Laplace's equation in polar coordinates

November 1, 2008

Laplace's equation in (x, y) coordinates

$$\nabla^2 f = 0$$

i.e.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

In polar coordinates...

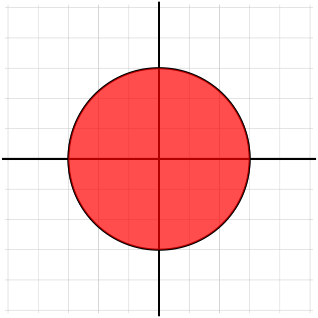
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2}$$

i.e. $\nabla^2 f = 0$ becomes

$$r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0$$

Why do it?

We use particular co-ordinate systems to solve problems with particular geometries.



Consider a circular plate with some temperature distribution on its boundary. The correct co-ordinates to use are polar co-ordinates.

Two new ideas

Check that the solution doesn't become infinite at some point in space. If it does get rid of the solution.

Check that the solution at $\theta = 2\pi$ is the same as the solution at $\theta = 0$. If the solution doesn't satisfy this condition get rid of it immediately.

Separation of variables

We look for solutions of the form

$$f(r,\theta) = R(r)T(\theta)$$

Substituting into

$$r^2\frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = 0$$

gives

$$r^2 R^{''}(r)T(\theta) + rR^{'}(r)T(\theta) + R(r)T^{''}(\theta) = 0$$

Separation of variables

Rearranging

$$r^2 R''(r)T(\theta) + rR'(r)T(\theta) + R(r)T''(\theta) = 0$$

gives

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)}$$

Each side is a function of a different variable, so both must be equal to a constant:

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = \text{const.}$$

Case 1: positive constant

We call the constant λ^2 and assume $\lambda \neq 0$

$$r^2 R''(r) + rR'(r) = \lambda^2 R(r)$$

i.e.

$$r^2 R''(r) + rR'(r) - \lambda^2 R(r) = 0$$

This is an ODE with coefficients which are powers matching the derivative. So we look for solutions of the form r^m.

Case 1: positive constant

Substituting $R(r) = r^m$ into

$$r^2 R''(r) + rR'(r) - \lambda^2 R(r) = 0$$

gives

$$m(m-1)r^m + mr^m - \lambda^2 r^m = 0 \quad \text{i.e.} \quad m(m-1) + m - \lambda^2 = 0$$

i.e.

$$m^2 - \lambda^2 = 0 \quad \text{i.e.} \quad m = \pm\lambda$$

So the general solution is

$$R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

Using some physical knowledge

Consider the general solution

$$R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

What happens to $r^{-\lambda}$ at $r = 0$? To avoid solutions which become infinite we discard the term $c_2 r^{-\lambda}$, leaving

$$R(r) = c_1 r^{\lambda}$$

Case 1: positive constant

The θ equation is:

$$T''(\theta) = -\lambda^2 T(\theta) \qquad \text{i.e.} \qquad T''(\theta) + \lambda^2 T(\theta) = 0$$

We know that this is solved by linear combinations of $\sin(\lambda\theta)$ and $\cos(\lambda\theta)$. Again we need some physical knowledge. θ is an angle, so the value of $T(\theta)$ must be the same at 0 and at 2π, i.e.

$$\cos(\lambda 0) = \cos(2\pi\lambda) \qquad \text{and} \qquad \sin(\lambda 0) = \sin(2\pi\lambda)$$

i.e.

$$1 = \cos(2\pi\lambda) \qquad \text{and} \qquad 0 = \sin(2\pi\lambda)$$

The first is only satisfied if λ is an integer n. In this case the second equation is also satisfied. I.e.

$$\lambda = n$$

Case 1: positive constant

So finally in the case of a positive constant we get the solution

$$f(r,\theta) = c_1 r^n (d_1 \cos\, n\theta + d_2 \sin\, n\theta)$$

Renaming some constants gives

$$f(r,\theta) = r^n (A \cos\, n\theta + B \sin\, n\theta)$$

Case 2: negative constant

We call the constant $-\mu^2$ and assume $\mu \neq 0$. Looking first at the θ equation:

$$T^{''}(\theta) = \mu^2\theta \qquad \text{i.e.} \qquad T^{''}(\theta) - \mu^2\theta = 0$$

