereby energy is dissipated and the amplitude reduces until the wave is fully dampened (video amplitude damping[1]).

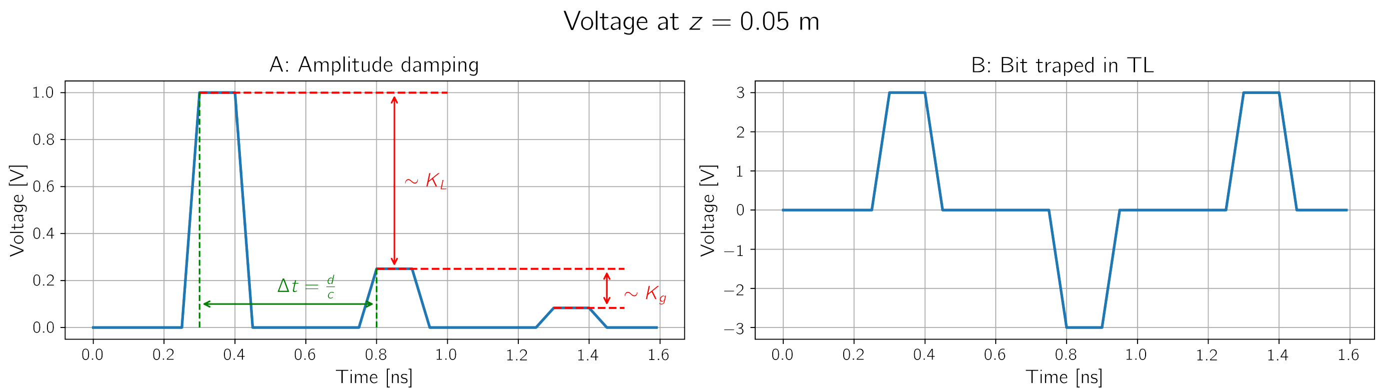


Figure 4: FDTD simulations of a voltage wave traveling trough a TL. This wave moves with constant speed c and is reflected at the load- and generator impedance with a damping factor K_L resp. K_g. In A, the bit amplitude decreases as it meets an impedance. In B, the bit is trapped in the TL ($K_L \approx -1$ and $K_g \approx 1$).

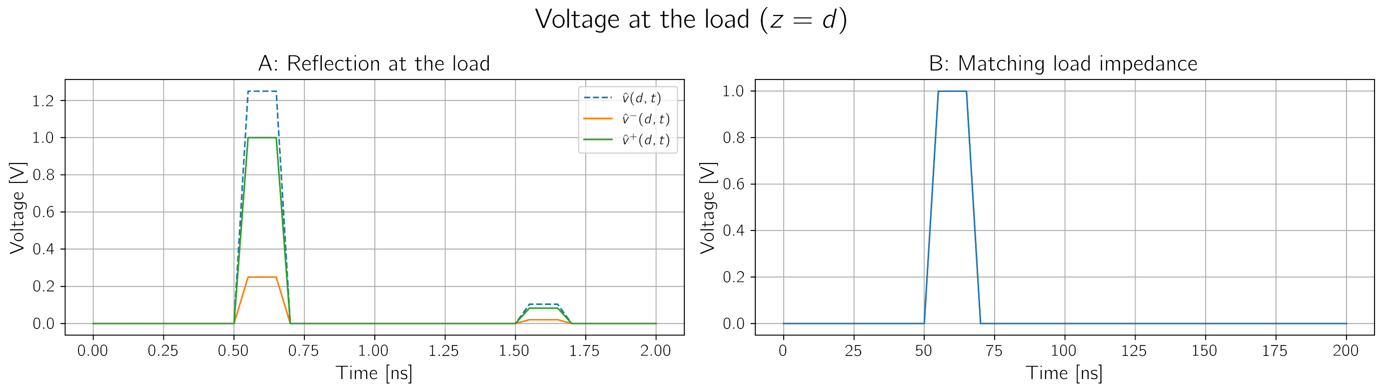


Figure 5: Plot of voltage wave (FDTD simulation) at the load and decomposed into its forward $\hat{v}^+$ and backward $\hat{v}^-$ component.

