Calculating the Pauli Matrix equivalent for Spin-1 Particles and further implementing it to calculate the Unitary Operators of the Harmonic Oscillator involving a Spin-1 System

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Here, we derive the Pauli Matrix Equivalent for Spin-1 particles (mainly Z-Boson and W-Boson). Pauli Matrices are generally associated with Spin-$\frac{1}{2}$ particles and it is used for determining the properties of many Spin-$\frac{1}{2}$ particles. But in our case, we try to expand its domain and attempt to implement it for calculating the Unitary Operators of the Harmonic oscillator involving the Spin-1 system and study it.

I. INTRODUCTION

Pauli Matrices are a set of three 2×2 complex matrices which are Hermitian and Unitary in nature and they occur in the Pauli Equation which takes into account the interaction of the spin of a particle with an external electromagnetic field. In Quantum Mechanics, each Pauli matrix is related to an angular momentum operator that corresponds to an observable describing the spin of a Spin-$\frac{1}{2}$ particle, in each of the three spatial directions. But we seldom need to deal with particles which are having spin more than $\frac{1}{2}$ i.e. Spin-1 Particles; Spin-$\frac{3}{2}$ Particles; Spin-2 Particles; etc. and for that we need to search for matrices which perform similar function to that of Pauli Matrices (in case of Spin-$\frac{1}{2}$ particles). In Spin-$\frac{1}{2}$ particles, the Pauli Matrices are in the form of:

$$X = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

$$Y = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{2}$$

$$Z = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

Also sometimes the Identity Matrix, I is referred to as the '*Zeroth*' Pauli Matrix and is denoted by σ_0. The above denoted Matrices are useful for Spin-$\frac{1}{2}$ particles like Fermions (Proton, Neutrons, Electrons, Quarks, etc.) and not for other particles. So, in this paper we will see how to further calculate the equivalent Pauli Matrix for Spin-1 particles and implement it to calculate the Unitary Operators of the Quantum Harmonic Oscillator involving a Spin-1 system.

*S*tructure:
In Section II, we do the Mathematical Modelling for the Equivalent Matrices and discuss the results for higher spin particles. Then, in Section III we revisit the harmonic oscillator in-context of our quantum world and discuss its analytical solution along with its features. In Section IV, we involve with the transformation of Unitary Operators and after that in Section V, we involve the implementation of the Spin-1 system into the Quantum Harmonic Oscillator with relevant transformations and generalization which ultimately leads to the derivation of the Hamiltonian of our system. Then in Section VI, we derive the Unitary Operators of our system using the informations from the previous sections and then in its following subsection we represent the 16×16 Matrix of the different Unitary Operators. At last, in Section VII, we discuss the Results of our project followed by the Conclusion and Acknowledgement.

II. MATHEMATICAL MODELLING

Let us assume that we have a Spin-s system for which the Eigenvalue S^2 is given by:

$$S^2 = s(s+1)\hbar^2 \tag{4}$$

Or

$$S = \sqrt{s(s+1)}\hbar$$

The eigenvalues of S_z are written $s_z\,\hbar$, where s_z is allowed to take the values $s, s-1, \cdots, -s+1, -s$ i.e. there are $2s+1$ distinct allowed values of s_z. We can represent the state of the particle by $(2s+1)$ different Wavefunctions which are in-turn denoted as $\psi_{s_z}(\mathbf{x'})$. Here $\psi_{s_z}(\mathbf{x'})$ is the Probability density for observing the particle at position $\mathbf{x}'$ with spin angular momentum $s_z\hbar$ in the z- direction. Now, by using the extended Pauli scheme, we can easily find out the Momentum operators and the Spin operators. The Spin Operator comes out in the form:

$$(\sigma_k)_{jl} = \frac{\langle s, j|S_k|s, l\rangle}{s\hbar} \tag{5}$$

Where, j and l are integers and $j, l \in (-s, +s)$. Now, to make our calculations easier, we continue with σ_z matrix. We know that:

$$S_z|s, j\rangle = j\hbar|s, j\rangle \tag{6}$$

∴ We can write:

$$(\sigma_3)_{jl} = \frac{\langle s,j|S_k|s,l\rangle}{s\hbar} = \frac{j}{s}\delta_{ij} \tag{7}$$

