

Tarea N°2 (Individual)

Cálculo III - FOGEC - FC - UV - 31 - 05 – 2022

Fecha para entregar: martes 7 de junio en horario de clases

1.- (15 Puntos)

Sea $f(x, y) = \begin{cases} \frac{2xy(x^2 - y^2)}{5(x^2 + y^2)} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$ una función

a) Hallar las funciones derivadas parciales, $f_x(x, y)$, $f_y(x, y)$

b) Determine $\frac{\partial^2 f(0,0)}{\partial x \partial y}$

2.- (15 Puntos)

Use la definición de diferenciabilidad para demostrar que la función escalar $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ definida por $f(x, y) = 2x^2y$, es diferenciable en cualquier punto $(a, b) \in \mathbb{R}^2$.

3.- (15 Puntos)

a) Determine si el sistema

$$\begin{cases} x + y + u + v + w = 0 \\ x^2 - y^2 + u^2 - 2v^2 + w^2 + 1 = 0 \\ x^3 + y^3 + u^4 - 3v^4 + 8w^4 + 2 = 0 \end{cases}$$

define implícitamente las funciones $u = u(x, y)$; $v = v(x, y)$ y $w = w(x, y)$ en el punto $p_0 = (x_0, y_0, u_0, v_0, w_0) = (1, -1, 1, -1, 0)$.

b) evaluar $\frac{dv}{dy}$ en $(1, -1)$

4.- (15 Puntos)

Sea $f(p, q) = p^n e^{pq^2}$. Halle el valor de la constante n , con $n \in \mathbb{N}$ de manera que satisfaga la ecuación:

$$\frac{\partial f}{\partial p} - \frac{1}{2} \frac{q}{p} \frac{\partial f}{\partial q} = 3p^2 e^{pq^2}$$

a)

$$\begin{aligned}f_x(x, y) &= \frac{\frac{2}{5}(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)2x}{(x^2 + y^2)^2} \\&= \frac{2}{5} \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} \\&= \frac{2}{5} \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= \frac{\frac{2}{5}(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)2y}{(x^2 + y^2)^2} \\&= \frac{2}{5} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} \\&= \frac{2}{5} \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}\end{aligned}$$

$$\begin{aligned}f_x(0,0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0,0)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0,0)}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{2(h^3 \cdot 0 - h \cdot 0^3)}{5(h^2 + 0^2)} - 0}{h} \\&= \lim_{h \rightarrow 0} \frac{0}{h} = 0\end{aligned}$$

$$\begin{aligned}f_y(0,0) &= \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{2(0^3 \cdot h - 0 \cdot h^3)}{5(0^2 + h^2)} - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0\end{aligned}$$

Por tanto, de lo anterior tenemos

$$f_x(x, y) = \begin{cases} \frac{2}{5} \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$$

y

$$f_y(x, y) = \begin{cases} \frac{2}{5} \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2} & \text{si } (x, y) \neq (0, 0) \\ 0 & \text{si } (x, y) = (0, 0) \end{cases}$$

b)

$$\begin{aligned} \frac{\partial^2 f(0,0)}{\partial x \partial y} &= \lim_{h \rightarrow 0} \frac{f_y(0+h, 0) - f_y(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2h(h^4 - 4h^2 \cdot 0^2 - 0^4)}{5(h^2 + 0^2)^2} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{2h^5}{5h^4}}{h} = \lim_{h \rightarrow 0} \frac{2h^5}{5h^5} = \frac{2}{5} \end{aligned}$$

2.- $f(x, y) = 2x^2y$

$$\begin{aligned} f_x(a, b) &= \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(a+h)^2b - 2a^2b}{h} = \lim_{h \rightarrow 0} \frac{2(a^2 + 2ah + h^2)b - 2a^2b}{h} \\ &= \lim_{h \rightarrow 0} \frac{2a^2b + 4abh + 2h^2b - 2a^2b}{h} \\ &= \lim_{h \rightarrow 0} \frac{4abh + 2h^2b}{h} = \lim_{h \rightarrow 0} (4ab + 2hb) = 4ab \\ f_y(a, b) &= \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{2a^2b + 2a^2h - 2a^2b}{h} = \lim_{h \rightarrow 0} \frac{2a^2h}{h} = \lim_{h \rightarrow 0} 2a^2 = 2a^2$$