2.4 Effect of varying τ_h and τ_r

The bit that is sent through the transmission line model is defined by the rising time (τ_r) and a bit time ($T_{bit} = \tau_h$). A falling time (τ_f) could be defined, but a symmetrical bit is assumed so $\tau_r = \tau_f$. The default settings are $\tau_h = 100 * 10^{-12}$ and $\tau_r = 40 * 10^{-12}$. For this values, the bit has a trapezium shape. This case is shown in figure 6 Letting these values vary a bit, one discovers a few interesting cases worth it to have look at.

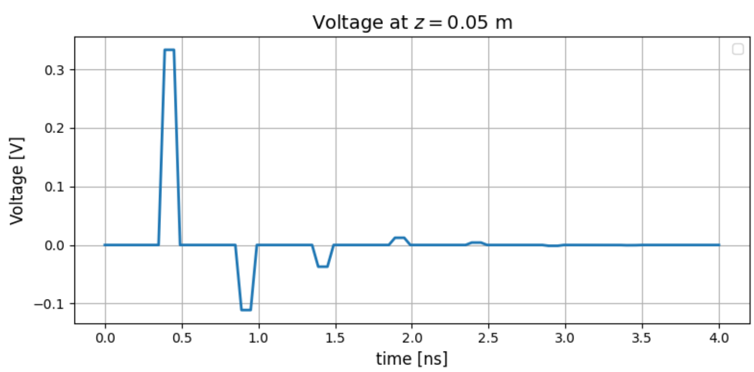


Figure 6: Bit behavior (voltage profile) under default settings

When τ_h becomes larger than 0.5 nanoseconds, interference between incident and reflected bits occur. The zero parts on the graph (in function of time) disappear and incidences and reflections influence each other, either positive or negative interference. This effect is shown in figure 7.

In the case of $\tau_h = \tau_r$, the bit has a triangular shape, as shown in figure 8a. A curious effect can be noticed in the voltage profile when τ_r approaches 0. One would expect the bit to have square wave shape, but instead some non converging effects occur displayed in figure 8b.

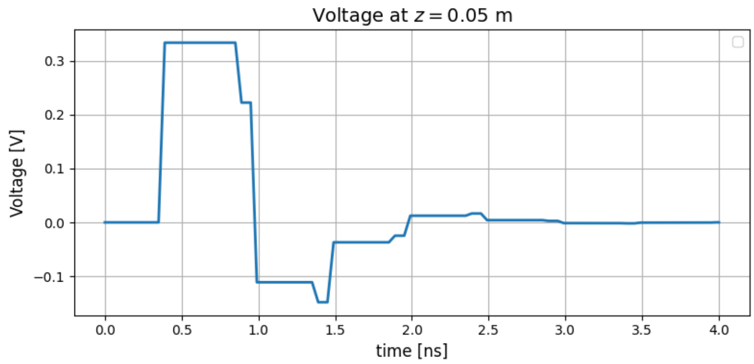
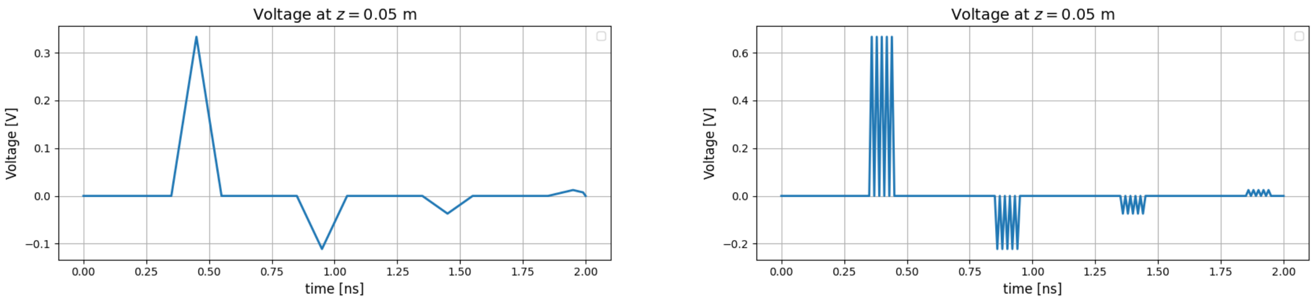


