Métodos Matemáticos de la Física II: Tarea 4 y 5

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1)

Solución: Sabemos que podemos calcular la serie de Fourier de una función con:

$$f(\theta) = \hat{f}_0 + \sum_{n=0}^{\infty}[a_n \cos n\theta + b_n \sin n\theta]$$

Tendremos que al ser periódica $f(-\theta) = f(\theta)$, lo que nos lleva a que los términos de seno al ser impares serán 0. Por lo ue tendremos:

$$f(\theta) = \hat{f}_0 + \sum_{n=0}^{\infty} a_n \cos n\theta$$

Ahora calcularemos $\hat{f}_0$:

$$\hat{f}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\theta^2-\pi^2)^2 d\theta = \frac{2}{2\pi}\int_{0}^{\pi}(\theta^2-\pi^2)^2 d\theta = \frac{1}{\pi}\int_{0}^{\pi}(\theta^4-2\theta^2\pi^2+\pi^4) d\theta = \frac{1}{\pi}\left(\frac{\pi^5}{5} - \frac{2\pi^2}{3}\pi^3 + \pi^5\right) = \pi^4\left(\frac{1}{5} - \frac{2}{3} + 1\right)$$

$$\hat{f}_0 = \pi^4\left(\frac{3-10+15}{15}\right)$$

$$\hat{f}_0 = \frac{8\pi^4}{15}$$

Ahora el coeficiente a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos n\theta(\theta^2-\pi^2)^2 d\theta = \frac{2}{\pi}\int_{0}^{\pi}\cos n\theta(\theta^4-2\theta^2\pi^2+\pi^4) d\theta = \frac{2}{\pi}\Bigg(\int_0^\pi \theta^4 \cos n\theta d\theta - 2\pi^2\int_0^\pi \theta^2 \cos n\theta d\theta + \pi^4\int_0^\pi \cos n\theta d\theta$$

$$a_n = \frac{2}{\pi}\left(\int_0^\pi \theta^4 \cos n\theta d\theta - 2\pi^2\int_0^\pi \theta^2 \cos n\theta d\theta\right)$$

Integrando por partes:

$$I_1 = \int_0^\pi \theta^4 \cos n\theta d\theta = \left[\theta^4\frac{\sin n\theta}{n} - \frac{4\theta^3\cos n\theta}{n^2} - \frac{12\theta^2\sin n\theta}{n^3} + \frac{24\theta\cos n\theta}{n^4} + \frac{24\sin n\theta}{n^5}\right]_0^\pi$$

$$I_1 = -\frac{4\pi^3(-1)^n}{n^2} + \frac{24\pi(-1)^n}{n^4}$$

$$I_2 = \int_0^\pi \theta^2 \cos n\theta d\theta = \left[\frac{\theta^2\sin n\theta}{n} - \frac{2\theta\cos n\theta}{n^2} - \frac{2\sin n\theta}{n^3}\right]_0^\pi$$

