

# 5

## CHAPTER

# Wave Properties of Matter and Quantum Mechanics I

*I thus arrived at the following overall concept which guided my studies:  
for both matter and radiations, light in particular, it is necessary to in-  
troduce the corpuscle concept and the wave concept at the same time.*

**Louis de Broglie, 1929**

*We regard quantum mechanics as a complete theory for which the fun-  
damental physical and mathematical hypotheses are no longer suscepti-  
ble of modification.*

**Werner Heisenberg and Max Born,  
paper delivered to the Solvay Congress, 1927**

Chapter 3 presented compelling evidence that light (electromagnetic radiation) must be particle-like to explain phenomena such as the photoelectric effect and Compton scattering. The emission and absorption of photons in atoms allow us to understand the optical spectra of hydrogen atoms.

In this chapter we discuss so many new, surprising results that an overview of them here is in order. For example, we already know that photons, as electromagnetic radiation, demonstrate wavelike properties. The only way we can interpret certain experimental observations is to conclude that wavelike properties are also exhibited by “particles” of matter. We begin the chapter by discussing experiments that prove that photons, in the form of x rays, behave as waves when passing through crystals. De Broglie’s suggestion that particles may also behave as waves was verified by the electron-scattering experiments of Davisson and Germer.

We then present a short review of wave phenomena, including a description of the localization of a particle in terms of a collection of waves. Physicists in the first part of the twentieth century had considerable difficulty understanding how wavelike and particle-like properties can occur in nature in the same entity. We now face the same hurdle. Niels Bohr’s principle of complementarity convinces us that *both* wavelike and particle-like properties are needed to give a complete description of matter (electrons, protons, and so on) and radiation (photons). We shall see that certain physical observables can only be expressed in terms of probabilities, with those probabilities determined by using wave functions  $\Psi(x, t)$ . Heisenberg’s uncertainty principle plays a major role in our understand-

ing of particle-like and wavelike behavior. This principle prohibits the precise, simultaneous knowledge of both momentum and position or of both energy and time. We will see that no experiment requires us to utilize both wave and particle properties *simultaneously*. Although modern quantum theory is applicable primarily at the atomic level, there are many macroscopic observations of its effects.

## 5.1 X-Ray Scattering

Following Röntgen's discovery of x rays in 1895, intense efforts were made to determine the nature and origin of the new penetrating radiation. Charles Barkla (Nobel Prize, 1917) made many x-ray measurements at Liverpool University during the early 1900s and is given credit for discovering that each element emits x rays of characteristic wavelengths and that x rays exhibit properties of polarization.

By 1912 it became clear that x rays were a form of electromagnetic radiation and must therefore have wave properties. However, because it had proved difficult to refract or diffract x rays as easily as visible light, it was suggested that their wavelengths must be much shorter than those of visible light. Max von Laue (1879–1960, Nobel Prize for Physics, 1914), a young theoretical physicist at the University of Munich, became interested in the nature of x rays primarily because of the presence at Munich of Röntgen and the theorist Arnold Sommerfeld (1868–1951), who would later play an important role in understanding atomic structure. Wilhelm Wien (1864–1928) and Sommerfeld, among others, estimated the wavelength of an x ray to be between  $10^{-10}$  and  $10^{-11}$  m. Knowing the distance between atoms in a crystal to be about  $10^{-10}$  m, Laue made the brilliant suggestion that x rays should scatter from the atoms of crystals. He suggested that, if x rays were a form of electromagnetic radiation, interference effects should be observed. From the study of optics, we know that wave properties are most easily demonstrated when the sizes of apertures or obstructions are about equal to or smaller than the wavelength of the light. We use gratings in optics to separate light by diffraction into different wavelengths. Laue suggested that crystals might act as three-dimensional gratings, scattering the waves and producing observable interference effects.

Laue designed the experiment and convinced two experimental physicists at Munich, Walter Friedrich and Paul Knipping, to perform the measurement. A schematic diagram of the transmission Laue process is shown in Figure 5.1 (page 164), along with one of Friedrich and Knipping's earliest experimental results. When they rotated the crystals, the positions and intensities of the diffraction maxima were shown to change. Laue performed the complicated analysis necessary to prove that x rays were scattered as waves from a three-dimensional crystal grating. Though the primary purpose of Laue's proposal was to prove the wave nature of x rays, he ended up also demonstrating the lattice structure of crystals, which led to the origin of solid-state physics and the development of modern electronics.

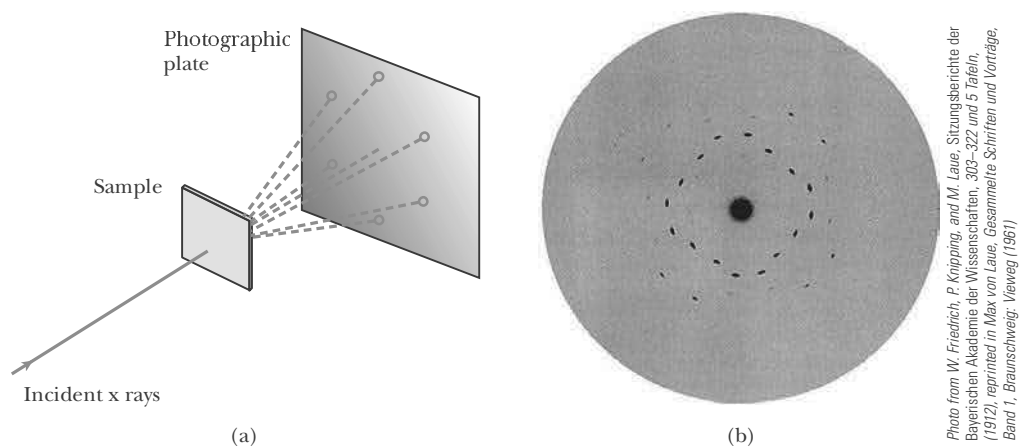
Two English physicists, William Henry Bragg and his son, William Lawrence Bragg, fully exploited the wave nature of x rays and simplified Laue's analysis. W. L. Bragg pointed out in 1912 that each of the images surrounding the bright central spot of the Laue photographs could be interpreted as the reflection of the incident x-ray beam from a unique set of planes of atoms within the crystal. Each dot in the pattern corresponds to a different set of planes in the crystal (see Figure 5.1b).



Photograph by Gen. Stab. Lit. Anst., courtesy AIP Emilio Segrè Visual Archives, Weber and Physics Today Collections.

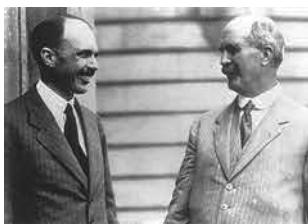
**Max von Laue** (1879–1960) was born, educated, and worked most of his life in Germany. After studying at Strasburg, Göttingen, and Munich, he received his doctorate in 1903 from the University of Berlin where he studied under Max Planck. He subsequently worked at several German universities and did his Nobel Prize-winning work on x-ray diffraction at Munich. He spent most of his productive career as a theoretical physicist in Berlin, where he had considerable influence on the development of scientific research in Germany.

**Laue proved wave nature of x rays and emphasized lattice structure of crystals**



**Figure 5.1** (a) Schematic diagram of Laue diffraction transmission method. A wide range of x-ray wavelengths scatters from a crystal sample. The x rays constructively interfere from certain planes, producing dots. (b) One of the first results of Friedrich and Knipping in 1912 showing the symmetric placement of *Laue dots* of x-ray scattering from ZnS. The analysis of these results by Laue, although complex, convincingly proved that x rays are waves.

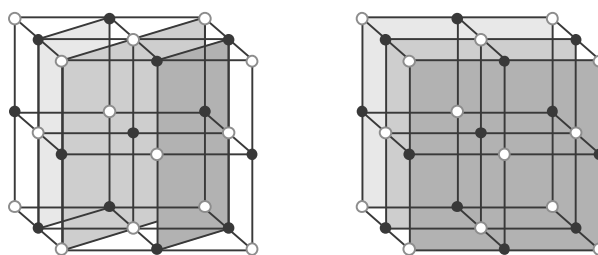
Courtesy Edgar Fahs Smith Memorial Collection, Department of Special Collections, University of Pennsylvania Library.



**William Lawrence Bragg** (1890–1971) (left) and **William Henry Bragg** (1862–1942) (right) were a son-father team, both of whom were educated at Cambridge. The father spent 22 years at the University of Adelaide in Australia, where his son was born. Both father and son initially studied mathematics but eventually changed to physics. The father was a skilled experimenter, and the son was able to conceptualize physical problems and express them mathematically. They did their important work on x-ray crystallography in 1912–1914 while the father was at the University of Leeds and the son was a graduate student at Cambridge working under J. J. Thomson. Both physicists had long and distinguished careers, with the son being director of the famous Cavendish Laboratory at Cambridge from 1938 to 1953. W. Lawrence Bragg received his Nobel Prize at age 25.

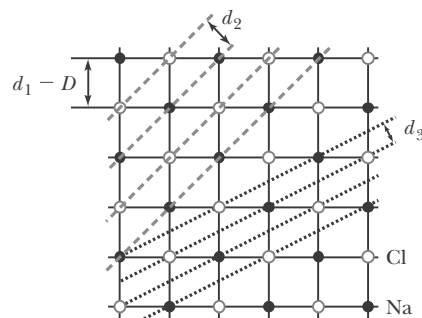
Is x-ray scattering from atoms within crystals consistent with what we know from classical physics? From classical electromagnetic theory we know that the oscillating electric field of electromagnetic radiation polarizes an atom, causing the positively charged nucleus and negatively charged electrons to move in opposite directions. The result is an asymmetric charge distribution, or electric dipole. The electric dipole oscillates at the same frequency as the incident wave and in turn reradiates electromagnetic radiation at the same frequency but in the form of spherical waves. These spherical waves travel throughout the matter and, in the case of crystals, may constructively or destructively interfere as the waves pass through different directions in the crystal.

If we consider x rays scattered from a simple rock salt crystal (NaCl, shown in Figure 5.2), we can, by following the Bragg simplification, determine conditions necessary for constructive interference. We study solids in Chapter 10, but for now note that the atoms of crystals like NaCl form lattice planes, called **Bragg planes**. We can see from Figure 5.3 that it is possible to have many Bragg planes in a crystal, each with different densities of atoms. Figure 5.4 shows an incident



**Figure 5.2** The crystal structure of NaCl (rock salt) showing two of the possible sets of lattice planes (Bragg planes).

Photo from W. Friedrich, P. Knipping, and M. Laue, *Sitzungsberichte der Bayerischen Akademie der Wissenschaften*, 303–322 und 5. Tafeln, (1912), reprinted in Max von Laue, *Gesammelte Schriften und Vorträge*, Band 1, Braunschweig: Vieweg (1961)



**Figure 5.3** Top view of NaCl (cubic crystal), indicating possible lattice planes.  $D$  is the interatomic spacing and the  $d_i$  are the distances between lattice planes.

plane wave of monochromatic x rays of wavelength  $\lambda$  scattering from two adjacent planes. There are two conditions for constructive interference of the scattered x rays:

1. The angle of incidence must equal the angle of reflection of the outgoing wave.
2. The difference in path lengths ( $2d \sin \theta$ ) shown in Figure 5.4 must be an integral number of wavelengths.

We will not prove condition 1 but will assume it.\* It is referred to as the *law of reflection* ( $\theta_{\text{incidence}} = \theta_{\text{reflection}}$ ), although the effect is actually due to diffraction and interference. Condition 2 will be met if

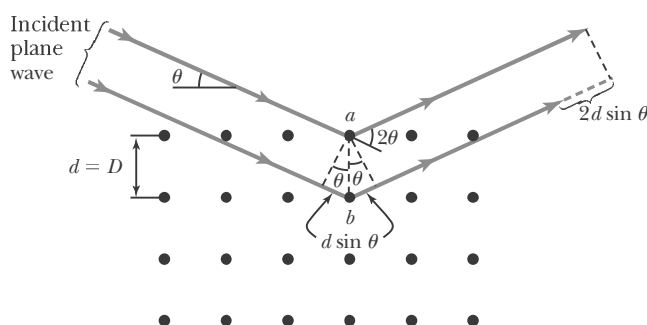
$$n\lambda = 2d \sin \theta \quad (n = \text{integer}) \quad (5.1)$$

**Conditions for constructive interference**

**Bragg's law**

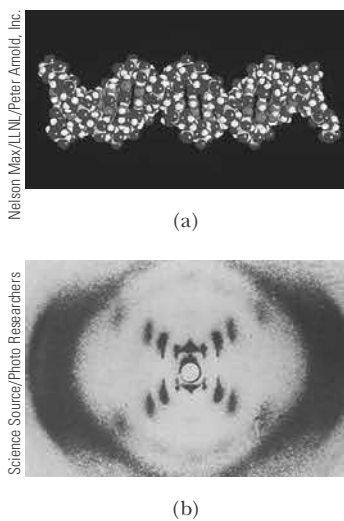
as can be seen from Figure 5.4, where  $D$  is the interatomic spacing (distance between atoms) and  $d$  is the distance between lattice planes. Equation (5.1) was first presented by W. L. Bragg in 1912 after he learned of Laue's results. The integer  $n$  is called the *order of reflection*, following the terminology of ruled diffraction gratings in optics. Equation (5.1) is known as **Bragg's law** and is useful for determining either the wavelength of x rays or the interplanar spacing  $d$  of the crystal if  $\lambda$  is already known.

W. H. Bragg and W. L. Bragg (who shared the 1915 Nobel Prize) constructed an apparatus similar to that shown in Figure 5.5 (page 166), called a *Bragg spectrometer*, and scattered x rays from several crystals. The intensity of the diffracted beam is determined as a function of scattering angle by rotating the crystal and the detector. The Braggs' studies opened up a whole new area of research that continues today.



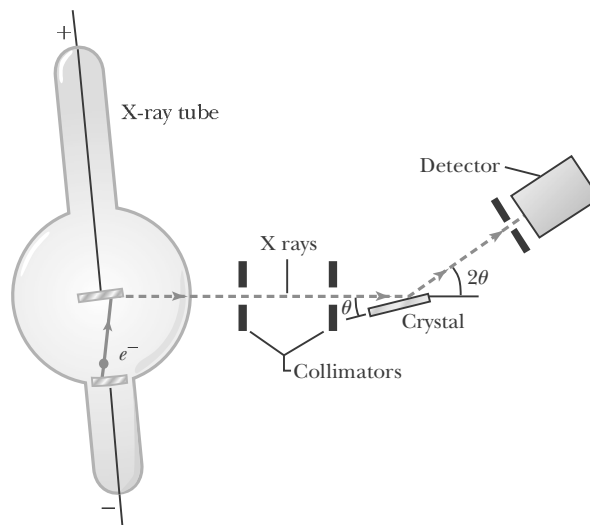
**Figure 5.4** Schematic diagram illustrating x-ray scattering from Bragg lattice planes. The path difference of the two waves illustrated is  $2d \sin \theta$ . Notice that the actual scattering angle from the incident wave is  $2\theta$ .

\*See L. R. B. Elton and D. F. Jackson, *American Journal of Physics* **34**, 1036 (1966), for a proof.



**Figure 5.6** (a) A computer graphic of the DNA double helix is shown. (b) This complex structure was understood only after hundreds of x-ray diffraction photos like this one by Rosalind Franklin were studied. Franklin, who worked at King's College in London in the early 1950s, produced the images of the DNA molecule that helped Watson and Crick unravel the DNA structure. Franklin died in 1958 at the age of 37, four years before the Nobel Prize was awarded to Watson and Crick.

### Powder technique

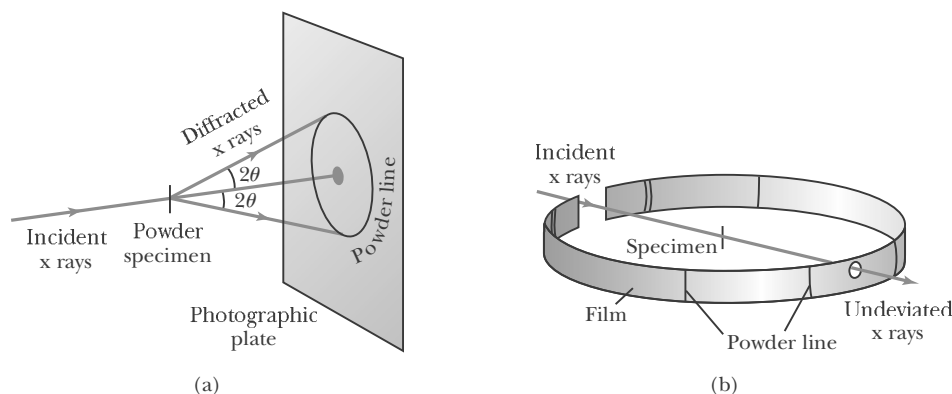


**Figure 5.5** Schematic diagram of Bragg spectrometer. X rays are produced by electron bombardment of metal target. The x rays are collimated by lead, scatter from a crystal, and are detected as a function of the angle  $2\theta$ .

Laue diffraction is primarily used to determine the orientation of single crystals by mounting the large crystals in a precisely known orientation. Radiation of many wavelengths (“white” light) is projected parallel to a high-symmetry direction of the crystal and, in the transmission method, produces arrays of interference maxima spots indicative of a particular plane in the crystal. These techniques are used to determine the complete structure of crystalline materials including a wide range of novel compounds from simple inorganic solids to complex macromolecules, such as proteins. Bragg and Laue x-ray diffraction techniques tell us almost everything we know about the structures of solids, liquids, and even complex molecules such as DNA (see Figure 5.6).

