

Inhoudsopgave

1	Krachten, momenten, spanningne en rekken	3
1.1	STATICA EN EVENWICHT VAN CONSTRUCTIES	3
1.1.1	Types ondersteuningen	3
1.1.2	Evenwicht van een constructie	3
1.2	INTUÏTIEF BEGRIP VAN SPANNINGEN EN REKKEN	3
1.3	SPANNINGEN	4
1.3.1	Definitie	4
1.3.2	Verband tussen spanningsvector $\vec{\phi}^{(n)}$ en spanningstensor $[\sigma]$	4
1.3.3	Vergelijkingen van het evenwicht	4
1.3.4	Transformatie van coördinaten en hoofdrichtingen	4
1.3.5	Kromlijnige coördinaten	5
1.4	REKKEN	5
1.4.1	Veralgemeende vervormingstoestand	5
1.4.2	Transformatie van coördinaten en hoofdrichtingen	5
1.4.3	Compatibiliteitsvoorwaarden	6
1.4.4	Kromlijnige coördinaten	6
1.5	LINEAIR ELASTISCH MATERIAALGEDRAG	7
1.5.1	Wet van Hooke	7
1.5.2	Bijzondere belastingsgevallen	7
1.5.3	Relaties tussen de elastische constanten	8
1.5.4	Kromlijnige coördinaten	8
1.6	OPLOSSING VAN HET LINEAIR ELASTISCH PROBLEEM	8
1.7	THERMISCHE SPANNINGEN	8
1.7.1	Vergelijkingen	8
1.8	ARBEID EN ELASTISCHE ENERGIE	9
1.8.1	Arbeid van een kracht	9
1.8.2	Arbeid van een moment	9
1.8.3	Wet van behoud van mechanische energie	9
2	Structureel gedrag	11
2.1	GEOMETRISCHE EIGENSCHAPPEN VAN DE DWARSDOORSNEDE	11
2.1.1	Opstellen vergelijkingen	11
2.1.2	Praktische berekening	12
2.2	NORMAALKRACHT, BUIGEND MOMENT EN DWARSKRACHT	13
2.2.1	Verband tussen q , V en M	13
2.2.2	Enkele referentiegevallen	13
2.3	VERBAND TUSSEN SNEDEKRACHTEN EN SPANNINGEN	15
2.3.1	Spanningen t.g.v. normaalkracht N	15
2.3.2	Spanningen t.g.v. buigend moment M	15
2.3.3	Spanningen t.g.v. dwarskracht V	15
2.4	VERPLAATSINGEN	15
2.4.1	Verplaatsingen t.g.v. de normaalkracht N	15
2.4.2	Verplaatsingen t.g.v. het buigend moment M	15
2.5	SINGULARITEITSFUNCTIES	16
2.6	INVLOED VAN DE KEUZE VAN HET ASSENSTELSEL	17

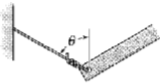






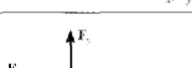
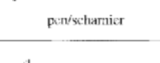
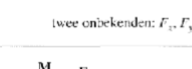
3	Tweedimensionale problemen	19
3.1	VLAKSPANNING EN VLAKVERVORMING	19
3.1.1	Vlakvervorming	19
3.1.2	Vlakvervorming	20
3.2	AXIAALSYMMETRISCHE BELASINGSGEVALLEN	21
3.2.1	Basisformules voor axiaalsymmetrie	21
3.2.2	Opstellen algemene vergelijkingen voor radiale belastingen	22
3.2.3	Trek- of drukspanning op de binnen- en buitenrand	22
3.2.4	Radiaal temperatuurveld	23
3.3	SPANNINGSCONCENTRATIES IN VLAKKE PLATEN	24

Hoofdstuk 1

Krachten, momenten, spanningne en rekken

1.1 STATICA EN EVENWICHT VAN CONSTRUCTIES

1.1.1 Types ondersteuningen

Type ondersteuning	Reacties
 kabel	 één onbekende: F
 rol	 één onbekende: F
 pen/scharnier	 twee onbekenden: F_x, F_y
 pen/scharnier	 twee onbekenden: F_x, F_y
 vaste ondersteuning	 drie onbekenden: F_x, F_y, M

1.1.2 Evenwicht van een constructie

$$\sum \mathbf{F} = 0$$

$$\sum \mathbf{M}_O = 0$$

1.2 INTUÏTIEF BEGRIP VAN SPANNINGEN EN REKKEN

$$\sigma = \frac{F}{A_0}$$

$$\varepsilon = \frac{\Delta L}{L_0}$$

$$\sigma = E \cdot \varepsilon$$

1.3 SPANNINGEN

1.3.1 Definitie

De spanningsvector

$$\vec{\phi}^{(n)} = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad (1.1)$$

De normaalspanning

$$\sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_n}{\Delta A} \quad (1.2)$$

De schuifspanning

$$\tau = \lim_{\Delta A \rightarrow 0} \frac{\Delta F_t}{\Delta A} \quad (1.3)$$

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad (1.4)$$

1.3.2 Verband tussen spanningsvector $\vec{\phi}^{(n)}$ en spanningstensor $[\sigma]$

Het verband tussen de spanningsvector en spanningstensor

$$\sigma_{ij} \cdot n_i = \phi_j^{(n)} \quad (i, j = x, y, z) \quad (1.5)$$

1.3.3 Vergelijkingen van het evenwicht

De vergelijkingen van het evenwicht

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0 \end{aligned} \quad (1.6)$$

Wet van de wederkerigheid der schuifspanningen

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned} \quad (1.7)$$

1.3.4 Transformatie van coördinaten en hoofdrichtingen

Transformatieregels

$$[\sigma'] = [a] \cdot [\sigma] \cdot [a]^\top \quad (1.8)$$

met

$$a_{rk} = \vec{e}'_r \cdot \vec{e}_k \quad r, k = x, y, z \quad (1.9)$$

Diagonaalsom van $[\sigma]$

$$\Sigma_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \quad (1.10)$$

Som van de hoofdminoren van $[\sigma]$

$$\Sigma_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \quad (1.11)$$

Determinant van $[\sigma]$

$$\Sigma_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{xz}\tau_{yz} \quad (1.12)$$

1.3.5 Kromlijnige coördinaten

Cilindercoördinaten

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} \quad (1.13)$$

De evenwichtsvergelijkingen

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial z} + F_r &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + F_\theta &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0 \end{aligned} \quad (1.14)$$

Bolcoördinaten

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{r\phi} & \tau_{\theta\phi} & \sigma_{\phi\phi} \end{bmatrix} \quad (1.15)$$

De evenwichtsvergelijkingen

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} + \frac{1}{r} (2\sigma_{rr} + \tau_{r\theta} \cot \theta - \sigma_{\theta\theta} - \sigma_{\phi\phi}) + F_r &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{1}{r} (3\tau_{r\theta} + \sigma_{\theta\theta} \cot \theta - \sigma_{\phi\phi} \cot \theta) + F_\theta &= 0 \\ \frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\tau_{r\phi} + 2\tau_{\theta\phi} \cot \theta) + F_\phi &= 0 \end{aligned} \quad (1.16)$$

