APPENDIX A
Summary of Vector Relations

A.1. Vector Algebra

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$
$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$
$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$
$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$
$$\mathbf{A}\times\mathbf{A} = \mathbf{0}$$
$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$
$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = [\mathbf{ABC}]$$
$$[\mathbf{ABC}] = [\mathbf{BCA}] = [\mathbf{CAB}] = -[\mathbf{ACB}] = -[\mathbf{BAC}] = -[\mathbf{CBA}]$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$
$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$

A.2. Vector Calculus

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$
$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$
$$\nabla\phi = \nabla\phi(u) = \phi'(u)\nabla u$$
$$\nabla\cdot(\mathbf{f}+\mathbf{g}) = \nabla\cdot\mathbf{f} + \nabla\cdot\mathbf{g}$$
$$\mathrm{div}(\mathrm{grad}\,\phi) = \nabla\cdot(\nabla\phi) = \nabla^2\phi$$
$$\nabla\times(\mathbf{f}+\mathbf{g}) = \nabla\times\mathbf{f} + \nabla\times\mathbf{g}$$
$$\nabla\times(\nabla\phi) = \mathbf{0}$$
$$\nabla\cdot(\nabla\times\mathbf{f}) = 0$$
$$(\mathbf{f}\times\nabla)\cdot\mathbf{g} = \mathbf{f}\cdot(\nabla\times\mathbf{g})$$
$$\nabla\cdot(\phi\mathbf{f}) = \phi\nabla\cdot\mathbf{f} + \mathbf{f}\cdot(\nabla\phi)$$
$$\nabla\times(\phi\mathbf{f}) = \phi\nabla\times\mathbf{f} + (\nabla\phi)\times\mathbf{f} = \phi\nabla\times\mathbf{f} - \mathbf{f}\times\nabla\phi$$
$$\nabla\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\nabla\times\mathbf{f}) - \mathbf{f}\cdot(\nabla\times\mathbf{g})$$
$$\nabla\times(\mathbf{f}\times\mathbf{g}) = \mathbf{f}(\nabla\cdot\mathbf{g}) - \mathbf{g}(\nabla\cdot\mathbf{f}) + (\mathbf{g}\cdot\nabla)\mathbf{f} - (\mathbf{f}\cdot\nabla)\mathbf{g}$$
$$\nabla(\mathbf{f}\cdot\mathbf{g}) = \mathbf{f}\times(\nabla\times\mathbf{g}) + \mathbf{g}\times(\nabla\times\mathbf{f}) + (\mathbf{f}\cdot\nabla)\mathbf{g} + (\mathbf{g}\cdot\nabla)\mathbf{f}$$
$$\mathrm{curl}(\mathrm{curl}\,\mathbf{f}) = \nabla\times(\nabla\times\mathbf{f}) = \nabla(\nabla\cdot\mathbf{f}) - \nabla^2\mathbf{f}$$
$$\nabla^2\mathbf{f} = \nabla(\nabla\cdot\mathbf{f}) - \nabla\times(\nabla\times\mathbf{f})$$
$$\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\,\nabla\phi - \phi\,\nabla\psi}{\psi^2}$$

$$\iiint_R \nabla\cdot\mathbf{f}\,dV = \oiint_S \mathbf{f}\cdot d\mathbf{S}$$
$$\iint_S \nabla\times\mathbf{f}\cdot d\mathbf{S} = \oint_C \mathbf{f}\cdot d\mathbf{r}$$
$$\iiint_R \nabla\phi\,dV = \oiint_S d\mathbf{S}\,\phi$$
$$\iint_S d\mathbf{S}\times\nabla\phi = \oint_C \phi\,d\mathbf{r}$$
$$\iiint_R \nabla\times\mathbf{f}\,dV = \oiint_S d\mathbf{S}\times\mathbf{f}$$
$$\iint_S (d\mathbf{S}\times\nabla)\times\mathbf{f} = \oint_C d\mathbf{r}\times\mathbf{f}$$

