Regla de la cadena

(Derivada de la compuesta de funciones de varias variables)

Sea $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ una función vectorial, tal que $f(x_1, \cdots, x_n) = \big(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\big) = (y_1, \cdots, y_m)$ y $g: V \subseteq \mathbb{R}^m \to \mathbb{R}$ una función escalar tal que $g(y_1, \cdots, y_m) = z$, si todas sus derivadas parciales existen y son continuas en sus dominios. Entonces la función compuesta, $g \circ f$ admite derivadas parciales continuas en $a \in U$ y se tiene además

$$\frac{\boldsymbol{\partial(g \circ f)}}{\boldsymbol{\partial x_j}}(\boldsymbol{a}) = \sum_{\boldsymbol{i=1}}^{\boldsymbol{m}} \left(\boldsymbol{g_{y_i}(b)}\right)(\boldsymbol{f_i})_{\boldsymbol{x_j}}(\boldsymbol{a})$$

donde $f(a) = b$.

Observación: $g \circ f: U \subseteq \mathbb{R}^n \to \mathbb{R}$, donde

1) $(g \circ f)(x_1, \cdots, x_n) = g\big(f(x_1, \cdots, x_n)\big)$

$$= g\big(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)\big)$$

$$= g(y_1, \cdots, y_m) = z \in \mathbb{R}$$

2) $\dfrac{\partial(g \circ f)}{\partial x_j}(a) = \displaystyle\sum_{i=1}^{m} \left(g_{y_i}(b)\right)(f_i)_{x_j}(a)$

$$= g_{y_1}(b)(f_1)_{x_j}(a) + g_{y_2}(b)(f_2)_{x_j}(a) + \cdots + g_{y_m}(b)(f_m)_{x_j}(a)$$

$$= \frac{\partial g(b)}{\partial y_1}\frac{\partial f_1(a)}{\partial x_j} + \frac{\partial g(b)}{\partial y_2}\frac{\partial f_2(a)}{\partial x_j} + \cdots + \frac{\partial g(b)}{\partial y_m}\frac{\partial f_m(a)}{\partial x_j}$$

Ejemplo Sean $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x - 1, x^2)$ y $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = 2x + 5y$. Se pide determinar $(g \circ f)_x(-2)$.

Solución

$$f(x) = (x-1, x^2)$$
$$g(x,y) = 2x + 5y$$

Método 1:

$$(g\circ f)(x) = g\big(f(x)\big) = g(x-1, x^2) = 2(x-1) + 5x^2$$
$$= 2x - 2 + 5x^2 = 5x^2 + 2x - 2$$
$$\Rightarrow (g\circ f)_x(x) = 10x + 2$$

Luego

$$(g\circ f)_x(-2) = -20 + 2 = -18$$

Método 2:

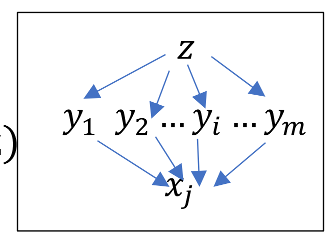
Dado que $f(x) = (x-1, x^2)$ sean $f_1(x) = x - 1$ y $f_2(x) = x^2$, y $g(x,y) = 2x + 5y$ luego

$$(g\circ f)_x(x) = \frac{\partial (g\circ f)}{\partial x}(x) = \frac{\partial g}{\partial x}(x,y)\frac{\partial f_1}{\partial x}(x) + \frac{\partial g}{\partial y}(x,y)\frac{\partial f_2}{\partial x}(x)$$
$$= (2)(1) + (5)(2x)$$
$$= 2 + 10x$$

Por tanto, $(g\circ f)_x(-2) = 2 + 10(-2) = -18$

3) Sean $z = g(y_1, \cdots, y_m) \in \mathbb{R}$ e $y_i = f_i(x_1, \cdots, x_n)$, con $i = 1, \cdots, m$ entonces

$$\frac{\partial z}{\partial x_j} = \sum_{i=1}^{m} \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} \text{(fórmula de Leibniz)}$$



Es decir

$$\frac{\partial z}{\partial x_j} = \frac{\partial z}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial z}{\partial y_2}\frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial z}{\partial y_m}\frac{\partial y_m}{\partial x_j}$$