1.4 REKKEN

1.4.1 Veralgemeende vervormingstoestand

Verband tussen rekken en verplaatsing

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned} \quad (1.17)$$

1.4.2 Transformatie van coördinaten en hoofdrichtingen

Transformatieregels

$$[\varepsilon'] = [a] \cdot [\varepsilon] \cdot [a]^\top \quad \text{met } a_{rk} = \vec{e}'_r \cdot \vec{e}_k \quad (1.18)$$

De rekstensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \quad (1.19)$$

Diagonaalsom van $[\varepsilon]$

$$i_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \quad (1.20)$$

Som van de hoofdminoren van $[\varepsilon]$

$$I_2 = \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xy}^2 - \varepsilon_{xz}^2 - \varepsilon_{yz}^2 \quad (1.21)$$

Determinant van $[\varepsilon]$

$$I_3 = \varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} - \varepsilon_{xx}\varepsilon_{yz}^2 - \varepsilon_{yy}\varepsilon_{xz}^2 - \varepsilon_{zz}\varepsilon_{xy}^2 + 2\varepsilon_{xy}\varepsilon_{xz}\varepsilon_{yz} \quad (1.22)$$

1.4.3 Compatibiliteitsvoorwaarden

$$\begin{aligned} \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x} \left(\frac{\partial \varepsilon_{xz}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial x} \right) \\ \frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} &= \frac{\partial}{\partial y} \left(\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{xz}}{\partial y} \right) \\ \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\ \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} &= \frac{1}{2} \left(\frac{\partial \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \right) \\ \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} &= \frac{1}{2} \left(\frac{\partial \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} \right) \\ \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} &= \frac{1}{2} \left(\frac{\partial \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \right) \end{aligned} \quad (1.23)$$

1.4.4 Kromlijnige coördinaten

Cilindercoördinaten

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta z} \\ \frac{1}{2}\gamma_{rz} & \frac{1}{2}\gamma_{\theta z} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\ \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} & \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\ \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) & \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) & \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (1.24)$$

Bolcoördinaten

$$\begin{aligned} [\varepsilon] &= \begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta\phi} \\ \frac{1}{2}\gamma_{rz} & \frac{1}{2}\gamma_{\theta\phi} & \varepsilon_{\phi\phi} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) \\ \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) & \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} & \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r} \cot \theta \right) \\ \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) & \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r} \cot \theta \right) & \frac{1}{r \sin \theta} + \frac{u_r}{r} + \frac{u_\theta}{r} \cot \theta \end{bmatrix} \end{aligned}$$

1.5 LINEAIR ELASTISCH MATERIAALGEDRAG

1.5.1 Wet van Hooke

Wet van Hooke

$$\begin{aligned}
 \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz})] \\
 \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz})] \\
 \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy})] \\
 \gamma_{xy} &= \frac{\tau_{xy}}{G} \\
 \gamma_{xz} &= \frac{\tau_{xz}}{G} \\
 \gamma_{yz} &= \frac{\tau_{yz}}{G}
 \end{aligned} \tag{1.25}$$

Geïnverteerde wet van Hooke

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})] \\
 \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz})] \\
 \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})] \\
 \tau_{xy} &= G\gamma_{xy} \\
 \tau_{xz} &= G\gamma_{xz} \\
 \tau_{yz} &= G\gamma_{yz}
 \end{aligned} \tag{1.26}$$

1.5.2 Bijzondere belastingsgevallen

Zuivere rek

(x-as volgens trekrichting)

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.27}$$

Zuivere afschuiving

(x-as volgens trekrichting)

$$[\sigma] = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \tag{1.28}$$

Hydrostatische belasting

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -|p| & 0 & 0 \\ 0 & -|p| & 0 \\ 0 & 0 & -|p| \end{bmatrix} \tag{1.29}$$

Torsie of wringing

(x-as in de langsrichting)

$$[\sigma] = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} \tag{1.30}$$

in polaire coördinaten

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\theta z} \\ 0 & \tau_{\theta z} & 0 \end{bmatrix} \quad (1.31)$$

1.5.3 Relaties tussen de elastische constanten

Verband tussen E , ν en G

$$G = \frac{E}{2(1 + \nu)} \quad (1.32)$$

Volumeverandering en compressiemodulus

Volumerek of dilatatie

$$\varepsilon_{\text{vol}} = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \quad (1.33)$$

Compressiemodulus of volume-elasticiteitsmodulus

$$K = \frac{-|p|}{\frac{\Delta V}{V}} = \frac{E}{3(1 - 2\nu)}, \quad \nu = \frac{1}{2} \quad (1.34)$$

1.5.4 Kromlijnige coördinaten

Cilindercoördinaten

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E} [\sigma_{xx} - \nu (\sigma_{\theta\theta} + \sigma_{zz})] \\ \varepsilon_{\theta\theta} &= \frac{1}{E} [\sigma_{\theta\theta} - \nu (\sigma_{xx} + \sigma_{zz})] \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{\theta\theta})] \\ \gamma_{x\theta} &= \frac{\tau_{x\theta}}{G} \\ \gamma_{xz} &= \frac{\tau_{rz}}{G} \\ \gamma_{\theta z} &= \frac{\tau_{\theta z}}{G} \end{aligned} \quad (1.35)$$

1.6 OPLOSSING VAN HET LINEAIR ELASTISCH PROBLEEM

Gebruik *vergelijkingen van het evenwicht*, *vergelijkingen tussen rek en verplaatsing* en de wet van Hooke samen met randvoorwaarden om het linear statisch probleem op te lossen.

1.7 THERMISCHE SPANNINGEN

1.7.1 Vergelijkingen

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} [\sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz})] + \alpha T \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz})] + \alpha T \\ \varepsilon_{zz} &= \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy})] + \alpha T \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \frac{\tau_{xz}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \end{aligned} \quad (1.36)$$

Geïnverteerd

$$\begin{aligned}
 \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})] - \frac{E}{1-2\nu}\alpha T \\
 \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz})] - \frac{E}{1-2\nu}\alpha T \\
 \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})] - \frac{E}{1-2\nu}\alpha T \\
 \tau_{xy} &= G\gamma_{xy} \\
 \tau_{xz} &= G\gamma_{xz} \\
 \tau_{yz} &= G\gamma_{yz}
 \end{aligned} \tag{1.37}$$