Figure 7: $\tau_h = 500 * 10^{-12}$



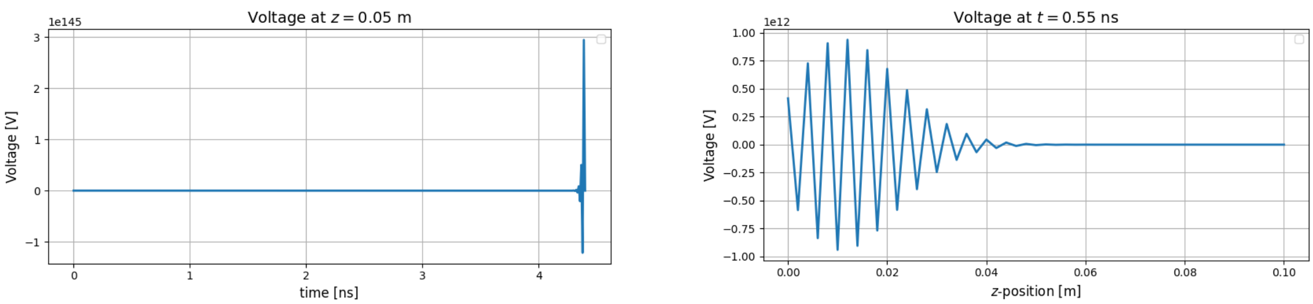
(a) $\tau_h = \tau_r$

(b) $\tau_r = 10^{-12}$

Figure 8

2.5 Effect of the Courant limit

In function of the FDTD model, a discretization was performed of course, both in time and space. To justify the model physically, the steps in both time and space have to be chosen small enough.
The step in space Δz has to be small enough to get rid of the wave phenomena. It cannot cross the boundary value of $\frac{\lambda_{min}}{10}$. The minimal wavelength λ_{min} is simple to calculate as the fraction of the wave speed and the maximal frequency. The maximal frequency is given by the bandwith of the signal: $\frac{1}{\pi\tau_r}$. Considering the default settings ($\tau_r = 40.0e-12s$ and $v = 200.0e6m/s$) is the maximal step in space equal to 2.5133 mm. Results with a bigger step in space can be ignored as they are physically incorrect.
The Courant limit is introduced to determine the maximal step in time. This limit states that Δt can not be bigger than $\frac{\Delta z}{v}$. Just as in the case of Δz, results for Δt bigger than this limit can be neglected since the time step Δt will be bigger than the time it takes for the wave to cover a distance Δz in that case. To illustrate this, the time step was brought up to $11e-12s$ and the none converging graph is shown in figure 9.



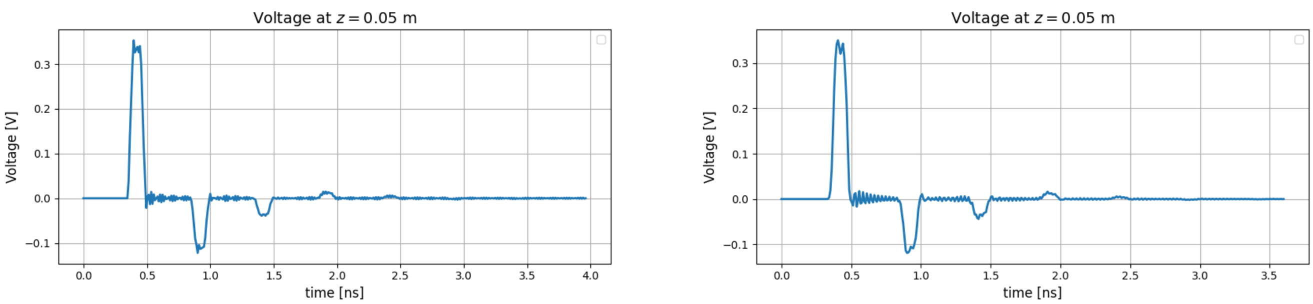
(a) Voltage for $\alpha = 1.1$ at $z = 0.05m$

(b) Voltage for $\alpha = 1.1$ at $t = 0.55ns$

Figure 9

Applying the default values, Δt exactly satisfies the Courant limit. This is the so called magic time step ($v\Delta z = \Delta t$). When a time step smaller than the magic one is chosen, Gibbs phenomena occur.
The Courant factor α, which also appears in the update equations, shows in what extend Gibbs phenomena are present. The maximal value of the Courant factor (to be physically relevant) is 1, which is also the perfect case.