$$I_2 = -\frac{2\pi(-1)^n}{n^2}$$

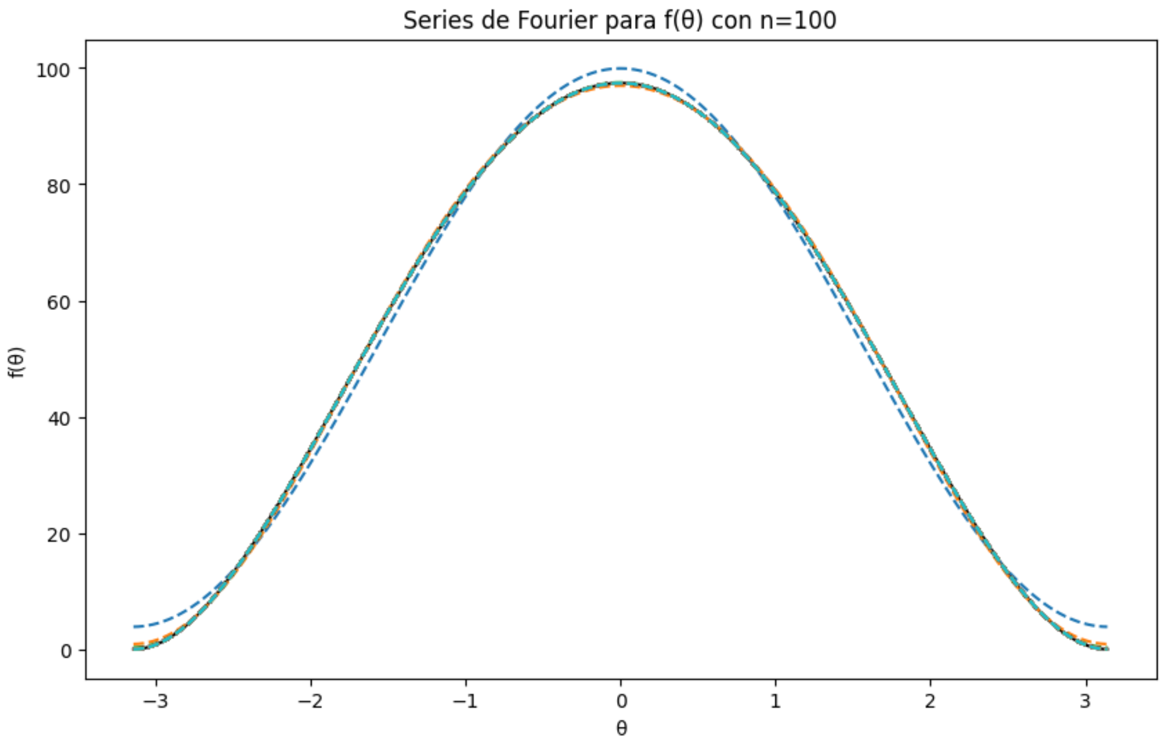
Por lo tanto tendremos:

$$a_n = \frac{2}{\pi}\left(-\frac{4\pi^3(-1)^n}{n^2} + \frac{24\pi(-1)^n}{n^4} - 2\pi^2(-\frac{2\pi(-1)^n}{n^2})\right) = \frac{2}{\pi}\left(\frac{24\pi(-1)^n}{n^4}\right)$$

$$a_n = \frac{48(-1)^n}{n^4}$$

Por lo tanto la serie de Fourier será:

$$f(\theta) = \frac{8\pi^4}{15} + \sum_{n=1}^{\infty} \frac{48(-1)^n}{n^4} \cos n\theta$$



2)

Solución: Seguiremos el mismo procedimiento anterior para encontrar la serie de Fourier.

$$\hat{f}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\theta} d\theta = \frac{1}{2\pi} \left(e^{\pi} - e^{-\pi} \right) = \frac{\sinh \pi}{\pi}$$

Ahora a_n:

$$a_n = \int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta$$

Integrando por partes

$$\int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta = \left[\cos n\theta e^{\theta} - \int_{-\pi}^{\pi} e^{\theta}(-n \sin n\theta) d\theta \right]_{-\pi}^{\pi} = \left[\cos n\theta e^{\theta} + \int_{-\pi}^{\pi} e^{\theta}(n \sin n\theta) d\theta \right]_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta = \left[\cos n\theta e^{\theta} + n \left(e^{\theta} \sin n\theta - \int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta \right) \right]_{-\pi}^{\pi} = \left[e^{\theta} \cos n\theta + n e^{\theta} \sin n\theta \right]_{-\pi}^{\pi} - n^2 \int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta$$

$$(1+n^2) \int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta = \left[e^{\theta} \cos n\theta + n e^{\theta} \sin n\theta \right]_{-\pi}^{\pi} = e^{\pi}(-1)^n - e^{-\pi}(-1)^n$$

$$\int_{-\pi}^{\pi} e^{\theta} \cos n\theta d\theta = \frac{2 \sinh \pi (-1)^n}{1+n^2}$$

Por lo que:

$$a_n = \frac{2\sinh\pi(-1)^n}{\pi(1+n^2)}$$

Ahora calcularemos b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\theta}\sin n\theta d\theta$$