1.8 ARBEID EN ELASTISCHE ENERGIE

1.8.1 Arbeid van een kracht

Verichte arbeid

$$dU_{\text{uitw}} = F \cdot dx \tag{1.38}$$

Totale arbeid over afstand x

$$U_{\text{uitw}} = \int_0^x F(x) \cdot dx \tag{1.39}$$

1.8.2 Arbeid van een moment

Verichte arbeid

$$dU_{\text{uitw}} = M \cdot d\theta \tag{1.40}$$

Totale arbeid over hoekverdraaiing

$$U_{\text{uitw}} = \int_0^\theta M(\theta) \cdot \theta \tag{1.41}$$

1.8.3 Wet van behoud van mechanische energie

$$U_{\text{uitw}} = U_{\text{inw}} \tag{1.42}$$

Hoofdstuk 2

Structureel gedrag

2.1 GEOMETRISCHE EIGENSCHAPPEN VAN DE DWARSDOORSNEDE

2.1.1 Opstellen vergelijkingen

Oppervlakte van de dwarsdoorsnede

$$A = \int \int dy' dz' \quad (2.1)$$

Statisch moment om de y' -as

$$S_{y'} = \int \int z' dy' dz' \quad (2.2)$$

Statisch moment om de z' -as

$$S_{z'} = \int \int y' dy' dz' \quad (2.3)$$

Ligging zwaartepunt

$$\begin{aligned} y'_o &= \frac{S_{z'}}{A} \\ z'_o &= \frac{S_{y'}}{A} \end{aligned} \quad (2.4)$$

Traagheidsmomenten van de doorsnede

$$\begin{aligned} I_{yy} &= \int \int z^2 dy dz \\ I_{zz} &= \int \int y^2 dy dz \\ I_{yz} &= - \int \int yz dy dz \end{aligned} \quad (2.5)$$

Stelling van Steiner (parallel axis theorem)

$$\begin{aligned} I_{y'y'} &= I_{yy} + A \cdot (z'_o)^2 \\ I_{z'z'} &= I_{zz} + A \cdot (y'_o)^2 \\ I_{y'z'} &= I_{yz} - A \cdot y'_o \cdot z'_o \end{aligned} \quad (2.6)$$

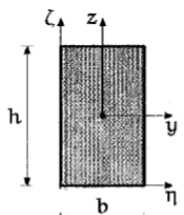
Rotatie van het assenstelsel

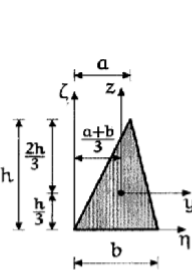
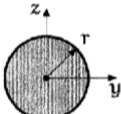
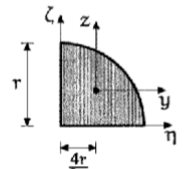
$$\begin{aligned} I_{y'y'} &= \cos^2 \alpha \cdot I_{yy} + \sin^2 \alpha \cdot I_{zz} + 2 \sin \alpha \cos \alpha \cdot I_{yz} \\ I_{z'z'} &= \sin^2 \alpha \cdot I_{yy} + \cos^2 \alpha \cdot I_{zz} - 2 \sin \alpha \cos \alpha \cdot I_{yz} \\ I_{y'z'} &= -\sin \alpha \cos \alpha \cdot I_{yy} + \sin \alpha \cos \alpha \cdot I_{zz} + (\cos^2 \alpha - \sin^2 \alpha) \cdot I_{yz} \end{aligned}$$

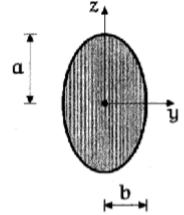
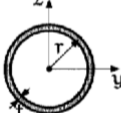
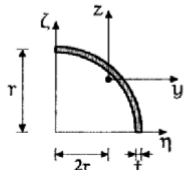
Voor de hoofdtraagheidsmomenten geldt

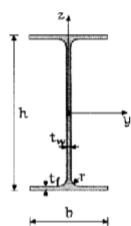
$$\tan 2\alpha = \frac{2I_{yz}}{I_{yy} - I_{zz}} \quad (2.7)$$

2.1.2 Praktische berekening

	$A = bh$	$S_{\eta} = \frac{bh^2}{2}$ $S_{\zeta} = \frac{hb^2}{2}$	$I_{yy} = \frac{bh^3}{12} ; I_{zz} = \frac{hb^3}{12}$ $I_{\eta\eta} = \frac{bh^3}{3} ; I_{\zeta\zeta} = \frac{hb^3}{3}$ $I_{\eta\zeta} = -\frac{b^2h^2}{4}$
---	----------	--	---

	$A = \frac{bh}{2}$	$S_{\eta} = \frac{bh^2}{6}$ $S_{\zeta} = \frac{bh}{6}(a+b)$	$I_{yy} = \frac{bh^3}{36}$ $I_{zz} = \frac{bh}{36}(b^2 - ab + a^2)$ $I_{yz} = -\frac{bh^2}{72}(2a - b)$ $I_{\eta\eta} = \frac{bh^3}{12}$ $I_{\zeta\zeta} = \frac{bh}{12}(b^2 + ab + a^2)$ $I_{\eta\zeta} = -\frac{bh^2}{24}(2a + b)$
	$A = \pi r^2$		$I_{yy} = I_{zz} = \frac{\pi r^4}{4}$
	$A = \frac{\pi}{4}r^2$	$S_{\eta} = S_{\zeta} = \frac{1}{3}r^3$	$I_{\eta\eta} = I_{\zeta\zeta} = \frac{\pi}{16}r^4$ $I_{\eta\zeta} = -\frac{1}{8}r^4$

	$A = \pi ab$		$I_{yy} = \frac{\pi}{4}ba^3$ $I_{zz} = \frac{\pi}{4}ab^3$
	$A = 2\pi rt$		$I_{yy} = I_{zz} = \pi r^3 t$
	$A = \frac{\pi}{2}rt$	$S_{\eta} = S_{\zeta} = r^2 t$	$I_{\eta\eta} = I_{\zeta\zeta} = \frac{\pi}{4}r^3 t$ $I_{\eta\zeta} = -\frac{1}{2}r^3 t$



Profiel	afmetingen (mm)					massa (kg/m)	A (mm ²)	I _{yy} (10 ⁴ mm ⁴)	W _y (10 ³ mm ³)	I _{zz} (10 ⁴ mm ⁴)	W _z (10 ³ mm ³)
	h	b	t _w	t _f	r						
IPE 80	80	46	3.8	5.2	5	6.0	764.3	80.1	20.0	8.49	3.69
IPE 100	100	55	4.1	5.7	7	8.26	1032	171	34.2	15.9	5.79
IPE 120	120	64	4.4	6.3	7	10.4	1321	318	53.0	27.7	8.65
IPE 140	140	73	4.7	6.9	7	12.9	1643	541	77.3	44.9	12.3
IPE 160	160	82	5.0	7.4	9	15.8	2009	869	109	68.3	16.7
IPE 180	180	91	5.3	8.0	9	18.8	2395	1317	146	101	22.2
IPE 200	200	100	5.6	8.5	12	22.4	2848	1943	194	142	28.5
IPE 220	220	110	5.9	9.2	12	26.2	3337	2771	252	205	37.3
IPE 240	240	120	6.2	9.8	15	30.7	3912	3891	324	284	47.3
IPE 270	270	135	6.6	10.2	15	36.1	4595	5790	429	420	62.2
IPE 300	300	150	7.1	10.7	15	42.2	5381	8356	557	604	80.5
IPE 330	330	160	7.5	11.5	18	49.15	6261	11770	713	788	98.5
IPE 360	360	170	8.0	12.7	18	57.1	7273	16270	904	1043	123
IPE 400	400	180	8.6	13.5	21	66.3	8446	23130	1156	1318	146
IPE 450	450	190	9.4	14.6	21	77.6	9882	33740	1500	1676	176
IPE 500	500	200	10.2	16.0	21	90.7	11530	48200	1928	2142	214
IPE 550	550	210	11.1	17.2	24	105.5	13440	67120	2441	2668	254
IPE 600	600	220	12.0	19.0	24	122.6	15600	92080	3069	3387	308