APPENDIX B
Vector Differential Operations in Orthogonal Coordinates

B.1. Rectangular Coordinates (x, y, z)

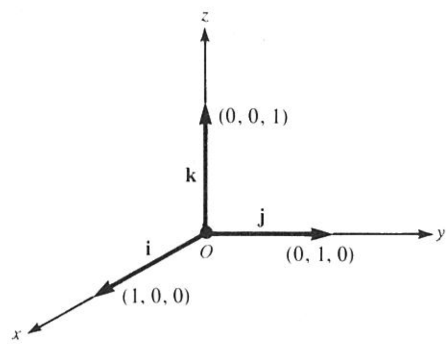
$$\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\mathbf{i} + \frac{\partial\psi}{\partial y}\mathbf{j} + \frac{\partial\psi}{\partial z}\mathbf{k}$$

$$\nabla\cdot\mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$



B.2. Cylindrical Coordinates (ρ, ϕ, z)

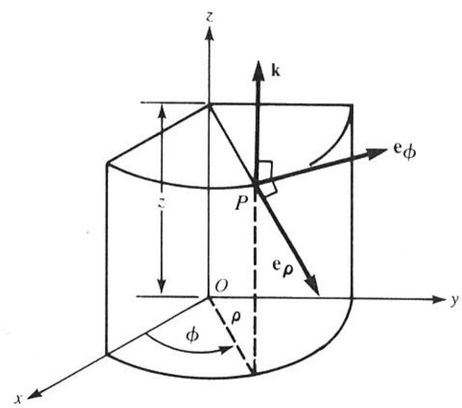
$$\mathbf{f} = f_\rho\mathbf{e}_\rho + f_\phi\mathbf{e}_\phi + f_3\mathbf{k}$$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\psi}{\partial z}\mathbf{k}$$

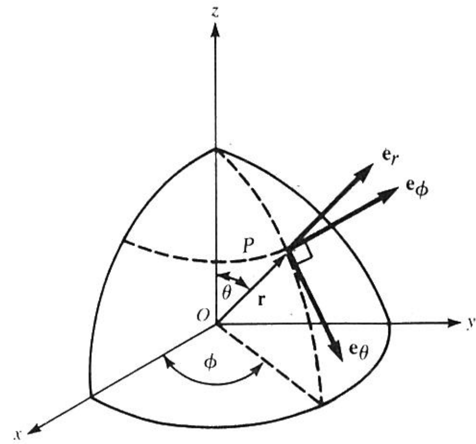
$$\nabla\cdot\mathbf{f} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_3}{\partial z}$$

$$\nabla\times\mathbf{f} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{k} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ f_\rho & f_\phi & f_3 \end{vmatrix}$$

$$= \frac{1}{\rho}\left(\frac{\partial f_3}{\partial\phi} - \frac{\partial f_\phi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial f_\rho}{\partial z} - \frac{\partial f_3}{\partial\rho}\right)\mathbf{e}_\phi + \frac{1}{\rho}\left(\frac{\partial f_\phi}{\partial\rho} - \frac{\partial f_\rho}{\partial\phi}\right)\mathbf{k}$$



B.3. Spherical Coordinates $(r,\ \theta,\ \phi)$



$$\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_\phi\mathbf{e}_\phi$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\mathbf{e}_\phi$$

$$\nabla\cdot\mathbf{f} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2 f_r) + r\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + r\frac{\partial f_\phi}{\partial\phi}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ f_r & rf_\theta & r\sin\theta f_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta f_\phi) - \frac{\partial f_\theta}{\partial\phi}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial f_r}{\partial\phi} - \frac{\partial}{\partial r}(rf_\phi)\right]\mathbf{e}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rf_\theta) - \frac{\partial f_r}{\partial\theta}\right]\mathbf{e}_\phi$$