2

Geometry of Riemann Spaces

The fact that the geometry of the space in which we live is Euclidean is a very basic daily experience. This may explain why it took so long before it was realised that this may actually not be correct, and that the question of the geometry of the space around us is a matter of empirical assessment. Early in the 19th century Gauss studied the geometry of curved surfaces, and showed that all references to a flat embedding space could be eliminated. In the same way Riemann formulated in 1854 the geometry of 3D spaces. He found that Euclidean geometry is merely one possibility out of many. Riemann's method could be generalized to spaces of arbitrary dimension. The geometry of these curved Riemann spaces is wholly described within the space itself, by the use of co-ordinates and the metric tensor. No embedding is required. These geometrical concepts gradually spread beyond the mathematical incrowd, and in the last quarter of the 19th century the idea that a fourth (spatial) dimension might exist had mesmerized the public's imagination, perhaps even more so than black holes did a century later. One of the products of that period was Abbott's famous Flatland.[1] The flatland analogy is nowadays a standard technique of teachers to explain some of the intricacies of curved spaces.

The theoretical framework of Riemann spaces is also the starting point for the mathematical formulation of GR. In this chapter we discuss the tools that any student should master in order to be able to deal with GR beyond the level of handwaving. In doing so we have deliberately chosen to stay close to intuition as that outweighs the merits of rigour, certainly on first acquaintance.

2.1 Definition

A Riemann space has the following properties:

[1] Abbott, E.A.: 1884, *Flatland: A Romance of many Dimensions, by a Square*, Seeley & Co. (London).

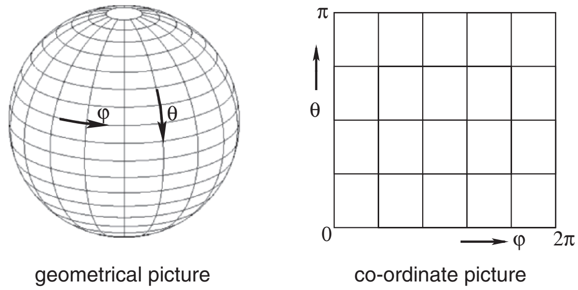

Fig. 2.1. A geometrical picture and the corresponding co-ordinate picture of the space defined by (2.2). Co-ordinate pictures will be frequently used.

1. Any point can be identified by a set of co-ordinates $\{x^\mu\}$; the number of independent x^μ is called the dimension.

2. It is possible to define continuously differentiable functions of $\{x^\mu\}$, in particular one-to-one co-ordinate transformations $\{x^\mu\} \leftrightarrow \{\overline{x}^\nu\}$.

3. There is a metric that specifies the distance $\mathrm{d}s^2$ between two nearby points x^μ and $x^\mu + \mathrm{d}x^\mu$:

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha \mathrm{d}x^\beta \quad ; \qquad g_{\alpha\beta} = g_{\beta\alpha} \ . \tag{2.1}$$

An antisymmetric part of $g_{\alpha\beta}$ does not contribute to $\mathrm{d}s^2$. Example: a spherical surface with radius 1 and co-ordinates θ, φ:

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \ . \tag{2.2}$$

Notation: $\mathrm{d}\theta^2 \equiv (\mathrm{d}\theta)^2$, $\mathrm{d}\varphi^2 \equiv (\mathrm{d}\varphi)^2$, but $\mathrm{d}s^2 = (\mathrm{d}s)^2$ only if $\mathrm{d}s^2 > 0$ as in (2.2). But the metric is in general not positive definite! In this simple case the geometrical structure may be visualised through embedding in an Euclidean space of one higher dimension, but for Riemann spaces of higher dimension this is no longer possible. Moreover, a Riemann space of dimension D cannot always be embedded in a flat space of dimension $D + 1$. It is often useful to draw a *co-ordinate picture* of a suitably chosen subspace, even though it contains no information on the geometry, see Fig. 2.1.

An important point is that the metric determines the local structure of the space, but reveals nothing about its global (topological) structure. A plane, a cone and a cylinder all have the same metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, but entirely different global structures.

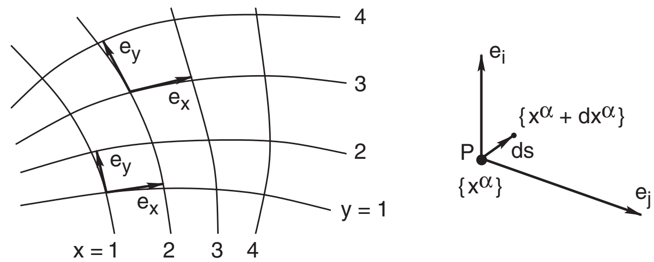

Fig. 2.2. Co-ordinate lines and base vectors spanning the tangent space. The choice of the co-ordinates is entirely free, and in practice dictated by the question which co-ordinates are the most expedient to use.

2.2 The tangent space

In each point we construct a set of base vectors tangent to the co-ordinate lines, as in Fig. 2.2. The arrow points towards increasing x^i. The base vectors span the flat tangent space, which has the same dimension as Riemann space. This construction evidently requires the existence of a flat embedding space, but that can be avoided as follows. Consider the curves $\{x^\alpha(p)\}$ through a point P in Riemann space (p = curve parameter), and construct $A^\sigma = [\mathrm{d}x^\sigma/\mathrm{d}p]_P$. These vectors A^σ span the abstract tangent space of P, which exists independent of any embedding. Usually, however, the abstract tangent space may be identified with the tangent space constructed in Fig. 2.2. For our discussion there is no real advantage in making the distinction and we shall work with the intuitive picture of Fig. 2.2.

We may use any metric we like in the tangent space, but there exists a preferred metric. Consider an infinitesimal section of Riemann space. This section is flat and virtually coincides with the tangent space. To an infinitesimal vector $\mathrm{d}\boldsymbol{s} = \mathrm{d}x^\alpha \boldsymbol{e}_\alpha$ *in the tangent space* we may therefore assign the length of the line element $\mathrm{d}s$ *in Riemann space*, i.e. we require $\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s} = \mathrm{d}s^2$:

$$\begin{aligned}\mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s} \;&=\; (\mathrm{d}x^\alpha\boldsymbol{e}_\alpha)\cdot(\mathrm{d}x^\beta\boldsymbol{e}_\beta) \;=\; \boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\beta\;\mathrm{d}x^\alpha\mathrm{d}x^\beta \\ &=\; g_{\alpha\beta}\;\mathrm{d}x^\alpha\mathrm{d}x^\beta \;, \end{aligned} \tag{2.3}$$

and it follows that

$$g_{\alpha\beta} \;\equiv\; \boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\beta \;. \tag{2.4}$$

Here $\cdot$ represents the vector inner product. This may be the usual inner product, for example when we deal with 2D surfaces embedded in a flat R_3. But in case of the Minkowski spacetime of SR, and in GR, the inner product is not positive definite, and we may have that $\boldsymbol{A}\cdot\boldsymbol{A} < 0$ (for spacelike vectors). By taking $\mathrm{d}x^\alpha = 1$ in (2.3) and all other $\mathrm{d}x^\beta = 0$ we see that $\boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\alpha = \mathrm{d}\boldsymbol{s}\cdot\mathrm{d}\boldsymbol{s}$ (no summation). It follows that the 'length' of $\boldsymbol{e}_\alpha$ corresponds to a jump $\Delta x^\alpha = 1$, at constant value of the other co-ordinates. Due to the curvature this is of course only approximately correct. These base vectors are called a co-ordinate basis because they are defined entirely by the co-ordinates and the metric. The length of the base vectors depends on the choice of the co-ordinates, and is in general a function of position. Consider for example polar co-ordinates in a plane, Fig. 2.3. The length of $\boldsymbol{e}_\mathrm{r}$ is constant, while $|\boldsymbol{e}_\varphi| \propto r$:

$$\mathrm{d}s^2 = \underset{\substack{\uparrow \\ \boldsymbol{e}_\mathrm{r}\cdot\boldsymbol{e}_\mathrm{r}}}{1}\cdot\mathrm{d}r^2 + \underset{\substack{\uparrow \\ \boldsymbol{e}_\varphi\cdot\boldsymbol{e}_\varphi}}{r^2}\mathrm{d}\varphi^2 \ . \tag{2.5}$$

Now that we have defined the basis we may construct finite vectors $\boldsymbol{A} = A^\alpha\boldsymbol{e}_\alpha$ in the tangent space through the usual parallelogram construction. These so called contravariant components A^α are the components of $\boldsymbol{A}$ along the basis.

