Special Theory of Relativity

2

CHAPTER

It was found that there was no displacement of the interference fringes, so that the result of the experiment was negative and would, therefore, show that there is still a difficulty in the theory itself. . . .

Albert Michelson, *Light Waves and Their Uses*, 1907

One of the great theories of physics appeared early in the twentieth century when Albert Einstein presented his special theory of relativity in 1905. We learned in introductory physics that Newton's laws of motion must be measured relative to some reference frame. A reference frame is called an **inertial frame** if Newton's laws are valid in that frame. If a body subject to no net external force moves in a straight line with constant velocity, then the coordinate system attached to that body defines an inertial frame. If Newton's laws are valid in one reference frame, then they are also valid in a reference frame moving at a uniform velocity relative to the first system. This is known as the **Newtonian principle of relativity** or **Galilean invariance**.

Inertial frame

Galilean invariance

Newton showed that it was not possible to determine absolute motion in space by any experiment, so he decided to use relative motion. In addition, the Newtonian concepts of time and space are completely separable. Consider two inertial reference frames, K and K′, that move along their x and x' axes, respectively, with uniform relative velocity $\vec{v}$ as shown in Figure 2.1. We show system K′ moving to the right with velocity $\vec{v}$ with respect to system K, which is fixed or

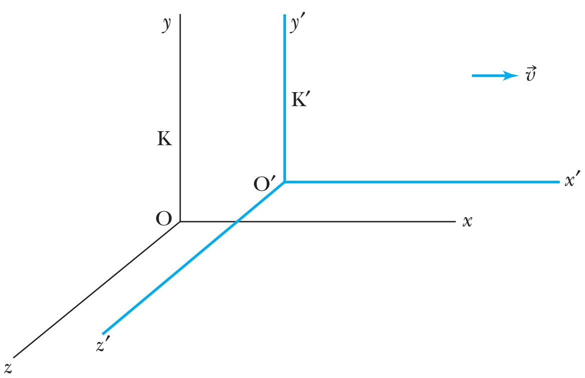


Figure 2.1 Two inertial systems are moving with relative speed v along their x axes. We show the system K at rest and the system K′ moving with speed v relative to the system K.

stationary somewhere. One result of the relativity theory is that there are no fixed, absolute frames of reference. We use the term *fixed* to refer to a system that is fixed on a particular object, such as a planet, star, or spaceship that itself is moving in space. The transformation of the coordinates of a point in one system to the other system is given by

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned} \tag{2.1}$$

Similarly, the inverse transformation is given by

$$\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \end{aligned} \tag{2.2}$$

Galilean transformation

where we have set $t = t'$ because Newton considered time to be absolute. Equations (2.1) and (2.2) are known as the **Galilean transformation**. Newton's laws of motion are invariant under a Galilean transformation; that is, they have the same form in both systems K and K′.

In the late nineteenth century Albert Einstein was concerned that although Newton's laws of motion had the same form under a Galilean transformation, Maxwell's equations did not. Einstein believed so strongly in Maxwell's equations that he showed there was a significant problem in our understanding of the Newtonian principle of relativity. In 1905 he published ideas that rocked the very foundations of physics and science. He proposed that space and time are not separate and that Newton's laws are only an approximation. This special theory of relativity and its ramifications are the subject of this chapter. We begin by presenting the experimental situation historically—showing why a problem existed and what was done to try to rectify the situation. Then we discuss Einstein's two postulates on which the special theory is based. The interrelation of space and time is discussed, and several amazing and remarkable predictions based on the new theory are shown.

As the concepts of relativity became used more often in everyday research and development, it became essential to understand the transformation of momentum, force, and energy. Here we study relativistic dynamics and the relationship between mass and energy, which leads to one of the most famous equations in physics and a new conservation law of mass-energy. Finally, we return to electromagnetism to investigate the effects of relativity. We learn that Maxwell's equations don't require change, and electric and magnetic effects are relative, depending on the observer. We leave until Chapter 15 our discussion of Einstein's general theory of relativity.

2.1 The Apparent Need for Ether

Thomas Young, an English physicist and physician, performed his famous experiments on the interference of light in 1802. A decade later, the French physicist and engineer Augustin Fresnel published his calculations showing the detailed understanding of interference, diffraction, and polarization. Because all known waves (other than light) require a medium in which to propagate (water waves have water, sound waves have, for example, air, and so on), it was naturally assumed that light also required a medium, even though light was apparently able to travel in vacuum through outer space. This medium was called the *luminiferous ether* or just **ether** for short, and it must have some amazing properties. The ether had to have such a low density that planets could pass through it, seemingly for eternity, with no apparent loss of orbit position. Its elasticity must be strong enough to pass waves of incredibly high speeds!

The concept of ether

The electromagnetic theory of light (1860s) of the Scottish mathematical physicist James Clerk Maxwell shows that the speed of light in different media depends only on the electric and magnetic properties of matter. In vacuum, the speed of light is given by $v = c = 1/\sqrt{\mu_0\epsilon_0}$, where μ_0 and ϵ_0 are the permeability and permittivity of free space, respectively. The properties of the ether, as proposed by Maxwell in 1873, must be consistent with electromagnetic theory, and the feeling was that to be able to discern the ether's various properties required only a sensitive enough experiment. The concept of ether was well accepted by 1880.

When Maxwell presented his electromagnetic theory, scientists were so confident in the laws of classical physics that they immediately pursued the aspects of Maxwell's theory that were in contradiction with those laws. As it turned out, this investigation led to a new, deeper understanding of nature. Maxwell's equations predict the velocity of light in a vacuum to be c. If we have a flashbulb go off in the moving system K′, an observer in system K′ measures the speed of the light pulse to be c. However, if we make use of Equation (2.1) to find the relation between speeds, we find the speed measured in system K to be $c + v$, where v is the relative speed of the two systems. However, Maxwell's equations don't differentiate between these two systems. Physicists of the late nineteenth century proposed that there must be one preferred inertial reference frame in which the ether was stationary and that in this system the speed of light was c. In the other systems, the speed of light would indeed be affected by the relative speed of the reference system. Because the speed of light was known to be so enormous, 3×10^8 m/s, no experiment had as yet been able to discern an effect due to the relative speed v. The ether frame would in fact be an absolute standard, from which other measurements could be made. Scientists set out to find the effects of the ether.

AIP/Emilio Segrè Visual Archives.

Albert A. Michelson (1852–1931) shown at his desk at the University of Chicago in 1927. He was born in Prussia but came to the United States when he was two years old. He was educated at the U.S. Naval Academy and later returned on the faculty. Michelson had appointments at several American universities including the Case School of Applied Science, Cleveland, in 1883; Clark University, Worcester, Massachusetts, in 1890; and the University of Chicago in 1892 until his retirement in 1929. During World War I he returned to the U.S. Navy, where he developed a rangefinder for ships. He spent his retirement years in Pasadena, California, where he continued to measure the speed of light at Mount Wilson.

2.2 The Michelson-Morley Experiment

The Earth orbits around the sun at a high orbital speed, about $10^{-4}c$, so an obvious experiment is to try to find the effects of the Earth's motion through the ether. Even though we don't know how fast the sun might be moving through the ether, the Earth's orbital *velocity* changes significantly throughout the year because of its change in direction, even if its orbital *speed* is nearly constant.

Albert Michelson (1852–1931) performed perhaps the most significant American physics experiment of the 1800s. Michelson, who was the first U.S. citizen to receive the Nobel Prize in Physics (1907), was an ingenious scientist who built an extremely precise device called an *interferometer*, which measures the phase difference between two light waves. Michelson used his interferometer to detect the difference in the speed of light passing through the ether in different directions. The basic technique is shown in Figure 2.2. Initially, it is assumed that one of the interferometer arms (AC) is parallel to the motion of the Earth through the ether. Light leaves the source S and passes through the glass plate at A. Because the back of A is partially silvered, part of the light is reflected,

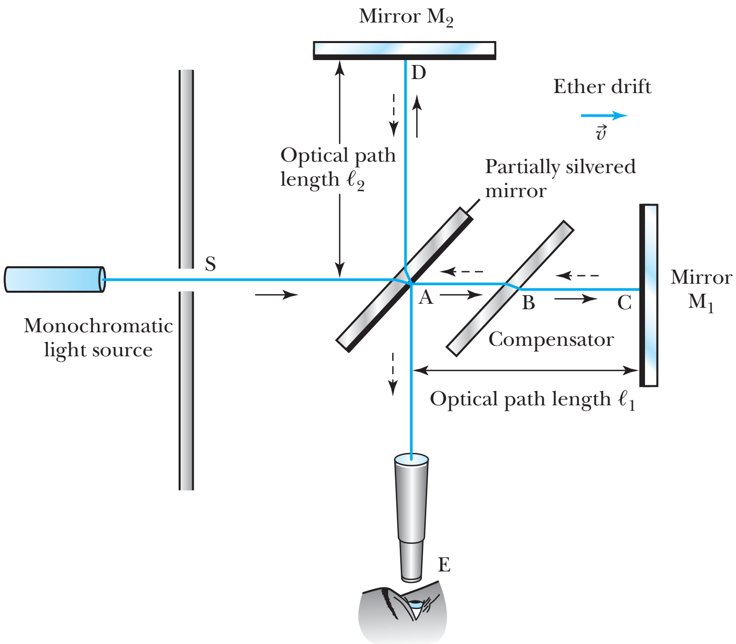
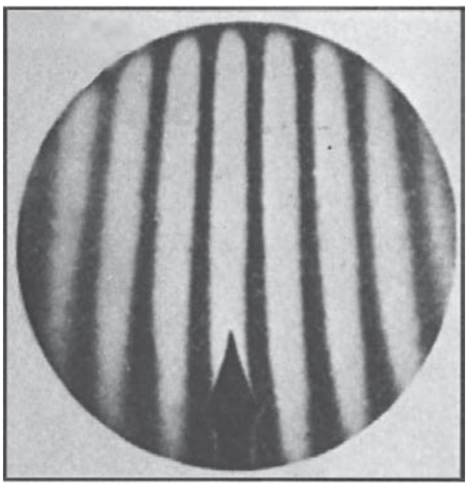


Figure 2.2 A schematic diagram of Michelson's interferometer experiment. Light of a single wavelength is partially reflected and partially transmitted by the glass at A. The light is subsequently reflected by mirrors at C and D, and, after reflection or transmission again at A, enters the telescope at E. Interference fringes are visible to the observer at E.

eventually going to the mirror at D, and part of the light travels through A on to the mirror at C. The light is reflected at the mirrors C and D and comes back to the partially silvered mirror A, where part of the light from each path passes on to the telescope and eye at E. The compensator is added at B to make sure both light paths pass through equal thicknesses of glass. Interference fringes can be found by using a bright light source such as sodium, with the light filtered to make it monochromatic, and the apparatus is adjusted for maximum intensity of the light at E. We will show that the fringe pattern should shift if the apparatus is rotated through 90° such that arm AD becomes parallel to the motion of the Earth through the ether and arm AC is perpendicular to the motion.

We let the optical path lengths of AC and AD be denoted by ℓ_1 and ℓ_2, respectively. The observed interference pattern consists of alternating bright and dark bands, corresponding to constructive and destructive interference, respectively (Figure 2.3). For constructive interference, the difference between the two



From L. S. Swenson, Jr., *Invention and Discovery* 43 (Fall 1987).

Figure 2.3 Interference fringes as they would appear in the eyepiece of the Michelson-Morley experiment.

path lengths (to and from the mirrors) is given by some number of wavelengths, $2(\ell_1 - \ell_2) = n\lambda$, where λ is the wavelength of the light and n is an integer.

The expected shift in the interference pattern can be calculated by determining the time difference between the two paths. When the light travels from A to C, the velocity of light according to the Galilean transformation is $c + v$, because the ether carries the light along with it. On the return journey from C to A the velocity is $c - v$, because the light travels opposite to the path of the ether. The total time for the round-trip journey to mirror M_1 is t_1:

$$t_1 = \frac{\ell_1}{c + v} + \frac{\ell_1}{c - v} = \frac{2c\ell_1}{c^2 - v^2} = \frac{2\ell_1}{c}\left(\frac{1}{1 - v^2/c^2}\right)$$

Now imagine what happens to the light that is reflected from mirror M_2. If the light is pointed directly at point D, the ether will carry the light with it, and the light misses the mirror, much as the wind can affect the flight of an arrow. If a swimmer (who can swim with speed v_2 in still water) wants to swim across a swiftly moving river (speed v_1), the swimmer must start heading upriver, so that when the current carries her downstream, she will move directly across the river. Careful reasoning shows that the swimmer's velocity is $\sqrt{v_2^2 - v_1^2}$ throughout her journey (Problem 4). Thus the time t_2 for the light to pass to mirror M_2 at D and back is

$$t_2 = \frac{2\ell_2}{\sqrt{c^2 - v^2}} = \frac{2\ell_2}{c}\frac{1}{\sqrt{1 - v^2/c^2}}$$

The time difference between the two journeys Δt is

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{\ell_2}{\sqrt{1 - v^2/c^2}} - \frac{\ell_1}{1 - v^2/c^2}\right) \tag{2.3}$$

We now rotate the apparatus by 90° so that the ether passes along the length ℓ_2 toward the mirror M_2. We denote the new quantities by primes and carry out an analysis similar to that just done. The time difference $\Delta t'$ is now

$$\Delta t' = t_2' - t_1' = \frac{2}{c}\left(\frac{\ell_2}{1 - v^2/c^2} - \frac{\ell_1}{\sqrt{1 - v^2/c^2}}\right) \tag{2.4}$$

Michelson looked for a shift in the interference pattern when his apparatus was rotated by 90°. The time difference is

$$\Delta t' - \Delta t = \frac{2}{c}\left(\frac{\ell_1 + \ell_2}{1 - v^2/c^2} - \frac{\ell_1 + \ell_2}{\sqrt{1 - v^2/c^2}}\right)$$

Because we know $c \gg v$, we can use the binomial expansion* to expand the terms involving v^2/c^2, keeping only the lowest terms.

$$\begin{aligned}\Delta t' - \Delta t &= \frac{2}{c}(\ell_1 + \ell_2)\left[\left(1 + \frac{v^2}{c^2} + \cdots\right) - \left(1 + \frac{v^2}{2c^2} + \cdots\right)\right] \\ &\approx \frac{v^2(\ell_1 + \ell_2)}{c^3} \end{aligned} \tag{2.5}$$

Michelson left his position at the U.S. Naval Academy in 1880 and took his interferometer to Europe for postgraduate studies with some of Europe's best physi-

*See Appendix 3 for the binomial expansion.

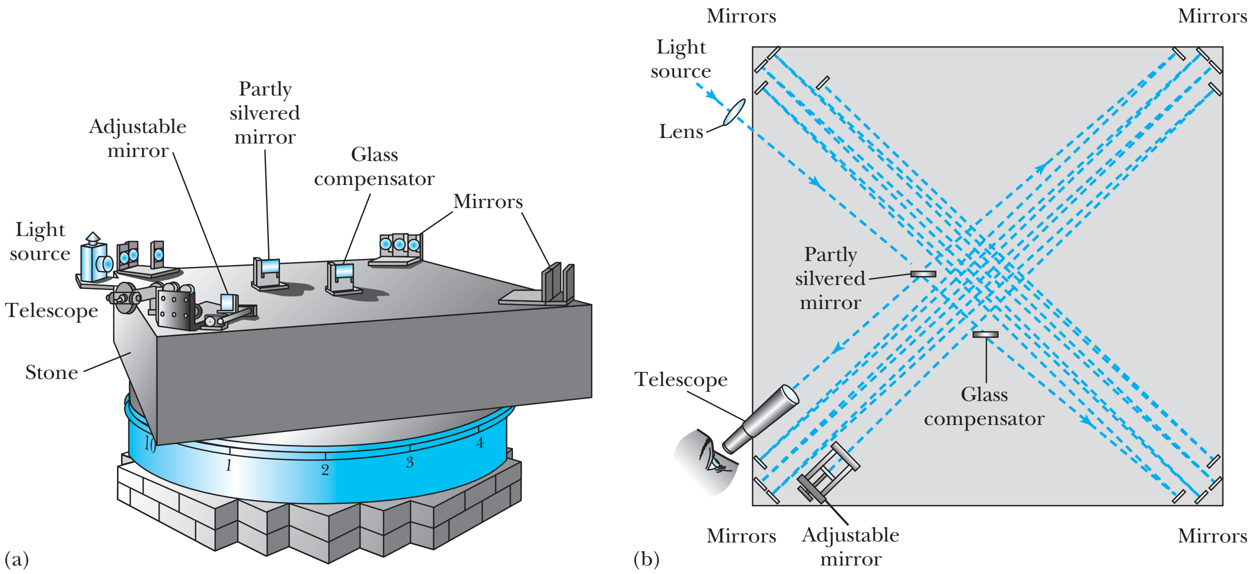


Figure 2.4 An adaptation of the Michelson and Morley 1887 experiment taken from their publication [A. A. Michelson and E. M. Morley, *Philosophical Magazine* **190**, 449 (1887)]. (a) A perspective view of the apparatus. To reduce vibration, the experiment was done on a massive soapstone, 1.5 m square and 0.3 m thick. This stone was placed on a wooden float that rested on mercury inside the annular piece shown underneath the stone. The entire apparatus rested on a brick pier. (b) The incoming light is focused by the lens and is both transmitted and reflected by the partly silvered mirror. The adjustable mirror allows fine adjustments in the interference fringes. The stone was rotated slowly and uniformly on the mercury to look for the interference effects of the ether.

Michelson in Europe

cists, particularly Hermann Helmholtz in Berlin. After a few false starts he finally was able to perform a measurement in Potsdam (near Berlin) in 1881. In order to use Equation (2.5) for an estimate of the expected time difference, the value of the Earth's orbital speed around the sun, 3×10^4 m/s, was used. Michelson's apparatus had $\ell_1 \approx \ell_2 \approx \ell = 1.2$ m. Thus Equation (2.5) predicts a time difference of 8×10^{-17} s. This is an exceedingly small time, but for a visible wavelength of 6×10^{-7} m, the period of one wavelength amounts to $T = 1/f = \lambda/c = 2 \times 10^{-15}$ s. Thus the time period of 8×10^{-17} s represents 0.04 fringes in the interference pattern. Michelson reasoned that he should be able to detect a shift of at least half this value but found none. Although disappointed, Michelson concluded that the hypothesis of the stationary ether must be incorrect.

The result of Michelson's experiment was so surprising that he was asked by several well-known physicists to repeat it. In 1882 Michelson accepted a position at the then-new Case School of Applied Science in Cleveland. Together with Edward Morley (1838–1923), a professor of chemistry at nearby Western Reserve College who had become interested in Michelson's work, he put together the more sophisticated experiment shown in Figure 2.4. The new experiment had an optical path length of 11 m, created by reflecting the light for eight round trips. The new apparatus was mounted on soapstone that floated on mercury to eliminate vibrations and was so effective that Michelson and Morley believed they could detect a fraction of a fringe shift as small as 0.005. With their new apparatus they expected the ether to produce a shift as large as 0.4 of a fringe. They reported in 1887 a *null result*—no effect whatsoever! The ether

Null result of Michelson-Morley experiment

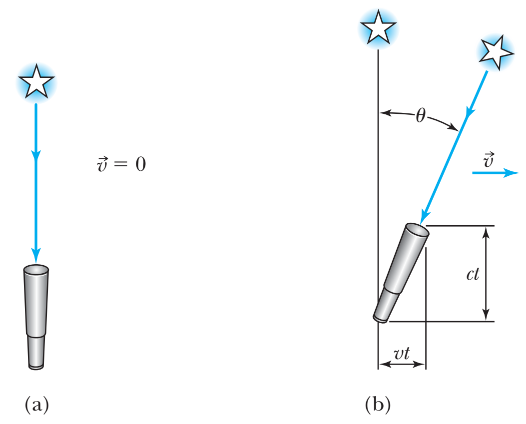


Figure 2.5 The effect of stellar aberration. (a) If a telescope is at rest, light from a distant star will pass directly into the telescope. (b) However, if the telescope is traveling at speed v (because it is fixed on the Earth, which has a motion about the sun), it must be slanted slightly to allow the starlight to enter the telescope. This leads to an apparent circular motion of the star as seen by the telescope, as the motion of the Earth about the sun changes throughout the solar year.

does not seem to exist. It is this famous experiment that has become known as the *Michelson-Morley experiment*.

The measurement so shattered a widely held belief that many suggestions were made to explain it. What if the Earth just happened to have a zero motion through the ether at the time of the experiment? Michelson and Morley repeated their experiment during night and day and for different seasons throughout the year. It is unlikely that at least sometime during these many experiments, the Earth would not be moving through the ether. Michelson and Morley even took their experiment to a mountaintop to see if the effects of the ether might be different. There was no change.

Ether drag

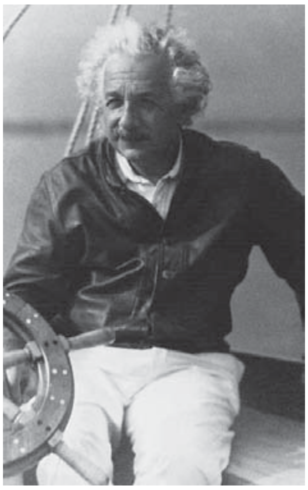
Stellar aberration

Of the many possible explanations of the null ether measurement, the one taken most seriously was the *ether drag* hypothesis. Some scientists proposed that the Earth somehow dragged the ether with it as the Earth rotates on its own axis and revolves around the sun. However, the ether drag hypothesis contradicts results from several experiments, including that of *stellar aberration* noted by the British astronomer James Bradley in 1728. Bradley noticed that the apparent position of the stars seems to rotate in a circular motion with a period of one year. The angular diameter of this circular motion with respect to the Earth is 41 seconds of arc. This effect can be understood by an analogy. From the viewpoint of a person sitting in a car during a rainstorm, the raindrops appear to fall vertically when the car is at rest but appear to be slanted toward the windshield when the car is moving forward. The same effect occurs for light coming from stars directly above the Earth's orbital plane. If the telescope and star are at rest with respect to the ether, the light enters the telescope as shown in Figure 2.5a. However, because the Earth is moving in its orbital motion, the apparent position of the star is at an angle θ as shown in Figure 2.5b. The telescope must actually be slanted at an angle θ to observe the light from the overhead star. During a time period t the starlight moves a vertical distance ct while the telescope moves a horizontal distance vt, so that the tangent of the angle θ is

$$\tan\theta = \frac{vt}{ct} = \frac{v}{c}$$

The orbital speed of the Earth is about 3×10^4 m/s; therefore, the angle θ is 10^{-4} rad or 20.6 seconds of arc, with a total opening of $2\theta = 41$ s as the Earth rotates—in agreement with Bradley's observation. The aberration reverses itself over the course of six months as the Earth orbits about the sun, in effect giving a circular motion to the star's position. This observation is in disagreement with the hypothesis of the Earth dragging the ether. If the ether were dragged with the Earth, there would be no need to tilt the telescope! The experimental observation of stellar aberration together with the null result of the Michelson and Morley experiment is enough evidence to refute the suggestions that the ether exists. Many other experimental observations have now been made that also confirm this conclusion.

The inability to detect the ether was a serious blow to reconciling the invariant form of the electromagnetic equations of Maxwell. There seems to be no single reference inertial system in which the speed of light is actually c. H. A. Lorentz and G. F. FitzGerald suggested, apparently independently, that the results of the Michelson-Morley experiment could be understood if length is contracted by the factor $\sqrt{1 - v^2/c^2}$ in the direction of motion, where v is the speed in the direction of travel. For this situation, the length ℓ_1, in the direction of motion, will be contracted by the factor $\sqrt{1 - v^2/c^2}$, whereas the length ℓ_2, perpendicular to v, will not. The result in Equation (2.3) is that t_1 will have the extra factor $\sqrt{1 - v^2/c^2}$, making Δt precisely zero as determined experimentally by Michelson. This contraction postulate, which became known as the *Lorentz-FitzGerald contraction,* was not proven from first principles using Maxwell's equations, and its true significance was not understood for several years until Einstein presented his explanation. An obvious problem with the Lorentz-FitzGerald contraction is that it is an ad hoc assumption that cannot be directly tested. Any measuring device would presumably be shortened by the same factor.



AIP/Emilio Segrè Visual Archives.

Albert Einstein (1879–1955), shown here sailing on Long Island Sound, was born in Germany and studied in Munich and Zurich. After having difficulty finding a position, he served seven years in the Swiss Patent Office in Bern (1902–1909), where he did some of his best work. He obtained his doctorate at the University of Zurich in 1905. His fame quickly led to appointments in Zurich, Prague, back to Zurich, and then to Berlin in 1914. In 1933, after Hitler came to power, Einstein left for the Institute for Advanced Study at Princeton University, where he became a U.S. citizen in 1940 and remained until his death in 1955. Einstein's total contributions to physics are rivaled only by those of Isaac Newton.

2.3 Einstein's Postulates

At the turn of the twentieth century, the Michelson-Morley experiment had laid to rest the idea of finding a preferred inertial system for Maxwell's equations, yet the Galilean transformation, which worked for the laws of mechanics, was invalid for Maxwell's equations. This quandary represented a turning point for physics.

Albert Einstein (1879–1955) was only two years old when Michelson reported his first null measurement for the existence of the ether. Einstein said that he began thinking at age 16 about the form of Maxwell's equations in moving inertial systems, and in 1905, when he was 26 years old, he published his startling proposal* about the principle of relativity, which he believed to be fundamental. Working without the benefit of discussions with colleagues outside his small circle of friends, Einstein was apparently unaware of the interest concerning the null result of Michelson and Morley.† Einstein instead looked at the problem in a more formal manner and believed that Maxwell's equations must be valid in

*In one issue of the German journal *Annalen der Physik* **17**, No. 4 (1905), Einstein published three remarkable papers. The first, on the quantum properties of light, explained the photoelectric effect; the second, on the statistical properties of molecules, included an explanation of Brownian motion; and the third was on special relativity. All three papers contained predictions that were subsequently confirmed experimentally.

†The question of whether Einstein knew of Michelson and Morley's null result before he produced his special theory of relativity is somewhat uncertain. For example, see J. Stachel, "Einstein and Ether Drift Experiments," *Physics Today* (May 1987), p. 45.

all inertial frames. With piercing insight and genius, Einstein was able to bring together seemingly inconsistent results concerning the laws of mechanics and electromagnetism with two postulates (as he called them; today we would call them laws). These postulates are

1. **The principle of relativity:** The laws of physics are the same in all inertial systems. There is no way to detect absolute motion, and no preferred inertial system exists.
2. **The constancy of the speed of light:** Observers in all inertial systems measure the same value for the speed of light in a vacuum.

