The Dimension of the Restricted Hitchin Component for a Triangle Group

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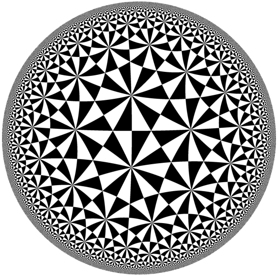
Background: Triangle Groups

Definition

$T(p, q, r)$ is the group of rotational symmetries of a tiling of the hyperbolic plane $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, and π/r, and admits a presentation

$$T(p, q, r) := \langle a, b, c \,|\, a^p = b^q = c^r = abc = 1 \rangle .$$



We start with the representation $\rho_0 : T(p, q, r) \to PSL(n, \mathbb{R})$, obtained by considering $T(p, q, r)$ as a subgroup of the orientation-preserving isometries of $\mathbb{H}^2$ and composing with the unique irreducible representation

$$PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R}).$$

Background: Hitchin Component

Definition

A *surface group* is the fundamental group of a compact surface.

- Hitchin (1992): worked out the dimensions of deformation spaces of surface groups in $PSL(n, \mathbb{R})$.

Definition

Consider a representation ρ_0 of an orbifold group $\pi_1(\Sigma)$ found by embedding $\pi_1(\Sigma)$ as a discrete subgroup of the order-preserving isometries of $\mathbb{H}^2$ and composing with the irreducible representation into $PSL(n, \mathbb{R})$. The ***Hitchin component*** $\mathcal{H}$ of the space of representations $\pi_1(\Sigma) \to PSL(n, \mathbb{R})$ is the connected component containing ρ_0.

Background: Hitchin Component

- Hitchin (1992): worked out the dimensions of deformation spaces of surface groups in $PSL(n, \mathbb{R})$.
- Choi and Goldman (2005): worked out the dimension of deformation spaces of orbifold groups in $n = 3$ case.
- Labourie (2006): representations in the Hitchin component are discrete and faithful.

$T(p, q, r)$ is the orbifold fundamental group of a sphere with 3 cone points of order p, q, and r, and contains a surface group as a subgroup of finite index. We examine deformations ρ in the same component of $PSL(n, \mathbb{R})$ as ρ_0; i.e., in the Hitchin component.

Deformations in $PSL(n, \mathbb{R})$

Definition

Let $n, k \in \mathbb{Z}$, $n, k \geq 2$, and let Q, R be the quotient and remainder of dividing n by k, i.e. $n = Qk + R$. Then

$$\sigma(n, k) := (n + R)Q + R.$$

Theorem (D.D. Long, M.B. Thistlethwaite 2017)

Let $\mathcal{H}$ be the Hitchin component for the degree n representation ρ_0 of the triangle group $T(p, q, r)$. Then

$$dim\mathcal{H} = (2n^2 + 1) - (\sigma(n, p) + \sigma(n, q) + \sigma(n, r)).$$

Deformations in $SL(n, \mathbb{R})$

Proof Idea

We discuss representations in $SL(n, \mathbb{R})$ of

$$U(p, q, r) = \left\langle \alpha, \beta, \gamma \,\middle|\, \alpha^{2p} = \beta^{2q} = \gamma^{2r} = \alpha\beta\gamma = z, z^2 = 1 \right\rangle,$$

with a diffeomorphic deformation space to that of $T(p, q, r)$ in $PSL(n, \mathbb{R})$.

For a generator $g \in \{\alpha, \beta, \gamma\}$ of order k in $U(p, q, r)$, consider deformations $\mathcal{D}$ of ρ_0 restricted to the cyclic subgroup $\langle g \rangle$.

$$\dim \mathcal{D} = \dim SL(n, \mathbb{R}) - \dim C\left(\rho_0(g)\right).$$

The image of g in $SL(2, \mathbb{R})$ has eigenvalues $\zeta := e^{\pi i/k}$ and $1/\zeta$, so $\rho_0(g)$ will have eigenvalues

$$\zeta^{-n+1}, \zeta^{-n+3}, \ldots, \zeta^{n-3}, \zeta^{n-1}.$$

Deformations in $SL(n, \mathbb{R})$

Proof Idea

$\rho_0(g)$ has eigenvalues

$$\zeta^{-n+1}, \zeta^{-n+3}, \ldots, \zeta^{n-3}, \zeta^{n-1}.$$

If $n = Qk + R$, R eigenvalues have multiplicity $Q + 1$ and $k - R$ have multiplicity Q.

So the dimension of the centralizer of $\rho_0(g)$ is

$$\dim C(\rho_0(g)) = R(Q+1)^2 + (k-R)Q^2 - 1 = (n+R)Q + R - 1,$$

and

$$\dim \mathcal{D} = n^2 - 1 - ((n+R)Q + R - 1) = n^2 - \sigma(n,p).$$

Deformations in $SL(n, \mathbb{R})$

Proof Idea

Define $\alpha_n := \rho_0(\alpha)$, $\beta_n := \rho_0(\beta)$, $\gamma_n := \rho_0(\gamma)$, *use* $[\ \ ]$ *to denote conjugacy classes, and define*

$$\mathcal{S} := \left\{ \alpha'\beta'\gamma' \,\middle|\, \alpha' \in [\alpha_n], \beta' \in [\beta_n], \gamma' \in [\gamma_n] \right\}.$$