If a single large crystal is not available, then many small crystals may be used. If these crystals are ground into a powdered form, the small crystals will then have random orientations. When a beam of x rays passes through the powdered crystal, the interference maxima appear as a series of rings. This technique, called **powder x-ray diffraction (XRD)**, is widely used to determine the structure of unknown solids, including the crystallographic structure and size. A schematic diagram of the powder techniques is shown in Figure 5.7a, along with the film arrangement to record powder photographs in Figure 5.7b. The lines indicated in part (b) are sections of rings called the *Debye-Scherrer pattern*, named after the discoverers. Figure 5.7c is a sequence of four photographs, each with an increasingly larger number of crystals, which indicates the progression from the Laue dots to the rings characteristic of the powder photographs.

Example 5.1 shows the value of x-ray crystallography and its tremendous usefulness. The technique pioneered by physicists in the first part of the twentieth century continues to be useful to many scientists in varied fields today.



**Figure 5.7** (a) Diagram showing the experimental arrangement of producing powder photographs from random-oriented crystals. (b) Film arrangement to record powder photographs. (c) The four photos show a progression of x-ray photographs for fluorite from a single crystal (clearly showing dots), through a few crystals, to a large number of crystals, which gives the rings the characteristic of an ideal powder photograph. (a) and (b) from N. F. M. Henry, H. Lipson, and W. A. Wooster, *The Interpretation of X-ray Diffraction Photographs*, London: MacMillan (1960).



## EXAMPLE 5.1

X rays scattered from rock salt (NaCl) are observed to have an intense maximum at an angle of  $20^\circ$  from the incident direction. Assuming  $n = 1$  (from the intensity), what must be the wavelength of the incident radiation?

**Strategy** We will use Equation (5.1) to find  $\lambda$ , but we need to know  $d$ , the lattice spacing, and the angle  $\theta$ . Notice that the angle between the incident beam and scattered wave for constructive interference is always  $2\theta$  (see Figures 5.4 and 5.5), and because  $2\theta = 20^\circ$ , we have  $\theta = 10^\circ$ . We can use the density of NaCl to help find  $d$ , because the volume taken up by one atom is  $d^3$ .

**Solution** In Section 4.1 we showed that

$$\frac{\text{Number of molecules}}{\text{Volume}} = \frac{N_A \rho}{M}$$

where  $N_A$  is Avogadro's number,  $\rho$  is the density, and  $M$  is the gram-molecular weight. For NaCl,  $\rho = 2.16 \text{ g/cm}^3$  and  $M = 58.5 \text{ g/mol}$ .

$$\frac{N_A \rho}{M} = \frac{\left(6.02 \times 10^{23} \frac{\text{molecules}}{\text{mol}}\right) \left(2.16 \frac{\text{g}}{\text{cm}^3}\right)}{58.5 \frac{\text{g}}{\text{mol}}}$$

$$\begin{aligned} \frac{N_A \rho}{M} &= 2.22 \times 10^{22} \frac{\text{molecules}}{\text{cm}^3} \\ &= 4.45 \times 10^{22} \frac{\text{atoms}}{\text{cm}^3} \\ &= 4.45 \times 10^{28} \frac{\text{atoms}}{\text{m}^3} \end{aligned}$$

Because NaCl has a cubic array, we take  $d$  as the distance between Na and Cl atoms, so we have a volume of  $d^3$  per atom.

$$\frac{1}{d^3} = 4.45 \times 10^{28} \frac{\text{atoms}}{\text{m}^3}$$

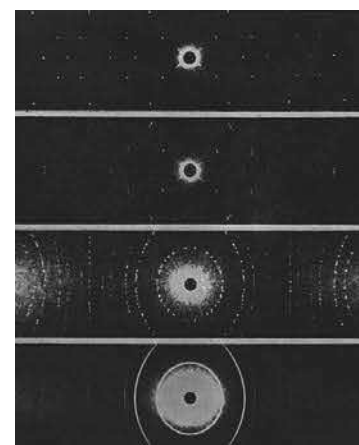
$$d = 2.82 \times 10^{-10} \text{ m} = 0.282 \text{ nm}$$

This technique of calculating the lattice spacing works for only a few cases because of the variety of crystal structures, many of which are noncubic.

We use Equation (5.1) to find  $\lambda$ .

$$\lambda = \frac{2d \sin \theta}{n} = \frac{(2)(0.282 \text{ nm})(\sin 10^\circ)}{1} = 0.098 \text{ nm}$$

which is a typical x-ray wavelength. NaCl is a useful crystal for determining x-ray wavelengths and for calibrating experimental apparatus.



From H. S. Lipson, *Crystals and X-Rays*, London: Wykeham (1970).



After serving in World War I, **Prince Louis V. de Broglie** (1892–1987) resumed his studies toward a doctoral degree at the University of Paris in 1924, where he reported his concept of matter waves as part of his doctoral dissertation. De Broglie spent his life in France where he enjoyed much success as an author and teacher.

## 5.2 De Broglie Waves

By 1920 it was established that x rays were electromagnetic radiation that exhibited wave properties. X-ray crystallography and its usefulness in studying the crystalline structure of atoms and molecules was being established. However, a detailed understanding of the atom was still lacking. Many physicists believed that a new, more general theory was needed to replace the rudimentary Bohr model of the atom. An essential step in this development was made by a young French graduate student, Prince Louis V. de Broglie, who began studying the problems of the Bohr model in 1920.

De Broglie was well versed in the work of Planck, Einstein, and Bohr. He was aware of the duality of nature expressed by Einstein in which matter and energy were not independent but were in fact interchangeable. De Broglie was particularly struck by the fact that photons (electromagnetic radiation) had both wave (x-ray crystallography) and corpuscular (photoelectric effect) properties. The concept of waves is needed to understand interference and diffraction (Section 5.1), but localized corpuscles are needed to explain phenomena like the photoelectric effect (Section 3.6) and Compton scattering (Section 3.8). If electromagnetic radiation must have *both wave and particle properties*, then why should material particles not have both wave and particle properties as well? According to de Broglie, the symmetry of nature encourages such an idea, and no laws of physics prohibit it.

When de Broglie presented his new hypothesis in a doctoral thesis to the University of Paris in 1924, it aroused considerable interest. De Broglie used Einstein's special theory of relativity together with Planck's quantum theory to establish the wave properties of particles. His fundamental relationship is the prediction

$$\lambda = \frac{h}{p} \quad (5.2)$$

That is, the wavelength to be associated with a particle is given by Planck's constant divided by the particle's momentum.

De Broglie was guided by the concepts of phase and group velocities of waves (see Section 5.4) to arrive at Equation (5.2). Recall that for a photon  $E = pc$ , and  $E = hf$ , so that

$$\begin{aligned} hf &= pc = p\lambda f \\ h &= p\lambda \end{aligned}$$

and

$$\lambda = \frac{h}{p} \quad (5.3)$$

De Broglie extended this relation for photons to all particles. Particle waves were called **matter waves** by de Broglie, and the wavelength expressed in Equation (5.2) is now called the **de Broglie wavelength** of a particle.

**Matter waves**

### EXAMPLE 5.2

Calculate the de Broglie wavelength of (a) a tennis ball of mass 57 g traveling 25 m/s (about 56 mph) and (b) an electron with kinetic energy 50 eV.

**Strategy** The calculation for both of these wavelengths is a straightforward application of Equation (5.2).

**Solution** (a) For the tennis ball,  $m = 0.057$  kg, so

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34} \text{ J} \cdot \text{s}}{(0.057 \text{ kg})(25 \text{ m/s})} = 4.7 \times 10^{-34} \text{ m}$$

(b) For the electron, it is more convenient to use eV units, so we rewrite the wavelength  $\lambda$  as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}} = \frac{hc}{\sqrt{2(mc^2)K}}$$

$$\lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{(2)(0.511 \times 10^6 \text{ eV})(50 \text{ eV})}} = 0.17 \text{ nm}$$

Note that because the kinetic energy of the electron is so small, we have used a nonrelativistic calculation. Calculations in modern physics are normally done using eV units, both because it is easier and also because eV values are more appropriate for atoms and nuclei (MeV, GeV) than are joules. The values of  $hc$  and some masses can be found inside the front cover.

How can we show whether such objects as the tennis ball or the electron in the previous example exhibit wavelike properties? The best way is to pass the objects through a slit having a width of the same dimension as the object's wavelength. We expect it to be virtually impossible to demonstrate interference or diffraction for the tennis ball, because we cannot find a slit as narrow as  $10^{-34}$  m. It is unlikely we will ever be able to demonstrate the wave properties of the tennis ball. But the de Broglie wavelength of the 50-eV electron, about 0.2 nm, is large enough that we should be able to demonstrate its wave properties. Because of their small mass, electrons can have a small momentum and in turn a large wavelength ( $\lambda = h/p$ ). Electrons offer our best chance of observing effects due to matter waves.

## Bohr's Quantization Condition

One of Bohr's assumptions concerning his hydrogen atom model was that the angular momentum of the electron-nucleus system in a stationary state is an integral multiple of  $h/2\pi$ . Let's now see if we can predict this result using de Broglie's result. Represent the electron as a standing wave in an orbit around the proton. The condition for a standing wave in this configuration is that the entire length of the standing wave must just fit around the orbit's circumference. We show an example of this in Figure 5.8. In order for it to be a correct standing wave, we must have

$$n\lambda = 2\pi r$$

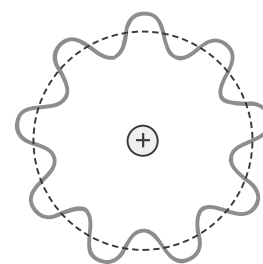
where  $r$  is the radius of the orbit. Now we use the de Broglie relation for the wavelength and obtain

$$2\pi r = n\lambda = n \frac{h}{p}$$

The angular momentum of the electron in this orbit is  $L = rp$ , so we have, using the above relation,

$$L = rp = \frac{nh}{2\pi} = n\hbar$$

We have arrived at Bohr's quantization assumption by simply applying de Broglie's wavelength for an electron in a standing wave. This result seemed to justify Bohr's assumption. De Broglie's wavelength theory for particles was a crucial step toward the new quantum theory, but experimental proof was lacking. As we will see in the next section, this was soon to come.



**Figure 5.8** A schematic diagram of standing waves in an electron orbit around a nucleus. An integral number of wavelengths fits in the orbit. Note that the electron does not “wobble” around the nucleus. The displacement from the dashed line represents its wave amplitude.





# Special Topic

## *Cavendish Laboratory*

Before the 1870s most of our scientific knowledge resulted from the research of people working in their own private laboratories. William Thomson (who would later become Lord Kelvin) established a laboratory at the University of Glasgow in the 1840s, and in the 1860s efforts began at both Oxford and Cambridge to build physical laboratories. In 1871 James Clerk Maxwell was called from his Scottish home to become the first Cavendish Professor at Cambridge University. Maxwell began planning and supervising the construction of the laboratory on Free School Lane in central Cambridge with an unexpected fervor while he gave regular lectures to students. The most important work of the day was to demonstrate the existence of Maxwell's electromagnetic waves, but they were "scooped" by Heinrich Hertz in Germany. Maxwell's successor was Lord Rayleigh, who published 50 papers during his five years at Cavendish before returning to his estate farm where he made most of his discoveries (including the noble gases) at his private laboratory.

The appointment of the young J. J. Thomson at age 28 as Cavendish Professor in 1884 was the beginning of a long and fruitful era in atomic physics. The discovery of the electron in 1897, the arrival of the young Ernest Rutherford from New Zealand as a student, and the early work of C. T. R. Wilson that led to the development of the cloud chamber all helped the Cavendish Laboratory expand, prosper, and grow in stature under Thomson's leadership. Thomson's 35-year leadership was remarkable in many ways, particularly in the manner he stepped down in 1919 upon the opportunity of attracting Rutherford back to Cavendish to be the next Professor.

During Rutherford's 19-year reign, the Cavendish became the most renowned center of science in the world. It attracted the best students, researchers, and visitors from all over the world. Rutherford was a team leader, and he surrounded himself with a collection of young physicists whom he called "his boys." By the end of the Rutherford era in 1937, the laboratory was mov-

ing into new directions with particle accelerators and cryogenic labs.

World War II would change the face of the Cavendish forever. Physicists spread out to perform wartime research, particularly on the development of the atomic bomb and radar, both of which played large roles in the allied victory. William Lawrence Bragg returned to Cambridge as Cavendish Professor to succeed Rutherford in 1937, and the field of x-ray crystallography flourished. The Cavendish scientists have had an uncanny ability to choose productive research areas. It has been said that the fields of molecular biology and radio astronomy started at the Cavendish in the late 1940s, and Bragg must be given credit for the foresight to support these fledgling subjects in the face of "Big Science" in the United States. Bragg's tenure as Cavendish Professor ended in 1953 just when Watson and Crick succeeded in discovering the DNA structure. Bragg also supported J. A. Ratcliffe and Martin Ryle, who had worked on radar at the Cavendish during the war, to construct the first radio telescope. This effort led to the discovery of quasars and pulsars.

When Sir Nevill Mott succeeded Bragg as Cavendish Professor in 1954, the lab made a turn toward solid state physics. Mott had worked on collision theory and nuclear problems in the 1930s but eventually turned to theoretical investigations of electronic systems. Brian Josephson did his pioneering theoretical work (see Chapter 10) on the supercurrent through a tunnel barrier while a student, graduating in 1964 with his Ph.D. In 1974 the Cavendish moved to a new site in West Cambridge. Condensed matter physics now accounts for the greater part of research at the Cavendish, but the groups in radio astronomy and high-energy physics are still important. The Cavendish Laboratory has set a standard that other laboratories can only hope to emulate.

We end with a list of Nobel Prizes awarded to those who did their most important work at the Cavendish Laboratory. The asterisks (for example, Rutherford and Rayleigh) indicate Nobel Prizes awarded primarily for work done elsewhere to people who are still widely associated with the Cavendish Laboratory.



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**Figure A** Upper left, the old Cavendish Laboratory on Free School Lane in Cambridge. The original building is to the left of the gate. The first four Cavendish professors: James Clerk Maxwell, upper right; Lord Rayleigh, bottom left; and Sir J. J. Thomson (left) and Lord Rutherford, bottom right.

### ***Cavendish Laboratory Nobel Prizes***

1904	Physics	Lord Rayleigh*	Density of gases, discovery of argon
1906	Physics	Sir J. J. Thomson	Investigations of electricity in gases
1908	Chemistry	Lord Rutherford*	Element disintegration
1915	Physics	Sir William Lawrence Bragg	X-ray analysis of crystals
1917	Physics	Charles G. Barkla	Secondary x rays
1922	Chemistry	Francis W. Aston	Isotopes discovery
1927	Physics	Charles T. R. Wilson	Cloud chamber
1928	Physics	Sir Owen W. Richardson	Thermionic emission
1935	Physics	Sir James Chadwick	Neutron discovery
1937	Physics	Sir George P. Thomson	Electron diffraction
1947	Physics	Sir Edward V. Appleton*	Upper atmosphere investigations
1948	Physics	Lord Patrick M. S. Blackett	Discoveries in nuclear physics
1951	Physics	Sir John D. Cockcroft and Ernest T. S. Walton	Nuclear transmutation
1962	Physiology or Medicine	Francis H. C. Crick and James D. Watson	DNA discoveries
1962	Chemistry	Max Perutz and Sir John Kendrew	Structures of globular proteins
1973	Physics	Brian D. Josephson	Supercurrent in tunnel barriers
1974	Physics	Sir Martin Ryle and Antony Hewish	Radio astrophysics, pulsars
1977	Physics	Sir Nevill F. Mott	Magnetic and disordered systems
1978	Physics	P. L. Kapitsa*	Low-temperature physics
1982	Chemistry	Sir Aaron Klug	Nucleic acid-protein complexes



AP/Emilio Segrè Visual Archives.

**Clinton J. Davisson** (1881–1958) is shown here in 1928 (right) looking at the electronic diffraction tube held by **Lester H. Germer** (1896–1971). Davisson received his undergraduate degree at the University of Chicago and his doctorate at Princeton. They performed their work at Bell Telephone Laboratory located in New York City. Davisson received the Nobel Prize in Physics in 1937.

## 5.3 Electron Scattering

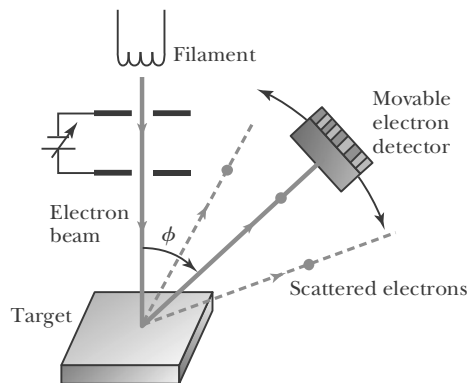
In 1925 a laboratory accident led to experimental proof for de Broglie's wavelength hypothesis. C. Davisson and L. H. Germer of Bell Telephone Laboratories (now part of Alcatel-Lucent) were investigating the properties of metallic surfaces by scattering electrons from various materials when a liquid air bottle exploded near their apparatus. Because the nickel target they were currently using was at a high temperature when the accident occurred, the subsequent breakage of their vacuum system caused significant oxidation of the nickel. The target had been specially prepared and was rather expensive, so they tried to repair it by, among other procedures, prolonged heating at various high temperatures in hydrogen and under vacuum to deoxidize it.

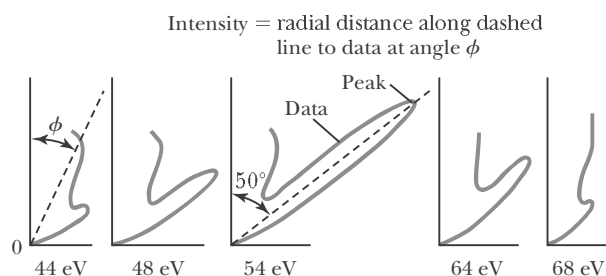
A simple diagram of the Davisson-Germer apparatus is shown in Figure 5.9. Upon putting the refurbished target back in place and continuing the experiments, Davisson and Germer found a striking change in the way electrons were scattering from the nickel surface. They had previously seen a smooth variation of intensity with scattering angle, but the new data showed large numbers of scattered electrons for certain energies at a given scattering angle. Davisson and Germer were so puzzled by their new data that after a few days, they cut open the tube to examine the nickel target. They found that the high temperature had modified the polycrystalline structure of the nickel. The many small crystals of the original target had been changed into a few large crystals as a result of the heat treatment. Davisson surmised it was this new crystal structure of nickel—the arrangement of atoms in the crystals, not the structure of the atoms—that had caused the new intensity distributions. Some 1928 experimental results of Davisson and Germer for 54-eV electrons scattered from nickel are shown in Figure 5.10. The scattered peak occurs for  $\phi = 50^\circ$ .