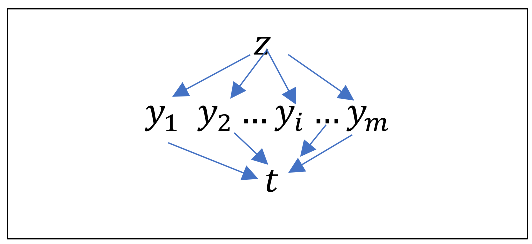
4.- Si $f: U \subseteq \mathbb{R} \to \mathbb{R}^m$ tiene funciones coordenadas $y_i = f_i(t)$ escalares de $\mathbb{R}$ en $\mathbb{R}$, diremos que

$$z = (g \circ f)(t) = g\big(f(t)\big) = g\big(f_1(t), \cdots, f_m(t)\big) = g(y_1, \cdots, y_m)$$

donde $g: V \subseteq \mathbb{R}^m \to \mathbb{R}$, es una función de una variable y con valores reales

$$\frac{dz}{dt} = \sum_{i=1}^{m} \frac{\partial z}{\partial y_i}\frac{dy_i}{dt}$$

$$= \frac{\partial z}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial z}{\partial y_2}\frac{dy_2}{dt} + \cdots + \frac{\partial z}{\partial y_m}\frac{dy_m}{dt}$$



Ejemplo 1

Sean

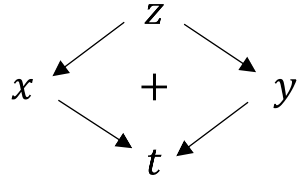
$$z = x^2 + 2xy$$

$$f(t) = \big(x = x(t), y = y(t)\big) = (3\cos t, 4sen\ t)$$

Se pide determinar $\dfrac{dz}{dt}$.

Solución

Puede controlarse la regla de la cadena mediante un diagrama donde en este caso z es la variable dependiente, $x\ e\ y$ las variables intermedias y t la variable independiente.



$$\boxed{\begin{aligned} z &= x^2 + 2xy \\ x &= 3\cos t \\ y &= 4sen\ t \end{aligned}}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Como $z = x^2 + 2xy$; $x = 3\cos t$ y $y = 4\ sen\ t$, entonces

$$\frac{dz}{dt} = (2x + 2y)\big(3(-sen\ t)\big) + 2x\ (4\cos t)$$

$$= -6(x + y)\ sen\ t + 8x\cos t$$

$$= -6(3\cos t + 4sen\ t)sen\ t + 24\cos t\cos t$$

$$= -18cost\ sent - 24sen^2t + 24\ cos^2\mathrm{t}$$

$$= -9sen\ 2t - 24sen^2t + 24cos^2t$$

Ejemplo 2

Sean $z = x^2y$, $x = e^{t^2}$ y $y = 2t + 1$ calcule $\dfrac{dz}{dt}$ para $t = 0.$

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & + & y \arrow[dl] \\ & t & \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt}(t) = (2xy)\big(2te^{t^2}\big) + (x^2)(2)$$

$$= 2e^{t^2}(2t + 1)\big(2te^{t^2}\big) + e^{2t^2}(2)$$

$$= 4\ te^{2t^2}(2t + 1) + 2e^{2t^2}$$

$$= 2e^{2t^2}(4t^2 + 2t + 1)$$

Luego

$$\frac{dz}{dt}(0) = 2e^{0}\left(4(0^{2})\right) + 2(0) + 1) = 2$$

Ejemplo 3

Sea $z = u^3 + 3v^2$ donde $u = \ln\,(xy)$ y $v = sen\ x - \cos y$. Hallar z_x y z_y .

Solución

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & + & v \arrow[dl] \\ & x & \end{tikzcd}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{y}{xy} = \frac{1}{x}}$$

Del diagrama tenemos

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

Para ello

$$\frac{\partial z}{\partial u} = 3u^2\ ;\ \frac{\partial u}{\partial x} = \frac{1}{x}$$

$$\frac{\partial z}{\partial v} = 6v\ ;\ \frac{\partial v}{\partial x} = \cos x$$

Luego

$$z_x = 3u^2\frac{1}{x} + 6v\ cos\ x$$

Por consiguiente

$$z_x = 3ln^2(xy)\frac{1}{x} + 6(sen\ x - \cos y)\cos x$$

$$\begin{tikzcd}
 & z \arrow[dl] \arrow[dr] \arrow[d, phantom, "+"] & \\
u \arrow[dr] & {} & v \arrow[dl] \\
 & y &
\end{tikzcd}$$

Del diagrama tenemos

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Como $z = u^3 + 3v^2$; $u = \ln(xy)$ y $v = sen\ x - \cos y$

