

# Structure of the Atom

# 4

## CHAPTER

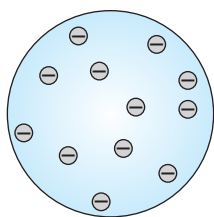
*Bohr's different. He's a football (U.S. soccer) player!*

*Ernest Rutherford, giving an uncharacteristic compliment to a theorist—Niels Bohr in this case.*

By the end of the nineteenth century most physicists and chemists (with a few notable exceptions) believed in an atomic theory of matter, even though no one had ever observed an atom directly. Origins of atomic theory date back to the Greek philosophers, who imagined atoms to be featureless hard spheres. Even though scientists of the late nineteenth century did not have technology to see things as small as atoms, they believed that atoms were composite structures having an internal structure. There are considerable similarities between how physicists addressed their atomic theories in the late nineteenth century and how elementary particle physicists still search for the underlying structure of the building blocks of matter.

We mention three pieces of evidence that physicists and chemists had in 1900 to indicate that the atom was not a fundamental unit. First, there seemed to be too many kinds of atoms, each belonging to a distinct chemical element. The original Greek idea was that there were four types of atoms—earth, air, water, and fire—which combined to make the various kinds of matter we observe. But the development of chemistry made it clear that there were at least 70 kinds of atoms, far too many for them all to be the ultimate elementary constituents of matter.

Second, it was found experimentally that atoms and electromagnetic phenomena were intimately related. For example, molecules can be dissociated into their component atoms by electrolysis. Some kinds of atoms form magnetic materials, and others form electrical conductors and insulators. All kinds of atoms emit light (which was known to be electromagnetic in nature) when they are heated, as well as when an electrical discharge passes through them. The visible light emitted by free or nearly free atoms of the chemical elements is not a continuum of frequencies but rather a discrete set of characteristic colors, so substances can be analyzed according to their chemical composition using their flame spectra. The existence of characteristic spectra (Section 3.3) pointed to an internal structure distinguishing the elements.



**Figure 4.1** Schematic of J. J. Thomson's model of the atom (later proved to be incorrect). The electrons are embedded in a homogeneous positively charged mass much like raisins in plum pudding. The electric force on the electrons is zero, so the electrons do not move around rapidly. The oscillations of the electrons give rise to electromagnetic radiation.

**Thomson's  
"plum-pudding"  
model of the atom**

Third, there was the problem of **valence**—why certain elements combine with some elements but not with others, and when they do combine, why they do so in varying proportions determined by the valences of the atoms. The characteristics of valence suggested that the forces between atoms are specific in nature, a characteristic that hinted at an internal atomic structure.

Finally, there were the discoveries of radioactivity, of x rays, and of the electron, all of which were at variance with earlier ideas of indivisible and elementary atoms. Because of these tantalizing indirect hints that the atom had a structure, the most exciting frontier of science in the early part of the twentieth century developed into an investigation of the atom and its internal composition.

The subject of this chapter is the beginning of quantum physics and its relation to the first cohesive theories of atomic structure. Although we now have a more complete theoretical framework with which to understand the early experiments than was available to scientists at the time, it is worth knowing some of the scientists' reasoning, both for historical interest and to illustrate how science progresses by trying to extend well-established ideas into unknown terrain.

In this chapter we discuss the atomic models of Thomson and Rutherford and learn how Rutherford discerned the correct structure of the atom by performing alpha-particle scattering experiments. We see that Bohr presented a model of the hydrogen atom based on the new quantum concept that correctly produced the Rydberg equation, and we study the successes and failures of Bohr's theory. We also learn the origin of characteristic x-ray spectra and the concept of atomic number. Finally, we show that electron scattering (the Franck-Hertz experiment) also confirmed the quantized structure of the atom.

## 4.1 The Atomic Models of Thomson and Rutherford

In the years immediately following J. J. Thomson's discovery of the electron in 1897, Thomson and others tried to unravel the mystery of the atomic structure. Scientists knew that electrons were much less massive than atoms and that for many atoms, the number of electrons was equal or slightly less than half the number representing atomic mass. The central question was, "How are the electrons arranged and where are the positive charges that make the atom electrically neutral?" (Note that protons had not been yet discovered.) Thomson proposed a model wherein the positive charges were spread uniformly throughout a sphere the size of the atom, with electrons embedded in the uniform background. His model, which was likened to raisins in plum pudding, is shown schematically in Figure 4.1. The arrangement of charges had to be in stable equilibrium. In Thomson's view, when the atom was heated, the electrons could vibrate about their equilibrium positions, thus producing electromagnetic radiation. The emission frequencies of this radiation would fall in the range of visible light if the sphere of positive charges were of diameter  $\sim 10^{-10}$  m, which was known to be the approximate size of an atom. Nevertheless, even though he tried for several years, Thomson was unable to calculate the light spectrum of hydrogen using his model.

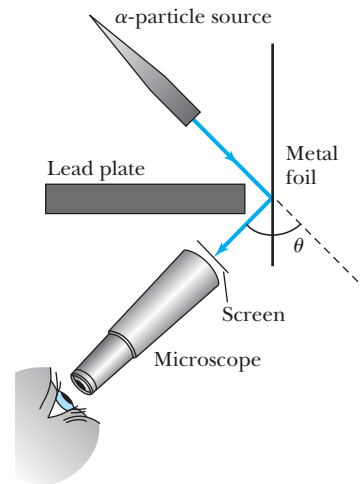
The small size of the atoms made it impossible to see directly their internal structure. In order to make further progress in deciphering atomic structure, a new approach was needed. The new direction was supplied by Ernest Rutherford, who was already famous for his Nobel Prize-winning work on radioactivity. Rutherford projected very small particles onto thin material, some of which

collided with atoms and eventually exited at various angles. Rutherford, assisted by Hans Geiger, conceived a new technique for investigating the structure of matter by scattering energetic alpha ( $\alpha$ ) particles\* (emitted by radioactive sources) from atoms. Together with a young student, Ernest Marsden, and working in Rutherford's lab, Geiger showed in 1909 that surprisingly many  $\alpha$  particles were scattered from thin gold-leaf targets at backward angles greater than  $90^\circ$  (see Figure 4.2).

Rutherford had pondered the structure of the atom for several years. He was well aware of Thomson's model because he had worked for Thomson at the Cavendish Laboratory as a research student from 1895 to 1898, after receiving his undergraduate education in his native New Zealand. Although he greatly respected Thomson, Rutherford could see that Thomson's model agreed neither with spectroscopy nor with Geiger's latest experiment with  $\alpha$  particles.

The experiments of Geiger and Marsden were instrumental in the development of Rutherford's model. A simple thought experiment with a .22-caliber rifle that fires a bullet into a thin black box is a model for understanding the problem. If the box contains a homogeneous material such as wood or water (as in Thomson's plum-pudding model), the bullet will pass through the box with little or no deviation in its path. However, if the box contains a few massive steel ball bearings, then occasionally a bullet will be deflected backward, similar to what Geiger and Marsden observed with  $\alpha$  scattering.

\*Rutherford had already demonstrated that the  $\alpha$  particle is an ionized helium atom.



**Figure 4.2** Schematic diagram of apparatus used by Geiger and Marsden to observe scattering of  $\alpha$  particles past  $90^\circ$ . “A small fraction of the  $\alpha$  particles falling upon a metal foil have their directions changed to such an extent that they emerge again at the side of incidence.” The scattered  $\alpha$  particle struck a scintillating screen where the brief flash was observed through the microscope. From H. Geiger and E. Marsden, *Proceedings of Royal Society (London)* 82, 495 (1909).



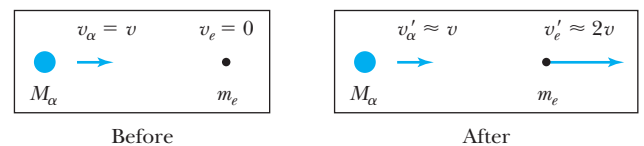
## EXAMPLE 4.1

Geiger and Marsden (1909) observed backward-scattered ( $\theta \geq 90^\circ$ )  $\alpha$  particles when a beam of energetic  $\alpha$  particles was directed at a piece of gold foil as thin as  $6.0 \times 10^{-7}$  m. Assuming an  $\alpha$  particle scatters from an electron in the foil, what is the maximum scattering angle?

**Strategy** We consider elastic scattering between the incident  $\alpha$  particle and an electron in the gold foil. The collision must obey the laws of conservation of momentum and energy. We find the maximum scattering angle corresponding to the maximum momentum change for the  $\alpha$  particle. Assume the incident  $\alpha$  particle has mass  $M_\alpha$  and velocity  $v_\alpha$ , and the mass of the electron is  $m_e$ . The maximum momentum transfer occurs when the  $\alpha$  particle hits the electron (at rest) head-on, as shown in Figure 4.3.

**Solution** Conservation of momentum (nonrelativistically) gives

$$M_\alpha \vec{v}_\alpha = M_\alpha \vec{v}'_\alpha + m_e \vec{v}'_e$$



**Figure 4.3** Schematic diagram (before and after) of an  $\alpha$  particle of speed  $v = v_\alpha$  and mass  $M_\alpha$  making a head-on collision with an electron initially at rest. Because the  $\alpha$  particle is so much more massive than the electron, the  $\alpha$  particle's velocity is hardly reduced.

Because the  $\alpha$  particle is so much more massive than the electron ( $M_\alpha/m_e \approx 4 \times 1837 \approx 7000$ ), the  $\alpha$  particle's velocity is hardly affected and  $v'_\alpha \approx v_\alpha$ . In an elastic collision with such unequal masses,  $v'_e \approx 2v_\alpha$  to conserve both energy and linear momentum (see Problem 3). Thus the maximum momentum change of the  $\alpha$  particle is simply

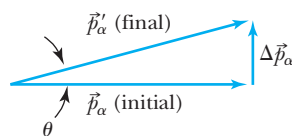
$$\Delta \vec{p}_\alpha = M_\alpha \vec{v}_\alpha - M_\alpha \vec{v}'_\alpha = m_e \vec{v}'_e$$

or for the head-on collision shown in Figure 4.3

$$\Delta p_{\max} = 2m_e v_\alpha$$

Although this maximum momentum change is along the direction of motion, let's determine an upper limit for the angular deviation  $\theta$  by letting  $\Delta p_{\max}$  be perpendicular to the direction of motion as shown in Figure 4.4. (This value of  $\theta$  is larger than can actually be observed because we know that the  $\Delta p_\alpha$  we calculated was for a head-on collision, and the  $\Delta p_\alpha$  for a glancing collision would be smaller.) Thus

$$\theta_{\max} = \frac{\Delta p_\alpha}{p_\alpha} = \frac{2m_e v_\alpha}{M_\alpha v_\alpha} = \frac{2m_e}{M_\alpha} = 2.7 \times 10^{-4} \text{ rad} = 0.016^\circ$$



**Figure 4.4** Vector diagram illustrating the change in momentum  $\Delta p_\alpha$  of the  $\alpha$  particle after scattering from the electron.

Thus it is impossible for an  $\alpha$  particle to be deflected through a large angle by a single encounter with an electron.

### Multiple scattering from electrons

What would happen if an  $\alpha$  particle were scattered by *many* electrons in the target? Multiple scattering is possible, and a calculation for random multiple scattering from  $N$  electrons results in an average scattering angle  $\langle \theta \rangle_{\text{total}} \approx \sqrt{N} \theta$ . The  $\alpha$  particle is as likely to scatter on one side of its direction as the other side for each collision. We can estimate the number of atoms across the thin gold layer of  $6 \times 10^{-7} \text{ m}$  used by Geiger and Marsden.

$$\begin{aligned} \frac{\text{Number of molecules}}{\text{cm}^3} &= \left( 6.02 \times 10^{23} \frac{\text{molecules}}{\text{mol}} \right) \left( \frac{1 \text{ mol}}{197 \text{ g}} \right) \left( 19.3 \frac{\text{g}}{\text{cm}^3} \right) \\ &= 5.9 \times 10^{22} \frac{\text{molecules}}{\text{cm}^3} = 5.9 \times 10^{28} \frac{\text{atoms}}{\text{m}^3} \end{aligned}$$

If there are  $5.9 \times 10^{28} \text{ atoms/m}^3$ , then each atom occupies  $(5.9 \times 10^{28})^{-1} \text{ m}^3$  of space. Assuming the atoms are equidistant, the distance  $d$  between centers is  $d = (5.9 \times 10^{28})^{-1/3} \text{ m} = 2.6 \times 10^{-10} \text{ m}$ . In the foil, then, there are

$$N = \frac{6 \times 10^{-7} \text{ m}}{2.6 \times 10^{-10} \text{ m}} = 2300 \text{ atoms}$$

along the  $\alpha$  particle's path. If we assume the  $\alpha$  particle interacts with one electron from each atom, then

$$\langle \theta \rangle_{\text{total}} = \sqrt{2300} (0.016^\circ) = 0.8^\circ$$

where we have used the result for  $\theta_{\max}$  from Example 4.1. Even if the  $\alpha$  particle scattered from all 79 electrons in each atom of gold,  $\langle \theta \rangle_{\text{total}} = 6.8^\circ$ .

Rutherford reported\* in 1911 that the experimental results were not consistent with  $\alpha$ -particle scattering from the atomic structure proposed by Thomson and that "it seems reasonable to suppose that the deflection through a large angle is due to a single atomic encounter." Rutherford proposed that an atom consisted mostly of empty space with a central charge, either positive or negative. Rutherford wrote in 1911, "Considering the evidence as a whole, it seems simplest to suppose that the atom contains a central charge distributed through a very small volume, and that the large single deflections are due to the central charge as a whole, and not to its constituents." Rutherford worked out the scattering expected for the  $\alpha$  particles as a function of angle, thickness of material,

### Rutherford's atomic model

velocity, and charge. Geiger and Marsden immediately began an experimental investigation of Rutherford's ideas and reported<sup>†</sup> in 1913, "we have completely verified the theory given by Prof. Rutherford." In that same year, Rutherford was the first to use the word **nucleus** for the central charged core and definitely decided that the core (containing most of the mass) was *positively* charged, surrounded by the negative electrons.

The popular conception of an atom today, often depicted as in Figure 4.5, is due to Rutherford. An extremely small positively charged core provides a Coulomb attraction for the negatively charged electrons flying at high speeds around the nucleus; this is the "solar system" or "planetary" model. We now know that the nucleus is composed of positively charged protons and neutral neutrons, each having approximately the same mass, and the electrons do not execute prescribed orbital paths.

## 4.2 Rutherford Scattering

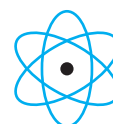
Rutherford's "discovery of the nucleus" laid the foundation for many of today's atomic and nuclear scattering experiments. By means of scattering experiments similar in concept to those of Rutherford and his assistants, scientists have elucidated the electron structure of the atom, the internal structure of the nucleus, and even the internal structures of the nuclear constituents, protons and neutrons. Rutherford's calculations and procedures are well worth studying in some detail because of their applicability to many areas of physical and biological science.

Scattering experiments help us study matter on an atomic scale, which is too small to be observed directly. The material to be studied is bombarded with rapidly moving particles (such as the 5- to 8-MeV  $\alpha$  particles used by Geiger and Marsden) in a well-defined and collimated beam. Although the present discussion is limited to charged-particle beams, the general procedure also applies to neutral particles such as neutrons; only the interaction between the beam particles and the target material is different.

The scattering of charged particles by matter is called **Coulomb** or **Rutherford scattering** when it takes place at low energies, where only the Coulomb force is important. At higher beam energies other forces (for example, nuclear interactions) may become important. A typical scattering experiment is diagrammed in Figure 4.6 (page 132). A charged particle of mass  $m$ , charge  $Z_1e$ , and speed  $v_0$  is incident on the target material or scatterer of charge  $Z_2e$ . The distance  $b$  is called the classical *impact parameter*; it is the closest distance of approach between the beam particle and scatterer if the projectile had continued in a straight line. The angle  $\theta$  between the incident beam direction and the direction of the deflected particle is called the *scattering angle*. Normally detectors are positioned at one or more scattering angles to count the particles scattered into the small cones of solid angle subtended by the detectors (see Figure 4.7, page 132).