The electrons were apparently being diffracted much like x rays, and Davisson, being aware of de Broglie's results, found that the Bragg law applied to their data as well. Davisson and Germer were able to vary the scattering angles for a given wavelength and vary the wavelength (by changing the electron accelerating voltage and thus the momentum) for a given angle.

The relationship between the incident electron beam and the nickel crystal scattering planes is shown in Figure 5.11. In the Bragg law,  $2\theta$  is the angle between the incident and exit beams. Therefore,  $\phi = \pi - 2\theta = 2\alpha$ . Because  $\sin \theta = \cos(\phi/2) = \cos \alpha$ , we have for the Bragg condition,  $n\lambda = 2d \cos \alpha$ .

**Figure 5.9** Schematic diagram of Davisson-Germer experiment. Electrons are produced by the hot filament, accelerated, and focused onto the target. Electrons are scattered at an angle  $\phi$  into a detector, which is movable. The distribution of electrons is measured as a function of  $\phi$ . The entire apparatus is located in a vacuum.





**Figure 5.10** Davisson and Germer data for scattering of electrons from Ni. The peak  $\phi = 50^\circ$  builds dramatically as the energy of the electron nears 54 eV. From C. J. Davisson, Franklin Institute Journal 205, 597–623 (1928).

However,  $d$  is the lattice plane spacing and is related to the interatomic distance  $D$  by  $d = D \sin \alpha$  so that

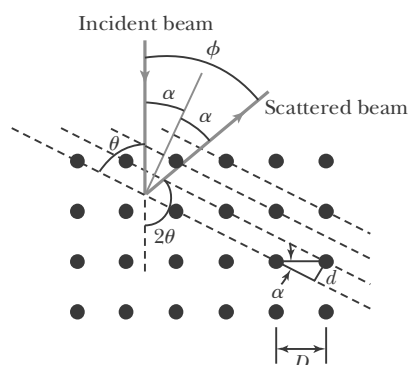
$$\begin{aligned} n\lambda &= 2d \sin \theta = 2D \sin \alpha \cos \alpha \\ n\lambda &= D \sin 2\alpha = D \sin \phi \end{aligned} \quad (5.4)$$

or

$$\lambda = \frac{D \sin \phi}{n} \quad (5.5)$$

For nickel the interatomic distance is  $D = 0.215$  nm. If the peak found by Davisson and Germer at  $50^\circ$  was  $n = 1$ , then the electron wavelength should be

$$\lambda = (0.215 \text{ nm})(\sin 50^\circ) = 0.165 \text{ nm}$$



**Figure 5.11** The scattering of electrons by lattice planes in a crystal. This figure is useful to compare the scattering relations  $n\lambda = 2d \sin \theta$  and  $n\lambda = D \sin \phi$  where  $\theta$  and  $\phi$  are the angles shown,  $D$  = interatomic spacing, and  $d$  = lattice plane spacing.



### EXAMPLE 5.3

Determine the de Broglie wavelength for a 54-eV electron used by Davisson and Germer.

**Strategy** We shall use the de Broglie wavelength Equation (5.2) to determine the wavelength  $\lambda$ . We need to find the momentum of a 54-eV electron, but because the energy is so low, we can do a nonrelativistic calculation. We shall do a

general calculation for the wavelength of any electron accelerated by a voltage of  $V_0$ .

**Solution** We write the kinetic energy K.E. in terms of the final momentum of the electron and the voltage  $V_0$  across which the electron is accelerated.

$$\frac{p^2}{2m} = \text{K.E.} = eV_0 \quad (5.6)$$

We find the momentum from this equation to be  $p = \sqrt{(2m)(eV_0)}$ . The de Broglie wavelength from Equation (5.2) is now

$$\begin{aligned}\lambda &= \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\sqrt{(2mc^2)(eV_0)}} \\ &= \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{(2)(0.511 \times 10^6 \text{ eV})(eV_0)}}\end{aligned}$$

$$\lambda = \frac{1.226 \text{ nm} \cdot \text{V}^{1/2}}{\sqrt{V_0}} \quad (5.7)$$

where the constants  $h$ ,  $c$ , and  $m$  have been evaluated and  $V_0$  is the voltage. For  $V_0 = 54 \text{ V}$ , the wavelength is

$$\lambda = \frac{1.226 \text{ nm} \cdot \text{V}^{1/2}}{\sqrt{54 \text{ V}}} = 0.167 \text{ nm}$$

We note that the value of the de Broglie wavelength 0.167 nm found in the previous example is in good agreement with that found experimentally (0.165 nm) by Davisson and Germer for the peak at  $50^\circ$ . This is an important result and shows that electrons have wavelike properties.

Shortly after Davisson and Germer reported their experiment, George P. Thomson (1892–1975), son of J.J. Thomson, reported seeing the effects of electron diffraction in transmission experiments. The first target was celluloid, and soon after that gold, aluminum, and platinum were used. The randomly oriented polycrystalline sample of beryllium produces rings (see Figure 5.12b). Davisson and Thomson received the Nobel Prize in 1937 for their investigations, which clearly showed that particles exhibited wave properties. In the next few years hydrogen and helium atoms were also shown to exhibit wave diffraction. An important modern measurement technique uses diffraction of neutrons to study the crystal and molecular structure of biologically important substances. All these experiments are consistent with the de Broglie hypothesis for the wavelength of a particle with mass.



### EXAMPLE 5.4

In introductory physics, we learned that a particle (ideal gas) in thermal equilibrium with its surroundings has a kinetic energy of  $3kT/2$ . Calculate the de Broglie wavelength for (a) a neutron at room temperature (300 K) and (b) a “cold” neutron at 77 K (liquid nitrogen).

**Strategy** In both of these cases we will use Equation (5.2) to find the de Broglie wavelength. First, we will need to determine the momentum, and we note in both cases the energies of the particles will be so low that we can perform a nonrelativistic calculation. Neutrons have a rest energy of almost 1000 MeV, and their kinetic energies at these temperatures will be quite low (0.026 eV at 300 K).

**Solution** We begin by finding the de Broglie wavelength of the neutron in terms of the temperature.

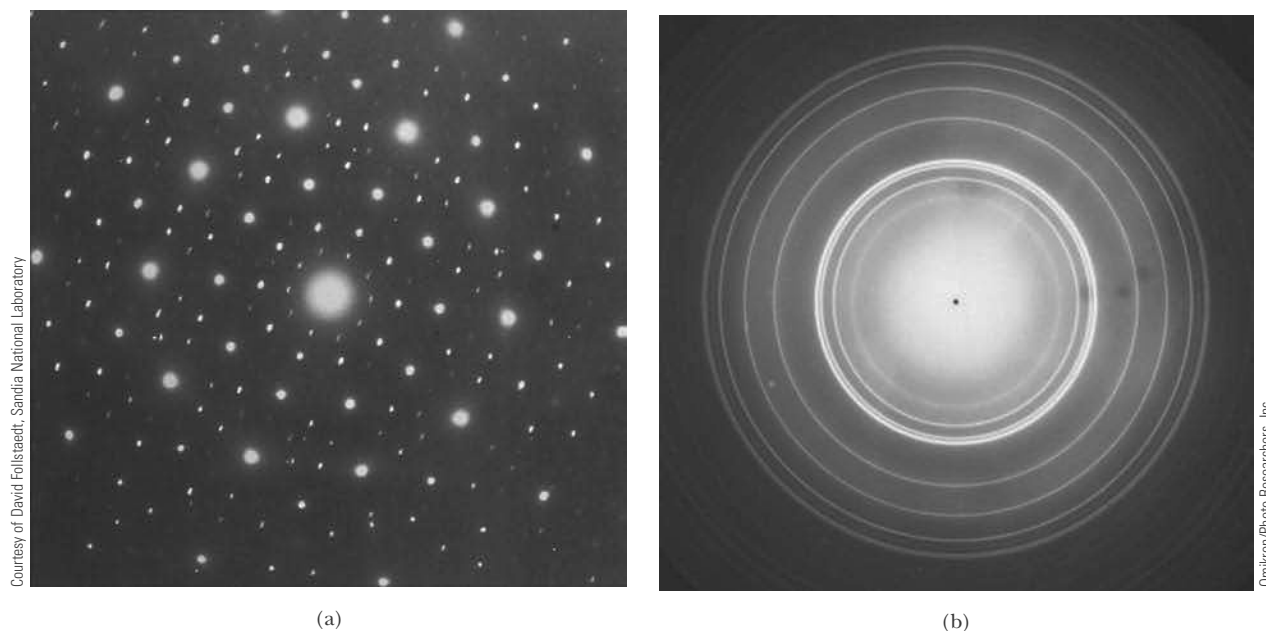
$$\begin{aligned}\frac{p^2}{2m} &= \text{K.E.} = \frac{3}{2}kT \\ p &= \sqrt{3mkT}\end{aligned} \quad (5.8)$$

$$\begin{aligned}\lambda &= \frac{h}{p} = \frac{h}{\sqrt{3mkT}} = \frac{hc}{\sqrt{3(mc^2)kT}} \\ &= \frac{1}{T^{1/2}} \frac{1240 \text{ eV} \cdot \text{nm}}{\sqrt{3(938 \times 10^6 \text{ eV})(8.62 \times 10^{-5} \text{ eV/K})}}\end{aligned}$$

It again has been convenient to use eV units.

$$\begin{aligned}\lambda &= \frac{2.52}{T^{1/2}} \text{ nm} \cdot \text{K}^{1/2} \\ \lambda(300 \text{ K}) &= \frac{2.52 \text{ nm} \cdot \text{K}^{1/2}}{\sqrt{300 \text{ K}}} = 0.145 \text{ nm} \\ \lambda(77 \text{ K}) &= \frac{2.52 \text{ nm} \cdot \text{K}^{1/2}}{\sqrt{77 \text{ K}}} = 0.287 \text{ nm}\end{aligned} \quad (5.9)$$

These wavelengths are thus suitable for diffraction by crystals. “Supercold” neutrons, used to produce even larger wavelengths, are useful because extraneous electric and magnetic fields do not affect neutrons nearly as much as electrons.



**Figure 5.12** Examples of transmission electron diffraction photographs. (a) Produced by scattering 120-keV electrons on the quasicrystal  $\text{Al}_{80}\text{Mn}_{20}$ . (b) Electron diffraction pattern on beryllium. Notice that the dots in (a) indicate that the sample was a crystal, whereas the rings in (b) indicate that a randomly oriented sample (or powder) was used.

## 5.4 Wave Motion

Because particles exhibit wave behavior, as shown in the last section for electron diffraction, it must be possible to formulate a wave description of particle motion. This is an essential step in our progress toward understanding the behavior of matter—the quantum theory of physics. Our development of quantum theory will be based heavily on waves, so we now digress briefly to review the physics of wave motion, which we shall soon apply to particles.

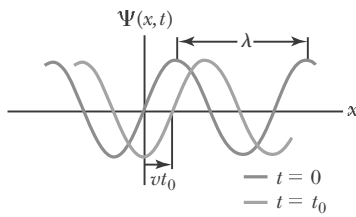
In introductory physics, we study waves of several kinds, including sound waves and electromagnetic waves (including light). The simplest form of wave has a sinusoidal form; at a fixed time (say,  $t = 0$ ) its spatial variation looks like

$$\Psi(x, t)|_{t=0} = A \sin\left(\frac{2\pi}{\lambda}x\right) \quad (5.10)$$

as shown in Figure 5.13 (p. 176). The function  $\Psi(x, t)$  represents the *instantaneous amplitude* or **displacement** of the wave as a function of position  $x$  and time  $t$ . In the case of a traveling wave moving down a string,  $\Psi$  is the displacement of the string from equilibrium; and in the case of electromagnetic radiation,  $\Psi$  is the magnitude of the electric field  $\vec{E}$  or magnetic field  $\vec{B}$ . The maximum displacement  $A$  is normally called the **amplitude**, but a better term for a harmonic wave such as we are considering may be **harmonic amplitude**.

As time increases, the position of the wave will change, so the general expression for the wave is

$$\Psi(x, t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \quad (5.11) \quad \text{Wave form}$$



**Figure 5.13** Wave form of a wave moving to the right at speed  $v$  shown at  $t = 0$  and  $t = t_0$ .

The position at time  $t = t_0$  is also shown in Figure 5.13. The **wavelength**  $\lambda$  is defined to be the distance between points in the wave with the same phase, for example, positive wave crests. The **period**  $T$  is the time required for a wave to travel a distance of one wavelength  $\lambda$ . Because the velocity [actually phase velocity, see Equation (5.17)] of the wave is  $v$ , we have  $\lambda = vT$ . The frequency  $f$  ( $= 1/T$ ) of a harmonic wave is the number of times a crest passes a given point (a complete *cycle*) per second. A traveling wave of the type described by Equation (5.11) satisfies the wave equation\*

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \quad (5.12)$$

If we use  $\lambda = vT$ , we can rewrite Equation (5.11) as

$$\Psi(x, t) = A \sin \left[ 2\pi \left( \frac{x}{\lambda} - \frac{t}{T} \right) \right] \quad (5.13)$$

#### Wave number and angular frequency

We can write Equation (5.13) more compactly by defining† the **wave number**  $k$  and **angular frequency**  $\omega$  by

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T} \quad (5.14)$$

Equation (5.13) then becomes

$$\Psi(x, t) = A \sin(kx - \omega t) \quad (5.15)$$

This is the mathematical description of a sine curve traveling in the positive  $x$  direction that has a displacement  $\Psi = 0$  at  $x = 0$  and  $t = 0$ . A similar wave traveling in the negative  $x$  direction has the form

$$\Psi(x, t) = A \sin(kx + \omega t) \quad (5.16)$$

The **phase velocity**  $v_{\text{ph}}$  is the velocity of a point on the wave that has a given phase (for example, the crest) and is given by

$$\text{Phase velocity} \quad v_{\text{ph}} = \frac{\lambda}{T} = \frac{\omega}{k} \quad (5.17)$$

If the wave does not have  $\Psi = 0$  at  $x = 0$  and  $t = 0$ , we can describe the wave using a **phase constant**  $\phi$ :

$$\text{Phase constant} \quad \Psi(x, t) = A \sin(kx - \omega t + \phi) \quad (5.18)$$

For example, if  $\phi = \pi/2$ , Equation (5.18) can be written

$$\Psi(x, t) = A \cos(kx - \omega t) \quad (5.19)$$

#### Principle of superposition

Observation of many kinds of waves has established the general result that when two or more waves traverse the same region, they act independently of each other. According to the **principle of superposition**, we add the displacements of

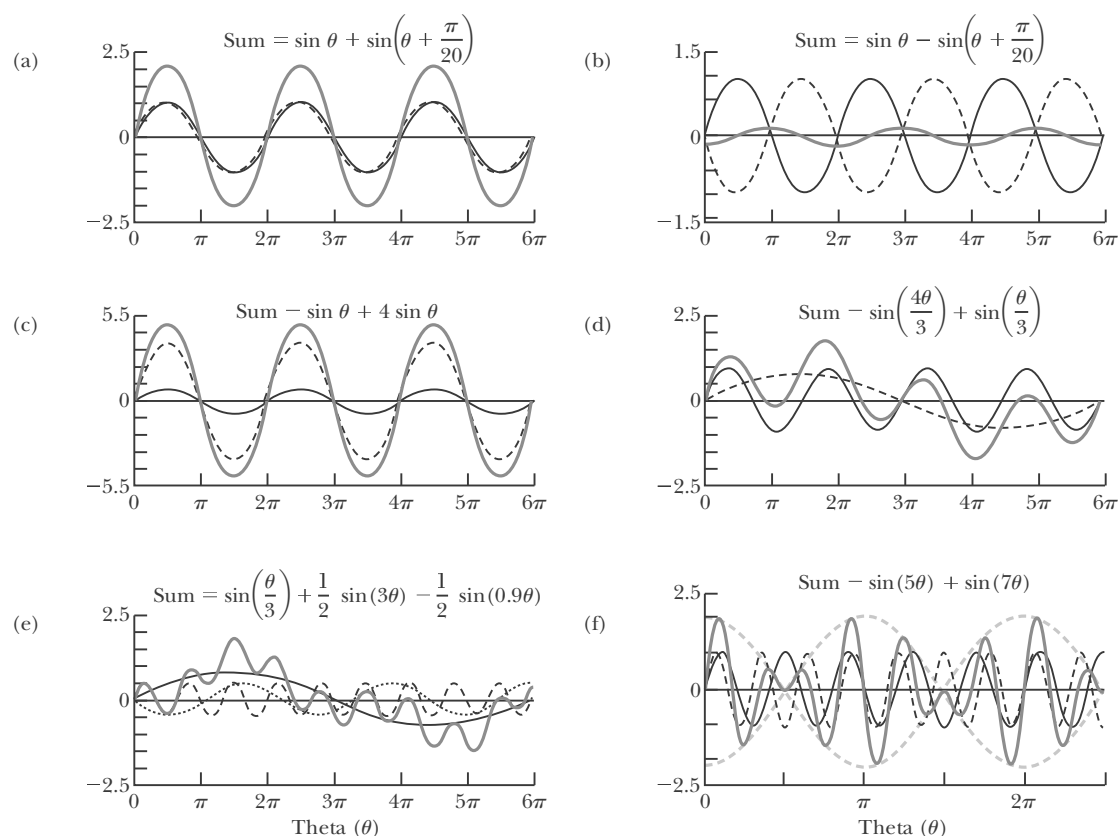
\*The derivation of the wave equation is presented in most introductory physics textbooks for a wave on a string, although it is often an optional section and might have been skipped. It would be worthwhile for the student to review its derivation now, especially the use of the partial derivatives.