Integrando por partes tenemos:

$$\int_{-\pi}^{\pi} e^{\theta}\sin n\theta d\theta = \left[e^{\theta}\sin n\theta - n\int_{-\pi}^{\pi} e^{\theta}\cos n\theta d\theta\right]_{-\pi}^{\pi} = \left[e^{\theta}\sin n\theta - ne^{\theta}\cos n\theta - n^2\int_{-\pi}^{\pi} e^{\theta}\sin n\theta\right]_{-\pi}^{\pi}$$

$$\int_{-\pi}^{\pi} e^{\theta}\sin n\theta = \left[\frac{\sin n\theta - ne^{\theta}\cos n\theta}{(1+n^2)}\right]_{-\pi}^{\pi} = \frac{n(-1)^n}{1+n^2}(e^{-\pi}-e^{\pi}) = \frac{n(-1)^n}{1+n^2}\sinh\pi$$

Por lo que nuestro b_n será:

$$b_n = \frac{n(-1)^n}{\pi(1+n^2)}\sinh\pi$$

Finalmente nuestra serie de Fourier será:

$$f(\theta) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(n^2+1)}\cos n\theta - \frac{n(-1)^n}{(n^2+1)}\sin n\theta\right]$$

Ahora para demostrar $\sum_{n=1}^{\infty}\frac{1}{n^2+1} = (\pi\coth\pi - 1)$ haremos la siguiente operación usando la expresión original y su expansión en series de Fourier:

$$f(\pi)+f(-\pi) = e^{\pi}+e^{-\pi} = \frac{2\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2+1} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2+1}$$

$$e^{\pi}+e^{-\pi} = \frac{2\sinh\pi}{\pi} + \frac{4\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1} \to \cosh\pi = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

$$\pi\coth\pi - 1 = 2\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

Finalmente obtenemos:

$$\frac{1}{2}(\pi\coth\pi - 1) = \sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

3)

Solución: Para encontrar la serie de Fourier de $f(\theta) = \theta e^{i\theta}$

$$\hat{f}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta e^{i\theta}d\theta = \frac{1}{2\pi}\left[\frac{\theta e^{i\theta}}{i} - \int_{-\pi}^{\pi}\frac{e^{i\theta}d\theta}{i}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\theta e^{i\theta}}{i} - \frac{e^{i\theta}d\theta}{i^2}\right]_{-\pi}^{\pi}$$

$$\hat{f}_0 = \frac{1}{2\pi}\left[\frac{\pi e^{i\pi}}{i} - \frac{e^{i\pi}}{i^2} - \frac{(-\pi)e^{-i\pi}}{i} + \frac{e^{-i\pi}}{i^2}\right] = \frac{1}{2\pi i}\left[\pi e^{i\pi} - \frac{e^{i\pi}}{i} + \pi e^{-i\pi} + \frac{e^{-i\pi}}{i}\right]$$

$$\hat{f}_0 = \frac{1}{2\pi i}\left[-2\sin\pi + \pi(2\cos\pi)\right] = \frac{1}{2\pi i}(-2\pi) = -\frac{1}{i}$$

$$\hat{f}_0 = i$$

Ahora calcularemosa_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta e^{i\theta} \cos n\theta d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{i\theta}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} \theta e^{i\theta(1+n)} d\theta + \int_{-\pi}^{\pi} \theta e^{i\theta(1-n)} d\theta\right]$$

$$I_1 = \int_{-\pi}^{\pi} \theta e^{i\theta(1+n)} d\theta = \left[\frac{\theta e^{i\theta(1+n)}}{i(1+n)} - \int_{-\pi}^{\pi} \frac{e^{i\theta(1+n)} d\theta}{i(1+n)}\right]_{-\pi}^{\pi} = \left[\frac{\theta e^{i\theta(1+n)}}{i(1+n)} - \frac{e^{i\theta(1+n)}}{(i(n+1))^2}\right]_{-\pi}^{\pi} = \left[\frac{\theta e^{i\theta(1+n)}}{i(1+n)} + \frac{e^{i\theta(1+n)}}{(n+1)^2}\right]_{-\pi}^{\pi}$$

$$I_1 = \frac{e^{i\pi(1+n)}}{(1+n)^2} - i\frac{\pi e^{i\pi(1+n)}}{(1+n)} - \frac{e^{-i\pi(1+n)}}{(1+n)^2} - i\frac{\pi e^{-i\pi(1+n)}}{1+n} = \frac{1}{(1+n)}\left[\frac{1}{(1+n)}\left(e^{i\pi(1+n)} - e^{-i\pi(1+n)}\right) - i\pi(e^{-i\pi(1+n)} + e^{-i\pi(1+n)})\right]$$