Ahora

$$\begin{aligned} f(a+h, b+k) &= 2(a+h)^2(b+k) \\ &= 2(a^2 + 2ah + h^2)(b+k) \\ &= 2a^2b + 2a^2k + 4abh + 4ahk + 2bh^2 + 2h^2k \end{aligned}$$

Por tanto,

$$\begin{aligned} &\lim_{(h,k) \rightarrow (0,0)} \frac{E(a+h, b+k)}{\|(h,k)\|} \\ &= \lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2 + k^2}} \end{aligned}$$

Donde

$$\begin{aligned} &f(a+h, b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k = \\ &\cancel{2a^2b} + \cancel{2a^2k} + \cancel{4abh} + 4ahk + 2bh^2 + 2h^2k - \cancel{2a^2b} - \cancel{4abh} - \cancel{2a^2k} \\ &= 4ahk + 2bh^2 + 2h^2k \end{aligned}$$

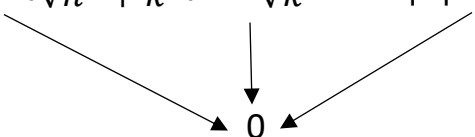
Luego

$$\begin{aligned} &\lim_{(h,k) \rightarrow (0,0)} \frac{E(a+h, b+k)}{\|(h,k)\|} \\ &= \lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2 + k^2}} \\ &= \lim_{(h,k) \rightarrow (0,0)} \frac{4ahk + 2bh^2 + 2h^2k}{\sqrt{h^2 + k^2}} \\ &= 2 \lim_{(h,k) \rightarrow (0,0)} \frac{2ahk + bh^2 + h^2k}{\sqrt{h^2 + k^2}} \\ &= 2 \left[2a \lim_{(h,k) \rightarrow (0,0)} \frac{hk}{\sqrt{h^2 + k^2}} + b \lim_{(h,k) \rightarrow (0,0)} \frac{h^2}{\sqrt{h^2 + k^2}} + \lim_{(h,k) \rightarrow (0,0)} \frac{h^2k}{\sqrt{h^2 + k^2}} \right] \\ &\quad (1) \qquad \qquad \qquad (2) \qquad \qquad \qquad (3) \end{aligned}$$

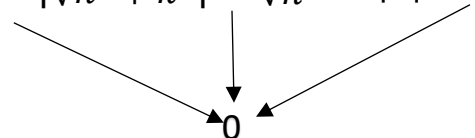
$$= 2[0 + 0 + 0] = 0$$

Pues

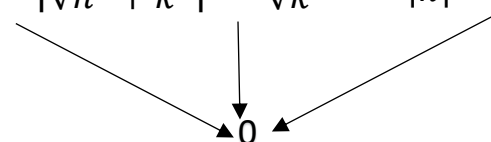
En (1)

$$0 \leq \left| \frac{hk}{\sqrt{h^2 + k^2}} \right| \leq \frac{|h||k|}{\sqrt{k^2}} = \frac{|h||k|}{|k|} = |h|$$


En (2)

$$0 \leq \left| \frac{h^2}{\sqrt{h^2 + k^2}} \right| \leq \frac{h^2}{\sqrt{h^2}} = \frac{|h|^2}{|h|} = |h|$$


En (3)

$$0 \leq \left| \frac{h^2 k}{\sqrt{h^2 + k^2}} \right| \leq \frac{h^2 |k|}{\sqrt{k^2}} = \frac{h^2 |k|}{|k|} = h^2$$