Here we have used the Orthonormality property of $|s,j\rangle$. Thus, σ_z is the suitably normalized diagonal matrix of the eigenvalues of S_z. The elements of σ_x and σ_y are most easily obtained by considering the ladder operators:

$$S_\pm = S_x \pm iS_y \tag{8}$$

Now, according to Eq.(5)-(8), we can write[9]:

$$S_+\,|s,j\rangle = [s\,(s+1) - j\,(j+1)]^{1/2}\,\hbar\,|s,j+1\rangle \tag{9}$$

and

$$S^-\,|s,j\rangle = [s\,(s+1) - j\,(j-1)]^{1/2}\,\hbar\,|s,j-1\rangle. \tag{10}$$

Now, by combining all the conditions and equations, we have:

$$\begin{aligned}(\sigma_1)_{j\,l} = & \frac{[s\,(s+1) - j\,(j-1)]^{1/2}}{2\,s}\,\delta_{j\ l+1} \\ & + \frac{[s\,(s+1) - j\,(j+1)]^{1/2}}{2\,s}\,\delta_{j\ l-1}\end{aligned} \tag{11}$$

and

$$\begin{aligned}(\sigma_2)_{j\,l} = & \frac{[s\,(s+1) - j\,(j-1)]^{1/2}}{2\,\mathrm{i}\,s}\,\delta_{j\ l+1} \\ & - \frac{[s\,(s+1) - j\,(j+1)]^{1/2}}{2\,\mathrm{i}\,s}\,\delta_{j\ l-1}\end{aligned} \tag{12}$$

∴ According to Eq.(7), Eq.(11) and Eq.(12), we have:

$$\sigma_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \tag{13}$$

$$\sigma_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix} \tag{14}$$

$$\sigma_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix} \tag{15}$$

Where, $\sigma_1, \sigma_2, \sigma_3$ are the Pauli Matrix equivalents for Spin-1 Particles.

III. HARMONIC OSCILLATOR IN BRIEF

The most common and familiar version of the Hamiltonian of the quantum harmonic oscillator in general can be written as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \tag{16}$$

where $\hat{H}$ is the Hamiltonian of the System, m is the mass of the particle, k is the bond stiffness (which is analogous to spring constant in classical mechanics), $\hat{x}$ is the position operator and

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{17}$$

is the momentum operator where $\hbar$ is the reduced Plank's constant.

The analytical solution of the Schrodinger wave equation is given by Ref.[1]:

$$\Psi = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\frac{1}{2^n\,n!}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{\zeta^2}{2}} e^{-\frac{\beta^2}{2}} H_{n_x}(\zeta)H_{n_y}(\beta)U(t) \tag{18}$$

Where;

$$\zeta = \sqrt{\frac{m\omega}{\hbar}}x \quad and \quad \beta = \sqrt{\frac{m\omega}{\hbar}}y \tag{19}$$

Here H_n is the nth order Hermite polynomial. U(t) is the unitary operator of the system showing its time evolution and is given by:

$$U(t) = exp(\frac{-itE_n}{\hbar}) = e^{\frac{-itE_n}{\hbar}} \tag{20}$$

where E_n are the allowed energy eigenvalues of the particle and are given by:

$$E_n = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar\omega = (n_x + n_y + 1)\hbar\omega \tag{21}$$

And the states corresponding to the various energy eigenvalues are orthogonal to each other and satisfy:

$$\int_{-\infty}^{+\infty}\psi_j\psi_x dx_i = 0 : \forall\, x_i \tag{22}$$

A much simpler approach to the harmonic oscillator problem lies in the use of ladder operator method where we make use of ladder operators i.e. the creation and annihilation operators $(\hat{a}, \hat{a}\dagger)$, to find the solution of the problem.