met de weerstandsmomenten gedefinieerd als

$$W_y = \frac{2 \cdot I_{yy}}{h}$$

$$W_z = \frac{2 \cdot I_{zz}}{b}$$

2.2 NORMAALKRACHT, BUIGEND MOMENT EN DWARSKRACHT

2.2.1 Verband tussen q, V en M

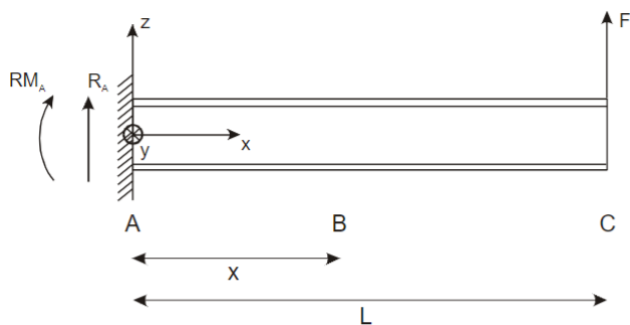
$$q = \frac{-dV}{dx}$$

$$V = \frac{dM}{dx}$$

$$q = \frac{-d^2M}{dx^2} \quad (2.8)$$

2.2.2 Enkele referentiegevallen

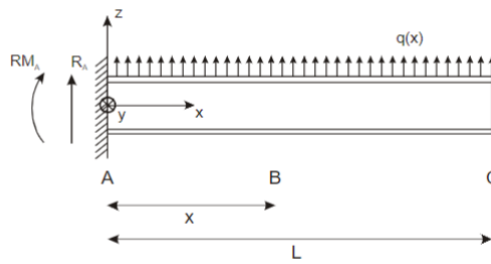
Ingekleemde balk met puntlast



$$V = F$$

$$M = -F \cdot (L - x)$$

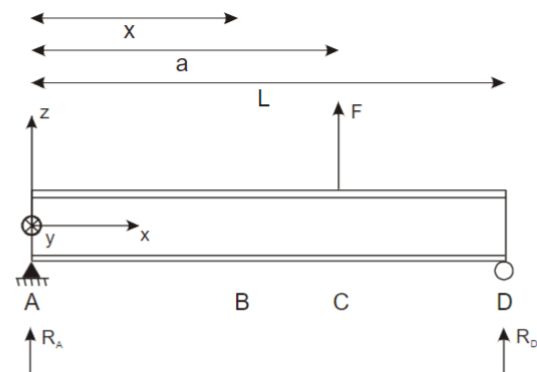
Ingekleemde balk met verdeelde belasting



$$V = q \cdot (L - x)$$

$$M = \frac{-q \cdot (L - x)^2}{2}$$

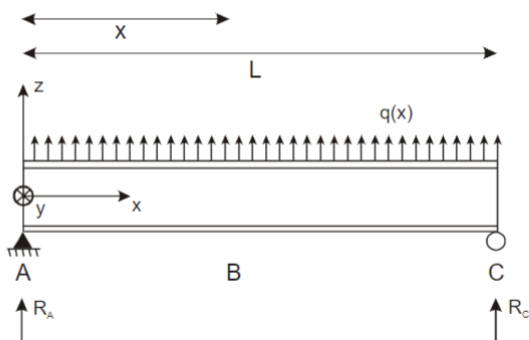
Balk op twee steunpunten met puntlast



$$V = \begin{cases} \frac{F \cdot (L - a)}{L} & , x < a \\ -\frac{F \cdot a}{L} & , x > a \end{cases}$$

$$M = \begin{cases} \frac{F \cdot x \cdot (L - a)}{L} & , x < a \\ \frac{F \cdot a \cdot (L - x)}{L} & , x > a \end{cases}$$

Balk op twee steunpunten met verdeelde belasting



$$V = q \cdot \left(\frac{L}{2} - x \right)$$

$$M = \frac{q \cdot x \cdot (L - x)}{2}$$

2.3 VERBAND TUSSEN SNEDEKRACHTEN EN SPANNINGEN

2.3.1 Spanningen t.g.v. normaalkracht N

De spanning t.g.v. de normaalkracht N

$$\sigma_{xx} = \frac{N}{A} \quad (2.9)$$

en de bijhorende rek

$$\varepsilon_{xx} = \frac{N}{E \cdot A} \quad (2.10)$$

2.3.2 Spanningen t.g.v. buigend moment M

$$\sigma_{xx} = \frac{M \cdot z}{I_{yy}} \quad (2.11)$$

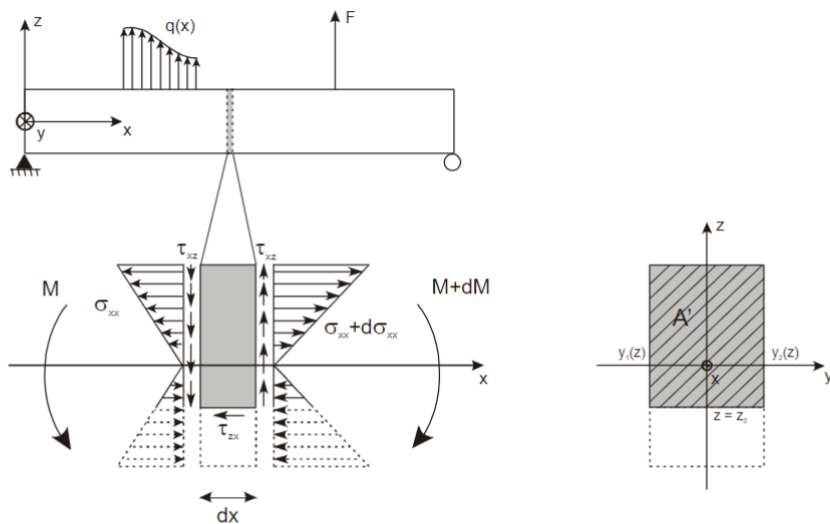
Kromtestraal

$$\frac{1}{R} = \frac{-M}{E \cdot I_{yy}} \quad (2.12)$$

2.3.3 Spanningen t.g.v. dwarskracht V

Formule van Jourawski

$$\tau_{xz}(z) = \frac{V}{I_{yy}} \cdot \frac{S_y(z)}{[y_2(z) - y_1(z)]} \quad (2.13)$$