3.- Consideremos

$$f(x, y, u, v, w) = x + y + u + v + w = 0$$

$$g(x, y, u, v, w) = x^2 - y^2 + u^2 - 2v^2 + w^2 + 1 = 0$$

$$h(x, y, u, v, w) = x^3 + y^3 + u^4 - 3v^4 + 8w^4 + 2 = 0$$

En el punto $p_0 = (x_0, y_0, u_0, v_0, w_0) = (1, -1, 1, -1, 0)$ se tiene

$$f(p_0) = 1 - 1 + 1 - 1 + 0 = 0$$

$$\begin{aligned} g(p_0) &= (1)^2 - (-1)^2 + (1)^2 - 2(-1)^2 + 0^2 + 1 \\ &= 1 - 1 + 1 - 2 + 1 = 0 \end{aligned}$$

$$\begin{aligned}
 h(p_0) &= (1)^3 + (-1)^3 + (1)^4 - 3(-1)^4 + 8(0)^4 + 2 \\
 &= 1 - 1 + 1 - 3 + 0 + 2 = 0
 \end{aligned}$$

Todas las derivadas parciales de f, g y h son continuas siempre. Se tiene, además

$$\begin{aligned}
 \frac{\partial(f, g, h)}{\partial(u, v, w)}(p_0) &= \begin{vmatrix} 1 & 1 & 1 \\ 2u & -4v & 2w \\ 4u^3 & -12v^3 & 32w^2 \end{vmatrix} \bigg|_{\substack{u=1 \\ v=-1 \\ w=0}} \\
 &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & -4 & 0 \\ 4 & 12 & 0 \end{vmatrix} \text{ (aplicando columna 3)} \\
 &= \begin{vmatrix} 2 & -4 \\ 4 & 12 \end{vmatrix} = 24 - 16 = 8 \neq 0
 \end{aligned}$$

Por tanto, por teorema de Cauchy Dini se concluye que el sistema define implícitamente las funciones $u = u(x, y)$; $v = v(x, y)$ y $w = w(x, y)$ en el punto $p_0 = (1, -1, 1, -1, 0)$.

b) Ahora evaluaremos $\frac{dv}{dy}$ en $(1, -1)$

$$\frac{dv}{dy} = - \frac{\frac{\partial(f, g, h)}{\partial(u, y, w)}}{\frac{\partial(f, g, h)}{\partial(u, v, w)}}$$

donde

$$\begin{aligned}
 \frac{\partial(f, g, h)}{\partial(u, y, w)}(p_0) &= \begin{vmatrix} 1 & 1 & 1 \\ 2u & -2y & 2w \\ 4u^3 & 3y^2 & 32w^2 \end{vmatrix} \bigg|_{\substack{u=1 \\ y=-1 \\ w=0}} \\
 &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 4 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} = 6 - 8 = -2 \\
 \frac{dv}{dy}(1, -1) &= -\frac{-2}{8} = \frac{1}{4}
 \end{aligned}$$

4.- Sea $f(p, q) = p^n e^{pq^2}$ entonces

$$\frac{\partial f}{\partial p} = np^{n-1}e^{pq^2} + p^n q^2 e^{pq^2} \dots (1)$$

$$\frac{\partial f}{\partial q} = 2pq p^n e^{pq^2} \dots (2)$$

De (1) y (2)

$$\frac{\partial f}{\partial p} - \frac{1}{2} \frac{q}{p} \frac{\partial f}{\partial q} = 3p^2 e^{pq^2}$$

$$np^{n-1}e^{pq^2} + p^n q^2 e^{pq^2} - \frac{1}{2} \frac{q}{p} 2pq p^n e^{pq^2} = 3p^2 e^{pq^2}$$

$$np^{n-1}e^{pq^2} + p^n q^2 e^{pq^2} - p^n q^2 e^{pq^2} = 3p^2 e^{pq^2}$$

$$np^{n-1}e^{pq^2} = 3p^2 e^{pq^2}$$

Entonces

$$np^{n-1} = 3p^2$$

$$np^{n-3} = 3$$

$$n = 3$$