$$I_1 = \frac{1}{(1+n)}\left[\frac{2i\sin(\pi(1+n))}{1+n} - i\pi(2\cos(\pi(1+n)))\right] = \frac{1}{(1+n)}(-2\pi i(-1)(-1)^n)$$

$$I_1 = -\frac{2\pi i(-1)^{n+1}}{(1+n)}$$

Ahora la segunda integral se resuelve de la misma manera:

$$I_2 = \int_{-\pi}^{\pi} \theta e^{i\theta(1-n)} d\theta = \left[\frac{e^{i\theta(1-n)}}{(1-n)^2} - \frac{\theta i e^{i\theta(1-n)}}{(1-n)}\right]_{-\pi}^{\pi} =$$

$$I_2 = -\frac{2\pi i(-1)^{n+1}}{(1-n)}$$

Por lo que nuestroa_n será:

$$a_n = \frac{1}{2\pi}\left(-\frac{2\pi i(-1)^{n+1}}{(1+n)} - \frac{2\pi i(-1)^{n+1}}{(1-n)}\right) = -\frac{2i(-1)^{n+1}}{(1-n^2)}$$

$$a_n = -\frac{2i(-1)^{n+1}}{(1-n^2)}$$

Ahora debemos calcular b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta e^{i\theta} \sin n\theta = \frac{1}{2\pi i}\left[\int_{-\pi}^{\pi} \theta e^{i\theta(1+n)} d\theta - \int_{-\pi}^{\pi} \theta e^{i\theta(1-n)} d\theta\right] = \frac{1}{2\pi i}\left[I_1 - I_2\right]$$

$$b_n = \frac{1}{2\pi i}\left(-\frac{2\pi i(-1)^{n+1}}{(1+n)} + \frac{2\pi i(-1)^{n+1}}{(1-n)}\right) = (-1)^{n+1}\left[\frac{1}{1-n} - \frac{1}{1+n}\right]$$

$$b_n = \frac{2n(-1)^{n+1}}{(1-n^2)}$$

Finalmente nuestra serie de Fourier será:

$$f(\theta) = i + 2\sum_{n=1}^{\infty}\left[-\frac{i(-1)^{n+1}}{(1-n^2)}\cos n\theta + \frac{2n(-1)^{n+1}}{(1-n^2)}\sin n\theta\right] = i + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(1-n^2)}\left[n\sin n\theta - i\cos n\theta\right]$$

Expresando seno y coseno de forma exponencial obtenenemos:

$$f(\theta) = i + 2\sum_{n=1}^{\infty}\frac{-i(-1)^{n+1}}{2(1-n)(1+n)}(e^{in\theta}(1+n) + e^{-in\theta}(1-n))$$

Finalmente obteniendo:

$$f(\theta) = i - i\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{e^{in\theta}}{1-n} + \frac{e^{-in\theta}}{1+n}\right)$$

Ahora para obtener las series reales de $g(\theta) = \theta\cos\theta$, tendremos:

$$\hat{g_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta\cos n\theta d\theta = \frac{1}{2\pi}\left[\theta\sin\theta + cos\theta\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(0)$$

$$\hat{g_0} = 0$$

Ahora calcularemos el coeficiente a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\cos\theta\cos n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\left(\frac{e^{in\theta}+e^{-in\theta}}{2}\right)$$

$$a_n = \frac{1}{4\pi}\left[\int_{-\pi}^{\pi}\theta e^{i\theta(1+n)}d\theta + \int_{-\pi}^{\pi}\theta e^{i\theta(1-n)}d\theta + \int_{-\pi}^{\pi}\theta e^{i\theta(n-1)}d\theta\int_{-\pi}^{\pi}\theta e^{-i\theta(1+n)}d\theta\right]$$

Usando los resultados anteriores podemos escribir:

$$a_n = \frac{1}{4\pi}\left[-2\pi i(-1)^{n+1}\left(\frac{1}{1+n}+\frac{1}{1-n}\right) + 2\pi i(-1)^{n+1}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\right] = -\frac{i(-1)^{n+1}}{2}\left(\frac{1}{n^2-1}+\frac{1}{1-n^2}\right)$$

$$a_n = 0$$

De forma similar tendremos esta forma para b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\cos\theta\sin n\theta d\theta = \frac{1}{4\pi i}\left[\frac{-2\pi i(-1)^{n+1}}{(n+1)} + \frac{2\pi i(-1)^{n+1}}{(1-n)} - \frac{2\pi i(-1)^{n+1}}{(n-1)} - \frac{2\pi i(-1)^{n+1}}{(n+1)}\right]$$