Depending on the functional form of the interaction between the particle and the scatterer, there will be a particular relationship between the impact parameter  $b$  and the scattering angle  $\theta$ . In the case of Coulomb scattering between

### The nucleus



**Figure 4.5** Solar or planetary model of the atom. Rutherford proposed that there is a massive, central core with a highly electric positive charge. According to Bohr, the electrons orbit around this nucleus. Although this is a common graphic, we now know this schematic is too simplistic.

### Basic scattering experiments

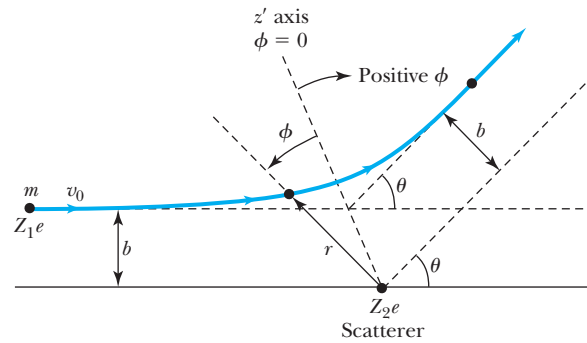
### Rutherford or Coulomb scattering

### Scattering angle

\*E. Rutherford, *Philosophical Magazine* **21**, 669 (1911).

<sup>†</sup>Hans Geiger and Ernest Marsden, *Philosophical Magazine* **25**, 604 (1913).

**Figure 4.6** Representation of Coulomb or Rutherford scattering. The projectile of mass  $m$  and charge  $Z_1e$  scatters from a particle of charge  $Z_2e$  at rest. The parameters  $r$  and  $\phi$ , which describe the projectile's orbit, are defined as shown. The angle  $\phi = 0$  corresponds to the position of closest approach. The impact parameter  $b$  and scattering angle  $\theta$  are also displayed.



a positively charged  $\alpha$  particle and a positively charged nucleus, the trajectories resemble those in Figure 4.7. When the impact parameter is small, the distance of closest approach  $r_{\min}$  is small, and the Coulomb force is large. Therefore, the scattering angle is large, and the particle is repelled backward. Conversely, for large impact parameters the particles never get close together, so the Coulomb force is small and the scattering angle is also small.

An important relationship for any interaction is that between  $b$  and  $\theta$ . We wish to find this dependence for the Coulomb force. We will make the same assumptions as Rutherford:

#### Scattering assumptions

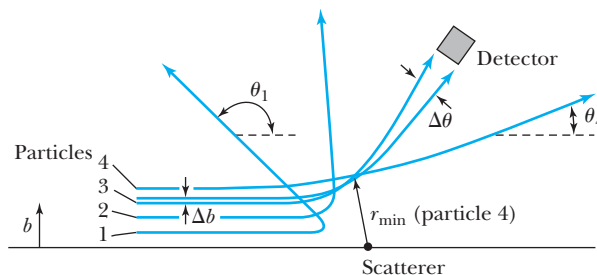
1. The scatterer is so massive that it does not significantly recoil; therefore the initial and final kinetic energies of the  $\alpha$  particle are practically equal.
2. The target is so thin that only a single scattering occurs.
3. The bombarding particle and target scatterer are so small that they may be treated as point masses and charges.
4. Only the Coulomb force is effective.

Assumption 1 means that  $K \equiv \text{K.E.}_{\text{initial}} \approx \text{K.E.}_{\text{final}}$  for the  $\alpha$  particle. For central forces such as the Coulomb force, the angular momentum,  $mv_0b$ , where  $v_0$  is the initial velocity of the particle, is also conserved (see Problem 52). This means that the trajectory of the scattered particle lies in a plane.

We define the instantaneous position of the particle by the angle  $\phi$  and the distance  $r$  from the force center, where  $\phi = 0$  (which defines the  $z'$  axis) when the distance  $r$  is a minimum, as shown in Figure 4.6. The change in momentum is equal to the impulse.

$$\Delta \vec{p} = \int \vec{F}_{\Delta p} dt \quad (4.1)$$

**Figure 4.7** The relationship between the impact parameter  $b$  and scattering angle  $\theta$ . Particles with small impact parameters approach the nucleus most closely ( $r_{\min}$ ) and scatter to the largest angles. Particles within the range of impact parameters  $\Delta b$  will be scattered within  $\Delta\theta$ .





where  $\vec{F}_{\Delta p}$  is the force along the direction of  $\Delta\vec{p}$ . The massive scatterer absorbs this (small) momentum change without gaining any appreciable kinetic energy (no recoil). We use the diagram of Figure 4.8 to show

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i \quad (4.2)$$

where the subscripts  $i$  and  $f$  indicate the initial and final values of the projectile's momentum, respectively. Because  $p_f \approx p_i = mv_0$ , the triangle between  $\vec{p}_f$ ,  $\vec{p}_i$ , and  $\Delta\vec{p}$  is isosceles. We redraw the triangle in Figure 4.8b, indicating the bisector of angle  $\theta$ . The magnitude  $\Delta p$  of the vector  $\Delta\vec{p}$  is now

$$\Delta p = 2mv_0 \sin \frac{\theta}{2} \quad (4.3)$$

The direction of  $\Delta\vec{p}$  is the  $z'$  axis (where  $\phi = 0$ ), so we need the component of  $\vec{F}$  along  $z'$  in Equation (4.1). The Coulomb force  $\vec{F}$  is along the instantaneous direction of the position vector  $\vec{r}$  (unit vector  $\hat{e}_r$ , where  $\hat{\phantom{x}}$  indicates a unit vector).

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{r^2} \hat{e}_r \quad (4.4)$$

and

$$F_{\Delta p} = F \cos \phi \quad (4.5)$$

where  $F_{\Delta p}$  is the component of the force  $\vec{F}$  along the direction of  $\Delta\vec{p}$  that we need.

Substituting the magnitudes from Equations (4.3) and (4.5) into the components of Equation (4.1) along the  $z'$  axis ( $\phi = 0$ ) gives

$$\Delta p = 2mv_0 \sin \frac{\theta}{2} = \int F \cos \phi \, dt = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \int \frac{\cos \phi}{r^2} \, dt$$

The instantaneous angular momentum must be conserved, so

$$mr^2 \frac{d\phi}{dt} = mv_0 b$$

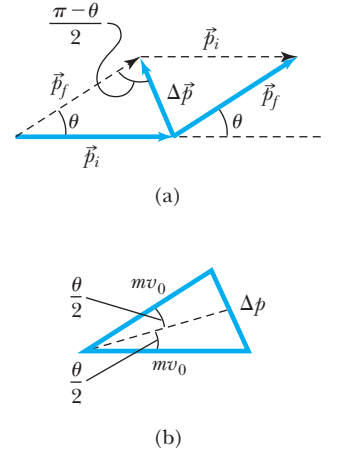
and

$$r^2 = \frac{v_0 b}{d\phi/dt}$$

Therefore,

$$\begin{aligned} 2mv_0 \sin \frac{\theta}{2} &= \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} \int_{\phi_i}^{\phi_f} \frac{\cos \phi}{v_0 b} \frac{d\phi}{dt} \, dt \\ &= \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 v_0 b} \int_{\phi_i}^{\phi_f} \cos \phi \, d\phi \end{aligned}$$

We let the initial angle  $\phi_i$  be on the negative side and the final angle  $\phi_f$  be on the positive side of the  $z'$  axis ( $\phi = 0$ , see Figure 4.6). Then we have  $\phi_i = -\phi_f$ , and  $-\phi_i + \phi_f + \theta = \pi$ , so  $\phi_i = -(\pi - \theta)/2$  and  $\phi_f = +(\pi - \theta)/2$ .



**Figure 4.8** (a) The scattering angle  $\theta$  and momentum change  $\Delta\vec{p}$  are determined from the initial and final values of the  $\alpha$  particle's momentum. (b) Because  $\vec{p}_f$ ,  $\vec{p}_i$ , and  $\vec{p}$  almost exactly form an isosceles triangle, we determine the magnitude of  $\Delta\vec{p}$  by bisecting the angle  $\theta$  and finding the length of the triangle leg opposite the angle  $\theta/2$ .



# Special Topic

## Lord Rutherford of Nelson

**E**rnest Rutherford\* was born of Scottish parents on August 30, 1871, near the town of Nelson, New Zealand. Rutherford, one of eleven children, obtained both a bachelor's and a master's degree in 1894 from the University of New Zealand in Christchurch. He constructed a magnetic detector that was able to receive electromagnetic waves over a distance of 60 feet through walls, quite a feat at the time.

In 1895 Rutherford won a competition to bring able men to British universities and went to work for the famous Professor J. J. Thomson of the Cavendish Laboratory at Cambridge University. Thomson was quite impressed by Rutherford's continuing research on the transmission and detection of "wireless waves" and encouraged him to publish his results and make presentations at scientific meetings. Rutherford also began investigations with Thomson on the effects of X-ray radiation from uranium in various gases. In 1898 Rutherford applied for and received a chaired appointment as Professor of Physics at McGill University in Montreal, Canada.

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\*This account is taken from A. S. Eve, *Rutherford*, New York: Macmillan (1939); and H. A. Boorse and L. Motz, eds., *The World of the Atom*, New York: Basic Books (1966).

Barely 27 years of age, he arrived in Montreal in 1898 to take up his new duties in the Macdonald Physics Laboratories. Rutherford's studies on radioactivity continued, and he had frequent correspondence and occasional visits with scientists from abroad. Early in 1900 he published a paper in *Philosophical Magazine* in which he named *alpha*, *beta*, and *gamma* as the three types of radiation from thorium and uranium. Rutherford attracted the aid of a young research chemist, Frederick Soddy of Oxford, who had arrived at McGill in 1900. Rutherford and Soddy discovered in 1902 that the elements, heretofore considered immutable, actually decayed to other elements. During the next few years Rutherford investigated  $\alpha$  particles and the radioactive decay chains of radium, thorium, and uranium.

In 1907 Rutherford returned to England as professor of physics at the University of Manchester where he did his greatest work. His first success was the proof that  $\alpha$  particles were indeed helium ions. In 1908 Rutherford received word that he had won the Nobel Prize for the work he and Soddy had done, but Rutherford was startled and amused to learn it was in chemistry, not physics.

It was during the next few years that Rutherford carried out his research into the nature of the atom that culminated with his discovery of the nucleus. In 1912 Rutherford wrote to a colleague, "Bohr, a Dane, has pulled out of Cambridge and turned up here to get some experience in radioactive work." That momentous trip resulted in the "Rutherford-Bohr atom."

$$\frac{8\pi\epsilon_0 m v_0^2 b}{Z_1 Z_2 e^2} \sin \frac{\theta}{2} = \int_{-(\pi-\theta)/2}^{+(\pi-\theta)/2} \cos \phi \, d\phi = 2 \cos \frac{\theta}{2}$$

We now solve this equation for the impact parameter  $b$ .

$$b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 m v_0^2} \cot \frac{\theta}{2}$$

This equation becomes

**Relation between  $b$  and  $\theta$**

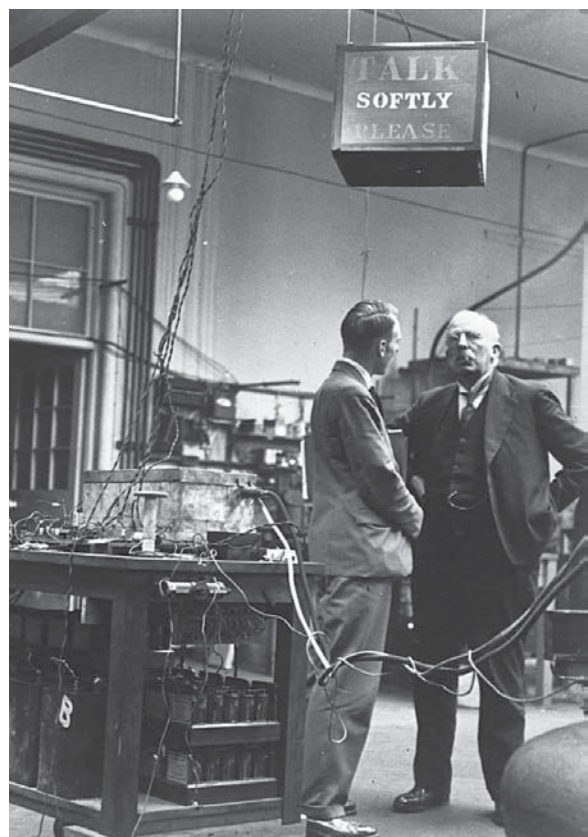
$$b = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \cot \frac{\theta}{2} \quad (4.6)$$



The work of Rutherford at Manchester together with Hans Geiger, Ernest Marsden, and Henry Moseley was to have dramatic consequences. Rutherford used the word *proton* to describe the fast hydrogen nuclei produced when he bombarded hydrogen and nitrogen with fast  $\alpha$  particles.

World War I broke up the family of research students working at Manchester, and in 1919 Rutherford accepted the Cavendish Professorship at Cambridge, the post just vacated by J. J. Thomson, who remained as Master of Trinity College. Being the successor at Cambridge to Maxwell, Rayleigh, and Thomson was no small feat, and Rutherford continued to do important research until the time of his death in 1937 at age 66. Rutherford was knighted in 1914, and in 1931 he was made a baron, choosing the town of Nelson near his boyhood home to become “Lord Rutherford of Nelson.” He was the greatest experimental physicist of his day, yet he was said to have “never made an enemy and never lost a friend.”

**Figure A** Ernest Rutherford, on the right, talking with J. A. Ratcliffe in the Cavendish Laboratory in 1936. The sign above Rutherford reads “TALK SOFTLY PLEASE” because the detectors being used were very sensitive to vibrations and noise. Rutherford, whose deep booming voice disturbed the detectors more than anyone else’s, didn’t seem to think the warning applied to him and was in a loud conversation when this photo was taken.



AFP/Emilio Segrè Visual Archives

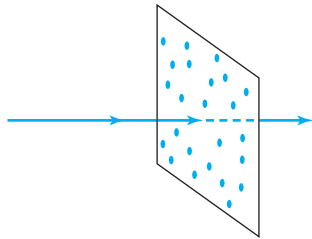
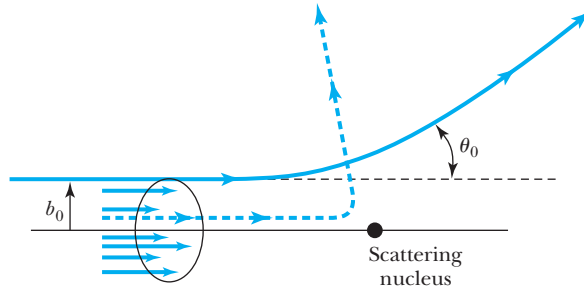
where  $K = mv_0^2/2$  is the kinetic energy of the bombarding particle. This is the fundamental relationship between the impact parameter  $b$  and scattering angle  $\theta$  that we have been seeking for the Coulomb force.