Entonces

$$\boxed{\frac{\partial u}{\partial y} = \frac{x}{xy} = \frac{1}{y}}$$

$$\frac{\partial z}{\partial u} = 3u^2\ ;\ \frac{\partial u}{\partial y} = \frac{1}{y}$$

$$\frac{\partial z}{\partial v} = 6v\ ;\ \frac{\partial v}{\partial y} = \operatorname{sen} y$$

Luego

$$z_y = 3u^2\frac{1}{y} + 6v\ sen\ y$$

$$z_y = 3ln^2(xy)\frac{1}{y} + 6(sen\ x - \cos y)\operatorname{sen} y$$

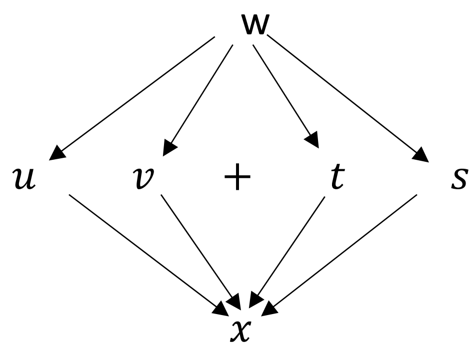
Ejemplo 4

Sean $w = 3u^5 - sen\ v + t^2 + s^{-1}$ donde $u = x^2 - y + z^2; v = xz^3 + 3; t = e^{2y-z+x}$ y $s = \cos(2z) - 9x$.

Hallar $\dfrac{\partial w}{\partial x}$.

Solución

Se tiene el sigiente diagrama



Luego la fórmula es

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial x}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial x}$$

Como $w=3u^5-sen\ v+t^2+s^{-1}$; $u=x^2-y+z^2$; $v=xz^3+3$; $t=e^{2y-z+x}$ y $s=\cos(2z)-9x$

Entonces

$$\frac{\partial w}{\partial x}=15u^4(2x)-(\cos v)(z^3)+2t\,e^{2y-z+x}-s^{-2}(-9)$$

$$\frac{\partial w}{\partial x}=30x(x^2-y+z^2)^4-z^3\cos(xz^3+3)+2e^{4y-2z+2x}+9(cos(2z)-9x)^{-2}$$

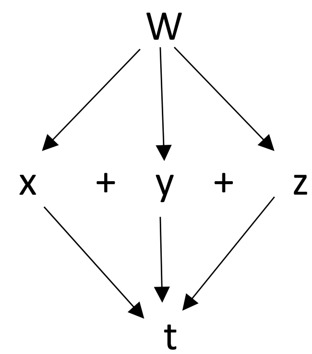
Ejemplo 5

Sea $w=xy-yz$ donde:

$$\begin{cases} x=t-1 \\ y=2t^3 \\ z=t^2+1 \end{cases}$$

Determinar $\dfrac{\partial w}{\partial t}$ para $t=1$

Solución



Del diagrama

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Como

$$w = xy - yz;\ x = t - 1$$

$$y = 2t^3 \text{ y } z = t^2 + 1$$

Entonces

$$\frac{\partial w}{\partial x} = y;\ \frac{\partial x}{\partial t} = 1$$

$$\frac{\partial w}{\partial y} = x - z;\ \frac{\partial y}{\partial t} = 6t^2$$

$$\frac{\partial w}{\partial z} = -y;\ \frac{\partial z}{\partial t} = 2t$$

Entonces

$$\frac{\partial w}{\partial t} = y(1) + (x - z)(6t^2) - y(2t)$$

Para $t = 1$ se tiene que

$$x = 0, y = 2 \text{ y } z = 2$$

luego

$$\frac{\partial w}{\partial t}(1) = 2(1) + (0 - 2)(6(1)^2) - 2(2)$$
$$= 2 - 12 - 4$$
$$= -14$$

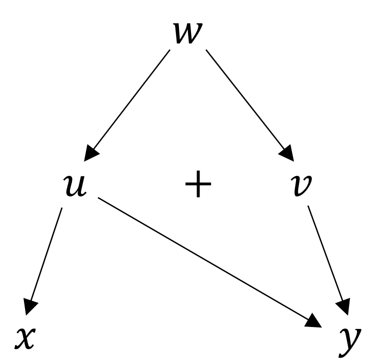
Ejemplo 6

a) Sea $w = f(u, v)$ donde $u = g(x, y)$; $v = h(y)$. Efectuar un diagrama para encontrar la expresión correspondiente a $\frac{\partial w}{\partial x}$ y $\frac{\partial w}{\partial y}$.

b) Use el resultado obtenido para determinar $\frac{\partial w}{\partial x}$ y $\frac{\partial w}{\partial y}$ para $w = u^2 + 2uv$; $u = x^2 - 2y$; $v = 4y^3$.