The next step is to define another (covariant) representation A_α of $\boldsymbol{A}$ by demanding that $\boldsymbol{A}\cdot\boldsymbol{A} = A_\alpha A^\alpha$, for every $\boldsymbol{A}$:

$$\boldsymbol{A}\cdot\boldsymbol{A} = (A^\alpha\boldsymbol{e}_\alpha)\cdot(A^\beta\boldsymbol{e}_\beta) = g_{\alpha\beta}\,A^\beta A^\alpha \equiv A_\alpha A^\alpha \ , \tag{2.6}$$

which leads to:

$$A_\alpha = g_{\alpha\beta}A^\beta \ . \tag{2.7}$$

In a more advanced treatment a distinction is made between tensors as geometrical objects, their contravariant representation located in an abstract tangent space, and the dual tangent space, in which the covariant representations reside. In the current, more primitive context the following interpretation suggests itself. Since $A_\gamma = g_{\gamma\beta}A^\beta = \boldsymbol{e}_\gamma\cdot\boldsymbol{e}_\beta A^\beta = (A^\beta\boldsymbol{e}_\beta)\cdot\boldsymbol{e}_\gamma = \boldsymbol{A}\cdot\boldsymbol{e}_\gamma$, it follows that A_γ is the projection of $\boldsymbol{A}$ on $\boldsymbol{e}_\gamma$. Hence, the contravariant components A^β are the components of $\boldsymbol{A}$ along the base vectors $\boldsymbol{e}_\beta$ (parallelogram construction), while the covariant component A_α is the projection of $\boldsymbol{A}$ on the base vector $\boldsymbol{e}_\alpha$, Fig. 2.3, right:

$$\text{contravariant } (A^\beta): \ \boldsymbol{A} = A^\beta\boldsymbol{e}_\beta \ , \tag{2.8}$$

$$\text{covariant } (A_\alpha): \ A_\alpha = \boldsymbol{A}\cdot\boldsymbol{e}_\alpha \ . \tag{2.9}$$

Finally, the concept of *index raising and lowering.* We can lower an index with the help of (2.7). The inverse operation of raising is defined as:

$$A^\gamma = g^{\gamma\alpha}A_\alpha \ . \tag{2.10}$$

The meaning of $g^{\gamma\alpha}$ can be gleaned from:

$$A^\gamma = g^{\gamma\alpha}A_\alpha = g^{\gamma\alpha}g_{\alpha\nu}A^\nu \ , \tag{2.11}$$

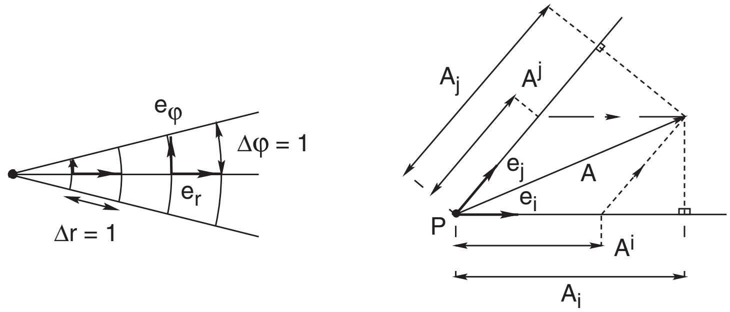

Fig. 2.3. Left: polar co-ordinates and the base vectors $\boldsymbol{e}_\mathrm{r}$ and $\boldsymbol{e}_\varphi$. Right: interpretation of the contravariant and covariant representation of a vector $\boldsymbol{A}$.

so that $g^{\gamma\alpha}g_{\alpha\nu} = \delta^\gamma_\nu$, i.e. $\{g^{\gamma\alpha}\}$ is the inverse of $\{g_{\alpha\nu}\}$. In summary:

$$\left.\begin{array}{lrcl} \text{index lowering :} & A_\alpha &=& g_{\alpha\beta}A^\beta \ , \\ \text{index raising :} & A^\gamma &=& g^{\gamma\nu}A_\nu \ , \\ & \{g^{\gamma\nu}\} &=& \{g_{\alpha\beta}\}^{-1} \ . \end{array}\right\} \tag{2.12}$$

We have silently adopted the *summation convention*: if an index occurs twice, once as a lower and once as an upper index, summation over that index is implied. Note that the rules for index raising and lowering are always valid, and have nothing to do with the question whether one is dealing with a tensor or not. The tensor concept is related to behaviour under co-ordinate transformations, which was not an issue above, and to which we turn our attention now.

2.3 Tensors

We are now in a position to do linear algebra in the tangent space, but we leave that aside and study the effect of co-ordinate transformations. Consider two overlapping sets of co-ordinates $\{x^\mu\}$ and $\{x^{\mu'}\}$. The notation is sloppy – it would be more appropriate to write $\{\overline{x}^\mu\}$ instead of $\{x^{\mu'}\}$, but $\{x^{\mu'}\}$ is much more expedient if used with care. A displacement $\delta x^{\mu'}$ is related to a displacement δx^ν through:

$$\delta x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\,\delta x^\nu \equiv x^{\mu'}_{\ ,\nu}\,\delta x^\nu \ . \tag{2.13}$$

Notation:

$$X_{,\nu} \equiv \frac{\partial X}{\partial x^\nu} \; ; \qquad X_{,\nu\rho} \equiv \frac{\partial^2 X}{\partial x^\nu \partial x^\rho} \; , \quad \text{etc.} \tag{2.14}$$

where X can be anything ($A_\alpha,\ g^{\alpha\beta}, ...$). We may freely interchange indices behind the comma: $X_{,\alpha\beta\gamma} = X_{,\alpha\gamma\beta} = X_{,\gamma\alpha\beta}$ etc.

Any set A^μ transforming according to (2.13) is called a contravariant tensor of rank 1:

$$A^{\mu'} = x^{\mu'}_{\ ,\nu} A^\nu \quad \leftrightarrow \quad A^\nu \ \text{ contravariant.} \tag{2.15}$$

Hence δx^ν is a contravariant tensor. Tensors of rank 1 are often referred to as vectors, and henceforth we shall use the word vector in this sense only. A function such as the temperature distribution $T(x)$ is called a *scalar*, a tensor of rank zero. Its value in a point is independent of the co-ordinate system, i.e. invariant for co-ordinate transformations: $T'(x') = T(x)$, where T' is the new function prescription. The derivative of a scalar Q,

$$B_\mu = \frac{\partial Q}{\partial x^\mu} \equiv Q_{,\mu} \tag{2.16}$$

transforms like $B_{\mu'} = Q'_{,\mu'} = Q_{,\nu}\, x^\nu_{\ ,\mu'} = x^\nu_{\ ,\mu'}\, B_\nu$. Every B_ν that transforms in this way is called a covariant vector or tensor of rank 1:

$$B_{\mu'} = x^\nu_{\ ,\mu'}\, B_\nu \quad \leftrightarrow \quad B_\nu \ \text{ covariant.} \tag{2.17}$$

From two covariant vectors we can form $T_{\mu\nu} = A_\mu B_\nu$, a covariant tensor of rank 2. More general tensors can be constructed through summation, $T_{\mu\nu} = A_\mu B_\nu + C_\mu D_\nu + ...$ This process may be continued: $T_{\alpha\beta} C^\gamma$ and $A_\mu C^\nu B_\rho$ are mixed tensors of rank 3 (provided T, A, B and C are tensors themselves). The indices of tensors of higher rank transform according to (2.15) resp. (2.17), for example:

$$T^{\alpha'}{}_{\beta'\gamma'}{}^{\delta'} = x^{\alpha'}_{\ ,\mu}\, x^\nu_{\ ,\beta'}\, x^\sigma_{\ ,\gamma'}\, x^{\delta'}_{\ ,\tau}\, T^\mu{}_{\nu\sigma}{}^\tau \; . \tag{2.18}$$

There is no other choice because (2.18) must hold for the special tensor $T^\alpha{}_{\beta\gamma}{}^\delta = P^\alpha Q_\beta R_\gamma S^\delta$, and the transformation rules for vectors have already been fixed! Note that we get a glimpse here of how the Lorentz transformations of SR will be generalised in GR: relation (1.7) of SR will be replaced by (2.15). This transformation is still locally linear, but different in each point of Riemann space as the $\{x^{\mu'}_{\ ,\nu}\}$ are functions of position. The single global Lorentz transformation will be replaced by a mesh of local Lorentz transformations.

The horizontal position of the indices is important: $T^\mu{}_\nu$ is different from $T_\nu{}^\mu$! The summation over double indices is called *contraction*. It lowers the rank by two. For example $T^\mu{}_\mu,\ T^\alpha{}_{\beta\alpha}{}^\gamma,\ P_{\alpha\beta}\, Q^{\beta\gamma},\ T^\alpha{}_{\beta\alpha}{}^\beta$ (double contraction). Double indices are *dummies*: $T^\alpha{}_\alpha = T^\mu{}_\mu$, dummies may occur only twice, once as an upper and once as a lower index. If you encounter expressions like $C^{\mu\mu}$, $P^{\alpha\beta}Q^{\alpha}{}_{\gamma}$ or $P_{\alpha\beta}\,Q^{\alpha\gamma}\,R_{\delta\alpha}$ then you have made a mistake somewhere!

Index raising and lowering, finally, is done by factors $g_{\alpha\beta}$ or $g^{\mu\nu}$ for each upper/lower index, e.g.:

$$\left.\begin{aligned} T^{\mu\nu} &= g^{\mu\alpha}\,T_{\alpha}{}^{\nu}\ , \\ T^{\alpha}{}_{\beta}{}^{\gamma\delta} &= g^{\alpha\mu}\,g_{\beta\nu}\,g^{\delta\sigma}\ T_{\mu}{}^{\nu\gamma}{}_{\sigma}\ , \quad \text{etc.} \end{aligned}\right\} \tag{2.19}$$

Again, like in (2.18), we have hardly any other choice here, because (2.19) must hold for the special tensors $T^{\mu\nu} = P^{\mu}Q^{\nu}$ and $T^{\alpha}{}_{\beta}{}^{\gamma\delta} = P^{\alpha}Q_{\beta}R^{\gamma}S^{\delta}$, and the rules for index raising and lowering for vectors have already been fixed. We are now in a position that we can raise and lower indices at liberty. We emphasise once more that the rules (2.12) and (2.19) for index gymnastics are generally valid, also for non-tensors. For example, $Q_{\mu\nu} = A_{\mu,\nu}$ is not a tensor (exercise 2.4), and yet $Q^{\mu}{}_{\nu} = g^{\mu\alpha}Q_{\alpha\nu}$.

