General Solution of Laplace's Equation in Spherical Coordinates

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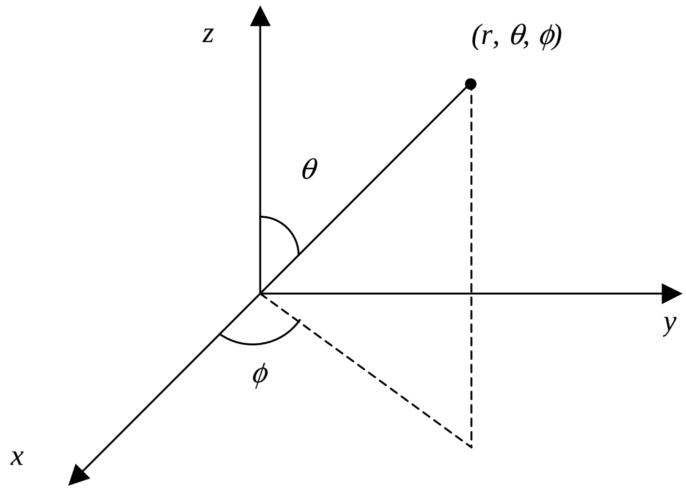
Given Laplace's equation is 3 dimensional spherical coordinates:

$$\nabla^2 u(r,\theta,\phi) = 0$$



To obtain the general solution of this equation in spherical coordinates, write the Laplacian operator in those coordinates (see Boas for a version of this).

$$\nabla^2 u(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

Now assume a separable solution of the form: $u(r,\theta,\phi) = R(r)P(\theta)\Phi(\phi)$

Plug this into the differential equation and perform the separation. Each of the differential operators only affects its own function (e.g. the r-derivatives don't affect P and Φ). This gives the intermediate result

$$\frac{P\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{RP}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

Now, divide through by the solution, $RP\Phi$, and move the ϕ dependence all to the right hand side of the equation. Also, to isolate the ϕ dependence, multiply through by $r^2\sin^2\theta$.

$$\frac{1}{R}\sin^2\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{P}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \qquad \text{(Eq. \#1)}$$

Azimuthal (ϕ) Separation:

The index m is the separation constant to separate the ϕ equation from r-θ. So we have the separated ϕ differential equation from Eq. #1. This ordinary differential equation is solved either by $\sin(m\phi)$ and $\cos(m\phi)$ or by the complex exponential function $\exp(\pm im\phi)$.

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \qquad \text{which has solutions} \qquad \Phi(\phi) \sim \exp(im\phi), \exp(-im\phi)$$

Note, physically we must have the same solution at ϕ and at ϕ + 2πn. This is called <u>*single valuedness*</u>. It requires that the index m be an integer.

Polar Angle (θ) Separation:

Now, back in Eq. #1, use the constant m and manipulate the r-θ equation on the LHS to separate r and θ dependencies. Divide through by $\sin^2\theta$ to isolate r from θ. This gives

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) = \frac{m^2}{\sin^2\theta}$$

Move the θ terms to the right hand side:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = l(l+1) \qquad \text{(Eq. \#2)}$$

The second index, l, is the separation constant for the r and θ equations. Look at the θ equation now. It can be written as

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{m^2}{\sin^2\theta}P = l(l+1)P$$

or it can be rewritten as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

At this point, make a change of variables by defining $\quad x = \cos\theta \quad$ and $dx = -\sin\theta d\theta$

Note that this x is just a new variable; it has nothing to do with the x-axis of the coordinates.

Since θ ranges between 0 and 180° (π radians), this forces x to vary between $-1 \le x \le 1$. Also note that $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$ and further that $\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$.

Putting all this together, we can change the θ-equation into a differential equation in x:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

This can be further manipulated into the final result (take the derivative indicated in the 1st term).

$$\left(1-x^2\right)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0$$

This is the associated Legendre equation, which has the associated Legendre polynomials as its solutions. Note that if $m = 0$, you get the original Legendre differential equation, with Legendre polynomials as its solutions. So we have

$$P=P_{lm}(x) \qquad \text{or} \qquad P=P_{lm}(\cos\theta)$$

Recall some of the properties of these polynomials:

$$P_{lm}(x)=\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l(x) \qquad \text{and} \qquad P_{lm}(x)=\frac{1}{2^l\,l!}\left(1-x^2\right)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l$$

and $\quad P_{l0}(x)=P_l(x)$

and also the orthonormality relation for the polynomials

$$\int_{-1}^{1}dxP_{lm}(x)P_{l'm'}(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\delta_{mm'}$$

Radial (r) Separation:

Finally, go back to Eq. #2, take the remaining r-equation and solve it. It has the form:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)R$$

Carry the derivative into the bracketed term. This gives the equation as

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}=l(l+1)R$$

To continue, assume a power law solution and plug-in to the equation:

let $R = r^n$ $\quad R' = nr^{n-1}$ and $\quad R'' = n(n-1)r^{n-2}$

Plugging these into the equation gives

$n(n-1)r^n + 2nr^n = l(l+1)r^n$ or the algebraic equation $n^2 - n + 2n = n^2 + n = l(l+1)$

Complete the square of the LHS in order to solve for *n*.

$$n^2 + n + \tfrac{1}{4} = l^2 + l + \tfrac{1}{4}$$

giving $\quad (n+\tfrac{1}{2})^2 = (l+\tfrac{1}{2})^2$

or finally $\quad n = \pm\left(l+\tfrac{1}{2}\right) - \tfrac{1}{2}$

So the permitted values of n are $\quad n = l$ and $-(l+1)$

This gives the radial solution as $\quad R(r) = ar^l + \dfrac{b}{r^{l+1}}$

General Solutions of Laplace's Equation in Spherical Coordinates:

Put everything together to write the general solution to Laplace's equation in spherical coordinates:

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right)P_{lm}(\cos\theta)e^{im\phi} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right)Y_l^m(\theta,\phi)$$

where the $Y_l^m(\theta,\phi)$ are the spherical harmonics.

If the problem has azimuthal symmetry (i.e. there is no azimuthal dependence in it), then this general solution simplifies to

$$u(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta)$$

The coefficients, a_l and b_l, can be found using the boundary conditions of the situation and inverting either of the series using the orthonormality of the appropriate polynomials.

Having done this separation once, from now on when encountering a spherical Laplace's problem, you can simply start with these general solutions and immediately begin to apply the boundary conditions to find the coefficients.