1 Introduction

My research interests lie in the intersection of **algebra**, **topology**, and **geometry**, in the study of representations of the fundamental groups of orbifolds and manifolds, particularly hyperbolic triangle groups. As we will see in Section 2, deformation spaces of these representations encode information about geometric structures on orbifolds. On a more algebraic note, finding explicit representations provides examples of Zariski dense surface subgroups of $SL(5,\mathbb{R})$ in $SL(5,\mathbb{Z})$, discussed in Section 3. First, however, we introduce some history and terminology surrounding hyperbolic triangle groups and higher Teichmüller theory.

A **hyperbolic triangle group** $T(p,q,r)$ is the group of rotational symmetries of a tiling of the hyperbolic plane $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, and π/r, and admits a presentation $T(p,q,r) := \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle$. Topologically, $T(p,q,r)$ is the orbifold fundamental group of a sphere with cone points of order p, q, and r. One motivation for studying triangle groups is that they contain surface groups as subgroups of finite index, and a result of Edmonds, Ewing, and Kulkarni in [4] determines precisely which surface groups are contained in a particular triangle group.

Studying representations of orbifolds is made considerably easier by using a concept introduced by Hitchin in 1992, namely a generalization of Teichmüller space that is now often referred to as the Hitchin component [5]. While Teichmüller space is generally understood as a space parameterizing conformal structures on a surface S, there is an equivalent characterization in the language of representations of the fundamental group of S. Conformal structures on S correspond to a quotient of $\mathbb{H}^2$ by the image of a faithful representation $\pi_1(S) \to PSL(2,\mathbb{R})$. Teichmüller space has the same dimension as the topologically connected component of these homomorphisms, modded out by conjugation in $PSL(2,\mathbb{R})$.

The **Hitchin component** of a surface S is a generalization of Teichmüller space that replaces $PSL(2,\mathbb{R})$ with $PSL(n,\mathbb{R})$ in the above description. In 2006, Labourie showed that representations in the Hitchin component are discrete and faithful [6]. Further, the Hitchin component also retains some of the geometric significance of the Teichmüller component. In 2005, for the $n = 3$ case, Goldman and Choi showed that the dimension of the Hitchin component for an orbifold Σ corresponds to the dimension of the deformation space of convex real projective structures on Σ [2]. Hitchin also proved that the dimension of the Hitchin component for a genus $g > 1$ surface is $(2g-2)\dim(PSL(n,\mathbb{R}))$ [5], but until very recently, less progress had been made for orbifolds in $PSL(n,\mathbb{R})$. In the $n = 3$ case, Choi and Goldman give a formula for the dimension of the Hitchin component in terms of the Euler characteristic and local orbifold features [2]; a November 2018 preprint of Alessandrini, Lee, and Schaffhauser resolves the question for orbifolds.

Recently, partial progress toward understanding the degree n Hitchin component for an orbifold was made by Long and Thistlethwaite, in the special case of an arbitrary hyperbolic triangle group $T(p,q,r)$ in $PSL(n,\mathbb{R})$ [9]. However, the definition of the Hitchin component extends beyond just $PSL(n,\mathbb{R})$ to the adjoint group of the split real form of any complex simple Lie group. In particular, one can define the Hitchin component in the case where $SL(n,\mathbb{R})$ is replaced by $Sp(2m)$ or $SO(m,m+1)$. Our contribution toward understanding the Hitchin component appears in Section 2, in the form of a formula for the Hitchin component for all hyperbolic triangle groups and degrees $n \geq 3$, with images in these symplectic or special orthogonal groups. While this was proven more generally in November

2018 by [1], the methods used for Section 2 are completely different and decidedly simpler, largely linear algebra and Lie theory accessible to a typical graduate student.