We are not able to select individual impact parameters  $b$  in a given experiment. When we put a detector at a particular angle  $\theta$ , we cover a finite  $\Delta\theta$ , which corresponds to a range of impact parameters  $\Delta b$ . The bombarding particles are incident at varied impact parameters all around the scatterer as shown in Figure 4.9 (page 136). All the particles with impact parameters less than  $b_0$  will be scattered at angles greater than  $\theta_0$ . Any particle with an impact parameter inside the area of the circle of area  $\pi b_0^2$  (radius  $b_0$ ) will be similarly scattered. For the case of Coulomb scattering, we denote the cross section by the symbol  $\sigma$ , where

$$\sigma = \pi b^2 \quad (4.7)$$

is the cross section for scattering through an angle  $\theta$  or more. The cross section

**Figure 4.9** All particles with impact parameters less than  $b_0$  will scatter at angles greater than  $\theta_0$ .



**Figure 4.10** The target is assumed to be so thin that all nuclei are exposed to the bombarding particles. No nucleus is hidden behind another.

$\sigma$  is related to the *probability* for a particle being scattered by a nucleus. If we have a target foil of thickness  $t$  with  $n$  atoms/volume, the number of target nuclei per unit area is  $nt$ . Because we assumed a thin target of area  $A$  and all nuclei are exposed as shown in Figure 4.10, the number of target nuclei is simply  $ntA$ . The value of  $n$  is the density  $\rho$  (g/cm<sup>3</sup>) times Avogadro's number  $N_A$  (molecules/mol) times the number of atoms/molecule  $N_M$  divided by the gram-molecular weight  $M_g$  (g/mol).

$$n = \frac{\rho \left( \frac{\text{g}}{\text{cm}^3} \right) N_A \left( \frac{\text{molecules}}{\text{mol}} \right) N_M \left( \frac{\text{atoms}}{\text{molecule}} \right)}{M_g \left( \frac{\text{g}}{\text{mol}} \right)} = \frac{\rho N_A N_M}{M_g} \frac{\text{atoms}}{\text{cm}^3} \quad (4.8)$$

The number of scattering nuclei per unit area is  $nt$ .

$$nt = \frac{\rho N_A N_M t}{M_g} \frac{\text{atoms}}{\text{cm}^2} \quad (4.9)$$

If we have a foil of area  $A$ , the number of target nuclei  $N_s$  is

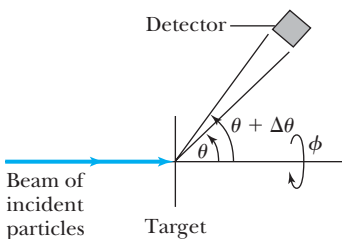
$$N_s = ntA = \frac{\rho N_A N_M t A}{M_g} \text{atoms} \quad (4.10)$$

The probability of the particle being scattered is equal to the total target area exposed for all the nuclei divided by the total target area  $A$ . If  $\sigma$  is the cross section for each nucleus, then  $ntA\sigma$  is the total area exposed by the target nuclei, and the fraction of incident particles scattered by an angle of  $\theta$  or greater is

$$f = \frac{\text{target area exposed by scatterers}}{\text{total target area}} = \frac{ntA\sigma}{A} = nt\sigma = nt\pi b^2 \quad (4.11)$$

$$f = \pi nt \left( \frac{Z_1 Z_2 e^2}{8\pi \epsilon_0 K} \right)^2 \cot^2 \frac{\theta}{2} \quad (4.12)$$

In a typical experiment, however, a detector is positioned over a range of angles from  $\theta$  to  $\theta + \Delta\theta$ , as shown in Figure 4.11. Thus we need to find the number of particles scattered between  $\theta$  and  $\theta + d\theta$  that corresponds to incident particles with impact parameters between  $b$  and  $b + db$  as displayed in Figure 4.12. The



**Figure 4.11** In most experiments, the detectors cover only a small angular range, from  $\theta$  to  $\theta + \Delta\theta$ , and measurements are made for different  $\theta$ . The detector also usually covers a small angular range in  $\phi$  (angle around beam direction). Because there is usually symmetry about the beam axis, the  $\phi$  angle is not normally varied.

### EXAMPLE 4.2

Find the fraction of 7.7-MeV  $\alpha$  particles that is deflected at an angle of  $90^\circ$  or more from a gold foil of  $10^{-6}$  m thickness.

**Strategy** We can use Equation (4.12) to calculate the fraction, but first we need to calculate  $n$ , the number of atoms/cm<sup>3</sup>. We do that using Equation (4.8).

**Solution** The density of gold is 19.3 g/cm<sup>3</sup>, and the atomic weight is 197 u. Equation (4.8) determines  $n$ .

$$n = \frac{\left(19.3 \frac{\text{g}}{\text{cm}^3}\right) \left(6.02 \times 10^{23} \frac{\text{molecules}}{\text{mol}}\right) \left(1 \frac{\text{atom}}{\text{molecule}}\right)}{197 \text{ g/mol}}$$

$$= 5.90 \times 10^{22} \frac{\text{atoms}}{\text{cm}^3} = 5.90 \times 10^{28} \frac{\text{atoms}}{\text{m}^3}$$

We insert this value of  $n$  into Equation (4.12) and find

$$f = \pi \left(5.90 \times 10^{28} \frac{\text{atoms}}{\text{m}^3}\right) (10^{-6} \text{ m})$$

$$\times \left[ \frac{(79)(2)(1.6 \times 10^{-19} \text{ C})^2 (9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)}{2(7.7 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})} \right]^2$$

$$\times (\cot 45^\circ)^2$$

$$= 4 \times 10^{-5}$$

One  $\alpha$  particle in 25,000 is deflected by  $90^\circ$  or greater.

fraction of the incident particles scattered between  $\theta$  and  $\theta + d\theta$  is  $df$ . The derivative of Equation (4.12) is

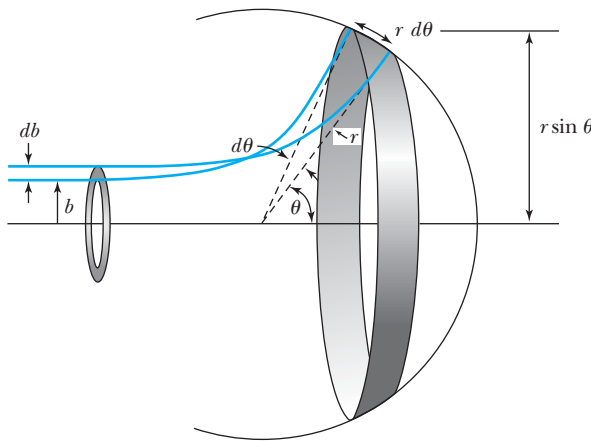
$$df = -\pi n t \left( \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \right)^2 \cot \frac{\theta}{2} \csc^2 \frac{\theta}{2} d\theta$$

If the total number of incident particles is  $N_i$ , the number of particles scattered into the ring of angular width  $d\theta$  is  $N_i |df|$ . The area  $dA$  into which the particles scatter is  $(r d\theta)(2\pi r \sin \theta) = 2\pi r^2 \sin \theta d\theta$ . Therefore, the number of particles scattered per unit area,  $N(\theta)$ , into the ring at scattering angle  $\theta$  is

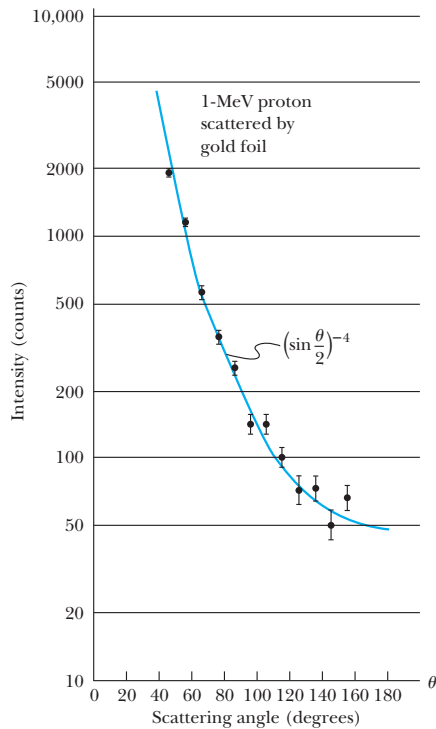
$$N(\theta) = \frac{N_i |df|}{dA} = \frac{N_i \pi n t \left( \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \right)^2}{2\pi r^2 \sin \theta d\theta} \cot \frac{\theta}{2} \csc^2 \frac{\theta}{2} d\theta$$

$$N(\theta) = \frac{N_i n t}{16} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{Z_1^2 Z_2^2}{r^2 K^2 \sin^4(\theta/2)}$$

(4.13) **Rutherford scattering equation**



**Figure 4.12** Particles over the range of impact parameters from  $b$  to  $b + db$  will scatter into the angular range  $\theta$  to  $\theta + d\theta$  (with  $db$  positive,  $d\theta$  will be negative).



Equation (4.13) is the famous **Rutherford scattering equation**. The important points are the following:

1. The scattering is proportional to the square of the atomic number of both the incident particle ( $Z_1$ ) and the target scatterer ( $Z_2$ ).
2. The number of scattered particles is inversely proportional to the square of the kinetic energy  $K$  of the incident particle.
3. The scattering is inversely proportional to the fourth power of  $\sin(\theta/2)$ , where  $\theta$  is the scattering angle.
4. The scattering is proportional to the target thickness for thin targets.

These specific predictions by Rutherford in 1911 were confirmed experimentally by Geiger and Marsden in 1913. The angular dependence is particularly characteristic and can be verified in a well-equipped undergraduate physics laboratory, as we see from some actual data shown in Figure 4.13.

**Figure 4.13** Results of undergraduate laboratory experiment of scattering 1-MeV protons from a gold target. The solid line shows the  $1/\sin^4(\theta/2)$  angular dependence of the data, verifying Rutherford's calculation.

### EXAMPLE 4.3

Calculate the fraction per  $\text{mm}^2$  area of 7.7-MeV  $\alpha$  particles scattered at  $45^\circ$  from a gold foil of thickness  $2.1 \times 10^{-7} \text{ m}$  at a distance of 1.0 cm from the target.

**Strategy** We use Equation (4.13) to determine the fraction per unit area  $N(\theta)/N_i$ . We calculated  $n = 5.90 \times 10^{28} \text{ atoms/m}^3$  in Example 4.2.

**Solution** We insert the values into Equation (4.13).

$$\begin{aligned} \frac{N(\theta)}{N_i} &= \frac{\left(5.90 \times 10^{28} \frac{\text{atoms}}{\text{m}^3}\right)(2.1 \times 10^{-7} \text{ m}) \left[ (1.6 \times 10^{-19} \text{ C})^2 \left( 9 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \right]^2}{16} \\ &\quad \times \frac{(2)^2 (79)^2}{(1.0 \times 10^{-2} \text{ m})^2 \left( 7.7 \text{ MeV} \times \frac{10^6 \text{ eV}}{\text{MeV}} \times \frac{1.6 \times 10^{-19} \text{ J}}{\text{eV}} \right)^2} \frac{1}{\sin^4(45^\circ/2)} \\ \frac{N(\theta)}{N_i} &= 3.2 \times 10^{-1} \text{ m}^{-2} = 3.2 \times 10^{-7} \text{ mm}^{-2} \end{aligned}$$

This is the theoretical basis for the experiment performed by Geiger and Marsden in 1913 to check the validity of Rutherford's calculation. Our calculated result agrees

with their experimental result of  $3.7 \times 10^{-7} \text{ mm}^{-2}$  when the experimental uncertainty is taken into account.



### CONCEPTUAL EXAMPLE 4.4

How can we find the distance of closest approach between a bombarding particle and a target scatterer of like charge?

**Solution** We can find this distance of closest approach for a given kinetic energy  $K$  and impact parameter  $b$ . The minimum separation occurs for a head-on collision. The bombarding particle turns around and scatters backward at  $180^\circ$ . At the instant the particle turns around, the entire kinetic energy has been converted into Coulomb potential energy. By setting the original (maximum) kinetic energy equal to

the Coulomb potential energy when  $r = r_{\min}$ , we can then solve the resulting equation for  $r_{\min}$ . Let  $K$  be the original kinetic energy of the bombarding particle.

$$K = \frac{(Z_1 e)(Z_2 e)}{4\pi\epsilon_0 r_{\min}} \quad (4.14)$$

We solve this equation to determine  $r_{\min}$ .

$$r_{\min} = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \quad (4.15)$$



### EXAMPLE 4.5

Rutherford found deviations from his Equation (4.13) at backward angles when he scattered 7.7-MeV  $\alpha$  particles ( $Z_1 = 2$ ) on aluminum ( $Z_2 = 13$ ). He suspected this was because the  $\alpha$  particle might be affected by approaching the nucleus so closely. Estimate the size of the nucleus based on these data.

**Strategy** We have just determined the distance of closest approach for the  $\alpha$  particle, which occurs for a head-on collision or scattering angle of  $180^\circ$ . We propose that the  $r_{\min}$  in this case is close to the sum of the  $\alpha$  particle ( ${}^4\text{He}$  nucleus) radius and the aluminum nuclear radius.

**Solution** We insert the values for the  $\alpha$  particle incident on aluminum into Equation (4.15) to find  $r_{\min}$ .

$$\begin{aligned} r_{\min} &= \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 K} \\ &= \frac{(2)(13)(1.60 \times 10^{-19} \text{ C})^2 (8.99 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2})}{(7.7 \text{ MeV})(1.60 \times 10^{-13} \text{ J/MeV})} \\ &= 4.9 \times 10^{-15} \text{ m} \end{aligned}$$

We find the sum of the  ${}^4\text{He}$  and aluminum nuclear radii to be about  $5 \times 10^{-15} \text{ m}$ .

We will see in Chapter 12 that aluminum's nuclear radius is about twice as large as that of  ${}^4\text{He}$ , and our approximate result here is in fair agreement with modern data. We now know that nuclear radii vary from  $1 \times 10^{-15}$  to  $8 \times 10^{-15} \text{ m}$ . Thus when  $\alpha$  particles scatter from aluminum, an  $\alpha$  particle may approach the nucleus close enough to be affected by the nuclear force (see Chapter 12).

## 4.3 The Classical Atomic Model

After Rutherford presented his calculations of charged-particle scattering in 1911 and the experimental verification by his group in 1913, it was generally conceded that the atom consisted of a small, massive, positively charged “nucleus” surrounded by moving electrons. Thomson's plum-pudding model was definitively excluded by the data. Actually, Thomson had previously considered a planetary model resembling the solar system (in which the planets move in elliptical orbits about the sun) but rejected it because, although both gravitational and Coulomb forces vary inversely with the square of the distance, the planets *attract* one another while orbiting around the sun, whereas the electrons would *repel* one another. Thomson considered this to be a fatal flaw from his knowledge of planetary theory.

In order to examine the failure of the planetary model, let us examine the simplest atom, hydrogen. We will assume circular electron orbits for simplicity rather than the more general elliptical ones. The force of attraction on the electron due to the nucleus (charge =  $+e$ ) is

$$\vec{F}_e = \frac{-1}{4\pi\epsilon_0} \frac{e^2}{r^2} \hat{e}_r \quad (4.16)$$

where the negative sign indicates the force is attractive and  $\hat{e}_r$  is a unit vector in the direction from the nucleus to the electron. This electrostatic force provides the centripetal force needed for the electron to move in a circular orbit at constant speed. Its radial acceleration is

$$a_r = \frac{v^2}{r} \quad (4.17)$$

where  $v$  is the tangential velocity of the electron. Newton's second law now gives

$$\frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} = \frac{mv^2}{r} \quad (4.18)$$

and

$$v = \frac{e}{\sqrt{4\pi\epsilon_0 mr}} \quad (4.19)$$

where we are using  $m$  without a subscript to be the electron's mass. When it is not clear what particle  $m$  refers to, we write the electron mass as  $m_e$ .



### EXAMPLE 4.6

Are we justified in using a nonrelativistic treatment for the speed of an electron in the hydrogen atom?

**Strategy** We use Equation (4.19) to calculate the electron's speed. If it is less than 1% of the speed of light, we are justified in using a nonrelativistic treatment. One difficulty is knowing the radius of the hydrogen atom. The size of an atom was thought to be about  $10^{-10}$  m, so we let  $r = 0.5 \times 10^{-10}$  m to estimate the electron's velocity.