Einstein's two postulates

The first postulate indicates that the laws of physics are the same in all coordinate systems moving with uniform relative motion to each other. Einstein showed that postulate 2 actually follows from the first one. He returned to the principle of relativity as espoused by Newton. Although Newton's principle referred only to the laws of mechanics, Einstein expanded it to include all laws of physics—including those of electromagnetism. We can now modify our previous definition of *inertial frames of reference* to be those frames of reference in which *all the laws of physics* are valid.

Inertial frames of reference revisited

Einstein's solution requires us to take a careful look at time. Return to the two systems of Figure 2.1 and remember that we had previously assumed that $t = t'$. We assumed that events occurring in system K′ and in system K could easily be synchronized. Einstein realized that each system must have its own observers with their own clocks and metersticks. *An event in a given system must be specified by stating both its space and time coordinates.* Consider the flashing of two bulbs fixed in system K as shown in Figure 2.6a. **M**ary, in system K′ (the **M**oving system) is beside **F**rank, who is in system K (the **F**ixed system), when the bulbs flash. As seen in Figure 2.6b the light pulses travel the same distance in system K and arrive at Frank *simultaneously*. Frank sees the two flashes at the same time. However, the two light pulses do not reach Mary simultaneously, because system K′ is moving to the right, and she has moved closer to the bulb on the right by the time the flash reaches her. The light flash coming from the left will reach her at some later time. Mary thus determines that the light on the right flashed before the one on the left, because she is at rest in her frame and both flashes approach her

Simultaneity

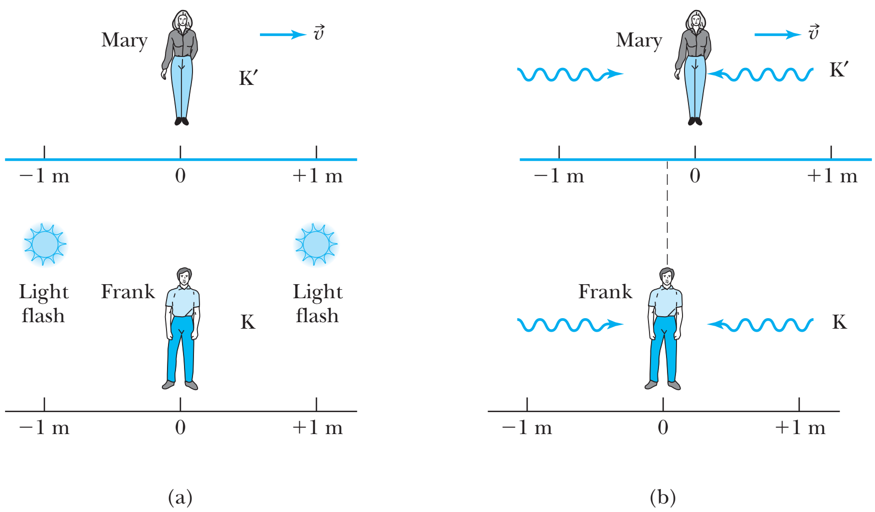


Figure 2.6 The problem of simultaneity. Flashbulbs positioned in system K at one meter on either side of Frank go off simultaneously in (a). Frank indeed sees both flashes simultaneously in (b). However, Mary, at rest in system K′ moving to the right with speed v, does not see the flashes simultaneously despite the fact that she was alongside Frank when the flashbulbs went off. During the finite time it took light to travel the one meter, Mary has moved slightly, as shown in exaggerated form in (b).

at speed c. We conclude that

> *Two events that are simultaneous in one reference frame (K) are not necessarily simultaneous in another reference frame (K′) moving with respect to the first frame.*

Synchronization of clocks

We must be careful when comparing the same event in two systems moving with respect to one another. Time comparison can be accomplished by sending light signals from one observer to another, but this information can travel only as fast as the finite speed of light. It is best if each system has its own observers with clocks that are synchronized. How can we do this? We place observers with clocks throughout a given system. If, when we bring all the clocks together at one spot at rest, all the clocks agree, then the clocks are said to be **synchronized**. However, we have to move the clocks relative to each other to reposition them, and this might affect the synchronization. A better way would be to flash a bulb halfway between each pair of clocks at rest and make sure the pulses arrive simultaneously at each clock. This will require many measurements, but it is a safe way to synchronize the clocks. We can determine the time of an event occurring far away from us by having a colleague at the event, with a clock fixed at rest, measure the time of the particular event, and send us the results, for example, by telephone or even by mail. If we need to check our clocks, we can always send light signals to each other over known distances at some predetermined time.

In the next section we derive the correct transformation, called the **Lorentz transformation**, that makes the laws of physics invariant between inertial frames of reference. We use the coordinate systems described by Figure 2.1. At $t = t' = 0$, the origins of the two coordinate systems are coincident, and the system K′ is traveling along the x and x' axes. For this special case, the Lorentz transformation equations are

Lorentz transformation equations

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (vx/c^2)}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{2.6}$$

Relativistic factor

We commonly use the symbols β and the *relativistic factor* γ to represent two longer expressions:

$$\beta = \frac{v}{c} \tag{2.7}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.8}$$

which allows the Lorentz transformation equations to be rewritten in compact form as

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \beta x/c) \end{aligned} \tag{2.6}$$

Note that $\gamma \geq 1$ ($\gamma = 1$ when $v = 0$).

2.4 The Lorentz Transformation

In this section we use Einstein's two postulates to find a transformation between inertial frames of reference such that all the physical laws, including Newton's laws of mechanics and Maxwell's electrodynamics equations, will have the same form. We use the fixed system K and moving system K′ of Figure 2.1. At $t = t' = 0$ the origins and axes of both systems are coincident, and system K′ is moving to the right along the x axis. A flashbulb goes off at the origins when $t = t' = 0$. According to postulate 2, the speed of light will be c in both systems, and the wavefronts observed in both systems must be spherical and described by

$$x^2 + y^2 + z^2 = c^2t^2 \tag{2.9a}$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{2.9b}$$

These two equations are inconsistent with a Galilean transformation because a wavefront can be spherical in only one system when the second is moving at speed v with respect to the first. The Lorentz transformation *requires* both systems to have a spherical wavefront centered on each system's origin.

Another clear break with Galilean and Newtonian physics is that we do not assume that $t = t'$. Each system must have its own clocks and metersticks as indicated in a two-dimensional system in Figure 2.7. Because the systems move only along their x axes, observers in both systems agree by direct observation that

$$y' = y$$

$$z' = z$$

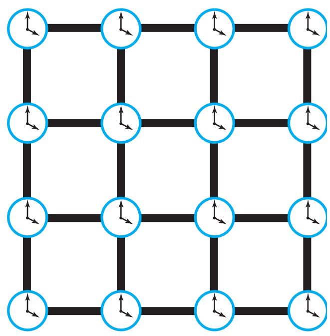


Figure 2.7 In order to make sure accurate event measurements can be obtained, synchronized clocks and uniform measuring sticks are placed throughout a system.

We know that the Galilean transformation $x' = x - vt$ is incorrect, but what is the correct transformation? We require a linear transformation so that each event in system K corresponds to one, and only one, event in system K′. The simplest *linear* transformation is of the form

$$x' = \gamma(x - vt) \tag{2.10}$$

We will see if such a transformation suffices. The parameter γ cannot depend on x or t because the transformation must be linear. The parameter γ must be close to 1 for $v \ll c$ in order for Newton's laws of mechanics to be valid for most of our measurements. We can use similar arguments from the standpoint of an observer stationed in system K′ to obtain an equation similar to Equation (2.10).

$$x = \gamma'(x' + vt') \tag{2.11}$$

Because postulate 1 requires that the laws of physics be the same in both reference systems, we demand that $\gamma' = \gamma$. Notice that the only difference between Equations (2.10) and (2.11) other than the primed and unprimed quantities being switched is that $v \to -v$, which is reasonable because according to the observer in each system, the other observer is moving either forward or backward.

According to postulate 2, the speed of light is c in both systems. Therefore, in each system the wavefront of the flashbulb light pulse along the respective x axes must be described by $x = ct$ and $x' = ct'$, which we substitute into Equations (2.10) and (2.11) to obtain

$$ct' = \gamma(ct - vt) \tag{2.12a}$$

and

$$ct = \gamma(ct' + vt') \tag{2.12b}$$

We divide each of these equations by c and obtain

$$t' = \gamma t\left(1 - \frac{v}{c}\right) \tag{2.13}$$

and

$$t = \gamma t'\left(1 + \frac{v}{c}\right) \tag{2.14}$$

We substitute the value of t from Equation (2.14) into Equation (2.13).

$$t' = \gamma^2 t'\left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right) \tag{2.15}$$

We solve this equation for γ^2 and obtain

$$\gamma^2 = \frac{1}{1 - v^2/c^2}$$

or

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.16}$$

In order to find a transformation for time t', we rewrite Equation (2.13) as

$$t' = \gamma\left(t - \frac{vt}{c}\right)$$

We substitute $t = x/c$ for the light pulse and find

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{t - vx/c^2}{\sqrt{1 - \beta^2}}$$

We are now able to write the complete Lorentz transformations as

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \beta^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (vx/c^2)}{\sqrt{1 - \beta^2}} \end{aligned} \tag{2.17}$$

The inverse transformation equations are obtained by replacing v by $-v$ as discussed previously and by exchanging the primed and unprimed quantities.

Inverse Lorentz transformation equations

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - \beta^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + (vx'/c^2)}{\sqrt{1 - \beta^2}} \end{aligned} \tag{2.18}$$

Notice that Equations (2.17) and (2.18) both reduce to the Galilean transformation when $v \ll c$. It is only for speeds that approach the speed of light that the Lorentz transformation equations become significantly different from the Galilean equations. In our studies of mechanics we normally do not consider such high speeds, and our previous results probably require no corrections. The laws of mechanics credited to Newton are still valid over the region of their applicability. Even for a speed as high as the Earth orbiting about the sun, 30 km/s, the value of the relativistic factor γ is 1.000000005. We show a plot of the relativistic parameter γ versus speed in Figure 2.8. As a rule of thumb, we should consider using the relativistic equations when $v/c > 0.1$ ($\gamma \approx 1.005$).

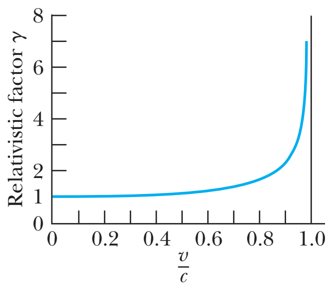


Figure 2.8 A plot of the relativistic factor γ as a function of speed v/c, showing that γ becomes large quickly as v approaches c.

Finally, consider the implications of the Lorentz transformation. The linear transformation equations ensure that a single event in one system is described by a single event in another inertial system. However, space and time are not separate. In order to express the position of x in system K′, we must use both x' and t'. We have also found that the Lorentz transformation does not allow a speed greater than c; the relativistic factor γ becomes imaginary in this case. We show later in this chapter that no object of nonzero mass can have a speed greater than c.

2.5 Time Dilation and Length Contraction

The Lorentz transformations have immediate consequences with respect to time and length measurements made by observers in different inertial frames. We shall consider time and length measurements separately and then see how they are related to one another.

Time Dilation

Consider again our two systems K and K′ with system K fixed and system K′ moving along the x axis with velocity $\vec{v}$ as shown in Figure 2.9a (p. 32). Frank lights a sparkler at position x_1 in system K. A clock placed beside the sparkler indicates the time to be t_1 when the sparkler is lit and t_2 when the sparkler goes out (Figure 2.9b). The sparkler burns for time T_0, where $T_0 = t_2 - t_1$. The time difference between two events occurring at the same position in a system as measured by a clock at rest in the system is called the **proper time**. We use the subscript zero on the time difference T_0 to denote the proper time.

Proper time

Now what is the time as determined by Mary who is passing by (but at rest in her own system K′)? All the clocks in both systems have been synchronized when the systems are at rest with respect to one another. The two events (sparkler lit and then going out) do not occur at the same place according to Mary. She is beside the sparkler when it is lit, but she has moved far away from the sparkler when it goes out (Figure 2.9b). Her friend Melinda, also at rest in system K′, is beside the sparkler when it goes out. Mary and Melinda measure the two times for the sparkler to be lit and to go out in system K′ as times t'_1 and t'_2. The Lorentz transformation relates these times to those measured in system K as

$$t'_2 - t'_1 = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

In system K the clock is fixed at x_1, so $x_2 - x_1 = 0$; that is, the two events occur at the same position. The time $t_2 - t_1$ is the proper time T_0, and we denote the time difference $t'_2 - t'_1 = T'$ as measured in the moving system K′:

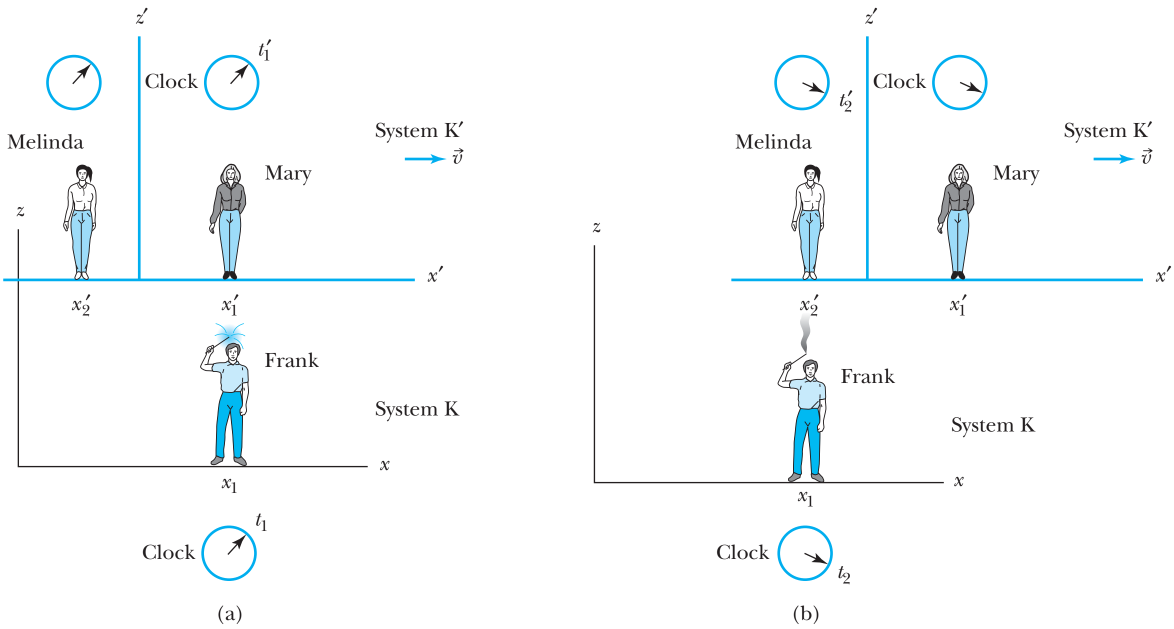


Figure 2.9 Frank measures the proper time for the time interval that a sparkler stays lit. His clock is at the same position in system K when the sparkler is lit in (a) and when it goes out in (b). Mary, in the moving system K′, is beside the sparkler at position x'_1 when it is lit in (a), but by the time it goes out in (b), she has moved away. Melinda, at position x'_2, measures the time in system K′ when the sparkler goes out in (b).

Time dilation

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0 \tag{2.19}$$

Thus the time interval measured in the moving system K′ is greater than the time interval measured in system K where the sparkler is at rest. This effect is known as **time dilation** and is a direct result of Einstein's two postulates. The time measured by Mary and Melinda in their system K′ for the time difference was greater than T_0 by the relativistic factor γ ($\gamma > 1$). The two events, sparkler being lit and then going out, did not occur at the same position ($x'_2 \neq x'_1$) in system K′ (see Figure 2.9b). This result occurs because of the absence of simultaneity. The events do not occur at the same space and time coordinates in the two systems. It requires three clocks to perform the measurement: one in system K and two in system K′.

Moving clocks run slow

The time dilation result is often interpreted by saying that *moving clocks run slow* by the factor γ^{-1}, and sometimes this is a useful way to remember the effect. The moving clock in this case can be any kind of clock. It can be the time that sand takes to pass through an hourglass, the time a sparkler stays lit, the time between heartbeats, the time between ticks of a clock, or the time spent in a class lecture. In all cases, the actual time interval on a moving clock is greater than the proper time as measured on a clock at rest. The proper time is always the smallest possible time interval between two events.

Each person will claim the clock in the other (moving) system is running slow. If Mary had a sparkler in her system K′ at rest, Frank (fixed in system K) would also measure a longer time interval on his clock in system K because the sparkler would be moving with respect to his system.

EXAMPLE 2.1

Show that Frank in the fixed system will also determine the time dilation result by having the sparkler be at rest in the system K′.

Strategy We should be able to proceed similarly to the derivation we did before when the sparkler was at rest in system K. In this case Mary lights the sparkler in the moving system K′. The time interval over which the sparkler is lit is given by $T_0' = t_2' - t_1'$, and the sparkler is placed at the position $x_1' = x_2'$ so that $x_2' - x_1' = 0$. In this case T_0' is the proper time. We use the Lorentz transformation from Equation (2.18) to determine the time difference $T = t_2 - t_1$ as measured by the clocks of Frank and his colleagues.

Solution We use Equation (2.18) to find $t_2 - t_1$:

$$T = t_2 - t_1 = \frac{(t_2' - t_1') + (v/c^2)(x_2' - x_1')}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{T_0'}{\sqrt{1 - v^2/c^2}} = \gamma T_0'$$

The time interval is still smaller in the system where the sparkler is at rest.

***Gedanken* experiments**

The preceding results naturally seem a little strange to us. In relativity we often carry out thought (or *gedanken* from the German word) experiments, because the actual experiments would be somewhat impractical. Consider the following *gedanken* experiment. Mary, in the moving system K′, flashes a light at her origin along her y' axis (Figure 2.10). The light travels a distance L, reflects off a mirror, and returns. Mary says that the total time for the journey is $T_0' = t_2' - t_1' = 2L/c$, and this is indeed the proper time, because the clock in K′ beside Mary is at rest.

What do Frank and other observers in system K measure? Let T be the round-trip time interval measured in system K for the light to return to the x axis. The light is flashed when the origins are coincident, as Mary passes by Frank with relative velocity v. When the light reaches the mirror in the system K′ at time $T/2$, the system K′ will have moved a distance $vT/2$ down the x axis. When the light is reflected back to the x axis, Frank will not even see the light return, because it will return a distance vT away, where another observer, Fred, is positioned. Because observers Frank and Fred have previously synchronized their clocks, they can still measure the total elapsed time for the light to be reflected from the mirror and return. According to observers in the K system, the total distance the light travels (as shown in Figure 2.10) is $2\sqrt{(vT/2)^2 + L^2}$. And according to postulate 2, the light must travel at the speed of light, so the total time interval T measured in system K is

$$T = \frac{\text{distance}}{\text{speed}} = \frac{2\sqrt{(vT/2)^2 + L^2}}{c}$$

As can be determined from above, $L = cT_0'/2$, so we have

$$T = \frac{2\sqrt{(vT/2)^2 + (cT_0'/2)^2}}{c}$$

which reduces to

$$T = \frac{T_0'}{\sqrt{1 - v^2/c^2}} = \gamma T_0'$$

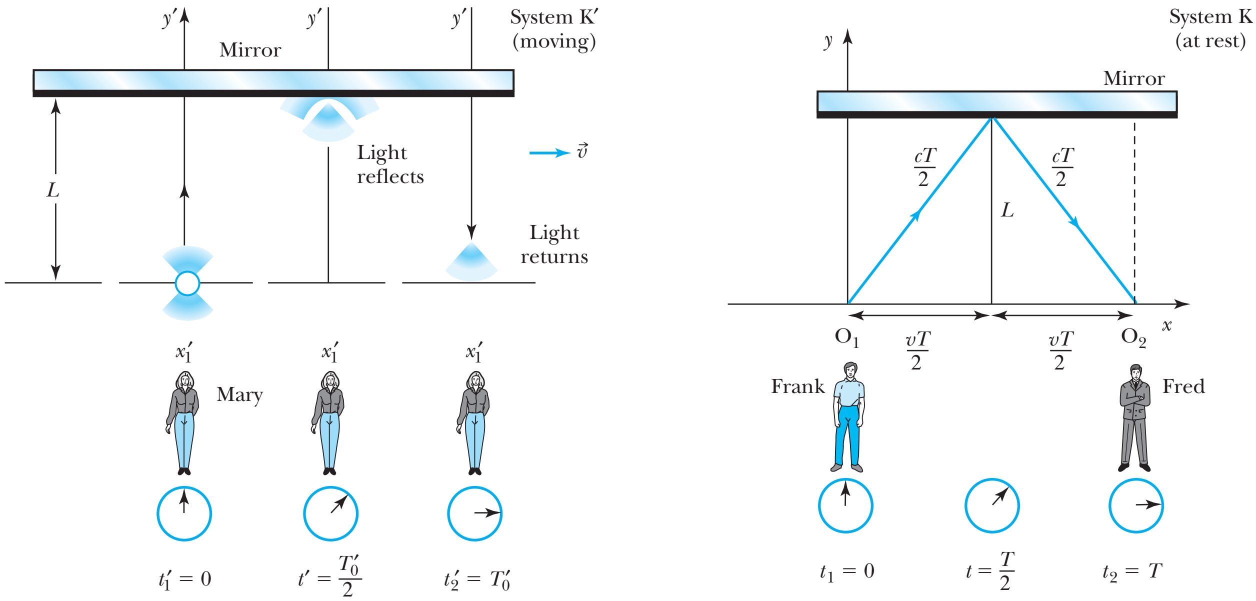


Figure 2.10 Mary, in system K′, flashes a light along her y' axis and measures the proper time $T'_0 = 2L/c$ for the light to return. In system K Frank will see the light travel partially down his x axis, because system K′ is moving. Fred times the arrival of the light in system K. The time interval T that Frank and Fred measure is related to the proper time by $T = \gamma T'_0$.

This is consistent with the earlier result. In this case $T > T'_0$. The proper time is always the shortest time interval, and we find that the clock in Mary's system K′ is "running slow."

EXAMPLE 2.2

It is the year 2150 and the United Nations Space Federation has finally perfected the storage of antiprotons for use as fuel in a spaceship. (Antiprotons are the antiparticles of protons. We discuss antiprotons in Chapter 3.) Preparations are under way for a manned spacecraft visit to possible planets orbiting one of the three stars in the star system Alpha Centauri, some 4.30 lightyears away. Provisions are placed on board to allow a trip of 16 years' total duration. How fast must the spacecraft travel if the provisions are to last? Neglect the period of acceleration, turnaround, and visiting times, because they are negligible compared with the actual travel time.

Strategy The time interval as measured by the astronauts on the spacecraft can be no longer than 16 years, because that is how long the provisions will last. However, from Earth we realize that the spacecraft will be moving at a high relative speed v to us, and that according to our clock in the stationary system K, the trip will last $T = 2L/v$, where L is the distance to the star.

Because provisions on board the spaceship will last for only 16 years, we let the proper time T'_0 in system K′ be 16 years. Using the time dilation result, we determine the relationship between T, the time measured on Earth, and the proper time T'_0 to be

$$T = \frac{2L}{v} = \frac{T'_0}{\sqrt{1 - v^2/c^2}} \tag{2.20}$$

We then solve this equation for the required speed v.

Solution A lightyear is a convenient way to measure large distances. It is the distance light travels in one year and is denoted by ly:

$$1 \text{ ly} = \left(3.00 \times 10^8 \frac{\text{m}}{\text{s}}\right)(1 \text{ year})\left(365 \frac{\text{days}}{\text{year}}\right)\left(24 \frac{\text{h}}{\text{day}}\right)\left(3600 \frac{\text{s}}{\text{h}}\right)$$
$$= 9.46 \times 10^{15} \text{ m}$$

Note that the distance of one lightyear is the speed of light, c, multiplied by the time of one year. The dimension of a lightyear works out to be length. In this case, the result is $4.30 \text{ ly} = c(4.30 \text{ y}) = 4.07 \times 10^{16}$ m.

We insert the appropriate numbers into Equation (2.20) and obtain

$$\frac{2(4.30 \text{ ly})(9.46 \times 10^{15} \text{ m/ly})}{v} = \frac{16 \text{ y}}{\sqrt{1 - v^2/c^2}}$$

The solution to this equation is $v = 0.473c = 1.42 \times 10^8$ m/s. The time interval as measured on Earth will be $\gamma T_0' = 18.2$ y. Notice that the astronauts will age only 16 years (their clocks run slow), whereas their friends remaining on Earth will age 18.2 years. Can this really be true? We shall discuss this question again in Section 2.8.

Length Contraction

Now let's consider what might happen to the length of objects in relativity. Let an observer in each system K and K′ have a meterstick at rest in his or her own respective system. Each observer lays the stick down along his or her respective x axis, putting the left end at x_ℓ (or x'_ℓ) and the right end at x_r (or x'_r). Thus, Frank in system K measures his stick to be $L_0 = x_r - x_\ell$. Similarly, in system K′, Mary measures her stick at rest to be $L'_0 = x'_r - x'_\ell = L_0$. Every observer measures a meterstick at rest in his or her own system to have the same length, namely one meter. The length as measured at rest is called the **proper length**.

Proper length

Let system K be at rest and system K′ move along the x axis with speed v. Frank, who is at rest in system K, measures the length of the stick moving in K′. The difficulty is to measure the ends of the stick simultaneously. We insist that Frank measure the ends of the stick at the same time so that $t = t_r = t_\ell$. The events denoted by (x, t) are (x_ℓ, t) and (x_r, t). We use Equation (2.17) and find

$$x'_r - x'_\ell = \frac{(x_r - x_\ell) - v(t_r - t_\ell)}{\sqrt{1 - v^2/c^2}}$$

The meterstick is at rest in system K′, so the length $x'_r - x'_\ell$ must be the proper length L'_0. Denote the length measured by Frank as $L = x_r - x_\ell$. The times t_r and t_ℓ are identical, as we insisted, so $t_r - t_\ell = 0$. Notice that the times of measurement by Mary in her system, t'_ℓ and t'_r, are *not* identical. It makes no difference when Mary makes the measurements in her own system, because the stick is at rest. However, it makes a big difference when Frank makes his measurements, because the stick is moving with speed v with respect to him. The measurements must be done simultaneously! With these results, the previous equation becomes

$$L'_0 = \frac{L}{\sqrt{1 - v^2/c^2}} = \gamma L$$

or, because $L'_0 = L_0$,

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma} \tag{2.21}$$

Length contraction

Notice that $L_0 > L$, so the moving meterstick shrinks according to Frank. This effect is known as **length** or **space contraction** and is characteristic of relative motion. This effect is also sometimes called the *Lorentz-FitzGerald contraction* because Lorentz and FitzGerald independently suggested the contraction as a way to solve the electrodynamics problem. This effect, like time dilation, is also reciprocal. Each observer will say that the other moving stick is shorter. There is no length contraction perpendicular to the relative motion, however, because $y' = y$ and $z' = z$. Observers in both systems can check the length of the other meterstick placed perpendicular to the direction of motion as the metersticks pass each other. They will agree that both metersticks are one meter long.