The lower the Courant factor becomes, the more Gibbs phenomena occur. This is illustrated in Figure 10 for α being respectively 0.99 and 0.90.



(a) $\alpha = 0.99$ (b) $\alpha = 0.90$

Figure 10

2.6 Resistive and Capacitive load

2.6.1 Adjusted boundary conditions

Adding a capacitance in parallel with the load as depicted in figure 11 gives rise to some adjustements to the voltage update function at $z = 0$.
Kirchoff's current law states now that

$$I_L^{m+\frac{1}{2}} = I_R^{m+\frac{1}{2}} + I_C^{m+\frac{1}{2}}. \tag{25}$$

Using Kirchoff's voltage law at the resistor gives

$$\begin{aligned} I_R^{m+\frac{1}{2}} &= \frac{V_N^{m+\frac{1}{2}}}{R_L} \\ &= \frac{V_N^m + V_N^{m+1}}{2R_L} \end{aligned} \tag{26}$$

The relation between the current and the voltage at the capitor is given by

$$\hat{i} = C\frac{\partial \hat{v}}{\partial t}. \tag{27}$$

For a first order FDM this turns into

$$I_C^{m+\frac{1}{2}} = C_L\frac{V_N^{m+1} - V_N^m}{\Delta t}. \tag{28}$$

Substituting (26) and (28) into (25) gives

$$I_L^{m+\frac{1}{2}} = \left(\frac{1}{2R_L} + \frac{C_L}{\Delta t}\right)V_N^{m+1} + \left(\frac{1}{2R_L} - \frac{C_L}{\Delta t}\right)V_N^m \tag{29}$$

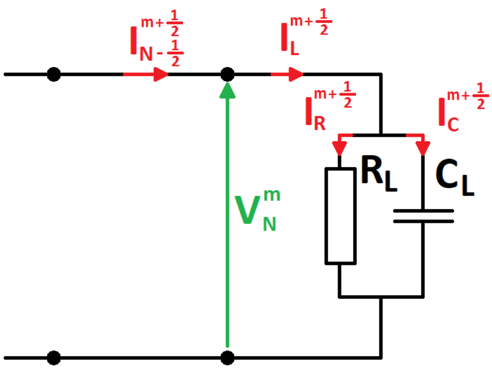


Figure 11: Load with capacitance

and can be rewritten as

$$\tilde{I}_L^{m+\frac{1}{2}} = \frac{R_c}{2Z_2} V_N^{m+1} + \frac{R_c}{2Z_1} V_N^m \tag{30}$$

with

$$\begin{aligned} Z_1 &= \left(\frac{1}{R_L} - 2\frac{C_L}{\Delta t}\right)^{-1} = \frac{R_L \Delta t}{\Delta t - 2C_L R_L} \\ Z_2 &= \left(\frac{1}{R_L} + 2\frac{C_L}{\Delta t}\right)^{-1} = \frac{R_L \Delta t}{\Delta t + 2C_L R_L} \end{aligned} \tag{31}$$

Substituting (30) into the adjusted voltage update function at $z = d$ yields

$$V_N^{m+1} = K_2' V_N^m + 2\kappa_2' \tilde{I}_{N-\frac{1}{2}}^{m+\frac{1}{2}}, \tag{32}$$

where

$$K_2' = \frac{Z_2}{Z_1} \frac{Z_1 - \alpha R_c}{Z_2 + \alpha R_c}, \tag{33}$$

$$\kappa_2' = \frac{\alpha Z_2}{Z_2 + \alpha R_c}. \tag{34}$$

are the new dimensionless constants.

2.6.2 Influence of C

First thing to notice is that when $C_L = 0$ then $K_2' = K_2$ and $\kappa_2' = \kappa_2$. This is a good sign since it stays consistent with the previous update equation.
Secondly in relations 31 there is a time step dependecy, hence the two dimensionless coefficients in the boundary update function at $z = d$ are also dependent on the implemented time step. Let's consider a well chosen time step and just look at the effect of changing C_L.