Solución

a) El diagrama es :



Luego se deduce que

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

b) Resolvamos ahora la otra parte del ejemplo

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x}$$

Como

$$w = u^2 + 2uv \,;\, u = x^2 - 2y \text{ y } v = 4y^3$$

Entonces

$$\frac{\partial w}{\partial x} = (2u + 2v)(2x) = 4(ux + vx)$$

$$\frac{\partial w}{\partial x} = (4(x^2 - 2y)x + 16y^3x)$$

$$\frac{\partial w}{\partial x} = 4x^3 - 8xy + 16xy^3$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial w}{\partial y} = (2u + 2v)(-2) + 2u\,(12y^2)$$

$$\frac{\partial w}{\partial y} = (2(x^2 - 2y) + 8y^3)(-2) + 24(x^2 - 2y)y^2$$

$$\frac{\partial w}{\partial y} = -4x^2 + 8y - 16y^3 + 24x^2y^2 - 48y^3$$

$$\frac{\partial w}{\partial y} = -4x^2 + 8y + 24x^2y^2 - 64y^3$$

■

Regla de la cadena desde una perspectiva más general

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^p \end{tikzcd}$$

$$g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$$

La derivada de la función compuesta $g \circ f$ se define de manera similar al caso de una composición de funciones de una variable.

$$(g \circ f)'(x_0) = g'\big(f(x_0)\big)f'(x_0)\ ;\ x_0 \in \mathbb{R}^n$$

Lo anterior es equivalente en terminos matriciales a

$$J(g \circ f)(x_0) = J\Big(g\big(f(x_0)\big)\Big)J\big(f(x_0)\big)$$

donde $J\big(f(x_0)\big)$

$$J\big(f(x_0)\big) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) \frac{\partial f_1}{\partial x_2}(x_0) \cdots \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) \frac{\partial f_2}{\partial x_2}(x_0) \cdots \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots \quad\quad \vdots \quad \ddots \quad \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) \frac{\partial f_m}{\partial x_2}(x_0) \cdots \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

Veamos el caso cuando $p = 1$.

En efecto, consideremos los $f_i = y_i$ como las funciones coordenadas de

$$f = (f_1, \dots \dots, f_m) = (y_1, \dots \dots, y_m).$$

En tanto la derivada de $g: \mathbb{R}^m \longrightarrow \mathbb{R}$ en $b = f(x_0)$ donde se denota por $(y_1, \dots \dots, y_m)$ a las variables de la función g es equivalente a la matriz jacobiana

$$J\big(g(b)\big) = \left[\frac{\partial g}{\partial y_1}(b)\,\frac{\partial g}{\partial y_2}(b)\cdots\frac{\partial g}{\partial y_m}(b)\right]$$

Entonces la derivada de la función compuesta $g \circ f$ en x_0 es la matriz

$$J(g \circ f)(x_0)$$

de orden $1 \text{ x } n$ que se obtiene como el producto de la matrices,

$$J\big(g(b)\big) \cdot J\big(f(x_0)\big)$$

Esto es,

$$J(g \circ f)(x_0) = \left[\frac{\partial}{\partial x_1}(g \circ f)(x_0)\,\frac{\partial}{\partial x_2}(g \circ f)(x_0)\cdots\frac{\partial}{\partial x_n}(g \circ f)(x_0)\right]$$

$$= \left[\frac{\partial g}{\partial y_1}\big(f(x_0)\big)\frac{\partial g}{\partial y_2}\big(f(x_0)\big)\cdots\frac{\partial g}{\partial y_m}\big(f(x_0)\big)\right]\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \cdots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