** Here $\hat{a}\dagger$ denotes the 'Creation' operator and $\hat{a}$ denotes the 'Annihilation' operator in Spin-1 system.

We can also the Hamiltonian in terms of the creation and annihilation operators $(\hat{a}, \hat{a}\dagger)$[7]:

$$\hat{H} = \hbar\omega(\hat{a}\hat{a}\dagger - \frac{1}{2}) = \hbar\omega(\hat{a}\hat{a}\dagger + \frac{1}{2})$$

Now the Hamiltonian for "a discrete quantum harmonic oscillator" is given by:

$$\hat{H} = \frac{(\hat{p}^d)^2}{2} + \frac{(\hat{x}^d)^2 + (\hat{y}^d)^2}{2} \quad (23)$$

where $\hat{p}^d$ is the discrete momentum operator and $\hat{x}^d$ and $\hat{y}^d$ are the discrete position operators in in x and y spatial dimension respectively. Also $\hat{p}^d$ can be expressed as:

$$\hat{p}^d = (F^d)^{-1} \cdot \hat{x}^d \cdot (F^d) \quad (24)$$

where F^d is the standard discrete Quantum Fourier Transform matrix[3]. Each element of F^d can be expressed as:

$$[F^d]_{j,k} = \frac{exp(2i\pi jk/N)}{\sqrt{N}} \quad (25)$$

Where $j, k \in [-\frac{N}{2}, ..., \frac{N}{2} - 1]$ and j= no. of rows in the matrix and k= no. of columns in the matrix.

IV. UNITARY OPERATOR TRANSFORMATIONS

For the sake of reducing mathematical complexity, let us assume $\hbar$, ω and m is unity (i.e.all are having value 1). So, we can write the Schrodinger equation as:

$$i\frac{\partial\Psi}{\partial t} = \hat{H}\Psi \quad (26)$$

which further implies:

$$\Psi(t) = \Psi(0)exp(-i\hat{H}t)$$

From the above, it is vivid that the unitary operator to be computed is $U(t) = exp(-i\hat{H}^d t)$ where $\hat{H}^d$ is the discretized Hamiltonian operator mentioned in Equation (3). So. the unitary operator is given by:

$$U(t) = exp(-it(\frac{(\hat{p}^d)^2}{2} + \frac{(\hat{x}^d)^2 + (\hat{y}^d)^2}{2})) \quad (27)$$

Or if we consider the X-dimension only, then we get the unitary operator as:

$$U_{\hat{x}}(t) = exp(\frac{-it}{2}((F^d)^{-1} \cdot \hat{(x^d)}^2 \cdot (F^d) + (\hat{x}^d)^2)) \quad (28)$$

Due to the discretization of space; the position operator $[\hat{x}^d]$, being a diagonal matrix, can be expanded by using the concept of Matrix exponential as Ref.[2]:

$$exp(-\frac{it}{2}[A]) = \mathbb{I} + \sum_{m=1}^{\infty}(-\frac{it}{2})^m \frac{[A]^m}{m!} \quad (29)$$

Here A is the corresponding Operator Matrix.

V. IMPLEMENTATION ON A SPIN-1 SYSTEM

The Hamiltonian of the full system is given by[4]:

$$\hat{H} = \hat{H}_{field} + \hat{H}_{atom} + \hat{H}_{int}$$

where $\hat{H}_{field}$ is the free Hamiltonian, $\hat{H}_{atom}$ is the atomic excitation Hamiltonian and $\hat{H}_{int}$ is the interaction Hamiltonian.