We can perform another *gedanken* experiment to arrive at the same result. This time we lay the meterstick along the x' axis in the moving system K′ (Figure 2.11a). The two systems K and K′ are aligned at $t = t' = 0$. A mirror is placed at the end of the meterstick, and a flashbulb goes off at the origin at $t = t' = 0$, sending a light pulse down the x' axis, where it is reflected and returned. Mary sees the stick at rest in system K′ and measures the proper length L_0 (which should of course be one meter). Mary uses the same clock fixed at $x' = 0$ for the time measurements. The stick is moving at speed v with respect to Frank in the fixed system K. The clocks at $x = x' = 0$ both read zero when the origins are aligned just when the flashbulb goes off. Notice the situation shown in system K

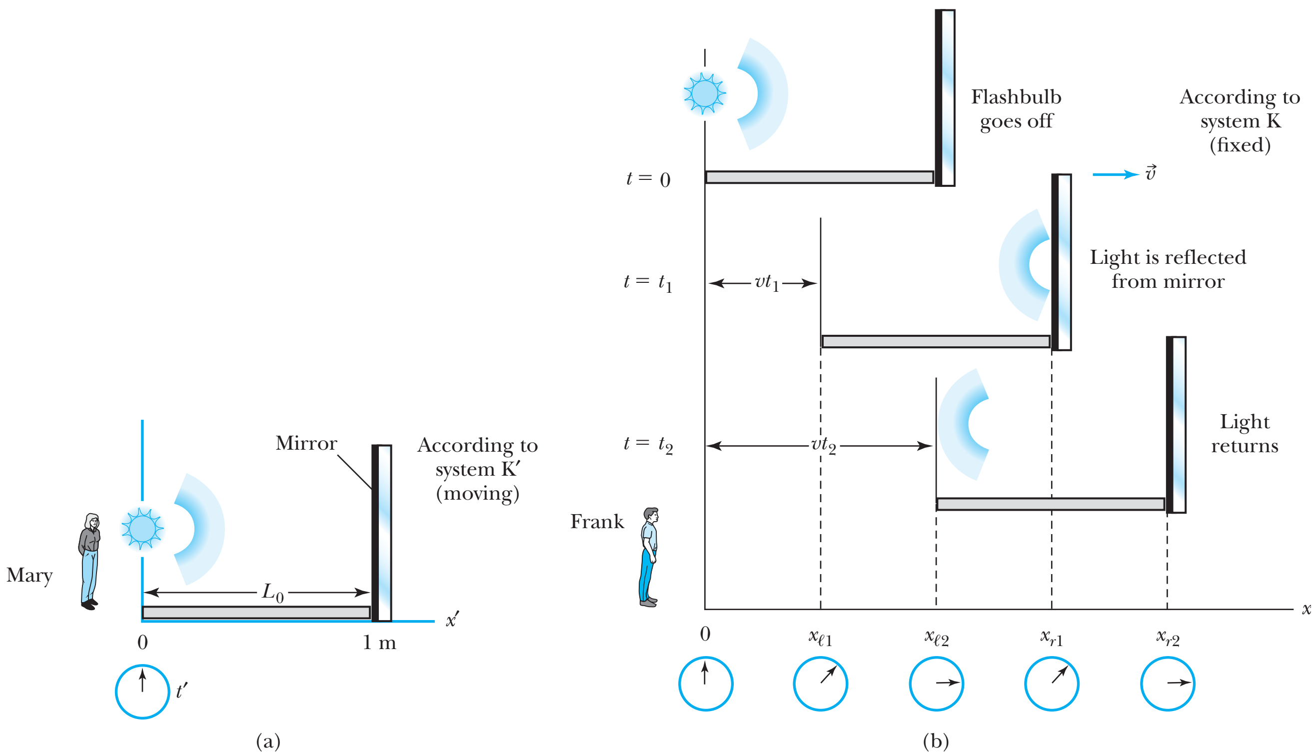


Figure 2.11 (a) Mary, in system K′, flashes a light down her x' axis along a stick at rest in her system of length L_0, which is the proper length. The time interval for the light to travel down the stick and back is $2L_0/c$. (b) Frank, in system K, sees the stick moving, and the mirror has moved a distance vt_1 by the time the light is reflected. By the time the light returns to the beginning of the stick, the stick has moved a total distance of vt_2. The times can be compared to show that the moving stick has been length contracted by $L = L_0\sqrt{1 - v^2/c^2}$.

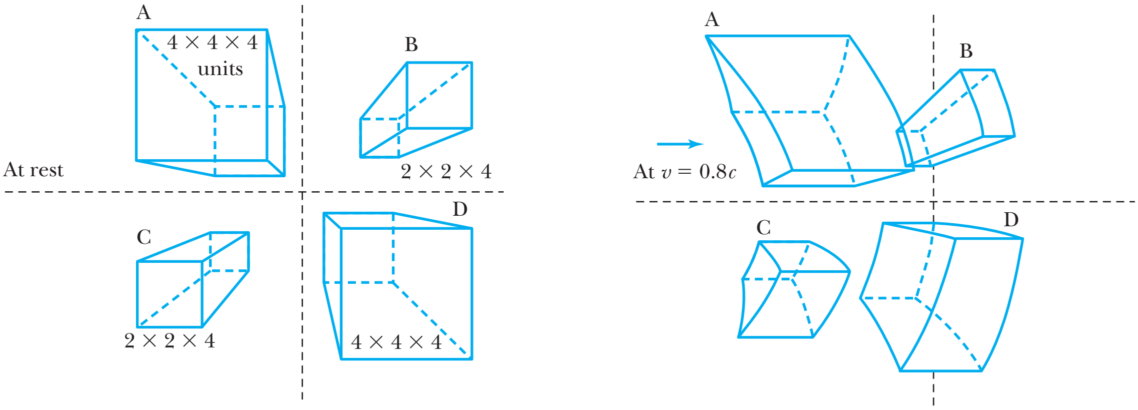


Figure 2.12 In this computer simulation, the rectangular boxes are drawn as if the observer were 5 units in front of the near plane of the boxes and directly in front of the origin. The boxes are shown at rest on the left. On the right side, the boxes are moving to the right at a speed of $v = 0.8c$. The horizontal lines are only length contracted, but notice that the vertical lines become hyperbolas. The objects appear to be slightly rotated in space. The objects that are further away from the origin appear earlier because they are photographed at an earlier time and because the light takes longer to reach the camera (or our eyes). *Reprinted with permission from* American Journal of Physics ***33****, 534 (1965), G. D. Scott and M. R. Viner. © 1965, American Association of Physics Teachers.*

(Figure 2.11b), where by the time the light reaches the mirror, the entire stick has moved a distance vt_1. By the time the light has been reflected back to the front of the stick again, the stick has moved a total distance vt_2. We leave the solution in terms of length contraction to Problem 18.

The effect of length contraction along the direction of travel may strongly affect the appearances of two- and three-dimensional objects. We see such objects when the light reaches our eyes, not when the light actually leaves the object. Thus, if the objects are moving rapidly, we will not see them as they appear at rest. Figure 2.12 shows the appearance of several such objects as they move. Note that not only do the horizontal lines become contracted, but the vertical lines also become hyperbolas. We show in Figure 2.13 a row of bars moving to the right with speed $v = 0.9c$. The result is quite surprising.

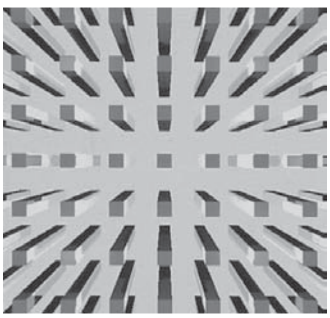
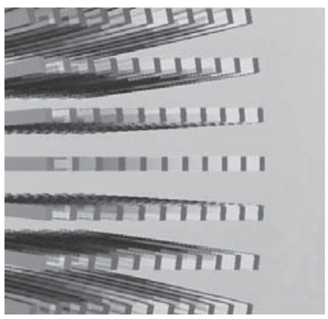
(a)

(b)


Figure 2.13 (a) An array of rectangular bars is seen from above at rest. (b) The bars are moving to the right at $v = 0.9c$. The bars appear to contract and rotate. *Quoted from P.-K. Hsuing and R. H. P. Dunn,* Science News ***137****, 232 (1990).*

EXAMPLE 2.3

Consider the solution of Example 2.2 from the standpoint of length contraction.

Strategy The astronauts have only enough provisions for a trip lasting 16 years. Thus they expect to travel for 8 years each way. If the star system Alpha Centauri is 4.30 lightyears away, it may appear that they need to travel at a velocity of $0.5c$ to make the trip. We want to consider this example as if the astronauts are at rest. Alpha Centauri will appear to be moving toward them, and the distance to the star system is length contracted. The distance measured by the astronauts will be less than 4.30 ly.

Solution The contracted distance according to the astronauts in motion is $(4.30 \text{ ly})\sqrt{1 - v^2/c^2}$. The velocity they need to make this journey is the contracted distance divided by 8 years.

$$v = \frac{\text{distance}}{\text{time}} = \frac{(4.30 \text{ ly})\sqrt{1 - v^2/c^2}}{8 \text{ y}}$$

If we divide by c, we obtain

$$\beta = \frac{v}{c} = \frac{(4.30 \text{ ly})\sqrt{1 - v^2/c^2}}{c(8 \text{ y})} = \frac{(4.30 \text{ ly})\sqrt{1 - v^2/c^2}}{(8 \text{ ly})}$$

$$8\beta = 4.30\sqrt{1 - \beta^2}$$

which gives

$$\beta = 0.473$$

$$v = 0.473c$$

which is just what we found in the previous example. The effects of time dilation and length contraction give identical results.

2.6 Addition of Velocities

A spaceship launched from a space station (see Figure 2.14) quickly reaches its cruising speed of $0.60c$ with respect to the space station when a band of asteroids is observed straight ahead of the ship. Mary, the commander, reacts quickly and orders her crew to blast away the asteroids with the ship's proton gun to avoid a catastrophic collision. Frank, the admiral on the space station, listens with apprehension to the communications because he fears the asteroids may eventually destroy his space station as well. Will the high-energy protons of speed $0.99c$ be able to successfully blast away the asteroids and save both the spaceship and

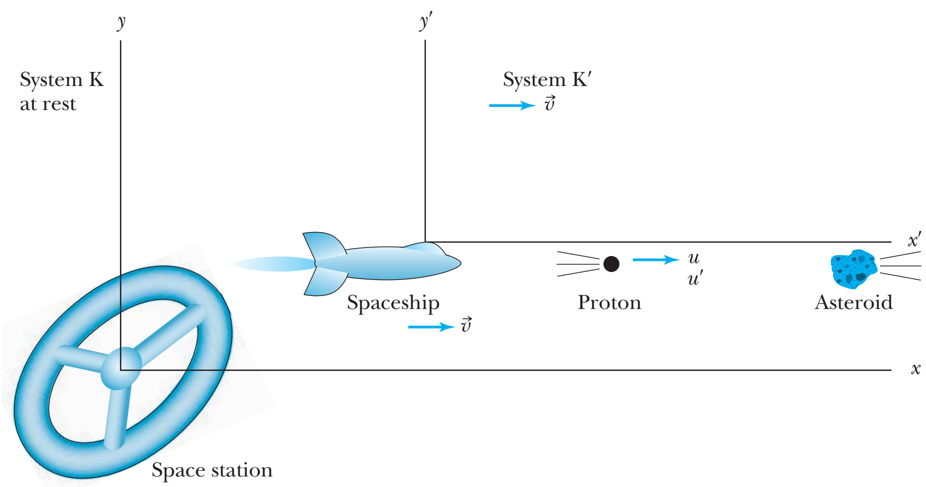


Figure 2.14 The space station is at rest at the origin of system K. The spaceship is moving to the right with speed v with respect to the space station and is in system K′. An asteroid is moving to the left toward both the spaceship and space station, so Mary, the commander of the spaceship, orders that the proton gun shoot protons to break up the asteroid. The speed of the protons is u and u' with respect to systems K and K′, respectively.

space station? If $0.99c$ is the speed of the protons with respect to the spaceship, what speed will Frank measure for the protons?

We will use the letter u to denote velocity of objects as measured in various coordinate systems. In this case, Frank (in the fixed, stationary system K on the space station) will measure the velocity of the protons to be u, whereas Mary, the commander of the spaceship (the moving system K′), will measure $u' = 0.99c$. We reserve the letter v to express the velocity of the coordinate systems with respect to each other. The velocity of the spaceship with respect to the space station is $v = 0.60c$.

Newtonian mechanics teaches us that to find the velocity of the protons with respect to the space station, we simply add the velocity of the spaceship with respect to the space station ($0.60c$) to the velocity of the protons with respect to the spaceship ($0.99c$) to determine the result $u = v + u' = 0.60c + 0.99c = 1.59c$. However, this result is not in agreement with the results of the Lorentz transformation. We use Equation (2.18), letting x be along the direction of motion of the spaceship (and high-speed protons), and take the differentials, with the results

$$\begin{aligned} dx &= \gamma(dx' + v\,dt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma[dt' + (v/c^2)\,dx'] \end{aligned} \tag{2.22}$$

Velocities are defined by $u_x = dx/dt$, $u_y = dy/dt$, $u'_x = dx'/dt'$, and so on. Therefore we determine u_x by

Relativistic velocity addition

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v\,dt')}{\gamma[dt' + (v/c^2)\,dx']} = \frac{u'_x + v}{1 + (v/c^2)u'_x} \tag{2.23a}$$

Similarly, u_y and u_z are determined to be

$$u_y = \frac{u'_y}{\gamma[1 + (v/c^2)u'_x]} \tag{2.23b}$$

$$u_z = \frac{u'_z}{\gamma[1 + (v/c^2)u'_x]} \tag{2.23c}$$

Equations (2.23) are referred to as the **Lorentz velocity transformations**. Notice that although the relative motion of the systems K and K′ is only along the x direction, the velocities along y and z are affected as well. This contrasts with the Lorentz transformation equations, where $y = y'$ and $z = z'$. However, the difference in velocities is simply ascribed to the transformation of time, which depends on v and x'. Thus, the transformations for u_y and u_z depend on v and u'_x. The inverse transformations for u'_x, u'_y, and u'_z can be determined by simply switching primed and unprimed variables and changing v to $-v$. The results are

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - (v/c^2)u_x} \\ u'_y &= \frac{u_y}{\gamma[1 - (v/c^2)u_x]} \\ u'_z &= \frac{u_z}{\gamma[1 - (v/c^2)u_x]} \end{aligned} \tag{2.24}$$

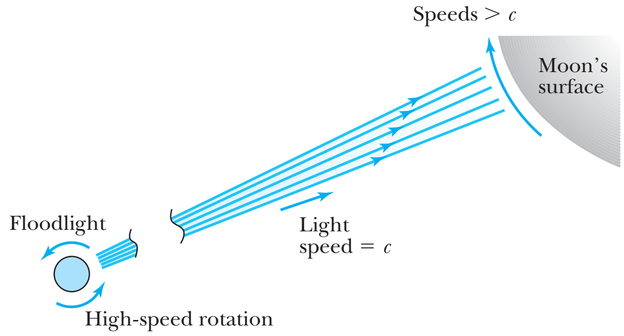


Figure 2.15 A floodlight revolving at high speeds can sweep a light beam across the surface of the moon at speeds exceeding c, but the speed of the light still does not exceed c.

Note that we found the velocity transformation equations for the situation corresponding to the inverse Lorentz transformation, Equations (2.18), before finding the velocity transformation for Equations (2.17).

What is the correct result for the speed of the protons with respect to the space station? We have $u'_x = 0.99c$ and $v = 0.60c$, so Equation (2.23a) gives us the result

$$u_x = \frac{0.990c + 0.600c}{1 + \dfrac{(0.600c)(0.990c)}{c^2}} = 0.997c$$

where we have assumed we know the speeds to three significant figures. Therefore, the result is a speed only slightly less than c. The Lorentz transformation does not allow a material object to have a speed greater than c. Only massless particles, such as light, can have speed c. If the crew members of the spaceship spot the asteroids far enough in advance, their reaction times should allow them to shoot down the uncharacteristically swiftly moving asteroids and save both the spaceship and the space station.

Although no particle with mass can carry energy faster than c, we can imagine a signal being processed faster than c. Consider the following *gedanken* experiment. A giant floodlight placed on a space station above the Earth revolves at 100 Hz, as shown in Figure 2.15. Light spreads out in the radial direction from the floodlight at speeds of c. On the surface of the moon, the light beam sweeps across at speeds far exceeding c (Problem 36). However, the light itself does not reach the moon at speeds faster than c. No energy is associated with the beam of light sweeping across the moon's surface. The energy (and linear momentum) is only along the radial direction from the space station to the moon.

EXAMPLE 2.4

Mary, the commander of the spaceship just discussed, is holding target practice for junior officers by shooting protons at small asteroids and space debris off to the side (perpendicular to the direction of spaceship motion) as the spaceship passes by. What speed will an observer in the space station measure for these protons?

Strategy We use the coordinate systems and speeds of the spaceship and proton gun as described previously. Let the direction of the protons now be perpendicular to the direction of the spaceship—along the y' direction. We already know in the spaceship's K′ system that $u'_y = 0.99c$ and $u'_x =$

$u'_z = 0$, and that the speed of the K′ system (spaceship) with respect to the space station is $v = 0.60c$. We use Equations (2.23) to determine u_x, u_y, and u_z and finally the speed u.

Solution To find the speeds in the system K, we first need to find γ.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.600^2}} = 1.25$$

Next we are able to determine the components of $\vec{u}$.

$$u_x(\text{protons}) = \frac{0 + 0.600c}{[1 + (0.600c)(0c)/c^2]} = 0.600c$$

$$u_y(\text{protons}) = \frac{0.990c}{1.25[1 + (0.600c)(0c)/c^2]} = 0.792c$$

$$u_z(\text{protons}) = \frac{0}{1.25[1 + (0.600c)(0c)/c^2]} = 0$$

$$u(\text{protons}) = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{(0.600c)^2 + (0.792c)^2}$$
$$= 0.994c$$

We have again assumed we know the velocity components to three significant figures. Mary and her junior officers only observe the protons moving perpendicular to their motion. However, because there are both u_x and u_y components, Frank (on the space station) sees the protons moving at an angle with respect to both his x and his y directions.

EXAMPLE 2.5

By the early 1800s experiments had shown that light slows down when passing through liquids. A. J. Fresnel suggested in 1818 that there would be a partial drag on light by the medium through which the light was passing. Fresnel's suggestion explained the problem of stellar aberration if the Earth was at rest in the ether. In a famous experiment in 1851, H. L. Fizeau measured the "ether" drag coefficient for light passing in opposite directions through flowing water. Let a moving system K′ be at rest in the flowing water and let v be the speed of the flowing water with respect to a fixed observer in K (see Figure 2.16). The speed of light in the water at rest (that is, in system K′) is u', and the speed of light as measured in K is u. If the index of refraction of the water is n, Fizeau found experimentally that

$$u = u' + \left(1 - \frac{1}{n^2}\right)v$$

which was in agreement with Fresnel's prediction. This result was considered an affirmation of the ether concept. The factor $1 - 1/n^2$ became known as *Fresnel's drag coefficient*. Show that this result can be explained using relativistic velocity addition *without the ether concept*.

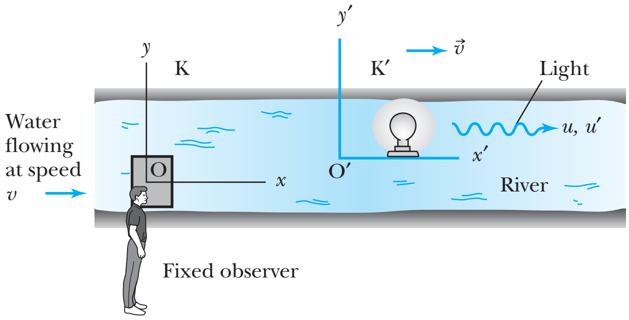


Figure 2.16 A stationary system K is fixed on shore, and a moving system K′ floats down the river at speed v. Light emanating from a source under water in system K′ has speed u, u' in systems K, K′, respectively.

Strategy We note from introductory physics that the velocity of light in a medium of index of refraction n is $u' = c/n$. We use Equation (2.23a) to solve for u.

Solution We have to calculate the speed only in the x-direction, so we dispense with the subscripts. We utilize Equation (2.23a) to determine

$$u = \frac{u' + v}{1 + u'v/c^2} = \frac{c/n + v}{1 + v/nc} = \frac{c}{n}\frac{\left(1 + \dfrac{nv}{c}\right)}{\left(1 + \dfrac{v}{nc}\right)}$$

Because $v \ll c$ in this case, we can expand the denominator $(1 + x)^{-1} = 1 - x + \cdots$ keeping only the lowest term in $x = v/c$. The above equation becomes

$$u = \frac{c}{n}\left(1 + \frac{nv}{c}\right)\left(1 - \frac{v}{nc} + \cdots\right)$$
$$= \frac{c}{n}\left(1 + \frac{nv}{c} - \frac{v}{nc} + \cdots\right)$$
$$= \frac{c}{n} + v - \frac{v}{n^2} = u' + \left(1 - \frac{1}{n^2}\right)v$$

which is in agreement with Fizeau's experimental result and Fresnel's prediction given earlier. This relativistic calculation is another stunning success of the special theory of relativity. There is no need to consider the existence of the ether.

2.7 Experimental Verification

We have used the special theory of relativity to describe some unusual phenomena. The special theory has also been used to make some startling predictions concerning length contraction, time dilation, and velocity addition. In this section we discuss only a few of the many experiments that have been done to confirm the special theory of relativity.

Muon Decay

When high-energy particles called *cosmic rays* enter the Earth's atmosphere from outer space, they interact with particles in the upper atmosphere (see Figure 2.17), creating additional particles in a *cosmic shower*. Many of the particles in the shower are π-*mesons* (pions), which decay into other unstable particles called *muons*. The properties of muons are described later when we discuss nuclear and particle physics. Because muons are unstable, they decay according to the radioactive decay law

Radioactive decay law

$$N = N_0 \exp\left(-\frac{(\ln 2)t}{t_{1/2}}\right) = N_0 \exp\left(-\frac{0.693t}{t_{1/2}}\right)$$

where N_0 and N are the number of muons at times $t = 0$ and $t = t$, respectively, and $t_{1/2}$ is the half-life of the muons. This means that in the time period $t_{1/2}$ half of the muons will decay to other particles. The half-life of muons (1.52×10^{-6} s) is long enough that many of them survive the trip through the atmosphere to the Earth's surface.

We perform an experiment by placing a muon detector on top of a mountain 2000 m high and counting the number of muons traveling at a speed near $v = 0.98c$ (see Figure 2.18a). Suppose we count 10^3 muons during a given time period t_0. We then move our muon detector to sea level (see Figure 2.18b), and we determine experimentally that approximately 540 muons survive the trip without decaying. We ignore any other interactions that may remove muons.

Classically, muons traveling at a speed of $0.98c$ cover the 2000-m path in 6.8×10^{-6} s, and according to the radioactive decay law, only 45 muons should survive the trip. There is obviously something wrong with the classical calculation, because we counted a factor of 12 more muons surviving than the classical calculation predicts.

Photo courtesy of NASA.

Figure 2.17 Much of what we know about muons in cosmic rays was learned from balloon flights carrying sophisticated detectors. This balloon is being prepared for launch in NASA's Ultra Long Duration Balloon program for a mission that may last up to 100 days. The payload will hang many meters below the balloon. Victor Hess began the first such balloon flights in 1912 (when he discovered cosmic rays), and much improved versions are still launched today from all over the world to study cosmic rays, the atmosphere, the sun, and the universe.

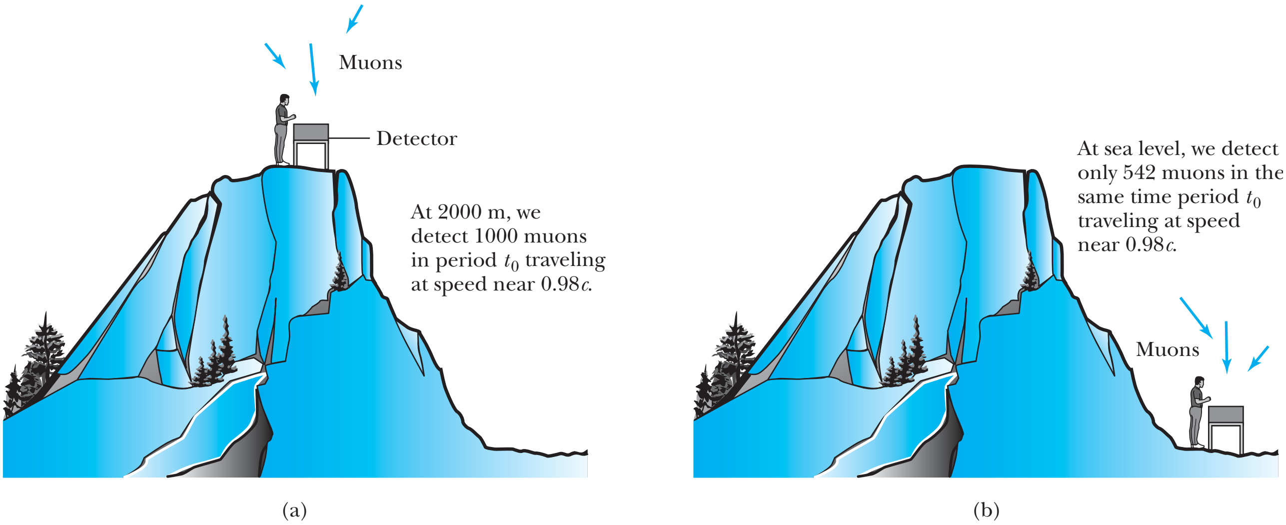


Figure 2.18 The number of muons detected with speeds near 0.98c is much different (a) on top of a mountain than (b) at sea level, because of the muon's decay. The experimental result agrees with our time dilation equation.