Exercise 2.1: The unit tensor is defined as $\delta^{\alpha}{}_{\beta} = 1$ for $\alpha = \beta$, otherwise 0. Prove that $\delta^{\alpha}{}_{\beta}$ is a tensor, and that $\delta^{\alpha}{}_{\beta} = \delta_{\beta}{}^{\alpha}$, so that we may write δ^{α}_{β} without risk of confusion. Show that $\delta_{\alpha\beta} = g_{\alpha\beta}$. Is $\eta_{\alpha\beta}$ a tensor? And $g_{\alpha\beta}$? One could define $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$, and 0 otherwise, but then $\delta_{\alpha\beta}$ is not a tensor.

Hint: $\delta^{\alpha'}{}_{\beta'}$ must be equal to $x^{\alpha'}{}_{,\nu}\,x^{\mu}{}_{,\beta'}\,\delta^{\nu}{}_{\mu}$, or $\delta^{\alpha'}{}_{\beta'} = x^{\alpha'}{}_{,\nu}\,x^{\nu}{}_{,\beta'} = x^{\alpha'}{}_{,\beta'}$ (chain rule) $= 1$ for $\alpha' = \beta'$ otherwise 0. Hence $\delta^{\alpha}{}_{\beta}$ is tensor. And $\delta_{\beta}{}^{\alpha} = g_{\beta\mu}\,g^{\alpha\nu}\delta^{\mu}{}_{\nu} = g_{\beta\mu}\,g^{\alpha\mu} = g^{\alpha\mu}g_{\mu\beta} = 1$ for $\alpha = \beta$, otherwise 0, i.e. identical to $\delta^{\alpha}{}_{\beta}$; $\delta_{\alpha\beta} = g_{\alpha\nu}\delta^{\nu}_{\beta} = g_{\alpha\beta}$; $\eta_{\alpha\beta}$ is a tensor in SR only, i.e. under Lorentz transformations; $g_{\alpha\beta}$ tensor: use (2.1), require that $\mathrm{d}s^2$ is also tensor in GR (invariant scalar), and $\mathrm{d}x^{\alpha}$ is tensor, then exercise 2.3. Other definition $\delta_{\alpha\beta}$: $\delta_{\alpha'\beta'} = x^{\nu}{}_{,\alpha'}\,x^{\mu}{}_{,\beta'}\,\delta_{\nu\mu}$? No, because the chain rule can no longer be used.

Exercise 2.2: If $T^{\alpha\beta}$ and $P^{\mu}{}_{\nu}$ are tensors then $P^{\mu}{}_{\mu}$ is a scalar, but $T^{\alpha\alpha}$ is not. The inner product $A_{\nu}B^{\nu}$ of two vectors is a scalar.

Hint: $P^{\mu'}{}_{\mu'} = x^{\mu'}{}_{,\alpha}\,x^{\beta}{}_{,\mu'}\,P^{\alpha}{}_{\beta}$, then the chain rule.

Exercise 2.3: Quotient theorem: If $A^{\lambda}P_{\lambda\mu\nu}$ is a tensor for arbitrary vector A^{λ}, then $P_{\lambda\mu\nu}$ is a tensor; $\mu\nu$ may be replaced with an arbitrary sequence of upper / lower indices.

Hint: $A^\lambda P_{\lambda\mu\nu}$ is a tensor, i.e. $A^{\lambda'} P_{\lambda'\mu'\nu'} = x^\alpha_{,\mu'}\, x^\beta_{,\nu'}\, A^\sigma P_{\sigma\alpha\beta}$ (λ' and σ are dummies!), then substitute $A^\sigma = x^\sigma_{,\lambda'}\, A^{\lambda'}$, etc.

Exercise 2.4: The derivative $A_{\mu,\nu}$ of a covariant vector A_μ is not a tensor, as it transforms according to:

$$A_{\mu',\nu'} = A_{\alpha,\beta}\, x^\alpha_{,\mu'}\, x^\beta_{,\nu'} + A_\alpha\, x^\alpha_{,\mu'\nu'} \; . \tag{2.20}$$

The problem is in the second term of (2.20). In SR only linear (Lorentz) transformations are allowed. In that case the second term is zero and $A_{\mu,\nu}$ is a tensor.

Hint: Start from $A_{\mu',\nu'} = (x^\alpha_{,\mu'}\, A_\alpha)_{,\nu'}$, then use the product rule.

Exercise 2.5: Prove $T_\alpha{}^\nu A_\nu = T_{\alpha\nu} A^\nu$; $T_\alpha{}^\alpha = T^\alpha{}_\alpha$; $g^\nu{}_\nu = 4$; $\eta^\nu{}_\nu = g^{00} - g^{11} - g^{22} - g^{33}$.

Hint: We know that $g^\nu{}_\nu = g^{\nu\alpha} g_{\alpha\nu} = \delta^\nu_\nu = 4$. The following may be illuminating: the scalar $g^\nu{}_\nu$ is invariant, compute in a freely falling frame: $g^\nu{}_\nu = \eta^\nu{}_\nu$, SR holds in that frame: $\eta^\nu{}_\nu = \eta^{\nu\alpha}\eta_{\alpha\nu} = 4$. But in GR: $\eta^\nu{}_\nu = g^{\nu\alpha}\eta_{\alpha\nu} =$ etc.

2.4 Parallel transport and Christoffel symbols

Consider a particle at position P in Riemann space, Fig. 2.4. The vectors associated with it (velocity, spin, ..) reside in the tangent space of P. At some later time the particle has moved to position Q, but the tangent space of Q does not coincide with that of P. To be able to do dynamics, we must develop a way to compare vectors in the different tangent spaces along the worldline of the particle. In other words, we need something against which to gauge the concept of 'change'. This is what parallel transport in GR is about.

Fig. 2.4 shows the curve $x^\sigma(p)$ in Riemann space. The vector $\boldsymbol{A}$ is always in the tangent space, but the tangent spaces of P, Q, R, .. are disjunct, and comparison of $\boldsymbol{A}(P)$ with $\boldsymbol{A}(Q)$ or $\boldsymbol{A}(R)$ is not possible. To this end we define a *connection* between tangent spaces, that is, a mathematical prescription telling us how a vector $\boldsymbol{A}(P)$ lies in the tangent space of Q if we 'transport' it along a given path from P to Q. This can be done in a variety of ways, but much of the mathematical freedom that we have is eliminated by the physical

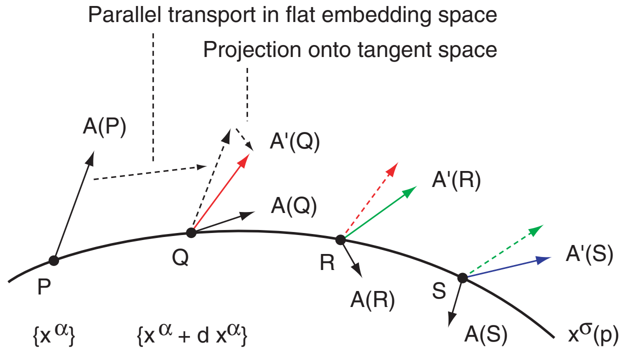

Fig. 2.4. Conceptual definition of parallel displacement of a vector along a curve $x^\sigma(p)$ in Riemann space: first an ordinary parallel displacement in the flat embedding space (resulting in the dashed arrows) followed by projection on the local tangent space. The process is repeated in infinitesimal steps.

requirement that we recover what we ordinarily do when we transport a vector parallel to itself in a flat space. Imagine the Riemann space embedded in a flat space of higher dimension. We know how to move $\boldsymbol{A}(P)$ around parallel to itself in this embedding space, because it is flat. Having arrived in Q, the result is projected onto the local tangent space. To order $O(\mathrm{d}x^\alpha)$ projection does not change the length of the vector: the projection angle γ is $O(\mathrm{d}x^\alpha)$, but $\cos\gamma = 1$ up to $O(\mathrm{d}x^\alpha)$. This process is now repeated with infinitesimal steps, and generates the coloured vector field $\boldsymbol{A}'$ in Fig. 2.4, starting from $\boldsymbol{A}(P)$. In this way we have generalized the concept of parallel transport to curved spaces, in such a way that it reduces to normal parallel transport for flat spaces. Not surprisingly, it is also the definition that turns out to work in GR. The result of the transport operation depends on the path, see Fig. 2.5. However, when $\boldsymbol{e}$ in Fig. 2.5 is parallel-transported along a small curve on the sphere there is virtually no change, because there is hardly any curvature felt (exercise 2.17).

We now formalise our intuitive approach. The difference $\mathrm{d}\boldsymbol{A} = \boldsymbol{A}(Q) - \boldsymbol{A}(P)$ is not defined, but up to order $O(\mathrm{d}x^\alpha)$ we have that $\mathrm{d}\boldsymbol{A} \simeq \boldsymbol{A}(Q) - \boldsymbol{A}'(Q)$, and this is useful as both vectors lie in the same tangent space. The vector $\mathrm{d}\boldsymbol{A}$ may be interpreted as the intrinsic change of $\boldsymbol{A}$, after correction for the 'irrelevant' change in the orientation of the tangent space:

$$\mathrm{d}\boldsymbol{A} \simeq \boldsymbol{A}(Q) - \boldsymbol{A}'(Q) \tag{2.21}$$

$$= \mathrm{d}(A^\mu \boldsymbol{e}_\mu) = (\mathrm{d}A^\mu)\boldsymbol{e}_\mu + A^\mu(\mathrm{d}\boldsymbol{e}_\mu) \; . \tag{2.22}$$

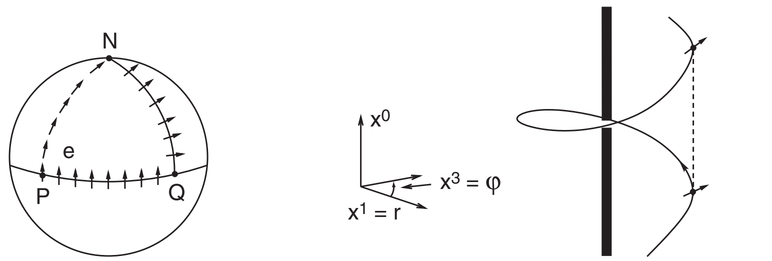

Fig. 2.5. Left: Parallel displacement of the vector $\boldsymbol{e}$ along PNQ and along PQ produces entirely different results. To the right, the geodesic precession of a top in orbit around a central mass, see text.