A. MODEL

We have modeled our system using Rabi Hamiltonian[5]. However, in our case we will be using somewhat modified version of Rabi Hamiltonian[6]:

$$H_s = \sum_{k=1}^{2}\omega_k a_k^\dagger a_k + \frac{\omega_0}{2}\sigma_3 + \sum_{k=1}^{2} g_k(e^{i\theta_k}a_k + e^{-i\theta_k}a_k^\dagger)\sigma_1 \quad (30)$$

Where ω_0 is the frequency of the main oscillator, ω_k is the frequency of the k-th environment oscillator; $a_k^\dagger$ and a_k are the creation and annihilation operators of the main system and the k-th environmental oscillator respectively. Whereas g_k's are the coupling constant for the interaction between the k-th environment oscillator and the main quantum oscillator. We set k=1 from now to prevent us from complicating the process.

For simplicity, we will consider the simplest case of our model and substitute k=1 in our original Hamiltonian [in Eq.(30)] to obtain the special case of our Hamiltonian which will be our working Hamiltonian from now:

$$H = \omega_1 a_1^\dagger a_1 + \frac{\omega_0}{2}\sigma_3 + g_1(e^{i\theta_1}a_1 + e^{-i\theta_1}a_1^\dagger)\sigma_1$$

For simplicity we will drop the sub-script 1 from our Hamiltonian and obtain:

$$H = \omega a^\dagger a + \frac{\omega_0}{2}\sigma_3 + g(e^{i\theta}a + e^{-i\theta}a^\dagger)\sigma_1 \quad (31)$$

B. RELEVANT TRANSFORMATION AND GENERALIZATION

Now, as our system involves Spin-1 particles, so the following commutation relations uphold:

$$[a_i, a_j^\dagger] \equiv a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij} \quad (32)$$

$$[a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0 \quad (33)$$

Here δ_{ij} is known as 'Kronecker delta'.

The operators used in the Hamiltonian can be transformed according to Holstein-Primakoff transformations[8] (i.e. it maps spin operators for a system of spin-S moments on a lattice to creation and annihilation operators) as:

$$\hat{S}_j^+ = \sqrt{(2S - \hat{n}_j)}\hat{a}_j \tag{34}$$

$$\hat{S}_j^- = \hat{a}_j^\dagger \sqrt{(2S - \hat{n}_j)} \tag{35}$$

where $\hat{a}_j^\dagger$ $(\hat{a}_j)$ is the creation (annihilation) operator at site j that satisfies the commutation relations mentioned above and $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ is the "Number Operator". Hence we can generalize the above equations as:

$$S^+ = \sqrt{(2S - a^\dagger a)}a \tag{36}$$

$$S^- = a^\dagger \sqrt{(2S - a^\dagger a)} \tag{37}$$

Where;

$$S_+ \equiv S_x + iS_y \qquad \text{and} \qquad S_- \equiv S_x - iS_y$$

Where;
S_x $(= \sigma_1)$, S_y $(= \sigma_2)$, S_z $(= \sigma_3)$ are the Pauli matrices for Spin-1 system (as mentioned in the previous section).

Now by using the above transformations; we can write our creation and annihilation operators in terms of Matrices as:

$$a^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{38}$$

Now the Hamiltonian for our coupled quantum harmonic oscillator in Eq.(31) can be decomposed as:

$$H = \omega a^\dagger a \otimes \mathbb{I} + \frac{\omega_0}{2}\mathbb{I} \otimes \sigma_3 + g(e^{i\theta}a + e^{-i\theta}a^\dagger) \otimes \sigma_1$$

Or the above equation can be written as:

$$H = \omega a^\dagger a \otimes \mathbb{I} + \frac{\omega_0}{2}\mathbb{I} \otimes S_3 + g(e^{i\theta}a + e^{-i\theta}a^\dagger) \otimes S_1 \tag{39}$$

Now, we will evaluate each term to simplify the expression of the Hamiltonian in the form of matrix. Here,

$$\omega a^\dagger a \otimes \mathbb{I} = \omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \omega a^\dagger a \otimes \mathbb{I} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \end{bmatrix} \tag{40}$$