†The term “wave number” has two common usages. Spectroscopists often use “wave number” as the reciprocal of the wavelength ( $1/\lambda$ ), so that it’s simply the number of waves that fit into a meter of length. The convention we adopt here ( $k = 2\pi/\lambda$ ) is also common and makes some of the formulas we use more compact and easier to use.

all waves present. A familiar example is the superposition of two sound waves of nearly equal frequencies: The phenomenon of beats is observed. Examples of superposition are shown in Figure 5.14. The net displacement depends on the harmonic amplitude, the phase, and the frequency of each of the individual waves. When we add waves at a given position and time, we simply add their instantaneous displacements. This can lead to constructive and destructive interference effects like we saw in x-ray scattering in Section 5.1.

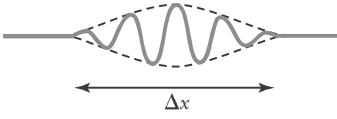
In quantum theory (or *quantum mechanics* as it is sometimes called to reflect its differences from *classical mechanics*), we will soon learn that we will use waves to represent a moving particle. How can we do that? In Figure 5.14 we see that when two waves are added together, we obtain regions of relatively large (and small) displacement. If we add many waves of different amplitudes and frequencies in particular ways, it is possible to obtain what is called a **wave packet**. The important property of the wave packet is that its net amplitude differs from zero

**Wave packet**



**Figure 5.14** Superposition of waves. The heavy blue line is the resulting wave. (a) Two waves of equal frequency and amplitude that are almost in phase. The result is a larger wave. (b) As in (a) but the two waves are almost out of phase. The result is a smaller wave. (c) Superposition of two waves with the same frequency, but different amplitudes. (d) Superposition of two waves of equal amplitude but different frequencies. (e) Superposition of three waves of different amplitudes and frequencies. (f) Superposition of two waves of almost the same frequency over many wavelengths, creating the phenomenon of beats. The blue dashed line indicates an envelope that denotes the maximum displacement of the combined waves.





**Figure 5.15** An idealized wave packet localized in space over a region  $\Delta x$  is the superposition of many waves of different amplitudes and frequencies.

only over a small region  $\Delta x$  as shown in Figure 5.15. We can localize the position of a particle in a particular region by using a wave packet description (see Problem 67 for a calculation of this effect).

Let us examine in detail the superposition of two waves. Assume both waves have the same harmonic amplitude  $A$  but different wave numbers ( $k_1$  and  $k_2$ ) and angular frequencies ( $\omega_1$  and  $\omega_2$ ). The superposition of the two waves is the sum

$$\begin{aligned}\Psi(x, t) &= \Psi_1(x, t) + \Psi_2(x, t) \\ &= A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t)\end{aligned}\quad (5.20)$$

$$\begin{aligned}&= 2A \cos\left[\frac{1}{2}(k_1 - k_2)x - \frac{1}{2}(\omega_1 - \omega_2)t\right] \cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right] \\ &= 2A \cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \cos(k_{av}x - \omega_{av}t)\end{aligned}\quad (5.21)$$

where  $\Delta k = k_1 - k_2$ ,  $\Delta \omega = \omega_1 - \omega_2$ ,  $k_{av} = (k_1 + k_2)/2$ , and  $\omega_{av} = (\omega_1 + \omega_2)/2$ . We display similar waves in Figure 5.14a–d, where the heavy solid line indicates the sum of the two waves. In Figure 5.14f the blue dashed line indicates an envelope, which denotes the maximum displacement of the combined waves. The combined (or summed) wave  $\Psi$  oscillates within this envelope with the wave number  $k_{av}$  and angular frequency  $\omega_{av}$ . The envelope is described by the first cosine factor of Equation (5.21), which has the wave number  $\Delta k/2$  and angular frequency  $\Delta \omega/2$ . The individual waves  $\Psi_1$  and  $\Psi_2$  each move with their own phase velocity:  $\omega_1/k_1$  and  $\omega_2/k_2$ . The combined wave has a phase velocity  $\omega_{av}/k_{av}$ . When combining many more than two waves, one obtains a pulse, or wave packet, which moves at the **group velocity**, as shown later. Only the group velocity, which describes the speed of the envelope ( $u_{gr} = \Delta \omega/\Delta k$ ), is important when dealing with wave packets.

In contrast to the pulse or wave packet, the combination of only two waves is not localized in space. However, for purposes of illustration, we can identify a “localized region”  $\Delta x = x_2 - x_1$  where  $x_1$  and  $x_2$  represent two consecutive points where the envelope is zero (or maximum, see Figure 5.14f). The term  $\Delta k \cdot x/2$  in Equation (5.21) must be different by a phase of  $\pi$  for the values  $x_1$  and  $x_2$ , because  $x_2 - x_1$  represents only one half of the wavelength of the envelope confining the wave.

$$\begin{aligned}\frac{1}{2}\Delta k x_2 - \frac{1}{2}\Delta k x_1 &= \pi \\ \Delta k(x_2 - x_1) - \Delta k \Delta x &= 2\pi\end{aligned}\quad (5.22)$$

Similarly, for a given value of  $x$  we can determine the time  $\Delta t$  over which the wave is localized and obtain

$$\Delta \omega \Delta t - 2\pi \quad (5.23)$$

The results of Equations (5.22) and (5.23) can be generalized for a case in which many waves form the wave packet (see Problem 67). The equations,  $\Delta k \Delta x - 2\pi$  and  $\Delta \omega \Delta t - 2\pi$ , are significant because they tell us that in order to know precisely the position of the wave packet envelope ( $\Delta x$  small), we must have a large range of wave numbers ( $\Delta k$  large). Similarly, to know precisely when the wave is at a given point ( $\Delta t$  small), we must have a large range of frequencies ( $\Delta \omega$  large). Equation (5.23) is the origin of the bandwidth relation, which is important in electronics. A particular circuit component must have a large bandwidth  $\Delta \omega$  in order for its signal to respond in a short time  $\Delta t$ .

If we are to treat particles as matter waves, we have to be able to describe the particle in terms of waves. An important aspect of a particle is its localization in space. That is why it is so important to form the wave packet that we have been discussing. We extend Equation (5.20) by summing over many waves with possibly different wave numbers, angular frequencies, and amplitudes.

$$\Psi(x, t) = \sum_i A_i \cos(k_i x - \omega_i t) \quad (5.24)$$

Such a result is called a **Fourier series**. When dealing with a continuous spectrum, it may be desirable to extend Equation (5.24) to the integral form called a **Fourier integral**.

$$\Psi(x, t) = \int \tilde{A}(k) \cos(kx - \omega t) dk \quad (5.25)$$

The amplitudes  $A_i$  and  $\tilde{A}(k)$  may be functions of  $k$ . The use of Fourier series and Fourier integrals is at a more advanced level of mathematics than we want to pursue now.\* We can, however, illustrate their value by one important example.

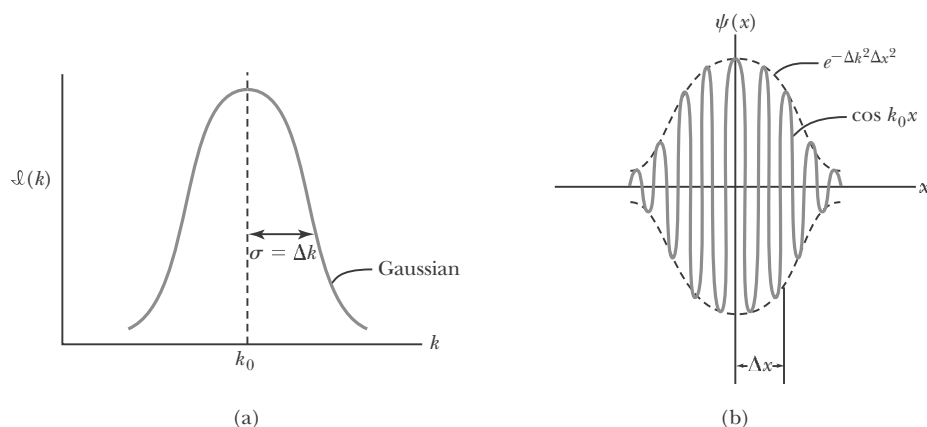
**Gaussian Wave Packet** Gaussian wave packets are often used to represent the position of particles, as illustrated in Figure 5.16, because the associated integrals are relatively easy to evaluate. At a given time  $t$ , say  $t = 0$ , a Gaussian wave can be expressed as

$$\Psi(x, 0) = \psi(x) = A e^{-\Delta k^2 x^2} \cos(k_0 x) \quad (5.26)$$

where  $\Delta k$  expresses the range of wave numbers used to form the wave packet. The  $\cos(k_0 x)$  term describes the wave oscillating inside the envelope bounded by the (Gaussian) exponential term  $e^{-\Delta k^2 x^2}$ . The intensity distribution  $\mathfrak{I}(k)$  for the wave numbers leading to Equation (5.26) is shown in Figure 5.16a. There is a high probability of a particular measurement of  $k$  being within one standard deviation of the mean value  $k_0$ . The function  $\psi(x)$  is shown in Figure 5.16b. For simplicity, let the constant  $A$  be 1. There is a good probability of finding the particle within the values of  $x = 0$  [ $\psi(x) = 1$ ] and  $x = \Delta x/2$  [ $\psi(x) = \exp(-\Delta k^2 \Delta x^2/4)$ ]. Roughly, the value of  $\psi(x)$  at the position  $x = \Delta x/2$  is about 0.6 (see Figure 5.16b), so we have

$$e^{-\Delta k^2 \Delta x^2/4} \approx 0.6$$

\*See John D. McGervey, *Introduction to Modern Physics*, Chap. 4, Orlando, FL: Academic Press (1983).



#### Fourier series and integral

#### Gaussian function

**Figure 5.16** The form of the probability distribution or intensity  $\mathfrak{I}(k)$  shown in (a) is taken to have a Gaussian shape with a standard deviation of  $\Delta k$  [determined when the function  $\exp[-(k - k_0)^2/2\sigma^2]$  has  $k - k_0 = \pm \Delta k$  and  $\Delta k = \sigma$ , the standard deviation]. This  $\mathfrak{I}(k)$  leads to  $\psi(x)$ , as shown in (b). The envelope for  $\psi(x)$  is described by the  $\exp(-\Delta k^2 x^2)$  term with the oscillating term  $\cos(k_0 x)$  contained by the envelope. At the given time  $t = 0$ , the wave packet (particle) is localized to the area  $x \approx 0 \pm \Delta x$  with wave numbers  $k \approx k_0 \pm \Delta k$ .

We take the logarithm of both sides and find

$$\frac{\Delta k^2 \Delta x^2}{4} \approx 0.5 \quad \text{or} \quad \Delta k \Delta x \approx 1.4 \quad (5.27)$$

This has been a rough calculation, and the result depends on the assumptions we have made. A more detailed calculation gives  $\Delta k \Delta x = 1/2$ . The important point is that with the Gaussian wave packet, we have arrived at a result similar to Equation (5.22), namely, that the product  $\Delta k \Delta x$  is on the order of unity. The localization of the wave packet over a small region  $\Delta x$  to describe a particle requires a large range of wave numbers; that is,  $\Delta k$  is large. Conversely, a small range of wave numbers cannot produce a wave packet localized within a small distance.

To complete our study of waves and the representation of particles by wave packets, we must be convinced that the superposition of waves is actually able to describe particles. We found earlier for the superposition of two waves that the group velocity,  $u_{\text{gr}} = \Delta\omega/\Delta k$ , represented the motion of the envelope. We can generalize this for the case of the wave packet and find that the wave packet moves with the group velocity  $u_{\text{gr}}$  given by

$$u_{\text{gr}} = \frac{d\omega}{dk} \quad (5.28)$$

Because the wave packet consists of many wave numbers, we should remember to evaluate this derivative at the center of the wave packet (that is,  $k = k_0$ ).

For a de Broglie wave, we have  $E = hf$  and  $p = h/\lambda$ . We can rewrite both of these equations in terms of  $\hbar$ .

$$E = hf = \hbar(2\pi f) = \hbar\omega \quad (5.29)$$

$$p = \frac{h}{\lambda} = \hbar \frac{2\pi}{\lambda} = \hbar k \quad (5.30)$$

where we have used the relations  $\omega = 2\pi f$  and  $k = 2\pi/\lambda$ . If we multiply the denominator and numerator in Equation (5.28) by  $\hbar$ , we have

$$u_{\text{gr}} = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dp}$$

We use the relativistic relation  $E^2 = p^2c^2 + m^2c^4$  and its derivative to find

$$2E dE = 2pc^2 dp$$

or

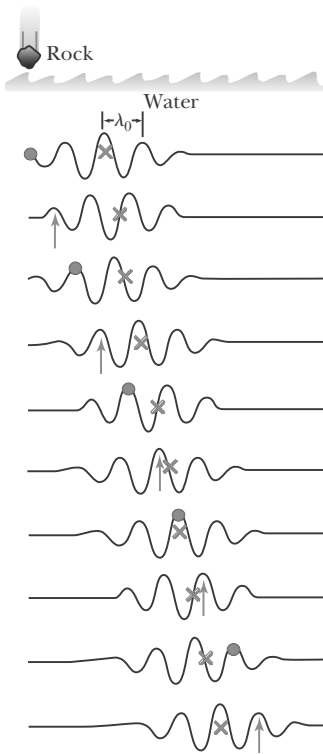
$$u_{\text{gr}} = \frac{dE}{dp} = \frac{pc^2}{E} \quad (5.31)$$

This is the velocity of a particle of momentum  $p$  and total energy  $E$ . Thus, it is plausible to assume that the group velocity of the wave packet can be associated with the velocity of a particle.

The phase velocity of a wave is represented by

$$v_{\text{ph}} = \lambda f = \frac{\omega}{k} \quad (5.32)$$

so that  $\omega = kv_{\text{ph}}$ .



**Figure 5.17** Progression with time of wave packet for which  $u_{\text{gr}} = v_{\text{ph}}/2$ . Note how the individual wave (arrow and dot alternately) moves through the wave packet (symbol X) with time.

Then, the group velocity is related to the phase velocity by

$$u_{\text{gr}} = \frac{d\omega}{dk} = \frac{d}{dk}(v_{\text{ph}}k) = v_{\text{ph}} + k \frac{dv_{\text{ph}}}{dk} \quad (5.33)$$

Thus, the group velocity may be greater or less than the phase velocity. A medium is called *nondispersive* when the phase velocity is the same for all frequencies and  $u_{\text{gr}} = v_{\text{ph}}$ . An example is electromagnetic waves in vacuum. Water waves are a good example of waves in a dispersive medium. When one throws a rock in a still pond, the envelope of the waves moves more slowly than the individual waves moving outward (Figure 5.17).

Dispersion plays an important role in the shape of wave packets. For example, in the case of the Gaussian wave packet shown in Figure 5.16 at  $t = 0$ , the wave packet will spread out as time progresses. A packet that is highly localized at one time will have its waves added together in a considerably different manner at another time because of the superposition of the waves.

Does the preceding discussion agree with our classical ideas? Consider a particle of mass  $m$  moving nonrelativistically with speed  $v$ . The phase velocity of this particle, if we treat it as a de Broglie wave, can be found by

$$v_{\text{ph}} = \lambda f = \frac{h}{p} \frac{E}{h} = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{mv}{2m} = \frac{v}{2} \quad (5.34)$$

The phase velocity is half of the particle's velocity, so the particle does not move with the phase velocity. In Problem 28 you will show that a relativistic treatment gives a different relationship between the phase and group velocities, but they are still not equal to each other.



### EXAMPLE 5.5

We just saw that the speed of a nonrelativistic particle of mass  $m$  is not equal to its phase velocity. Show that the particle speed is equal to the group velocity.

**Strategy** We can use the relation for the group velocity in Equation (5.28) or (5.31). Either should work, and using both equations will be instructive.

**Solution** First, we look at Equation (5.31) for our nonrelativistic particle:

$$u_{\text{gr}} = \frac{pc^2}{E} = \frac{mvc^2}{mc^2} = v \quad (5.35)$$

In order to use Equation (5.28) we utilize the results in Equations (5.29) and (5.30) for  $\omega$  and  $k$ .

$$u_{\text{gr}} = \frac{d\omega}{dk} = \frac{d(E/\hbar)}{d(p/\hbar)} = \frac{dE}{dp} = \frac{d}{dp} \frac{p^2}{2m} = \frac{2p}{2m} = v$$

We agree that the particle, when acting as a wave, moves with the group velocity, not the phase velocity.



### EXAMPLE 5.6

Newton showed that deep-water waves have a phase velocity of  $\sqrt{g\lambda/2\pi}$ . Find the group velocity of such waves and discuss the motion.

**Strategy** We use Equation (5.33) to relate the group and phase velocities, but first we need to find the phase velocity  $v_{\text{ph}}$  in terms of  $k$ .

**Solution** If we use  $\lambda = 2\pi/k$ , we have

$$v_{\text{ph}} = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{k}} = \sqrt{g}k^{-1/2}$$

Now we can take the necessary derivative for Equation (5.33).

$$u_{\text{gr}} = \sqrt{\frac{g}{k}} + k \frac{d}{dk} \left( \sqrt{g}k^{-1/2} \right) = \sqrt{\frac{g}{k}} + k\sqrt{g} \left( -\frac{1}{2}k^{-3/2} \right)$$

$$u_{\text{gr}} = \sqrt{\frac{g}{k}} - \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_{\text{ph}}$$

The group velocity is determined to be one half of the phase velocity. Such an effect can be observed by throwing a rock in a still pond. As the circular waves move out, the individual waves seem to run right through the wave crests and then disappear (see Figure 5.17).