Because the classical calculation does not agree with the experimental result, we should consider a relativistic calculation. The muons are moving at a speed of 0.98c with respect to us on Earth, so the effects of time dilation will be dramatic. In the muon rest frame, the time period for the muons to travel 2000 m (on a clock fixed with respect to the mountain) is calculated from Equation (2.19) to be $(6.8/5.0) \times 10^{-6}$ s, because $\gamma = 5.0$ for $v = 0.98c$. For the time $t = 1.36 \times 10^{-6}$ s, the radioactive decay law predicts that 538 muons will survive the trip, in agreement with the observations. An experiment similar to this was performed by B. Rossi and D. B. Hall* in 1941 on the top of Mount Washington in New Hampshire.

It is useful to examine the muon decay problem from the perspective of an observer traveling with the muon. This observer would not measure the distance from the top of the 2000-m mountain to sea level to be 2000 m. Rather, this observer would say that the distance is contracted and is only $(2000\text{ m})/5.0 = 400$ m. The time to travel the 400-m distance would be $(400\text{ m})/0.98c = 1.36 \times 10^{-6}$ s according to a clock at rest with a muon. Using the radioactive decay law, an observer traveling with the muons would still predict 538 muons to survive. Therefore, we obtain the identical result whether we consider time dilation or space contraction, and both are in agreement with the experiment, thus confirming the special theory of relativity.

Atomic Clock Measurement

In an atomic clock, an extremely accurate measurement of time is made using a well-defined transition in the ^{133}Cs atom ($f = 9{,}192{,}631{,}770$ Hz). In 1971 two American physicists, J. C. Hafele and Richard E. Keating (Figure 2.19), used four

*B. Rossi and D. B. Hall, *Physical Review* **50**, 223 (1941). An excellent, though now dated, film recreating this experiment (*Time Dilation—An Experiment with μ-mesons* by D. H. Frisch and J. H. Smith) is available from the Education Development Center, Newton, Mass. See also D. H. Frisch and J. H. Smith, *American Journal of Physics* **31**, 342 (1963).

Figure 2.19 Joseph Hafele and Richard Keating are shown unloading one of their atomic clocks and the associated electronics from an airplane in Tel Aviv, Israel, during a stopover in November 1971 on their round-the-world trip to test special relativity.

cesium beam atomic clocks to test the time dilation effect. They flew the four portable cesium clocks eastward and westward on regularly scheduled commercial jet airplanes around the world and compared the time with a reference atomic time scale at rest at the U.S. Naval Observatory in Washington, D.C. (Figure 2.20).

The trip eastward took 65.4 hours with 41.2 flight hours, whereas the westward trip, taken a week later, took 80.3 hours with 48.6 flight hours. The comparison with the special theory of relativity is complicated by the rotation of the Earth and by a gravitational effect arising from the general theory of relativity. The actual relativistic predictions and experimental observations for the time differences* are

Travel	Predicted	Observed
Eastward	-40 ± 23 ns	-59 ± 10 ns
Westward	275 ± 21 ns	273 ± 7 ns

A negative time indicates that the time on the moving clock is less than the reference clock. The moving clocks lost time (ran slower) during the eastward trip, but gained time (ran faster) during the westward trip. This occurs because of the rotation of the Earth, indicating that the flying clocks ticked faster or slower than the reference clocks on Earth. The special theory of relativity is verified within the experimental uncertainties.

*See J. C. Hafele and R. E. Keating, *Science* **177**, 166–170 (1972).

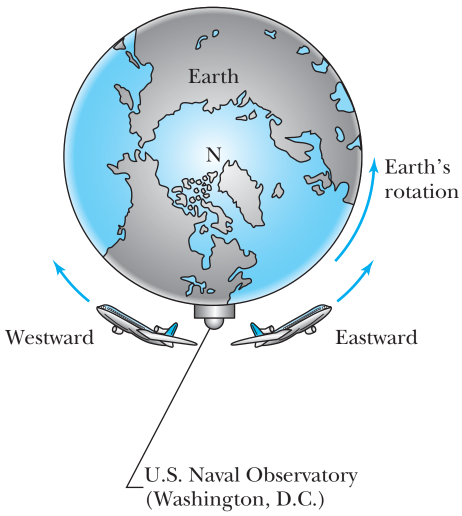


Figure 2.20 Two airplanes took off (at different times) from Washington, D.C., where the U.S. Naval Observatory is located. The airplanes traveled east and west around Earth as it rotated. Atomic clocks on the airplanes were compared with similar clocks kept at the observatory to show that the moving clocks in the airplanes ran slower.

EXAMPLE 2.6

In 1985 the space shuttle *Challenger* flew a cesium clock and compared its time with a fixed clock left on Earth. The shuttle orbited at approximately 330 km above Earth with a speed of 7712 m/s (~17,250 mph). (a) Calculate the expected time lost per second for the moving clock and compare with the measured result of -295.02 ± 0.29 ps/s, which includes a predicted effect due to general relativity of 35.0 ± 0.06 ps/s. (b) How much time would the clock lose due to special relativity alone during the entire shuttle flight that lasted for 7 days?

Strategy This should be a straightforward application of the time dilation effect, but we have the complicating fact that the space shuttle is moving in a noninertial system (orbiting around Earth). We don't want to consider this now, so we make the simplifying assumption that the space shuttle travels in a straight line with respect to Earth and the two events in the calculations are the shuttle passing the starting point (launch) and the ending point (landing). We are not including the effects of general relativity.

We know the orbital speed of the shuttle with respect to Earth, which allows us to determine β and the relativistic factor γ. We let T be the time measured by the clock fixed on Earth. Then we can use the time dilation effect given by Equation (2.19) to determine the proper time T_0' measured by the clock in the space shuttle. The time difference is $\Delta T = T - T_0'$. We have $T_0' = T\sqrt{1-\beta^2}$ and $\Delta T = T - T_0' = T(1 - \sqrt{1-\beta^2})$. For part (b) we need to find the total time lost for the moving clock for 7 days.

Solution (a) We have $\beta = v/c = (7712 \text{ m/s})/(2.998 \times 10^8 \text{ m/s}) = 2.572 \times 10^{-5}$. Because β is such a small quantity, we can use a power series expansion of the square root $\sqrt{1-\beta^2}$, keeping only the lowest term in β^2 for ΔT.

$$\Delta T = T\left[1 - \left(1 - \frac{\beta^2}{2} + \cdots\right)\right] = \frac{\beta^2 T}{2}$$

Now we have

$$\frac{\Delta T}{T} = \frac{\beta^2}{2} = \frac{1}{2}(2.572 \times 10^{-5})^2 = 330.76 \times 10^{-12}$$

In this case ΔT is positive, which indicates that the space shuttle clock lost this fraction of time, so the moving clock lost 330.76 ps for each second of motion.

How does this compare with the measured time? The total measured result was a loss of 295.02 ± 0.29 ps/s, but we must add the general relativity prediction of 35.0 ± 0.06 ps/s to the measured value to obtain the result due only to special relativity. So the measured special relativity result is close to 330.02 ps/s, which differs from our calculated result by only 0.2%!

(b) The total time of the seven-day mission was 6.05×10^5 s, so the total time difference between clocks is $(330.76 \times 10^{-12})(6.05 \times 10^5 \text{ s}) = 0.2$ ms, which is easily detected by cesium clocks.

Velocity Addition

An interesting test of the velocity addition relations was made by T. Alväger and colleagues* at the CERN nuclear and particle physics research facility on the border of Switzerland and France. They used a beam of almost 20-GeV (20×10^9 eV) protons to strike a target to produce neutral pions (π^0) having energies of more than 6 GeV. The π^0 ($\beta \approx 0.99975$) have a very short half-life and soon decay into two γ rays. In the rest frame of the π^0 the two γ rays go off in opposite directions. The experimenters measured the velocity of the γ rays going in the forward direction in the laboratory (actually 6°, but we will assume 0° for purposes of calculation because there is little difference). The Galilean addition of velocities would require the velocity of the γ rays to be $u = 0.99975c + c = 1.99975c$, because the velocity of γ rays is already c. However, the relativistic velocity addition, in which

Pion decay experiment

*See T. Alväger, F. J. M. Farley, J. Kjellman, and I. Wallin, *Physics Letters* **12**, 260 (1964). See also article by J. M. Bailey, *Arkiv Fysik* **31**, 145 (1966).

$v = 0.99975c$ is the velocity of the π^0 rest frame with respect to the laboratory and $u' = c$ is the velocity of the γ rays in the rest frame of the π^0, predicts the velocity u of the γ rays measured in the laboratory to be, according to Equation (2.23a),

$$u = \frac{c + 0.99975c}{1 + \frac{(0.99975c)(c)}{c^2}} = c$$

The experimental measurement was accomplished by measuring the time taken for the γ rays to travel between two detectors placed about 30 m apart and was in excellent agreement with the relativistic prediction, but not the Galilean one. We again have conclusive evidence of the need for the special theory of relativity.

Testing Lorentz Symmetry

Although we have mentioned only three rather interesting experiments, physicists performing experiments with nuclear and particle accelerators have examined thousands of cases that verify the correctness of the concepts discussed here. Quantum electrodynamics (QED) includes special relativity in its framework, and QED has been tested to one part in 10^{12}.

Lorentz symmetry requires the laws of physics to be the same for all observers, and Lorentz symmetry is important at the very foundation of our description of fundamental particles and forces. Lorentz symmetry, together with the principles of quantum mechanics that are discussed in much of the remainder of this book, form the framework of relativistic quantum field theory. Many interactions that could be added to our best theories of physics (see the Standard Model in Chapter 14) are excluded, because they would violate Lorentz symmetry. In just the past two decades, physicists have conceived and performed many experiments that test Lorentz symmetry, but no violations have been discovered to date. For example, tests done with electrons have shown no violations to one part in 10^{32}, with neutrons one part in 10^{31}, and with protons one part in 10^{27}. These are phenomenal numbers, but many more experiments are currently underway, and more are planned. Several physicists have proposed in recent years that some theories of quantum gravity imply that Lorentz symmetry is not valid. They suggest a violation may occur at very small distances around 10^{-35} m. Direct investigation at these small distances is not now possible, because the energy required is huge (10^{28} eV), but such effects may be observed in highly energetic events in outer space. To date, no verified experiments have found a violation of Lorentz symmetry, but interest remains high.*

2.8 Twin Paradox

One of the most interesting topics in relativity is the twin (or clock) paradox. Almost from the time of publication of Einstein's famous paper in 1905, this subject has received considerable attention, and many variations exist. Let's summarize the paradox. Suppose twins, Mary and Frank, choose different career paths. Mary (the **M**oving twin) becomes an astronaut and Frank (the **F**ixed twin) a stockbroker. At age 30, Mary sets out on a spaceship to study a star system 8 ly from Earth. Mary travels at very high speeds to reach the star and returns during her life span.

---

*See "Lorentz Invariance on Trial," Maxim Pospelov and Michael Romalis, *Physics Today* (July 2004) p. 40. See also *Scientific American* (September 2004) Special Issue on "Beyond Einstein."

According to Frank's understanding of special relativity, Mary's biological clock ticks more slowly than his own, so he claims that Mary will return from her trip younger than he. The paradox is that Mary similarly claims that it is Frank who is moving rapidly with respect to her, so that when she returns, Frank will be the younger. To complicate the paradox further one could argue that because nature cannot allow both possibilities, it must be true that symmetry prevails and that the twins will still be the same age. Which is the correct solution?

Who is the younger twin?

The correct answer is that Mary returns from her space journey as the younger twin. According to Frank, Mary's spaceship takes off from Earth and quickly reaches its travel speed of $0.8c$. She travels the distance of 8 ly to the star system, slows down and turns around quickly, and returns to Earth at the same speed. The accelerations (positive and negative) take negligible times compared to the travel times between Earth and the star system. According to Frank, Mary's travel time to the star is 10 years $[(8 \text{ ly})/0.8c = 10 \text{ y}]$ and the return is also 10 years, for a total travel time of 20 years, so that Frank will be $30 + 10 + 10 \text{ y} = 50$ years old when Mary returns. However, because Mary's clock is ticking more slowly, her travel time to the star is only $10\sqrt{1 - 0.8^2} \text{ y} = 6$ years. Frank calculates that Mary will only be $30 + 6 + 6 \text{ y} = 42$ years old when she returns with respect to his own clock at rest.

The important fact here is that Frank's clock is in an inertial system* during the entire trip; however, Mary's clock is not. As long as Mary is traveling at constant speed away from Frank, both of them can argue that the other twin is aging less rapidly. However, when Mary slows down to turn around, she leaves her original inertial system and eventually returns in a completely different inertial system. Mary's claim is no longer valid, because she does not remain in the same inertial system. There is also no doubt as to who is in the inertial system. Frank feels no acceleration during Mary's entire trip, but Mary will definitely feel acceleration during her reversal time, just as we do when we step hard on the brakes of a car. The acceleration at the beginning and the deceleration at the end of her trip present little problem, because the fixed and moving clocks could be compared if Mary were just passing by Frank each way. It is Mary's acceleration at the star system that is the key. If we invoke the two postulates of special relativity, there is no paradox. The instantaneous rate of Mary's clock is determined by her instantaneous speed, but she must account for the acceleration effect when she turns around. A careful analysis of Mary's entire trip using special relativity, including acceleration, will be in agreement with Frank's assessment that Mary is younger. Mary returns to Earth rich as well as famous, because her stockbroker brother has invested her salary wisely during the 20-year period (for which she only worked 12 years!).

Mary is both younger and rich

We follow A. P. French's excellent book, *Special Relativity,* to present Table 2.1 (page 48), which analyzes the twin paradox. Both Mary and Frank send out signals at a frequency f (as measured by their own clock). We include in the table the various journey timemarks and signals received during the trip, with one column for the twin Frank who stayed at home and one for the astronaut twin Mary who went on the trip. Let the total time of the trip as measured on Earth be T. The speed of Mary's spaceship is v (as measured on Earth), which gives a relativistic factor γ. The distance Mary's spaceship goes before turning around (as measured on Earth) is L. Much of this table is best analyzed by using spacetime (see the next section) and the Doppler effect (see Section 2.10).

*The rotating and orbiting Earth is only an approximate inertial system.

Table 2.1 Twin Paradox Analysis

Item	Measured by Frank (remains on Earth)	Measured by Mary (traveling astronaut)
Time of total trip	$T = 2L/v$	$T' = 2L/\gamma v$
Total number of signals sent	$fT = 2fL/v$	$fT' = 2fL/\gamma v$
Frequency of signals received at beginning of trip f'	$f\sqrt{\frac{1-\beta}{1+\beta}}$	$f\sqrt{\frac{1-\beta}{1+\beta}}$
Time of detecting Mary's turnaround	$t_1 = L/v + L/c$	$t_1' = L/\gamma v$
Number of signals received at the rate f'	$f't_1 = \frac{fL}{v}\sqrt{1-\beta^2}$	$f't_1' = \frac{fL}{v}(1-\beta)$
Time for remainder of trip	$t_2 = L/v - L/c$	$t_2' = L/\gamma v$
Frequency of signals received at end of trip f''	$f\sqrt{\frac{1+\beta}{1-\beta}}$	$f\sqrt{\frac{1+\beta}{1-\beta}}$
Number of signals received at rate f''	$f''t_2 = \frac{fL}{v}\sqrt{1-\beta^2}$	$f''t_2' = \frac{fL}{v}(1+\beta)$
Total number of signals received	$2fL/\gamma v$	$2fL/v$
Conclusion as to other twin's measure of time taken	$T' = 2L/\gamma v$	$T = 2L/v$

After A. French, *Special Relativity,* New York: Norton (1968), p. 158.

2.9 Spacetime

When describing events in relativity, it is sometimes convenient to represent events on a **spacetime** diagram as shown in Figure 2.21. For convenience we use only one spatial coordinate x and specify position in this one dimension. We use ct instead of time so that both coordinates will have dimensions of length. Spacetime diagrams were first used by H. Minkowski in 1908 and are often called **Minkowski diagrams**. We have learned in relativity that we must denote both space and time to specify an event. This is the origin of the term *fourth dimension* for time. The events for A and B in Figure 2.21 are denoted by the respective coordinates (x_A, ct_A) and (x_B, ct_B), respectively. The line connecting events A and B is the path from A to B and is called a **worldline**. A spaceship launched from $x = 0$, $ct = 0$ with constant velocity v has the worldline shown in Figure 2.22: a straight line with slope c/v. For example, a light signal sent out from the origin with speed c is represented on a spacetime graph with a worldline that has a slope $c/c = 1$, so that line makes an angle of 45° with both the x and ct axes. Any real motion in the spacetime diagram cannot have a slope of less than 1 (angle with the x axis $< 45°$), because that motion would have a speed greater than c. The Lorentz transformation does not allow such a speed.

Spacetime (Minkowski) diagrams

Worldline

Let us consider two events that occur at the same time ($ct = 0$) but at different positions, x_1 and x_2. We denote the events (x, ct) as $(x_1, 0)$ and $(x_2, 0)$, and we show them in Figure 2.23 in an inertial system with an origin fixed at $x = 0$ and $ct = 0$. How can we be certain that the two events happen simultaneously if they occur at different positions? We must devise a method that will allow us to determine experimentally that the events occurred simultaneously. Let us place clocks at positions x_1 and x_2 and place a flashbulb at position x_3 halfway between x_1 and x_2. The two clocks have been previously synchronized and keep identical time. At time $t = 0$, the flashbulb explodes and sends out light signals from position x_3. The light signals proceed along their worldlines as shown in Figure 2.23. The two light signals arrive at positions x_1 and x_2 at identical times t as shown on the spacetime diagram. By using such techniques we can be sure that events occur simultaneously in our inertial reference system.

But what about other inertial reference systems? We realize that the two events will not be simultaneous in a reference system K′ moving at speed v with respect to our (x, ct) system. Because the two events have different spatial coordinates, x_1 and x_2, the Lorentz transformation will preclude them from occurring at the same time t' simultaneously in the moving coordinate systems. We can see this by supposing that events 1, 2, and 3 take place on a spaceship moving with velocity v. The worldlines for x_1 and x_2 are the two slanted lines beginning at x_1 and x_2 in Figure 2.24. However, when the flashbulb goes off, the light signals from x_3 still proceed at 45° in the (x, ct) reference system. The light signals intersect the worldlines from positions x_1 and x_2 at different times, so we do not see the events as being simultaneous in the moving system. Spacetime diagrams can be useful in showing such phenomena.

Anything that happened earlier in time than $t = 0$ is called the *past* and anything that occurs after $t = 0$ is called the *future*. The spacetime diagram in Figure 2.25a shows both the past and the future. Notice that only the events within the shaded area below $t = 0$ can affect the present. Events outside this area cannot affect the present because of the limitation $v \le c$; this region is called *elsewhere*. Similarly, the present cannot affect any events occurring outside the shaded area above $t = 0$, again because of the limitation of the speed of light.

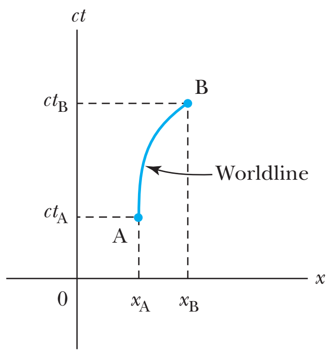


Figure 2.21 A spacetime diagram is used to specify events. The worldline denoting the path from event A to event B is shown.

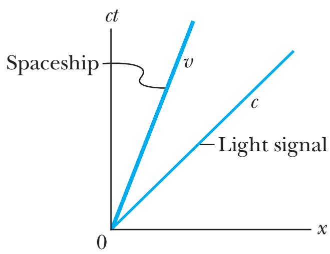


Figure 2.22 A light signal has the slope of 45° on a spacetime diagram. A spaceship moving along the x axis with speed v is a straight line on the spacetime diagram with a slope c/v.

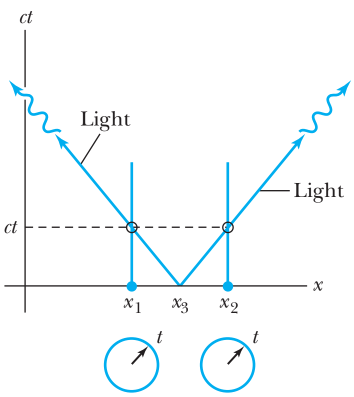


Figure 2.23 Clocks positioned at x_1 and x_2 can be synchronized by sending a light signal from a position x_3 halfway between. The light signals intercept the worldlines of x_1 and x_2 at the same time t.

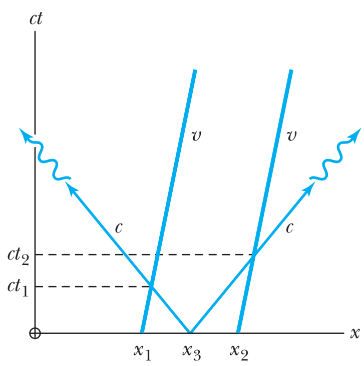


Figure 2.24 If the positions x_1 $(= x_1')$ and x_2 $(= x_2')$ of the previous figure are on a moving system K′ when the flashbulb goes off, the times will not appear simultaneously in system K, because the worldlines for x_1' and x_2' are slanted.

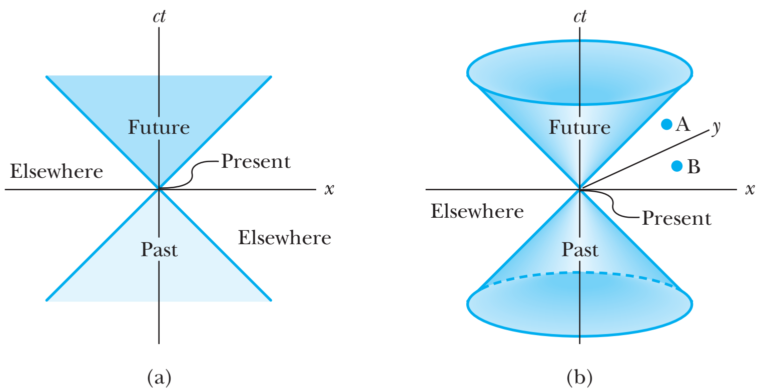


Figure 2.25 (a) The spacetime diagram can be used to show the past, present, and future. Only causal events are placed inside the shaded area. Events outside the shaded area below $t = 0$ cannot affect the present. (b) If we add an additional spatial coordinate y, a space cone can be drawn. The present cannot affect event A, but event B can.

Light cone

If we add another spatial coordinate y to our spacetime coordinates, we will have a cone as shown in Figure 2.25b, which we refer to as the **light cone**. All causal events related to the present ($x = 0$, $ct = 0$) must be within the light cone. In Figure 2.25b, anything occurring at present ($x = 0$, $ct = 0$) cannot possibly affect an event at position A; however, the event B can easily affect event A because A would be within the range of light signals emanating from B.

Invariant quantities

Invariant quantities have the same value in all inertial frames. They serve a special role in physics because their values do not change from one system to another. For example, the speed of light c is invariant. We are used to defining distances by $d^2 = x^2 + y^2 + z^2$, and in Euclidean geometry, we obtain the same result for d^2 in any inertial frame of reference. Is there a quantity, similar to d^2, that is also invariant in the special theory? If we refer to Equations (2.9), we have similar equations in both systems K and K′. Let us look more carefully at the quantity s^2 defined as

$$s^2 = x^2 - (ct)^2 \tag{2.25a}$$

and also

$$s'^2 = x'^2 - (ct')^2 \tag{2.25b}$$

If we use the Lorentz transformation for x and t, we find that $s^2 = s'^2$, so **s^2 is an invariant quantity.** This relationship can be extended to include the two other spatial coordinates, y and z, so that*

$$s^2 = x^2 + y^2 + z^2 - (ct)^2 \tag{2.26}$$

For simplicity, we will sometimes continue to use only the single spatial coordinate x.

If we consider two events, we can determine the quantity Δs^2 where

$$\Delta s^2 = \Delta x^2 - c^2\,\Delta t^2 \tag{2.27}$$

Spacetime interval

between the two events, and we find that it is invariant in any inertial frame. The quantity Δs is known as the **spacetime interval** between two events. There are three possibilities for the invariant quantity Δs^2.

Lightlike

1. **$\Delta s^2 = 0$:** In this case $\Delta x^2 = c^2\,\Delta t^2$, and the two events can be connected only by a light signal. The events are said to have a **lightlike** separation.
2. **$\Delta s^2 > 0$:** Here we must have $\Delta x^2 > c^2\,\Delta t^2$, and no signal can travel fast enough to connect the two events. The events are not causally connected

* Some authors use the negative of the expression here in Equation (2.26).

and are said to have a **spacelike** separation. In this case we can always find an inertial frame traveling at a velocity less than c in which the two events can occur simultaneously in time but at different places in space.

Spacelike

3. **$\Delta s^2 < 0$:** Here we have $\Delta x^2 < c^2\,\Delta t^2$, and the two events can be causally connected. The interval is said to be **timelike**. In this case we can find an inertial frame traveling at a velocity less than c in which the two events occur at the same position in space but at different times. The two events can never occur simultaneously.

Timelike

EXAMPLE 2.7

Draw the spacetime diagram for the motion of the twins discussed in Section 2.8. Draw light signals being emitted from each twin at annual intervals and count the number of light signals received by each twin from the other.

Strategy We shall let Mary leave Earth at the origin $(x, ct) = (0, 0)$. She will return to Earth at $x = 0$, but at a later time $ct = 20$ ly. Her worldlines will be described by two lines of slope $+c/v$ and $-c/v$, whereas Frank's worldline remains fixed at $x = 0$. Frank's and Mary's signals have slopes of ± 1 on the spacetime diagram. We pay close attention to when the light signals sent out by Frank and Mary reach their twin's worldlines.

Solution We show in Figure 2.26 (page 52) the spacetime diagram. The line representing Mary's trip has a slope $c/0.8c = 1.25$ on the outbound trip and -1.25 on the return trip. During the trip to the star system, Mary does not receive the second annual light signal from Frank until she reaches the star system. This occurs because the light signal takes considerable time to catch up with Mary. However, during the return trip Mary receives Frank's light signals at a rapid rate, receiving the last one (number 20) just as she returns. Because Mary's clock is running slow, we see the light signals being sent less often on the spacetime diagram in the fixed system. Mary sends out her sixth annual light signal when she arrives at the star system. However, this signal does not reach Frank until the 18th year! During the remaining two years, however, Frank receives Mary's signals at a rapid rate, finally receiving all 12 of them. Frank receives the last 6 signals during a time period of only 2 years.

A 3-vector $\vec{R}$ can be defined using Cartesian coordinates x, y, z in three-dimensional Euclidean space. Another 3-vector $\vec{R}'$ can be determined in another Cartesian coordinate system using x', y', z' in the new system. So far in introductory physics we have discussed translations and rotations of axes between these two systems. We have learned that there are two geometries in Newtonian spacetime. One is the three-dimensional Euclidean geometry in which the space interval is $d\ell^2 = dx^2 + dy^2 + dz^2$, and the other is a one-dimensional time interval dt. Minkowski pointed out that both space and time by themselves will not suffice under a Lorentz transformation, and only a union of both will be independent and useful.