Simplificando obtendremos:

$$b_n = \frac{2n(-1)^{n+1}}{(1-n^2)}$$

Por lo tanto tendremos que la expansión en series de Fourier de $e^{\theta}\cos n\theta$, es una función real.

$$g(\theta) = 2\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{1-n^2}\sin n\theta$$

Ahora en el caso de $h(\theta) = \theta\sin\theta$

$$\hat{h_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta\sin\theta d\theta = \frac{1}{2\pi}\left[-\theta\cos\theta + \sin\theta\right]_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi-\pi)$$

$$\hat{h_0} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\sin\theta\cos n\theta d\theta = \frac{(-1)^{n+1}}{2}\left[\frac{1}{n-1}-\frac{1}{1+n}-\frac{1}{1-n}-\frac{1}{1+n}\right] = \frac{(-1)^{n+1}}{2}\left[\frac{-2}{1+n}+\frac{2-2n}{(1-n)(n-1)}\right]$$

$$a_n = \frac{2(-1)^{n+1}}{(n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\sin\theta\sin n\theta d\theta = \frac{-i(-1)^{n+1}}{2}\left[\frac{-1}{n+1}+\frac{1}{1-n}+\frac{1}{n-1}+\frac{1}{n+1}\right] = -\frac{i(-1)^{n+1}}{2}\left[\frac{n-1+1-n}{(n-1)(n+1)}\right]$$

$$b_n = 0$$

Por lo que nuestra serie real para $\theta\sin\theta$ será:

$$h(\theta) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos n\theta$$

4)

Solución: Para la función diente de sierra $f(\theta) = \theta^2$ tendremos:

$$\hat{f}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{1}{2\pi}\frac{2\pi^3}{3}$$

$$\hat{f}_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \cos n\theta d\theta = \frac{1}{\pi}\left[\frac{\theta^2 \sin n\theta}{n} - \int_{-\pi}^{\pi}\frac{2\theta \sin n\theta d\theta}{n}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\theta^2 \sin n\theta}{n} - \frac{2}{n}\left(-\frac{\theta\cos n\theta}{n} + \int_{-\pi}^{\pi}\frac{\cos n\theta d\theta}{n}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\theta^2 \sin n\theta}{n} - \frac{2}{n}\left(-\frac{\theta\cos n\theta}{n} + \frac{1}{n}\frac{\sin n\theta}{n}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\theta^2 \sin n\theta}{n} + \frac{2\theta\cos n\theta}{n^2} - \frac{2\sin n\theta}{n^3}\right]_{-\pi}^{\pi}$$

Evaluando en π y $-\pi$ los términos de seno se anulan. Por lo que tendremos:

$$a_n = \frac{1}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2}\right] = \frac{4\pi(-1)^n}{n^2}$$

$$a_n = \frac{4\pi(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \sin n\theta d\theta = \frac{1}{\pi}\left[-\frac{\theta^2\cos n\theta}{n} - \int_{-\pi}^{\pi}\left(-\frac{\cos n\theta}{n}\right)2\theta d\theta\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[-\frac{\theta^2\cos n\theta}{n} + \frac{2}{n}\left(\int_{-\pi}^{\pi}\theta\cos n\theta d\theta\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\theta^2\cos n\theta}{n} + \frac{2}{n}\left(\int_{-\pi}^{\pi}\theta\cos n\theta d\theta\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[-\frac{\theta^2\cos n\theta}{n} + \frac{2}{n}\left(\frac{\theta\sin n\theta}{n} - \int_{-\pi}^{\pi}\frac{\sin n\theta d\theta}{n}\right)\right]_{-\pi}^{\pi}$$

$$b_n = b_n = \frac{1}{\pi}\left[-\frac{\theta^2\cos n\theta}{n} + \frac{2\theta\sin n\theta}{n^2} + \frac{2\cos n\theta}{n^3}\right]_{-\pi}^{\pi}$$

De manera similar los términos de seno se anularán.