## 5.5 Waves or Particles?

By this point it is not unusual for a student to be a little confused. We have learned that electromagnetic radiation behaves sometimes as waves (as in interference and diffraction) and other times as particles (as in the photoelectric and Compton effects). We have been presented evidence in this chapter that particles also behave as waves (electron diffraction). Can all this really be true? If a particle is a wave, what is waving? In the preceding section we learned that, at least mathematically, we can describe particles by using wave packets. Can we represent matter as waves and particles simultaneously? And can we represent electromagnetic radiation as waves and particles simultaneously? We must answer these questions about the **wave-particle duality** before proceeding with our study of quantum theory.

**Double-Slit Experiment with Light** To better understand the differences and similarities of waves and particles, we analyze Young's double-slit diffraction experiment, which is studied in detail in introductory physics courses (lectures and labs) to show the interference character of light. Figure 5.18a shows a schematic diagram of the experiment. This experiment is easily performed with the use of a low-power laser. With both slits open, a nice interference pattern is observed, with bands of maxima and minima. When one of the slits is covered, this interference pattern is changed, and a rather broad peak is observed (see

**Figure 5.18** (a) Schematic diagram of Young's double-slit experiment. This experiment is easily performed with a laser as the light source (and  $\ell \gg d$ , where  $d$  = distance between slit centers). (b) The solid line indicates the interference pattern due to both slits. If either of the slits is covered, single-slit diffraction gives the result shown in the dashed curve.

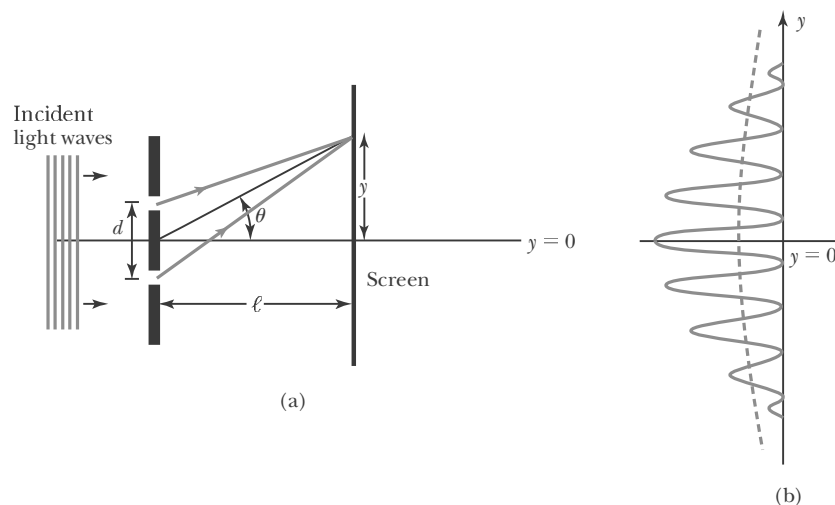
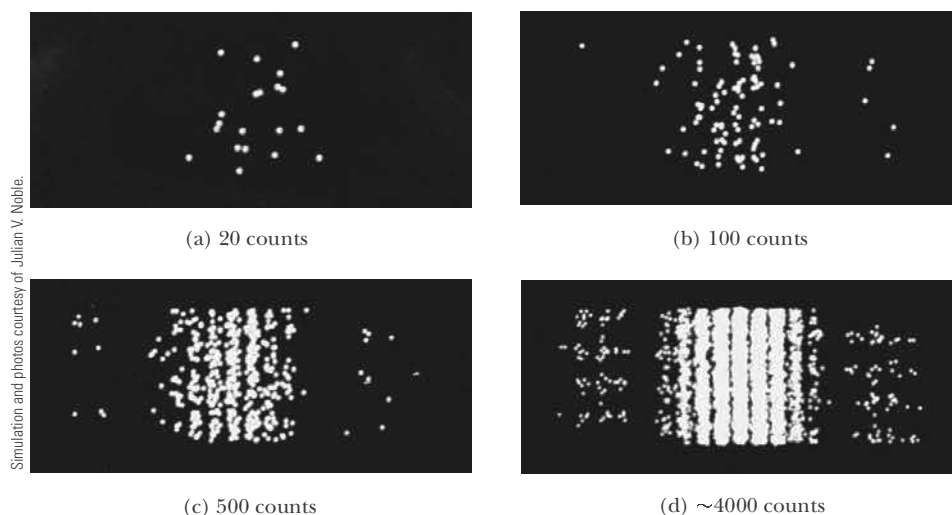


Figure 5.18b). Thus, we conclude that the double-slit interference pattern is due to light passing through *both* slits—a wave phenomenon.

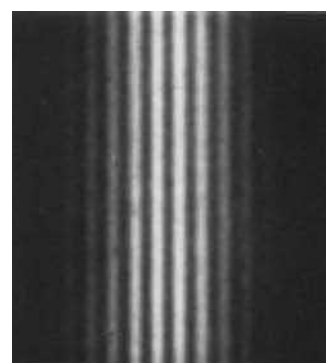
However, if the light intensity is reduced, and we observe the pattern on a screen, we learn that the light arriving on the screen produces flashes at various points, entirely representative of particle behavior! If we take pictures of the screen after varying lengths of time, the pictures will look like those shown in Figure 5.19. Eventually the interference pattern characteristic of wave behavior emerges. There is therefore no contradiction in this experiment. If we want to know the precise location of the light (photon), we must use the particle description and not the wave description.

**Electron Double-Slit Experiment** Now let us examine a similar double-slit experiment that uses electrons rather than light. If matter also behaves as waves, then shouldn't the same experimental results be obtained if we use electrons rather than light? The answer is yes, and physicists did not doubt the eventual result. This experiment is not as easy to perform as the similar one with light. The difficulty arises in constructing slits narrow enough to exhibit wave phenomena. This requires  $\lambda \sim a$ , where  $a$  is the slit width. For light of  $\lambda = 600$  nm, slits can be produced mechanically. However, for electrons of energy 50 keV,  $\lambda = 5 \times 10^{-3}$  nm, which is smaller than a hydrogen atom ( $\sim 0.1$  nm). Nevertheless, in 1961 C. Jönsson\* of Tübingen, Germany, succeeded in showing double-slit interference effects for electrons (Figure 5.20) by constructing very narrow slits and using relatively large distances between the slits and the observation screen. Copper slits were made by electrolytically depositing copper on a polymer strip printed on silvered glass plates. This experiment demonstrated that precisely the same behavior occurs for both light (waves) and electrons (particles). We have seen similar behavior previously from the Debye-Scherrer rings produced by the diffraction of x rays (waves) and electrons (particles).

\*C. Jönsson, *American Journal of Physics* **42**, 4 (1974), translation of *Zeitschrift für Physik* **161**, 454 (1961).



**Figure 5.19** Computer simulation of Young's double-slit interference experiment for light or electrons. This calculation was performed for slit width  $a = 4\lambda$  and slit distance  $d = 20\lambda$ . The four pictures are for increasing number of counts: 20, 100, 500, 4000. The interference pattern has clearly emerged for 500 counts.



**Figure 5.20** Demonstration of electron interference using two slits similar in concept to Young's double-slit experiment for light. This result by Claus Jönsson clearly shows that electrons exhibit wave behavior (see also Example 5.7).



### EXAMPLE 5.7

In the experiment by Jönsson, 50-keV electrons impinged on slits of width 500 nm separated by a distance of 2000 nm. The observation screen was located 350 mm beyond the slits. What was the distance between the first two maxima?

**Strategy** The equation specifying the orders of maxima and the angle  $\theta$  from incidence is (see Figure 5.18) given by

$$d \sin \theta = n\lambda \quad (5.36)$$

The first maximum is of order  $n = 0$  and occurs for  $\theta = 0$ . The next maximum, at angle  $\theta$ , occurs for  $n = 1$ :

$$\sin \theta = \frac{\lambda}{d} = \frac{\lambda}{2000 \text{ nm}}$$

We need to determine the wavelength  $\lambda$  to find the angle  $\theta$ . Once we know  $\theta$ , we can find the distance between the two maxima on the observation screen.

**Solution** We have already calculated the wavelength for electrons of energy  $eV_0$  in Equation (5.7).

$$\lambda = \frac{1.226 \text{ nm} \cdot \text{V}^{1/2}}{\sqrt{50 \times 10^3 \text{ V}}} = 5.48 \times 10^{-3} \text{ nm}$$

Because 50 keV may be too high an energy for a nonrelativistic calculation such as that done in Equation (5.7), we should perform a relativistic calculation to be certain. We first find the momentum and insert that into  $\lambda = h/p$ .

$$\begin{aligned} (pc)^2 - E^2 &= E_0^2 - (K + E_0)^2 = E_0^2 \\ &= (50 \times 10^3 \text{ eV} + 0.511 \times 10^6 \text{ eV})^2 \\ &\quad - (0.511 \times 10^6 \text{ eV})^2 \\ &= (0.231 \times 10^6 \text{ eV})^2 \end{aligned}$$

Now we can determine the wavelength.

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.231 \times 10^6 \text{ eV}} = 5.36 \times 10^{-3} \text{ nm}$$

Therefore, we find the more accurate relativistic value to be somewhat less (2%) than the nonrelativistic value. Now we can determine the angle.

$$\sin \theta = \frac{5.36 \times 10^{-3} \text{ nm}}{2000 \text{ nm}} = 2.68 \times 10^{-6}$$

The distance of the first maximum along the screen is  $y = \ell \tan \theta$ , but for such a small angle,  $\sin \theta \approx \tan \theta$ .

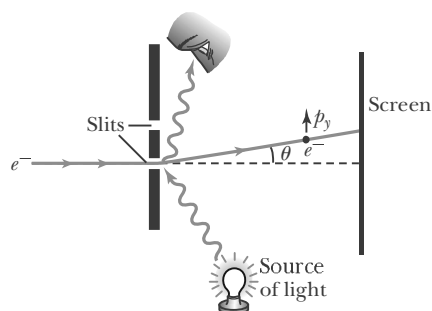
$$\begin{aligned} y &= \ell \tan \theta \approx \ell \sin \theta = 350 \text{ mm}(2.68 \times 10^{-6}) \\ &= 9.38 \times 10^{-4} \text{ mm} \frac{10^6 \text{ nm}}{\text{mm}} = 938 \text{ nm} \end{aligned}$$

Such a diffraction pattern is too small to be viewed by the naked eye. Jönsson magnified the pattern by a series of electronic lenses and then observed a fluorescent screen with a 10-power optical microscope to see the diffraction pattern as expected.

**Another Gedanken Experiment** If we were to cover one of the slits in the preceding Jönsson experiment, the double-slit interference pattern would be destroyed—just as it was when light was used. But our experience tells us the electron is a particle, and we believe that it can go through only one of the slits. Let's devise a *gedanken* experiment, shown in Figure 5.21, to determine which slit the electron went through. We set up a light shining on the double slit and use a powerful microscope to look at the region. After the electron passes through one of the slits, light bounces off the electron; we observe the reflected light, so we know which slit the electron came through.

To do this experiment, we need to use light having wavelength smaller than the slit separation  $d$ , in order to determine which slit the electron went through. We use a subscript “ph” to denote variables for light (photon). Therefore, we have  $\lambda_{\text{ph}} < d$ . The momentum of the photon is

$$p_{\text{ph}} = \frac{h}{\lambda_{\text{ph}}} > \frac{h}{d}$$



**Figure 5.21** An attempt to determine which slit an electron passes through in the double-slit experiment. A powerful light source scatters a photon from the electron, and the scattered photon is observed. The motion of the electron is affected.

For us to show the interference effects for the electrons passing through the slits, the electrons must also have a wavelength on the order of the slit separation  $d$ ,  $\lambda_{\text{el}} \sim d$ . The momentum of the electrons will be on the order of

$$p_{\text{el}} = \frac{h}{\lambda_{\text{el}}} \sim \frac{h}{d}$$

The difficulty is that the momentum of the photons used to determine which slit the electron went through is sufficiently great to strongly modify the momentum of the electron itself, thus changing the direction of the electron! The attempt to identify which slit the electron is passing through will in itself change the interference pattern. We will take a closer look at this experiment in Section 5.6. In trying to determine which slit the electron went through, we are examining the particle-like behavior of the electron. When we are examining the interference pattern of the electron, we are using the wavelike behavior of the electron.

Bohr resolved this dilemma by pointing out that the particle-like and wave-like aspects of nature are *complementary*. Both are needed—they just can't be observed simultaneously.

**Bohr's principle of complementarity:** *It is not possible to describe physical observables simultaneously in terms of both particles and waves.*

**Principle of complementarity**

**Physical observables** are those quantities such as position, velocity, momentum, and energy that can be experimentally measured. In any given instance we must use either the particle description or the wave description. Usually the choice is clear. The interference pattern of the double-slit experiment suggests that the light (or electron) had to go through both slits, and we must use the wave description. In our description of nature, we cannot describe phenomena by displaying both particle and wave behavior at the same time.

**Physical observables**

By the use of the principle of complementarity, we can better understand the **wave-particle duality** problem, which has been plaguing us. It is not unusual for students to feel uncomfortable with this duality, which does not exist in classical physics. However, as a “principle” and not a “law,” the complementarity principle does seem to describe nature, and, as such, we use it. We must pay close attention to the fact that we do not use waves and particles simultaneously to describe a particular phenomenon. Experiments dictate what actually happens in nature, and we must draw up a set of rules to describe our observations. These rules naturally lead to a probability interpretation of experimental observations. If we set up a series of small detectors along the screen in the electron double-slit experiment, we can speak of the probability of the electron being detected by one of the detectors. The interference pattern can guide us in our probability determinations. But once the electron has been registered by one of the detec-

**Solution of wave-particle duality**



tors, the probability of its being seen in the other detectors is zero. Matter and radiation propagation is described by wavelike behavior, but matter and radiation interact (that is, undergo creation/annihilation or detection) as particles.

## 5.6 Uncertainty Principle

In Section 5.4, when we discussed the superposition of waves, we learned that to localize a wave packet over a small region  $\Delta x$ , we had to use a large range  $\Delta k$  of wave numbers. For the case of two waves, we found in Equation (5.22) that  $\Delta k \Delta x = 2\pi$ . If we examine a Gaussian wave packet closely, we would find that the product  $\Delta k \Delta x = 1/2$ . The minimum value of the product  $\Delta k \Delta x$  is obtained when Gaussian wave packets are used.

In Section 5.4 we learned that it is impossible to measure simultaneously, with no uncertainty, the precise values of  $k$  and  $x$  for the same particle. The wave number  $k$  may be rewritten as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = p \frac{2\pi}{h} = \frac{p}{\hbar} \quad (5.37)$$

and

$$\Delta k = \frac{\Delta p}{\hbar} \quad (5.38)$$

so that, in the case of the Gaussian wave packet,

$$\Delta k \Delta x = \frac{\Delta p}{\hbar} \Delta x = \frac{1}{2}$$

or

$$\Delta p \Delta x = \frac{\hbar}{2} \quad (5.39)$$

for Gaussian wave packets.

The relationship in Equation (5.39) was first presented in 1927 by the German physicist Werner Heisenberg, who won the Nobel Prize for Physics in 1932. This uncertainty applies in all three dimensions, so we should put a subscript on  $\Delta p$  to indicate the  $x$  direction  $\Delta p_x$ . Heisenberg's **uncertainty principle** can therefore be written

Heisenberg uncertainty  
principle for  $p_x$  and  $x$

$$\Delta p_x \Delta x > \frac{\hbar}{2} \quad (5.40)$$

which establishes limits on the simultaneous knowledge of the values of  $p_x$  and  $x$ .<sup>\*</sup> The limits on  $\Delta p_x$  and  $\Delta x$  represent the lowest possible limits on the uncertainties in knowing the values of  $p_x$  and  $x$ , no matter how good an experimental measurement is made. It is possible to have a greater uncertainty in the values of  $p_x$  and  $x$ , but it is not possible to know them with more precision than allowed by the uncertainty principle. The uncertainty principle does not apply to the products of  $\Delta p_z$  and  $\Delta x$  or to that of  $\Delta p_y$  and  $\Delta z$ . The value of  $\Delta p_z \Delta x$  can be zero. Equation (5.40) is true not only for specific waves such as water or sound, but for matter waves as

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<sup>\*</sup>In some representations of the uncertainty principle, the factor  $\frac{1}{2}$  is absent. Our form represents the lower limit of uncertainty.

well. It is a consequence of the de Broglie wavelength of matter. If we want to know the position of a particle very accurately, then we must accept a large uncertainty in the momentum of the particle. Similarly, if we want to know the precise value of a particle's momentum, it is not possible to specify the particle's location precisely. The uncertainty principle represents another sharp digression with classical physics, where it is assumed that it is possible to specify simultaneously and precisely both the particle's position and momentum. Because of the small value of  $\hbar$ , the uncertainty principle becomes important only on the atomic level.

Consider a particle for which the location is known within a width of  $\ell$  along the  $x$  axis. We then know the position of the particle to within a distance  $\Delta x < \ell/2$ . The uncertainty principle specifies that  $\Delta p$  is limited by

$$\Delta p \geq \frac{\hbar}{2\Delta x} \geq \frac{\hbar}{\ell} \quad (5.41)$$

Because  $p = mv$ , we have  $\Delta p = m\Delta v$ , and

$$\Delta v = \frac{\Delta p}{m} \geq \frac{\hbar}{m\ell} \quad (5.42)$$

These results have some interesting implications. For example, consider a particle with low energy. What is the minimum kinetic energy such a particle can have? We can use nonrelativistic equations, so we have  $K = p^2/2m$ . Equation (5.41) indicates there is an uncertainty in the momentum, so we can assume the minimum value of the momentum will be at least as large as its uncertainty and  $p_{\min} > \Delta p$  to find the minimum value of the kinetic energy  $K_{\min}$ .

$$K_{\min} = \frac{p_{\min}^2}{2m} \geq \frac{(\Delta p)^2}{2m} \geq \frac{\hbar^2}{2m\ell^2} \quad (5.43)$$

Note that this equation indicates that if we are uncertain as to the exact position of a particle, for example, an electron somewhere inside an atom of diameter  $\ell$ , the particle can't have zero kinetic energy.