Here, $\mathrm{d}\boldsymbol{A}$ has been split into two contributions: the change $\mathrm{d}A^\mu \equiv A^\mu(Q) - A^\mu(P)$ of the contravariant components of $\boldsymbol{A}$, and a contribution from the change of the base vectors. On general grounds we anticipate $\mathrm{d}\boldsymbol{e}_\mu$ to be proportional to both $\{\mathrm{d}x^\beta\}$ and $\{\boldsymbol{e}_\alpha\}$:

$$\mathrm{d}\boldsymbol{e}_\mu = \Gamma^\alpha_{\ \mu\beta}\,\mathrm{d}x^\beta\,\boldsymbol{e}_\alpha \ . \tag{2.23}$$

$\Gamma^\alpha_{\ \mu\beta}$ is called the *Christoffel symbol of the second kind*, and as may be expected it is intimately related to the metric tensor:

$$\Gamma^\mu_{\ \nu\sigma} = \tfrac{1}{2} g^{\mu\lambda}\,(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}) \equiv g^{\mu\lambda}\,\Gamma_{\lambda\nu\sigma} \ . \tag{2.24}$$

The $=$ sign is proved in § 2.5. The $\equiv$ sign defines the *Christoffel symbol of the first kind*, simply by raising one index with $g^{\mu\lambda}$. According to (2.23) the Christoffel symbols define the connection between the base vectors of the tangent spaces at different positions. As pointed out above, there exist more general connection coefficients than (2.24), but these play no role in GR.

Insert (2.23) in (2.22) and rename the dummy-indices:

$$\mathrm{d}\boldsymbol{A} = (\mathrm{d}A^\mu + \Gamma^\mu_{\ \nu\sigma}\,A^\nu\,\mathrm{d}x^\sigma)\,\boldsymbol{e}_\mu \equiv (\mathrm{D}A^\mu)\,\boldsymbol{e}_\mu \ . \tag{2.25}$$

The right hand side defines the intrisic change $\mathrm{D}A^\mu$, which apparently obeys the following equation:

$$\frac{\mathrm{D}A^\mu}{\mathrm{D}p} = \frac{\mathrm{d}A^\mu}{\mathrm{d}p} + \Gamma^\mu_{\ \nu\sigma}\,A^\nu\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}p} \qquad \text{(contravariant);} \tag{2.26}$$

$$\frac{\mathrm{D}A_\mu}{\mathrm{D}p} = \frac{\mathrm{d}A_\mu}{\mathrm{d}p} - \Gamma^\nu_{\ \mu\sigma}\,A_\nu\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}p} \qquad \text{(covariant).} \tag{2.27}$$

For the second relation (2.27) see exercise 2.8. We may apply these equations in two ways. For a *given* vector field we may compute $\mathrm{D}A^\mu$ or $\mathrm{D}A_\mu$ for a displacement $\mathrm{d}p$ along $x^\sigma(p)$. On the other hand, one may solve $\mathrm{D}A^\mu/\mathrm{D}p = 0$ or $\mathrm{D}A_\mu/\mathrm{D}p = 0$ starting from an initial value $A^\mu(P)$ or $A_\mu(P)$, and construct a vector field along $x^\sigma(p)$ for which $\mathrm{d}\boldsymbol{A} = \boldsymbol{A} - \boldsymbol{A}' = 0$. Parallel transport of a vector along $x^\sigma(p)$ is therefore described by the differential equation

$$\frac{\mathrm{D}A_\mu}{\mathrm{D}p} = 0 \qquad \text{or} \qquad \frac{\mathrm{D}A^\mu}{\mathrm{D}p} = 0 \; . \tag{2.28}$$

We mention a few properties of the Christoffel symbols. They are symmetrical in the last two indices:

$$\Gamma^\mu_{\ \nu\sigma} = \Gamma^\mu_{\ \sigma\nu} \ ; \qquad \Gamma_{\lambda\nu\sigma} = \Gamma_{\lambda\sigma\nu} \; . \tag{2.29}$$

By interchanging the indices in (2.24) we may infer $\Gamma_{\nu\lambda\sigma}$, and on adding that to $\Gamma_{\lambda\nu\sigma}$ one obtains

$$\Gamma_{\lambda\nu\sigma} + \Gamma_{\nu\lambda\sigma} = g_{\lambda\nu,\sigma} \; . \tag{2.30}$$

The Christoffel symbol transforms according to

$$\Gamma^{\mu'}_{\ \nu'\sigma'} = \Gamma^\rho_{\ \alpha\beta}\, x^{\mu'}_{\ ,\rho}\, x^\alpha_{\ ,\nu'}\, x^\beta_{\ ,\sigma'} + x^{\mu'}_{\ ,\rho}\, x^\rho_{\ ,\nu'\sigma'} \; . \tag{2.31}$$

The proof is for diehards (see literature). The first term is what we would expect if the Christoffel symbol were a tensor, but the second term makes that it is actually not a tensor. The concept of parallel transport will be used in § 2.5 to define geodesics.

In SR the velocity and spin vector of a particle on which no forces are exerted are constant. They are transported parallel along the ‘straight’ orbit of the particle. The idea of GR is that a particle under the influence of gravity moves freely in a *curved* spacetime. A natural generalisation is that velocity and spin vector of the particle can be found by parallel transport along the orbit in spacetime. In this way we are able to understand the geodesic precession of a top. Fig. 2.5 shows a co-ordinate picture, with $x^0 = ct$ on the vertical axis and polar co-ordinates $x^1 = r$ and $x^3 = \varphi$ in the horizontal plane. The worldline of the top orbiting the central object (vertical bar) is a spiral. The spin 4-vector (whose spatial part is directed along the spin axis) is parallel-transported along the worldline. After one revolution the top has returned to same spatial position, but because spacetime is not flat – not visible in a co-ordinate picture – the spin vector has changed its direction. At this point one may wonder how the effect is related to the Thomas precession. We refer to Ch. 8 for a more general treatment, from which both Thomas precession and geodesic precession emerge in the appropriate limit.

Exercise 2.6: The length of a vector remains constant under parallel transport:

$$\mathrm{d}\, A^\nu A_\nu \;=\; \mathrm{d}\,(g_{\mu\nu} A^\mu A^\nu) = 0 \;.$$

Hint: First attempt: d = D = intrinsic change: $\mathrm{D}A^\nu A_\nu = (\mathrm{D}A^\nu)A_\nu + A^\nu(\mathrm{D}A_\nu) = 0$, because $\mathrm{D}A^\nu = (\mathrm{D}A^\nu/\mathrm{D}p)\,\mathrm{d}p = 0$, etc. But (2.27) must still be proven, and for that we need $\mathrm{d}\,A^\nu A_\nu = 0$. Second attempt: d = total change: $\mathrm{d}\,g_{\mu\nu}A^\mu A^\nu = 2A_\nu \mathrm{d}A^\nu + A^\mu A^\nu\, g_{\mu\nu,\sigma}\,\mathrm{d}x^\sigma$; (2.26): $\mathrm{d}A^\nu = -\Gamma^\nu_{\ \mu\sigma}\, A^\mu \mathrm{d}x^\sigma$; exercise 2.5 and (2.30): $2A_\nu \mathrm{d}A^\nu = -2\Gamma_{\nu\mu\sigma}\, A^\nu A^\mu \mathrm{d}x^\sigma = -g_{\nu\mu,\sigma}\, A^\nu A^\mu \mathrm{d}x^\sigma$.

Exercise 2.7: Prove that $\mathrm{d}\, A^\nu B_\nu \;=\; 0$ under parallel transport.

Hint: The length of $A^\nu + B^\nu$ is constant.

Exercise 2.8: For parallel transport of a covariant vector:

$$\mathrm{d}B_\mu \;=\; \Gamma^\nu_{\ \mu\sigma}\, B_\nu\, \mathrm{d}x^\sigma \;. \tag{2.32}$$

Hint: $0 \;=\; \mathrm{d}\, A^\mu B_\mu \;=\; A^\mu\, \mathrm{d}B_\mu \;+\; B_\mu\, \mathrm{d}A^\mu$, and $\mathrm{d}A^\mu$ is known.

Exercise 2.9: Prove that

$$\Gamma^\mu_{\ \nu\mu} \;=\; g_{,\nu}/2g \;=\; \tfrac{1}{2}\big(\log|g|\big)_{,\nu} \;\;; \qquad g \;=\; \det\{g_{\alpha\beta}\}\;. \tag{2.33}$$

Hint: (2.24): $\Gamma^\mu_{\ \nu\mu} = \frac{1}{2} g^{\lambda\mu} g_{\lambda\mu,\nu}$. For a matrix M we have that $\mathrm{Tr}\,(M^{-1}M_{,\nu}) = (\mathrm{Tr}\,\log M)_{,\nu} = (\log\det M)_{,\nu}$. Take $M = \{g_{\alpha\beta}\}$.