Similarly,

$$\frac{\omega_0}{2}\mathbb{I} \otimes S_z = \frac{\omega_0}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \frac{\omega_0}{2}\mathbb{I} \otimes S_z = \begin{bmatrix} \frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\omega_0}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\omega_0}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega_0}{2} \end{bmatrix} \tag{41}$$

Finally,

$$g(e^{i\theta}b + e^{-i\theta}b^\dagger) \otimes S_x = \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & e^{i\theta} & 0 \\ e^{-i\theta} & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{g}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

Substituting the above values in Eq.(39), we get the value of H (a 9×9 matrix) as:

$$H = \begin{bmatrix} \frac{\omega_0}{2} & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega_0}{2} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & (\omega+\frac{\omega_0}{2}) & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \omega & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & (\omega-\frac{\omega_0}{2}) & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & (\omega+\frac{\omega_0}{2}) & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & (\omega-\frac{\omega_0}{2}) \end{bmatrix}$$

VI. DERIVATION OF UNITARY OPERATORS

Clearly, we know that for a system with Hamiltonian H, the unitary operator is given by:

$$U = e^{-iHt} \tag{43}$$

Where H is the Hamiltonian of the system derived in the previous section.

But to find the unitary operator compatible, we need to change the form of our Hamiltonian and write it as a sum of two matrices whose corresponding unitary operators are relatively easier to compute:

$$H = X + Y$$

Where,

$$X = \begin{bmatrix} \frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega_0}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\omega+\frac{\omega_0}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\omega-\frac{\omega_0}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\omega+\frac{\omega_0}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\omega-\frac{\omega_0}{2}) \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 & \frac{ge^{i\theta}}{\sqrt{2}} \\ 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{ge^{i\theta}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{ge^{-i\theta}}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we have,

$$U = e^{-iXt}.e^{-iYt}$$

$$\Longrightarrow U = U_x(t).U_y(t)$$

where $U_x(t) = e^{-iXt}$ and $U_y(t) = e^{-iYt}$. First we will compute $U_y(t)$, then $U_x(t)$. We can see that $U_y(t)$ can be expanded using Taylor series of expansion of the exponential function as:

$$U_y(t) = exp(-itY) = \mathbb{I} + \sum_{m=1}^{\infty}(-it)^m\frac{Y^m}{m!}$$

$$\Longrightarrow U_y(t) = \mathbb{I} + (-it)^1\frac{Y}{1!} + (-it)^2\frac{Y^2}{2!} + (-it)^3\frac{Y^3}{3!} + (-it)^4\frac{Y^4}{4!} + (-it)^5\frac{Y^5}{5!} + ...$$

Now, for simplicity, let us denote $\frac{g}{\sqrt{2}} = g^{'}$

$$\Longrightarrow U_y(t) = [1 + \frac{(-itg^{'})^2}{2!} + \frac{(-itg^{'})^4}{4!} + ...]\mathbb{I} + [\frac{(-itg^{'})}{1!} + \frac{(-itg^{'})^3}{3!} + \frac{(-itg^{'})^5}{5!} + ...]M$$

Where;

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 & e^{i\theta} & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 & 0 & 0 & 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta} & 0 & e^{-i\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\theta} & 0 & 0 & 0 & 0 \end{bmatrix}$$

(** We can observe that [Y^2, Y^4, Y^6,....] will give Identity matrices whereas [Y^1, Y^3, Y^5,...] will give the same

Qubit states	Results after $\mathrm{U}_y(t) acts$
$\|0000\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0000\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0100\rangle)$
$\|0001\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0001\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}(\|0011\rangle + 0101))$
$\|0010\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0010\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0100\rangle)$
$\|0011\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0011\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0001\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0111\rangle)$
$\|0100\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0100\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}(\|0110\rangle + \|1000\rangle) - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}(\|0000\rangle + \|0010\rangle))$
$\|0101\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0101\rangle - i\sin\frac{gt}{\sqrt{2}}e^{-i\theta}\|0111\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0001\rangle)$
$\|0110\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0110\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0100\rangle)$
$\|0111\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|0111\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}(\|0011\rangle + 0101))$
$\|1000\rangle$	$(\cos\frac{gt}{\sqrt{2}}\|1000\rangle - i\sin\frac{gt}{\sqrt{2}}e^{i\theta}\|0100\rangle)$