$$b_n = \frac{1}{\pi}\left[-\frac{\pi^2(-1)^n}{n} + \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{3} - \frac{2(-1)^n}{3}\right]$$

$$b_n = 0$$

Por lo tanto la serie de Fourier será:

$$f(\theta) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\theta$$

Ahora, usando la identidad de Parseval.

$$\int_{-\pi}^{\pi}[f(\theta)]^2\, d\theta = \int_{-\pi}^{\pi}\theta^4 d\theta = \left[\frac{\theta^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{5}$$

Ahora tendremos que:

$$\frac{2\pi^5}{5} = \pi\left(\frac{(2\hat{f}_0)^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right)$$

Desarrollando obtenemos:

$$\frac{2\pi^4}{5} = \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4} \to \frac{2\pi^4}{5} - \frac{4\pi^4}{18} = 16\sum_{n=1}^{\infty}\frac{1}{n^4} \to \frac{8\pi^4}{45} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Dividiendo por 16 obtenemos:

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

5)

Solución: Al ser una ecuación diferencial homogénea, una de sus soluciones será:

$$y_n(x) = exp\left(ix\frac{n\pi}{L}\right)$$

Con $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ Por lo tanto tendremos:

$$y_n(x) = exp\left(ix\frac{n\pi}{L}\right) = exp\left(ix\sqrt{\lambda_n}\right) = A\cos(\sqrt{\lambda_n}x) + B\sin(\sqrt{\lambda_n}x)$$

Usando la condición de contorno $y(0) = 0$

$$A = 0$$

Si tenemos: $y_n(x) = B\sin(\sqrt{\lambda_n}x)$ y usamos le segunda condición de contorno $y(1) + y'(1) = 0$

$$B\sin(\sqrt{\lambda_n}) + B\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) = 0$$

Si suponemos $B \neq 0$

$$\sin(\sqrt{\lambda_n}) + \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) = 0 \rightarrow \tan(\sqrt{\lambda_n}) + \sqrt{\lambda_n} = 0$$

$$-\tan(\sqrt{\lambda_n}) = \sqrt{\lambda_n}$$

Esta ecuación tiene una infinidad de soluciones λ_n que corresponden a los puntos donde la tangente se anula. Por lo tanto pueden existir $\lambda_1, \lambda_2, ...$ valores propios, tales que $\lambda_1 < \lambda_2 < \lambda_3 < ...$

Si tenemos $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ para $n \rightarrow \infty$ tendremos que se comportará de forma que la componente del seno en la tangente tiene ceros en $n\pi$ y la componente de coseno varía lentamente cerca de esos puntos, por lo que para $n \rightarrow \infty$

$$\lambda_n \approx (n\pi)^2$$

Por lo que los valores λ_n para n muy grande, crecen asintóticamente como $n^2\pi^2$

6)

a)

Dividiendo por $(1 - x^2)$

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{n(n+1)}{(1-x^2)}y = 0$$

De aquí podemos concluir que:

$$p(x) = -\frac{2x}{(1-x^2)}$$

$$q(x) = \frac{1}{(1-x^2)}$$

Reescribiendo la ecuación:

$$y'' - \frac{2x}{(1-x^2)}y' = -\frac{n(n+1)}{(1-x^2)}y$$

Y si tenemos la forma Sturm-Liouville:

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = \lambda w(x)y$$

Igualando las partes derechas:

$$-\frac{n(n+1)}{(1-x^2)}y = \lambda w(x)y$$

De aquí podemos concluir que:

$$w(x) = \frac{1}{x^2 - 1}$$
$$\lambda = n(n+1)$$

Ahora reescribiendola en la forma SL:

$$-\frac{d}{dx}\left[-\frac{2x}{(1-x^2)}\frac{dy}{dx}\right] + \frac{1}{(1-x^2)}y = \frac{n(n+1)}{(x^2-1)}y$$

b)

Dividiendo por x:

$$y'' + \frac{b-x}{x}y' - \frac{a}{x}y = 0$$

De aquí concluimos que:

$$p(x) = \frac{b-x}{x}$$
$$q(x) = \frac{a}{x}$$

Igualando con la expresión de SL:

$$\frac{a}{x}y = \lambda w(x) y$$

De aquí concluimos que:

$$w(x) = \frac{a}{x}$$
$$\lambda = 1$$

Escribiendo en la forma SL:

$$-\frac{d}{dx}\left[\frac{b-x}{x}\frac{dy}{dx}\right] + \frac{a}{x}y = \frac{a}{x}y$$

c)