2.5 Geodesics

Intuitively, a geodesic is a line that is 'as straight as possible' on a curved surface. We say that a curve $x^\mu(p)$ is a geodesic when the tangent vector $\mathrm{d}x^\mu/\mathrm{d}p$ remains a tangent vector under parallel transport along $x^\mu(p)$. Therefore $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}p$ must satisfy (2.28), and we arrive at the geodesic equation:

$$\frac{\mathrm{D}}{\mathrm{D}p}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}p}\right) \;=\; 0 \quad\to\quad \ddot{x}^\mu \;+\; \Gamma^\mu_{\ \nu\sigma}\,\dot{x}^\nu\,\dot{x}^\sigma \;=\; 0\;, \tag{2.34}$$

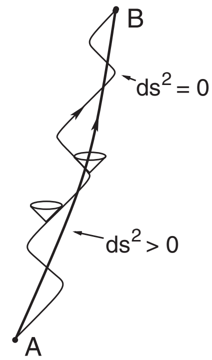

Fig. 2.6. A timelike geodesic connecting events A and B is the curve with the *maximum* possible interval length between A and B, see text.

with $\dot{} = \mathrm{d}/\mathrm{d}p$. For timelike geodesics[2] the parameter p in (2.34) is proportional to the interval length s. Proof: according to exercise 2.6 the length of $\dot{x}^\alpha = \mathrm{d}x^\alpha/\mathrm{d}p$ is constant along $x^\mu(p)$, i.e. $\dot{x}^\alpha \dot{x}_\alpha = g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta \equiv (\mathrm{d}s/\mathrm{d}p)^2$ is constant. For timelike geodesics $\mathrm{d}s^2 > 0$, and we may take the square root to conclude that $\mathrm{d}s = \mathrm{const}\cdot\mathrm{d}p$. Later, when GR is cast into the geometrical framework developed here, this result will be connected to proper time (a physical concept that does not yet exist here): $\mathrm{d}s = c\mathrm{d}\tau$, so that

$$\mathrm{d}p \;\propto\; \mathrm{d}s \;\propto\; \mathrm{d}\tau \qquad \text{for timelike geodesics.} \tag{2.35}$$

This is important as it implies that we may, for timelike geodesics, replace the curve parameter p in (2.34) by the interval length s or the proper time τ.

Intuitively, a geodesic is also the shortest possible route between two points. For a positive definite metric this is indeed the case, but $\mathrm{d}s^2$ can be positive as well as negative in GR. Assuming that the interval $\int\mathrm{d}s = \int\dot{s}\,\mathrm{d}p$ of a timelike geodesic is an extremum (see below), it is easy to see that it should be a maximum: there always exists an arbitrarily nearby worldline that has a smaller $\int\mathrm{d}s$, by letting it jump more or less from light-cone to light-cone, as in Fig. 2.6 (see e.g. Wald (1984) § 9.3). The construction of Fig. 2.6 fails for spacelike geodesics.

[2] In an analogy with (1.4) we speak of a timelike (spacelike) worldline or geodesic when $\mathrm{d}s^2 > 0$ ($\mathrm{d}s^2 < 0$). A null worldline or null geodesic has $\mathrm{d}s^2 = 0$. For spacelike and null geodesics p can no longer be interpreted as an interval length.

Eq. (2.34) may also be derived from a variational principle.[3] The simplest is $\delta\int \dot{s}\,\mathrm{d}p = 0$, and this is equivalent to $\delta\int F(\dot{s})\,\mathrm{d}p = 0$ provided F is monotonous, $F' \neq 0$. We choose $\delta\int \dot{s}^2\,\mathrm{d}p = 0$, or

$$\delta\int L\,\mathrm{d}p = 0\ ; \quad L(x^\alpha, \dot{x}^\beta) = (\mathrm{d}s/\mathrm{d}p)^2 = g_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta\ . \tag{2.36}$$

The solution is determined by the Euler-Lagrange equations (Appendix C)

$$\frac{\partial L}{\partial x^\lambda} = \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\partial L}{\partial \dot{x}^\lambda}\right)\ . \tag{2.37}$$

Now, $\partial L/\partial x^\lambda = g_{\alpha\beta,\lambda}\,\dot{x}^\alpha\dot{x}^\beta$ because only $g_{\alpha\beta}$ depends on $\{x^\mu\}$. By using $\partial\dot{x}^\alpha/\partial\dot{x}^\lambda = \delta^\alpha_\lambda$ one gets $\partial L/\partial\dot{x}^\lambda = 2g_{\alpha\lambda}\,\dot{x}^\alpha$. Substitute this in (2.37):

$$\begin{aligned} g_{\alpha\beta,\lambda}\,\dot{x}^\alpha\,\dot{x}^\beta &= 2(g_{\alpha\lambda}\,\dot{x}^\alpha)^{\cdot} \\ &= 2(g_{\alpha\lambda,\beta}\,\dot{x}^\beta\,\dot{x}^\alpha + g_{\alpha\lambda}\,\ddot{x}^\alpha)\ , \end{aligned}$$

or

$$g_{\alpha\lambda}\,\ddot{x}^\alpha + \tfrac{1}{2}(2g_{\lambda\alpha,\beta} - g_{\alpha\beta,\lambda})\,\dot{x}^\alpha\,\dot{x}^\beta = 0\ . \tag{2.38}$$

Now comes a frequently used trick: renaming of dummy indices: $2g_{\lambda\alpha,\beta}\cdot\dot{x}^\alpha\,\dot{x}^\beta = g_{\lambda\alpha,\beta}\,\dot{x}^\alpha\,\dot{x}^\beta + g_{\lambda\beta,\alpha}\,\dot{x}^\beta\,\dot{x}^\alpha = (g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha})\,\dot{x}^\alpha\,\dot{x}^\beta$. Substitution in (2.38) and multiplication with $g^{\mu\lambda}$ gives:

$$\ddot{x}^\mu + \tfrac{1}{2}g^{\mu\lambda}\,(g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda})\,\dot{x}^\alpha\,\dot{x}^\beta = 0\ . \tag{2.39}$$

This is of the form of (2.34) and the factor multiplying $\dot{x}^\alpha\,\dot{x}^\beta$ must be equal to $\Gamma^\mu_{\ \alpha\beta}$, which proves (2.24). Variational calculus is a very efficient tool for this type of problem. Without much difficulty, it permits us to find the geodesic equation directly from the metric, and from this equation one may just read the Christoffel symbols $\Gamma^\mu_{\ \nu\sigma}$. This is usually a lot faster than calculating them from (2.24), and this method is therefore highly recommended.

The following result is very helpful when analysing the dynamics of a test particle in GR (assuming that its orbit is a geodesic), because it allows us to find *constants of the motion*. From the text below (2.37) we see that $\partial L/\partial x^\lambda$ vanishes if $g_{\alpha\beta,\lambda} = 0$. And then eq. (2.37) says that $\partial L/\partial\dot{x}^\lambda = 2g_{\alpha\lambda}\dot{x}^\alpha$ is constant. In terms of the *4-velocity* $u^\mu = \mathrm{d}x^\mu/\mathrm{d}p$ we have found that the covariant 4-velocity $u_\lambda = g_{\lambda\alpha}u^\alpha$ is constant:

$$g_{\alpha\beta,\lambda} = 0 \quad \rightarrow \quad u_\lambda = g_{\lambda\nu}\,\dot{x}^\nu = \text{constant} \tag{2.40}$$

with $^{\cdot} = \mathrm{d}/\mathrm{d}p$. The fact that u_λ is a constant along a geodesic if the metric is independent of x^λ – doesn't that ring a bell?

[3] Here we switch to another definition of geodesics without proving its equivalence with (2.34).

Exercise 2.10: Show that the geodesics of the Lorentz metric ($g_{\alpha\beta} = \eta_{\alpha\beta}$) are straight lines.

Exercise 2.11: Show that the variational problem (2.36) is equivalent to $\delta \int F(L)\, \mathrm{d}p = 0$ if F is monotonous, $F' \neq 0$.

Hint: Write down (2.37) with $L \to F(L)$; use $\partial F(L)/\partial x^\lambda = F'\, \partial L/\partial x^\lambda$, and $(F'\, \partial L/\partial \dot{x}^\lambda)^{\cdot} = (F')^{\cdot}\, \partial L/\partial \dot{x}^\lambda + F'(\partial L/\partial \dot{x}^\lambda)^{\cdot}$. But $(F')^{\cdot} = F'' \mathrm{d}L/\mathrm{d}p = 0$ (L is constant on $x^\mu(p)$ because $\dot{x}^\alpha \dot{x}_\alpha$ is).

2.6 The covariant derivative

For a given vector field A^μ that is not restricted to the curve $x^\sigma(p)$ we can elaborate $\mathrm{d}A^\mu/\mathrm{d}p$ in (2.26) as $\mathrm{d}A^\mu/\mathrm{d}p = A^\mu{}_{,\sigma}\, \dot{x}^\sigma$, because we are able to compute derivatives in other directions than along the curve. This leads to the introduction of the covariant derivative

$$\frac{\mathrm{D}A^\mu}{\mathrm{D}p} = \left(A^\mu{}_{,\sigma} + \Gamma^\mu_{\nu\sigma}\, A^\nu\right) \dot{x}^\sigma \equiv A^\mu{}_{:\sigma}\, u^\sigma \ , \tag{2.41}$$

where $u^\sigma = \dot{x}^\sigma = \mathrm{d}x^\sigma/\mathrm{d}p$ and

$$A^\mu{}_{:\sigma} \equiv A^\mu{}_{,\sigma} + \Gamma^\mu_{\nu\sigma}\, A^\nu \tag{2.42}$$

is the *covariant derivative* of A^μ. It may be regarded as the 'intrinsic derivative', the derivative after correction for the meaningless change in orientation of the base vectors. In a similar way we may obtain the covariant derivative of a covariant vector from (2.27):

$$A_{\mu:\sigma} = A_{\mu,\sigma} - \Gamma^\nu_{\mu\sigma}\, A_\nu \ . \tag{2.43}$$

Important is that both $A^\mu{}_{:\sigma}$ and $A_{\mu:\sigma}$ are tensors if A^μ is a vector, even though neither of the two terms on the right hand sides of (2.42) and (2.43) are tensors themselves. The proof is a matter of combining relations (2.20) and (2.31), and is left to the reader.