TABLE I. Operator $U_y(t)$ acting on Qubit States.

matrix which is given above as M. So we differentiate them in two groups.)

$$\implies U_y(t) = \cos g^{'}t\mathbb{I} - iM\sin g^{'}t$$

$$\implies U_y(t) = \cos\frac{gt}{\sqrt{2}}\mathbb{I} - iM\sin\frac{gt}{\sqrt{2}} \tag{44}$$

Now for Spin-1 particles, we need to use a 4-qubit system but for implementing a 4-qubit system we must require a 16×16 matrix because any matrix of order $N \times N$ must satisfy the condition $N = 2^n$ (where n= number of qubits). But we can express the above equation in form of a 16×16 matrix (which we have shown in the next sub-section), instead of a 9×9 matrix, by adding 1 diagonally seven times and placing 0 in other positions. In our situation we need only nine of the sixteen 4-qubit states (mentioned in Table (I)) because for the other seven states we will get the same Unitary matrix as result (i.e. without any change).

In the above segment, we computed the $U_y(t)$ operator. In order to compute $U_x(t)$ which is equal to e^{-iXt}, we first expand the expression using the Taylor expansion of the exponential function just like we did in earlier case as:

$$U_x(t) = exp(-itX) = \mathbb{I} + \sum_{m=1}^{\infty}(-it)^m\frac{X^m}{m!}$$

$$\implies U_x(t) = \mathbb{I} + (-it)^1\frac{X}{1!} + (-it)^2\frac{X^2}{2!} + (-it)^3\frac{X^3}{3!} + (-it)^4\frac{X^4}{4!} + (-it)^5\frac{X^5}{5!} + ...$$

Therefore by using the above equation, we can express $U_x(t)$ in terms of e as:

$\mathrm{U}_{\hat{x}}(t)[1,1]$	exp(-($\frac{\omega_0}{2}$)it)
$\mathrm{U}_{\hat{x}}(t)[2,2]$	exp(-(0)it)
$\mathrm{U}_{\hat{x}}(t)[3,3]$	exp(($\frac{\omega_0}{2}$)it)
$\mathrm{U}_{\hat{x}}(t)[4,4]$	exp(-($\omega + \frac{\omega_0}{2}$)it)
$\mathrm{U}_{\hat{x}}(t)[5,5]$	exp(-(ω)it)
$\mathrm{U}_{\hat{x}}(t)[6,6]$	exp(-($\omega - \frac{\omega_0}{2}$)it)
$\mathrm{U}_{\hat{x}}(t)[7,7]$	exp(-($\omega + \frac{\omega_0}{2}$)it)
$\mathrm{U}_{\hat{x}}(t)[8,8]$	exp(-(ω)it)
$\mathrm{U}_{\hat{x}}(t)[9,9]$	exp(-($\omega - \frac{\omega_0}{2}$)it)

In this case also we will consider a 16×16 matrix (in place of a 9×9 matrix) because of same reason mentioned before and also we will construct the matrix in the same pattern as mentioned in case of $U_y(t)$ operator. It is easy to observe as X is a diagonal matrix, each diagonal element of $U_x(t)$ makes an exact Taylor expansion of the exponential function

(**The 16×16 matrix for both $U_y(t)$ and $U_x(t)$ operators are mentioned in the next sub-section.)