**Solution** Equation (4.19) gives

$$\begin{aligned} v &\approx \frac{(1.6 \times 10^{-19} \text{ C})(9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)^{1/2}}{(9.11 \times 10^{-31} \text{ kg})^{1/2}(0.5 \times 10^{-10} \text{ m})^{1/2}} \\ &\approx 2.2 \times 10^6 \text{ m/s} < 0.01c \end{aligned}$$

This justifies a nonrelativistic treatment.

The kinetic energy of the system is due to the electron,  $K = mv^2/2$ . The nucleus is so massive compared with the electron ( $m_{\text{proton}} = 1836m$ ) that the nucleus may be considered to be at rest. The potential energy  $V$  is simply  $-e^2/4\pi\epsilon_0 r$ , so the total mechanical energy is

$$E = K + V = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad (4.20)$$

If we substitute for  $v$  from Equation (4.19), we have

$$E = \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = \frac{-e^2}{8\pi\epsilon_0 r} \quad (4.21)$$

The total energy is negative, indicating a bound system.



Thus far, the classical atomic model seems plausible. The problem arises when we consider that the electron is accelerating due to its circular motion about the nucleus. We know from classical electromagnetic theory that an accelerated electric charge continuously radiates energy in the form of electromagnetic radiation. If the electron is radiating energy, then the total energy  $E$  of the system, Equation (4.21), must decrease continuously. In order for this to happen, the radius  $r$  must decrease. The electron will continuously radiate energy as the electron orbit becomes smaller and smaller until the electron crashes into the nucleus! This process, displayed in Figure 4.14, would occur in about  $10^{-9}$  s (see Problem 18).

Thus the classical theories of Newton and Maxwell, which had served Rutherford so well in his analysis of  $\alpha$ -particle scattering and had thereby enabled him to discover the nucleus, also led to the failure of the planetary model of the atom. Physics had reached a decisive turning point like that encountered in 1900 with Planck's revolutionary hypothesis of the quantum behavior of radiation. In the early 1910s, however, the answer would not be long in coming, as we shall see in the next section.

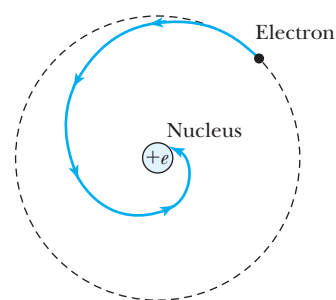
## 4.4 The Bohr Model of the Hydrogen Atom

Shortly after receiving his Ph.D. from the University of Copenhagen in 1911, the 26-year-old Danish physicist Niels Bohr traveled to Cambridge University to work with J. J. Thomson. He subsequently went to the University of Manchester to work with Rutherford for a few months in 1912 where he became particularly involved in the mysteries of the new Rutherford model of the atom. Bohr returned to the University of Copenhagen in the summer of 1912 with many questions about atomic structure. Like several others, he believed that a fundamental length about the size of an atom ( $10^{-10}$  m) was needed for an atomic model. This fundamental length might somehow be connected to Planck's new constant  $h$ . The pieces finally came together during the fall and winter of 1912-1913 when Bohr learned of new precise measurements of the hydrogen spectrum and of the empirical formulas describing them. He set out to find a fundamental basis from which to derive the Balmer formula [Equation (3.12)], the Rydberg equation [Equation (3.13)], and Ritz's combination principles (see Problem 19).

Bohr was well acquainted with Planck's work on the quantum nature of radiation. Like Einstein, Bohr believed that quantum principles should govern more phenomena than just the blackbody spectrum. He was impressed by Einstein's application of the quantum theory to the photoelectric effect and to the specific heat of solids (see Chapter 9 for the latter) and wondered how the quantum theory might affect atomic structure.

In 1913, following several discussions with Rutherford during 1912 and 1913, Bohr published the paper\* "On the Constitution of Atoms and Molecules." He subsequently published several other papers refining and restating his "assumptions" and their predicted results. We will generally follow Bohr's papers in our discussion.

**Planetary model is doomed.**



**Figure 4.14** The electromagnetic radiation of an orbiting electron in the planetary model of the atom will cause the electron to spiral inward until it crashes into the nucleus.

\*Niels Bohr, *Philosophical Magazine* **26**, 1 (1913) and **30**, 394 (1915).

Bohr assumed that electrons moved around a massive, positively charged nucleus. We will assume for simplicity (as did Bohr at first) that the electron orbits are circular rather than elliptical and that the nuclear mass is so much greater than the electron's mass that it may be taken to be infinite. The electron has charge  $-e$  and mass  $m$  and revolves around a nucleus of charge  $+e$  in a circle of radius  $a$ . The size of the nucleus is small compared with the atomic radius  $a$ .

Bohr's model may best be summarized by the following “general assumptions” of his 1915 paper:

- A. Certain “stationary states” exist in atoms, which differ from the classical stable states in that the orbiting electrons do not continuously radiate electromagnetic energy. The stationary states are states of definite total energy.
- B. The emission or absorption of electromagnetic radiation can occur only in conjunction with a transition between two stationary states. The frequency of the emitted or absorbed radiation is proportional to the difference in energy of the two stationary states (1 and 2):

$$E = E_1 - E_2 = hf$$

where  $h$  is Planck's constant.

- C. The dynamical equilibrium of the system in the stationary states is governed by classical laws of physics, but these laws do not apply to transitions between stationary states.
- D. The mean value  $K$  of the kinetic energy of the electron-nucleus system is given by  $K = nhf_{\text{orb}}/2$ , where  $f_{\text{orb}}$  is the frequency of rotation. For a circular orbit, Bohr pointed out that this assumption is equivalent to the angular momentum of the system in a stationary state being an integral multiple of  $h/2\pi$ . (This combination of constants occurs so often that we give it a separate symbol,  $\hbar \equiv h/2\pi$ , pronounced “h bar.”)

These four assumptions were all that Bohr needed to derive the Rydberg equation. Bohr believed that Assumptions A and C were self-evident because atoms were stable: atoms exist and do not continuously radiate energy (therefore Assumption A). It also seemed that the classical laws of physics could not explain the observed behavior of the atom (therefore Assumption C).

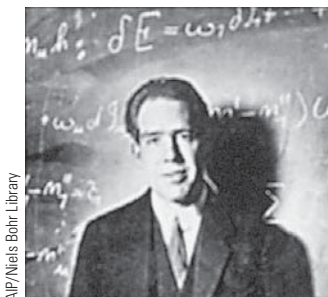
Bohr later stated (1915) that Assumption B “appears to be necessary in order to account for experimental facts.” Assumption D was the hardest for Bohr's critics to accept. It is central to the derivation of the binding energy of the hydrogen atom in terms of fundamental constants; hence Bohr restated and defended it in several ways in his papers. We have emphasized here the quantization of angular momentum aspect of Assumption D. This leads to a particularly simple derivation of the Rydberg equation.

Bohr chose his four assumptions to keep as much as possible of classical physics by introducing just those new ideas that were needed to explain experimental data. Bohr's recognition that something new was needed and his attempt to tie this to Planck's quantum hypothesis represented an advance in understanding perhaps even greater than Einstein's theory of the photoelectric effect.

Let us now proceed to derive the Rydberg equation using Bohr's assumptions. The total energy (potential plus kinetic) of a hydrogen atom was derived previously in Equation (4.21). For circular motion, the magnitude of the angular momentum  $L$  of the electron is

$$L = |\vec{r} \times \vec{p}| = mvr$$

### Bohr's general assumptions



AP/Niels Bohr Library

**Niels Bohr** (1885–1962) was more than just a discoverer of modern physics theories. Born in Denmark, he was the son of a university professor and began high school at about the time Planck announced his results. After his education in Denmark, Bohr traveled to England in 1911 where he worked first with J. J. Thomson and later with Ernest Rutherford. Bohr nurtured many young theoretical physicists in his Institute of Theoretical Physics (now called the Niels Bohr Institute) formed in Copenhagen in 1921, the year before Bohr won the Nobel Prize.

Assumption D states this should equal  $n\hbar$ :

$$L = mvr = n\hbar \quad (4.22a)$$

where  $n$  is an integer called the **principal quantum number**. We solve the previous equation for the velocity and obtain

**Principal quantum number**

$$v = \frac{n\hbar}{mr} \quad (4.22b)$$

Equation (4.19) yields an independent relation between  $v$  and  $r$ . If we determine  $v^2$  from Equations (4.19) and (4.22b) and set them equal, we find

$$v^2 = \frac{e^2}{4\pi\epsilon_0 mr} = \frac{n^2\hbar^2}{m^2 r^2} \quad (4.23)$$

From Equation (4.23) we see that only certain values of  $r$  are allowed.

$$r_n = \frac{4\pi\epsilon_0 n^2 \hbar^2}{me^2} \equiv n^2 a_0 \quad (4.24)$$

where the **Bohr radius**  $a_0$  is given by

$$\begin{aligned} a_0 &= \frac{4\pi\epsilon_0 \hbar^2}{me^2} \\ &= \frac{(1.055 \times 10^{-34} \text{ J}\cdot\text{s})^2}{\left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(9.11 \times 10^{-31} \text{ kg})(1.6 \times 10^{-19} \text{ C})^2} \\ &= 0.53 \times 10^{-10} \text{ m} \end{aligned}$$

**Bohr radius**

Notice that the smallest diameter of the hydrogen atom is  $2r_1 = 2a_0 \approx 10^{-10} \text{ m}$ , the suspected (now known) size of the hydrogen atom! Bohr had found the fundamental length  $a_0$  that he sought in terms of the fundamental constants  $\epsilon_0$ ,  $\hbar$ ,  $e$ , and  $m$ . This fundamental length is determined for the value  $n = 1$ . Note from Equation (4.24) that the atomic radius is now quantized. The quantization of various physical values arises because of the principal quantum number  $n$ . The value  $n = 1$  gives the radius of the hydrogen atom in its lowest energy state (called the “ground” state). The values of  $n > 1$  determine other possible radii where the hydrogen atom is in an “excited” state.

The energies of the stationary states can now be determined from Equations (4.21) and (4.24).

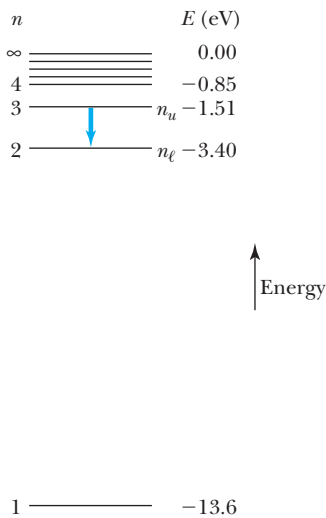
$$E_n = -\frac{e^2}{8\pi\epsilon_0 r_n} = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2} \equiv -\frac{E_0}{n^2} \quad (4.25)$$

**Quantized energy states**

The lowest energy state ( $n = 1$ ) is  $E_1 = -E_0$  where

$$E_0 = \frac{e^2}{8\pi\epsilon_0 a_0} = \frac{e^2}{(8\pi\epsilon_0)} \frac{me^2}{4\pi\epsilon_0 \hbar^2} = \frac{me^4}{2\hbar^2 (4\pi\epsilon_0)^2} = 13.6 \text{ eV} \quad (4.26)$$

This is the experimentally measured ionization energy of the hydrogen atom. Bohr’s Assumptions C and D imply that the atom can exist only in “stationary states” with definite, quantized energies  $E_n$ , displayed in the **energy-level diagram** of Figure 4.15 (page 144). Emission of a quantum of light occurs when the atom is in an excited state (quantum number  $n = n_u$ ) and decays to a lower energy



**Figure 4.15** The energy-level diagram of the hydrogen atom. The principal quantum numbers  $n$  are shown on the left, with the energy of each level indicated on the right. The ground-state energy is  $-13.6$  eV; negative total energy indicates a bound, attractive system. When an atom is in an excited state (for example,  $n_u = 3$ ) and decays to a lower stationary state (for example,  $n_\ell = 2$ ), the hydrogen atom must emit the energy difference in the form of electromagnetic radiation; that is, a photon emerges.

#### Bohr predicted new hydrogen wavelengths

#### Absorption and emission spectrum

state ( $n = n_\ell$ ). A transition between two energy levels is schematically illustrated in Figure 4.15. According to Assumption B we have

$$hf = E_u - E_\ell \quad (4.27)$$

where  $f$  is the frequency of the emitted light quantum (photon). Because  $\lambda f = c$ , we have

$$\begin{aligned} \frac{1}{\lambda} &= \frac{f}{c} = \frac{E_u - E_\ell}{hc} \\ &= \frac{-E_0}{hc} \left( \frac{1}{n_u^2} - \frac{1}{n_\ell^2} \right) = \frac{E_0}{hc} \left( \frac{1}{n_\ell^2} - \frac{1}{n_u^2} \right) \end{aligned} \quad (4.28)$$

where

$$\frac{E_0}{hc} = \frac{me^4}{4\pi c \hbar^3 (4\pi\epsilon_0)^2} \equiv R_\infty \quad (4.29)$$

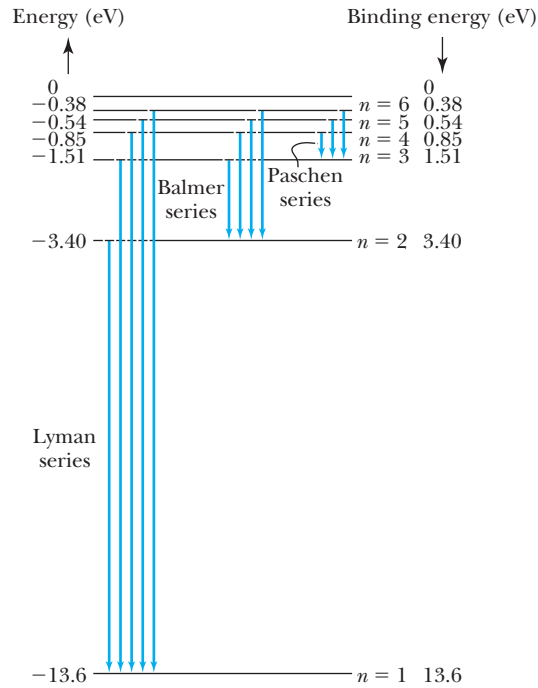
This constant  $R_\infty$  is called the **Rydberg constant** (for an infinite nuclear mass). Equation (4.28) becomes

$$\frac{1}{\lambda} = R_\infty \left( \frac{1}{n_\ell^2} - \frac{1}{n_u^2} \right) \quad (4.30)$$

which is similar to the Rydberg equation (3.13). The value of  $R_\infty = 1.097373 \times 10^7 \text{ m}^{-1}$  calculated from Equation (4.29) agrees well with the experimental values given in Chapter 3, and we will obtain an even more accurate result in the next section.

Bohr's model predicts the frequencies (and wavelengths) of all possible transitions in atomic hydrogen. Several of the series are shown in Figure 4.16. The Lyman series represents transitions to the lowest state with  $n_\ell = 1$ ; the Balmer series results from downward transitions to the stationary state  $n_\ell = 2$ ; and the Paschen series represents transitions to  $n_\ell = 3$ . As mentioned in Section 3.3, not all of these series were known experimentally in 1913, but it was clear that Bohr had successfully accounted for the known spectral lines of hydrogen.