The general solution of this equation consists of linear combinations of $e^{\mu\theta}$ and $e^{-\mu\theta}$. As before, since θ is an angle we require $T(0) = T(2\pi)$, i.e.

$$e^{\mu 0} = 1 = e^{2\pi\mu} \qquad \text{and} \qquad e^{-\mu 0} = 1 = e^{-2\pi\mu}$$

This only has solution $\mu = 0$ whereas we have assumed that $\mu \neq 0$. **So there are no physically meaningful solutions with a negative constant.**

Case 3: zero constant

The θ equation becomes

$$T^{''}(\theta) = 0$$

i.e.

$$T(\theta) = c_3\theta + c_4$$

Insisting that $T(2\pi) = T(0)$ gives

$$c_3 2\pi + c_4 = c_3 0 + c_4$$

i.e.

$$c_3 = 0$$

Case 3: zero constant

The r equation becomes

$$r^2 R''(r) + rR'(r) = 0$$

We try r^m giving

$$m(m-1) + m = 0 \quad \text{i.e.} \quad m^2 = 0 \quad \text{i.e.} \quad m = 0$$

This is a repeated root, so the two solutions are

$$r^0 \quad \text{and} \quad \ln\, r$$

giving

$$R(r) = d_3 r^0 + d_4 \ln\, r$$

Case 3: zero constant

What happens to $\ln r$ as $r \to 0$? As before we get rid of unphysical behaviour and get

$$R(r) = d_3 r^0 = d_3$$

So in the case of a zero constant, both $R(r)$ and $T(\theta)$ are just constants so we can write

$$R(r)T(\theta) = C$$

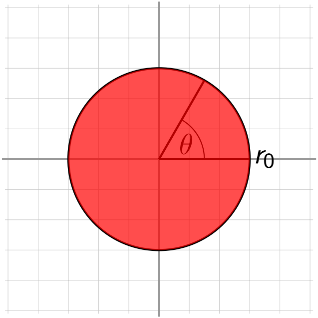
The "general solution"

We thus get the general solution:

$$f(r, \theta) = C + \sum_n r^n (A_n \cos\, n\theta + B_n \sin\, n\theta)$$

A circular plate with radius r_0

Consider a circular plate with radius r_0 and some temperature $T = g(\theta)$ along the boundary (i.e. at $r = r_0$).



outer perimeter:

$$f(r_0, \theta) = g(\theta)$$

Substituting in the boundary conditions

$$f(r,\theta) = C + \sum_n r^n (A_n \cos\ n\theta + B_n \sin\ n\theta)$$

Substituting in the boundary condition:

$$g(\theta) = f(r_0,\theta) = C + \sum_n {r_0}^n (A_n \cos\ n\theta + B_n \sin\ n\theta)$$

Remember that the Fourier series for $g(\theta)$ is

$$g(\theta) = \frac{1}{2}a_0 + \sum_n (a_n \cos\ n\theta + b_n \sin\ n\theta)$$

Substituting in the boundary conditions

Comparing

$$g(\theta) = f(r_0, \theta) = C + \sum_n {r_0}^n (A_n \cos\, n\theta + B_n \sin\, n\theta)$$

and

$$g(\theta) = \frac{1}{2}a_0 + \sum_n (a_n \cos\, n\theta + b_n \sin\, n\theta)$$

gives

$$C = \frac{1}{2}a_0, \quad A_n = \frac{a_n}{r_0^n}, \quad B_n = \frac{b_n}{r_0^n}$$

where a_n and b_n are the Fourier coefficients for $g(\theta)$.

Substituting in the boundary conditions

So the full solution is:

$$f(r,\theta) = \frac{1}{2}a_0 + \sum_n \left(\frac{r}{r_0}\right)^n (a_n \cos\, n\theta + b_n \sin\, n\theta)$$

where a_n and b_n are the Fourier coefficients for $g(\theta)$.