Again, we operate this operator on different 4-qubits states (in our situation we need only nine of the sixteen 4-qubit states because for the other seven states we will get the same Unitary matrix as result.) and then study the results for the same given in Table(IV):

Qubit states	Results after $\mathrm{U}_x(t) acts$
$\|0000\rangle$	$\mathrm{e}^{(-\frac{\omega_0}{2})it}\|0000\rangle$
$\|0001\rangle$	$\mathrm{e}^{(0)it}\|0000\rangle$
$\|0010\rangle$	$\mathrm{e}^{(\frac{\omega_0}{2})it}\|0000\rangle$
$\|0011\rangle$	$\mathrm{e}^{-(\omega+\frac{\omega_0}{2})it}\|0011\rangle$
$\|0100\rangle$	$\mathrm{e}^{(-\omega)it}\|0100\rangle$
$\|0101\rangle$	$\mathrm{e}^{-(\omega-\frac{\omega_0}{2})it}\|0101\rangle$
$\|0110\rangle$	$\mathrm{e}^{-(\omega+\frac{\omega_0}{2})it}\|0110\rangle$
$\|0111\rangle$	$\mathrm{e}^{(-\omega)it}\|0111\rangle$
$\|1000\rangle$	$\mathrm{e}^{-(\omega-\frac{\omega_0}{2})it}\|1000\rangle$

TABLE II. Operator $U_x(t)$ acting on Qubit States.

From the above table we can see the effect of $U_x(t)$ operator acting on the different 4-qubit states.

Now, we know how to implement both the parts of our Unitary operator and the complete unitary matrix (16×16) can be implemented by by operating both the operations in series. In this way we can easily calculate our Unitary operators for Spin-1 system and also verify the effectiveness of Pauli Matrices equivalent to Spin-1 system/ particles.

UNITARY OPERATOR MATRIX REPRESENTATIONS

The 16×16 Matrix representation of the Unitary operators $U_y(t)$ and $U_x(t)$ are:

$$U_y(t) = \begin{bmatrix}
A & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & B & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & A & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C & 0 & C & 0 & A & 0 & B & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & 0 & A & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C & 0 & C & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Where,

A $= \cos(\frac{gt}{\sqrt{2}})$; B $= -i\sin(\frac{gt}{\sqrt{2}})e^{i\theta}$ and C $= -i\sin(\frac{gt}{\sqrt{2}})e^{-i\theta}$.

$$U_x(t) = \begin{bmatrix}
S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Where,

P $= e^{-(\omega+\frac{\omega_0}{2})it}$; Q $= e^{(-\omega)it}$; R $= e^{-(\omega-\frac{\omega_0}{2})it}$; S $= e^{(-\frac{\omega_0}{2})it}$ and $\frac{1}{s} = \frac{1}{e^{(-\frac{\omega_0}{2})it}} = e^{(\frac{\omega_0}{2})it}$

DATA AVAILABILITY

Further information regarding process of implementing the Unitary Operators on higher spin particles like Spin-$\frac{3}{2}$ particles, Spin-2 particles etc. can be made available upon reasonable number of requests.

VII. RESULTS

In first part of our paper, we extend the idea of Pauli Matrices from Spin-$\frac{1}{2}$ particles to Spin-1 particles and see the implementation of the equivalent matrices. We can also see that equivalent Pauli Matrices can also be found for higher spin particles like Spin-$\frac{3}{2}$ particles, Spin-2 particles etc.

In the final part, we introduce a coupled quantum harmonic oscillator to a Spin-1 system (Z-Bosonic/ W-Bosonic system etc.) and try to implement its unitary operator to the system using our previous section's knowledge.

CONCLUSION

Now, we understand that we can further implement the idea of Pauli equivalent matrices on higher spin particles and derive the unitary operators of the Quantum Harmonic Oscillators using those informations in a much simpler yet effective manner. We conclude with one last remark that there are various processes for finding the Pauli equivalent matrices for higher spin particles but we have used a much simpler yet effective process to find the values and implement it further onto a Quantum Harmonic Oscillator for finding its Unitary Operators.

Also it will be better to notice that I didn't disentangle the qubit states in the results which I got after $U_y(t)$ acts on the qubits because it would be necessary in case of Quantum simulation of a circuit which we are not involving and we are keeping all the results in the entangled state.

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