The frequencies of the photons in the emission spectrum of an element are directly proportional to the differences in energy of the stationary states. When we pass white light (composed of all visible photon frequencies) through atomic hydrogen gas, we find that certain frequencies are absent. This pattern of dark lines is called an **absorption spectrum**. The missing frequencies are *precisely* the ones observed in the corresponding **emission spectrum**. In absorption, certain photons of light are absorbed, giving up energy to the atom and enabling the electron to move from a lower ( $\ell$ ) to a higher ( $u$ ) stationary state. Equations (4.27) and (4.30) describe the frequencies and wavelengths of the absorbed photons. The atom will remain in the excited state for only a short time (on the order of  $10^{-10}$  s) before emitting a photon and returning to a lower stationary state. Thus, at ordinary temperatures practically all hydrogen atoms exist in the lowest possible energy state,  $n = 1$ , and only the absorption spectral lines of the Lyman series are normally observed. However, these lines are not in the visible region. The sun produces electromagnetic radiation over a wide range of wavelengths, including the visible region. When sunlight passes through the sun's



**Figure 4.16** Transitions between many of the stationary states in the hydrogen atom are indicated. Transitions (ultraviolet) to the  $n = 1$  state from the higher-lying states are called the Lyman series. The transitions shown to the  $n = 2$  state (Balmer series) were discovered first because they are in the visible wavelength range. The Paschen series (transitions to  $n = 3$ ) are in the infrared. The energies of each state as well as the binding energies are noted.

outer atmosphere, its hydrogen atoms absorb the wavelengths of the Balmer series (visible region), and the absorption spectrum has dark lines at the known wavelengths of the Balmer series.

We can determine the electron's velocity in the Bohr model from Equations (4.22b) and (4.23).

$$v_n = \frac{n\hbar}{mr_n} = \frac{n\hbar}{mn^2a_0} = \frac{1}{n} \frac{\hbar}{ma_0} \quad (4.31)$$

or

$$v_n = \frac{1}{n} \frac{e^2}{4\pi\epsilon_0\hbar}$$

The value of  $v_1$  is  $\hbar/ma_0 = 2.2 \times 10^6$  m/s, which is less than 1% of the speed of light. We define the dimensionless quantity ratio of  $v_1$  to  $c$  as

$$\alpha \equiv \frac{v_1}{c} = \frac{\hbar}{ma_0c} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \quad (4.32)$$

**Fine structure constant**

This ratio is called the **fine structure constant**. It appears often in atomic structure calculations.

We insert a word of caution at this point. Bohr's atomic model of quantized energy levels represented a significant step forward in understanding the structure of the atom. Although it had many successes, we know now that, in principle, it is wrong. We will discuss some of its successes and failures in the next section and discuss the correct quantum theory in Chapter 6. Nevertheless, Bohr's atomic model is useful in our first attempt in understanding the structure of the atom.



### EXAMPLE 4.7

Determine the longest and shortest wavelengths observed in the Paschen series for hydrogen. Which are visible?

**Strategy** We use Equation (4.30) to determine the wavelengths. The lowest energy state  $n_\ell$  (see Figure 4.16) is 3. We calculate the wavelengths for  $n_u = 4$  and  $\infty$  to obtain the extreme longest (maximum) and shortest (minimum) wavelengths.

**Solution** We insert the values of  $n$  into Equation (4.30) with the Rydberg constant to obtain

$$\frac{1}{\lambda_{\max}} = (1.0974 \times 10^7 \text{ m}^{-1}) \left( \frac{1}{3^2} - \frac{1}{4^2} \right) = 5.335 \times 10^5 \text{ m}^{-1}$$

$$\lambda_{\max} = 1875 \text{ nm}$$

The maximum wavelength, 1875 nm, is not visible and is in the infrared.

$$\frac{1}{\lambda_{\min}} = (1.0974 \times 10^7 \text{ m}^{-1}) \left( \frac{1}{3^2} - \frac{1}{\infty} \right) = 1.219 \times 10^6 \text{ m}^{-1}$$

$$\lambda_{\min} = 820 \text{ nm}$$

The minimum wavelength, 820 nm, is not visible and also in the infrared.

## The Correspondence Principle

Early in the 1900s physicists had trouble reconciling well-known and well-understood classical physics results with the new quantum ones. Sometimes completely different results were valid in their own domains. For example, there were two radiation laws: one used classical electrodynamics to determine the properties of radiation from an accelerated charge, but another explanation was expressed in Bohr's atomic model. Physicists proposed various kinds of correspondence principles to relate the new modern results with the old classical ones that had worked so well in their own domain. In his 1913 paper Bohr proposed perhaps the best *correspondence principle* to guide physicists in developing new theories. This principle was refined several times over the next few years.

### Bohr's correspondence principle

*Bohr's correspondence principle: In the limits where classical and quantum theories should agree, the quantum theory must reduce to the classical result.*

To illustrate this principle, let us examine the predictions of the two radiation laws. The frequency of the radiation produced by the atomic electrons in the Bohr model of the hydrogen atom should agree with that predicted by classical electrodynamics in a region where the finite size of Planck's constant is unimportant—for large quantum numbers  $n$  where quantization effects are minimized. To see how this works we recall that classically the frequency of the radiation emitted is equal to the orbital frequency  $f_{\text{orb}}$  of the electron around the nucleus:

$$f_{\text{classical}} = f_{\text{orb}} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \frac{v}{r} \quad (4.33a)$$

where for circular motion the angular velocity is  $\omega = v/r$ . If we substitute for  $v$  from Equation (4.19), we find

$$f_{\text{classical}} = \frac{1}{2\pi} \left( \frac{e^2}{4\pi\epsilon_0 m r^3} \right)^{1/2} \quad (4.33b)$$

We make the connection to the Bohr model by inserting the orbital radius  $r$  from Equation (4.24) into Equation (4.33b). We then know the classical



frequency in terms of fundamental constants and the principal quantum number  $n$ .

$$f_{\text{classical}} = \frac{me^4}{4\epsilon_0^2 h^3} \frac{1}{n^3} \quad (4.34)$$

In the Bohr model, the nearest we can come to continuous radiation is a cascade of transitions from a level with principal quantum number  $n + 1$  to the next lowest and so on:

$$n + 1 \rightarrow n \rightarrow n - 1 \rightarrow \dots$$

The frequency of the transition from  $n + 1 \rightarrow n$  is

$$\begin{aligned} f_{\text{Bohr}} &= \frac{E_0}{h} \left[ \frac{1}{n^2} - \frac{1}{(n+1)^2} \right] \\ &= \frac{E_0}{h} \left[ \frac{n^2 + 2n + 1 - n^2}{n^2(n+1)^2} \right] = \frac{E_0}{h} \left[ \frac{2n+1}{n^2(n+1)^2} \right] \end{aligned}$$

which for large  $n$  becomes

$$f_{\text{Bohr}} \approx \frac{2nE_0}{hn^4} = \frac{2E_0}{hn^3}$$

If we substitute  $E_0$  from Equation (4.26), the result is

$$f_{\text{Bohr}} = \frac{me^4}{4\epsilon_0^2 h^3} \frac{1}{n^3} = f_{\text{classical}} \quad (4.35)$$

**Equivalence of Bohr  
and classical frequencies**

so the frequencies of the radiated energy agree between classical theory and the Bohr model for large values of the quantum number  $n$ . Bohr's correspondence principle is verified for large orbits, where classical and quantum physics should agree.

By 1915, as Bohr's model gained widespread acceptance, the critics of the quantum concept were finding it harder to gain an audience. Bohr had demonstrated the necessity of Planck's quantum constant in understanding atomic structure, and Einstein's conception of the photoelectric effect was generally accepted as well. The *assumption* of quantized angular momentum  $L_n = n\hbar$  led to the quantization of other quantities  $r$ ,  $v$ , and  $E$ . We collect the following three equations here for easy reference.

$$\text{Orbital radius} \quad r_n = \frac{4\pi\epsilon_0\hbar^2}{me^2} n^2 = n^2 a_0 \quad (4.24)$$

$$\text{Velocity} \quad v_n = \frac{n\hbar}{mr_n} \quad (4.22b)$$

$$\text{Energy} \quad E_n = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2} \quad (4.25)$$

## 4.5 Successes and Failures of the Bohr Model

As we briefly mentioned in the previous section, the Bohr atomic model was a first step in understanding the structure of the atom. We discuss the correct description of the hydrogen atom in Chapter 7 after we introduce quantum

theory in Chapter 6. Wavelength measurements for the atomic spectrum of hydrogen are precise and exhibit a small disagreement with the Bohr model results just presented. These disagreements can be corrected by looking more carefully at our original assumptions, one of which was to assume an infinite nuclear mass.

## Reduced Mass Correction

The electron and hydrogen nucleus actually revolve about their mutual center of mass as shown in Figure 4.17. This is a two-body problem, and our previous analysis should be in terms of  $r_e$  and  $r_M$  instead of just  $r$ . A straightforward analysis derived from classical mechanics shows that this two-body problem can be reduced to an equivalent one-body problem in which the motion of a particle of mass  $\mu_e$  moves in a central force field around the center of mass. The only change required in the results of Section 4.4 is to replace the electron mass  $m_e$  by its **reduced mass**  $\mu_e$  where

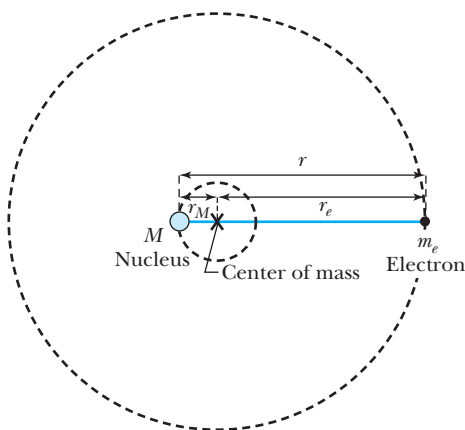
Reduced mass

$$\mu_e = \frac{m_e M}{m_e + M} = \frac{m_e}{1 + \frac{m_e}{M}} \quad (4.36)$$

and  $M$  is the mass of the nucleus (see Problem 53). In the case of the hydrogen atom,  $M$  is the proton mass, and the correction for the hydrogen atom is  $\mu_e = 0.999456 m_e$ . This difference can be measured experimentally. The Rydberg constant for infinite nuclear mass,  $R_\infty$ , defined in Equation (4.29), should be replaced by  $R$ , where

$$R = \frac{\mu_e}{m_e} R_\infty = \frac{1}{1 + \frac{m_e}{M}} R_\infty = \frac{\mu_e e^4}{4\pi c \hbar^3 (4\pi\epsilon_0)^2} \quad (4.37)$$

The Rydberg constant for hydrogen is  $R_H = 1.096776 \times 10^7 \text{ m}^{-1}$ .



**Figure 4.17** Because the nucleus does not actually have an infinite mass, the electron and nucleus rotate about a common center of mass that is located very near the nucleus. This diagram is a very simplistic view of a hydrogen atom.



### EXAMPLE 4.8

Calculate the wavelength for the  $n_u = 3 \rightarrow n_\ell = 2$  transition (called the  $H_\alpha$  line) for the atoms of hydrogen, deuterium, and tritium.

**Strategy** We use Equation (4.30) but with  $R_\infty$  replaced by the Rydberg constant expressed in Equation (4.37). In order to use Equation (4.37) we will need the masses for hydrogen, deuterium, and tritium.

**Solution** The following masses are obtained by subtracting the electron mass from the atomic masses given in Appendix 8.

$$\text{Proton} = 1.007276 \text{ u}$$

$$\text{Deuteron} = 2.013553 \text{ u}$$

$$\text{Triton (tritium nucleus)} = 3.015500 \text{ u}$$

The electron mass is  $m_e = 0.0005485799 \text{ u}$ . The Rydberg constants are

$$R_H = \frac{1}{1 + \frac{0.0005486}{1.00728}} R_\infty = 0.99946 R_\infty \quad \text{Hydrogen}$$

$$R_D = \frac{1}{1 + \frac{0.0005486}{2.01355}} R_\infty = 0.99973 R_\infty \quad \text{Deuterium}$$

$$R_T = \frac{1}{1 + \frac{0.0005486}{3.01550}} R_\infty = 0.99982 R_\infty \quad \text{Tritium}$$

The calculated wavelength for the  $H_\alpha$  line is

$$\frac{1}{\lambda} = R \left( \frac{1}{2^2} - \frac{1}{3^2} \right) = 0.13889 R$$

$$\lambda(H_\alpha, \text{hydrogen}) = 656.47 \text{ nm}$$

$$\lambda(H_\alpha, \text{deuterium}) = 656.29 \text{ nm}$$

$$\lambda(H_\alpha, \text{tritium}) = 656.23 \text{ nm}$$

Deuterium was discovered when two closely spaced spectral lines of hydrogen near 656.4 nm were observed in 1932. These proved to be the  $H_\alpha$  lines of atomic hydrogen and deuterium.

The Bohr model may be applied to any single-electron atom (hydrogen-like) even if the nuclear charge is greater than 1 proton charge ( $+e$ ), for example  $\text{He}^+$  and  $\text{Li}^{++}$ . The only change needed is in the calculation of the Coulomb force, where  $e^2$  is replaced by  $Ze^2$  to account for the nuclear charge of  $+Ze$ . The Rydberg Equation (4.30) now becomes

$$\frac{1}{\lambda} = Z^2 R \left( \frac{1}{n_\ell^2} - \frac{1}{n_u^2} \right) \quad (4.38)$$

where the Rydberg constant is given by Equation (4.37). Bohr applied his model to the case of singly ionized helium,  $\text{He}^+$ . We emphasize that Equation (4.38) is valid only for single-electron atoms (H,  $\text{He}^+$ ,  $\text{Li}^{++}$ , and so on) and does not apply to any other atoms (for example He, Li,  $\text{Li}^+$ ). Charged atoms, such as  $\text{He}^+$ ,  $\text{Li}^+$ , and  $\text{Li}^{++}$ , are called *ions*.

In his original paper of 1913, Bohr predicted the spectral lines of  $\text{He}^+$  although they had not yet been identified in the lab. He showed that certain lines (generally ascribed to hydrogen) that had been observed by Pickering in stellar spectra, and by Fowler in vacuum tubes containing both hydrogen and helium, could be identified as singly ionized helium. Bohr showed that the wavelengths predicted for  $\text{He}^+$  with  $n_\ell = 4$  are almost identical to those of H for  $n_\ell = 2$ , except that  $\text{He}^+$  has *additional lines* between those of H (see Problem 35). The correct explanation of this fact by Bohr gave credibility to his model.



### EXAMPLE 4.9

Calculate the shortest wavelength that can be emitted by the  $\text{Li}^{++}$  ion.

**Strategy** The shortest wavelength occurs when the electron changes from the highest state (unbound,  $n_u = \infty$ ) to the lowest state ( $n_\ell = 1$ ). We use Equation (4.38) to calculate the wavelength.

**Solution** Equation (4.38) gives

$$\frac{1}{\lambda} = (3)^2 R \left( \frac{1}{1^2} - \frac{1}{\infty} \right) = 9R$$

$$\lambda = \frac{1}{9R} = 10.1 \text{ nm}$$

When we let  $n_u = \infty$ , we have what is known as the *series limit*, which is the shortest wavelength possibly emitted for each of the named series.

## Other Limitations

### Fine structure

As the level of precision increased in optical spectrographs, it was observed that each of the lines, originally believed to be single, actually could be resolved into two or more lines, known as **fine structure**. Arnold Sommerfeld adapted the special theory of relativity (assuming some of the electron orbits were elliptical) to Bohr's hypotheses and was able to account for some of the "splitting" of spectral lines. Subsequently it has been found that other factors (especially the electron's *spin*, or *intrinsic angular momentum*) also affect the fine structure of spectral lines.

It was soon observed that external magnetic fields (the Zeeman effect) and external electric fields (the Stark effect) applied to the radiating atoms affected the spectral lines, splitting and broadening them. Although classical electromagnetic theory could quantitatively explain the (normal) Zeeman effect (see Chapter 7), it was unable to account for the Stark effect; for this the quantum model of Bohr and Sommerfeld was necessary.