2.4 VERPLAATSINGEN

2.4.1 Verplaatsingen t.g.v. de normaalkracht N

$$\Delta L = \int_0^L \frac{N}{A \cdot E} dx \quad (2.14)$$

2.4.2 Verplaatsingen t.g.v. het buigend moment M

$$\frac{d^2 u}{dx^2} = \frac{-M}{E \cdot I_{yy}} \quad (2.15)$$

$$q = \frac{-dV}{dx} = \frac{-d^2 M}{dx^2} = -EI_{yy} \frac{d^3 \alpha}{dx^3} = EI_{yy} \frac{d^4 u}{dx^4} \quad (2.16)$$

Ingekleemde balk met puntlast

$$u(x) = \frac{F}{E \cdot I_{yy}} \cdot \left(L \cdot \frac{x^2}{2} - \frac{x^3}{6} \right) \quad (2.17)$$

Ingekleemde balk met verdeelde belasting

$$u(x) = \frac{q}{E \cdot I_{yy}} \cdot \left[\frac{(L-x)^4}{24} + \frac{L^3}{6} \cdot x - \frac{L^4}{24} \right] \quad (2.18)$$

Balk op twee steunpunten met puntlast

$$u(x) = \begin{cases} \frac{F}{E \cdot I_{yy}} \cdot \left[-\frac{(L-a)}{L} \cdot \frac{x^3}{6} + \frac{a \cdot (L-a) \cdot (2L-a)}{6L} \cdot x \right], & x < a \\ \frac{F}{E \cdot I_{yy}} \cdot \left[-\frac{a \cdot (L-x)^3}{6L} - \frac{a \cdot (L-a) \cdot (L+a)}{6L} \cdot x + \frac{a \cdot (L-a) \cdot (L+a)}{6} \right], & x > a \end{cases} \quad (2.19)$$

Balk op twee steunpunten met verdeelde belasting

$$u(x) = \frac{q}{E \cdot I_{yy}} \cdot \left[\frac{1}{24} \cdot x^4 - \frac{L}{12} \cdot x^3 + \frac{L^3}{24} \cdot x \right] \quad (2.20)$$

2.5 SINGULARITEITSFUNCTIES

Voor de verdeelde belasting $q(x)$

$$\forall n \in \mathbb{N}: \quad \langle x-a \rangle^n = \begin{cases} 0, & x < a \\ (x-a)^n, & x > a \end{cases} \quad (2.21)$$

met volgende regels

$$\begin{aligned} \frac{d}{dx} \langle x-a \rangle^n &= n \cdot \langle x-a \rangle^{n-1} \\ \int \langle x-a \rangle^n dx &= \frac{\langle x-a \rangle^{n+1}}{n+1} + C \end{aligned} \quad (2.22)$$


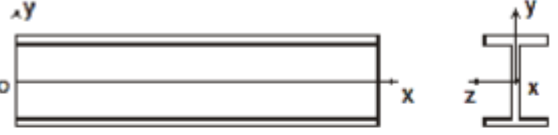
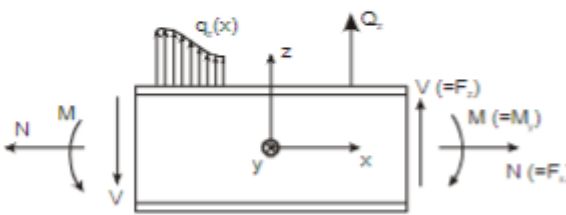
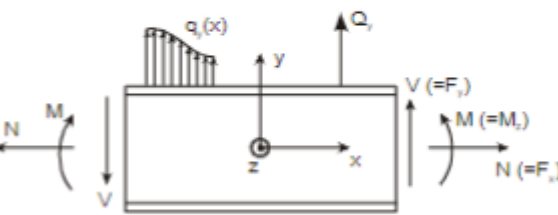
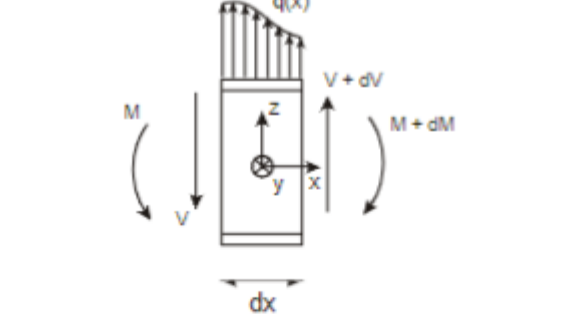
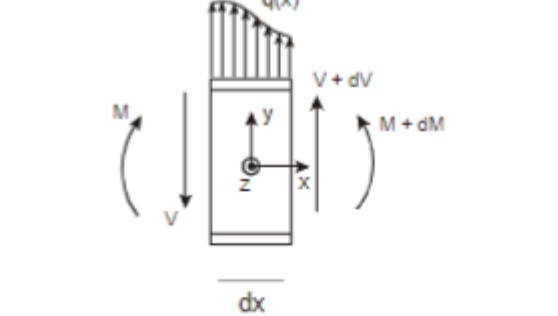
En voor geconcentreerde puntlasten F of koppes K

$$\begin{aligned} q &= F \cdot \langle x-a \rangle^{-1} = \begin{cases} 0, & x < a \\ F, & x > a \end{cases} \\ q &= K \cdot \langle x-a \rangle^{-2} = \begin{cases} 0, & x < a \\ K, & x > a \end{cases} \end{aligned} \quad (2.23)$$

met rekenregels $\forall n \in \mathbb{N}$

$$\begin{aligned} \frac{d}{dx} \langle x-a \rangle^{-n} &= n \cdot \langle x-a \rangle^{-n-1} \\ \int \langle x-a \rangle^{-n} dx &= \frac{\langle x-a \rangle^{n+1}}{-n+1} \end{aligned} \quad (2.24)$$

2.6 INVLOED VAN DE KEUZE VAN HET ASSENSTELSEL

Keuze in deze cursus	Alternatieve keuze
	
	
	
$q = -\frac{dV}{dx}$ $V = \frac{dM}{dx}$ $q = -\frac{d^2M}{dx^2}$	$q = -\frac{dV}{dx}$ $V = -\frac{dM}{dx}$ $q = \frac{d^2M}{dx^2}$
Positief koppel K: $q = K \cdot \langle x - a \rangle^{-2}$	Positief koppel K: $q = -K \cdot \langle x - a \rangle^{-2}$

Hoofdstuk 3

Tweedimensionale problemen

3.1 VLAKSPANNING EN VLAKVERVORMING

3.1.1 Vlakvervorming

Algemeen

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.1)$$

Uitwerking van de wet van Hooke:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \\ \varepsilon_{zz} &= \frac{-\nu}{E} (\sigma_{xx} + \sigma_{yy}) = \frac{-\nu}{1-\nu} (\varepsilon_{xx} + \varepsilon_{yy}) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \quad (3.2)$$