Tenemos que la ecuación es homogénea:

$$y'' + 4y' + (4+\lambda) + y = 0$$

Consideremos una solución de la forma: $y_h(x) = exp(ix\sqrt{\lambda_n})$ Ahora calculando su primera y segunda derivada tendremos:

$$y_h'(x) = i\sqrt{\lambda_n}exp(ix\sqrt{\lambda_n})$$
$$y_h''(x) = (i\sqrt{\lambda_n})^2 exp(ix\sqrt{\lambda_n})$$

Reemplazando:

$$(i\sqrt{\lambda_n})^2 exp(ix\sqrt{\lambda_n}) + 4i\sqrt{\lambda_n}exp(ix\sqrt{\lambda_n}) + 4exp(ix\sqrt{\lambda_n}) + \lambda_n exp(ix\sqrt{\lambda_n}) = 0$$

Usando la condición de contorno $y(0) = 0$

$$(i\sqrt{\lambda_n})^2 + 4i\sqrt{\lambda_n} + 4 + \lambda_n = 0 \to -\sqrt{\lambda_n} + 4i + \frac{4}{\sqrt{\lambda_n}} + \sqrt{\lambda_n} = 0$$

$$\sqrt{\lambda_n} = -\frac{1}{i} \to \lambda_n = \pm 1$$

Por lo tanto los valores propios serán:

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Y las funciones propias serán.

$$y_1(x) = e^{ix}$$
$$y_2(x) = e^{-x}$$

7)

Solución: Si tomamos $u = e^{-x^2}$ y tomamos su derivada $u' = -2xe^{-x^2}$

$$u' + 2xu = 0$$

Usando la formula de Leibniz para la derivada $n+1$ veces del producto, obtendremos:

$$(xu)^{n+1} = xu^{(n+1)} + (n+1)u^{(n)}$$

$$u^{(n+2)} + 2xu^{(n+1)} + 2(n+1)u^{(n)} = 0$$

Si definimos $v = (-1)^n u^{(n)}$

$$v'' + 2v' + 2(n+1)v = 0$$

Si definimos $y = ve^{x^2} \to v = ye^{-x^2}$ y si tomamos sus derivadas:

$$v' = e^{-x^2}(y' - 2xy)$$

$$v'' = -2xe^{-x^2}(y' - 2xy) + e^{-x^2}(y'' - 2y - 2xy') = e^{-x^2}(y'' - 4xy' + y(4x^2 - 2))$$

Reemplazando y multiplicando por e^{x^2}

$$y'' - 4xy' + (4x^2 - 2)y + 2x(y' - 2xy) + 2(n+1)y = 0$$

$$y'' - 2xy' + 2ny = 0$$

La cual es la ecuacún de Hermite, si tomamos:

$$y(x) = H_n(x) = e^{x^2}v = (-1)^n e^{x^2} u^{(n)} = (-1)e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

Por lo que:

$$H_n(x) = (-1)e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

8)

Solución:

$$I = \int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx$$

Si consideramos $n \geq m$

$$I = (-1)^n \int_{-\infty}^{\infty} H_m(x)\frac{d^n}{dx^n}(e^{-x^2})$$

Integrando por partes:

$$I = (-1)^{n+1}2m \int_{-\infty}^{\infty} H_{m-1}(x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2})$$

Integrando por parte m veces:

$$I = (-1)^m(-1)^n 2^m m! \int_{-\infty}^{\infty} H_0(x)\frac{d^{n-m}}{dx^{n-m}}(e^{-x^2})$$

Si $n > m$

$$I = (-1)^{n+m}2^m m! \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}}(e^{-x^2}) = \left[(-1)^{n+m}2^m m! \frac{d^{n-m-1}}{dx^{n-m-1}}(e^{-x^2})\right]_{-\infty}^{\infty} = 0$$

Si $n = m$

$$I = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi}$$

Resumiendo:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{mn}$$

Si definimos las funciones:

$$\varphi_n(x) = \frac{H_n(x) e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}$$

Es claro que $\{\varphi_n\}_n$ es un conjunto ortogonal.

$$\int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) dx = \delta_{mn}$$