Next follow a few definitions. The covariant derivative of a product XY of two tensors is:

$$(XY)_{:\sigma} = X_{:\sigma}\, Y + X\, Y_{:\sigma} \ . \tag{2.44}$$

For example:

$$
\begin{aligned}
(A_\mu B_\nu)_{:\sigma} &= (A_{\mu,\sigma} - \Gamma^\alpha_{\ \mu\sigma} A_\alpha) B_\nu + A_\mu (B_{\nu,\sigma} - \Gamma^\alpha_{\ \nu\sigma} B_\alpha) \\
&= (A_\mu B_\nu)_{,\sigma} - \Gamma^\alpha_{\ \mu\sigma} A_\alpha B_\nu - \Gamma^\alpha_{\ \nu\sigma} A_\mu B_\alpha \ . \qquad (2.45)
\end{aligned}
$$

Accordingly, we define the covariant derivative of a covariant second rank tensor as:

$$
T_{\mu\nu:\sigma} = T_{\mu\nu,\sigma} - \Gamma^\alpha_{\ \mu\sigma} T_{\alpha\nu} - \Gamma^\alpha_{\ \nu\sigma} T_{\mu\alpha} \ . \qquad (2.46)
$$

The recipe for tensors of higher rank should be clear by now. For example, if we need an expression for $T_\alpha{}^\beta{}_{\gamma:\sigma}$, then we merely have to work out $(P_\alpha Q^\beta R_\gamma)_{:\sigma}$ as in (2.44) and (2.45). The general pattern is $T^*_{\ *:\sigma} = T^*_{\ *,\sigma} \pm \Gamma$-term for every index. For a scalar:

$$
Q_{:\sigma} = Q_{,\sigma} \ . \qquad (2.47)
$$

Covariant derivatives do not commute, unlike normal derivatives ($X_{,\alpha\beta} = X_{,\beta\alpha}$ for every X). We calculate $B_{\mu:\nu:\sigma}$ by substituting $T_{\mu\nu} = B_{\mu:\nu}$ in (2.46):

$$
B_{\mu:\nu:\sigma} = B_{\mu:\nu,\sigma} - \Gamma^\alpha_{\ \mu\sigma} B_{\alpha:\nu} - \Gamma^\alpha_{\ \nu\sigma} B_{\mu:\alpha} \ , \qquad (2.48)
$$

which should be elaborated further with (2.43). After that, interchange ν and σ and subtract. The result of a somewhat lengthy calculation is:

$$
B_{\mu:\nu:\sigma} - B_{\mu:\sigma:\nu} = B_\alpha R^\alpha_{\ \mu\nu\sigma} \qquad (2.49)
$$

with

$$
R^\alpha_{\ \mu\nu\sigma} = \Gamma^\alpha_{\ \mu\sigma,\nu} - \Gamma^\alpha_{\ \mu\nu,\sigma} + \Gamma^\tau_{\ \mu\sigma} \Gamma^\alpha_{\ \tau\nu} - \Gamma^\tau_{\ \mu\nu} \Gamma^\alpha_{\ \tau\sigma} \ . \qquad (2.50)
$$

$R^\alpha_{\ \mu\nu\sigma}$ is called the RIEMANN tensor. It is a tensor because (2.49) is valid for every vector B_α and because the left hand side is a tensor. Then apply the quotient theorem. Apparently, covariant derivatives commute only if $R^\alpha_{\ \mu\nu\sigma} = 0$. The Riemann tensor plays a crucial role in GR because it contains all information about the curvature of space. Note the remarkable fact that according to (2.49) the difference of two consecutive covariant differentiations is proportional to the vector itself. The explanation is given in the next section.

Exercise 2.12: Show that

$$
T^{\mu\nu}{}_{:\sigma} = T^{\mu\nu}{}_{,\sigma} + \Gamma^\mu_{\ \alpha\sigma} T^{\alpha\nu} + \Gamma^\nu_{\ \alpha\sigma} T^{\mu\alpha} \ . \qquad (2.51)
$$

Great care is needed in using these relations. For example, let $T^{\mu\nu}$ be diagonal. Then it seems evident that $T^{1\mu}{}_{:\mu} = T^{11}{}_{:1}$, but that is not the case. Why not?

Hint: Write out $(A^\mu B^\nu)_{:\sigma}$ as in (2.45). It is due to the action of the invisible dummy index α.

Exercise 2.13: An important property is that the metric tensor behaves as a constant under covariant differentiation:

$$g_{\mu\nu:\sigma} = 0 \ . \tag{2.52}$$

Hint: Use (2.46) and (2.30).

Exercise 2.14: Prove the following compact form of the geodesic equation:

$$u^\sigma u_{\mu:\sigma} = 0 \qquad \text{or} \qquad u^\sigma u^\mu{}_{:\sigma} = 0 \ . \tag{2.53}$$

Hint: The last relation is just $0 = \mathrm{D}u^\mu/\mathrm{D}p = (2.41)$; the first relation with (2.52): $0 = g_{\lambda\mu}u^\mu{}_{:\sigma}u^\sigma = (g_{\lambda\mu}u^\mu)_{:\sigma}u^\sigma =$ etc.

Exercise 2.15: A reminder of the linear algebra aspects of tensor calculus. Given a 2D Riemann space with co-ordinates x, y, a metric and two vectors in the tangent space of the point (x, y):

$$\mathrm{d}s^2 = \mathrm{d}x^2 + 4\mathrm{d}x\mathrm{d}y + \mathrm{d}y^2 \ ; \quad A^\alpha = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \ ; \quad B_\alpha = \begin{pmatrix} y \\ x \end{pmatrix} \ .$$

Write down $g_{\mu\nu}$ and $g^{\mu\nu}$ and show that all Christoffel symbols are zero. Compute A_ν and $B^\nu{}_{:\nu}$.

Hint: $g_{11} = g_{22} = 1$; $g_{12} = g_{21} = 2$, use (2.24) for the Christoffel symbols;

$$g^{\mu\nu} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \ ; \quad A_\mu = \begin{pmatrix} 9 \\ 6 \end{pmatrix} \ ; \quad B^\nu{}_{:\nu} = \frac{4}{3} \ .$$

The Γ's being zero we have $B^\nu{}_{:\nu} = B^\nu{}_{,\nu}$.

2.7 Riemann tensor and curvature

The metric tensor does not tell us whether a space is flat, because the use of ‘strange’ co-ordinates is not prohibited. For example $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2$

(planar polar co-ordinates) defines a flat space, but (2.2) defines a curved space. The metric tensor contains apparently a mix of information on co-ordinates and curvature. The intrinsic curvature properties are determined by the Riemann tensor. We shall illustrate this by transporting a vector A^μ parallel to itself along two different paths to the same final position, see Fig. 2.7. According to (2.26), $\mathrm{d}A^\mu = -f_\sigma(x)\mathrm{d}x^\sigma$ with $f_\sigma(x) = \Gamma^\mu{}_{\nu\sigma}A^\nu$ (the upper index μ is omitted for brevity as it does not change). The difference of the two final vectors is:

$$\begin{aligned}
\mathrm{d}A^\mu &= A_1^\mu - A_2^\mu \\
&= -f_\sigma(x)\mathrm{d}\xi^\sigma - f_\sigma(x+\mathrm{d}\xi)\mathrm{d}\eta^\sigma + f_\sigma(x)\mathrm{d}\eta^\sigma + f_\sigma(x+\mathrm{d}\eta)\mathrm{d}\xi^\sigma \\
&\simeq -f_\sigma\mathrm{d}\xi^\sigma - f_\sigma\mathrm{d}\eta^\sigma - f_{\sigma,\lambda}\,\mathrm{d}\xi^\lambda\mathrm{d}\eta^\sigma + f_\sigma\mathrm{d}\eta^\sigma + f_\sigma\mathrm{d}\xi^\sigma + f_{\sigma,\lambda}\,\mathrm{d}\eta^\lambda\mathrm{d}\xi^\sigma \\
&= (f_{\sigma,\lambda} - f_{\lambda,\sigma})\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda\ .
\end{aligned} \tag{2.54}$$

Now substitute $f_\sigma = \Gamma^\mu{}_{\nu\sigma}A^\nu = A^\mu{}_{:\sigma} - A^\mu{}_{,\sigma}$. The terms $A^\mu{}_{,\sigma}$ cancel, and after some index gymnastics we arrive at (exercise 2.16):

$$\begin{aligned}
\mathrm{d}A^\mu &= (A^\mu{}_{:\sigma,\lambda} - A^\mu{}_{:\lambda,\sigma})\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda \\
&= g^{\mu\nu}(A_{\nu:\sigma,\lambda} - A_{\nu:\lambda,\sigma})\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda \\
&= g^{\mu\nu}(A_{\nu:\sigma:\lambda} - A_{\nu:\lambda:\sigma})\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda \\
&= g^{\mu\nu}R_{\alpha\nu\sigma\lambda}\,A^\alpha\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda \\
&= g^{\mu\nu}R_{\nu\alpha\lambda\sigma}\,A^\alpha\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda \\
&= R^\mu{}_{\alpha\lambda\sigma}\,A^\alpha\,\mathrm{d}\xi^\sigma\mathrm{d}\eta^\lambda\ .
\end{aligned} \tag{2.55}$$

On account of (2.24) the Christoffel symbols vanish identically in a flat space with rectangular co-ordinates, since $g_{\mu\nu}$ has only constant elements. Therefore the Riemann tensor (2.50) is zero as well. The transformation properties of a tensor then ensure that $R^\alpha{}_{\mu\nu\sigma}$ is zero in a flat space for any choice of the co-ordinates.[4] In that case parallel transport along a closed path leaves a vector unchanged.[5] But in a curved space the orientation of the vector will have

[4] Contrary to the Christoffel symbols, which are not tensors. For example, the Christoffel symbols vanish in rectangular co-ordinates in a plane, but *not* in polar co-ordinates.