We can form a four-dimensional space or four-vector using the four components x, y, z, ict. The equivalent of Equation (2.27) becomes

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 - c^2dt^2 \\ ds'^2 &= dx'^2 + dy'^2 + dz'^2 - c^2dt'^2 \\ ds^2 &= ds'^2 \end{aligned} \tag{2.28}$$

We previously noted that ds^2 (actually Δs^2) can be positive, negative, or zero. With the four-vector formalism we only have the *spacetime* geometry, not separate geometries for space and time. The spacetime distances $ds^2 = ds'^2$ are invariant

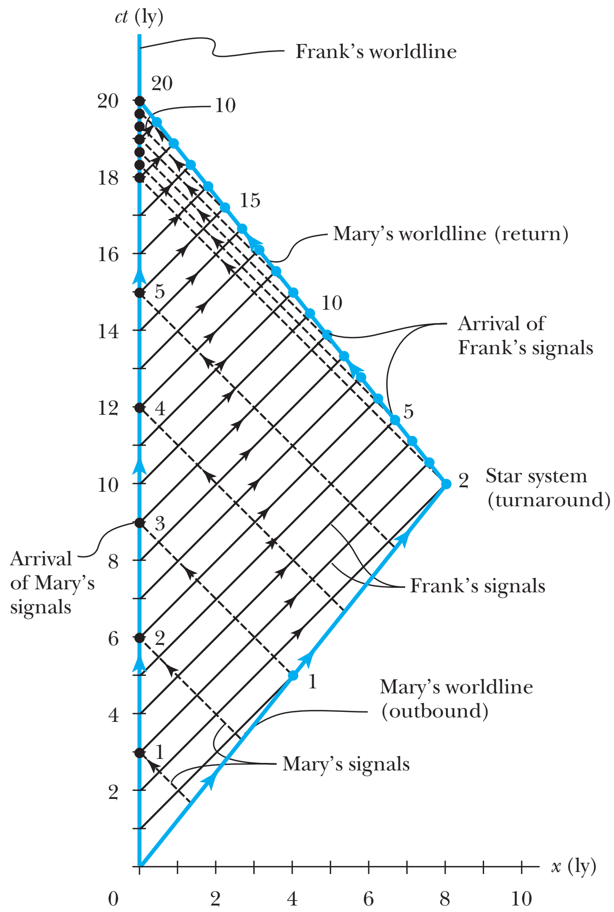


Figure 2.26 The spacetime diagram for Mary's trip to the star system and back. Notice that Frank's worldline is a vertical line at $x = 0$, and Mary's two worldlines have the correct slope given by the magnitude c/v. The black dashed lines represent light signals sent at annual intervals from Mary to Frank. Frank's annual signals to Mary are solid black. The solid dots denote the time when the light signals arrive.

under the Lorentz transformation. In Section 2.12 we will learn how the energy and momentum of a particle are connected. Similar to the spacetime four-vector, there is an energy-momentum four-vector, and the invariant quantity is the mass.

The four-vector formalism gives us equations that produce form-invariant quantities under appropriate Lorentz transformations. It allows the mathematical construction of relativistic physics to be somewhat easier. However, the penalty is that we would have to stop and learn matrix algebra and perhaps even about *tensors* and, eventually, *spinors*. At this point in our study there is little to be gained in understanding about relativity. Another disadvantage in utilizing four-vectors at this point is that there is no general agreement among authors as to terminology. Sometimes ict is term 0 of the four-vector (ict, x, y, z with x, y, z being terms 1, 2, 3), and sometimes it is described as term 4 (x, y, z, ict). Sometimes the formalism is arranged such that the imaginary number $i = \sqrt{-1}$ doesn't appear. We have chosen not to use four-vectors.

2.10 Doppler Effect

You may have already studied the Doppler effect of sound in introductory physics. It causes an increased frequency of sound as a source such as a train (with whistle blowing) approaches a receiver (our eardrum) and a decrease in frequency as the source recedes. A change in sound frequency also occurs when the source is fixed and the receiver is moving. The change in frequency of the sound wave depends on whether the source or receiver is moving. On first thought it seems that the Doppler effect in sound violates the principle of relativity, until we realize that there is in fact a special frame for sound waves. Sound waves depend on media such as air, water, or a steel plate to propagate. For light, however, there is no such medium. It is only relative motion of the source and receiver that is relevant, and we expect some differences between the relativistic Doppler effect for light waves and the normal Doppler effect for sound. It is not possible for a source of light to travel faster than light in a vacuum, but it is possible for a source of sound to travel faster than the speed of sound. Similarly, in a medium such as water in which light travels slower than c, a light source can travel faster than the speed of light.

Consider a source of light (for example, a star) and a receiver (an astronomer) approaching one another with a relative velocity v. First we consider the receiver fixed (Figure 2.27a) in system K and the light source in system K′ moving toward the receiver with velocity v. The source emits n waves during the time interval T. Because the speed of light is always c and the source is moving with velocity v, the total distance between the front and rear of the wave train emitted during the time interval T is

$$\text{Length of wave train} = cT - vT$$

Because there are n waves emitted during this time period, the wavelength must be

$$\lambda = \frac{cT - vT}{n}$$

and the frequency, $f = c/\lambda$, is

$$f = \frac{cn}{cT - vT} \tag{2.29}$$

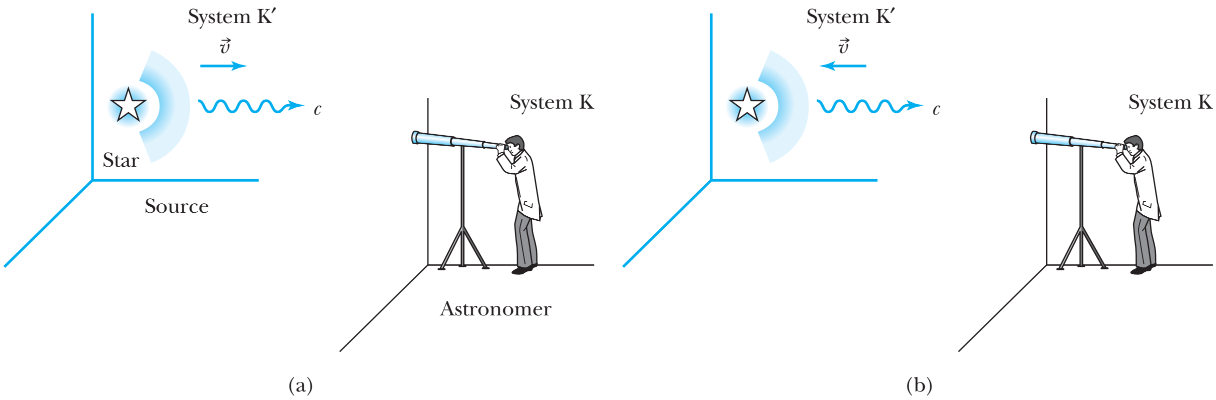


Figure 2.27 (a) The source (star) is approaching the receiver (astronomer) with velocity v while it emits starlight signals with speed c. (b) Here the source and receiver are receding with velocity v. The Doppler effect for light is different than that for sound, because of relativity and no medium to carry the light waves.

Special Topic

Applications of the Doppler Effect

The Doppler effect is not just a curious result of relativity. It has many practical applications, three of which are discussed here, and others are mentioned in various places in this text.

Astronomy

Perhaps the best-known application is in astronomy, where the Doppler shifts of known atomic transition frequencies determine the relative velocities of astronomical objects with respect to us. Such measurements continue to be used today to find the distances of such unusual objects as quasars (objects having incredibly large masses that produce tremendous amounts of radiation; see Chapter 16). The Doppler effect has been used to discover other effects in astronomy, for example, the rate of rotation of Venus and the fact that Venus rotates in the opposite direction of Earth—the sun rises in the west on Venus. This was determined by observing light reflected from both sides of Venus—on one side it is blueshifted and on the other side it is redshifted, as shown in Figure A. The same technique has been used to determine the rate of rotation of stars.

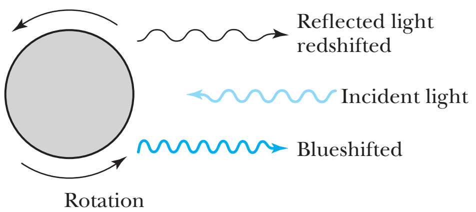


Figure A

Radar

The Doppler effect is nowhere more important than it is in radar. When an electromagnetic radar signal reflects off of a moving target, the so-called *echo* signal will be shifted in frequency by the Doppler effect. Very small frequency shifts can be determined by examining the beat frequency of the echo signal with a reference signal. The frequency shift is proportional to the radial component of the target's velocity. Navigation radar is quite complex, and ingenious techniques have been devised to determine the target position and velocity using multiple radar beams. By using pulsed Doppler radar it is possible to separate moving targets from stationary targets, called clutter.

Doppler radar is also extensively used in meteorology. Vertical motion of airdrafts, sizes and motion of raindrops, motion of thunderstorms, and detailed patterns of wind distribution have all been studied with Doppler radar.

X rays and gamma rays emitted from moving atoms and nuclei have their frequencies shifted by the Doppler effect. Such phenomena tend to broaden radiation frequencies emitted by stationary atoms and nuclei and add to the natural spectral widths observed.

In its rest frame, the source emits n waves of frequency f_0 during the proper time T_0'.

$$n = f_0 T_0' \tag{2.30}$$

The proper time interval T_0' measured on the clock at rest in the moving system is related to the time interval T measured on a clock fixed by the receiver in system K by

$$T_0' = \frac{T}{\gamma} \tag{2.31}$$

where γ is the relativistic factor of Equation (2.16). The clock moving with the source measures the proper time because it is present with both the beginning and end of the wave.

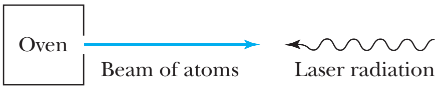


Figure B

Laser Cooling

In order to perform fundamental measurements in atomic physics, it is useful to limit the effects of thermal motion and to isolate single atoms. A method taking advantage of the Doppler effect can slow down even neutral atoms and eventually isolate them. Atoms emitted from a hot oven will have a spread of velocities. If these atoms form a beam as shown in Figure B, a laser beam impinging on the atoms from the right can slow them down by transferring momentum.

Atoms have characteristic energy levels that allow them to absorb and emit radiation of specific frequencies. Atoms moving with respect to the laser beam will "see" a shift in the laser frequency because of the Doppler effect. For example, atoms moving toward the laser beam will encounter light with high frequency, and atoms moving away from the laser beam will encounter light with low frequency. Even atoms moving in the same direction within the beam of atoms will see slightly different frequencies depending on the velocities of the various atoms. Now, if the frequency of the laser beam is tuned to the precise frequency seen by the faster atoms so that those atoms can be excited by absorbing the radiation, then those faster atoms will be slowed down by absorbing the momentum of the laser radiation. The slower atoms will "see" a laser beam that has been Doppler shifted to a lower frequency than is needed to absorb the radiation, and these atoms are not as likely to absorb the laser radiation. The net effect is that the atoms as a whole are *slowed down* and their *velocity spread is reduced*.

As the atoms slow down, they see that the Doppler-shifted frequencies of the laser change, and the atoms no longer absorb the laser radiation. They continue with the same lower velocity and velocity spread. The lower temperature limits reached by Doppler cooling depend on the atom, but typical values are on the order of hundreds of microkelvins. Doppler cooling is normally accompanied by intersecting laser beams at different angles; an "optical molasses" can be created in which atoms are essentially trapped. Further cooling is obtained by other techniques including "Sisyphus" and evaporative cooling, among others. In a remarkable series of experiments by various researchers, atoms have been cooled to temperatures approaching 10^{-10} K. The 1997 Nobel Prize in Physics was awarded to Steven Chu, Claude Cohen-Tannoudji, and William Phillips for these techniques. An important use of laser cooling is for atomic clocks. See http://www.nist.gov/physlab/div847/grp50/primary-frequency-standards.cfm for a good discussion. See also Steven Chu, "Laser Trapping of Neutral Particles," *Scientific American* **266**, 70 (February 1992). In Chapter 9 we will discuss how laser cooling is used to produce an ultracold state of matter known as a Bose-Einstein condensate.

We substitute the proper time T_0' from Equation (2.31) into Equation (2.30) to determine the number of waves n. Then n is substituted into Equation (2.29) to determine the frequency.

$$f = \frac{cf_0T/\gamma}{cT - vT}$$

$$= \frac{1}{1 - v/c}\frac{f_0}{\gamma} = \frac{\sqrt{1 - v^2/c^2}}{1 - v/c}f_0$$

where we have inserted the equation for γ. If we use $\beta = v/c$, we can write the previous equation as

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0 \quad \text{Source and receiver approaching} \tag{2.32}$$

It is straightforward to show that Equation (2.32) is also valid when the source is fixed and the receiver approaches it with velocity v. It is the relative velocity v, of course, that is important (Problem 49).

But what happens if the source and receiver are receding from each other with velocity v (see Figure 2.27b)? The derivation is similar to the one just done, except that the distance between the beginning and end of the wave train becomes

$$\text{Length of wave train} = cT + vT$$

because the source and receiver are receding rather than approaching. This change in sign is propagated throughout the derivation (Problem 50), with the final result

$$f = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_0 \quad \text{Source and receiver receding} \tag{2.33}$$

Equations (2.32) and (2.33) can be combined into one equation if we agree to use a + sign for β $(+v/c)$ when the source and receiver are approaching each other and a − sign for β $(-v/c)$ when they are receding. The final equation becomes

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0 \quad \text{Relativistic Doppler effect} \tag{2.34}$$

The Doppler effect is useful in many areas of science including astronomy, atomic physics, and nuclear physics. One of its many applications includes an effective radar system for locating airplane position and speed (see Special Topic, “Applications of the Doppler Effect”).

Elements absorb and emit characteristic frequencies of light due to the existence of particular atomic levels. We will learn more about this later. Scientists have observed these characteristic frequencies in starlight and have observed shifts in the frequencies. One reason for these shifts is the Doppler effect, and the frequency changes are used to determine the speed of the emitting object with respect to us.

Redshifts

This is the source of the **redshifts** of starlight caused by objects moving away from us. These data have been used to ascertain that the universe is expanding. The farther away the star, the higher the redshift. This observation is what led Harlow Shapley and Edwin Hubble to the idea that the universe started with a Big Bang.*

So far in this section we have only considered the source and receiver to be directly approaching or receding. Of course, it is also possible for the two to be moving at an angle with respect to one another, as shown in Figure 2.28. We omit the derivation here† but present the results. The angles θ and θ' are the angles the light signals make with the x axes in the K and K′ systems. They are related by

*Excellent references are “The Cosmic Distance Scale” by Paul Hodge, *American Scientist* **72**, 474 (1984), and “Origins” by S. Weinberg, *Science* **230**, 15 (1985). This subject is discussed in Chapter 16.

†See Robert Resnick, *Introduction to Special Relativity*, New York: Wiley (1968).

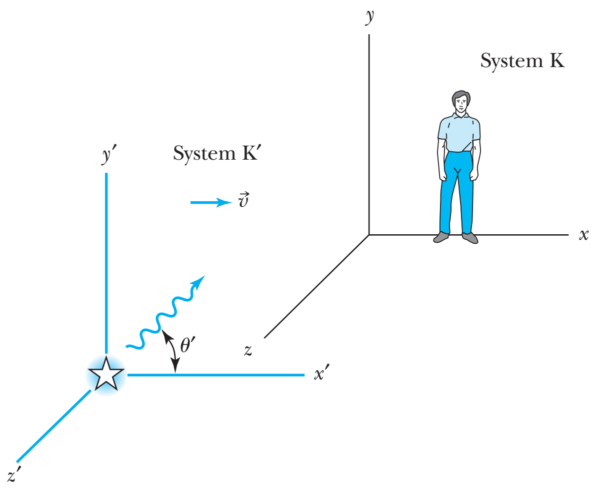


Figure 2.28 The light signals in system K′ are emitted at an angle θ' from the x' axis and remain in the $x'y'$ plane.

$$f \cos\theta = \frac{f_0(\cos\theta' + \beta)}{\sqrt{1-\beta^2}} \tag{2.35}$$

and

$$f \sin\theta = f_0 \sin\theta' \tag{2.36}$$

The generalized Doppler shift equation becomes

$$f = \frac{1 + \beta\cos\theta'}{\sqrt{1-\beta^2}} f_0 \tag{2.37}$$

Note that Equation (2.37) gives Equation (2.32) when $\theta' = 0$ (source and receiver approaching) and gives Equation (2.33) when $\theta' = 180°$ (source and receiver receding). This situation is known as the *longitudinal Doppler effect.*

When $\theta = 90°$ the emission is purely transverse to the direction of motion, and we have the *transverse Doppler effect*, which is purely a relativistic effect that does not occur classically. The transverse Doppler effect is directly due to time dilation and has been verified experimentally. Equations (2.35) through (2.37) can also be used to understand stellar aberration.

EXAMPLE 2.8

In Section 2.8 we discussed what happened when Mary traveled on a spaceship away from her twin brother Frank, who remained on Earth. Analyze the light signals sent out by Frank and Mary by using the relativistic Doppler effect.

Strategy We will use Equation (2.34) for both the outbound and return trip to analyze the frequency of the light signals sent and received. During the outbound trip the source (Frank) and receiver (Mary) are receding so that $\beta = -0.8$. For the return trip, we have $\beta = +0.8$. The frequency f_0 will be the signals that Frank sends; the frequency f will be those that Mary receives.

Solution First, we analyze the frequency of the light signals that Mary receives from Frank. Equation (2.34) gives

$$f = \frac{\sqrt{1 + (-0.8)}}{\sqrt{1 - (-0.8)}} f_0 = \frac{f_0}{3}$$

Because Frank sends out signals annually, Mary will receive the signals only every 3 years. Therefore during the 6-year trip in Mary's system to the star system, she will receive only 2 signals.

During the return trip, $\beta = 0.8$ and Equation (2.34) gives

$$f = \frac{\sqrt{1 + 0.8}}{\sqrt{1 - 0.8}} f_0 = 3f_0$$

so that Mary receives 3 signals each year for a total of 18 signals during the return trip. Mary receives a total of 20 annual light signals from Frank, and she concludes that Frank has aged 20 years during her trip.

Now let's analyze the light signals that Mary sends Frank. During the outbound trip the frequency at which Frank receives signals from Mary will also be $f_0/3$. During the 10 years that it takes Mary to reach the star system on his clock, he will receive 10/3 signals—3 signals plus 1/3 of the time to the next one. Frank continues to receive Mary's signals at the rate $f_0/3$ for another 8 years, because that is how long it takes the sixth signal she sent him to reach Earth. Therefore, for the first 18 years of her journey, according to his own clock he receives $18/3 = 6$ signals. Frank has no way of knowing that Mary has turned around and is coming back until he starts receiving signals at frequency $3f_0$. During Mary's return trip Frank will receive signals at the frequency $3f_0$ or 3 per year. However, in his system, Mary returns 2 years after he has received her sixth signal and turned around to come back. During this 2-year period he will receive 6 more signals, so he concludes she has aged a total of only 12 years.

Notice that this analysis is in total agreement with the spacetime diagram of Figure 2.26 and is somewhat easier to obtain. Although geometrical constructions like spacetime diagrams are sometimes useful, an analytical calculation is usually easier.

2.11 Relativistic Momentum

Newton's second law, $\vec{F} = d\vec{p}/dt$, keeps its same form under a Galilean transformation, but we might not expect it to do so under a Lorentz transformation. There may be similar transformation difficulties with the conservation laws of linear momentum and energy. We need to take a careful look at our previous definition of linear momentum to see whether it is still valid at high speeds. According to Newton's second law, for example, an acceleration of a particle already moving at very high speeds could lead to a speed greater than the speed of light. That would be in conflict with the Lorentz transformation, so we expect that Newton's second law might somehow be modified at high speeds.

Because physicists believe the conservation of linear momentum is fundamental, we begin by considering a collision that has no external forces. Frank (**F**ixed or stationary system) is at rest in system K holding a ball of mass m. Mary (**M**oving system) holds a similar ball in system K′ that is moving in the x direction with velocity v with respect to system K as shown in Figure 2.29a. Frank throws his ball along his y axis, and Mary throws her ball with exactly the same speed along her negative y' axis. The two balls collide in a perfectly elastic collision, and each of them catches their own ball as it rebounds. Each twin measures the speed of his or her own ball to be u_0 both before and after the collision.

We show the collision according to both observers in Figure 2.29. Consider the conservation of momentum according to Frank as seen in system K. The velocity of the ball thrown by Frank has components in his own system K of

$$\begin{aligned} u_{Fx} &= 0 \\ u_{Fy} &= u_0 \end{aligned} \tag{2.38}$$

If we use the definition of momentum, $\vec{p} = m\vec{v}$, the momentum of the ball thrown by Frank is entirely in the y direction:

$$p_{Fy} = mu_0 \tag{2.39}$$

Because the collision is perfectly elastic, the ball returns to Frank with speed u_0 along the $-y$ axis. The change of momentum of his ball as observed by Frank in system K is

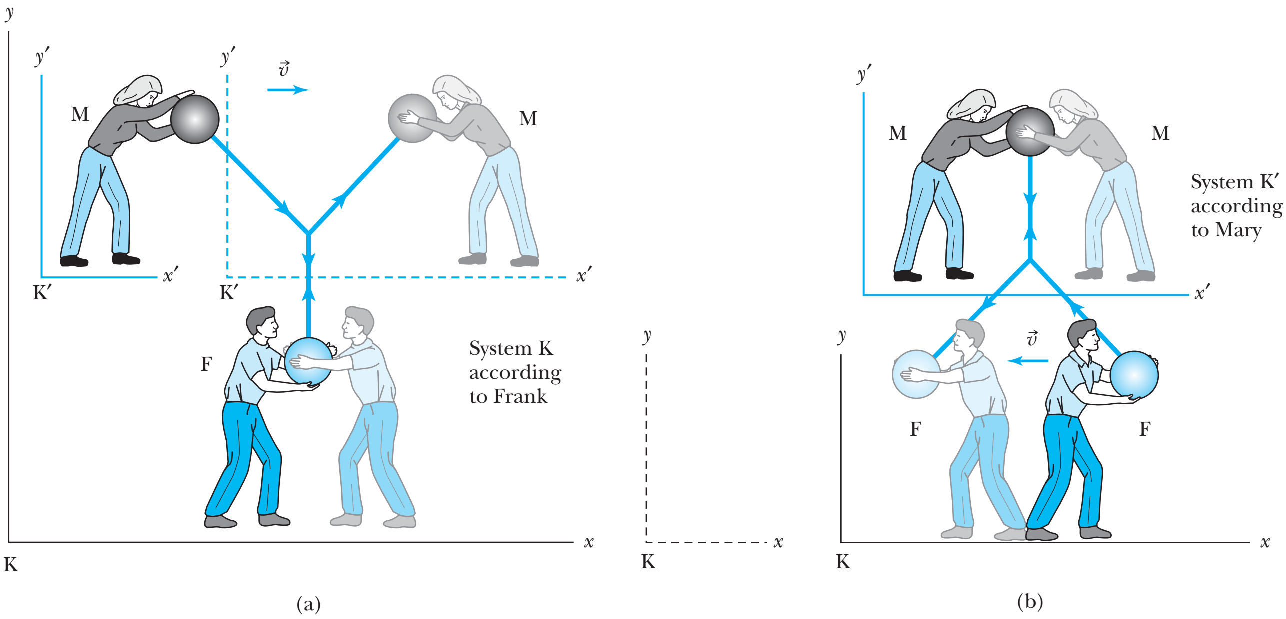


Figure 2.29 Frank is in the fixed K system, and Mary is in the moving K′ system. Frank throws his ball along his $+y$ axis, and Mary throws her ball along her $-y'$ axis. The balls collide. The event is shown in Frank's system in (a) and in Mary's system in (b). (Because it is awkward to show the twins as they catch the ball, we have drawn them faintly and in a reversed position.)

$$\Delta p_F = \Delta p_{Fy} = -2mu_0 \tag{2.40}$$

In order to confirm the conservation of linear momentum, we need to determine the change in the momentum of Mary's ball *as measured by Frank*. We will let the primed speeds be measured by Mary and the unprimed speeds be measured by Frank (except that u_0 is always the speed of the ball as measured by the twin in his or her own system). Mary measures the initial velocity of her own ball to be $u'_{Mx} = 0$ and $u'_{My} = -u_0$, because she throws it along her own $-y'$ axis. To determine the velocity of Mary's ball as measured by Frank, we need to use the velocity transformation equations of Equation (2.23). If we insert the appropriate values for the speeds just discussed, we obtain

$$\begin{aligned} u_{Mx} &= v \\ u_{My} &= -u_0\sqrt{1 - v^2/c^2} \end{aligned} \tag{2.41}$$

Before the collision, the momentum of Mary's ball as measured by Frank becomes

$$\begin{aligned} \text{Before} \quad p_{Mx} &= mv \\ \text{Before} \quad p_{My} &= -mu_0\sqrt{1 - v^2/c^2} \end{aligned} \tag{2.42}$$

For a perfectly elastic collision, the momentum after the collision is

$$\begin{aligned} \text{After} \quad p_{Mx} &= mv \\ \text{After} \quad p_{My} &= +mu_0\sqrt{1 - v^2/c^2} \end{aligned} \tag{2.43}$$

The change in momentum of Mary's ball according to Frank is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2} \tag{2.44}$$

The conservation of linear momentum requires the total change in momentum of the collision, $\Delta p_F + \Delta p_M$, to be zero. The addition of Equations (2.40) and (2.44) clearly does not give zero. *Linear momentum is not conserved if we use the conventions for momentum from classical physics even if we use the velocity transformation equations from the special theory of relativity*. There is no problem with the x direction, but there is a problem with the y direction along the direction the ball is thrown in each system.

Difficulty with classical linear momentum

Rather than abandon the conservation of linear momentum, let us look for a modification of the definition of linear momentum that preserves both it and Newton's second law. We follow a procedure similar to the one we used in deriving the Lorentz transformation; we assume the simplest, most reasonable change that may preserve the conservation of momentum. We assume that the classical form of momentum $m\vec{u}$ is multiplied by a factor that may depend on velocity. Let the factor be $\Gamma(u)$. Our trial definition for linear momentum now becomes

$$\vec{p} = \Gamma(u)m\vec{u} \tag{2.45}$$

In Example 2.9 we show that momentum is conserved in the collision just described for the value of $\Gamma(u)$ given by

$$\Gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{2.46}$$

Notice that the *form* of Equation (2.46) is the same as that found earlier for the Lorentz transformation. We even give $\Gamma(u)$ the same symbol: $\Gamma(u) = \gamma$. However, this γ is different; it contains the speed of the particle u, whereas the Lorentz transformation contains the relative speed v between the two inertial reference frames. This distinction should be kept in mind because it can cause confusion. Because the usage is so common among physicists, we will use γ for both purposes. However, when there is any chance of confusion, we will write out $1/\sqrt{1 - u^2/c^2}$ and use $\gamma = 1/\sqrt{1 - v^2/c^2}$ for the Lorentz transformation. We will write out $1/\sqrt{1 - u^2/c^2}$ often to avoid confusion.