Geïnverteerde wet van Hooke

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)} [\varepsilon_{xx} + \nu \varepsilon_{yy}] \\ \sigma_{yy} &= \frac{E}{(1-\nu^2)} [\varepsilon_{yy} + \nu \varepsilon_{xx}] \end{aligned} \quad (3.3)$$

$$\tau_{xy} = G \gamma_{xy} \quad (3.4)$$

De transformatieformules voor de spanningen:

$$\begin{aligned} \sigma'_{xx} &= \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + \tau_{xy} 2 \sin \theta \cos \theta \\ \sigma'_{yy} &= \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - \tau_{xy} 2 \sin \theta \cos \theta \\ \tau'_{xy} &= (\sigma_{yy} - \sigma_{xx}) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \end{aligned} \quad (3.5)$$

waarvoor de hoofdrichtingen worden bepaald door

$$\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_{yy} - \sigma_{xx}} \quad (3.6)$$

Deze zijn

$$\begin{aligned} \sigma_I &= \frac{\sigma_{xx} + \sigma_{yy} + \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4(\tau_{xy})^2}}{2} \\ \sigma_{II} &= \frac{\sigma_{xx} + \sigma_{yy} - \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4(\tau_{xy})^2}}{2} \end{aligned} \quad (3.7)$$

Cirkel van Mohr

Cirkel met middelpunt

$$M = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}, 0 \right) \quad (3.8)$$

en straal

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + (\tau_{xy})^2} \quad (3.9)$$

Vlakspanning met thermische effecten

Uitwerking van de wet van Hooke:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) + \alpha T \\ \varepsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) + \alpha T \\ \varepsilon_{zz} &= \frac{-\nu}{E} (\sigma_{xx} + \sigma_{yy}) + \alpha T = \frac{-\nu}{1-\nu} (\varepsilon_{xx} + \varepsilon_{yy}) + \frac{1+\nu}{1-\nu} \alpha T \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \quad (3.10)$$

Geïnverteerde wet van Hooke

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-\nu^2)} [\varepsilon_{xx} + \nu \varepsilon_{yy}] - \frac{E}{1-\nu} \alpha T \\ \sigma_{yy} &= \frac{E}{(1-\nu^2)} [\varepsilon_{yy} + \nu \varepsilon_{xx}] - \frac{E}{1-\nu} \alpha T \\ \tau_{xy} &= G \gamma_{xy} \end{aligned} \quad (3.11)$$

3.1.2 Vlakvervorming

Algemeen

De spanningstensor

$$[\sigma] = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2} \gamma_{xy} & 0 \\ \frac{1}{2} \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.12)$$

Uitwerking van de wet van Hooke:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1+\nu}{E} ((1-\nu)\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} &= \frac{1+\nu}{E} ((1-\nu)\sigma_{yy} - \nu\sigma_{xx}) \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \quad (3.13)$$

Geïnverteerde wet van Hooke

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}] \\ \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}] \\ \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)} [\nu(\varepsilon_{xx} + \varepsilon_{yy})] = \nu(\sigma_{xx} + \sigma_{yy}) \\ \tau_{xy} &= G \gamma_{xy} \end{aligned} \quad (3.14)$$

De transformatieformules voor de spanningen:

$$\begin{aligned} \varepsilon'_{xx} &= \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta \\ \varepsilon'_{yy} &= \varepsilon_{xx} \sin^2 \theta + \varepsilon_{yy} \cos^2 \theta - \gamma_{xy} \sin \theta \cos \theta \\ \gamma'_{xy} &= (\varepsilon_{yy} - \varepsilon_{xx}) \sin \theta \cos \theta + \frac{\gamma_{xy}}{2} (\cos^2 \theta - \sin^2 \theta) \end{aligned} \quad (3.15)$$

waarvoor de hoofdrichtingen worden bepaald door

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_{yy} - \varepsilon_{xx}} \quad (3.16)$$

Deze zijn

$$\begin{aligned} \varepsilon_I &= \frac{\varepsilon_{xx} + \varepsilon_{yy} + \sqrt{(\varepsilon_{xx} - \varepsilon_{yy})^2 + (\gamma_{xy})^2}}{2} \\ \varepsilon_{II} &= \frac{\varepsilon_{xx} + \varepsilon_{yy} - \sqrt{(\varepsilon_{xx} - \varepsilon_{yy})^2 + (\gamma_{xy})^2}}{2} \end{aligned} \quad (3.17)$$

Cirkel van Mohr

Cirkel met middelpunt

$$M = \left(\frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}, 0 \right) \quad (3.18)$$

en straal

$$R = \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2} \right)^2 + \left(\frac{\gamma_{xy}}{2} \right)^2} \quad (3.19)$$

Vlakspanning met thermische effecten

Uitwerking van de wet van Hooke:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1 + \nu}{E} ((1 - \nu)\sigma_{xx} - \nu\sigma_{yy}) + (1 + \nu)\alpha T \\ \varepsilon_{yy} &= \frac{1 + \nu}{E} ((1 - \nu)\sigma_{yy} - \nu\sigma_{xx}) + (1 + \nu)\alpha T \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \quad (3.20)$$

Geïnverteerde wet van Hooke

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_{xx} + \nu\varepsilon_{yy}] - \frac{E}{1 - 2\nu}\alpha T \\ \sigma_{yy} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_{yy} + \nu\varepsilon_{xx}] - \frac{E}{1 - 2\nu}\alpha T \\ \sigma_{zz} &= \frac{E}{(1 + \nu)(1 - 2\nu)} [\nu(\varepsilon_{xx} + \varepsilon_{yy})] - \frac{E}{1 - 2\nu}\alpha T = \nu(\sigma_{xx} + \sigma_{yy}) - E\alpha T \\ \tau_{xy} &= G\gamma_{xy} \end{aligned} \quad (3.21)$$

3.2 AXIAALSYMMETRISCHE BELASINGSGEVALLEN

3.2.1 Basisformules voor axiaalsymmetrie

De spanningstensor

$$[\sigma] = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_{zz} \end{bmatrix} \quad (3.22)$$

De rektensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{rr} & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta z} \\ \frac{1}{2}\gamma_{rz} & \frac{1}{2}\gamma_{\theta z} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{du_r}{dr} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & \frac{du_z}{dz} \end{bmatrix} \quad (3.23)$$

waaruit volgt dat

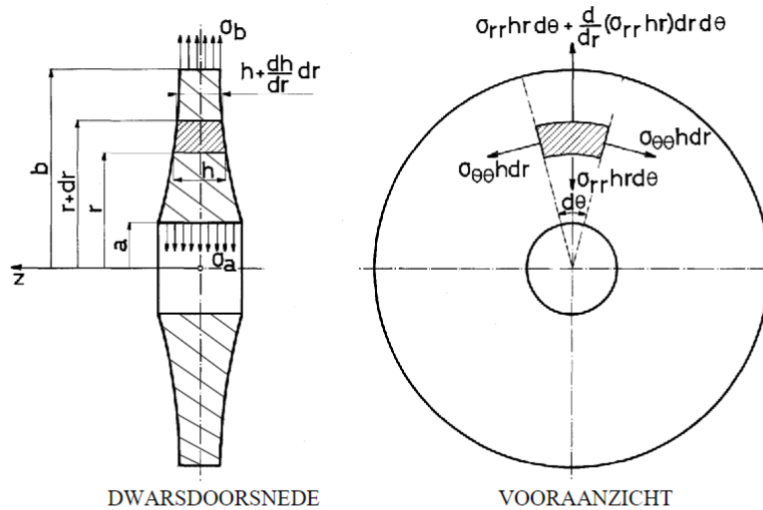
$$u_r = r \cdot \varepsilon_{\theta\theta} \quad (3.24)$$

De wet van Hooke, algemeen met thermische spanningen, wordt gegeven door

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E} [\sigma_{rr} - \nu (\sigma_{\theta\theta} + \sigma_{zz})] + \alpha T \\ \varepsilon_{yy} &= \frac{1}{E} [\sigma_{\theta\theta} - \nu (\sigma_{xx} + \sigma_{zz})] + \alpha T\end{aligned}\quad (3.25)$$

$$\varepsilon_{zz} = \frac{1}{E} [\sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy})] + \alpha T \quad (3.26)$$

3.2.2 Opstellen algemene vergelijkingen voor radiale belastingen



In het algemeen

$$\frac{d}{dr}(\sigma_{rr} \cdot h \cdot r) - \sigma_{\theta\theta} \cdot h = 0 \quad (3.27)$$

en in het bijzonder voor constante dikte h

$$\frac{d}{dr}(\sigma_{rr}) - \sigma_{\theta\theta} = 0 \quad (3.28)$$

3.2.3 Trek- of drukspanning op de binnen- en buitenrand

Schijf in vlakspanning belast σ_a en/of σ_b

De spanningen

$$\begin{aligned}\sigma_{rr} &= \sigma_a \frac{a^2}{a^2 - b^2} \left(1 - \frac{b^2}{r^2}\right) - \sigma_b \frac{b^2}{a^2 - b^2} \left(1 - \frac{a^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \sigma_a \frac{a^2}{a^2 - b^2} \left(1 + \frac{b^2}{r^2}\right) - \sigma_b \frac{b^2}{a^2 - b^2} \left(1 + \frac{a^2}{r^2}\right)\end{aligned}\quad (3.29)$$

De radiale verplaatsing

$$u_r(r) = \frac{\sigma_a}{E} \frac{a^2}{a^2 - b^2} \left[(1 - \nu)r + (1 + \nu)\frac{b^2}{r} \right] - \frac{\sigma_b}{E} \frac{b^2}{a^2 - b^2} \left[(1 - \nu)r + (1 + \nu)\frac{a^2}{r} \right] \quad (3.30)$$

Verticale rek

$$\varepsilon_{zz} = \frac{-\nu}{E} (\sigma_{rr} + \sigma_{\theta\theta}) = \frac{-2\nu}{E} \frac{a^2 \sigma_a - b^2 \sigma_b}{a^2 - b^2} \quad (3.31)$$

Voor volle schijf geldt

$$\begin{aligned}\sigma_{rr} &= \sigma_{\theta\theta} = \sigma_b \\ u_r(r) &= \frac{1 - \nu}{E} \sigma_b r\end{aligned}\quad (3.32)$$

Schijf in vlakvervorming belast met σ_a en/of σ_b

De spanningen

$$\begin{aligned}\sigma_{rr} &= \sigma_a \frac{a^2}{a^2 - b^2} \left(1 - \frac{b^2}{r^2}\right) - \sigma_b \frac{b^2}{a^2 - b^2} \left(1 - \frac{a^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \sigma_a \frac{a^2}{a^2 - b^2} \left(1 + \frac{b^2}{r^2}\right) - \sigma_b \frac{b^2}{a^2 - b^2} \left(1 + \frac{a^2}{r^2}\right)\end{aligned}\quad (3.33)$$

De radiale verplaatsing

$$u_r(r) = \frac{\sigma_a}{E}(1 + \nu) \frac{a^2}{a^2 - b^2} \left[(1 - 2\nu)r + \frac{b^2}{r} \right] - \frac{\sigma_b}{E}(1 + \nu) \frac{b^2}{a^2 - b^2} \left[(1 - 2\nu)r + \frac{a^2}{r} \right] \quad (3.34)$$

Verticale rek

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = 2\nu \frac{a^2 \sigma_a - b^2 \sigma_b}{a^2 - b^2} \quad (3.35)$$

Voor volle schijf geldt

$$\begin{aligned}\sigma_{rr} &= \sigma_{\theta\theta} = \sigma_b \\ u_r(r) &= \frac{(1 + \nu)(1 - 2\nu)}{E} \sigma_b r\end{aligned}\quad (3.36)$$

3.2.4 Radiaal temperatuurveld**Schijf in vlakspanning met radiaal temperatuurveld $T(r)$**

spanningen

$$\begin{aligned}\sigma_{rr} &= E\alpha \frac{r^2 - a^2}{b^2 - a^2} \frac{1}{r^2} \int_a^b T(r) r dr - E\alpha \frac{1}{r^2} \int_a^r T(r) r dr \\ \sigma_{\theta\theta} &= E\alpha \frac{r^2 + a^2}{b^2 - a^2} \frac{1}{r^2} \int_a^b T(r) r dr + E\alpha \frac{1}{r^2} \int_a^r T(r) r dr - E\alpha T(r)\end{aligned}\quad (3.37)$$

Radiale verplaatsing

$$u_r(r) = \alpha \frac{1}{b^2 - a^2} \left[(1 - \nu)r + (1 + \nu) \frac{a^2}{r} \right] \int_a^b T(r) r dr + \alpha \frac{(1 + \nu)}{r} \int_a^r T(r) r dr \quad (3.38)$$

Verticale rek

$$\varepsilon_{zz} = \frac{-\nu}{E} (\sigma_{rr} + \sigma_{\theta\theta}) + \alpha T(r) = \frac{-2\nu}{b^2 - a^2} \alpha \int_a^b T(r) r dr + (1 + \nu) \alpha T(r) \quad (3.39)$$