AIP/Emilio Segrè Visual Archives

**Werner Heisenberg** (1901–1976) was born in Germany, where he spent his entire career at various universities including Munich, Leipzig, and Berlin. He was appointed director of the Kaiser Wilhelm Institute in Berlin in 1942, the highest scientific position in Germany. After World War II Heisenberg spent much of his effort supporting research and opportunities for young physicists and speaking out against the atomic bomb.

## EXAMPLE 5.8

Calculate the momentum uncertainty of (a) a tennis ball constrained to be in a fence enclosure of length 35 m surrounding the court and (b) an electron within the smallest diameter of a hydrogen atom.

**Strategy** We will use Equation (5.40) to find  $\Delta p_x$ . The position uncertainty  $\Delta x$  is approximately half of the enclosure.

**Solution** (a) If we insert the uncertainty of the location of the tennis ball,  $\Delta x = (35 \text{ m})/2$ , into Equation (5.40), we have

$$\Delta p_x \geq \frac{1}{2} \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34} \text{ J} \cdot \text{s}}{2(35 \text{ m})/2} = 3 \times 10^{-36} \text{ kg} \cdot \text{m/s}$$

We will have no problem specifying the momentum of the tennis ball!

(b) The diameter of the hydrogen atom in its lowest energy state (smallest radius) is  $2a_0$ . We arbitrarily take the uncertainty  $\Delta x$  to be half the diameter or equal to the radius,  $\Delta x = a_0$ .

$$\Delta x = a_0 = 0.529 \times 10^{-10} \text{ m}$$

$$\Delta p_x \geq \frac{1}{2} \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34} \text{ J} \cdot \text{s}}{2(0.529 \times 10^{-10} \text{ m})} \\ = 1 \times 10^{-24} \text{ kg} \cdot \text{m/s}$$

This may seem like a small momentum, but for an electron with a mass of about  $10^{-30} \text{ kg}$ , it corresponds to a speed of about  $10^6 \text{ m/s}$ , which is not insignificant! Note that this is comparable to the speed of the electron in the first Bohr orbit [Equation (4.31)].

**EXAMPLE 5.9**

Treat the hydrogen atom as a one-dimensional entity of length  $2a_0$  and determine the electron's minimum kinetic energy.

**Strategy** We will use the uncertainty principle to determine  $K_{\min}$ . Equation (5.43) gives us the minimum kinetic energy for a particle known to be located within a distance  $\ell$ .

**Solution** Equation (5.43) gives

$$K_{\min} = \frac{\hbar^2}{2m\ell^2} = \frac{(\hbar c)^2}{2mc^2\ell^2}$$

$$= \frac{(197 \text{ eV} \cdot \text{nm})^2}{(2)(0.511 \times 10^6 \text{ eV})(2 \times 0.0529 \text{ nm})^2} = 3.4 \text{ eV}$$

A calculation considering three dimensions would give a result about twice this value. This simple calculation gives a reasonable value for the kinetic energy of the ground state electron of the hydrogen atom.

**Energy-Time Uncertainty Principle** Equation (5.40) is not the only form of the uncertainty principle. We can find another form by using Equation (5.23) from our study of wave motion. When we superimposed two waves to form a wave packet we found  $\Delta\omega \Delta t = 2\pi$ . If we evaluate this same product using Gaussian packets, we will find

$$\Delta\omega \Delta t = \frac{1}{2} \quad (5.44)$$

just as we did for the product  $\Delta k \Delta x$ . A relationship like this is easy to understand. If we are to localize a wave packet in a small time  $\Delta t$  (instead of over an infinite time as for a single wave), we must include the frequencies of many waves to have them cancel everywhere but over the time interval  $\Delta t$ . Because  $E = hf$ , we have for each wave

$$\Delta E = h \Delta f = h \frac{\Delta\omega}{2\pi} = \hbar \Delta\omega$$

Therefore

$$\Delta\omega = \frac{\Delta E}{\hbar} \quad \text{and} \quad \Delta\omega \Delta t = \frac{\Delta E}{\hbar} \Delta t = \frac{1}{2}$$

We can therefore obtain another form of Heisenberg's uncertainty principle:

**Heisenberg uncertainty principle for energy and time**

$$\Delta E \Delta t > \frac{\hbar}{2} \quad (5.45)$$

Other *conjugate variables* similar to  $p_x$  and  $x$  in Equation (5.40) also form uncertainty principle relations. The product of conjugate variables (such as  $p_x$  and  $x$  or  $E$  and  $t$ ) must have the same dimensions as Planck's constant. Conjugate variable pairs include the angular momentum  $L$  and angle  $\theta$ , as well as the rotational inertia  $I$  and angular velocity  $\omega$ . Similar uncertainty relations can be written for them.

We once again must emphasize that the uncertainties expressed in Equations (5.40) and (5.45) are intrinsic. They are not due to our inability to construct better measuring equipment. No matter how well we can measure, no matter how accu-

rate an instrument we build, and no matter how long we measure, we can never do any better than the uncertainty principle allows. Many people, including Einstein, have tried to think of situations in which it is violated, but they have not succeeded. At the 1927 Solvay conference Bohr and Einstein had several discussions about the uncertainty principle. Every morning at breakfast Einstein would present a new *gedanken* experiment that would challenge the uncertainty principle. In his careful, deliberate manner, Bohr would refute each objection. Eventually Einstein conceded—he could not provide a valid example of contradiction. These discussions continued off and on into the 1930s, because Einstein had difficulty accepting the idea that the quantum theory could give a complete description of physical phenomena. He believed that quantum theory could give a statistical description of a collection of particles but could not describe the motion of a single particle. Einstein presented several paradoxes to support his ideas. Bohr was able to analyze each paradox and present a reasonable answer. Bohr stressed his complementarity principle, which precludes a simultaneous explanation in terms of waves and particles, as well as Heisenberg's uncertainty principle.

Let's return to the previous discussion of determining which slit an electron passes through in the double-slit experiment (see Figure 5.21). We again shine light on the electrons passing through the slits and look with a powerful microscope. This time we will use the uncertainty principle and make a more detailed calculation. Photons from the shining light bounce off the electron as the electron passes through one of the slits. Photons then scatter into the microscope where we observe them. We must be able to locate the electron's position in  $y$  to at least within  $\Delta y < d/2$  (where  $d$  is the distance between the two slits) to know which slit each electron went through. If the position of the electron is uncertain to less than  $d/2$ , then according to the uncertainty principle, the electron's momentum must be uncertain to at least  $\Delta p_y > \hbar/d$ . Just by scattering photons off the electrons to know which slit the electron went through, we introduce an uncertainty in the electron's momentum. This uncertainty has been caused by the measurement itself.

Consider an electron originally moving in a particular direction; let us choose  $\theta = 0$  for convenience. By scattering the photon from the electron we now have an uncertainty in the angle  $\theta$  due to the “kick” given the electron by the photon in the measurement process. The uncertainty in the electron's angle due to a possible momentum change along the  $y$  axis is  $\Delta\theta = \Delta p_y/p$ , but because  $p = h/\lambda$ , we have

$$\Delta\theta = \frac{\Delta p_y}{p} = \frac{(\Delta p_y)\lambda}{h} = \frac{(\hbar)\lambda}{hd} = \frac{\lambda}{2\pi d}$$

According to Equation (5.36) the first interference maximum will be at  $\sin \theta = \lambda/d$  and the first minimum at  $\sin \theta = \lambda/2d$ . For small angle scattering,  $\sin \theta \approx \theta$ , and the angle of the first minimum is  $\theta_{\min} \approx \lambda/2d$ . Note that the position of the first minimum is on the same order as our uncertainty in  $\Delta\theta$ , so the interference pattern is washed out. If we insist on identifying the electrons as particles and knowing which slit the electrons pass through, the wave characteristics of the electron disappear. We cannot simultaneously treat the electron as both a particle and a wave. This limitation seems to be a fundamental characteristic of the laws of nature. Only the smallness of Planck's constant  $h$  keeps us from encountering this limitation in everyday life.

Niels Bohr tried to turn this limitation into a philosophical principle. When he was awarded the Danish Order of the Elephant, he featured on his coat of arms (see Figure 5.22) the Chinese yin-yang symbol, which stands for the two

### Bohr and Einstein discussions



AIP/Niels Bohr Library, Margarethe Bohr Collection.

**Figure 5.22** Niels Bohr's coat of arms was designed in 1947 when he was awarded the Danish Order of the Elephant. This award was normally given only to royalty and foreign presidents. Bohr chose the Chinese yin-yang symbol because it stands for the two opposing but inseparable elements of nature. The translation of the Latin motto is “Opposites are complementary.” It was hung near the king's coat of arms in the church of Frederiksborg Castle at Hillerød.

opposing but inseparable elements in nature. The Latin motto on the center of the coat of arms means “Opposites are complementary.”



### EXAMPLE 5.10

Calculate the minimum kinetic energy of an electron that is localized within a typical nuclear radius of  $6 \times 10^{-15}$  m.

**Strategy** Let's assume the minimum electron momentum is equal to that determined by the uncertainty principle for an electron constrained within the distance  $\Delta x$  equal to the nuclear radius ( $\Delta x = \pm r$ ). We can then determine the minimum electron energy from the minimum momentum.

**Solution** Given that  $\Delta x \approx r = 6 \times 10^{-15}$  m, we have

$$\begin{aligned}\Delta p &\geq \frac{\hbar}{2\Delta x} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{1.2 \times 10^{-14} \text{ m}} \\ &\geq (5.48 \times 10^{-2} \text{ eV} \cdot \text{s/m}) \left( \frac{3 \times 10^8 \text{ m/s}}{c} \right) \\ &\geq 1.64 \times 10^7 \text{ eV}/c\end{aligned}$$

Because we assumed that the momentum  $p$  is at least as large as the uncertainty in  $p$ , we have

$$p \approx \Delta p \geq 1.64 \times 10^7 \text{ eV}/c$$

Because we don't yet know the electron's energy, let's be careful and calculate it relativistically.

$$\begin{aligned}E^2 &= (pc)^2 + E_0^2 \\ &= \left[ \left( 1.64 \times 10^7 \frac{\text{eV}}{c} \right) c \right]^2 + (0.511 \text{ MeV})^2 \\ &= (16.4 \text{ MeV})^2 + (0.511 \text{ MeV})^2 \\ E &= 16.4 \text{ MeV} \\ \text{K.E.} = E - E_0 &= 16.4 \text{ MeV} - 0.51 \text{ MeV} \\ &= 15.9 \text{ MeV}\end{aligned}$$

Note that because  $E \gg E_0$ , a relativistic calculation was needed.



### CONCEPTUAL EXAMPLE 5.11

We found in the last example that if an electron is confined within the size of a nuclear radius, the uncertainty principle suggests that the minimum kinetic energy of the electron must have a minimum value of about 16 MeV. What does this indicate about the possibility of electrons existing within the nucleus?

**Solution** The value of 16 MeV for the electron's kinetic energy is larger than that observed for electrons emitted from nuclei in beta decay. We conclude that electrons are not confined within the nucleus. Electrons emitted from the nucleus (during beta decay) must actually be created when they are emitted.



### EXAMPLE 5.12

An atom in an excited state normally remains in that state for a very short time ( $\sim 10^{-8}$  s) before emitting a photon and returning to a lower energy state. The “lifetime” of the excited state can be regarded as an uncertainty in the time  $\Delta t$  associated with a measurement of the energy of the state.

This, in turn, implies an “energy width,” namely, the corresponding energy uncertainty  $\Delta E$ . Calculate (a) the characteristic “energy width” of such a state and (b) the uncertainty ratio of the frequency  $\Delta f/f$  if the wavelength of the emitted photon is 300 nm.

**Strategy** (a) We use the uncertainty principle, Equation (5.45), to determine  $\Delta E$  because we know  $\Delta t$ .

(b) We can determine  $\Delta f$  from the energy uncertainty  $\Delta E$  by using  $E = hf$ :  $\Delta E = h \Delta f$ . We can determine the frequency by  $f = c/\lambda$ .

**Solution** (a) Equation (5.45) gives

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{(2)(10^{-8} \text{ s})} = 3.3 \times 10^{-8} \text{ eV}$$

This is a small energy, but many excited energy states have such energy widths. For stable ground states,  $\tau = \infty$ , and  $\Delta E = 0$ . For excited states in the nucleus, the lifetimes can be as short as  $10^{-20}$  s (or shorter) with energy widths of 100 keV (or more).

(b) The frequency is found to be

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8 \text{ m/s}}{300 \times 10^{-9} \text{ m}} = 10^{15} \text{ Hz} \quad (5.46)$$

The uncertainty  $\Delta f$  is

$$\Delta f = \frac{\Delta E}{h} = \frac{3.3 \times 10^{-8} \text{ eV}}{4.136 \times 10^{-15} \text{ eV} \cdot \text{s}} = 8 \times 10^6 \text{ Hz} \quad (5.47)$$

The uncertainty ratio of the frequency  $\Delta f/f$  is

$$\frac{\Delta f}{f} = \frac{8 \times 10^6 \text{ Hz}}{10^{15} \text{ Hz}} = 8 \times 10^{-9}$$

Modern instruments are capable of measuring ratios approaching  $10^{-17}$ , or 1 Hz in a frequency of  $10^{17}$  Hz! Experimental physicists have managed to improve this ratio by an irregular factor of 100 every three years over the past two decades. The experimental limitations are considerably better than needed to measure the energy widths.

## 5.7 Probability, Wave Functions, and the Copenhagen Interpretation

We learned in elementary physics that the instantaneous wave intensity of electromagnetic radiation (light) is  $\epsilon_0 c E^2$  where  $E$  is the electric field. Thus the probability of observing light is proportional to the square of the electric field. In the double-slit light experiment we can be assured that the electric field of the light wave is relatively large at the bright spots on the screen and small in the region of the dark places.

If Young's double-slit experiment is performed with very low intensity levels of light, individual flashes can be seen on the observing screen. We show a simulation of the experiment in Figure 5.19. After only 20 flashes (Figure 5.19a) we cannot make any prediction as to the eventual pattern, but we still know that the *probability* of observing a flash is proportional to the square of the electric field. We now briefly review this calculation that is normally given in introductory physics courses. If the distance from the central ray along the screen we are observing in an experiment like that depicted in Figure 5.18a is denoted by  $y$ , the probability for the photon to be found between  $y$  and  $y + dy$  is proportional to the intensity of the wave ( $E^2$ ) times  $dy$ . For Young's double-slit experiment, the value of the electric field  $\vec{E}$  produced by the two interfering waves is large where the flash is likely to be observed and small where it is not likely to be seen. By counting the number of flashes we relate the energy flux  $I$  (called the intensity) of the light to the number flux,  $N$  per unit area per unit time, of photons having energy  $hf$ . In the wave description, we have  $I = \epsilon_0 c \langle E^2 \rangle$ , and in what appears to be the particle description,  $I = Nh f$ . The flux of photons  $N$ , or the probability  $P$  of observing the photons, is proportional to the average value of the square of the electric field  $\langle E^2 \rangle$ .

How can we interpret the probability of finding the electron in the wave description?

**Wave function** First, let's remember that the localization of a wave can be accomplished by using a wave packet. We used a function  $\Psi(x, t)$  to denote the superposition of many waves to describe the wave packet. We call this function  $\Psi(x, t)$  the **wave function**. In the case of light, we know that the electric field  $\vec{E}$  and magnetic field  $\vec{B}$  satisfy a wave equation. In electrodynamics either  $\vec{E}$  or  $\vec{B}$  serves as the wave function  $\Psi$ . For particles (say electrons) a similar behavior occurs. In this case the wave function  $\Psi(x, t)$  determines the probability, just as the flux of photons  $N$  arriving at the screen and the electric field  $\vec{E}$  determined the probability in the case of light.

**Probability density** For matter waves having a de Broglie wavelength, it is the wave function  $\Psi$  that determines the likelihood (or probability) of finding a particle at a particular position in space at a given time. The value of the wave function  $\Psi$  has no physical significance itself, and as we will see later, it can have a **complex** value (containing both real and imaginary numbers). The quantity  $|\Psi|^2$  is called the **probability density** and represents the probability of finding the particle in a given unit volume at a given instant of time.

In general,  $\Psi(x, y, z, t)$  is a complex quantity and depends on the spatial coordinates  $x, y$ , and  $z$  as well as time  $t$ . The complex nature will be of no concern to us: we use  $\Psi$  times its complex conjugate  $\Psi^*$  when finding probabilities. We are interested here in only a single dimension  $y$  along the observing screen and for a given time  $t$ . In this case  $\Psi^*\Psi dy = |\Psi|^2 dy$  is the probability of observing an electron in the interval between  $y$  and  $y + dy$  at a given time, and we call this  $P(y) dy$ .

$$P(y) dy = |\Psi(y, t)|^2 dy \quad (5.48)$$

**Normalization** Because the electron has to have a probability of unity of being observed *somewhere* along the screen, we integrate the probability density over all space by integrating over  $y$  from  $-\infty$  to  $\infty$ . This process is called **normalization**.

$$\int_{-\infty}^{\infty} P(y) dy = \int_{-\infty}^{\infty} |\Psi(y, t)|^2 dy = 1 \quad (5.49)$$

Max Born (Nobel Prize, 1954), one of the founders of the quantum theory, first proposed this probability interpretation of the wave function in 1926. The determination of the wave function  $\Psi(x, t)$  is discussed in much more detail in the next chapter.