If

$$\Pi : [\alpha_n] \times [\beta_n] \times [\gamma_n] \to \mathcal{S}$$

is the product map, then $\mathcal{H}$ *is diffeomorphic to* $\Pi^{-1}(\alpha_n\beta_n\gamma_n)$ *and*

$$dim\mathcal{H} = \dim[\alpha_n] + \dim[\beta_n] + \dim[\gamma_n] - \dim\mathcal{S}.$$

To complete the proof, it suffices to show that $\mathcal{S}$ *contains a neighborhood of the identity in* $SL(n, \mathbb{R})$*.*

Subgroups of $SL(n,\mathbb{R})$

Observation

For any generator α of $U(p,q,r)$, $\alpha_n = \rho_0(\alpha)$ satisfies $\alpha_n^T F \alpha_n = F$,

$$F = \begin{pmatrix} 0 & & & & & \binom{n-1}{0} \\ & & & & -\binom{n-1}{1} & \\ & & & \binom{n-1}{2} & & \\ & & -\binom{n-1}{3} & & & \\ & \iddots & & & & \\ \pm\binom{n-1}{n-1} & & & & & 0 \end{pmatrix}.$$

So $\alpha_n \in \begin{cases} SO(m, m+1) & \text{for } n = 2m+1 \text{ odd} \\ Sp(2m) & \text{for } n = 2m \text{ even} \end{cases}.$

Which deformations also lie in these subgroups of $SL(n,\mathbb{R})$?

Subgroups of $SL(n, \mathbb{R})$

Definition

The *restricted Hitchin component* $\mathcal{H}_G$ for a group G is the set of all representations in the Hitchin component whose images are contained within the group G.

For the remainder of the talk, G will denote either $SO(m, m+1)$ or $Sp(2m)$, depending on n.

Definition

For positive integers n, k, write $n = Qk + R$.

If $n_{\mathcal{E}} := n \pmod 2$, $k_{\mathcal{E}} := k \pmod 2$, and $Q_{\mathcal{E}} := Q \pmod 2$, then

$$\sigma_G(n, k) := \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q + 1\right)\right).$$

Subgroups of $SL(n, \mathbb{R})$

Theorem (W.)

If $n = 2m$ or $2m + 1$, let $G = Sp(2m)$ or $SO(m, m + 1)$ respectively, and let $\mathcal{H}_G$ be the restricted Hitchin component for the representation ρ_0 of the triangle group $T(p, q, r)$ into G. Then

$$dim\mathcal{H}_G = \dim G - (\sigma_G(n, p) + \sigma_G(n, q) + \sigma_G(n, r)) \,.$$

Proof Idea

If g is a generator in $U(p, q, r)$ of order $2k$, its image in $SL(2, \mathbb{R})$ can be conjugated to $diag(\zeta, \overline{\zeta})$, where $\zeta := -e^{\pi i/k}$. If D is its image in $SL(n, \mathbb{R})$,

$$D = (-1)^{n-1} diag\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right),$$

and $D_i = \overline{D_{n-i+1}}$.

Subgroups of $SL(n, \mathbb{R})$

Proof Idea

So we can write D as a direct sum decomposition of up to one identity matrix, up to one negative identity matrix, and some (possibly zero) number of matrices of the form

$$D_\omega = \mathit{diag}\left(\omega, \ldots, \omega, \overline{\omega}, \ldots, \overline{\omega}\right), \ \omega \in \mathbb{C} \setminus \mathbb{R}.$$

Further, this decomposition is compatible with breaking F into a direct sum of (anti-)symmetric matrices. So we can write the dimension of the centralizer as a sum of centralizers for scalar matrices and matrices of the form D_ω for $\omega \notin \mathbb{R}$.

Finding the dimension of the latter reduces to a straightforward matrix computation, so that if D_ω is a $2d \times 2d$ matrix, the dimension of its centralizer is d^2.

Subgroups of $SL(n, \mathbb{R})$

Proof Idea

The problem of finding the dimension of the centralizer of D is reduced to counting multiplicities of 1, -1, and pairs of complex conjugates. Since

$$D = (-1)^{n-1} diag \left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)} \right),$$

± 1 *can only be diagonal entries of D if n is odd (forcing the middle entry to be* 1*), or if k divides one of* $n-1, n-3, \ldots, n-2m+1$.

The number of real entries depends on the parity of

- *n: determines whether the middle entry must be* 1
- *k: helps determine whether k can divide any power(s) above*
- *Q: determines multiplicity of* ± 1 *entries*

Subgroups of $SL(n, \mathbb{R})$

Proof Idea

Breaking the argument into cases based on the parity of n, then subcases for the parity of k, then sub-subcases for the parity of Q, we obtain a series of equations for the dimension of the centralizer of D, which can be combined into

$$\dim C_G(g) = \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q + 1\right)\right).$$

So the deformation space of D has dimension $\dim G - \dim C_G(g)$*, and arguments nearly identical to those in Long and Thistlethwaite allow us to combine the dimensions of the deformation spaces into*

$$dim \mathcal{H}_G = \dim G - \left(\sigma_G(n, p) + \sigma_G(n, q) + \sigma_G(n, r)\right).$$

References

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[2] N.J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449 - 473.

[3] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Inventiones Math. **165** (2006), 51 - 114.

[4] D.D. Long and M.B. Thistlethwaite, *The Dimension of the Hitchin Component for Triangle Groups*, `web.math.ucsb.edu/~long/pubpdf/slnr_submitted.pdf`