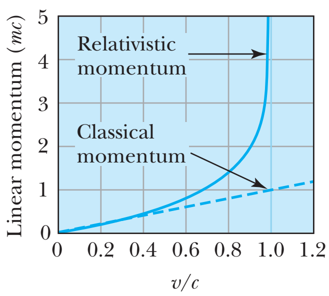


Figure 2.30 The linear momentum of a particle of mass m is plotted versus its velocity (v/c) for both the classical and relativistic momentum results. As $v \to c$ the relativistic momentum becomes quite large, but the classical momentum continues its linear rise. The relativistic result is the correct one.

We can make a plausible determination for the correct form of the momentum if we use the proper time discussed previously to determine the velocity. The momentum becomes

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt}\frac{dt}{d\tau} \tag{2.47}$$

We retain the velocity $\vec{u} = d\vec{r}/dt$ as used classically, where $\vec{r}$ is the position vector. All observers do not agree as to the value of $d\vec{r}/dt$, but they do agree as to the value of $d\vec{r}/d\tau$, where $d\tau$ is the proper time measured in the moving system K$'$. The value of $dt/d\tau$ ($= \gamma$) is obtained from Equation (2.31), where the speed u is used in the relation for γ to represent the relative speed of the moving (Mary's) frame and the fixed (Frank's) frame.

Relativistic momentum

The definition of the **relativistic momentum** becomes, from Equation (2.47),

$$\vec{p} = m\frac{d\vec{r}}{dt}\gamma$$

$$\vec{p} = \gamma m\vec{u} \quad \text{Relativistic momentum} \tag{2.48}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{2.49}$$

This result for the relativistic momentum reduces to the classical result for small values of u/c. The classical momentum expression is good to an accuracy of 1% as long as $u < 0.14c$. We show both the relativistic and classical momentum in Figure 2.30.

Some physicists like to refer to the mass in Equation (2.48) as the *rest mass* m_0 and call the term $m = \gamma m_0$ the *relativistic mass*. In this manner the classical form of momentum, $m\vec{u}$, is retained. The mass is then imagined to increase at high speeds. Most physicists prefer to keep the concept of mass as an invariant, intrinsic property of an object. We adopt this latter approach and will use the term *mass* exclusively to mean *rest mass*. Although we may use the terms *mass* and *rest mass* synonymously, we will not use the term *relativistic mass*. The use of relativistic mass too often leads the student into mistakenly inserting the term into classical expressions where it does not apply.

Rest and relativistic mass

EXAMPLE 2.9

Show that linear momentum is conserved for the collision just discussed and shown in Figure 2.29.

Strategy We use the relativistic momentum to modify the expressions obtained for the momentum of the balls thrown by Frank and Mary. We will then check to see whether momentum is conserved according to Frank. We leave to Problem 62 the question of whether momentum is conserved according to Mary's system.

Solution From Equation (2.39), the momentum of the ball thrown by Frank becomes

$$p_{Fy} = \gamma m u_0 = \frac{m u_0}{\sqrt{1 - u_0^2/c^2}}$$

For an elastic collision, the magnitude of the momentum for this ball is the same before and after the collision. After the collision, the momentum will be the negative of this value, so the change in momentum becomes, from Equation (2.40),

$$\Delta p_F = \Delta p_{Fy} = -2\gamma m u_0 = -\frac{2 m u_0}{\sqrt{1 - u_0^2/c^2}} \tag{2.50}$$

Now we consider the momentum of Mary's ball as measured by Frank. Even with the addition of the γ factor for the momentum in the x direction, we still have $\Delta p_{Mx} = 0$. We must look more carefully at Δp_{My}. First, we find the speed of the ball thrown by Mary as measured by Frank. We use Equations (2.41) to determine

$$u_M = \sqrt{u_{Mx}^2 + u_{My}^2} = \sqrt{v^2 + u_0^2(1 - v^2/c^2)} \tag{2.51}$$

The relativistic factor γ for the momentum for this situation is

$$\gamma = \frac{1}{\sqrt{1 - u_M^2/c^2}}$$

The value of p_{My} is now found by modifying Equation (2.42) with this value of γ.

$$p_{My} = -\gamma m u_0 \sqrt{1 - v^2/c^2} = \frac{-m u_0 \sqrt{1 - v^2/c^2}}{\sqrt{1 - u_M^2/c^2}}$$

We insert the value of u_M from Equation (2.51) into this equation to give

$$p_{My} = \frac{-m u_0 \sqrt{1 - v^2/c^2}}{\sqrt{(1 - u_0^2/c^2)(1 - v^2/c^2)}} = \frac{-m u_0}{\sqrt{1 - u_0^2/c^2}} \tag{2.52}$$

The momentum after the collision will still be the negative of this value, so the change in momentum becomes

$$\Delta p_M = \Delta p_{My} = \frac{2 m u_0}{\sqrt{1 - u_0^2/c^2}} \tag{2.53}$$

The change in the momentum of the two balls as measured by Frank is given by the sum of Equations (2.50) and (2.53):

$$\Delta p = \Delta p_F + \Delta p_M = 0$$

Thus Frank indeed finds that momentum is conserved. Mary should also determine that linear momentum is conserved (see Problem 62).

2.12 Relativistic Energy

We now turn to the concepts of energy and force. When forming the new theories of relativity and quantum physics, physicists resisted changing the well-accepted ideas of classical physics unless absolutely necessary. In this same spirit we also choose to keep intact as many definitions from classical physics as possible and let experiment dictate when we are incorrect. In practice, the concept of force is best defined by its use in Newton's laws of motion, and we retain here the classical definition of force as used in Newton's second law. In the previous section we studied the concept of momentum and found a relativistic expression in Equation (2.48). Therefore, we modify Newton's second law to include our new definition of linear momentum, and force becomes

Relativistic force

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{u}) = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}\right) \tag{2.54}$$

Aspects of this force will be examined in the problems (see Problems 55–58).

Introductory physics presents kinetic energy as the work done on a particle by a net force. We retain here the same definitions of kinetic energy and work. The work W_{12} done by a force $\vec{F}$ to move a particle from position 1 to position 2 along a path $\vec{s}$ is defined to be

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = K_2 - K_1 \tag{2.55}$$

where K_1 is defined to be the kinetic energy of the particle at position 1.

For simplicity, let the particle start from rest under the influence of the force $\vec{F}$ and calculate the final kinetic energy K after the work is done. The force is related to the dynamic quantities by Equation (2.54). The work W and kinetic energy K are

$$W = K = \int \frac{d}{dt}(\gamma m\vec{u}) \cdot \vec{u}\, dt \tag{2.56}$$

where the integral is performed over the differential path $d\vec{s} = \vec{u}\, dt$. Because the mass is invariant, it can be brought outside the integral. The relativistic factor γ depends on u and cannot be brought outside the integral. Equation (2.56) becomes

$$K = m\int dt \frac{d}{dt}(\gamma\vec{u}) \cdot \vec{u} = m\int u\, d(\gamma u)$$

The limits of integration are from an initial value of 0 to a final value of γu.

$$K = m\int_0^{\gamma u} u\, d(\gamma u) \tag{2.57}$$

The integral in Equation (2.57) is straightforward if done by the method of integration by parts. The result, called the *relativistic kinetic energy*, is

Relativistic kinetic energy

$$K = \gamma mc^2 - mc^2 = mc^2\left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right) = mc^2(\gamma - 1) \tag{2.58}$$

Equation (2.58) does not seem to resemble the classical result for kinetic energy, $K = \frac{1}{2}mu^2$. However, if it is correct, we expect it to reduce to the classical result for low speeds. Let's see whether it does. For speeds $u \ll c$, we expand γ in a binomial series as follows:

$$K = mc^2\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - mc^2$$

$$= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2$$

where we have neglected all terms of power $(u/c)^4$ and greater, because $u \ll c$. This gives the following equation for the relativistic kinetic energy at low speeds:

$$K = mc^2 + \frac{1}{2}mu^2 - mc^2 = \frac{1}{2}mu^2 \qquad (2.59)$$

which is the expected classical result. We show both the relativistic and classical kinetic energies in Figure 2.31. They diverge considerably above a velocity of $0.6c$.

A common mistake students make when first studying relativity is to use either $\frac{1}{2}mu^2$ or $\frac{1}{2}\gamma mu^2$ for the relativistic kinetic energy. It is important to *use only Equation (2.58) for the relativistic kinetic energy*. Although Equation (2.58) looks much different from the classical result, it is the only correct one, and *neither* $\frac{1}{2}mu^2$ *nor* $\frac{1}{2}\gamma mu^2$ *is a correct relativistic result*.

Equation (2.58) is particularly useful when dealing with particles accelerated to high speeds. For example, the fastest speeds produced in the United States have been in the 3-kilometer-long electron accelerator at the Stanford Linear Accelerator Laboratory. This accelerator produces electrons with a kinetic energy of 8×10^{-9} J (50 GeV) or 50×10^9 eV. The electrons have speeds so close to the speed of light that the tiny difference from c is difficult to measure directly. The speed of the electrons is inferred from the relativistic kinetic energy of Equation (2.58) and is given by $0.99999999995c$. Such calculations are difficult to do with calculators because of significant-figure limitations. As a result, we use kinetic energy or momentum to express the motion of a particle moving near the speed of light and rarely use its speed.

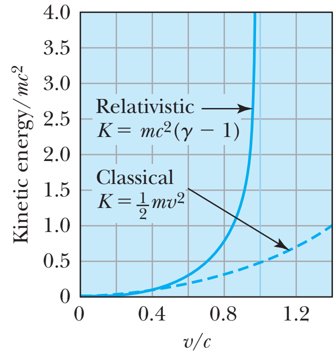


Figure 2.31 The kinetic energy as a fraction of rest energy (K/mc^2) of a particle of mass m is shown versus its velocity (v/c) for both the classical and relativistic calculations. Only the relativistic result is correct. Like the momentum, the kinetic energy rises rapidly as $v \to c$.

CONCEPTUAL EXAMPLE 2.10

Determine whether an object with mass can ever have the speed of light.

Solution If we examine Equation (2.58), we see that when $u \to c$, the kinetic energy $K \to \infty$. Because there is not an infinite amount of energy available, we agree that no object with mass can have the speed of light. The classical and relativistic speeds for electrons are shown in Figure 2.32 as a function of their kinetic energy. Physicists have found that experimentally it does not matter how much energy we give an object having mass. Its speed can never quite reach c.

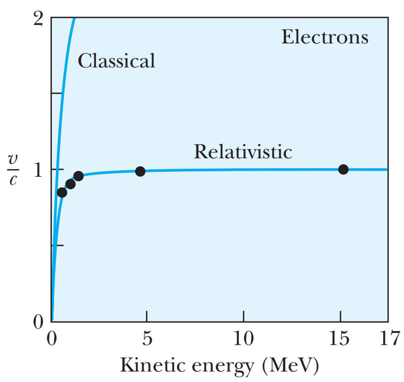


Figure 2.32 The velocity (v/c) of electrons is shown versus kinetic energy for both classical (incorrect) and relativistic calculations. The experimentally measured data points agree with the relativistic results. *Adapted with permission from* American Journal of Physics *32, 551 (1964), W. Bertozzi. © 1964 American Association of Physics Teachers.*

EXAMPLE 2.11

Electrons used to produce medical x rays are accelerated from rest through a potential difference of 25,000 volts before striking a metal target. Calculate the speed of the electrons and determine the error in using the classical kinetic energy result.

Strategy We calculate the speed from the kinetic energy, which we determine both classically and relativistically and then compare the results. In order to determine the correct speed of the electrons, we must use the relativistically correct kinetic energy given by Equation (2.58). The work done to accelerate an electron across a potential difference V is given by qV, where q is the charge of the particle. The work done to accelerate the electron from rest is the final kinetic energy K of the electron.

Solution The kinetic energy is given by

$$K = W = qV = (1.6 \times 10^{-19}\ \text{C})(25 \times 10^{3}\ \text{V})$$
$$= 4.0 \times 10^{-15}\ \text{J}$$

We first determine γ from Equation (2.58) and from that, the speed. We have

$$K = (\gamma - 1)mc^2 \tag{2.60}$$

From this equation, γ is found to be

$$\gamma = 1 + \frac{K}{mc^2} \tag{2.61}$$

The quantity mc^2 for the electron is determined to be

$$mc^2(\text{electron}) = (9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})^2$$
$$= 8.19 \times 10^{-14}\ \text{J}$$

The relativistic factor is then $\gamma = 1 + [(4.0 \times 10^{-15}\ \text{J})/(8.19 \times 10^{-14}\ \text{J})] = 1.049$. Equation (2.8) can be rearranged to determine β^2 as a function of γ^2, where $\beta = u/c$.

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2} = \frac{(1.049)^2 - 1}{(1.049)^2} = 0.091 \tag{2.62}$$

The value of β is 0.30, and the correct speed, $u = \beta c$, is 0.90×10^8 m/s.

We determine the error in using the classical result by calculating the velocity using the nonrelativistic expression. The nonrelativistic expression is $K = \frac{1}{2}mu^2$, and the speed is given by

$$u = \sqrt{\frac{2(4.0 \times 10^{-15}\ \text{J})}{9.11 \times 10^{-31}\ \text{kg}}}$$
$$= 0.94 \times 10^8\ \text{m/s} \quad \text{(nonrelativistic)}$$

The (incorrect) classical speed is about 4% greater than the (correct) relativistic speed. Such an error is significant enough to be important in designing electronic equipment and in making test measurements. Relativistic calculations are particularly important for electrons, because they have such a small mass and are easily accelerated to speeds very close to c.

Total Energy and Rest Energy

We rewrite Equation (2.58) in the form

$$\gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = K + mc^2 \tag{2.63}$$

The term mc^2 is called the **rest energy** and is denoted by E_0.

Rest energy

$$E_0 = mc^2 \tag{2.64}$$

This leaves the sum of the kinetic energy and rest energy to be interpreted as the **total energy** of the particle. The total energy is denoted by E and is given by

Total energy

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{E_0}{\sqrt{1 - u^2/c^2}} = K + E_0 \tag{2.65}$$

Equivalence of Mass and Energy

These last few equations suggest the equivalence of mass and energy, a concept attributed to Einstein. The result that energy = mc^2 is one of the most famous equations in physics. Even when a particle has no velocity, and thus no kinetic energy, we still believe that the particle has energy through its mass, $E_0 = mc^2$. Nuclear reactions are certain proof that mass and energy are equivalent. The concept of motion as being described by *kinetic energy* is preserved in relativistic dynamics, but a particle with no motion still has energy through its mass.

In order to establish the equivalence of mass and energy, we must modify two of the conservation laws that we learned in classical physics. Mass and energy are no longer two separately conserved quantities. We must combine them into one law of the **conservation of mass-energy**. We will see ample proof during the remainder of this book of the validity of this basic conservation law.

Conservation of mass-energy

Even though we often say "energy is turned into mass" or "mass is converted into energy" or "mass and energy are interchangeable," what we mean is that mass and energy are *equivalent*; this is important to understand. Mass is another form of energy, and we use the terms *mass-energy* and *energy* interchangeably. This is not the first time we have had to change our understanding of energy. In the late eighteenth century it became clear that heat was another form of energy, and the nineteenth-century experiments of James Joule showed that heat loss or gain was related to work.

Consider two blocks of wood, each of mass m and having kinetic energy K, moving toward each other as shown in Figure 2.33. A spring placed between them is compressed and locks in place as they collide. Let's examine the conservation of mass-energy. The energy before the collision is

$$\text{Mass-energy before:} \qquad E = 2mc^2 + 2K \tag{2.66}$$

and the energy after the collision is

$$\text{Mass-energy after:} \qquad E = Mc^2 \tag{2.67}$$

where M is the (rest) mass of the system. Because energy is conserved, we have $E = 2mc^2 + 2K = Mc^2$, and the new mass M is greater than the individual masses $2m$. The kinetic energy went into compressing the spring, so the spring has increased

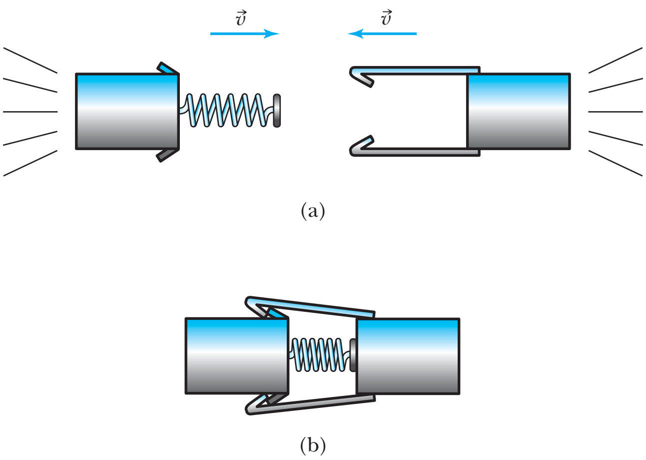


Figure 2.33 (a) Two blocks of wood, one with a spring attached and both having mass m, move with equal speeds v and kinetic energies K toward a head-on collision. (b) The two blocks collide, compressing the spring, which locks in place. The system now has increased mass, $M = 2m + 2K/c^2$, with the kinetic energy being converted into the potential energy of the spring.

potential energy. Kinetic energy has been converted into mass, the result being that the potential energy of the spring has caused the system to have more mass. We find the difference in mass ΔM by setting the previous two equations for energy equal and solving for $\Delta M = M - 2m$.

$$\Delta M = M - 2m = \frac{2K}{c^2} \tag{2.68}$$

Linear momentum is conserved in this head-on collision.

The fractional mass increase in this case is quite small and is given by $f_r = \Delta M/2m$. If we use Equation (2.68), we have

$$f_r = \frac{M - 2m}{2m} = \frac{2K/c^2}{2m} = \frac{K}{mc^2} \tag{2.69}$$

For typical masses and kinetic energies of blocks of wood, this fractional increase in mass is too small to measure. For example, if we have blocks of wood of mass 0.1 kg moving at 10 m/s, Equation (2.69) gives

$$f_r = \frac{\frac{1}{2}mv^2}{mc^2} = \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}\frac{(10 \text{ m/s})^2}{(3 \times 10^8 \text{ m/s})^2} = 6 \times 10^{-16}$$

where we have used the nonrelativistic expression for kinetic energy because the speed is so low. This very small numerical result indicates that questions of mass increase are inappropriate for macroscopic objects such as blocks of wood and automobiles crashing into one another. Such small increases cannot now be measured, but in the next section, we will look at the collision of two high-energy protons, in which considerable energy is available to create additional mass. Mass-energy relations are essential in such reactions.

Relationship of Energy and Momentum

Physicists believe that linear momentum is a more fundamental concept than kinetic energy. There is no conservation of kinetic energy, whereas the conservation of linear momentum in isolated systems is inviolate as far as we know. A more fundamental result for the total energy in Equation (2.65) might include momentum rather than kinetic energy. Let's proceed to find a useful result. We begin with Equation (2.48) for the relativistic momentum written in magnitude form only.

$$p = \gamma mu = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

We square this result, multiply by c^2, and rearrange the result.

$$\begin{aligned} p^2c^2 &= \gamma^2 m^2 u^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) = \gamma^2 m^2 c^4 \beta^2 \end{aligned}$$

We use Equation (2.62) for β^2 and find

$$\begin{aligned} p^2c^2 &= \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) \\ &= \gamma^2 m^2 c^4 - m^2 c^4 \end{aligned}$$

The first term on the right-hand side is just E^2, and the second term is E_0^2. The last equation becomes

$$p^2c^2 = E^2 - E_0^2$$

We rearrange this last equation to find the result we are seeking, a relation between energy and momentum.

Momentum-energy relation

$$\boxed{E^2 = p^2c^2 + E_0^2} \tag{2.70}$$

or

$$\boxed{E^2 = p^2c^2 + m^2c^4} \tag{2.71}$$

Equation (2.70) is a useful result to relate the total energy of a particle with its momentum. The quantities $(E^2 - p^2c^2)$ and m are invariant quantities. Note that when a particle's velocity is zero and it has no momentum, Equation (2.70) correctly gives E_0 as the particle's total energy.

Massless Particles

Equation (2.70) can also be used to determine the total energy for particles having zero mass. For example, Equation (2.70) predicts that the total energy of a photon is

$$E = pc \qquad \text{Photon} \tag{2.72}$$

The energy of a photon is completely due to its motion. It has no rest energy, because it has no mass.

We can show that the previous relativistic equations correctly predict that the speed of a photon must be the speed of light c. We use Equations (2.65) and (2.72) for the total energy of a photon and set the two equations equal.

$$E = \gamma mc^2 = pc$$

If we insert the value of the relativistic momentum from Equation (2.48), we have

$$\gamma mc^2 = \gamma muc$$

The fact that $u = c$ follows directly from this equation after careful consideration of letting $m \to 0$ and realizing that $\gamma \to \infty$.

Massless particles must travel at the speed of light

$$u = c \qquad \text{Massless particle} \tag{2.73}$$

CONCEPTUAL EXAMPLE 2.12

Tachyons are postulated particles that travel faster than the speed of light. (The word tachyon is derived from the Greek word *tachus*, which means "speedy.") They were first seriously proposed and investigated in the 1960s. Use what we have learned thus far in this chapter and discuss several properties that tachyons might have.

Solution Let's first examine Equation (2.65) for energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \tag{2.65}$$

Because $u > c$, the energy must be imaginary if the mass is real, or conversely, if we insist that energy be real, we must have an imaginary mass! For purposes of discussion, we will henceforth assume that energy is real and tachyon mass is imaginary. Remember that ordinary matter must always travel at speed less than c, light must travel at the speed of light, and tachyons must always have speed greater than c. In order to slow down a tachyon, we must give it *more* energy, according to Equation (2.65). Note that the energy must become infinite if we want to slow down a tachyon to speed c. If the tachyon's energy is reduced, it speeds up!

Because tachyons travel faster than c, we have a problem with causality. Consider a tachyon leaving Earth at time $t = 0$ that arrives at a distant galaxy at time T. A spaceship traveling at speed less than c from Earth to the galaxy could conceivably find that the tachyon arrived at the galaxy before it left Earth!

It has been proposed that tachyons might be created in high-energy particle collisions or in cosmic rays. No confirming evidence has been found. Tachyons, if charged, could also be detected from *Cerenkov radiation*. When we refer to speed c, we always mean in a vacuum. When traveling in a medium, the speed must be less than c. When particles have speed greater than light travels in a medium, characteristic electromagnetic radiation is emitted. The effect of the blue glow in swimming pool nuclear reactors is due to this Cerenkov radiation.

2.13 Computations in Modern Physics

We were taught in introductory physics that the international system of units is preferable when doing calculations in science and engineering. This is generally true, but in modern physics we sometimes use other units that are more convenient for atomic and subatomic scales. In this section we introduce some of those units and demonstrate their practicality through several examples. Recall that the work done in accelerating a charge through a potential difference is given by $W = qV$. For a proton, with charge $e = 1.602 \times 10^{-19}$ C, accelerated across a potential difference of 1 V, the work done is

$$W = (1.602 \times 10^{-19})(1 \text{ V}) = 1.602 \times 10^{-19} \text{ J}$$

In modern physics calculations, the amount of charge being considered is almost always some multiple of the electron charge. Atoms and nuclei all have an exact multiple of the electron charge (or neutral). For example, some charges are proton ($+e$), electron ($-e$), neutron (0), pion (0, $\pm e$), and a singly ionized carbon atom ($+e$). The work done to accelerate the proton across a potential difference of 1 V could also be written as

$$W = (1\ e)(1 \text{ V}) = 1 \text{ eV}$$

Use eV for energy

where e stands for the electron charge. Thus eV, pronounced "electron volt," is also a unit of energy. It is related to the SI (*Système International*) unit joule by the two previous equations.

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \tag{2.74}$$

The eV unit is used more often in modern physics than the SI unit J. The term eV is often used with the SI prefixes where applicable. For example, in atomic and solid state physics, eV itself is mostly used, whereas in nuclear physics MeV (10^6 eV, *mega*-electron-volt) and GeV (10^9 eV, *giga*-electron-volt) are predominant, and in particle physics GeV and TeV (10^{12} eV, *tera*-electron-volt) are used. When we speak of a particle having a certain amount of energy, the common usage is to refer to the kinetic energy. A 6-GeV proton has a *kinetic* energy of 6 GeV, not a *total* energy of 6 GeV. Because the rest energy of a proton is about 1 GeV, this proton would have a total energy of about 7 GeV.

Like the SI unit for energy, the SI unit for mass, kilogram, is a very large unit of mass in modern physics calculations. For example, the mass of a proton is only 1.6726×10^{-27} kg. Two other mass units are commonly used in modern physics. First, the rest energy E_0 is given by Equation (2.64) as mc^2. The rest energy of the proton is given by

$$E_0(\text{proton}) = (1.67 \times 10^{-27}\ \text{kg})(3 \times 10^8\ \text{m/s})^2 = 1.50 \times 10^{-10}\ \text{J}$$
$$= 1.50 \times 10^{-10}\ \text{J}\frac{1\ \text{eV}}{1.602 \times 10^{-19}\ \text{J}} = 9.38 \times 10^8\ \text{eV}$$

The rest energies of the elementary particles are usually quoted in MeV or GeV. (To five significant figures, the rest energy of the proton is 938.27 MeV.) Because $E_0 = mc^2$, the mass is often quoted in units of MeV/c^2; for example, the mass of the proton is given by 938.27 MeV/c^2. We will find that the mass unit of MeV/c^2 is quite useful. The masses of several elementary particles are given on the inside of the front book cover. Although we will not do so, research physicists often quote the mass in units of just eV (or MeV, etc.).

Use MeV/c^2 for mass

The other commonly used mass unit is the (unified) **atomic mass unit**. It is based on the definition that the mass of the neutral carbon-12 (^{12}C) atom is exactly 12 u, where u is one atomic mass unit.* We obtain the conversion between kilogram and atomic mass units u by comparing the mass of one ^{12}C atom.