The use of wave functions  $\Psi(x, y, z, t)$  rather than the classical positions  $x(t)$ ,  $y(t)$ ,  $z(t)$  represents a clean break between classical and modern physics. Physicists have developed a set of rules and procedures in quantum theory to determine physical observables like position, momentum, and energy (see Section 6.2).

## The Copenhagen Interpretation

Erwin Schrödinger and Werner Heisenberg worked out independent and separate mathematical models for the quantum theory in 1926. We examine Schrödinger's theory in Chapter 6, because it is somewhat easier to understand and is based on waves. Paul Dirac reported his relativistic quantum theory in 1928. Today there is little disagreement about the mathematical formalism of quantum theory. That is not the case regarding its interpretation.

We want to examine the *Copenhagen interpretation*, because it is the mainstream interpretation of quantum theory. Werner Heisenberg announced his uncertainty

principle in early 1927 while he was a lecturer in Bohr's Institute of Theoretical Physics. At first Bohr, the mentor, thought Heisenberg's uncertainty principle was too narrow, and he pointed out a mistake in Heisenberg's paper concerning a *gedanken* experiment about a gamma-ray microscope used by Heisenberg to prove his point. Heisenberg, the 25-year-old rising star, strongly objected at first to Bohr's opinion and refused Bohr's suggestion to withdraw his paper on the uncertainty principle. Bohr and Heisenberg had many discussions in 1927 formulating the interpretation of quantum mechanics now known as the "Copenhagen interpretation," "Copenhagen school," or sometimes unkindly as "Copenhagen orthodoxy." It was strongly supported by Max Born and Wolfgang Pauli (profiled in Chapter 8).

There are various formulations of the interpretation, but it is generally based on the following:

1. The uncertainty principle of Heisenberg
2. The complementarity principle of Bohr
3. The statistical interpretation of Born, based on probabilities determined by the wave function

Together these three concepts form a logical interpretation of the physical meaning of quantum theory. According to the Copenhagen interpretation, physics depends on the outcomes of measurement. Consider a single electron passing through the two-slit experiment. We can determine precisely where the electron hits the screen by noting a flash. The Copenhagen interpretation rejects arguments about where the electron was between the times it was emitted in the apparatus (and subsequently passed through the two slits) and when it flashed on the screen. The measurement process itself randomly chooses one of the many possibilities allowed by the wave function, and the wave function instantaneously changes to represent the final outcome. Bohr argued that it is not the task of physics to find out how nature is, because we can never understand the quantum world or assign physical meaning to the wave function. Bohr and Heisenberg argued that measurement outcomes are the only reality in physics.

Many physicists objected (and some still do!) to the Copenhagen interpretation for widely varying reasons. One of the basic objections is to its nondeterministic nature. Some also object to the vague measurement process that converts probability functions into nonprobabilistic measurements. Famous physicists who objected to the Copenhagen interpretation were Albert Einstein, Max Planck, Louis de Broglie, and Erwin Schrödinger. Einstein and Schrödinger never accepted the Copenhagen interpretation. Einstein was particularly bothered by the reliance on probabilities, and he wrote Born in 1926 that "God does not throw dice." Nonetheless, it is fair to say that the great majority of physicists today accept the Copenhagen interpretation as the primary interpretation of quantum mechanics. In the past decade physicists have used feedback systems to demonstrate that quantum indeterminism can be reduced by guiding the outcome of a probabilistic quantum process toward a deterministic outcome.\*

Several paradoxes have been proposed by physicists to refute the Copenhagen interpretation. They include the famous Schrödinger cat paradox,<sup>†</sup> the



AP Emilio Segrè Visual Archives

**Max Born** (1882–1970) was born a German in what is now Poland. After studying at several European universities he received his degree in 1907 from the University of Göttingen. After visiting several universities and serving in World War I, he became a professor at Göttingen in 1921 where he did his most important work on the statistical meaning of the new quantum theory (Nobel Prize in Physics, 1954). He and his student, Werner Heisenberg, collaborated on the matrix mechanics version of quantum mechanics. Born, a Jew, was forced to emigrate from Germany in 1933, and after visiting Italy, Cambridge, and India, he settled at the University of Edinburgh in 1936, from which he retired in 1953.

\*See, for example, J. M. Geremia, J. K. Stockton, and H. Mabuchi, *Science* **304**, 270 (2004).

<sup>†</sup>Schrödinger published an essay in 1935, "The Present Situation in Quantum Mechanics," in which he described a thought experiment where a cat in a closed box either lived or died according to whether a quantum event occurred. The paradox was that it was not possible to know whether the cat was dead or alive until an observer opened the box, an apparent contradiction to the intuitive notion that the cat is either alive or dead at any moment.



Einstein-Podolsky-Rosen paradox,\* and Bell's theorem (or inequality).† Space does not allow us to describe these paradoxes (see Problems 49-51). A Princeton University graduate student, Hugh Everett III, announced an alternate interpretation to the Copenhagen view in 1957. In Everett's "Many Worlds" interpretation, the concept of parallel universes is invoked—in itself such a weird idea that it has not gained wide acceptance, but it overcomes some objections to the Copenhagen interpretation. Since 1957 there have been several versions of the Many Worlds interpretation presented, and some physicists prefer it over the Copenhagen interpretation. Nevertheless, the Copenhagen interpretation remains the favored interpretation.

## 5.8 Particle in a Box

Let's now consider the situation of a particle of mass  $m$  trapped in a one-dimensional box of width  $\ell$ . We have already used the uncertainty principle in Equation (5.43) to calculate the minimum kinetic energy of such a particle. Now let's determine the possible energies of such a particle. Because of our discussion in the previous section we want to use the wave nature of the particle in this determination.

First, what is the most probable location of the particle in the state with the lowest energy at a given time, say  $t = 0$ , so that  $\Psi(x, 0) = \Psi(x)$ ? To find the probable location, we will treat the particle as a sinusoidal wave. The particle cannot be physically outside the confines of the box, so the amplitude of the wave motion must vanish at the walls and beyond. In the language of the wave function, its probability of being outside is zero, so the wave function must vanish outside. The wave function must be continuous, and the probability distribution can have only one value at each point in the box. For the probability to vanish at the walls, we must have an integral number of half wavelengths  $\lambda/2$  fit into the box. Note that all the possible waves shown in Figure 5.23 fit this requirement.

The requirement of an integral number of half wavelengths  $\lambda/2$  means that

$$\frac{n\lambda}{2} = \ell \quad \text{or} \quad \lambda_n = \frac{2\ell}{n} \quad (n = 1, 2, 3, \dots) \quad (5.50)$$

The possible wavelengths are quantized, and the wave shapes will have  $\sin(n\pi x/\ell)$  factors. If we treat the problem nonrelativistically and assume there is no potential energy, the energy  $E$  of the particle is

$$E = \text{K.E.} = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

If we insert the values for  $\lambda_n$ , we have

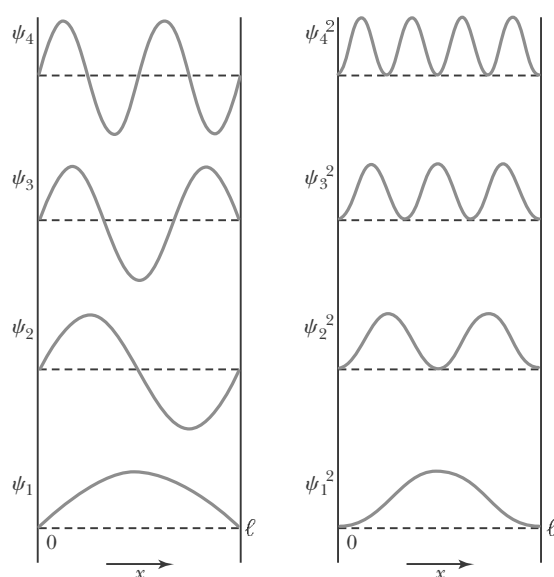
$$E_n = \frac{h^2}{2m} \left( \frac{n}{2\ell} \right)^2 = n^2 \frac{h^2}{8m\ell^2} \quad (n = 1, 2, 3, \dots) \quad (5.51)$$

Therefore, the possible energies of the particle are quantized, and each of these energies  $E_n$  is a possible energy level. Note that the lowest energy is  $E_1 = h^2/8m\ell^2$ . Because we assumed the potential energy to be zero,  $E_n$  is also equal to the kinetic energy. We previously found in Equation (5.43) a value for  $K_{\min} = \hbar^2/2m\ell^2$ ,

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\*A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? *Physical Review* **47**, 777 (1935).

†J. S. Bell, On the Einstein Podolsky Rosen paradox, *Physics* **1**, 195 (1964).



**Figure 5.23** Possible ways of fitting waves into a one-dimensional box of length  $\ell$ . The left side shows the wave functions for the four lowest energy values. The right side shows the corresponding probability distributions.

which differs by a factor  $1/\pi^2$  from the value of  $E_1$ . Why the difference? Equation (5.51) is based on the wave theory of physics, and as we shall see in the next chapter, it is a better calculation than the result in Equation (5.43), which used approximations.

The probability of observing the particle between  $x$  and  $x + dx$  in each state is  $P_n dx \propto |\psi_n(x)|^2 dx$ . Notice that  $E_0 = 0$  is not a possible state, because  $n = 0$  corresponds to  $\psi_0 = 0$ . The lowest energy level is therefore  $E_1$ , with a probability density  $P_1 \propto |\psi_1(x)|^2$ , shown in Figure 5.23. The most probable location for the particle in the lowest energy state is in the middle of the box.

This particle-in-a-box model is more important than it might seem. It is our first application of what we call “quantum theory” or “quantum mechanics.” Notice how the quantization of energy arises from the need to fit a whole number of half-waves into the box and how we obtained the corresponding probability densities of each of the states. The concept of energy levels, as first discussed in the Bohr model, has surfaced in a natural way by using waves. The procedure followed is the same as finding the allowed modes of standing waves inside the box. We can use all the results that we learned about waves in elementary physics.



### EXAMPLE 5.13

Find the quantized energy levels of an electron constrained to move in a one-dimensional atom of size 0.1 nm.

**Strategy** We previously found the minimum kinetic energy of an electron in a similar situation in Example 5.9. In the present case we want to use quantum theory, so we use Equation (5.51) for the energy levels.

**Solution** We use Equation (5.51) and insert the appropriate values for  $m$  and  $\ell$ .

$$\begin{aligned} E_n &= n^2 \frac{h^2}{8m\ell^2} = n^2 \frac{h^2 c^2}{8mc^2 \ell^2} \\ &= n^2 \frac{(1239.8 \text{ eV} \cdot \text{nm})^2}{(8)(0.511 \times 10^6 \text{ eV})(0.1 \text{ nm})^2} \\ &= n^2(38 \text{ eV}) \end{aligned}$$

The first three energy levels are  $E_1 = 38 \text{ eV}$ ,  $E_2 = 152 \text{ eV}$ , and  $E_3 = 342 \text{ eV}$ .

## Summary

Max von Laue suggested the scattering of x rays from matter, thereby firmly establishing the wave nature of x rays and the lattice structure of crystals. W. H. Bragg and W. L. Bragg exploited the wave behavior of x rays by utilizing x-ray scattering to determine the spacing  $d$  between crystal planes according to Bragg's law:

$$n\lambda = 2d \sin \theta \quad (5.1)$$

In an important conceptual leap, de Broglie suggested that particles might also exhibit wave properties, with a wavelength  $\lambda$  determined by their momentum:

$$\lambda = \frac{h}{p} \quad \text{de Broglie wavelength} \quad (5.2)$$

Davisson and Germer, and G. P. Thomson independently, demonstrated the wave characteristics of particles by diffracting low-energy electrons from crystals.

Particles may be described using waves by representing them as wave packets, the superposition of many waves of different amplitudes and frequencies. The group velocity  $u_{\text{gr}} = d\omega/dk$  represents the speed of the particle described by the wave packet.

Niels Bohr proposed a principle of complementarity, stating that it is not possible to describe physical behavior

simultaneously in terms of both particles and waves. We must use either one form of description or the other, thus resolving (or avoiding) the wave-particle duality problem.

We describe particles exhibiting wave behavior by using wave functions  $\Psi$ , which in general may be complex-valued functions of space and time. The probability of observing a particle between  $x$  and  $x + dx$  at time  $t$  is  $|\Psi(x, t)|^2 dx$ .

Werner Heisenberg pointed out that it is not possible to know simultaneously both the exact momentum and position of a particle or to know its precise energy at a precise time. These relationships

$$\Delta p_x \Delta x \geq \hbar/2 \quad (5.40)$$

$$\Delta E \Delta t \geq \hbar/2 \quad (5.45)$$

are called *Heisenberg's uncertainty principle* and are consistent with Bohr's complementarity principle. No experiment, regardless of how clever, can measure  $p$ ,  $x$ ,  $E$ , and  $t$  better than the uncertainties expressed in Equations (5.40) and (5.45). The mainstream interpretation of quantum theory is the Copenhagen interpretation, which depends on the uncertainty principle, the complementarity principle, and the statistical interpretation.

Energy levels naturally arise when a particle in a box is considered to have wave behavior.

## Questions

1. In 1900, did it seem clear that x rays were electromagnetic radiation? Give reasons for your answer. Was it important to perform further experiments to verify the characteristics of x rays?
2. In the early 1900s it was found that x rays were more difficult to refract or diffract than visible light. Why did this lead researchers to suppose that the wavelengths of x rays were shorter rather than longer than those of light?
3. What determines whether a given photon is an x ray? Could an x ray have a wavelength longer than some ultraviolet light?
4. For a single crystal, transmission x-ray scattering will produce dots. However, if there are randomly oriented crystals, as in powder, concentric rings appear. Explain the difference qualitatively.
5. How many particles do you think might be shown experimentally to exhibit wavelike properties? List at least four and discuss possible experiments to show this behavior.
6. Why are neutrons more widely used than protons for studying crystal structure? What about using a hydrogen atom?
7. Why are "cold" neutrons useful for studying crystal structure? How could one obtain "cold" neutrons?
8. It has been said that many experimental discoveries are made as a result of accidents (for example, that of Davisson and Germer). This statement may have some truth, but what traits and abilities must good scientists possess to take advantage of their accidents?
9. Images taken with transmission electron microscopes are produced by passing very high energy electrons (40–100 keV) through matter. Why are the images always in black and white (or any two colors)?
10. Are the following phenomena wave or particle behaviors? Give your reasoning. (a) Television picture, (b) rainbows, (c) football sailing between goal posts, (d) telescope observing the moon, (e) police radar.
11. The experiment by Jönsson that showed the wavelike properties of electrons passing through a double slit

- is considered a pedagogically interesting experiment but not a landmark experiment. Why do you suppose this is true?
12. Can you think of an experiment other than those mentioned in this chapter that might show the wave-like properties of particles? Discuss it.
  13. Why doesn't the uncertainty principle restriction apply between the variables  $p_x$  and  $x$ ?
  14. How does the uncertainty principle apply to a known stable atomic system that apparently has an infinite lifetime? How well can we know the energy of such a system?
  15. According to the uncertainty principle, can a particle having a kinetic energy of exactly zero be confined somewhere in a box of length  $L$ ? Explain.
  16. What is similar about the conjugate variable pairs  $(p_x, x)$ ,  $(E, t)$ ,  $(L, \theta)$ , and  $(I, \omega)$ ?
  17. What are the dimensions of the wave function  $\Psi(x, t)$  that describes matter waves? Give your reasoning.
  18. Soon after their discovery, Davisson and Germer were using their experimental technique to describe new crystal structures of nickel. Do you think they were justified? Explain how you think their results allowed them to make such statements.
  19. Albert Einstein was a dissenter to the Copenhagen Interpretation and what it represented until the day he died. In a letter to Max Born in December 1926, Einstein wrote, "The theory yields a lot, but it hardly brings us any closer to the secret of the Old One. In any case I am convinced that God does not throw dice." What do you think Einstein meant by this statement? Who or what is the Old One?
  20. It has been said that the energy-time version of the uncertainty principle allows a violation of the conservation of energy. The argument is that the uncertainty  $\Delta E$  allows the possibility that we may not know that energy conservation is violated during a small time  $\Delta t$ . Discuss arguments pro and con concerning this possibility.
  21. The Fifth Solvay Congress, held in Brussels in October 1927, was dedicated to the quantum theory. A photo taken of the famous participants is often reproduced. Identify at least 10 participants and discuss what their contributions were to quantum physics, either experimentally or theoretically.
  22. Summarize the discussions that Einstein and Bohr had at the 1927 Solvay Congress. List at least three objections that Einstein had to the Copenhagen interpretation of quantum mechanics and give Bohr's explanation.

## Problems

Note: The more challenging problems have their problem numbers shaded by a blue box.

### 5.1 X-Ray Scattering

1. X rays scattered from a crystal have a first-order diffraction peak at  $\theta = 12.5^\circ$ . At what angle will the second- and third-order peaks appear?
2. X rays of wavelength 0.207 nm are scattered from NaCl. What is the angular separation between first- and second-order diffraction peaks? Assume scattering planes that are parallel to the surface.
3. Potassium chloride is a crystal with lattice spacing of 0.314 nm. The first peak for Bragg diffraction is observed to occur at  $12.8^\circ$ . What energy x rays were diffracted? What other order peaks can be observed ( $\theta \leq 90^\circ$ )?
4. A cubic crystal with interatomic spacing of 0.24 nm is used to select  $\gamma$  rays of energy 100 keV from a radioactive source containing a continuum of energies. If the incident beam is normal to the crystal, at what angle do the 100-keV  $\gamma$  rays appear?

### 5.2 De Broglie Waves

5. Calculate the de Broglie wavelength of a 1.2-kg rock thrown with a speed of 6.0 m/s into a pond. Is this

wavelength similar to that of the water waves produced? Explain.