Although the Bohr model was a great step forward in the application of the new quantum theory to understanding the tiny atom, it soon became apparent that the model had its limitations:

1. It could be successfully applied only to single-electron atoms ( $\text{H}$ ,  $\text{He}^+$ ,  $\text{Li}^{++}$ , and so on).
2. It was not able to account for the intensities or the fine structure of the spectral lines.
3. Bohr's model could not explain the binding of atoms into molecules.

### Limitations of Bohr model

We discuss in Chapter 7 the full quantum mechanical theory of the hydrogen atom, which accounts for all of these phenomena. The Bohr model was an ad hoc theory to explain the hydrogen spectral lines. Although it was useful in the beginnings of quantum physics, we now know that the Bohr model does not correctly describe atoms. Despite its flaws, Bohr's model should not be denigrated. It was the first step from a purely classical description of the atom to the correct quantum explanation. As usually happens in such tremendous changes of understanding, Bohr's model simply did not go far enough—he retained too many classical concepts. Einstein, many years later, noted\* that Bohr's achievement "appeared to me like a miracle and appears as a miracle even today."

\*P. A. Schillp, ed., *Albert Einstein, Philosopher-Scientist*, La Salle, IL: The Open Court, 1949.

## 4.6 Characteristic X-Ray Spectra and Atomic Number

By 1913 when Bohr's model was published, little progress had been made in understanding the structure of many-electron atoms. It was believed that the general characteristics of the Bohr-Rutherford atom would prevail. We discussed the production of x rays from the bombardment of various materials by electrons in Section 3.7. It was known that an x-ray tube with an anode made from a given element produced a continuous spectrum of bremsstrahlung x rays on which are superimposed several peaks with frequencies characteristic of that element (see Figure 3.19).

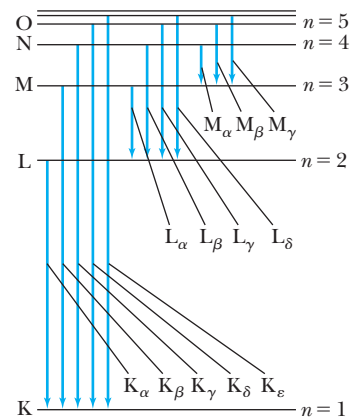
We can now understand these **characteristic x-ray wavelengths** by adopting Bohr's *electron shell* hypothesis. Bohr's model suggests that an electron shell based on the radius  $r_n$  can be associated with each of the principal quantum numbers  $n$ . Electrons with lower values of  $n$  are more tightly bound to the nucleus than those with higher values. The radii of the electron orbits increase in proportion to  $n^2$  [Equation (4.24)]. A specific energy is associated with each value of  $n$ . We may assume that when we add electrons to a fully ionized many-electron atom, the inner shells (low values of  $n$ ) are filled before the outer shells. We have not yet discussed how many electrons each shell contains or even why electrons tend to form shells. Historically, the shells were given letter names: the  $n = 1$  shell was called the K shell,  $n = 2$  was the L shell, and so on. The shell structure of an atom is indicated in Figure 4.18. In heavy atoms with many electrons, we may suppose that several shells contain electrons. What happens when a high-energy electron in an x-ray tube collides with one of the K-shell electrons (we shall call these *K electrons*) in a target atom? If enough energy can be transferred to the K electron to dislodge it from the atom, the atom will be left with a vacancy in its K shell. The atom is most stable in its lowest energy state or *ground state*, so it is likely that an electron from one of the higher shells will change its state and fill the inner-shell vacancy at lower energy, emitting radiation as it changes its state. When this occurs in a heavy atom we call the electromagnetic radiation emitted an *x ray*, and it has the energy

$$E(\text{x ray}) = E_u - E_\ell \quad (4.39)$$

The process is precisely analogous to what happens in an excited hydrogen atom. The photon produced when the electron falls from the L shell into the K shell is called a  $K_\alpha$  x ray; when it falls from the M shell into the K shell, the photon is called a  $K_\beta$  x ray. This scheme of x-ray identification is diagrammed in Figure 4.18. The relative positions of the energy levels of the various shells differ for each element, so the characteristic x-ray energies of the elements are simply the energy differences between the shells. The two strong peaks in the molybdenum spectrum of Figure 3.19 are the  $K_\alpha$  and  $K_\beta$  x rays.

This simple description of the electron shells, which will be modified later in Chapters 7 and 8 by the full quantum mechanical treatment, was not understood by early 1913. The experimental field of x-ray detection was beginning to flourish (see Section 3.3), and the precise identification of the wavelengths of characteristic x rays was possible. In 1913 H. G. J. Moseley, working in Rutherford's Manchester laboratory, was engaged in cataloguing the characteristic x-ray spectra of a series of elements. He concentrated on the K- and L-shell x rays produced in an x-ray tube. Physicists in Rutherford's Manchester lab had already fully accepted the concept of the atomic number, although there was no firm experimental

### Characteristic x-ray wavelengths



**Figure 4.18** Historically, the stationary states were also given letter identifications: K shell ( $n = 1$ ), L shell ( $n = 2$ ), M shell ( $n = 3$ ), and so on. The x rays emitted when an atom changes energy states are given different names depending on the initial and final states. The Greek letter subscripts indicate the value of  $\Delta n$  and the roman letters the value of  $n$  for the final state.

### Significance of atomic number

evidence for doing so. Most of the European physicists still believed that **atomic weight**  $A$  was the important factor, and the periodic table of elements was so structured. The **atomic number**  $Z$  is the number of protons in the nucleus. The makeup of the nucleus was unknown at the time, so  $Z$  was related to the positive charge of the nucleus.

Moseley compared the frequencies of the characteristic x rays with the then supposed atomic number of the elements and found empirically an amazing linear result when he plotted the atomic number  $Z$  versus the square root of the measured frequency as shown in Figure 4.19:

$$f_{K_\alpha} = \frac{3cR}{4}(Z - 1)^2 \quad (4.40)$$

This result holds for the  $K_\alpha$  x rays, and a similar result was found for the L shell. The data shown in Figure 4.19 are known as a *Moseley plot*. Moseley began his work in 1913 in Manchester and, after moving to Oxford late in 1913, completed the investigation in early 1914. Although it is clear that Bohr and Moseley discussed physics and even corresponded after Bohr left for Copenhagen, Moseley does not mention Bohr's model in his 1914 paper. Thus, it is not known whether Bohr's ideas had any influence on Moseley's work.

Using Bohr's model we can understand Moseley's empirical result, Equation (4.40). If a vacancy occurs in the K shell, there is still one electron remaining in the K shell. (We will see in Chapter 8 that, at most, two electrons can occupy the K shell.) An electron in the L shell will feel an effective charge of  $(Z - 1)e$  due to  $+Ze$  from the nucleus and  $-e$  from the remaining K-shell electron, because the L-shell orbit is normally outside the K-shell orbit. The other electrons outside the K shell hardly affect the L-shell electron. The x ray produced when a transition occurs from the  $n = 2$  to the  $n = 1$  shell has the wavelength, from Equation (4.38), of

$$\frac{1}{\lambda_{K_\alpha}} = R(Z - 1)^2 \left( \frac{1}{1^2} - \frac{1}{2^2} \right) = \frac{3}{4} R(Z - 1)^2 \quad (4.41)$$

or

$$f_{K_\alpha} = \frac{c}{\lambda_{K_\alpha}} = \frac{3cR}{4}(Z - 1)^2 \quad (4.42)$$

which is precisely the equation Moseley found describing the  $K_\alpha$ -shell x rays. We can write Equation (4.41) in a more general form for the K series of x-ray wavelengths:

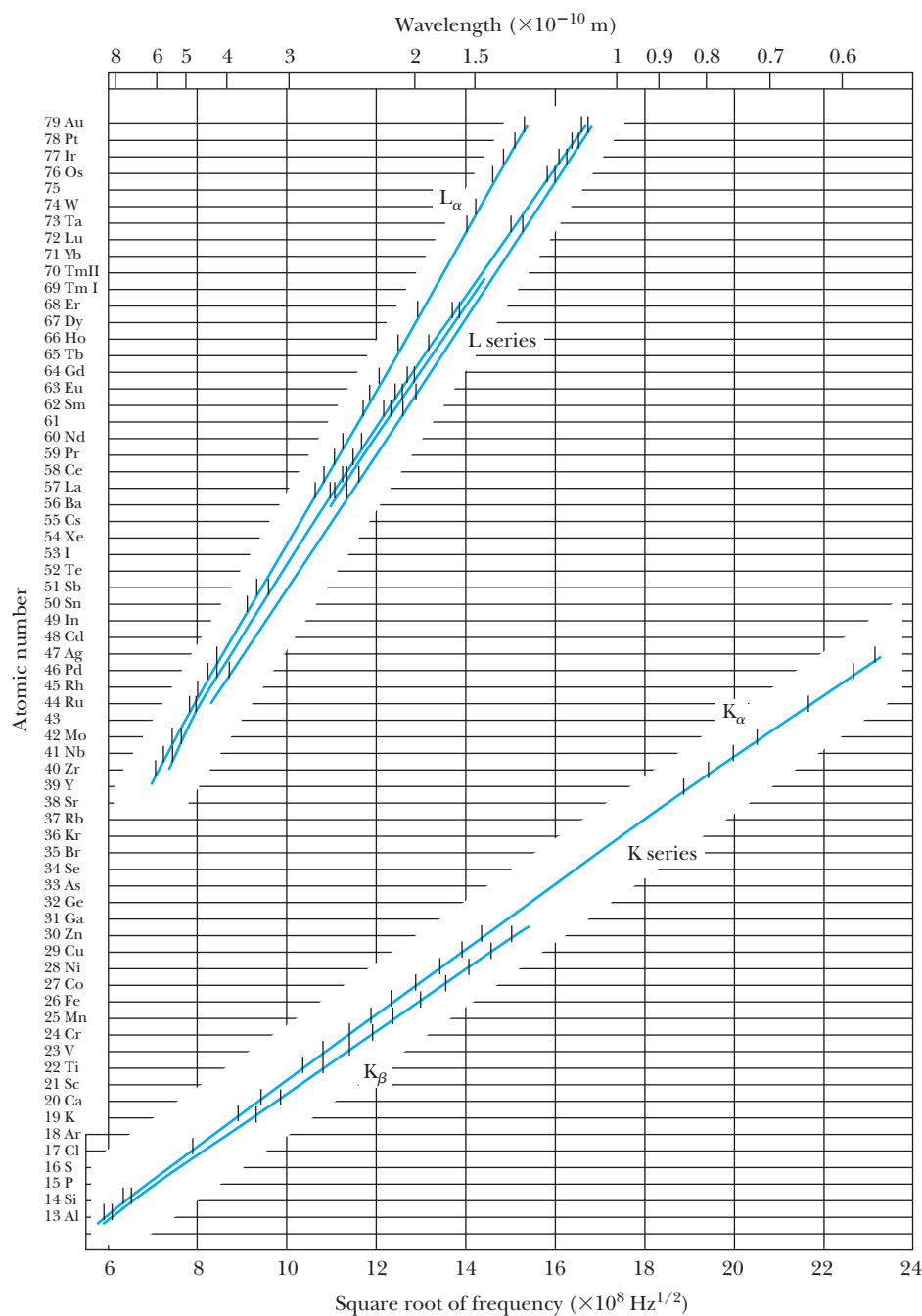
$$\frac{1}{\lambda_K} = R(Z - 1)^2 \left( \frac{1}{1^2} - \frac{1}{n^2} \right) = R(Z - 1)^2 \left( 1 - \frac{1}{n^2} \right) \quad (4.43)$$

Moseley correctly concluded that the atomic number  $Z$  was the determining factor in the ordering of the periodic table, and this reordering was more consistent with chemical properties than one based on atomic weight. It put potassium ( $Z = 19$ ,  $A = 39.10$ ) after argon ( $Z = 18$ ,  $A = 39.95$ ) by atomic number rather than the reverse by atomic weight. Moseley concluded that the atomic number of an element should be identified with the number of positive units of electricity in the nucleus (that is, the number of protons). He tabulated all the atomic numbers between Al ( $Z = 13$ ) and Au ( $Z = 79$ ) and pointed out there were still three elements ( $Z = 43$ , 61, and 75) yet to be discovered! The element promethium ( $Z = 61$ ) was finally discovered around 1940.



**Henry G. J. Moseley** (1887–1915), shown here working in 1910 in the Balliol-Trinity laboratory of Oxford University, was a brilliant young experimental physicist with varied interests. Unfortunately, he was killed in action at the young age of 27 during the English expedition to the Dardanelles. Moseley volunteered and insisted on combat duty in World War I, despite the attempts of Rutherford and others to keep him out of action.





**Figure 4.19** Moseley's original data indicating the relationship between the atomic number  $Z$  and the characteristic x-ray frequencies. Notice the missing entries for elements  $Z = 43$ ,  $61$ , and  $75$ , which had not yet been identified. There are also a few errors in the atomic number designations for the elements. © From H. G. J. Moseley, *Philosophical Magazine* (6), 27, 703 (1914).

Moseley's research helped put the Rutherford-Bohr model of the atom on a firmer footing. It clarified the importance of the electron shells for all the elements, not just for hydrogen. It also helped show that the atomic number was the significant factor in the ordering of the periodic table, not the atomic weight.



### EXAMPLE 4.10

Moseley found experimentally that the equation describing the frequency of the  $L_\alpha$  spectral line was

$$f_{L_\alpha} = \frac{5}{36} cR(Z - 7.4)^2 \quad (4.44)$$

How can the Bohr model explain this result? What is the general form for the L-series wavelengths  $\lambda_L$ ?

**Strategy** We follow the general procedure that we used to find Equation (4.42). The  $L_\alpha$  x ray results from a transition from the M shell ( $n_u = 3$ ) to the L shell ( $n_\ell = 2$ ). There may be several electrons in the L shell and two electrons in the K shell that shield the nuclear charge  $+Ze$  from the M-shell electron making the transition to the L shell. Let's assume the effective charge that the electron sees is  $+Z_{\text{eff}}e$ . Then we can use Equation (4.38) to find both  $Z_{\text{eff}}$  and the general form for the  $\lambda_L$  series of wavelengths.

**Solution** We replace  $Z$  by  $Z_{\text{eff}}$  in Equation (4.38) and find

$$f_{L_\alpha} = \frac{c}{\lambda_{L_\alpha}} = cRZ_{\text{eff}}^2 \left( \frac{1}{2^2} - \frac{1}{3^2} \right) \quad (4.45)$$

$$f_{L_\alpha} = \frac{5cRZ_{\text{eff}}^2}{36}$$

According to Moseley's data the effective charge  $Z_{\text{eff}}$  must be  $Z - 7.4$ . This result is within the spirit of the Bohr model, which applied primarily to hydrogen-like atoms.

We rewrite Equation (4.45) to determine  $\lambda_L$  for the entire series:

$$\frac{1}{\lambda_L} = RZ_{\text{eff}}^2 \left( \frac{1}{2^2} - \frac{1}{n^2} \right) = R(Z - 7.4)^2 \left( \frac{1}{4} - \frac{1}{n^2} \right) \quad (4.46)$$

## 4.7 Atomic Excitation by Electrons

All the evidence for the quantum theory discussed so far has involved quanta of electromagnetic radiation (photons). In particular, the Bohr model explained measured optical spectra of certain atoms. Spectroscopic experiments were typically performed by exciting the elements, for example, in a high-voltage discharge tube, and then examining the emission spectra.

The German physicists James Franck and Gustav Hertz decided to study electron bombardment of gaseous vapors to study the phenomenon of ionization. They set out in 1914 explicitly to study the possibility of transferring a part of an electron's kinetic energy to an atom. Their measurements would provide a distinctive new technique for studying atomic structure.