However, while there has been some progress in understanding the geometric significance of the Hitchin component for large n (see Section 5 of [12] for a summary of this progress), much in this area is still unknown. Even so, computing specific representations in the Hitchin component is interesting from an algebraic perspective. By continuously deforming a representation of $T(3,3,4) \to SL(5,\mathbb{R})$ in the Hitchin component, we can obtain other discrete, faithful representations. This, along with our earlier discussion of surface subgroups as well as some number theoretic considerations, allows us to find infinite families of Zariski dense surface subgroups of $SL(5,\mathbb{Z})$ in Section 3. In Section 4, we draw connections between the projects in Sections 2 and 3, and provide ideas for future work based on those projects.

2 Deformation Spaces for Triangle Groups in Sp and SO

Consider the representation ρ_0 that takes each rotation of the upper half plane in a triangle group, writes it as a Möbius transformation with coefficient matrix in $PSL(2,\mathbb{R})$, and sends that matrix through the unique irreducible representation $PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$. This base representation is contained in the Hitchin component, and forms the starting point for Long and Thistlethwaite's work in [9]. Further, for each generator $g \in T(p,q,r)$, we note that $\rho_0(g)$ is contained in $Sp(2m)$ if n is even, and $SO(m,m+1)$ if n is odd. Naturally, one might wonder what happens to the dimension of the Hitchin component when restricted to one of these matrix subgroups of $SL(n,\mathbb{R})$.

Unfortunately, the approach of Long and Thistlethwaite in [9], which counts multiplicities of eigenvalues for each generator, does not translate directly to the Hitchin component for $Sp(2m)$ and $SO(m,m+1)$, as it does not preserve the additional group relations. However, counting multiplicities of real eigenvalues and pairs of complex conjugate eigenvalues separately allows for an approach that does not destroy group relations. The cost is that counting multiplicities becomes considerably more difficult, and requires breaking the argument into cases based on the degree n of the representation, the order k of the generator, and the quotient Q of integer division $n\backslash k$. Doing so, we obtain the following for the dimension of the centralizer of a cyclic generator (where R is the remainder of $n\backslash k$):

$$\sigma_G(n,k) := \frac{1}{2}\left((n+R)Q + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right).$$

The rest of the approach used to prove the theorem below primarily follows the Lie theory used in [9], with some additional difficulty in finding a diagonal basis for the adjoint operator in the Lie algebra for $SO(m,m+1)$ and $Sp(2m)$ compared to $SL(n,\mathbb{R})$.

Theorem 2.1. *If $n = 2m$ or $2m+1$, let $G = Sp(2m)$ or $SO(m,m+1)$ respectively, and let $\mathcal{H}_G$ be the restriction of the Hitchin component to representations of the triangle group $T(p,q,r)$ lying in G. Then*

$$dim\mathcal{H}_G = 2\dim G - \left(\sigma_G(n,p) + \sigma_G(n,q) + \sigma_G(n,r)\right).$$

3 Infinitely Many Zariski Dense Surface Subgroups of $SL(5,\mathbb{Z})$

Recently, Long and Thistlethwaite produced a one-parameter family of discrete, faithful, Zariski dense representations $\rho_k : T(3,3,4) \to SL(5,\mathbb{Z})$ that remain pairwise non-conjugate when restricted to surface subgroups of $T(3,3,4)$ [8]. The triangle group $T(3,3,4)$ is of particular interest because, due to the result mentioned earlier from [4], it has the fundamental groups of all genus $g \geq 2$ orientable surfaces as finite-index subgroups.

Using Theorem 1.1 of [9], we observe that the Hitchin component for $T(3,3,4)$ in $SL(5,\mathbb{R})$ is 2-dimensional. As a result, this 1-parameter family cannot possibly exhaust all representations in the Hitchin component, and one might hope for further infinite families. Using some of the machinery developed by Cooper, Long, and Thistlethwaite in [3] and applied to triangle groups in [7] and [8], we produce two additional infinite families.

Theorem 3.1. *The two families of representations of the triangle group given below satisfy:*

(a) τ_k and υ_k are discrete and faithful for every $k \in \mathbb{R}$.

(b) For each $k \in \mathbb{Z}_{\geq 0}$, the images of τ_k and υ_k are Zariski dense and pairwise nonconjugate in $SL(5,\mathbb{R})$.

