Vector potential and electric field of a rotating charged sphere

Solution: *a. The charge density is,*

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\delta(r-R) \ ,$$

giving the current density,

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}) = \rho(\mathbf{r})[\vec{\omega}\times\mathbf{r}]$$

Now, we have at the surface,

$$\mathbf{r} = R\hat{\mathbf{e}}_r = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ .$$

With $\vec{\omega} = \omega\hat{\mathbf{e}}_z = \omega(0,0,1)$ *we obtain,*

$$\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) = \sin\theta(-\sin\phi, \cos\phi, 0) = \sin\theta\hat{\mathbf{e}}_\phi$$

and therefore the current density,

$$\mathbf{j}(\mathbf{r}) = \frac{Q\omega}{4\pi R}\sin\theta\delta(r-R)\hat{\mathbf{e}}_\phi$$

b. For the potential vector we have,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' = \frac{Q}{(4\pi)^2R^2}\vec{\omega}\times\int d^3\mathbf{r}\ '\delta(r'-R)\frac{\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \ .$$

For the integration we let $\mathbf{r}$ *go towards* z *(in this case, the direction of* $\vec{\omega}$ *is no longer the* z*-axis). Now we have,* $|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2rr'\cos\theta'}$ *e* $\mathbf{r}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$. *A integration over* ϕ' *then gives for the* x *and* y *components the value 0. Denoting* $\cos'$ *by* u' *and replacing* $\hat{\mathbf{e}}_z$, *we get again, after integration for* $\hat{\mathbf{e}}_r$,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{2\pi Q}{(4\pi)^2R^2}(\vec{\omega}\times\hat{\mathbf{e}}_r)\int r'^3\delta(r'-R)\int_{-1}^{+1}\frac{u'du'}{\sqrt{r^2+r'^2-2rr'u'}}dr' \\ &= \frac{\mu_0 QR}{8\pi}(\vec{\omega}\times\hat{\mathbf{e}}_r)\int_{-1}^{+1}\frac{u'du'}{\sqrt{r^2+R^2-2rRu'}} \ .\end{aligned}$$

The integral can be resolved by partial integration,

$$\begin{aligned}I &= \int_{-1}^{+1}\frac{u'du'}{\sqrt{r^2+R^2-2rRu'}} = -\frac{1}{rR}\left[u'\sqrt{r^2+R^2-2rRu'}\right]_{-1}^{+1} + \frac{1}{rR}\int_{-1}^{+1}\sqrt{r^2+R^2-2rRu'} \\ &= -\frac{1}{rR}(|r-R|+|r+R|) + \frac{1}{rR}\left[-\frac{2}{3}\frac{1}{2rR}(r^2+R^2-2rRu')^{3/2}\right]_{-1}^{+1} \\ &= -\frac{1}{rR}(|r-R|+|r+R|) - \frac{1}{3r^2R^2}(|r-R|^3-|r+R|^3) \ .\end{aligned}$$

For $r > R$ *we get from this,*

$$I = \frac{2R}{3r^2}$$

while for $r < R$

$$I = \frac{2r}{3R^2}$$

With this we get for the potential vector,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 QR}{12\pi}(\vec{\omega} \times \hat{\mathbf{e}}_r) \begin{cases} \frac{r}{R^2} & \textit{if} \;\; r < R \\ \frac{R}{r^2} & \textit{se} \;\; r > R \end{cases} = \frac{\mu_0 QR}{12\pi} \omega \sin\theta \hat{\mathbf{e}}_\phi \begin{cases} \frac{r}{R^2} & \textit{if} \;\; r < R \\ \frac{R}{r^2} & \textit{se} \;\; r > R \end{cases}$$

where we use again that $\vec{\omega} = \omega \hat{\mathbf{e}}_z$ *e* $\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = \sin\theta \hat{\mathbf{e}}_\phi$. *Obviously,* $\mathbf{A}$ *has only the component* ϕ.

b. Now the magnetic induction is $\mathbf{B} = \nabla \times \mathbf{A}$. *Using the rotation in spherical coordinates (1.91), we get immediately,*

$$\mathbf{B} = \hat{\mathbf{e}}_r \left[\frac{1}{r \sin\theta} \frac{\partial A_\phi \sin\theta}{\partial \theta} \right] - \hat{\mathbf{e}}_\theta \left[\frac{1}{r} A_\phi + \frac{\partial A_\phi}{\partial r} \right] = \frac{\mu_0 Q \omega}{6\pi R} \begin{cases} (\hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta) & \textit{se} \;\; r < R \\ \frac{R^3}{r^3} (\hat{\mathbf{e}}_r \cos\theta + \frac{1}{2} \hat{\mathbf{e}}_\theta \sin\theta) & \textit{se} \;\; r > R \end{cases}$$

With $\hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta = \hat{\mathbf{e}}_z$ *we find that, curiously, the magnetic field is homogeneous within the layer.*