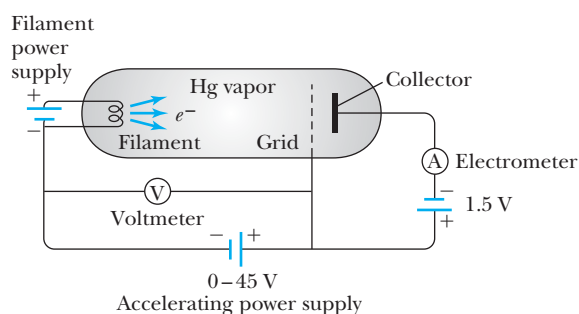
An experimental arrangement similar to that used by Franck and Hertz is shown in Figure 4.20. This particular arrangement is one actually used in a typical undergraduate physics laboratory experiment. Electrons are emitted thermionically from a hot cathode (filament) and are then accelerated by an electric field with its intensity determined by a variable (0- to 45-V) power supply. After passing through a grid consisting of wire mesh, the electrons are subjected to a decelerating voltage (typically 1.5 V) between grid and anode (collector). If the electrons have greater than 1.5 eV after passing through the grid, they will have enough energy to reach the collector and be registered as current in an extremely sensitive ammeter (called an *electrometer*). A voltmeter measures the accelerating voltage  $V$ . The experiment consists of measuring the current  $I$  in the electrometer as a function of  $V$ .

The accelerating electrons pass through a region containing mercury (Hg) vapor (a monatomic gas). Franck and Hertz found that as long as the accelerating voltage  $V$  was below about 5 V (that is, the maximum kinetic energy of the electrons was below 5 eV), the electrons apparently did not lose energy. The



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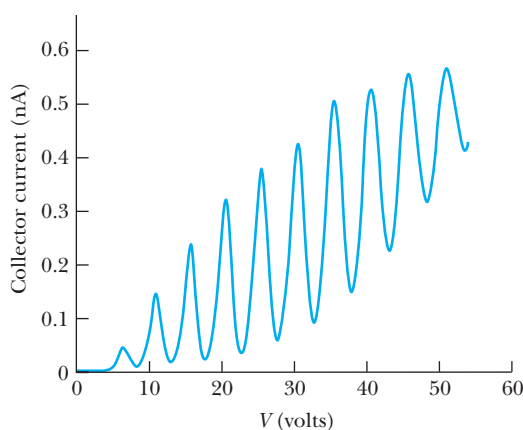
**James Franck** (1882–1964), shown here on the left with Gustav Hertz in Tübingen, Germany, in 1926, came to America in 1935 to avoid Nazi persecution and became an important American scientist who trained many experimental physicists. Gustav Hertz (1887–1975), the nephew of Heinrich Hertz who discovered electromagnetic waves, worked in German universities and industrial labs before going to the Soviet Union in 1945. They received the Nobel Prize for Physics for the experiment named after them (Franck-Hertz experiment) in 1925.



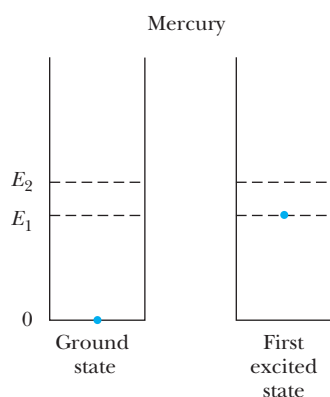
**Figure 4.20** Schematic diagram of apparatus used in an undergraduate physics laboratory for the Franck-Hertz experiment. The hot filament produces electrons, which are accelerated through the mercury vapor toward the grid. A decelerating voltage between grid and collector prevents the electrons from registering in the electrometer unless the electron has a certain minimum energy.

electron current registered in the electrometer continued to increase as  $V$  increased. However, as the accelerating voltage increased above 5 V, there was a sudden drop in the current (see Figure 4.21, which was constructed using data taken by students performing this experiment). As the accelerating voltage continued to increase above 5 V, the current increased again, but suddenly dropped above 10 V. Franck and Hertz first interpreted this behavior as the onset of ionization of the Hg atom; that is, an atomic electron is given enough energy to remove it from the Hg, leaving the atom ionized. They later realized that the Hg atom was actually being excited to its first excited state.

We can explain the experimental results of Franck and Hertz within the context of Bohr's picture of *quantized atomic energy levels*. In the most popular representation of atomic energy states, we say that the atom, when all the electrons are in their lowest possible energy states, is the **ground state**. We define this energy  $E_0$  to be zero. The first quantized energy state above the ground state is called the first excited state, and it has energy  $E_1$ . The energy difference  $E_1 - 0 = E_1$  is called the **excitation energy** of the state  $E_1$ . We show the position of one



**Figure 4.21** Data from an undergraduate student's Franck-Hertz experiment using apparatus similar to that shown in Figure 4.20. The energy difference between peaks is about 5 V, but the first peak is not at 5 V because of the work function differences of the metals used for the filament and grid.



**Figure 4.22** A valence electron is shown in the ground state of mercury on the left. On the right side the electron has been elevated to the first excited state after a bombarding electron scattered inelastically from the mercury atom.

electron in an energy-level diagram of Hg in Figure 4.22 in both the ground state and first excited state. The first excited state of Hg is at an excitation energy of 4.88 eV. As long as the accelerating electron's kinetic energy is below 4.88 eV, no energy can be transferred to Hg because not enough energy is available to excite an electron to the next energy level in Hg. The Hg atom is so much more massive than the electron that almost no kinetic energy is transferred to the recoil of the Hg atom; the collision is *elastic*. The electron can only bounce off the Hg atom and continue along a new path with about the same kinetic energy. If the electron gains at least 4.88 eV of kinetic energy from the accelerating potential, it can transfer 4.88 eV to an electron in Hg, promoting it to the first excited state. This is an *inelastic* collision. A bombarding electron that has lost energy in an inelastic collision then has too little energy (after it passes the grid) to reach the collector. Above 4.88 V, the current dramatically drops because the inelastically scattered electrons no longer reach the collector.

When the accelerating voltage is increased to 7 or 8 V, even electrons that have already made an inelastic collision have enough remaining energy to reach the collector. Once again the current increases with  $V$ . However, when the accelerating voltage reaches 9.8 V, the electrons have enough energy to excite two Hg atoms in successive inelastic collisions, losing 4.88 eV in each ( $2 \times 4.88 \text{ eV} = 9.76 \text{ eV}$ ). The current drops sharply again. As we see in Figure 4.21, even with student apparatus it is possible to observe several successive excitations as the accelerating voltage is increased. Notice that the energy differences between peaks are typically 4.9 eV. The first peak does not occur at 4.9 eV because of the difference in the work functions between the dissimilar metals used as cathode and anode. Other highly excited states in Hg can also be excited in an inelastic collision, but the probability of exciting them is much smaller than that for the first excited state. Franck and Hertz, however, were able to detect them.

The Franck-Hertz experiment convincingly proved the quantization of atomic electron energy levels. The bombarding electron's kinetic energy can change only by certain discrete amounts determined by the atomic energy levels of the mercury atom. They performed the experiment with gases of several other elements and obtained similar results.

### CONCEPTUAL EXAMPLE 4.11

Would it be experimentally possible to observe radiation emitted from the first excited state of Hg after it was produced by an electron collision?

**Solution** If the collision of the bombarding electron with the mercury atom is elastic, mercury will be left in its ground state. If the collision is inelastic, however, the mercury atom will end up in its excited state at 4.9 eV (see Figure 4.22). The mercury atom will not exist long in its first excited state

and should decay quickly ( $\sim 10^{-8} \text{ s}$ ) back to the ground state. Franck and Hertz considered this possibility and looked for x rays. They observed no radiation emitted when the electron's kinetic energy was below about 5 V, but as soon as the current dropped as the voltage went past 5 V, indicating excitation of Hg, an emission line of wavelength 254 nm (ultraviolet) was observed. Franck and Hertz set  $E = 4.88 \text{ eV} = hf = (hc)/\lambda$  and showed that the value of  $h$  determined from  $\lambda = 254 \text{ nm}$  was in good agreement with values of Planck's constant determined by other means.

We have learned in this chapter about the Rutherford-Bohr concept of the atom. Rutherford showed that the atom consisted of an object with most of the mass in the positively charged nucleus. Electrons apparently orbit the nucleus. Bohr was able to derive the important Rydberg equation by proposing his quantized shell model of the atom and explaining how electrons can have stable orbits around the nucleus. The experiment of Franck and Hertz confirmed the quantized shell behavior. Nevertheless, it was clear that Bohr's model was primarily effective for hydrogen-like atoms and that a full and complete description for the majority of the atomic elements was lacking. Before pursuing that in Chapters 6–8, we must first return to investigate the wave properties of matter in Chapter 5, where even more surprises await us.

## Summary

Rutherford proposed a model of the atom consisting of a massive, compact (relative to the size of the atom), positively charged nucleus surrounded by electrons. His assistants, Geiger and Marsden, performed scattering experiments with energetic alpha particles and showed that the number of backward-scattered  $\alpha$  particles could be accounted for only if the model were correct. The relation between the impact parameter  $b$  and scattering angle  $\theta$  for Coulomb scattering is

$$b = \frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 K} \cot \frac{\theta}{2} \quad (4.6)$$

Rutherford's equation for the number of particles scattered at angle  $\theta$  is

$$N(\theta) = \frac{N_i n t}{16} \left( \frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{Z_1^2 Z_2^2}{r^2 K^2 \sin^4(\theta/2)} \quad (4.13)$$

where the dependence on charges  $Z_1 e$  and  $Z_2 e$ , the kinetic energy  $K$ , the target thickness  $t$ , and the scattering angle  $\theta$  were verified experimentally. The classical planetary atomic model predicts the rapid demise of the atom because of electromagnetic radiation.

Niels Bohr was able to derive the empirical Rydberg formula for the wavelengths of the optical spectrum of hydrogen by using his "general assumptions." This led to the quantization of various physical parameters of the hydrogen atom, including the radius,  $r_n = n^2 a_0$ , where  $a_0 = 0.53 \times 10^{-10}$  m, and the energy,  $E_n = -E_0/n^2$ , where  $E_0 = 13.6$  eV.

The Rydberg equation

$$\frac{1}{\lambda} = R \left( \frac{1}{n_\ell^2} - \frac{1}{n_u^2} \right)$$

gives the wavelengths, where  $n_\ell$  and  $n_u$  are the quantum numbers for the lower and upper stationary states, respectively. The Bohr model could explain the optical spectra of hydrogen-like atoms such as  $\text{He}^+$  and  $\text{Li}^{++}$ , but could not account for the characteristics of many-electron atoms. This indicated that the model was incomplete and only approximate. Bohr's correspondence principle relates quantum theories to classical ones in the limit of large quantum numbers.

By examining the characteristic x-ray spectra of the chemical elements, Moseley proved the fundamental significance of the atomic number. We can derive the empirical Moseley relation

$$f_{K_\alpha} = \frac{3cR}{4} (Z - 1)^2 \quad (4.40)$$

from the structure of the atom proposed by Rutherford, together with Bohr's model of hydrogen-like energy levels.

Another way of studying atomic structure is by using electron scattering rather than photon or optical methods. Franck and Hertz were able to confirm the quantized structure of the atom and determine a value of Planck's constant  $h$  in good agreement with other methods.

## Questions

1. Thomson himself was perhaps the biggest critic of the model referred to as “plum pudding.” He tried for years to make it work. What experimental data could he not predict? Why couldn’t he make the planetary model of Rutherford-Bohr work?
2. Does it seem fortuitous that most of the successful physicists who helped unravel the secrets of atomic structure (Thomson, Rutherford, Bohr, Geiger, and Moseley) worked either together or in close proximity in England? Why do you suppose we don’t hear names of physicists working on this idea in other European countries or in the United States?
3. Could the Rutherford scattering of  $\alpha$  particles past  $90^\circ$  be due to scattering from electrons collected together (say,  $100 e^-$ ) in one place over a volume of diameter  $10^{-15}$  m? Explain.
4. In an intense electron bombardment of the hydrogen atom, significant electromagnetic radiation is produced in all directions upon decay. Which emission line would you expect to be most intense? Why?
5. Why are peaks due to higher-lying excited states in the Franck-Hertz experiment not more observable?
6. As the voltage increases above 5 V in the Franck-Hertz experiment, why doesn’t the current suddenly jump back up to the value it had below 5 V?
7. Using Hg gas in the Franck-Hertz experiment, approximately what range of voltages would you expect for the first peak? Explain.
8. When are photons likely to be emitted in the Franck-Hertz experiment?
9. Is an electron most strongly bound in an H,  $\text{He}^+$ , or  $\text{Li}^{++}$  atom? Explain.
10. Why do we refer to atoms as being in the “ground” state or “stationary”? What does an “excited” state mean?
11. What lines would be missing for hydrogen in an absorption spectrum? What wavelengths are missing for hydrogen in an emission spectrum?
12. Why can’t the Bohr model be applied to the neutral He atom? What difficulties do you think Bohr had in modifying his model for He?
13. Describe how the hydrogen atom might absorb a photon of energy less than 13.6 eV. Describe a process by which a 9.8-eV photon might be absorbed. What about a 15.2-eV photon?

## Problems

Note: The more challenging problems have their problem numbers shaded by a blue box.

### 4.1 The Atomic Models of Thomson and Rutherford

1. In Thomson’s plum-pudding model, devise an atomic composition for carbon that consists of a pudding of charge  $+6e$  along with six electrons. Try to configure a system in which the charged particles move only about points in stable equilibrium.
2. How large an error (in percent) in the velocity do we make by treating the velocity of a 7.7-MeV alpha particle nonrelativistically?
3. In Example 4.1, show that the electron’s velocity must be  $v_e' \approx 2v_\alpha$  in order to conserve energy and linear momentum.
4. Thomson worked out many of the calculations for multiple scattering. If we find an average scattering angle of  $1^\circ$  for alpha-particle scattering, what would be the probability that the alpha particle could scatter by as much as  $80^\circ$  because of multiple scattering? The probability for large-angle scattering is  $\exp(-(\theta/\langle\theta\rangle)^2)$ . Geiger and Marsden found that about 1 in 8000  $\alpha$  particles were deflected past  $90^\circ$ . Can multiple scat-

tering explain the experimental results of Geiger and Marsden? Explain.

### 4.2 Rutherford Scattering

5. Calculate the impact parameter for scattering a 7.7-MeV  $\alpha$  particle from gold at an angle of (a)  $1^\circ$  and (b)  $90^\circ$ .
6. A beam of 8.0-MeV  $\alpha$  particles scatters from a thin gold foil. What is the ratio of the number of  $\alpha$  particles scattered to angles greater than  $1^\circ$  to the number scattered to angles greater than  $2^\circ$ ?
7. For aluminum ( $Z = 13$ ) and gold ( $Z = 79$ ) targets, what is the ratio of an alpha particle scattering at any angle for equal numbers of scattering nuclei per unit area?
8. What fraction of 5-MeV  $\alpha$  particles will be scattered through angles greater than  $8^\circ$  from a gold foil ( $Z = 79$ , density =  $19.3 \text{ g/cm}^3$ ) of thickness  $10^{-8}$  m?
9. In an experiment done by scattering 5.5-MeV  $\alpha$  particles from a thin gold foil, students find that 10,000  $\alpha$  particles are scattered at an angle greater than  $50^\circ$ . (a) How many of these  $\alpha$  particles will be scattered greater than  $90^\circ$ ? (b) How many will be scattered between  $70^\circ$  and  $80^\circ$ ?