(c) For each surface subgroup $\Gamma \leq T(3,3,4)$, there are subsequences $\{\tau_{k_i}\}_{i\in\mathbb{N}}$ and $\{\upsilon_{k_i}\}_{i\in\mathbb{N}}$ such that $\tau_{k_i}(\Gamma)$ and $\upsilon_{k_i}(\Gamma)$ are pairwise non-conjugate surface subgroups in $SL(5,\mathbb{Z})$.

$$\tau_k, \upsilon_k : T(3,3,4) = \langle a, b | a^3 = b^3 = (ab)^4 = 1 \rangle \to SL(5,\mathbb{R})$$

$$\tau_k(a) = \begin{bmatrix} -1 & -1 & 0 & 0 & 3 \\ 4 & -3 & 3 & 0 & 3 \\ 3 & 0 & 1 & 0 & -3 \\ -2(3+4k+8k^2) & 0 & 0 & 1 & -2 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix}, \upsilon_k(a) = \begin{bmatrix} -1 & 1 & 0 & 0 & 3 \\ -1 & -3 & -3 & 0 & -3 \\ 1 & 1 & 1 & 0 & 0 \\ -2(7+12k+8k^2) & 5 & 5 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\tau_k(b) = \begin{bmatrix} 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & -3+4k \\ -1 & 0 & 0 & -2k & -4-8k^2 \\ 0 & 0 & 0 & 2k(3+4k+8k^2) & 1+4k+32k^2+32k^3+64k^4 \\ 0 & 0 & 0 & -1-2k-4k^2 & -1-6k-8k^2-16k^3 \end{bmatrix},$$

$$\upsilon_k(b) = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & -1 & -1 & 6-4k \\ -1 & 1 & 0 & -1-2k & 2+2k-8k^2 \\ 0 & 0 & 0 & 4(-2-k+3k^2+4k^3) & 19-18k-44k^2+64k^4 \\ 0 & 0 & 0 & -3-6k-4k^2 & 7+4k-12k^2-16k^3 \end{bmatrix}$$

The computer methods used involve a directed form of Newton's method, which is used in [8] to find the same two-parameter family of representations $\rho_{(u,v)} : T(3,3,4) \to SL(5,\mathbb{C})$ that we use as a starting point. These representations are not necessarily in the Hitchin component, but the trace field is $\mathbb{Q}(u,v)(\alpha)$, where α is the square root of a polynomial in the parameters u and v. If we plot the integer coordinates (u,v) that make α an integer, coordinates "missing" from [8] appear and can be grouped into two single-parameter families by polynomial interpolation. Even restricting to this new parameterization, though, matrix entries appear in $\mathbb{Q}[i]$ rather than $\mathbb{Z}$. To address this, we use an approach developed in

[8] that relies on the dimension of certain nullspaces to find a basis over $\mathbb{Q}$, followed by an ad-hoc conjugation to produce matrices in $SL(5,\mathbb{Z})$, as desired.

Once the families above are found, it remains to show that each one produces infinitely many pairwise nonconjugate surface subgroups in $SL(5,\mathbb{Z})$ for each of the surface groups found in $T(3,3,4)$. This is the result of a delicate argument using quadratic reciprocity from number theory, along with Dirichlet's theorem on arithmetic progressions.

4 Connections and Future Directions

While both of the previous sections concern Hitchin representations of hyperbolic triangle groups, there are further connections between these two projects. Theorem 2.1 can be used to show that the dimension of the Hitchin component of $T(3,3,4)$ in $SO(2,3)$ is 0, meaning that the only Hitchin representation with image in $SO(2,3)$ is the base representation. There is experimental evidence to suggest that this happens in only a few isolated instances; of considerably more interest is the case where the dimension of the Hitchin component for $SL(n,\mathbb{R})$ matches that of $SO(m,m+1)$ or $Sp(2m)$. In that case, there are no representations of $T(p,q,r)$ with Zariski dense images in the Hitchin component, since the closure of each image will be contained in a proper Lie subgroup of $SL(n,\mathbb{R})$.