Observe que, por ejemplo, el j-ésimo elemento de $J(g \circ f)(x_0)$ es

$$\frac{\partial(g \circ f)}{\partial x_j}(x_0) =$$

se obtiene multiplicando la matriz $J\Big(g\big(f(x_0)\big)\Big)$ $por\ la\ j - ésima\ columna\ de\ J\big(f(x_0)\big)$

$$= g_{y_1}(b)(f_1)_{x_j}(x_0) + g_{y_2}(b)(f_2)_{x_j}(x_0) + \cdots + g_{y_m}(b)(f_m)_{x_j}(x_0)$$

$$= \frac{\partial g}{\partial y_1}(b)\frac{\partial f_1}{\partial x_j}(x_0) + \frac{\partial g}{\partial y_2}(b)\frac{\partial f_2}{\partial x_j}(x_0) + \cdots + \frac{\partial g}{\partial y_m}(b)\frac{\partial f_m}{\partial x_j}(x_0)$$

$$= \sum_{i=1}^{m}\left(\frac{\partial g}{\partial y_i}(b)\right)\frac{\partial f_i}{\partial x_j}(x_0) = \sum_{i=1}^{m}\Big(g_{y_i}(b)\Big)(f_i)_{x_j}(x_0)$$

fórmula que ya conocemos (ver página 1).

Ejemplo

Considremos las funciones $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ y $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ dadas por

$$f(x,y) = (xy, 5x, y^3)$$
$$g(x,y,z) = (3x^2 + y^2 + z^2, 5xyz)$$

La composición es $g \circ f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, donde se denota por $(g \circ f)_1$ y $(g \circ f)_2$ a las funciones coordenadas de $g \circ f$. Entonces la matriz jacobiana de $g \circ f$ es:

$$\begin{aligned} J(g \circ f)(x,y) &= \begin{bmatrix} \frac{\partial}{\partial x}(g \circ f)_1(x,y) & \frac{\partial}{\partial y}(g \circ f)_1(x,y) \\ \frac{\partial}{\partial x}(g \circ f)_2(x,y) & \frac{\partial}{\partial y}(g \circ f)_2(x,y) \end{bmatrix} \\ &= Jg(f(x,y))\, J(f(x,y) \end{aligned}$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial x}\big(f(x,y)\big) & \frac{\partial g_1}{\partial y}\big(f(x,y)\big) & \frac{\partial g_1}{\partial z}\big(f(x,y)\big) \\ \frac{\partial g_2}{\partial x}\big(f(x,y)\big) & \frac{\partial g_2}{\partial y}\big(f(x,y)\big) & \frac{\partial g_2}{\partial z}\big(f(x,y)\big) \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

Sustituyendo las derivadas parciales respectivas de

$$g_1(x,y,z) = 3x^2 + y^2 + z^2 \;\; ; \;\; g_2(x,y,z) = 5xyz$$
$$f_1(x,y) = xy \;\; ; \;\; f_2(x,y) = 5x \text{ y } f_3(x,y) = y^3$$

Nos queda,

$$= \begin{bmatrix} 6x & 2y & 2z \\ 5yz & 5xz & 5xy \end{bmatrix}_{f(x,y)} \begin{bmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6x & 2y & 2z \\ 5yz & 5xz & 5xy \end{bmatrix}_{(xy,5x,y^3)} \begin{bmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6xy & 2(5x) & 2(y^3) \\ 5(5x)(y^3) & 5(xy)y^3 & 5(xy)(5x) \end{bmatrix} \begin{bmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6xy & 10x & 2y^3 \\ 25xy^3 & 5xy^4 & 25x^2y \end{bmatrix} \begin{bmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6xy^2 + 50x & 6x^2y + 6y^5 \\ 50xy^4 & 100x^2y^3 \end{bmatrix}$$

También se puede llegar a este resultado si antes hacemos explícita la composición $\boxed{g(x,y,z) = (3x^2 + y^2 + z^2, 5xyz)}$

$$(g \circ f)(x,y) = g\big(f(x,y)\big) = g(xy, 5x, y^3)$$

$$= \big(3(xy)^2 + (5x)^2 + (y^3)^2, 5(xy)(5x)(y^3)\big)$$

$$= (3x^2y^2 + 25x^2 + y^6, 25x^2y^4)$$

Y luego derivamos directamente. En efecto, sean

$$(g \circ f)_1(x,y) = 3x^2y^2 + 25x^2 + y^6 \;;$$

$$(g \circ f)_2(x,y) = 25x^2y^4$$

$$J(g \circ f)(x,y) = \begin{bmatrix} \dfrac{\partial}{\partial x}(g \circ f)_1(x,y) & \dfrac{\partial}{\partial y}(g \circ f)_1(x,y) \\ \dfrac{\partial}{\partial x}(g \circ f)_2(x,y) & \dfrac{\partial}{\partial y}(g \circ f)_2(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 6xy^2 + 50x & 6x^2y + 6y^5 \\ 50xy^4 & 100x^2y^3 \end{bmatrix}$$