① a) Soluciones homogéneas : $\sin(x+\phi)$, $\cos(x+\phi)$

Condiciones de contorno $y_1(0)=0 \Rightarrow y_1 = \sin x$

$y_2'(1)=0 \Rightarrow y_2 = \cos(x-1)$

$$A = [p(t)[y_2' y_1 - y_1' y_2]]^{-1} \qquad p(t)=1$$

$$= [1\cdot(-\sin(x-1)\cdot \sin x - \cos x \cdot \cos(x-1))]^{-1}$$

$$= -\sin^2 x \cos(-1) - \sin(-1)\cos x \sin x$$
$$- \cos^2 x \cos(-1) + \sin(-1)\cos x \sin x$$

$$= -\cos(-1)$$

$$G = \begin{cases} A y_1(x) y_2(t) & 0 \le x < t \\ A y_2(x) y_1(t) & t \le x \le 1 \end{cases}$$

$$\therefore G = -\frac{1}{\cos(-1)}\cdot[\sin x \cos(t-1)] \qquad 0 \le x < t$$

$$-\frac{1}{\cos(-1)}\cdot[\cos(x-1)\sin t] \qquad t \le x \le 1$$

b) Soluciones homogéneas: $\exp(x)$, $\exp(-x)$

$y_1(-\infty)$ finito $\rightarrow y_1 = \exp(x)$

$y_2(+\infty)$ finito $\rightarrow y_2 = \exp(-x) \qquad p(t)=1$

$$A = [1\cdot(-\exp(-x)\cdot\exp(x) - \exp(x)\cdot\exp(-x))]^{-1}$$

$$= -\frac{1}{2}$$

$$G = -\frac{1}{2}\exp(x)\cdot\exp(-t) = -\frac{1}{2}\exp(x-t) \qquad -\infty < x < t$$

$$-\frac{1}{2}\exp(t-x) \qquad t < x < +\infty$$

② $\mathcal{L}\psi = \vec{\nabla}\cdot[p\vec{\nabla}\psi] + q\psi$

$\langle \phi | \mathcal{L}\psi \rangle = \langle \mathcal{L}\phi | \psi \rangle$ ← Hermitiano

$$\langle \phi | \mathcal{L}\psi \rangle = \int_V \phi^* \vec{\nabla}\cdot[p\vec{\nabla}\psi]\,d\tau + \int_V \phi^* q\psi\,d\tau \quad ⓐ$$

$$\int_V \phi^* \vec{\nabla}\cdot[p\vec{\nabla}\psi]\,d\tau = \underbrace{\int_V \vec{\nabla}\cdot[\phi^* p\vec{\nabla}\psi]\,d\tau}_{\substack{\text{integral de superficie} \\ = 0 \text{ por la condición de Dirichlet}}} - \int_V \vec{\nabla}\phi^* \cdot p\vec{\nabla}\psi\,d\tau \quad ⓒ$$

$$\langle \mathcal{L}\phi | \psi \rangle = \int_V \vec{\nabla}\cdot[p\vec{\nabla}\phi^*]\,\psi\,d\tau + \int q\phi^*\psi\,d\tau \quad ⓑ$$

$$\int_V \psi\,\vec{\nabla}\cdot[p\vec{\nabla}\phi^*]\,d\tau = \underbrace{\int_V \vec{\nabla}\cdot[\psi p\vec{\nabla}\phi^*]\,d\tau}_{\substack{\text{integral de superficie} \\ = 0}} - \int_V \vec{\nabla}\psi\cdot p\vec{\nabla}\phi^*\,d\tau \quad ⓓ$$

Integrales ⓐ y ⓑ son iguales, ⓒ y ⓓ también

$\therefore\ \langle \phi | \mathcal{L}\psi \rangle = \langle \psi | \mathcal{L}\phi \rangle$ hermitiano ✓

Problema 3: $g(x,t) = g(u+v,t) = g(u,t)\, g(v,t)$

(a)
$$g(u+v,t) = \sum_{n=-\infty}^{\infty} J_n(u+v)\, t^n = g(u,t)\, g(v,t) = \sum_{s=-\infty}^{\infty} J_s(u)\, t^s \sum_{p=-\infty}^{\infty} J_p(v)\, t^p$$
$$= \sum_{s} \sum_{p} J_s(u)\, J_p(v)\, t^{s+p} \quad ; \quad \text{haciendo } n = s+p$$
$$\Rightarrow \sum_{n=-\infty}^{\infty} J_n(u+v)\, t^n = \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_s(u)\, J_{n-s}(v)\, t^n$$

comparando,
$$\boxed{J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u)\, J_{n-s}(v)}$$

(b) Escribamos el resultado anterior como
$$J_n(u+v) = \sum_{s=-\infty}^{-1} J_s(u)\, J_{n-s}(v) + J_0(u)\, J_n(v) + \sum_{s=1}^{\infty} J_s(u)\, J_{n-s}(v)$$

para $n=0$
$$J_0(u+v) = J_0(u)\, J_0(v) + \underbrace{\sum_{s=-\infty}^{-1} J_s(u)\, J_{-s}(v)}_{s \to -s} + \sum_{s=1}^{\infty} J_s(u)\, J_{-s}(v)$$