Temperatuurveld voor constante binnenrand en buitenrand temperatuur

$$\begin{aligned}T(r) &= (T_a - T_0) + (T_b - T_a) \frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)} \\ \Rightarrow \int_a^r T(r) r dr &= \frac{1}{4} (T_b - T_a) \frac{a^2}{\ln\left(\frac{b}{a}\right)} \left[\left(\frac{r}{a}\right)^2 \left(2 \ln\left(\frac{r}{a}\right) - 1\right) + 1 \right] + \frac{1}{2} (T_a - T_0) (r^2 - a^2)\end{aligned}\quad (3.40)$$

waardoor de spanningen geschreven kunnen worden als

$$\begin{aligned}\sigma_{rr} &= \frac{E\alpha(T_b - T_a)}{2 \ln\left(\frac{b}{a}\right)} \left[\frac{r^2 - a^2}{r^2} \frac{b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) - \ln\left(\frac{r}{a}\right) \right] \\ \sigma_{\theta\theta} &= \frac{E\alpha(T_b - T_a)}{2 \ln\left(\frac{b}{a}\right)} \left[\frac{r^2 + a^2}{r^2} \frac{b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) - \ln\left(\frac{r}{a}\right) - 1 \right]\end{aligned}\quad (3.41)$$

Schijf in vlakvervorming met radiaal temperatuurveld $T(r)$

spanningen

$$\begin{aligned}\sigma_{rr} &= \frac{E}{1-\nu} \frac{r^2 - a^2}{b^2 - a^2} \frac{\alpha}{r^2} \int_a^b T(r) r dr - \frac{E}{1-\nu} \frac{\alpha}{r^2} \int_a^r T(r) r dr \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu} \frac{r^2 + a^2}{b^2 - a^2} \frac{\alpha}{r^2} \int_a^b T(r) r dr + \frac{E}{1-\nu} \frac{\alpha}{r^2} \int_a^r T(r) r dr - \frac{E}{1-\nu} \alpha T(r)\end{aligned}\quad (3.42)$$

Radiale verplaatsing

$$u_r(r) = \alpha \frac{1+\nu}{1-\nu} \frac{1}{b^2 - a^2} \left[(1-2\nu)r + \frac{a^2}{r} \right] \int_a^b T(r) r dr + \alpha \frac{1+\nu}{1-\nu} \frac{1}{r} \int_a^r T(r) r dr \quad (3.43)$$

Verticale spanning

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) - E\alpha T(r) = \frac{2\nu E}{1-\nu} \frac{\alpha}{b^2 - a^2} \int_a^b T(r) r dr - \frac{E}{1-\nu} \alpha T(r) \quad (3.44)$$

Temperatuurveld voor constante binnenrand en buitenrand temperatuur geeft voor de spanningen

$$\begin{aligned}\sigma_{rr} &= \frac{E\alpha(T_b - T_a)}{2(1-\nu) \ln\left(\frac{b}{a}\right)} \left[\frac{r^2 - a^2}{r^2} \frac{b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) - \ln\left(\frac{r}{a}\right) \right] \\ \sigma_{\theta\theta} &= \frac{E\alpha(T_b - T_a)}{2(1-\nu) \ln\left(\frac{b}{a}\right)} \left[\frac{r^2 + a^2}{r^2} \frac{b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) - \ln\left(\frac{r}{a}\right) - 1 \right]\end{aligned}\quad (3.45)$$

$$\sigma_{zz} = \frac{E\alpha(T_b - T_a)}{(1-\nu) \ln\left(\frac{b}{a}\right)} \left[\nu \frac{b^2}{b^2 - a^2} \ln\left(\frac{b}{a}\right) - \ln\left(\frac{r}{a}\right) - \frac{\nu}{2} \right] - E\alpha(T_a - T_0) \quad (3.46)$$

Lange buis met vrije uiteinden met radiaal temperatuurveld $T(r)$

Spanningen

$$\begin{aligned}\sigma_{rr} &= \frac{E}{1-\nu} \frac{r^2 - a^2}{b^2 - a^2} \frac{\alpha}{r^2} \int_a^b T(r) r dr - \frac{E}{1-\nu} \frac{\alpha}{r^2} \int_a^r T(r) r dr \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu} \frac{r^2 + a^2}{b^2 - a^2} \frac{\alpha}{r^2} \int_a^b T(r) r dr + \frac{E}{1-\nu} \frac{\alpha}{r^2} \int_a^r T(r) r dr - \frac{E}{1-\nu} \alpha T(r) \\ \sigma_{zz} &= \nu(\sigma_{rr} + \sigma_{\theta\theta}) - E\alpha T(r) = \frac{2\nu E}{1-\nu} \frac{\alpha}{b^2 - a^2} \int_a^b T(r) r dr - \frac{E}{1-\nu} \alpha T(r)\end{aligned}\quad (3.47)$$

De rek en verplaatsing worden dan gegeven door

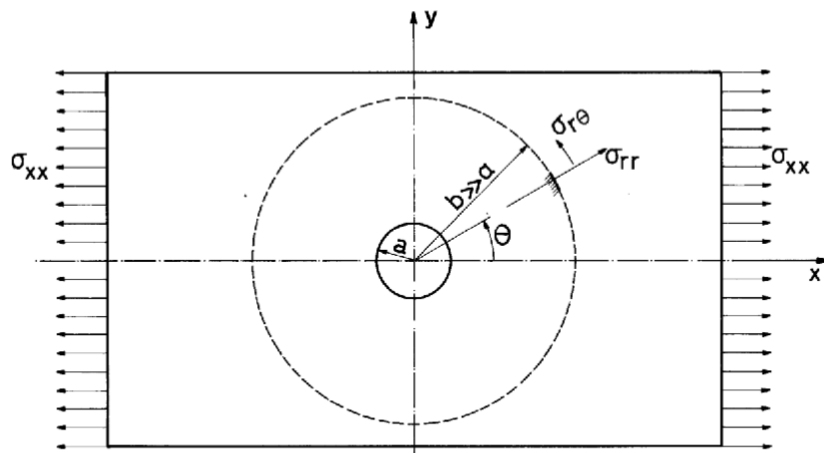
$$\begin{aligned}\varepsilon_{zz} &= \frac{2\alpha}{b^2 - a^2} \int_a^b T(r) r dr \\ u_r(r) &= \frac{\alpha}{r} \frac{1}{1-\nu} \left[(1+\nu) \int_a^r T(r) r dr + \frac{(1-3\nu)r^2 + (1+\nu)a^2}{b^2 - a^2} \int_a^b T(r) r dr \right]\end{aligned}\quad (3.48)$$

3.3 SPANNINGSCONCENTRATIES IN VLAKKE PLATEN

De spanningen worden gegeven door

$$\begin{aligned}\sigma_{rr} &= \frac{\sigma_{xx}}{2} \left[1 - \frac{a^2}{r^2} + \cos 2\theta \left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4} \right) \right] \\ \sigma_{\theta\theta} &= \frac{\sigma_{xx}}{2} \left[1 + \frac{a^2}{r^2} - \cos 2\theta \left(1 + 3\frac{a^4}{r^4} \right) \right]\end{aligned}\quad (3.49)$$

$$\tau_{r\theta} = \frac{-\sigma_{xx}}{2} \sin 2\theta \left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4} \right) \quad (3.50)$$



Figuur 4.11 Spanningsconcentratie aan een ronde opening in een plaat [9].