Atomic mass unit

$$\text{Mass}(^{12}\text{C atom}) = \frac{12\ \text{g/mol}}{6.02 \times 10^{23}\ \text{atoms/mol}}$$
$$= 1.99 \times 10^{-23}\ \text{g/atom} \tag{2.75}$$
$$\text{Mass}(^{12}\text{C atom}) = 1.99 \times 10^{-26}\ \text{kg} = 12\ \text{u/atom}$$

Therefore, the conversion is (when properly done to 6 significant figures)

$$1\ \text{u} = 1.66054 \times 10^{-27}\ \text{kg} \tag{2.76}$$
$$1\ \text{u} = 931.494\ \text{MeV}/c^2 \tag{2.77}$$

We have added the conversion from atomic mass units to MeV/c^2 for completeness.

From Equations (2.70) and (2.72) we see that a convenient unit of momentum is energy divided by the speed of light, or eV/c. We will use the unit eV/c for momentum when appropriate. Remember also that we often quote β (= v/c) for velocity, so that c itself is an appropriate unit of velocity.

*To avoid confusion between velocity and atomic mass unit, we will henceforth use v for velocity when the possibility exists for confusing the mass unit u with the velocity variable u.

EXAMPLE 2.13

A 2.00-GeV proton hits another 2.00-GeV proton in a head-on collision. (a) Calculate v, β, p, K, and E for each of the initial protons. (b) What happens to the kinetic energy?

Strategy (a) By the convention just discussed, a 2.00-GeV proton has a kinetic energy of 2.00 GeV. We use Equation (2.65) to determine the total energy and Equation (2.70) to determine momentum if we know the total energy. To determine β and v, it helps to first determine the relativistic factor γ, which we can use Equation (2.65) to find. Then we use Equation (2.62) to find β and v. These are all typical calculations that are performed when doing relativistic computations.

Solution (a) We use $K = 2.00$ GeV and the proton rest energy, 938 MeV, to find the total energy from Equation (2.65),

$$E = K + E_0 = 2.00 \text{ GeV} + 938 \text{ MeV} = 2.938 \text{ GeV}$$

The momentum is determined from Equation (2.70).

$$p^2c^2 = E^2 - E_0^2 = (2.938 \text{ GeV})^2 - (0.938 \text{ GeV})^2$$
$$= 7.75 \text{ GeV}^2$$

The momentum is calculated to be

$$p = \sqrt{7.75(\text{GeV}/c)^2} = 2.78 \text{ GeV}/c$$

Notice how naturally the unit of GeV/c arises in our calculation.

In order to find β we first find the relativistic factor γ. There are several ways to determine γ; one is to compare the rest energy with the total energy. From Equation (2.65) we have

$$E = \gamma E_0 = \frac{E_0}{\sqrt{1 - u^2/c^2}}$$

$$\gamma = \frac{E}{E_0} = \frac{2.938 \text{ GeV}}{0.938 \text{ GeV}} = 3.13$$

We use Equation (2.62) to determine β.

$$\beta = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} = \sqrt{\frac{3.13^2 - 1}{3.13^2}} = 0.948$$

The speed of a 2.00-GeV proton is $0.95c$ or 2.8×10^8 m/s.

(b) When the two protons collide head-on, the situation is similar to the case when the two blocks of wood collided head-on with one important exception. The time for the two protons to interact is less than 10^{-20} s. If the two protons did momentarily stop at rest, then the two-proton system would have its mass increased by an amount given by Equation (2.68), $2K/c^2$ or 4.00 GeV/c^2. The result would be a highly excited system. In fact, the collision between the protons happens very quickly, and there are several possible outcomes. The two protons may either remain or disappear, and new additional particles may be created. Two of the possibilities are

$$p + p \rightarrow p + p + p + \bar{p} \tag{2.78}$$

$$p + p \rightarrow \pi^+ + d \tag{2.79}$$

where the symbols are p (proton), $\bar{p}$ (antiproton), π (pion), and d (deuteron). We will learn more about the possibilities later when we study nuclear and particle physics. Whatever happens must be consistent with the conservation laws of charge, energy, and momentum, as well as with other conservation laws to be learned. Such experiments are routinely done in particle physics. In the analysis of these experiments, the equivalence of mass and energy is taken for granted.

Binding Energy

The equivalence of mass and energy becomes apparent when we study the binding energy of atoms and nuclei that are formed from individual particles. For example, the hydrogen atom is formed from a proton and electron bound together by the electrical (Coulomb) force. A deuteron is a proton and neutron bound together by the nuclear force. The potential energy associated with the force keeping the system together is called the **binding energy** E_B. The binding energy is the work required to pull the particles out of the bound system into separate, free particles at rest. The conservation of energy is written as

$$M_{\text{bound system}}c^2 + E_B = \sum_i m_i c^2 \tag{2.80}$$

where the m_i values are the masses of the free particles. The binding energy is *the difference between the rest energy of the individual particles and the rest energy of the combined, bound system.*

$$E_B = \sum_i m_i c^2 - M_{\text{bound system}}c^2 \tag{2.81}$$

For the case of two final particles having masses m_1 and m_2, we have

$$E_B = (m_1 + m_2 - M_{\text{bound system}})c^2 = \Delta M c^2 \tag{2.82}$$

where ΔM is the difference between the final and initial masses.

When two particles (for example, a proton and neutron) are bound together to form a composite (like a deuteron), part of the rest energy of the individual particles is lost, resulting in the binding energy of the system. The rest energy of the combined system must be reduced by this amount. The deuteron is a good example. The rest energies of the particles are

$$\text{Proton} \quad E_0 = 1.007276c^2\ \text{u} = 938.27\ \text{MeV}$$

$$\text{Neutron} \quad E_0 = 1.008665c^2\ \text{u} = 939.57\ \text{MeV}$$

$$\text{Deuteron} \quad E_0 = 2.01355c^2\ \text{u} = 1875.61\ \text{MeV}$$

The binding energy E_B is determined from Equation (2.81) to be

$$E_B(\text{deuteron}) = 938.27\ \text{MeV} + 939.57\ \text{MeV} - 1875.61\ \text{MeV} = 2.23\ \text{MeV}$$

CONCEPTUAL EXAMPLE 2.14

Why can we ignore the 13.6 eV binding energy of the proton and electron when making mass determinations for nuclei, but not the binding energy of a proton and neutron?

Solution The binding energy of the proton and electron in the hydrogen atom is only 13.6 eV, which is so much smaller than the 1-GeV rest energy of a neutron and proton that it can be neglected when making mass determinations. The deuteron binding energy of 2.23 MeV, however, represents a much larger fraction of the rest energies and is extremely important. The binding energies of heavy nuclei such as uranium can be more than 1000 MeV, and even that much energy is not large enough to keep uranium from decaying to lighter nuclei. The Coulomb repulsion between the many protons in heavy nuclei is mostly responsible for their instability. Nuclear stability is addressed in Chapter 12.

EXAMPLE 2.15

What is the minimum kinetic energy the protons must have in the head-on collision of Equation (2.79), $p + p \rightarrow \pi^+ + d$, in order to produce the positively charged pion and deuteron? The mass of π^+ is 139.6 MeV/c^2.

Strategy For the minimum kinetic energy K required, we need just enough energy to produce the rest energies of the final particles. We let the final kinetic energies of the pion and deuteron be zero. Because the collision is head-on, the momentum will be zero before and after the collision, so the pion and deuteron will truly be at rest with no kinetic energy. We use the conservation of energy to determine the kinetic energy.

Solution Conservation of energy requires

$$m_pc^2 + K + m_pc^2 + K = m_dc^2 + m_{\pi^+}c^2$$

The rest energies of the proton and deuteron were given in this section, so we solve the previous equation for the kinetic energy.

$$\begin{aligned} K &= \tfrac{1}{2}(m_dc^2 + m_{\pi^+}c^2 - 2m_pc^2) \\ &= \tfrac{1}{2}[1875.6\ \text{MeV} + 139.6\ \text{MeV} - 2(938.3\ \text{MeV})] \\ &= 69\ \text{MeV} \end{aligned}$$

Nuclear experiments like this are normally done with fixed targets, not head-on collisions, and much more energy than 69 MeV is required, because linear momentum must also be conserved.

EXAMPLE 2.16

The atomic mass of the ^{4}He atom is 4.002603 u. Find the binding energy of the ^{4}He nucleus.

Strategy This is a straightforward application of Equation (2.81), and we will need to determine the atomic masses.

Solution Equation (2.81) gives

$$E_B(^4\text{He}) = 2m_pc^2 + 2m_nc^2 - M_{\text{He}}\, c^2$$

Later we will learn to deal with atomic masses in cases like this, but for now we will subtract the two electron masses from the atomic mass of ^{4}He to obtain the mass of the ^{4}He nucleus. The mass of the electron is given on the inside of the front cover, along with the masses of the proton and neutron.

$$M_{^4\text{He}}(\text{nucleus}) = 4.002603\ \text{u} - 2(0.000549\ \text{u})$$
$$= 4.001505\ \text{u}$$

We determine the binding energy of the ^{4}He nucleus to be

$$E_B(^4\text{He}) = [2(1.007276\,\text{u}) + 2(1.008665\,\text{u}) - 4.001505\,\text{u}]c^2$$
$$= 0.0304\,c^2\text{u}$$

$$E_B(^4\text{He}) = (0.0304\,c^2\text{u})\frac{931.5\ \text{MeV}}{c^2\ \text{u}} = 28.3\ \text{MeV}$$

The binding energy of the ^{4}He nucleus is large, almost 1% of its rest energy.

EXAMPLE 2.17

The molecular binding energy is called the *dissociation energy*. It is the energy required to separate the atoms in a molecule. The dissociation energy of the NaCl molecule is 4.24 eV. Determine the fractional mass increase of the Na and Cl atoms when they are not bound together in NaCl. What is the mass increase for a mole of NaCl?

Strategy Binding energy is a concept that applies to various kinds of bound objects, including a nucleus, an atom, a molecule, and others. We can use Equation (2.82) in the present case to find ΔM, the change in mass, in terms of the binding energy E_B/c^2. We then divide ΔM by M to find the fractional mass increase.

Solution From Equation (2.82) we have $\Delta M = E_B/c^2$ (the binding energy divided by c^2) as the mass difference between the molecule and separate atoms. The mass of NaCl is 58.44 u. The fractional mass increase is

$$f_r = \frac{\Delta M}{M} = \frac{E_B/c^2}{M} = \frac{4.24\ \text{eV}/c^2}{58.44\ \text{u}}\,\frac{c^2\ \text{u}}{931\ \text{MeV}}\,\frac{1\ \text{MeV}}{10^6\ \text{eV}}$$
$$= 7.8 \times 10^{-11}$$

One mole of NaCl has a mass of 58.44 g, so the mass decrease for a mole of NaCl is $f_r \times 58.44$ g or only 4.6×10^{-9} g. Such small masses cannot be directly measured, which is why nonconservation of mass was not observed for chemical reactions—the changes are too small.

EXAMPLE 2.18

A positively charged sigma particle (symbol Σ^+) produced in a particle physics experiment decays very quickly into a neutron and positively charged pion before either its energy or momentum can be measured. The neutron and pion are observed to move in the same direction as the Σ^+ was originally moving, with momenta of 4702 MeV/c and 169 MeV/c, respectively. What was the kinetic energy of the Σ^+ and its mass?

Strategy The decay reaction is

$$\Sigma^+ \rightarrow n + \pi^+$$

where n is a neutron. Obviously the Σ^+ has more mass than the sum of the masses of n and π^+, or the decay would not occur. We have to conserve both momentum and energy for this reaction. We use Equation (2.70) to find the total energy of the neutron and positively charged pion, but in order to determine the rest energy of Σ^+, we need to know the momentum. We can determine the Σ^+ momentum from the conservation of momentum.

Solution The rest energies of n and π^+ are 940 MeV and 140 MeV, respectively. The total energies of E_n and E_{π^+} are, from $E = \sqrt{p^2c^2 + E_0^2}$,

$$E_n = \sqrt{(4702 \text{ MeV})^2 + (940 \text{ MeV})^2} = 4795 \text{ MeV}$$

$$E_{\pi^+} = \sqrt{(169 \text{ MeV})^2 + (140 \text{ MeV})^2} = 219 \text{ MeV}$$

The sum of these energies gives the total energy of the reaction, 4795 MeV + 219 MeV = 5014 MeV, both before and after the decay of Σ^+. Because all the momenta are along the same direction, we must have

$$p_{\Sigma^+} = p_n + p_{\pi^+} = 4702 \text{ MeV}/c + 169 \text{ MeV}/c$$
$$= 4871 \text{ MeV}/c$$

This must be the momentum of the Σ^+ before decaying, so now we can find the rest energy of Σ^+ from Equation (2.70).

$$E_0^2(\Sigma^+) = E^2 - p^2c^2 = (5014 \text{ MeV})^2 - (4871 \text{ MeV})^2$$
$$= (1189 \text{ MeV})^2$$

The rest energy of the Σ^+ is 1189 MeV, and its mass is 1189 MeV/c^2.

We find the kinetic energy of Σ^+ from Equation (2.65).

$$K = E - E_0 = 5014 \text{ MeV} - 1189 \text{ MeV} = 3825 \text{ MeV}$$

2.14 Electromagnetism and Relativity

We have been concerned mostly with the kinematical and dynamical aspects of the special theory of relativity strictly from the mechanics aspects. However, recall that Einstein first approached relativity through electricity and magnetism. He was convinced that Maxwell's equations were invariant (have the same form) in all inertial frames. Einstein wrote in 1952,

> *What led me more or less directly to the special theory of relativity was the conviction that the electromagnetic force acting on a body in motion in a magnetic field was nothing else but an electric field.*

Einstein's conviction about electromagnetism

Einstein was convinced that magnetic fields appeared as electric fields observed in another inertial frame. That conclusion is the key to electromagnetism and relativity.

Maxwell's equations and the Lorentz force law are invariant in different inertial frames. In fact, with the proper Lorentz transformations of the electric and magnetic fields (from relativity theory) together with Coulomb's law (force between stationary charges), Maxwell's equations can be obtained. We will not attempt that fairly difficult mathematical task here, nor do we intend to obtain the Lorentz transformation of the electric and magnetic fields. These subjects are studied in more advanced physics classes. However, we will show qualitatively that the magnetic force that one observer sees is simply an electric force according to an observer in another inertial frame. The electric field arises from charges, whereas the magnetic field arises from *moving* charges.

Magnetism and electricity are relative

Electricity and magnetism were well understood in the late 1800s. Maxwell predicted that all electromagnetic waves travel at the speed of light, and he combined electricity, magnetism, and optics into one successful theory. This classical theory has withstood the onslaught of time and experimental tests.* There were, however, some troubling aspects of the theory when it was observed from different Galilean frames of reference. In 1895 H. A. Lorentz "patched up" the diffi-

*The meshing of electricity and magnetism together with quantum mechanics, called the *theory of quantum electrodynamics* (QED), is one of the most successful theories in physics.

culties with the Galilean transformation by developing a new transformation that now bears his name, the Lorentz transformation. However, Lorentz did not understand the full implication of what he had done. It was left to Einstein, who in 1905 published a paper titled "On the Electrodynamics of Moving Bodies," to fully merge relativity and electromagnetism. Einstein did not even mention the famous Michelson-Morley experiment in this classic 1905 paper, which we take as the origin of the special theory of relativity, and the Michelson-Morley experiment apparently played little role in his thinking. Einstein's belief that *Maxwell's equations describe electromagnetism in any inertial frame* was the key that led Einstein to the Lorentz transformations. Maxwell's assertion that all electromagnetic waves travel at the speed of light and Einstein's postulate that the speed of light is invariant in all inertial frames seem intimately connected.

We now proceed to discuss qualitatively the relative aspects of electric and magnetic fields and their forces. Consider a positive test charge q_0 moving to the right with speed v outside a neutral, conducting wire as shown in Figure 2.34a in the frame of the inertial system K, where the positive charges are at rest and the negative electrons in the wire have speed v to the right. The conducting wire is long and has the same number of positive ions and conducting electrons. For simplicity, we have taken the electrons and the positive charges to have the same speed, but the argument can be generalized.

What is the force on the positive test charge q_0 outside the wire? The total force is given by the Lorentz force

$$\vec{F} = q_0(\vec{E} + \vec{v} \times \vec{B}) \tag{2.83}$$

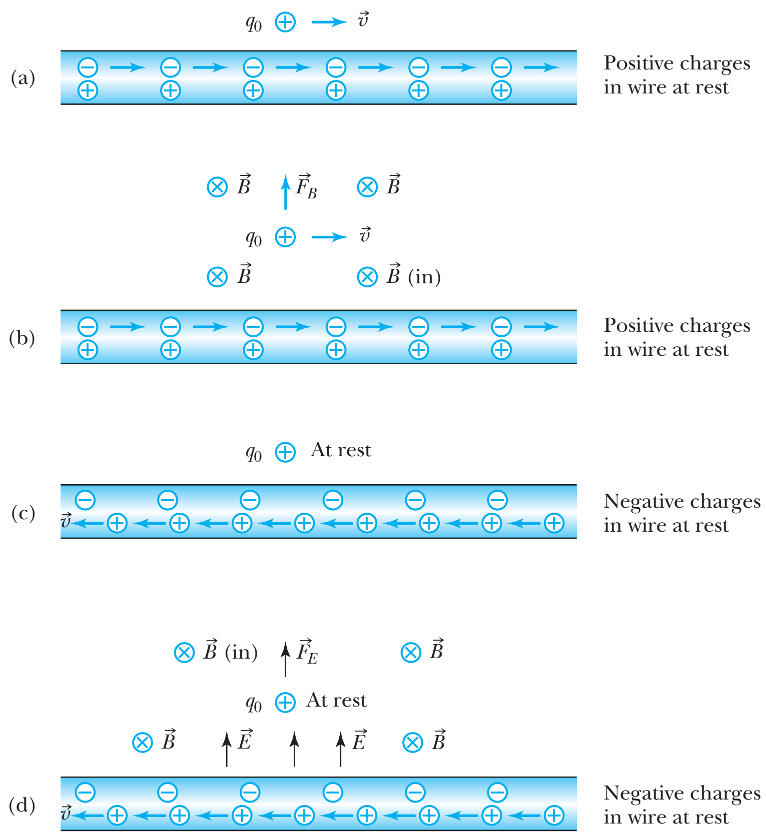


Figure 2.34 (a) A positive charge q_0 is placed outside a neutral, conducting wire. The figure is shown in the system where the positive charges in the wire are at rest. Note that the charge q_0 has the same velocity as the electrons. (b) The moving electrons produce a magnetic field, which causes a force $\vec{F}_B$ on q_0. (c) This is similar to (a), but in this system the electrons are at rest. (d) Now there is an abundance of positive charges due to length contraction, and the resulting electric field repels q_0. There is also a magnetic field, but this causes no force on q_0, which is at rest in this system.

and can be due to an electric field, a magnetic field, or both. Because the total charge inside the wire is zero, the electric force on the test charge q_0 in Figure 2.34a is also zero. But we learned in introductory physics that the moving electrons in the wire (current) produce a magnetic field $\vec{B}$ at the position of q_0 that is into the page (Figure 2.34b). The moving charge q_0 *will be repelled upward by the magnetic force* ($q_0\vec{v} \times \vec{B}$) due to the magnetic field of the wire.

Let's now see what happens in a different inertial frame K′ moving at speed v to the right with the test charge (see Figure 2.34c). Both the test charge q_0 and the negative charges in the conducting wire are at rest in system K′. In this system an observer at the test charge q_0 observes the same density of negative ions in the wire as before. However, in system K′ the positive ions are now moving to the left with speed v. Due to length contraction, the positive ions will appear to be closer together to a stationary observer in K′. Because the positive charges appear to be closer together, there is a higher density of positive charges than of negative charges in the conducting wire. The result is an electric field as shown in Figure 2.34d. The test charge q_0 will now be *repelled* in the presence of the electric field. What about the magnetic field now? The moving charges in Figure 2.34c also produce a magnetic field that is into the page, but this time the charge q_0 is at rest with respect to the magnetic field, so charge q_0 feels no magnetic force.

What appears as a magnetic force in one inertial frame (Figure 2.34b) appears as an electric force in another (Figure 2.34d). Electric and magnetic fields are *relative* to the coordinate system in which they are observed. The Lorentz contraction of the moving charges accounts for the difference. This example can be extended to two conducting wires with electrons moving, and a similar result will be obtained (see Problem 86). It is this experiment, on the force between two parallel, conducting wires, in which current is defined. Because charge is defined using current, the experiment is also the basis of the definition of the electric charge.

We have come full circle in our discussion of the special theory of relativity. The laws of electromagnetism represented by Maxwell's equations have a special place in physics. The equations themselves are invariant in different inertial systems; only the interpretations as electric and magnetic fields are relative.

Summary

Efforts by Michelson and Morley proved in 1887 that either the elusive ether does not exist or there must be significant problems with our understanding of nature.

Albert Einstein solved the problem in 1905 by applying two postulates:

1. The principle of relativity: The laws of physics are the same in all inertial systems.
2. The constancy of the speed of light: Observers in all inertial systems measure the same value for the speed of light in vacuum.

Einstein's two postulates are used to derive the Lorentz transformation relating the space and time coordinates of events viewed from different inertial systems. If system K′ is moving at speed v along the $+x$ axis with respect to system K, the two sets of coordinates are related by

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (vx/c^2)}{\sqrt{1-\beta^2}} \end{aligned} \tag{2.17}$$

The inverse transformation is obtained by switching the primed and unprimed quantities and changing v to $-v$.

The time interval between two events occurring at the same position in a system as measured by a clock at rest is called the proper time T_0. The time interval T' between the same two events measured by a moving observer is related to the proper time T_0 by the time dilation effect.

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} \tag{2.19}$$

We say that moving clocks run slow, because the shortest time is always measured on clocks at rest.

The length of an object measured by an observer at rest relative to the object is called the proper length L_0. The length of the same object measured by an observer who sees the object moving at speed v is L, where

$$L = L_0\sqrt{1 - v^2/c^2} \tag{2.21}$$

This effect is known as length or space contraction, because moving objects are contracted in the direction of their motion.

If u and u' are the velocities of an object measured in systems K and K′, respectively, and v is the relative velocity between K and K′; the relativistic addition of velocities (Lorentz velocity transformation) is

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{u'_x + v}{1 + (v/c^2)u'_x} \\ u_y &= \frac{u'_y}{\gamma[1 + (v/c^2)u'_x]} \\ u_z &= \frac{u'_z}{\gamma[1 + (v/c^2)u'_x]} \end{aligned} \tag{2.23}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.8}$$

The Lorentz transformation has been tested for a hundred years, and no violation has yet been detected. Nevertheless, physicists continue to test its validity, because it is one of the most important results in science.

Spacetime diagrams are useful to represent events geometrically. Time may be considered to be a fourth dimension for some purposes. The spacetime interval for two events defined by $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\,\Delta t^2$ is an invariant between inertial systems.

The relativistic Doppler effect for light frequency f is given by

$$f = \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}} f_0 \tag{2.34}$$

where β is positive when source and receiver are approaching one another and negative when they are receding.

The classical form for linear momentum is replaced by the special relativity form:

$$\vec{p} = \gamma m\vec{u} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} \tag{2.48}$$

The relativistic kinetic energy is given by

$$K = \gamma mc^2 - mc^2 = mc^2\left(\frac{1}{\sqrt{1 - u^2/c^2}} - 1\right) \tag{2.58}$$

The *total energy* E is given by

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \frac{E_0}{\sqrt{1 - u^2/c^2}} = K + E_0 \tag{2.65}$$

where $E_0 = mc^2$. This equation denotes the equivalence of mass and energy. The laws of the conservation of mass and of energy are combined into one conservation law: the conservation of mass-energy.

Energy and momentum are related by

$$E^2 = p^2c^2 + E_0^2 \tag{2.70}$$

In the case of massless particles (for example, the photon), $E_0 = 0$, so $E = pc$. Massless particles must travel at the speed of light.

The electron volt, denoted by eV, is equal to 1.602×10^{-19} J. The unified atomic mass unit u is based on the mass of the ^{12}C atom.

$$1\ \text{u} = 1.66054 \times 10^{-27}\ \text{kg} = 931.494\ \text{MeV}/c^2 \tag{2.76, 2.77}$$

Momentum is often quoted in units of eV/c, and the velocity is often given in terms of β $(= v/c)$.

The difference between the rest energy of individual particles and the rest energy of the combined, bound system is called the binding energy.

Maxwell's equations are invariant under transformations between any inertial reference frames. What appears as electric and magnetic fields is relative to the reference frame of the observer.

Questions

1. Michelson used the motion of the Earth around the sun to try to determine the effects of the ether. Can you think of a more convenient experiment with a higher speed that Michelson might have used in the 1880s? What about today?
2. If you wanted to set out today to find the effects of the ether, what experimental apparatus would you want to use? Would a laser be included? Why?
3. For what reasons would Michelson and Morley repeat their experiment on top of a mountain? Why would they perform the experiment in summer and winter?
4. Does the fact that Maxwell's equations do not need to be modified because of the special theory of relativity, whereas Newton's laws of motion do, mean that Maxwell's work is somehow greater or more significant than Newton's? Explain.
5. The special theory of relativity has what effect on measurements done today? (a) None whatsoever, because any correction would be negligible. (b) We need to consider the effects of relativity when objects move close to the speed of light. (c) We should always make a correction for relativity because Newton's laws are basically wrong. (d) It doesn't matter, because we can't make measurements where relativity would matter.
6. Why did it take so long to discover the theory of relativity? Why didn't Newton figure it out?
7. Can you think of a way you can make yourself older than those born on your same birthday?
8. Will metersticks manufactured on Earth work correctly on spaceships moving at high speed? Explain.
9. Devise a system for you and three colleagues, at rest with you, to synchronize your clocks if your clocks are too large to move and are separated by hundreds of miles.
10. In the experiment to verify time dilation by flying the cesium clocks around the Earth, what is the order of the speed of the four clocks in a system fixed at the center of the Earth, but not rotating?
11. Can you think of an experiment to verify length contraction directly? Explain.
12. Would it be easier to perform the muon decay experiment in the space station orbiting above Earth and then compare with the number of muons on Earth? Explain.
13. On a spacetime diagram, can events above $t = 0$ but not in the shaded area in Figure 2.25 affect the future? Explain.
14. Why don't we also include the spatial coordinate z when drawing the light cone?
15. What would be a suitable name for events connected by $\Delta s^2 = 0$?
16. Is the relativistic Doppler effect valid only for light waves? Can you think of another situation in which it might be valid?
17. In Figure 2.22, why can a real worldline not have a slope less than one?
18. Explain how in the twin paradox, we might arrange to compare clocks at the beginning and end of Mary's journey and not have to worry about acceleration effects.
19. In each of the following pairs, which is the more massive: a relaxed or compressed spring, a charged or uncharged capacitor, or a piston-cylinder when closed or open?
20. In the fission of ^{235}U, the masses of the final products are less than the mass of ^{235}U. Does this make sense? What happens to the mass?
21. In the fusion of deuterium and tritium nuclei to produce a thermonuclear reaction, where does the kinetic energy that is produced come from?
22. Mary, the astronaut, wants to travel to the star system Alpha Centauri, which is 4.3 lightyears away. She wants to leave on her 30th birthday, travel to Alpha Centauri but not stop, and return in time for her wedding to Vladimir on her 35th birthday. What is most likely to happen? (a) Vladimir is a lucky man, because he will marry Mary after she completes her journey. (b) Mary will have to hustle to get in her wedding gown, and the wedding is likely to be watched by billions of people. (c) It is a certainty that Mary will not reach Alpha Centauri if she wants to marry Vladimir as scheduled. (d) Mary does reach Alpha Centauri before her 35th birthday and sends a radio message to Vladimir from Alpha Centauri that she will be back on time. Vladimir is relieved to receive the message before the wedding date.
23. A salesman driving a very fast car was arrested for driving through a traffic light while it was red, according to a policeman parked near the traffic light. The salesman said that the light was actually green to him, because it was Doppler shifted. Is he likely to be found innocent? Explain.