[5] Conversely, if the Riemann tensor is zero, it can be proven that there exist co-ordinates so that $g_{\mu\nu}$ is constant which implies that the space is flat, see e.g. Dirac (1975) § 12.

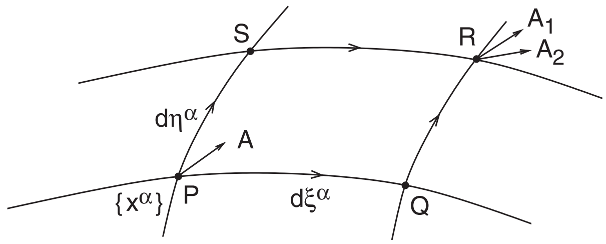

Fig. 2.7. Parallel transport of the vector $\boldsymbol{A}$ from P to R along path 1 (PQR) and path 2 (PSR) produces a different result.

changed. Once this is accepted intuitively, it is clear that the difference $\mathrm{d}A^{\mu}$ must be proportional to the length of the vector, which explains the factor A^{α} in (2.55). The derivation in (2.55) shows that the difference $\mathrm{d}A^{\mu}$ is also proportional to the difference of two consecutive covariant differentiations, and this explains why this difference is proportional to the vector itself, as in (2.49).

There are several other ways to illustrate the relation between the Riemann tensor and curvature. One is the equation for the *geodesic deviation*, see exercise 2.18. Another is the relation between *Gaussian curvature* and the Riemann tensor. Gaussian curvature refers to surfaces embedded in a flat 3D space. The curvature κ in a point P of a curve on the surface is defined as the inverse radius of the osculating circle at P. Each point has two principal curvatures κ_1 and κ_2, and the Gaussian curvature $K \equiv \kappa_1\kappa_2$ is an invariant determined by the geometry of the surface, which has several interesting properties.[6] Turning now to Riemann spaces, take two orthogonal unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the tangent space of a point P which are not null. Now consider those geodesics in Riemann space that are tangent in P to the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. These geodesics subtend, locally around P, a 2D curved subspace of Riemann space. The Gaussian curvature of this 2D space at P is $R_{\alpha\mu\nu\sigma}e_1^{\alpha}e_2^{\mu}e_1^{\nu}e_2^{\sigma}$, apart from the sign.[7]

The Riemann tensor obeys several symmetry relations that reduce the number of independent components from n^4 to $n^2(n^2-1)/12$ (see literature). In 4 dimensions $R^{\alpha}{}_{\nu\rho\sigma}$ has only 20 independent components, and all contractions of $R^{\alpha}{}_{\nu\rho\sigma}$ are either zero or equal, apart from the sign. We choose

[6] E.g. Gauss's theorem on integral curvature: the sum of the three interior angles of a geodesic triangle (bounded by 3 geodesics) equals π plus the surface integral of K.

[7] For a proof of these statements see e.g. Robertson and Noonan (1969) p. 216.

the *Ricci tensor*:[8]

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\nu\alpha} \qquad \text{(RICCI)}. \tag{2.56}$$

The explicit expression follows from (2.50):

$$R_{\mu\nu} \;=\; \Gamma^{\alpha}_{\;\mu\alpha,\nu} \;-\; \Gamma^{\alpha}_{\;\mu\nu,\alpha} \;-\; \Gamma^{\alpha}_{\;\mu\nu}\,\Gamma^{\beta}_{\;\alpha\beta} \;+\; \Gamma^{\alpha}_{\;\mu\beta}\,\Gamma^{\beta}_{\;\nu\alpha}\;. \tag{2.57}$$

We infer from (2.33) that $\Gamma^{\alpha}_{\;\mu\alpha,\nu} = \frac{1}{2}\big(\log|g|\,\big)_{,\mu\nu}$ so that all terms in (2.57) are symmetric in μ and in ν. Hence $R_{\mu\nu}$ is symmetric:

$$R_{\mu\nu} = R_{\nu\mu}\;. \tag{2.58}$$

We may contract once more:

$$R \;\equiv\; R^{\nu}{}_{\nu} = g^{\nu\mu}R_{\mu\nu} = R^{\alpha\beta}{}_{\beta\alpha}\;. \tag{2.59}$$

R is called the *total curvature*. Finally we introduce the *Einstein tensor* $G_{\mu\nu}$:

$$G_{\mu\nu} \;=\; R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \qquad \text{(EINSTEIN)}. \tag{2.60}$$

The Einstein tensor will be useful later because its divergence is zero:

$$G^{\mu\nu}{}_{:\nu} \;=\; (R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)_{:\nu} \;=\; 0\;. \tag{2.61}$$

Riemann, Ricci en Einstein tensor contain at most second derivatives of $g_{\alpha\beta}$. By substituting (2.24) in (2.50) we get:

$$\begin{aligned} R^{\alpha}{}_{\mu\nu\sigma} \;=\; & \tfrac{1}{2}g^{\alpha\beta}\left(g_{\beta\sigma,\mu\nu} \;-\; g_{\mu\sigma,\beta\nu} \;-\; g_{\beta\nu,\mu\sigma} \;+\; g_{\mu\nu,\beta\sigma}\right) \\ & + \; g^{\alpha\beta}\left(\Gamma_{\tau\beta\sigma}\,\Gamma^{\tau}_{\;\mu\nu} \;-\; \Gamma_{\tau\beta\nu}\,\Gamma^{\tau}_{\;\mu\sigma}\right)\;. \end{aligned} \tag{2.62}$$

The corresponding expressions for $R_{\mu\nu}$ and for $G_{\mu\nu}$ can be found from this by contraction. The first term contains all second-order derivatives. The first-order derivatives are in the second term. The proofs of (2.61) and (2.62) can be found in the literature, but are not important here.

Exercise 2.16: Provide the missing details of the derivation of (2.55).

Hint: Second = sign: $A^{\mu}{}_{:\sigma,\lambda} \;=\; (g^{\mu\nu}A_{\nu})_{:\sigma,\lambda} \;=\; (g^{\mu\nu}A_{\nu:\sigma})_{,\lambda} \;=\; g^{\mu\nu}{}_{,\lambda}\,A_{\nu:\sigma} + g^{\mu\nu}A_{\nu:\sigma,\lambda}$, but A^{μ} is parallel transported, hence $A_{\nu:\sigma} = 0$, etc. Third = sign: $A_{\nu:\sigma:\lambda} \;=\; A_{\nu:\sigma,\lambda}$ from (2.48). Fifth = sign: $R_{\alpha\nu\sigma\lambda} \;=\; R_{\nu\alpha\lambda\sigma}$ is a symmetry relation of the Riemann tensor.

[8] Other authors define $R_{\mu\nu} \;=\; R^{\alpha}{}_{\mu\alpha\nu}$, another source of sign differences. For a complete classification of all sign conventions see the red pages in Misner et al. (1971). In terms of this classification we follow the $-\,+\,-$ convention.

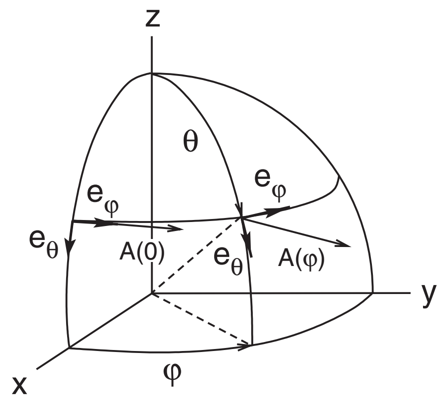

Fig. 2.8. Parallel transport of a vector $\boldsymbol{A}$ over the surface of a sphere with radius $r = 1$, see exercise 2.17.

Exercise 2.17: Consider a 2D spherical surface with radius $r = 1$, see Fig. 2.8. Calculate the Christoffel symbols and the total curvature R. Convince yourself that $R \propto r^{-2}$. Show that a vector $\boldsymbol{A}$ will rotate in the tangent space as it is parallel-transported along a circle $\theta = \theta_0$. Try to understand this with the intuitive definition of parallel transport in § 2.4. Start in $(\theta, \varphi) = (\theta_0, 0)$ with $(A^\theta, A^\varphi) = (0, 1/\sin\theta_0)$. Show that $A^i A_i$ is always 1, i.e. $|\boldsymbol{A}| \equiv 1$, and that after one full revolution $\boldsymbol{A}$ has rotated over an angle $2\pi\cos\theta_0$. Discuss the limiting cases $\theta_0 = \pi/2$ (geodesic!) and $\theta_0 \ll 1$.