Example 1

The temperature varies periodically around the perimeter

$$g(\theta) = 1 + \sin\ \theta$$

Here the Fourier series is in the function itself, so

$$a_0 = 2, \quad a_n = 0\ (n \geq 1), \quad b_1 = 1, \quad b_n = 0\ (n \geq 2)$$

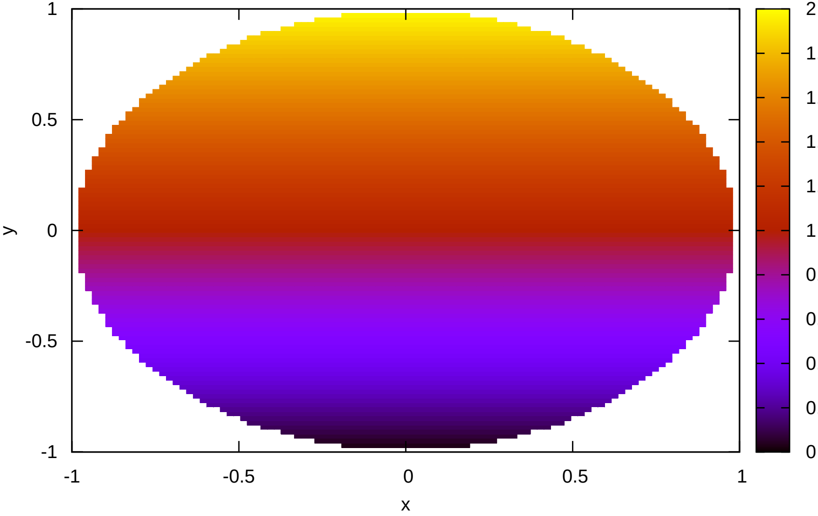
The general solution

$$f(r, \theta) = \frac{1}{2} a_0 + \sum_n \left(\frac{r}{r_0} \right)^n (a_n \cos\ n\theta + b_n \sin\ n\theta)$$

reduces to:

$$f(r, \theta) = 1 + \left(\frac{r}{r_0} \right) \sin\ \theta$$

What the solution looks like



Example 2

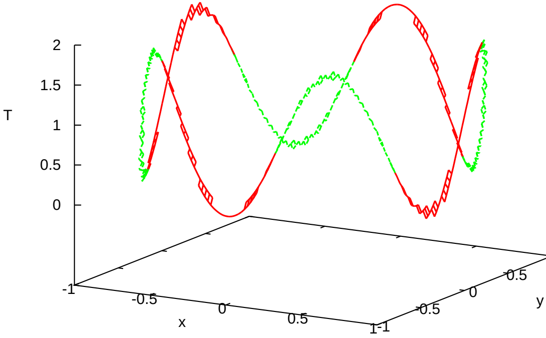
The temperature varies periodically around the perimeter, oscillating faster

$$g(\theta) = 1 + \sin 5\theta$$

Here the Fourier series is in the function itself, so

$$a_0 = 2, \quad a_n = 0 \ (n \geq 1)$$

$$b_5 = 1, \quad b_n = 0 \ (n \neq 5)$$



Example 2

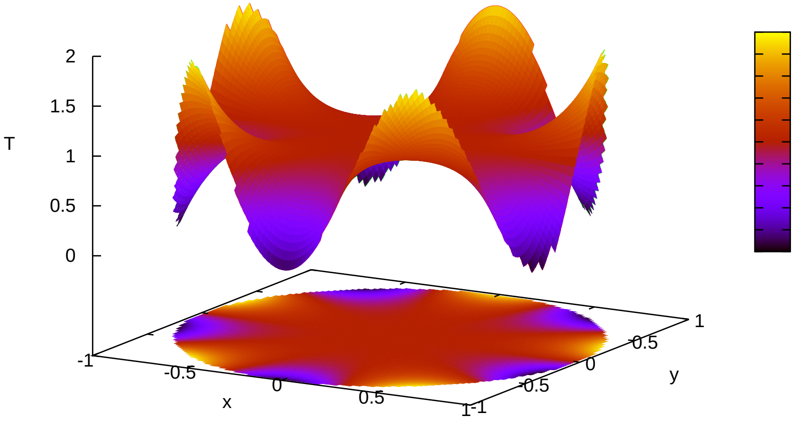
The general solution

$$f(r,\theta) = \frac{1}{2}a_0 + \sum_n \left(\frac{r}{r_0}\right)^n (a_n \cos\, n\theta + b_n \sin\, n\theta)$$