- $C_L \to 0$: $K_2' \to K_2$ and $\kappa_2' \to \kappa_2$, initial situation
- $C_L \to \infty$: $K_2' \to 1$ and $\kappa_2' \to 0$, fully reflected wave
- $C_L \to \frac{\Delta t}{2R_L}$: $K_2' \to \frac{Z_2}{Z_2 + \alpha R_C}$ and $\kappa_2' \to \frac{2\alpha Z_2}{Z_2 + \alpha R_C}$, this shows that the division by zero for Z_1 does not lead to any problems.

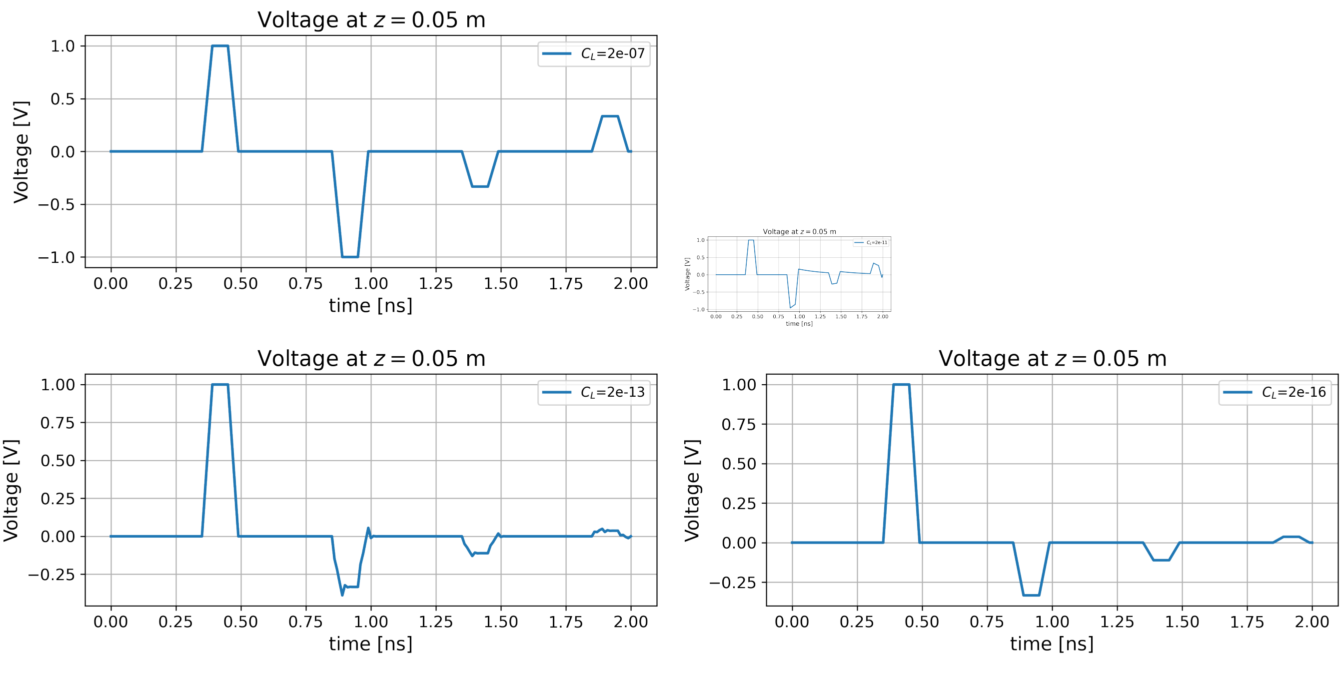


Figure 12: Different values for the load capacitance

From figure 12 the following obesrvations can be made:

- For large values of the capacitance no energy loss will occur at the load, i.e. the amplitude of the voltage does not change when passing the load. The capacitor has enough capacitance to store the whole bit and return it to the transmision line.
- For intermediate values for the capacitance, i.e. $C_L = 2e - 11$, the decharging of the capacitor can clearly be seen (exponetial drop).
- When the capacitance gets very low its effect gets negligible.

3 Biomedical application

Since this report was written by three students in biomedical engineering, this section looks shortly in the application of transmission line theory in this field of engineering. A small study of literature was conducted to gather some insight in this topic. When one thinks of electric signals related to the human body, of course the nervous system immediately comes to mind. Neuro-engineering is a very active field within biomedical engineering. The foundation of modelling nerve impulses was built by Hodgkin and Huxley in 1951. They constructed a model to replicate the specific way