Problema 1
$$\langle \epsilon \rangle = \frac{\sum_{m=1} m\epsilon_0 e^{-\frac{m\epsilon_0}{KT}}}{\sum_{m=0}^{\infty} e^{-m\frac{\epsilon_0}{KT}}}$$; Hagamos $x = e^{-\frac{\epsilon_0}{KT}}$

a)

i) $$\sum_{m=0}^{\infty} e^{-m\frac{\epsilon_0}{KT}} = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x} = \frac{1}{1-e^{-\epsilon_0/KT}}$$

$(\beta \equiv \frac{1}{KT})$

ii) $$\sum_{m=1}^{\infty} m\epsilon_0 e^{-\frac{m\epsilon_0}{KT}} \equiv \sum_{m=1}^{\infty} m\epsilon_0 e^{-\beta m\epsilon_0} = -\frac{\partial}{\partial\beta}\sum_{m=0}^{\infty} e^{-\beta\epsilon_0 m} = -\frac{\partial}{\partial\beta}\left(\frac{1}{1-e^{-\beta\epsilon_0}}\right) = \frac{\epsilon_0 e^{-\beta\epsilon_0}}{(1-e^{-\beta\epsilon_0})^2}$$

$$\therefore \langle\epsilon\rangle = \frac{\frac{\epsilon_0 e^{-\beta\epsilon_0}}{(1-e^{-\beta\epsilon_0})^{\cancel{2}}}}{\frac{1}{\cancel{(1-e^{-\beta\epsilon_0})}}} = \frac{\epsilon_0 e^{-\beta\epsilon_0}}{1-e^{-\beta\epsilon_0}} = \frac{\epsilon_0}{e^{\beta\epsilon_0}-1} \, //$$

b) $\epsilon_0 << KT \rightarrow \beta\epsilon_0 << 1 \Rightarrow e^{\beta\epsilon_0} \simeq 1 + \beta\epsilon_0 + \ldots \Rightarrow e^{\beta\epsilon_0} - 1 \simeq \beta\epsilon_0$

$$\therefore \langle\epsilon\rangle \simeq \frac{\epsilon_0}{\beta\epsilon_0} = \frac{1}{\beta} = KT \, //$$

Problema 2 : Formemos los productos

a) $M_x M_y = \frac{1}{2}\begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}$; $M_y M_x = \frac{1}{2}\begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix}$; $M_x M_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$; $M_z M_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

$M_y M_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$; $M_z M_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$

Luego, $[A,B] = AB - BA$

$[M_x, M_y] = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i M_z$ //

$[M_y, M_z] = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = i M_x$ //

$[M_z, M_x] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = i M_y$ //

$\Rightarrow [M_i, M_j] = i\epsilon_{ijk} M_k$ //

b) $M_x^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$; $M_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$; $M_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\rightarrow M_x^2 + M_y^2 + M_z^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\mathbb{1}_3 = M^2$ //

c) • $[M^2, M_i] = 2[\mathbb{1}_3, M_i] = 2(\mathbb{1}_3 M_i - M_i \mathbb{1}_3) = 2(M_i - M_i) = 0$ //

• $[M_z, L^+] = [M_z, M_x + iM_y] = \underbrace{[M_z, M_x]}_{iM_y} + i\underbrace{[M_z, M_y]}_{-iM_x} = iM_y + M_x = L^+$ //

• $[L^+, L^-] = [M_x + iM_y, M_x - iM_y] = \underbrace{[M_x, M_x]}_{=0} - i\underbrace{[M_x, M_y]}_{iM_z} + i\underbrace{[M_y, M_x]}_{-iM_z} + \underbrace{[M_y, M_y]}_{=0}$

$= 2M_z$ //

PROBLEMA 3

Podemos escribir $\vec{\nabla}\cdot(\varphi\vec{E}) = \vec{\nabla}\varphi\cdot\vec{E} + \varphi\vec{\nabla}\cdot\vec{E}$

$$= -E^2 + \rho\varphi/\varepsilon_0$$

Integramos la ecuación sobre todo el espacio:

$$\int \vec{\nabla}\cdot(\varphi\vec{E})\,d\tau = -\int E^2\,d\tau + \frac{1}{\varepsilon_0}\int\rho\varphi\,d\tau$$

Por la ley de Gauss tenemos:

$$\int_V \vec{\nabla}\cdot(\varphi\vec{E})\,d\tau = \int_{\partial V}\varphi\vec{E}\cdot d\vec{\sigma} = 0$$ ya que $\varphi \sim \frac{1}{r}$

$$\Rightarrow \quad \varepsilon_0\int E^2\,d\tau = \int\rho\varphi\,d\tau$$

PROBLEMA 4

a) $u = x + y \quad v = x/y$

$yv = x \Rightarrow u = yv + y = y(v+1) \Rightarrow y = \frac{u}{v+1}$

$$x = \frac{uv}{v+1}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \frac{\partial x}{\partial u} = \frac{v}{v+1}$$

$$\frac{\partial x}{\partial v} = \frac{u}{v+1} - \frac{uv}{(v+1)^2} = \frac{u}{(v+1)^2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v+1} \qquad \frac{\partial y}{\partial v} = -\frac{u}{(v+1)^2}$$

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$= \frac{v}{v+1} \cdot \left(-\frac{u}{(v+1)^2}\right) - \frac{u}{(v+1)^2} \cdot \frac{1}{(v+1)}$$

$$= -\frac{uv}{(v+1)^3} - \frac{u}{(v+1)^3} = -\frac{u(v+1)}{(v+1)^3} = -\frac{u}{(v+1)^2}$$

b) $$J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1/y \\ 1 & -x/y^2 \end{vmatrix} = -\frac{x}{y^2} - \frac{y}{y^2}$$

$$= -\frac{x+y}{y^2} = -u \cdot \frac{(v+1)^2}{u^2} = -\frac{(v+1)^2}{u}$$

$$\therefore J^{-1} = \frac{1}{J}$$