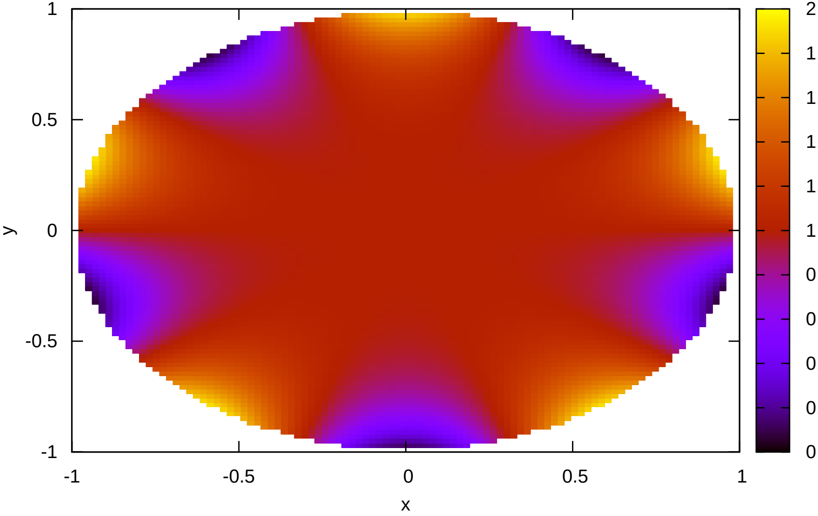
reduces to:

$$f(r,\theta) = 1 + \left(\frac{r}{r_0}\right)^5 \sin\, 5\theta$$

What the solution looks like



What the solution looks like

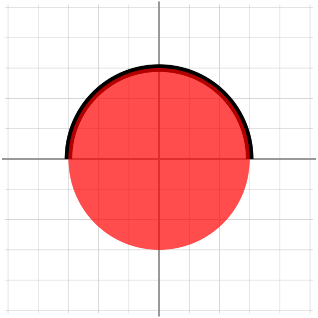


Example 3

Consider

$$g(\theta) = \begin{cases} 0 & -\pi \leq \theta < 0 \\ 1 & 0 \leq \theta < \pi \end{cases}$$

So that the top half of the plate is held at a temperature of 1 while the bottom half is held at a temperature of 0



Example 1

Calculating the Fourier series of

$$g(\theta) = \begin{cases} 0 & -\pi \leq \theta < 0 \\ 1 & 0 \leq \theta < \pi \end{cases}$$

gives

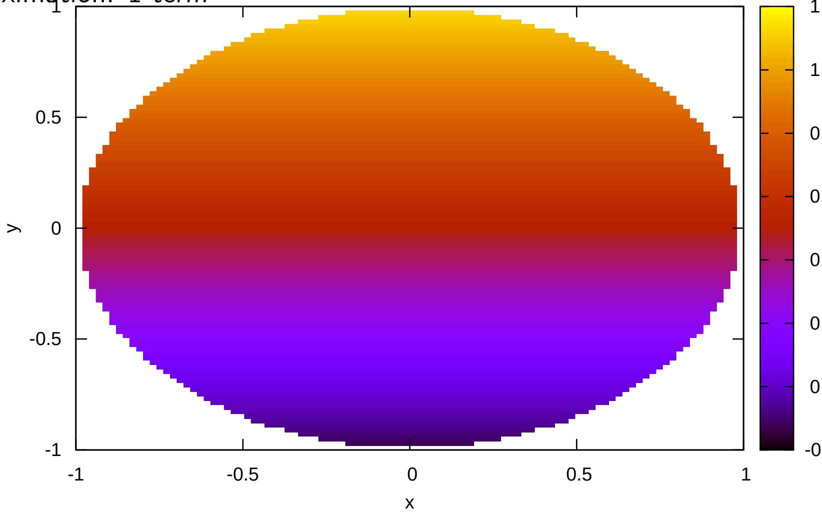
$$a_0 = 1, \quad a_n = 0 \text{ (for } n \geq 2), \quad b_n = \begin{cases} 0 & n \text{ even} \\ (2/n\pi) & n \text{ odd} \end{cases}$$

So the full solution is

$$f(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \left(\frac{r}{r_0} \right)^n \frac{\sin\, n\theta}{n}$$

