Oscilaciones acopladas

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A pequeñas amplitudes

$$L = \frac{1}{2}\dot{\vec{q}}^T \hat{M}\dot{\vec{q}} - \frac{1}{2}\vec{q}^T \hat{K}\vec{q} \qquad \text{con } \dot{\vec{q}} = \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

$$\vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

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$$\left.\begin{array}{l} \hat{M}^T = \hat{M} \\ \hat{K}^T = \hat{K} \end{array}\right\} \text{Matrices simétricas}$$

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$$L = \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j=1}^{n} K_{ij} q_i q_j$$

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Ecuación de movimiento para q_l

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \frac{\partial L}{\partial q_l} = 0$$

luego $\displaystyle \frac{\partial L}{\partial \dot{q}_l} = \frac{1}{2}\sum_{i,j} M_{ij}\frac{\partial}{\partial \dot{q}_l}(\dot{q}_i\dot{q}_j)$; Obs. $\displaystyle \frac{\partial \dot{q}_i}{\partial \dot{q}_l} = \delta_{il}$, etc.

$$= \frac{1}{2}\sum_{i,j} M_{ij}\left[\delta_{il}\dot{q}_j + \delta_{jl}\dot{q}_i\right]$$

$$\therefore \frac{\partial L}{\partial \dot{q}_{\ell}} = \frac{1}{2}\sum_{i,j} M_{ij}\,\delta_{i\ell}\,\dot{q}_j + \frac{1}{2}\sum_{i,j} M_{ij}\,\delta_{j\ell}\,\dot{q}_i$$

$$= \frac{1}{2}\sum_{j} M_{\ell j}\,\dot{q}_j + \frac{1}{2}\sum_{i} M_{i\ell}\,\dot{q}_i$$

$$= \frac{1}{2}\sum_{i} M_{\ell i}\,\dot{q}_i + \frac{1}{2}\sum_{i} M_{i\ell}\,\dot{q}_i$$

$$= \frac{1}{2}\sum_{i} \underbrace{(M_{\ell i} + M_{i\ell})}_{2M_{\ell i}}\,\dot{q}_i$$

$$\frac{\partial L}{\partial \dot{q}_{\ell}} = \sum_{i=1}^{n} M_{\ell i}\,\dot{q}_i \Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{\ell}}\right) = \sum_{i=1}^{n} M_{\ell i}\,\ddot{q}_i$$

análogamente

$$\frac{\partial L}{\partial q_{\ell}} = -\sum_{i} K_{\ell i}\, q_i$$

La ecuación de movimiento es entonces

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{\ell}}\right) - \frac{\partial L}{\partial q_{\ell}} = 0 \Longrightarrow \sum_{i=1}^{m} M_{\ell i}\,\ddot{q}_i + \sum_{i=1}^{n} K_{\ell i}\, q_i = 0$$

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$$\sum_{i=1}^{n} \left[M_{ei}\ddot{q}_i - K_{ei} q_i \right] = 0$$

Bajo condición de pequeña oscilación (M.A.S.)

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$$\ddot{q}_i = -\omega^2 q_i \qquad (\omega^2 = \text{incógnita})$$

$$\therefore \quad \sum_{i=1}^{n} \left[-\omega^2 M_{ei} + K_{ei} \right] q_i = 0$$

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Matricialmente se reescribe como:

$$\left[-\omega^2 \hat{M} + \hat{K} \right] \vec{q} = 0$$

$$\Downarrow$$

$$\left[\begin{pmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{pmatrix} - \omega^2 \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} \right] \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = 0$$

La solución no trivial se determina a partir de evaluar $|\hat{K} - \omega^2 \hat{M}| = 0$ lo que permite hallar las frecuencias de los modos normales.