En la 1ª somatoria hacemos el cambio $s \to -s$
$$\therefore J_0(u+v) = J_0(u)\, J_0(v) + \sum_{s=1}^{\infty} \left(J_{-s}(u)\, J_s(v) + J_s(u)\, J_{-s}(v) \right)$$

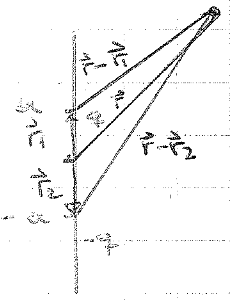
Ya que $s \in \mathbb{Z} \Rightarrow J_{-s}(u) = (-1)^s J_s(u) \wedge J_s(v) = (-1)^s J_{-s}(v)$

$$J_0(u+v) = J_0(u)\, J_0(v) + \sum_{s=1}^{\infty} \left[(-1)^{2s} + 1 \right] J_s(u)\, J_{-s}(v)$$

$$\Rightarrow \boxed{J_0(u+v) = J_0(u)\, J_0(v) + 2 \sum_{s=1}^{\infty} J_s(u)\, J_{-s}(v)}$$

Problema 4 El potencial en un punto $\vec{r}$ debido a las cargas

(a)



$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_1|} + \frac{1}{4\pi\epsilon_0}\frac{(-q)}{|\vec{r}-\vec{r}_2|} \quad ; \quad \vec{r}_1 = a\hat{k} \; ; \; \vec{r}_2 = -a\hat{k}$$

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\left[\left(r^2+a^2-2ar\cos\theta\right)^{-1/2} - \left(r^2+a^2-2ar\cos(\pi-\theta)\right)^{-1/2}\right]$$

$$\cos(\pi-\theta) = -\cos\theta$$

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\left[\left(r^2+a^2-2ar\cos\theta\right)^{-1/2} - \left(r^2+a^2+2ar\cos\theta\right)^{-1/2}\right]$$

Para $r > a$: $\Phi(r) = \dfrac{q}{4\pi\epsilon_0}\left\{ r^{-1}\left[1-2\left(\frac{a}{r}\right)\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2} - r^{-1}\left[1-2\left(-\frac{a}{r}\right)\cos\theta + \left(-\frac{a}{r}\right)^2\right]^{-1/2}\right\}$

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{\sqrt{1-2\left(\frac{a}{r}\right)\cos\theta + \left(\frac{a}{r}\right)^2}} - \frac{1}{\sqrt{1-2\left(-\frac{a}{r}\right)\cos\theta + \left(\frac{a}{r}\right)^2}}\right]$$

$$* \quad g(x,t) = \left(1-2xt+t^2\right)^{-1/2} = \sum_{l=0}^{\infty} P_l(x)\, t^l$$

$$\therefore \Phi(r) = \frac{q}{4\pi\epsilon_0 r}\left\{\sum_{l=0}^{\infty} P_l(\cos\theta)\cdot\left(\frac{a}{r}\right)^l - \sum_{l=0}^{\infty}(-1)^l P_l(\cos\theta)\left(\frac{a}{r}\right)^l\right\}$$

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}\left\{\sum_{l=0}^{\infty}\left(1-(-1)^l\right)P_l(\cos\theta)\frac{a^l}{r^l}\right\}$$

con l par se anulan los términos
$l \to 2l-1$

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r}\cdot 2\sum_{l=1}^{\infty} P_{2l-1}(\cos\theta)\frac{a^{2l-1}}{r^{2l-1}}$$

$$\Rightarrow \boxed{\Phi(r) = \frac{q}{2\pi\epsilon_0}\sum_{l=1}^{\infty}\frac{a^{2l-1}}{r^{2l}}P_{2l-1}(\cos\theta)} \quad r > a$$

Para $r < a$ $\Rightarrow$ $\boxed{\Phi(r) = \dfrac{q}{2\pi\epsilon_0}\displaystyle\sum_{l=1}^{\infty}\frac{r^{2l-1}}{a^{2l}}P_{2l-1}(\cos\theta)}$ $\quad r < a$

(b) Dipolo $r \gg a$; $qa = p/2 \;\wedge\; a \to 0$

$$\Phi(r) = \frac{q}{2\pi\epsilon_0}\left(\frac{a}{r^2}P_1(\cos\theta) + \frac{a^3}{r^4}P_3(\cos\theta) + \frac{a^5}{r^6}P_5(\cos\theta) + \dots\right) = \frac{qa}{2\pi\epsilon_0}\left(\frac{P_1(\cos\theta)}{r^2} + \frac{a^2 P_3(\cos\theta)}{r^4} + \dots\right)$$

Tomando $qa = p/2$ y $a \to 0$

$$\boxed{\Phi(r) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}}$$