6. Calculate the de Broglie wavelength of a typical nitrogen molecule in the atmosphere on a hot summer day ( $37^\circ\text{C}$ ). Compare this with the diameter (less than 1 nm) of the molecule.
7. Transmission electron microscopes that use high-energy electrons accelerated over a range from 40 to 100 kV are employed in many applications including the study of biological samples (like a virus) and nanoscience research and development (alloy particles and carbon nanotubes, for example). What would be the spatial limitation for this range of electrons? It is often true that resolution is limited by the optics of the lens system, not by the intrinsic limitation due to the de Broglie wavelength.
8. A 3.0-MV transmission electron microscope has been in operation at Osaka University in Japan for several years. The higher-energy electrons allow for deeper sample penetration and extremely high resolution. What is the resolution limit for these electrons?
9. Work out Example 5.2b strictly using SI units of m, J, kg, and so on, and compare with the method of the example using eV units.

10. Assume that the total energy  $E$  of an electron greatly exceeds its rest energy  $E_0$ . If a photon has a wavelength equal to the de Broglie wavelength of the electron, what is the photon's energy? Repeat the problem assuming  $E = 2E_0$  for the electron.
11. Determine the de Broglie wavelength of a particle of mass  $m$  and kinetic energy  $K$ . Do this for both (a) a relativistic and (b) a nonrelativistic particle.
12. The Stanford Linear Accelerator accelerated electrons to an energy of 50 GeV. What is the de Broglie wavelength of these electrons? What fraction of a proton's diameter ( $d \approx 2 \times 10^{-15}$  m) can such a particle probe?
13. Find the kinetic energy of (a) photons, (b) electrons, (c) neutrons, and (d)  $\alpha$  particles that have a de Broglie wavelength of 0.13 nm.
14. Find the de Broglie wavelength of neutrons in equilibrium at the temperatures (a) 5.0 K and (b) 0.010 K.
15. An electron initially at rest is accelerated across a potential difference of 3.00 kV. What are its wavelength, momentum, kinetic energy, and total energy?
16. What is the wavelength of an electron with kinetic energy (a) 40 eV, (b) 400 eV, (c) 4.0 keV, (d) 40 keV, (e) 0.40 MeV, and (f) 4.0 MeV? Which of these energies are most suited for study of the NaCl crystal structure?
17. Calculate the de Broglie wavelength of (a) an oxygen ( $O_2$ ) molecule darting around the room at 480 m/s and (b) an *Escherichia coli* bacterium of mass  $6.5 \times 10^{-13}$  kg, which has been measured to move at a speed of  $1.0 \times 10^{-5}$  m/s.
18. (a) What is the de Broglie wavelength of the 1.0-TeV protons accelerated in the Fermilab Tevatron accelerator? These high-energy protons are needed to probe elementary particles (see Chapter 14). (b) Repeat for the 7.0-TeV protons produced at CERN.

### 5.3 Electron Scattering

19. In an electron-scattering experiment, an intense reflected beam is found at  $\phi = 32^\circ$  for a crystal with an interatomic distance of 0.23 nm. What is the lattice spacing of the planes responsible for the scattering? Assuming first-order diffraction, what are the wavelength, momentum, kinetic energy, and total energy of the incident electrons?
20. Davisson and Germer performed their experiment with a nickel target for several electron bombarding energies. At what angles would they find diffraction maxima for 48-eV and 64-eV electrons?
21. A beam of 2.0-keV electrons incident on a crystal is refracted and observed (by transmission) on a screen 35 cm away. The radii of three concentric rings on the screen, all corresponding to first-order diffraction, are 2.1 cm, 2.3 cm, and 3.2 cm. What is the lattice-plane spacing corresponding to each of the three rings?
22. A beam of thermal neutrons (kinetic energy = 0.025 eV) scatters from a crystal with interatomic spacing 0.45 nm. What is the angle of the first-order Bragg peak?

### 5.4 Wave Motion

23. Generating plants in some power systems drop 10% of their load when the AC frequency changes by 0.30 Hz from the standard of 60 Hz. How often must the reading be monitored in order for the automatic operating system to be able to take corrective action? Let the time between measurements be at least half that determined by the bandwidth relation.
24. Consider electrons of kinetic energy 6.0 eV and 600 keV. For each electron, find the de Broglie wavelength, particle speed, phase velocity (speed), and group velocity (speed).
25. A wave, propagating along the  $x$  direction according to Equation (5.18), has a maximum displacement of 4.0 cm at  $x = 0$  and  $t = 0$ . The wave speed is 5.0 cm/s, and the wavelength is 7.0 cm. (a) What is the frequency? (b) What is the wave's amplitude at  $x = 10$  cm and  $t = 13$  s?
26. A wave of wavelength 4.0 cm has a wave speed of 4.2 cm/s. What is its (a) frequency, (b) period, (c) wave number, and (d) angular frequency?
27. Two waves are traveling simultaneously down a long Slinky. They can be represented by  $\Psi_1(x, t) = 0.0030 \sin(6.0x - 300t)$  and  $\Psi_2(x, t) = 0.0030 \sin(7.0x - 250t)$ . Distances are measured in meters and time in seconds. (a) Write the expression for the resulting wave. (b) What are the phase and group velocities? (c) What is  $\Delta x$  between two adjacent zeros of  $\Psi$ ? (d) What is  $\Delta k \Delta x$ ?
28. A wave packet describes a particle having momentum  $p$ . Starting with the relativistic relationship  $E^2 = p^2 c^2 + E_0^2$ , show that the group velocity is  $\beta c$  and the phase velocity is  $c/\beta$  (where  $\beta = v/c$ ). How can the phase velocity physically be greater than  $c$ ?
29. For waves in shallow water the phase velocity is about equal to the group velocity. What is the dependence of the phase velocity on the wavelength?
30. Find the group and phase velocities of 10-MeV protons and 10-MeV electrons (see Problem 28).
31. Use Equation (5.25) with  $\hat{A}(k) = A_0$  for the range of  $k = k_0 - \Delta k/2$  to  $k_0 + \Delta k/2$ , and  $\hat{A}(k) = 0$  elsewhere, to determine  $\Psi(x, 0)$ , that is, at  $t = 0$ . Sketch the envelope term, the oscillating term, and  $|\Psi(x, 0)|^2$ . Approximately what is the width  $\Delta x$  over the full-width-half-maximum part of  $|\Psi(x, 0)|^2$ ? What is the value of  $\Delta k \Delta x$ ?
32. Use Equation (5.31) to show that  $u_{gr}$  correctly represents the velocity of the particle both relativistically and classically.

### 5.5 Waves or Particles?

33. Light of intensity  $I_0$  passes through two sets of apparatus. One contains one slit and the other two slits. The slits have the same width. What is the ratio of the outgoing intensity amplitude for the central peak for the two-slit case compared to the single slit?

34. Design a double-slit electron-scattering experiment using 1.0-keV electrons that will provide the first maximum at an angle of  $1^\circ$ . What will be the slit separation  $d$ ?
35. You want to design an experiment similar to the one done by Jönsson that does not require magnification of the interference pattern in order to be seen. Let the two slits be separated by 2000 nm. Assume that you can discriminate visually between maxima that are as little as 0.3 mm apart. You have at your disposal a lab that allows the screen to be placed 80 cm away from the slits. What energy electrons will you require? Do you think such low-energy electrons will represent a problem? Explain.

### 5.6 Uncertainty Principle

36. A proton is confined in a uranium nucleus of radius  $7.2 \times 10^{-15}$  m. Determine the proton's minimum kinetic energy according to the uncertainty principle if the proton is confined to a one-dimensional box that has length equal to the nuclear diameter.
37. A neutron is confined in a deuterium nucleus (deuteron) of diameter  $3.1 \times 10^{-15}$  m. Use the energy-level calculation of a one-dimensional box to calculate the neutron's minimum kinetic energy. What is the neutron's minimum kinetic energy according to the uncertainty principle?
38. What is the ratio uncertainty of the velocities ( $\Delta v/v$ ) of (a) an electron and (b) a proton confined to a one-dimensional box of length 1.8 nm?
39. Show that the uncertainty principle can be expressed in the form  $\Delta L \Delta \theta > \hbar/2$ , where  $\theta$  is the angle and  $L$  the angular momentum. For what uncertainty in  $L$  will the angular position of a particle be completely undetermined?
40. Some physics theories indicate that the lifetime of the proton is about  $10^{36}$  years. What would such a prediction say about the energy of the proton?
41. What is the bandwidth  $\Delta \omega$  of an amplifier for radar if it amplifies a pulse of width 2.4  $\mu$ s?
42. Find the minimum uncertainty in the speed of a bacterium having mass  $3.0 \times 10^{-15}$  kg if we know the position of the bacterium to within 1.0  $\mu$ m, that is, to about its own size.
43. An atom in an excited state of 4.7 eV emits a photon and ends up in the ground state. The lifetime of the excited state is  $1.0 \times 10^{-13}$  s. (a) What is the energy uncertainty of the emitted photon? (b) What is the spectral line width (in wavelength) of the photon?
44. An electron microscope is designed to resolve objects as small as 0.14 nm. What energy electrons must be used in this instrument?
45. Rayleigh's criterion is used to determine when two objects are barely resolved by a lens of diameter  $d$ . The angular separation must be greater than  $\theta_R$  where

$$\theta_R = 1.22 \frac{\lambda}{d}$$

In order to resolve two objects 4000 nm apart at a distance of 20 cm with a lens of diameter 5 cm, what energy (a) photons or (b) electrons should be used? Is this consistent with the uncertainty principle?

46. Calculate the de Broglie wavelength of a 5.5-MeV  $\alpha$  particle emitted from an  $^{241}\text{Am}$  nucleus. Could this particle exist inside the  $^{241}\text{Am}$  nucleus (diameter  $\approx 1.6 \times 10^{-14}$  m)? Explain.
47. Show that the minimum energy of a simple harmonic oscillator is  $\hbar\omega/2$ . What is the minimum energy in joules for a mass of 28 g oscillating on a spring with a spring constant of 8.2 N/m?

### 5.7 Probability, Wave Functions, and the Copenhagen Interpretation

48. The wave function of a particle in a one-dimensional box of width  $L$  is  $\psi(x) = A \sin(\pi x/L)$ . If we know the particle must be somewhere in the box, what must be the value of  $A$ ?
49. Write a cogent description of the Schrödinger cat paradox. Discuss variations of the paradox and the current status of its experimental verification.
50. Write a cogent description of the Einstein-Podolsky-Rosen paradox. Discuss variations of the paradox and the current status of its experimental verification.
51. Write a cogent description of the Bell inequality. Discuss variations and the current status of its experimental verification.

### 5.8 Particle in a Box

52. Write down the normalized wave functions for the first three energy levels of a particle of mass  $m$  in a one-dimensional box of width  $L$ . Assume there are equal probabilities of being in each state.
53. A particle in a one-dimensional box of length  $L$  has a kinetic energy much greater than its rest energy. What is the ratio of the following energy levels  $E_n$ :  $E_2/E_1$ ,  $E_3/E_1$ ,  $E_4/E_1$ ? How do your answers compare with the nonrelativistic case?

### General Problems

54. Consider a wave packet having the product  $\Delta p \Delta x = \hbar$  at a time  $t = 0$ . What will be the width of such a wave packet after the time  $m(\Delta x)^2/\hbar$ ?
55. Analyze the Gaussian wave packet carefully and show that  $\Delta k \Delta x = 1/2$ . You must justify the assumptions you make concerning uncertainties in  $k$  and  $x$ . Take the Gaussian form given in Equation (5.26). (Hint: the linear "spread" of the wave packet  $\Delta x$  is given by one standard deviation, at which point the probability amplitude ( $|\Psi|^2$ ) has fallen to one half its peak value.)
56. An electron emitted in the beta decay of bismuth-210 has a mean kinetic energy of 390 keV. (a) Find the de Broglie wavelength of the electron. (b) Would such an electron be useful in a Davisson-Germer type scattering experiment? Address this question by deter-

- mining the angle at which a first-order diffraction maximum would be found using the same nickel target as Davisson and Germer.
57. Electrons produced at the Thomas Jefferson National Accelerator Facility have a maximum energy of 6.0 GeV. (a) What is the de Broglie wavelength of each electron? (b) In what part of the electromagnetic spectrum do you find a photon of comparable wavelength?
  58. The artificially created nuclear isotope tritium ( ${}^3\text{H}$ ) is important in many applications. This isotope undergoes beta decay, emitting an electron with a mean kinetic energy of 5.7 keV. (a) What is the de Broglie wavelength of such an emitted electron? (b) Is it likely that the electron existed inside the 3.4-fm-diameter nucleus just before it was emitted? Explain.
  59. As you saw in Chapter 4, the size of the hydrogen atom grows in proportion to  $n^2$ , where  $n$  is the quantum state. For an atom in the  $n = 10$  state, model the electron as confined to a one-dimensional box of length equal to the atom's diameter. Find the minimum energy of the electron in this box.
  60. An oboe player tunes the orchestra with the "Concert A" note, which has a frequency of 440 Hz. If she plays the note for 2.5 s, what minimum range of frequencies is heard during this time?
  61. As we learned in Section 3.9, an electron and positron can annihilate each other completely and form two gamma rays. (a) If the electron and positron were initially at rest, what are the wavelengths of the two emitted gamma rays? (b) Repeat if the electron and positron were each traveling at a speed of  $0.30c$  measured in the lab and collided head-on.
  62. Most of the particles known to physicists are unstable. For example, the lifetime of the neutral pion,  $\pi^0$ , is about  $8.4 \times 10^{-17}$  s. Its mass is  $135.0 \text{ MeV}/c^2$ . (a) What is the energy width of the  $\pi^0$  in its ground state? (b) What is the relative uncertainty  $\Delta m/m$  of the pion's mass?
  63. The range of the nuclear strong force is believed to be about  $1.2 \times 10^{-15} \text{ m}$ . An early theory of nuclear physics proposed that the particle that "mediates" the strong force (similar to the photon mediating the electromagnetic force) is the pion. Assume that the pion moves at the speed of light in the nucleus, and calculate the time  $\Delta t$  it takes to travel between nucleons. Assume that the distance between nucleons is also about  $1.2 \times 10^{-15} \text{ m}$ . Use this time  $\Delta t$  to calculate the energy  $\Delta E$  for which energy conservation is violated during the time  $\Delta t$ . This  $\Delta E$  has been used to estimate the mass of the pion. What value do you determine for the mass? Compare this value with the measured value of  $135 \text{ MeV}/c^2$  for the neutral pion.
  64. The planes of atoms in a particular cubic crystal lie parallel to the surface, 0.80 nm apart. X rays having wavelength 0.50 nm are directed at an angle  $\theta$  to the surface. (a) For what values of  $\theta$  will there be a strong reflection? (b) What energy electrons could give the same result?
  65. Aliens visiting Earth are fascinated by baseball. They are so advanced that they have learned how to vary  $\hbar$  to make sure that a pitcher cannot throw a strike with any confidence. Assume the width of the strike zone is 0.38 m, the speed of the baseball is 35 m/s, the mass of the baseball is 145 g, and the ball travels a distance of 18 m. What value of  $\hbar$  is required? (*Hint:* there are two uncertainties here: the width of the strike zone and the transverse momentum of the pitched ball.)
  66. Neutrons from nuclear reactors are used in neutron diffraction experiments to measure interplanar spacings of a crystal lattice. The interplanar spacing can be measured as an indication of strain in the sample. Neutrons are particularly useful because they are less destructive than x rays and are able to penetrate deep into the sample. Their magnetic moment allows their use to study magnetic properties of matter. To study a particular polycrystalline sample with a planar spacing of 0.156 nm, a detector is mounted at an angle of  $26^\circ$  from the incident neutron beam. What energy neutrons from the reactor must be used in this experiment? An accelerator-based spallation neutron source is in operation at Oak Ridge National Laboratory.
  67. Use a computer program to produce a wave packet using the function  $\psi_n = A_n \cos(2\pi nx)$  where the integer  $n$  ranges from 9 to 15. Let the amplitude  $A_{12} = 1$  with the amplitudes  $A_n$  decreasing symmetrically by  $1/2, 1/3, 1/4$  on either side of  $A_{12}$  (for example,  $A_{10} = 1/3$  and  $A_{15} = 1/4$ ). (a) Plot the wave packet  $\psi = \sum_n \psi_n$  versus  $x$  and each wave  $\psi_n$  over a wide enough range in  $x$  to see repeatable behavior for the wave packet. (b) Where is the wave packet centered? Over what value of  $x$  is the wave packet repeated?
  68. Most elementary particles (see Chapter 14) are not stable, and physicists have measured their mean lifetime  $\tau$ . Consider the uncertainty that this places on their mass-energy. The energy spread  $\Gamma$  is the full width of the particle's energy distribution at half its maximum value. (a) If we relate  $\tau = \Delta t$  and  $\Gamma = 2 \Delta E$ , what is the relation  $\Gamma\tau$  in terms of the uncertainty principle? (b) What is the energy spread in the mass-energy of the following particles with their mean lifetimes in parentheses: neutron (886 s), charged pion  $\pi^+$  or  $\pi^-$  ( $2.6 \times 10^{-8} \text{ s}$ ), and upsilon ( $1.2 \times 10^{-20} \text{ s}$ )?
  69. "Ultrafast" lasers produce bursts of light that last only on the order of 10 fs. Because of the uncertainty principle, such short bursts have a relatively large uncertainty in frequency and wavelength. A particular ultrafast laser produces a 10-fs burst of light from a 532-nm laser. (a) Find the uncertainty  $\Delta f$  in the light's frequency and the ratio  $\Delta f/f$ . (b) What is the range  $\Delta\lambda$  of wavelengths produced? (c) Compare your answer to part (b) with the original wavelength and with the length of the light pulse that is generated in 10 fs.
  70. An ultrafast laser (see the preceding problem) has a central wavelength of 550 nm. What pulse duration would result in a spread of wavelengths that just covered the visible spectrum, 400 nm to 700 nm?



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