10. Students want to construct a scattering experiment using a powerful source of 5.5-MeV  $\alpha$  particles to scatter from a gold foil. They want to be able to count 1 particle/s at  $50^\circ$ , but their detector is limited to a maximum count rate of 2000 particles/s. Their detector subtends a small angle. Will their experiment work without modifying the detector if the other angle they want to measure is  $6^\circ$ ? Explain.
11. The nuclear radii of aluminum and gold are approximately  $r = 3.6$  fm and  $7.0$  fm, respectively. The radii of protons and alpha particles are  $1.3$  fm and  $2.6$  fm, respectively. (a) What energy  $\alpha$  particles would be needed in head-on collisions for the nuclear surfaces to just touch? (This is about where the nuclear force becomes effective.) (b) What energy protons would be needed? In both (a) and (b), perform the calculation for aluminum and for gold.
12. Consider the scattering of an alpha particle from the positively charged part of the Thomson plum-pudding model. Let the kinetic energy of the  $\alpha$  particle be  $K$  (nonrelativistic) and let the atomic radius be  $R$ . (a) Assuming that the maximum transverse Coulomb force acts on the  $\alpha$  particle for a time  $\Delta t = 2R/v$  (where  $v$  is the initial speed of the  $\alpha$  particle), show that the largest scattering angle we can expect from a single atom is
- $$\theta = \frac{2Z_2e^2}{4\pi\epsilon_0KR}$$
- (b) Evaluate  $\theta$  for an 8.0-MeV  $\alpha$  particle scattering from a gold atom of radius  $0.135$  nm.
13. Using the results of the previous problem, (a) find the average scattering angle of a 10-MeV  $\alpha$  particle from a gold atom ( $R \approx 10^{-10}$  m) for the positively charged part of the Thomson model. (b) How does this compare with the scattering from the electrons?

#### 4.3 The Classical Atomic Model

14. The radius of a hydrogen nucleus is believed to be about  $1.2 \times 10^{-15}$  m. (a) If an electron rotates around the nucleus at that radius, what would be its speed according to the planetary model? (b) What would be the total mechanical energy? (c) Are these reasonable?
15. Make the (incorrect) assumption that the nucleus is composed of electrons and that the protons are outside. (a) If the size of an atom were about  $10^{-10}$  m, what would be the speed of a proton? (b) What would be the total mechanical energy? (c) What is wrong with this model?
16. Calculate the speed and radial acceleration for a ground-state electron in the hydrogen atom. Do the same for the  $\text{He}^+$  ion and the  $\text{Li}^{++}$  ion.
17. Compute and compare the electrostatic and gravitational forces in the classical hydrogen atom, assuming a radius  $5.3 \times 10^{-11}$  m.
18. Calculate the time, according to classical laws, it would take the electron of the hydrogen atom to radiate its energy and crash into the nucleus. [Hint: The radiated power  $P$  is given by  $(1/4\pi\epsilon_0)(2Q^2/3c^3)(d^2\vec{r}/dt^2)^2$  where  $Q$  is the charge,  $c$  the speed of light, and  $\vec{r}$  the position vector of the electron from the center of the atom.]

#### 4.4 The Bohr Model of the Hydrogen Atom

19. The Ritz combination rules express relationships between observed frequencies of the optical emission spectra. Prove one of the more important ones:
- $$f(\mathbf{K}_\alpha) + f(\mathbf{L}_\alpha) = f(\mathbf{K}_\beta)$$
- where  $\mathbf{K}_\alpha$  and  $\mathbf{K}_\beta$  refer to the Lyman series and  $\mathbf{L}_\alpha$  to the Balmer series of hydrogen (Figure 4.18).
20. (a) Calculate the angular momentum in  $\text{kg} \cdot \text{m}^2/\text{s}$  for the lowest electron orbit in the hydrogen atom. Compare the result with Planck's constant  $h$ . (b) Repeat for an electron in the  $n = 2$  state of hydrogen.
21. Use the known values of  $\epsilon_0$ ,  $h$ ,  $m$ , and  $e$  to calculate the following to five significant figures:  $hc$  (in  $\text{eV} \cdot \text{nm}$ ),  $e^2/4\pi\epsilon_0$  (in  $\text{eV} \cdot \text{nm}$ ),  $mc^2$  (in  $\text{keV}$ ),  $a_0$  (in  $\text{nm}$ ), and  $E_0$  (in  $\text{eV}$ ).
22. What is the total mechanical energy for a ground-state electron in H,  $\text{He}^+$ , and  $\text{Li}^{++}$  atoms? For which atom is the electron most strongly bound? Why?
23. A hydrogen atom in an excited state absorbs a photon of wavelength  $410$  nm. What were the initial and final states of the hydrogen atom?
24. A hydrogen atom in an excited state emits a photon of wavelength  $95$  nm. What are the initial and final states of the hydrogen atom?
25. What is the binding energy of the electron in the ground state of (a) deuterium, (b)  $\text{He}^+$ , and (c)  $\text{Be}^{+++}$ ?
26. The **isotope shift** of spectral lines refers to the shift in wavelengths (or frequencies) due to the different isotopic masses of given elements. Find the isotope shifts for each of the four visible Balmer series wavelengths for deuterium and tritium compared with hydrogen.
27. Find the isotope shift (see Problem 26) of the ground-state energy for deuterium and tritium compared with the ground-state energy of hydrogen. Express the answer in  $\text{eV}$ .
28. Describe the visible absorption spectra for (a) a hydrogen atom and (b) an ionized helium atom,  $\text{He}^+$ .
29. A hydrogen atom exists in an excited state for typically  $10^{-8}$  s. How many revolutions would an electron make in an  $n = 3$  state before decaying?
30. Light from a Nd:Yag laser with a wavelength of  $397$  nm is incident upon a hydrogen atom in the  $n = 2$  state at rest. What is the highest state to which hydrogen can be excited?
31. A muonic atom consists of a muon (mass  $m = 106 \text{ MeV}/c^2$  and charge  $q = -e$ ) in place of an electron. For the muon in a hydrogen atom, what is (a) the smallest radius and (b) the binding energy of the

muon in the ground state? (c) Calculate the series limit of the wavelength for the first three series.

32. Positronium is an atom composed of an electron and a positron (mass  $m = m_e$ , charge  $q = +e$ ). Calculate the distance between the particles and the energy of the lowest energy state of positronium. (*Hint*: what is the reduced mass of the two particles? See Problem 53.)
33. (a) Find the Bohr radius of the positronium atom described in the previous problem. (b) Find the wavelength for the transition from  $n_u = 2$  to  $n_\ell = 1$  for positronium.
34. What is the difference in the various Bohr radii  $r_n$  for the hydrogen atom (a) between  $r_1$  and  $r_2$ , (b) between  $r_5$  and  $r_6$ , and (c) between  $r_{10}$  and  $r_{11}$ ? (d) Show that for *Rydberg atoms* (hydrogen atoms with large  $n$ , discussed in Chapter 8) the difference between successive radii is approximately  $2na_0$ .

#### 4.5 Successes and Failures of the Bohr Model

35. Compare the Balmer series of hydrogen with the series where  $n_\ell = 4$  for the ionized helium atom  $\text{He}^+$ . What is the difference between the wavelengths of the  $L_\alpha$  and  $L_\beta$  line of hydrogen and the  $n_u = 6$  and 8 of  $\text{He}^+$ ? Is there a wavelength of the Balmer series that is very similar to any wavelength values where  $n_\ell = 4$  in  $\text{He}^+$ ? Explain.
36. Calculate the Rydberg constant for the single-electron (hydrogen-like) ions of helium, potassium, and uranium. Compare each of them with  $R_\infty$  and determine the percentage difference.
37. In 1896 Pickering found lines from the star Zeta Pup-pis that had not been observed on Earth. Bohr showed in 1913 that the lines were due to  $\text{He}^+$ . Show that an equation giving these wavelengths is

$$\frac{1}{\lambda} = R \left( \frac{1}{n_\ell^2} - \frac{1}{n_u^2} \right)$$

What value should the Rydberg constant  $R$  have in this case?

#### 4.6 Characteristic X-Ray Spectra and Atomic Number

38. What wavelengths for the  $L_\alpha$  lines did Moseley predict for the missing  $Z = 43$ , 61, and 75 elements? (See Example 4.10.)
39. If the resolution of a spectrograph is  $\Delta\lambda = 10^{-12}$  m, would it be able to separate the  $K_\alpha$  lines for lead and bismuth? Explain.
40. Determine the correct equation to describe the  $K_\beta$  frequencies measured by Moseley. Compare that with Moseley's equation for  $K_\alpha$  frequencies. Does the result agree with the data in Figure 4.19? Explain.
41. Calculate the  $K_\alpha$  and  $K_\beta$  wavelengths for He and Li.
42. (a) Calculate the ratio of  $K_\alpha$  wavelengths for uranium and carbon. (b) Calculate the ratio of  $L_\alpha$  wavelengths for platinum and calcium.

43. Calculate the three longest wavelengths and the series limit for the molybdenum atom.

44. An unknown element is used as a target in an x-ray tube. Measurements show that the characteristic spectral lines with the longest wavelengths are 0.155 nm and 0.131 nm. What is the element? (*Hint*: you will find the answer to Problem 40 to be useful.)

#### 4.7 Atomic Excitation by Electrons

45. If an electron of 45 eV had a head-on collision with an Hg atom at rest, what would be the kinetic energy of the recoiling Hg atom? Assume an elastic collision.
46. In the Franck-Hertz experiment, explain why the small potential difference between the grid and collector plate is useful. Redraw the data of Figure 4.21 the way the data would appear without this small retarding potential.
47. Calculate the value of Planck's constant determined by Franck and Hertz when they observed the 254-nm ultraviolet radiation using Hg vapor.
48. Consider an element having excited states at 3.6 eV and 4.6 eV used as a gas in the Franck-Hertz experiment. Assume that the work functions of the materials involved cancel out. List all the possible peaks that *might* be observed with electron scattering up to an accelerating voltage of 18 V.

#### General Problems

49. The redshift measurements of spectra from magnesium and iron are important in understanding distant galaxies. What are the  $K_\alpha$  and  $L_\alpha$  wavelengths for magnesium and iron?
50. In the early 1960s the strange optical emission lines from starlike objects that also produced tremendous radio signals confused scientists. Finally, in 1963 Maarten Schmidt of the Mount Palomar observatory discovered that the optical spectra were just those of hydrogen but redshifted because of the tremendous velocity of the object with respect to Earth. The object was moving away from Earth at a speed of 50,000 km/s! Compare the wavelengths of the normal and redshifted spectral lines for the  $K_\alpha$  and  $K_\beta$  lines of the hydrogen atom.
51. A beam of 8.0-MeV  $\alpha$  particles scatters from a gold foil of thickness 0.32  $\mu\text{m}$ . (a) What fraction of the  $\alpha$  particles is scattered between  $1.0^\circ$  and  $2.0^\circ$ ? (b) What is the ratio of  $\alpha$  particles scattered through angles greater than  $1^\circ$  to the number scattered through angles greater than  $10^\circ$ ? Greater than  $90^\circ$ ?
52. In Rutherford scattering we noted that angular momentum is conserved. The angular momentum of the incident  $\alpha$  particle relative to the target nucleus is  $mv_0b$  where  $m$  is the mass,  $v_0$  is the initial velocity of the  $\alpha$  particle, and  $b$  is the impact parameter. Start with

$\vec{L} = \vec{r} \times \vec{p}$  and show that angular momentum is conserved, and the magnitude is given by  $mv_0b$  along the entire path of the  $\alpha$  particle while it is scattered by the Coulomb force from a gold nucleus.

53. The proton (mass  $M$ ) and electron (mass  $m$ ) in a hydrogen atom actually rotate about their common center of mass as shown in Figure 4.17. The distance  $r = r_e + r_M$  is still defined to be the electron-nucleus distance. Show that Equation (4.24) is only modified by substituting for  $m$  by

$$\mu = \frac{m}{1 + m/M}$$

54. In Bohr's Assumption D, he assumed the mean value  $K$  of the kinetic energy of the electron-nucleus system to be  $nhf_{\text{orb}}/2$  where  $f_{\text{orb}}$  is the orbital frequency of the electron around the nucleus. Calculate  $f_{\text{orb}}$  in the ground state in the following ways: (a) Use  $f_{\text{classical}}$  in Equation (4.34). (b) Use Equation (4.33a), but first determine  $v$  and  $r$ . (c) Show that the mean value  $K$  is equal to the absolute value of the electron-nucleus system total energy and that this is 13.6 eV. Use this value of  $K$  to determine  $f_{\text{orb}}$  from the relation for  $K$  stated above.
55. Show that the quantization of angular momentum  $L = n\hbar$  follows from Bohr's Assumption D that the mean value  $K$  of the kinetic energy of the electron-nucleus system is given by  $K = nhf_{\text{orb}}/2$ . Assume a circular orbit.
56. (a) Calculate the energies of the three lowest states of positronium. (b) Determine the wavelengths of the  $K_\alpha$ ,  $K_\beta$ ,  $L_\alpha$ , and  $L_\beta$  transitions.
57. Careful measurements of light from a distant galaxy show that the longest observed wavelength in the Lyman series of hydrogen is 137.15 nm. If the galaxy is moving directly away from us, what is its velocity?
58. Consider a two-electron atom in which the electrons, orbiting a nucleus of charge  $+Ze$ , follow Bohr-like orbits of the same radius  $r$ , with the electrons always on opposite sides of the nucleus. (a) Show that the net force on each electron is toward the nucleus and has magnitude

$$F = \frac{e^2}{4\pi\epsilon_0 r^2} \left( Z - \frac{1}{4} \right)$$

(b) Use the fact that this is the centripetal force to show that the square of each electron's orbital speed  $v$  is given by

$$v^2 = \frac{e^2}{4\pi\epsilon_0 m r} \left( Z - \frac{1}{4} \right)$$

(c) Use the result of part (b) along with Bohr's rule that the angular momentum of each of the two electrons is  $L = \hbar$  in the ground state to show that

$$r = \frac{\epsilon_0 \hbar^2}{\pi m e^2} \left( Z - \frac{1}{4} \right)$$

(d) Show that the atom's total energy (kinetic plus potential) is

$$E = -\frac{me^4}{8\epsilon_0^2 \hbar^2} \left( 2Z - \frac{1}{2} \right) \left( Z - \frac{1}{4} \right)$$

(e) The energy needed to remove both electrons is just the negative of the energy you found in part (d). Compute the energy needed to remove both electrons in helium, and then repeat for  $\text{Li}^+$ . Compare your results with the experimental values of 79.0 eV and 198 eV, respectively.

59. It may be argued on theoretical grounds that the radius of the hydrogen atom should depend only on the fundamental constants  $\hbar$ ,  $e$ , the electrostatic force constant  $k = 1/4\pi\epsilon_0$ , and  $m$  (the electron's mass). Use dimensional analysis to show that the combination of these factors that yields a result with dimensions of length is  $\hbar^2/kme^2$ . Discuss this result in relation to Equation (4.24).
60. A Rydberg atom (discussed in more detail in Chapter 8) is a single-electron atom with a large quantum number  $n$ . Rydberg states are close together in energy (see Figure 4.15), so transitions between adjacent Rydberg states produce long-wavelength photons. Consider a transition from a state  $n + 1$  to a state  $n$  in hydrogen. (a) Starting with Equation (4.30), use the binomial expansion to show that this transition produces a photon with wavelength approximately  $n^3/2R$ . (b) Obtain the same result as in part (a), this time starting with Equation (4.25) and computing  $dE/dn$ . The result,  $dE/dn$ , can then be approximated by  $\Delta E/\Delta n$ , with  $\Delta n = 1$  for this transition and  $\Delta E = hc/\lambda$  for the emitted photon. (c) Using the approximate expression you derived in (a) and (b), compute the wavelength for a transition from  $n = 101$  to  $n = 100$  in hydrogen. (Use  $R_\infty$  and ignore the reduced-mass correction.) Compare your answer with the exact wavelength for this transition, computed using Equation (4.30).
61. (a) Calculate the  $K_\alpha$  and  $K_\beta$  x-ray wavelengths for molybdenum and compare the results with those shown in the graph in Figure 3.19. (b) Why don't the L-series x rays show up in that graph?