Problems

Note: The more challenging problems have their problem numbers shaded by a blue box.

2.1 The Need for Ether

1. Show that the form of Newton's second law is invariant under the Galilean transformation.
2. Show that the definition of linear momentum, $p = mv$, has the same form $p' = mv'$ under a Galilean transformation.

2.2 The Michelson-Morley Experiment

3. Show that the equation for t_2 in Section 2.2 expresses the time required for the light to travel to the mirror D and back in Figure 2.2. In this case the light is traveling perpendicular to the supposed direction of the ether. In what direction must the light travel to be reflected by the mirror if the light must pass through the ether?
4. A swimmer wants to swim straight across a river with current flowing at a speed of $v_1 = 0.350$ m/s. If the swimmer swims in still water with speed $v_2 = 1.25$ m/s, at what angle should the swimmer point upstream from the shore, and at what speed will the swimmer swim across the river?
5. Show that the time difference $\Delta t'$ given by Equation (2.4) is correct when the Michelson interferometer is rotated by 90°.
6. In the 1887 experiment by Michelson and Morley, the length of each arm was 11 m. The experimental limit for the fringe shift was 0.005 fringes. If sodium light was used with the interferometer ($\lambda = 589$ nm), what upper limit did the null experiment place on the speed of the Earth through the expected ether?
7. Show that if length is contracted by the factor $\sqrt{1 - v^2/c^2}$ in the direction of motion, then the result in Equation (2.3) will have the factor needed to make $\Delta t = 0$ as needed by Michelson and Morley.

2.3 Einstein's Postulates

8. Explain why Einstein argued that the constancy of the speed of light (postulate 2) actually follows from the principle of relativity (postulate 1).
9. Prove that the constancy of the speed of light (postulate 2) is inconsistent with the Galilean transformation.

2.4 The Lorentz Transformation

10. Use the spherical wavefronts of Equations (2.9) to derive the Lorentz transformation given in Equations (2.17). Supply all the steps.
11. Show that both Equations (2.17) and (2.18) reduce to the Galilean transformation when $v \ll c$.
12. Determine the ratio $\beta = v/c$ for the following: (a) A car traveling 95 km/h. (b) A commercial jet airliner traveling 240 m/s. (c) A supersonic airplane traveling at Mach 2.3 (Mach number $= v/v_{\text{sound}}$). (d) The space station, traveling 27,000 km/h. (e) An electron traveling 25 cm in 2 ns. (f) A proton traveling across a nucleus (10^{-14} m) in 0.35×10^{-22} s.
13. Two events occur in an inertial system K as follows:

    Event 1: $x_1 = a, \quad t_1 = 2a/c, \; y_1 = 0, \; z_1 = 0$
    Event 2: $x_2 = 2a, \quad t_2 = 3a/2c, \; y_2 = 0, \; z_2 = 0$

    In what frame K′ will these events appear to occur at the same time? Describe the motion of system K′.
14. Is there a frame K′ in which the two events described in Problem 13 occur at the same place? Explain.
15. Find the relativistic factor γ for each of the parts of Problem 12.
16. An event occurs in system K′ at $x' = 2$ m, $y' = 3.5$ m, $z' = 3.5$ m, and $t' = 0$. System K′ and K have their axes coincident at $t = t' = 0$, and system K′ travels along the x axis of system K with a speed $0.8c$. What are the coordinates of the event in system K?
17. A light signal is sent from the origin of a system K at $t = 0$ to the point $x = 3$ m, $y = 5$ m, $z = 10$ m. (a) At what time t is the signal received? (b) Find (x', y', z', t') for the receipt of the signal in a frame K′ that is moving along the x axis of K at a speed of $0.8c$. (c) From your results in (b) verify that the light traveled with a speed c as measured in the K′ frame.

2.5 Time Dilation and Length Contraction

18. Show that the experiment depicted in Figure 2.11 and discussed in the text leads directly to the derivation of length contraction.
19. A rocket ship carrying passengers blasts off to go from New York to Los Angeles, a distance of about 5000 km. (a) How fast must the rocket ship go to have its own length shortened by 1%? (b) Ignore effects of general relativity and determine how much time the rocket ship's clock and the ground-based clocks differ when the rocket ship arrives in Los Angeles.
20. Astronomers discover a planet orbiting around a star similar to our sun that is 20 lightyears away. How fast must a rocket ship go if the round trip is to take no longer than 40 years in time for the astronauts aboard? How much time will the trip take as measured on Earth?
21. Particle physicists use particle track detectors to determine the lifetime of short-lived particles. A muon has a mean lifetime of 2.2 μs and makes a track 9.5 cm long before decaying into an electron and two neutrinos. What was the speed of the muon?

22. The Apollo astronauts returned from the moon under the Earth's gravitational force and reached speeds of almost 25,000 mi/h with respect to Earth. Assuming (incorrectly) they had this speed for the entire trip from the moon to Earth, what was the time difference for the trip between their clocks and clocks on Earth?

23. A clock in a spaceship is observed to run at a speed of only 3/5 that of a similar clock at rest on Earth. How fast is the spaceship moving?

24. A spaceship of length 40 m at rest is observed to be 20 m long when in motion. How fast is it moving?

25. The Concorde traveled 8000 km between two places in North America and Europe at an average speed of 375 m/s. What is the total difference in time between two similar atomic clocks, one on the airplane and one at rest on Earth during a one-way trip? Consider only time dilation and ignore other effects such as Earth's rotation.

26. A mechanism on Earth used to shoot down geosynchronous satellites that house laser-based weapons is finally perfected and propels golf balls at $0.94c$. (Geosynchronous satellites are placed 3.58×10^4 km above the surface of the Earth.) (a) What is the distance from the Earth to the satellite, as measured by a detector placed inside the golf ball? (b) How much time will it take the golf ball to make the journey to the satellite in the Earth's frame? How much time will it take in the golf ball's frame?

27. Two events occur in an inertial system K at the same time but 4 km apart. What is the time difference measured in a system K′ moving parallel to these two events when the distance separation of the events is measured to be 5 km in K′?

28. Imagine that in another universe the speed of light is only 100 m/s. (a) A person traveling along an interstate highway at 120 km/h ages at what fraction of the rate of a person at rest? (b) This traveler passes by a meterstick at rest on the highway. How long does the meterstick appear?

29. In another universe where the speed of light is only 100 m/s, an airplane that is 40 m long at rest and flies at 300 km/h will appear to be how long to an observer at rest?

30. Two systems K and K′ synchronize their clocks at $t = t' = 0$ when their origins are aligned as system K′ passes by system K along the x axis at relative speed $0.8c$. At time $t = 3$ ns, Frank (in system K) shoots a proton gun having proton speeds of $0.98c$ along his x axis. The protons leave the gun at $x = 1$ m and arrive at a target 120 m away. Determine the event coordinates (x, t) of the gun firing and of the protons arriving as measured by observers in both systems K and K′.

2.6 Addition of Velocities

31. A spaceship is moving at a speed of $0.84c$ away from an observer at rest. A boy in the spaceship shoots a proton gun with protons having a speed of $0.62c$. What is the speed of the protons measured by the observer at rest when the gun is shot (a) away from the observer and (b) toward the observer?

32. A proton and an antiproton are moving toward each other in a head-on collision. If each has a speed of $0.8c$ with respect to the collision point, how fast are they moving with respect to each other?

33. Imagine the speed of light in another universe to be only 100 m/s. Two cars are traveling along an interstate highway in opposite directions. Person 1 is traveling 110 km/h, and person 2 is traveling 140 km/h. How fast does person 1 measure person 2 to be traveling? How fast does person 2 measure person 1 to be traveling?

34. In the Fizeau experiment described in Example 2.5, suppose that the water is flowing at a speed of 5 m/s. Find the difference in the speeds of two beams of light, one traveling in the same direction as the water and the other in the opposite direction. Use $n = 1.33$ for water.

35. Three galaxies are aligned along an axis in the order A, B, C. An observer in galaxy B is in the middle and observes that galaxies A and C are moving in opposite directions away from him, both with speeds $0.60c$. What is the speed of galaxies B and C as observed by someone in galaxy A?

36. Consider the *gedanken* experiment discussed in Section 2.6 in which a giant floodlight stationed 400 km above the Earth's surface shines its light across the moon's surface. How fast does the light flash across the moon?

2.7 Experimental Verification

37. A group of scientists decide to repeat the muon decay experiment at the Mauna Kea telescope site in Hawaii, which is 4205 m above sea level. They count 10^4 muons during a certain time period. Repeat the calculation of Section 2.7 and find the classical and relativistic number of muons expected at sea level. Why did they decide to count as many as 10^4 muons instead of only 10^3?

38. Consider a reference system placed at the U.S. Naval Observatory in Washington, D.C. Two planes take off from Washington Dulles Airport, one going eastward and one going westward, both carrying a cesium atomic clock. The distance around the Earth at 39° latitude (Washington, D.C.) is 31,000 km, and Washington rotates about the Earth's axis at a speed of 360 m/s. Calculate the predicted differences between the clock left at the observatory and the two clocks in the airplanes (each traveling at 300 m/s) when the airplanes return to Washington. Include the rotation of the Earth but no general relativistic effects. Compare with the predictions given in the text.

2.8 Twin Paradox

39. Derive the results in Table 2.1 for the frequencies f' and f''. During what time period do Frank and Mary receive these frequencies?

40. Derive the results in Table 2.1 for the time of the total trip and the total number of signals sent in the frame of both twins. Show your work.

2.9 Spacetime

41. Use the Lorentz transformation to prove that $s^2 = s'^2$.

42. Prove that for a timelike interval, two events can never be considered to occur simultaneously.

43. Prove that for a spacelike interval, two events cannot occur at the same place in space.

44. Given two events, (x_1, t_1) and (x_2, t_2), use a spacetime diagram to find the speed of a frame of reference in which the two events occur simultaneously. What values may Δs^2 have in this case?

45. (a) Draw on a spacetime diagram in the fixed system a line expressing all the events in the moving system that occur at $t' = 0$ if the clocks are synchronized at $t = t' = 0$. (b) What is the slope of this line? (c) Draw lines expressing events occurring for the four times t'_4, t'_3, t'_2, and t'_1 where $t'_4 < t'_3 < 0 < t'_2 < t'_1$. (d) How are these four lines related geometrically?

46. Consider a fixed and a moving system with their clocks synchronized and their origins aligned at $t = t' = 0$. (a) Draw on a spacetime diagram in the fixed system a line expressing all the events occurring at $t' = 0$. (b) Draw on this diagram a line expressing all the events occurring at $x' = 0$. (c) Draw all the worldlines for light that pass through $t = t' = 0$. (d) Are the x' and ct' axes perpendicular? Explain.

47. Use the results of the two previous problems to show that events simultaneous in one system are not simultaneous in another system moving with respect to the first. Consider a spacetime diagram with x, ct and x', ct' axes drawn such that the origins coincide and the clocks were synchronized at $t = t' = 0$. Then consider events 1 and 2 that occur simultaneously in the fixed system. Are they simultaneous in the moving system?

2.10 Doppler Effect

48. An astronaut is said to have tried to get out of a traffic violation for running a red light ($\lambda = 650$ nm) by telling the judge that the light appeared green ($\lambda = 540$ nm) to her as she passed by in her high-powered transport. If this is true, how fast was the astronaut going?

49. Derive Equation (2.32) for the case where the source is fixed but the receiver approaches it with velocity v.

50. Do the complete derivation for Equation (2.33) when the source and receiver are receding with relative velocity v.

51. A spacecraft traveling out of the solar system at a speed of $0.95c$ sends back information at a rate of 1400 kHz. At what rate do we receive the information?

52. Three radio-equipped plumbing vans are broadcasting on the same frequency f_0. Van 1 is moving east of van 2 with speed v, van 2 is fixed, and van 3 is moving west of van 2 with speed v. What is the frequency of each van as received by the others?

53. Three radio-equipped plumbing vans are broadcasting on the same frequency f_0. Van 1 is moving north of van 2 with speed v, van 2 is fixed, and van 3 is moving west of van 2 with speed v. What frequency does van 3 hear from van 2; from van 1?

54. A spaceship moves radially away from Earth with acceleration 29.4 m/s^2 (about $3g$). How much time does it take for the sodium streetlamps ($\lambda = 589$ nm) on Earth to be invisible (with a powerful telescope) to the human eye of the astronauts? The range of visible wavelengths is about 400 to 700 nm.

2.11 Relativistic Momentum

55. Newton's second law is given by $\vec{F} = d\vec{p}/dt$. If the force is always perpendicular to the velocity, show that $\vec{F} = m\gamma\vec{a}$, where $\vec{a}$ is the acceleration.

56. Use the result of the previous problem to show that the radius of a particle's circular path having charge q traveling with speed v in a magnetic field perpendicular to the particle's path is $r = p/qB$. What happens to the radius as the speed increases as in a cyclotron?

57. Newton's second law is given by $\vec{F} = d\vec{p}/dt$. If the force is always parallel to the velocity, show that $\vec{F} = \gamma^3 m\vec{a}$.

58. Find the force necessary to give a proton an acceleration of 10^{19} m/s^2 when the proton has a velocity (along the same direction as the force) of (a) $0.01c$, (b) $0.1c$, (c) $0.9c$, and (d) $0.99c$.

59. A particle having a speed of $0.92c$ has a momentum of 10^{-16} kg · m/s. What is its mass?

60. A particle initially has a speed of $0.5c$. At what speed does its momentum increase by (a) 1%, (b) 10%, (c) 100%?

61. The Bevatron accelerator at the Lawrence Berkeley Laboratory accelerated protons to a kinetic energy of 6.3 GeV. What magnetic field was necessary to keep the protons traveling in a circle of 15.2 m? (See Problem 56.)

62. Show that linear momentum is conserved in Example 2.9 as measured by Mary.

2.12 Relativistic Energy

63. Show that $\frac{1}{2}m\gamma v^2$ does not give the correct kinetic energy.

64. How much ice must melt at 0°C in order to gain 2 g of mass? Where does this mass come from? The heat of fusion for water is 334 J/g.

65. Physicists at the Stanford Linear Accelerator Center (SLAC) bombarded 9-GeV electrons head-on with 3.1-GeV positrons to create B mesons and anti-B mesons. What speeds did the electron and positron have when they collided?

66. The Tevatron accelerator at the Fermi National Accelerator Laboratory (Fermilab) outside Chicago boosts protons to 1 TeV (1000 GeV) in five stages (the numbers given in parentheses represent the total kinetic energy at the end of each stage): Cockcroft-Walton (750 keV), Linac (400 MeV), Booster (8 GeV), Main ring or injector (150 GeV), and finally the Tevatron itself (1 TeV). What is the speed of the proton at the end of each stage?

67. Calculate the momentum, kinetic energy, and total energy of an electron traveling at a speed of (a) $0.020c$, (b) $0.20c$, and (c) $0.90c$.

68. The total energy of a body is found to be twice its rest energy. How fast is it moving with respect to the observer?

69. A system is devised to exert a constant force of 8 N on an 80-kg body of mass initially at rest. The force pushes the mass horizontally on a frictionless table. How far does the body have to be pushed to increase its mass-energy by 25%?

70. What is the speed of a proton when its kinetic energy is equal to twice its rest energy?

71. What is the speed of an electron when its kinetic energy is (a) 10% of its rest energy, (b) equal to the rest energy, and (c) 10 times the rest energy?

72. Derive the following equation:

$$\beta = \frac{v}{c} = \sqrt{1 - \left(\frac{E_0}{E_0 + K}\right)^2}$$

73. Prove that $\beta = pc/E$. This is a useful relation to find the velocity of a highly energetic particle.

74. A good rule of thumb is to use relativistic equations whenever the kinetic energies determined classically and relativistically differ by more than 1%. Find the speeds when this occurs for (a) electrons and (b) protons.

75. How much mass-energy (in joules) is contained in a peanut weighing 0.1 ounce? How much mass-energy do you gain by eating 10 ounces of peanuts? Compare this with the food energy content of peanuts, about 100 kcal per ounce.

76. Calculate the energy needed to accelerate a spaceship of mass 10,000 kg to a speed of $0.3c$ for intergalactic space exploration. Compare this with a projected annual energy usage on Earth of 10^{21} J.

77. Derive Equation (2.58) for the relativistic kinetic energy and show all the steps, especially the integration by parts.

78. A test automobile of mass 1000 kg moving at high speed crashes into a wall. The average temperature of the car is measured to rise by 0.5°C after the wreck. What is the change in mass of the car? Where does this change in mass come from? (Assume the average specific heat of the automobile is close to that of steel, $0.11\ \text{cal} \cdot \text{g}^{-1} \cdot {}^\circ\text{C}^{-1}$.)

2.13 Computations in Modern Physics

79. A helium nucleus has a mass of 4.001505 u. What is its binding energy?

80. A free neutron is an unstable particle and beta decays into a proton with the emission of an electron. How much kinetic energy is available in the decay?

81. The Large Hadron Collider at Europe's CERN facility is designed to produce 7.0 TeV (that is, 7.0×10^{12} eV) protons. Calculate the speed, momentum, and total energy of the protons.

82. What is the kinetic energy of (a) an electron having a momentum of 40 GeV/c? (b) a proton having a momentum of 40 GeV/c?

83. A muon has a mass of 106 MeV/c^2. Calculate the speed, momentum, and total energy of a 200-MeV muon.

84. The reaction $^2\text{H} + {}^2\text{H} \rightarrow n + {}^3\text{He}$ (where n is a neutron) is one of the reactions useful for producing energy through nuclear fusion. (a) Assume the deuterium nuclei (^{2}H) are at rest and use the atomic mass units of the masses in Appendix 8 to calculate the mass-energy imbalance in this reaction. (Note: You can use atomic masses for this calculation, because the electron masses cancel out.) This amount of energy is given up when this nuclear reaction occurs. (b) What percentage of the initial rest energy is given up?

85. The reaction $^2\text{H} + {}^3\text{H} \rightarrow n + {}^4\text{He}$ is one of the reactions useful for producing energy through nuclear fusion. (a) Assume the deuterium (^{2}H) and tritium (^{3}H) nuclei are at rest and use the atomic mass units of the masses in Appendix 8 to calculate the mass-energy imbalance in this reaction. This amount of energy is given up when this nuclear reaction occurs. (b) What percentage of the initial rest energy is given up?

2.14 Electromagnetism and Relativity

86. Instead of one positive charge outside a conducting wire, as was discussed in Section 2.14 and shown in Figure 2.34, consider a second conducting wire parallel to the first one. Both wires have positive and negative charges, and the wires are electrically neutral. Assume that in both wires the positive charges travel to the right and negative charges to the left. (a) Consider an inertial frame moving with the negative charges of wire 1. Show that the second wire is attracted to the first wire in this frame. (b) Now consider an inertial frame moving with the positive charges of the second wire. Show that the first wire is attracted to the second. (c) Use this argument to show that electrical and magnetic forces are relative.

General Problems

87. An Ω^- particle has rest energy 1672 MeV and mean lifetime 8.2×10^{-11} s. It is created and decays in a particle track detector and leaves a track 24 mm long. What is the total energy of the Ω^- particle?

88. Show that the following form of Newton's second law satisfies the Lorentz transformation. Assume the force is parallel to the velocity.

$$F = m\frac{dv}{dt}\frac{1}{[1 - (v^2/c^2)]^{3/2}}$$

89. Use the results listed in Table 2.1 to find (a) the number of signals Frank receives at the rate f' and the time at which Frank detects Mary's turnaround, and (b) the number of signals Mary receives at the rate f' and her clock reading when she turns around. (c) From Frank's perspective, find the time for the remainder of the trip (after he detects Mary's turnaround), the number of signals he receives at the rate f'', the total number of signals he receives, and Mary's age, based on that total number of signals. (d) From Mary's perspective, find the time for the remainder of the trip (after her turnaround), the number of signals she receives at the rate f'', the total number of signals she receives, and Frank's age, based on that total number of signals.

90. For the twins Frank and Mary described in Section 2.8, consider Mary's one-way trip at a speed of $0.8c$ to the star system 8 lightyears from Earth. Compute the spacetime interval s in the fixed frame and s' in the moving frame, and compare the results.

91. Frank and Mary are twins. Mary jumps on a spaceship and goes to the star system Alpha Centauri (4.30 lightyears away) and returns. She travels at a speed of $0.8c$ with respect to Earth and emits a radio signal every week. Frank also sends out a radio signal to Mary once a week. (a) How many signals does Mary receive from Frank before she turns around? (b) At what time does the frequency of signals Frank receives suddenly change? How many signals has he received at this time? (c) How many signals do Frank and Mary receive for the entire trip? (d) How much time does the trip take according to Frank and to Mary? (e) How much time does each twin say the other twin will measure for the trip? Do the answers agree with those for (d)?

92. A police radar gun operates at a frequency of 10.5 GHz. The officer, sitting in a patrol car at rest by the highway, directs the radar beam toward a speeding car traveling 80 mph directly away from the patrol car. What is the frequency shift of the reflected beam, relative to the original radar beam?

93. A spaceship moving $0.80c$ direction away from Earth fires a missile that the spaceship measures to be moving at $0.80c$ perpendicular to the ship's direction of travel. Find the velocity components and speed of the missile as measured by Earth.

94. An electron has a total energy that is 250 times its rest energy. Determine its (a) kinetic energy, (b) speed, and (c) momentum.

95. A proton moves with a speed of $0.90c$. Find the speed of an electron that has (a) the same momentum as the proton, and (b) the same kinetic energy.

96. A high-speed K^0 meson is traveling at a speed of $0.90c$ when it decays into a π^+ and a π^- meson. What are the greatest and least speeds that the mesons may have?

97. Frank and Mary are twins, and Mary wants to travel to our nearest star system, Alpha Centauri (4.30 lightyears away). Mary leaves on her 30th birthday and intends to return to Earth on her 52nd birthday. (a) Assuming her spaceship returns from Alpha Centauri without stopping, how fast must her spaceship travel? (b) How old will Frank be when she returns?

98. The International Space Federation constructs a new spaceship that can travel at a speed of $0.995c$. Mary, the astronaut, boards the spaceship to travel to Barnard's star, which is the second nearest star to our solar system after Alpha Centauri and is 5.98 lightyears away. After reaching Barnard's star, the spaceship travels slowly around the star system for three years doing research before returning back to Earth. (a) How much time does her journey take? (b) How much older is her twin Frank than Mary when she returns?

99. A powerful laser on Earth rotates its laser beam in a circle at a frequency of 0.030 Hz. (a) How fast does the spot that the laser makes on the moon move across the moon's landscape? (b) With what rotation frequency should the laser rotate if the laser spot moves across the moon's landscape at speed c?

100. The Lockheed SR-71 Blackbird may be the fastest non-research airplane ever built; it traveled at 2200 miles/hour (983 m/s) and was in operation from 1966 to 1990. Its length is 32.74 m. (a) By what percentage would it appear to be length contracted while in flight? (b) How much time difference would occur on an atomic clock in the plane compared to a similar clock on Earth during a flight of the Blackbird over its range of 3200 km?

101. A spaceship is coming directly toward you while you are in the International Space Station. You are told that the spaceship is shining sodium light (with an intense yellow doublet of wavelengths 588.9950 and 589.5924 nm). You have an apparatus that can resolve two closely spaced wavelengths if the difference is $\Delta\lambda < 0.55$ nm. If you find that you can just resolve the doublet, how fast is the spaceship traveling with respect to you?

102. Quasars are among the most distant objects in the universe and are moving away from us at very high speeds, as discussed in Chapter 16. Astrophysicists use the redshift parameter z to determine the redshift of such rapidly moving objects. The parameter z is determined by observing a wavelength λ' of a known spectral line of wavelength λ_{source} on Earth; $z = \Delta\lambda/\lambda_{\text{source}} = (\lambda' - \lambda_{\text{source}})/\lambda_{\text{source}}$. Find the speed of two quasars having z values of 1.9 and 4.9.

103. One possible decay mode of the neutral kaon is $K^0 \rightarrow \pi^0 + \pi^0$. The rest energies of the K^0 and π^0 are 498 MeV and 135 MeV, respectively. The kaon is initially at rest when it decays. (a) How much energy is released in the decay? (b) What are the momentum and relative directions of the two neutral pions (π^0)?

104. The sun radiates energy at a rate of 3.9×10^{26} W. (a) At what rate is the sun losing mass? (b) At that rate, how much time would it take to exhaust the sun's fuel supply? The sun's mass is 2.0×10^{30} kg, and you may assume that the reaction producing the energy is about 0.7% efficient. Compare your answer with the sun's expected remaining lifetime, about 5 Gy.

105. One way astrophysicists have identified "extrasolar" planets orbiting distant stars is by observing redshifts or blueshifts in the star's spectrum due to the fact that the star and planet each revolve around their common center of mass. (See *Scientific American,* August 2010, p. 41.) Consider a star the size of our sun (mass $= 1.99 \times 10^{30}$ kg), with a planet the size of Jupiter (1.90×10^{27} kg) in a circular orbit of radius 7.79×10^{11} m and a period of 11.9 years. (a) Find the speed of the star revolving around the system's center of mass. (b) Assume that Earth is in the planet's orbital plane, so that at one point in its orbit the star is moving directly toward Earth, and at the opposite point it moves directly away from Earth. How much is 550-nm light redshifted and blueshifted at those two extreme points?

106. Small differences in the wavelengths in the sun's spectrum are detected when measurements are taken from different parts of the sun's disk. Specifically, measurements of the 656-nm line in hydrogen taken from opposite sides on the sun's equator—one side approaching Earth and the other receding—differ from each other by 0.0090 nm. Use this information to find the rotational period of the sun's equator. Express your answer in days. (The sun's equatorial radius is 6.96×10^8 m.)