The methods used by Long and Thistlethwaite to determine the dimension of the Hitchin component in [9] (and by extension, those used in Section 2) rely heavily on the algebraic structure of triangle groups. It is likely possible to use a similar approach for other classes of orbifold groups, particularly considering the narrow characterization of local behavior for 2-orbifolds in a proposition from Chapter 13 of Thurston's notes [11].

Question 1. Can the approach in Section 2 be extended to a general 2-orbifold?

Our discussions have focused on the Hitchin component, but there is another side to the story told so far. If we allow $SL(n,\mathbb{R})$ to have complex entries, we get $SL(n,\mathbb{C})$. The anti-holomorphic involution that takes entries in $\mathbb{C}$ to their complex conjugates has fixed points in $SL(n,\mathbb{R})$, but another anti-holomorphic involution has fixed points in $SU(\ell,m)$; our base representation $\sigma_0 : T(p,q,r) \to SL(n,\mathbb{R})$ has its image in $SL(n,\mathbb{R}) \cap SU(\ell,m)$. Further, these two Lie groups have the same dimension, leading naturally to our final question:

Question 2. What can be said about the deformation space for an orbifold group into SU?

These deformations would no longer be in the Hitchin component, but may have interesting geometric or topological properties of their own. A computational foray into this deformation space as in Section 3 may also produce interesting results.

Finally, a recent experience in the 2018 AMS MRC on Number Theoretic Methods in Hyperbolic Geometry has taken my interest in triangle groups in a new direction. In particular, Michelle Chu, Samantha Fairchild, and I have been examining the following question posted by Reid in [10]:

Question 3. Is a Hurwitz surface determined up to isometry by its length spectrum?

Hurwitz surfaces are surfaces with maximal possible symmetry (for a fixed genus), and their Fuchsian groups are a normal subgroup of the $(2,3,7)$ triangle group. Preliminary investigation suggests that the representation theory of triangle groups could contribute to an answer to this question.

References

[1] D. Alessandrini, G.-S. Lee, and F. Schaffhauser, *Hitchin components for orbifolds*, arXiv:1811.05366 (2018).

[2] S. Choi and W.M. Goldman, *The deformation spaces of convex $\mathbb{R}P^2$-structures on 2-orbifolds*, Amer. J. Math **127** (2005), no. 5, 1019–1102.

[3] D. Cooper, D.D. Long, and M.B. Thistlethwaite, *Computing varieties of representations of hyperbolic 3-manifolds into* $\mathrm{SL}(4, \mathbb{R})$, Experiment. Math. **15** (2006), no. 3, 291–305. MR 2264468

[4] A.L. Edmonds, J.H. Ewing, and R.S. Kulkarni, *Torsion free subgroups of Fuchsian groups and tessellations of surfaces*, Invent. Math. **69** (1982), no. 3, 331–346.

[5] N.J. Hitchin, *Lie groups and teichmüller space*, Topology **31** (1992), no. 3, 449–473.

[6] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), 51–114.

[7] D.D. Long, A.W. Reid, and M.B. Thistlethwaite, *Zariski dense surface subgroups in* $\mathrm{SL}(3, \mathbf{Z})$, Geom. Topol. **15** (2011), no. 1, 1–9.

[8] D.D. Long and M.B. Thistlethwaite, *Zariski dense surface subgroups in* $SL(4, \mathbb{Z})$, Experiment. Math. **15** (2016), 291–305.

[9] D.D. Long and M.B. Thistlethwaite, *The dimension of the hitchin component for triangle groups*, To appear, Geom. Dedicata (2018), `http://web.math.ucsb.edu/~long/pubpdf/slnr_submitted.pdf`.

[10] Alan W. Reid, *Traces, lengths, axes and commensurability*, Annales de la Faculté des sciences de Toulouse : Mathématiques **Ser. 6, 23** (2014), no. 5, 1103–1118 (en). MR 3294604

[11] W.P. Thurston, *The geometry and topology of three-manifolds*, Princeton University Math Dept. Notes (1979), `http://library.msri.org/books/gt3m/`.

[12] A. Wienhard, *An invitation to higher Teichmüller theory*, arXiv:1803.06870 (2018).