Hint: (2.2): $g_{11} = 1$, $g_{22} = \sin^2\theta$ $(\theta = 1,\ \varphi = 2)$. Do *not* use (2.24), but rather (2.37) with $L(\theta, \dot\theta, \dot\varphi) = \dot\theta^2 + \sin^2\theta\,\dot\varphi^2$:

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= \Big(\frac{\partial L}{\partial \dot\theta}\Big)^{\cdot} \ \rightarrow\ \ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = 0\ ;\\
\frac{\partial L}{\partial \varphi} &= \Big(\frac{\partial L}{\partial \dot\varphi}\Big)^{\cdot} \ \rightarrow\ \ddot\varphi + 2\cot\theta\,\dot\theta\dot\varphi = 0\ .
\end{aligned}$$

By comparing with (2.34) we may just read the Γ's: $\Gamma^1_{\ 22} = -\sin\theta\cos\theta\,$; $\Gamma^2_{\ 12} = \cot\theta$ (double product!). All other Γ's are zero. (2.33) $\rightarrow$ $\Gamma^\alpha_{\ \mu\alpha,\nu} = (\log\sin\theta)_{,\mu\nu}$ $\rightarrow$ $\Gamma^\alpha_{\ 1\alpha,1} = -1/\sin^2\theta$. And $\Gamma^\alpha_{\ 11,\alpha} = 0\,$; $\Gamma^\alpha_{\ 22,\alpha} = -(\sin\theta\cdot\cos\theta)_{,\theta} = \sin^2\theta - \cos^2\theta$. Algebra: $R_{11} = -1$ and $R_{22} = -\sin^2\theta$. Finally $R = g^{\mu\nu}R_{\mu\nu} = g^{11}R_{11} + g^{22}R_{22} = R_{11} + (1/\sin^2\theta)\,R_{22} = -2$. For a sphere with radius r: $R = -2/r^2$ (minus sign due to sign convention).
Parallel transport: p is proportional to the arc length (why?), so choose $p = \varphi$; (2.28)+(2.26): $A^\mu_{\ ,\varphi} + \Gamma^\mu_{\ \nu\sigma}A^\nu x^\sigma_{\ ,\varphi} = 0$ with $x^1_{\ ,\varphi} = \mathrm{d}\theta/\mathrm{d}\varphi = 0$ and $x^2_{\ ,\varphi} = \mathrm{d}\varphi/\mathrm{d}\varphi = 1$:

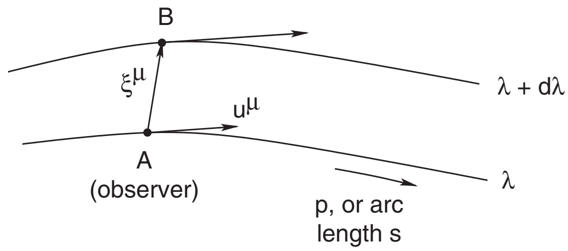

Fig. 2.9. The geodesic deviation.

$$A^{\theta}_{,\varphi} \;=\; \sin\theta_0\cos\theta_0\,A^{\varphi}\ ; \qquad A^{\varphi}_{,\varphi} \;=\; -\cot\theta_0\,A^{\theta}\ .$$

Eliminate A^{φ}: $A^{\theta}_{,\varphi\varphi} + \cos^2\theta_0\,A^{\theta} = 0$, same equation holds for A^{φ}. Harmonic oscillator with frequency $\cos\theta_0$. Solution for given initial value:

$$A^{\theta} \;=\; \sin(\varphi\cos\theta_0)\ ; \qquad A^{\varphi} \;=\; \cos(\varphi\cos\theta_0)/\sin\theta_0\ .$$

$\boldsymbol{A}$ rotates clockwise when looking down on the tangent space from outside; $\theta_0 = \pi/2$: $A^{\theta} \equiv 0$ and $A^{\varphi} \equiv 1/\sin\theta_0 = 1$, therefore $\boldsymbol{A}$ remains a tangent vector; $\theta_0 \ll 1$ (small circle around the north pole): in that case the tangent space is always almost parallel to the equatorial plane, with base vectors $\boldsymbol{x}$ en $\boldsymbol{y}$, and $\boldsymbol{e}_\theta \simeq \boldsymbol{x}\cos\varphi + \boldsymbol{y}\sin\varphi$ and $\boldsymbol{e}_\varphi \simeq (\boldsymbol{y}\cos\varphi - \boldsymbol{x}\sin\varphi)\sin\theta_0$. For $\theta_0 \ll 1$ it follows that $\boldsymbol{A} = A^{\theta}\boldsymbol{e}_\theta + A^{\varphi}\boldsymbol{e}_\varphi \simeq \boldsymbol{y}$, so that $\boldsymbol{A}$ remains virtually unchanged with respect to a fixed frame.

Exercise 2.18: Given a set of geodesics $x^{\mu}(p,\lambda)$ where p is the curve parameter and λ labels different geodesics (λ is constant along one geodesic). Consider two neighbouring geodesics λ and $\lambda + \delta\lambda$. The points A and B are connected by the vector $\xi^{\mu} = x^{\mu}(p,\lambda+\delta\lambda) - x^{\mu}(p,\lambda) \simeq (\partial x^{\mu}/\partial\lambda)\delta\lambda \equiv e^{\mu}\delta\lambda$. Prove that:

$$\frac{\mathrm{D}^2\xi^{\mu}}{\mathrm{D}p^2} \;=\; R^{\mu}{}_{\alpha\beta\nu}\,u^{\alpha}u^{\beta}\xi^{\nu}\ ; \qquad u^{\alpha} = \dot{x}^{\alpha} = \frac{\partial x^{\alpha}}{\partial p}\ . \tag{2.63}$$

This is the equation for the geodesic deviation, that will play an important role later. In a flat space the Riemann tensor is zero, and then ξ^{μ} is a linear function of p, and for timelike geodesics also a linear function of the arc length s, as expected. In a curved space however this is no longer the case. For example, on a sphere $\xi^{\mu}(s)$ will be something like a sine-function.

Hint: The proof comes in three steps:

$$(a) \qquad \frac{\partial e^\mu}{\partial p} = \frac{\partial^2 x^\mu}{\partial p\,\partial\lambda} = \frac{\partial u^\mu}{\partial\lambda} = u^\mu{}_{,\alpha}\,\frac{\partial x^\alpha}{\partial\lambda} = u^\mu{}_{,\alpha}\,e^\alpha\ ;$$

$$\begin{aligned}
(b) \quad e^\mu{}_{:\alpha}\,u^\alpha &\equiv \frac{\mathrm{D}e^\mu}{\mathrm{D}p} = \frac{\partial e^\mu}{\partial p} + \Gamma^\mu_{\alpha\beta}\,e^\alpha u^\beta \\
&= u^\mu{}_{,\alpha}\,e^\alpha + \Gamma^\mu_{\alpha\beta}\,e^\alpha u^\beta = u^\mu{}_{:\alpha}\,e^\alpha\ ;
\end{aligned}$$

$$\begin{aligned}
(c) \quad \frac{\mathrm{D}^2 e^\mu}{\mathrm{D}p^2} &\equiv (e^\mu{}_{:\alpha}\,u^\alpha)_{:\beta}\,u^\beta = (u^\mu{}_{:\alpha}\,e^\alpha)_{:\beta}\,u^\beta \\
&= u^\mu{}_{:\alpha}\,e^\alpha{}_{:\beta}\,u^\beta + u^\mu{}_{:\alpha:\beta}\,e^\alpha u^\beta \\
&= u^\mu{}_{:\alpha}\,u^\alpha{}_{:\beta}\,e^\beta + u^\mu{}_{:\alpha:\beta}\,u^\alpha e^\beta + \big(u^\mu{}_{:\alpha:\beta} - u^\mu{}_{:\beta:\alpha}\big)\,u^\beta e^\alpha \\
&= \big(u^\mu{}_{:\alpha}\,u^\alpha\big)_{:\beta}\,e^\beta + g^{\mu\nu}\big(u_{\nu:\alpha:\beta} - u_{\nu:\beta:\alpha}\big)\,u^\beta e^\alpha \\
&= g^{\mu\nu}\,u_\sigma\,R^\sigma{}_{\nu\alpha\beta}\,u^\beta e^\alpha \\
&= R_\sigma{}^\mu{}_{\alpha\beta}\,u^\sigma u^\beta e^\alpha \\
&= R^\mu{}_{\sigma\beta\alpha}\,u^\sigma u^\beta e^\alpha\ .
\end{aligned}$$

In (c) we have twice used (b), next $u^\mu{}_{:\beta:\alpha}\,e^\alpha u^\beta = u^\mu{}_{:\alpha:\beta}\,e^\beta u^\alpha$ is added and substracted again, and then (2.53) and (2.49). The last = sign is a symmetry relation of the Riemann tensor. Because $\delta\lambda$ is constant, the equation also holds for $\xi^\mu = e^\mu\delta\lambda$.

Exercise 2.19: Be aware of some inconsistencies in the notation. We encountered one in exercise 2.12. Meet two more here. In § 2.2 and § 2.3 it was stressed that the rules for index raising and lowering are always valid. Does that mean that

$$g^{\mu\alpha}g_{\alpha\lambda,\nu} \stackrel{?}{=} g^\mu{}_{\lambda,\nu}\ ; \tag{2.64}$$

$$g_{\mu\alpha}\dot{u}^\alpha \stackrel{?}{=} \dot{u}_\mu\ . \tag{2.65}$$

Hint: In exercise 2.12 the trouble was caused by a hidden index; here we discover that the symbols without derivative had already been defined; one way to see that (2.64) cannot be correct is to note that $g^\mu{}_{\lambda,\nu} \equiv \delta^\mu_{\lambda,\nu} = 0$, and since $\det\{g^{\mu\alpha}\} \neq 0 \rightarrow g_{\alpha\lambda,\nu} = 0 \ \rightarrow g_{\alpha\lambda} = \text{const}$. Instead, $0 = (g^{\mu\alpha}g_{\alpha\lambda})_{,\nu} = g^{\mu\alpha}{}_{,\nu}\,g_{\alpha\lambda} + g^{\mu\alpha}g_{\alpha\lambda,\nu}$, etc. Likewise, u_μ is *defined* as $g_{\mu\alpha}u^\alpha$ so that $\dot{u}_\mu = (g_{\mu\alpha}u^\alpha)^{\cdot} = g_{\mu\alpha,\sigma}\,u^\sigma u^\alpha + g_{\mu\alpha}\dot{u}^\alpha$. Also correct is $\dot{u}_\mu = u_{\mu,\alpha}\dot{x}^\alpha = u_{\mu,\alpha}u^\alpha$.