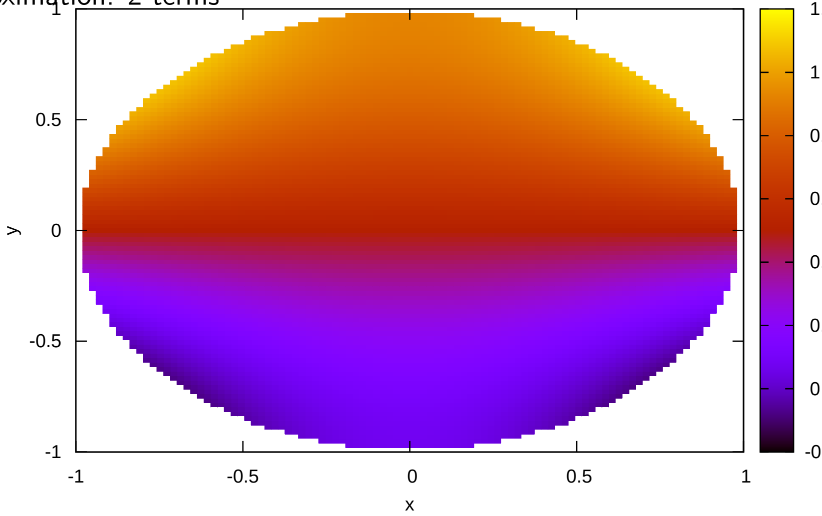
What the solution looks like

Approximation: 1 term



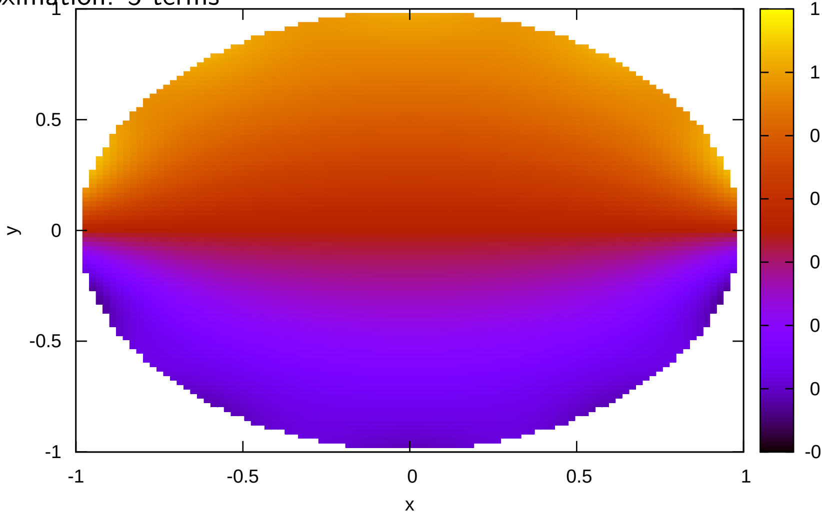
What the solution looks like

Approximation: 2 terms



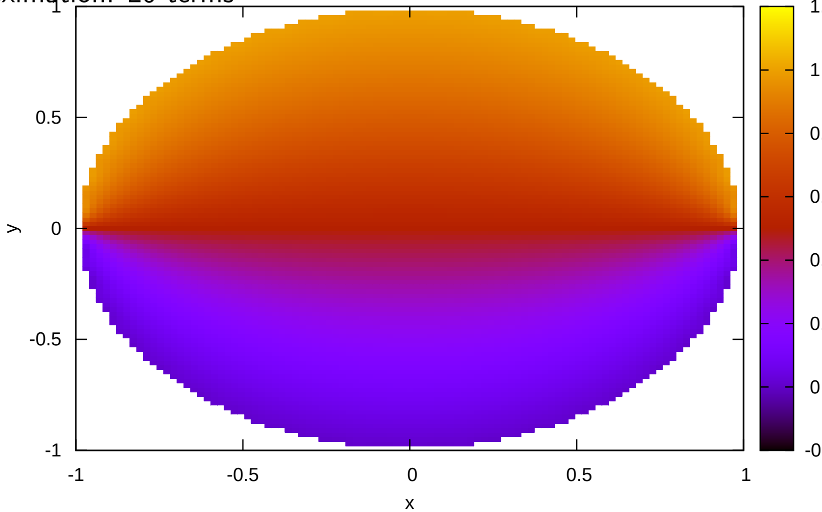
What the solution looks like

Approximation: 5 terms



What the solution looks like

Approximation: 20 terms



What the solution looks like

Approximation: 60 terms