The Dimension of the Restricted Hitchin Component for a Triangle Group

Elise A. Weir

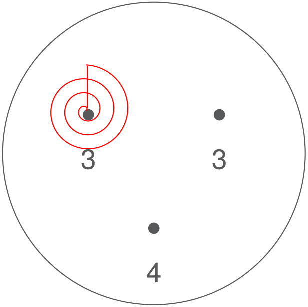
University of Tennessee, Knoxville

February 6th, 2018

THE UNIVERSITY OF TENNESSEE KNOXVILLE

Triangle Groups

- A *surface group* is the fundamental group of a compact surface.
- An orbifold is a generalization of a surface that looks locally like the quotient of $\mathbb{R}^2$ by the linear action of a finite group.
- A sphere with cone points of order 3, 3, and 4:

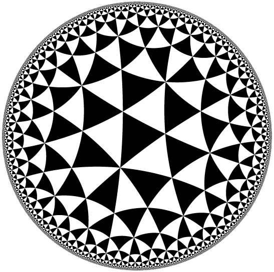


Triangle Groups

Definition

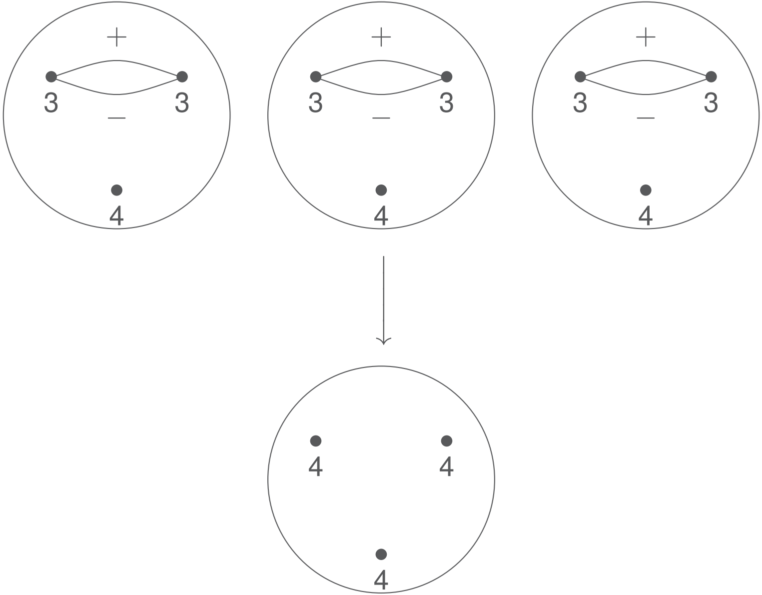
$T(p,q,r)$ is the group of rotational symmetries of a tiling of the hyperbolic plane $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, and π/r, and admits a presentation

$$T(p,q,r) := \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle .$$

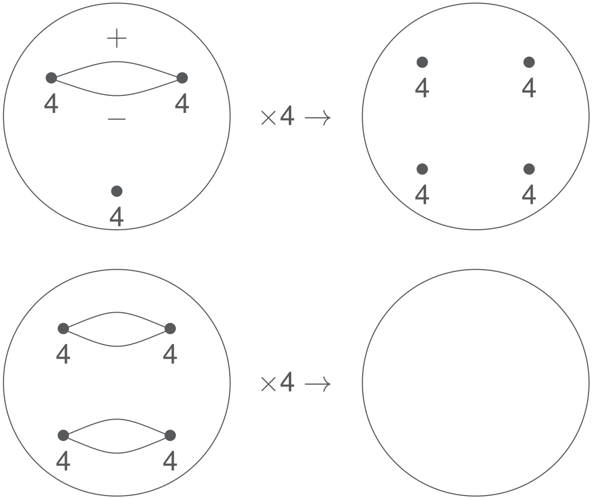


$T(p,q,r)$ is the orbifold fundamental group of a sphere with 3 cone points of order p, q, and r, and contains a surface group as a subgroup of finite index.

Surface Groups in $T(3, 3, 4)$



Surface Groups in $T(3, 3, 4)$



The Hitchin Component

Definition

For an orbifold group $\pi_1(\Sigma)$, the ***Hitchin component*** $\mathcal{H}$ of the space of representations $\pi_1(\Sigma) \to PSL(n, \mathbb{R})$ is the connected component containing ρ_0, a “base" representation we will construct shortly.

Theorem (N.J. Hitchin, 1992)

For a compact oriented surface S, the Hitchin component of $\pi_1(S)$ into $PSL(n, \mathbb{R})$ has dimension $\chi(S)(1 - n^2)$.

- Choi and Goldman (2005): dimensions of deformation spaces of orbifold groups in $n = 3$ case.
- Labourie (2006): representations in the Hitchin component are discrete and faithful.

Representations of Triangle Groups in $PSL(n, \mathbb{R})$

- Embed an orbifold group $\pi_1(\Sigma)$ as a discrete subgroup of the order-preserving isometries of $\mathbb{H}^2$.
- $\phi : T(p, q, r) \to PSL(2, \mathbb{R})$ takes a rotation to its coefficient matrix:

$$\left(z \mapsto \frac{fz + g}{hz + k} \right) \longmapsto \begin{pmatrix} f & g \\ h & k \end{pmatrix}.$$

- We compose $\phi : T \to PSL(2, \mathbb{R})$ with the unique irreducible representation $\phi_n : PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$.
- To construct ϕ_n, we will use the 2×2 matrix to assign a substitution, and examine where it sends basis elements of degree $n - 1$ homogeneous polynomials in two variables:

$$\{x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}\}$$

Representations of Triangle Groups in $PSL(n, \mathbb{R})$

- For simplicity, consider the $n = 3$ case with $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$.
- The substitution $x \mapsto x + 3y$, $y \mapsto 2x + 7y$ takes

$$\begin{array}{lll} x^2 & \mapsto (x+3y)^2 & = x^2 + 6xy + 9y^2 \\ xy & \mapsto (x+3y)(2x+7y) & = 2x^2 + 13xy + 21y^2 \\ y^2 & \mapsto (2x+7y)^2 & = 4x^2 + 28xy + 49y^2 \end{array}.$$

- So $\phi_n \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 13 & 28 \\ 9 & 21 & 49 \end{pmatrix}$.
- Our base representation $\rho_0 : T(p, q, r) \to PSL(n, \mathbb{R})$ will be defined by $\rho_0 := \phi_n \circ \phi$.

Deformations in $PSL(n, \mathbb{R})$

Definition

Let $n, k \in \mathbb{Z}$, $n, k \geq 2$, and let Q, R be the quotient and remainder of dividing n by k, i.e. $n = Qk + R$. Then

$$\sigma(n, k) := (n + R)Q + R.$$

Theorem (D.D. Long, M.B. Thistlethwaite 2018)

Let $\mathcal{H}$ be the Hitchin component for the degree n representation ρ_0 of the triangle group $T(p, q, r)$. Then

$$dim\mathcal{H} = (2n^2 + 1) - (\sigma(n, p) + \sigma(n, q) + \sigma(n, r)).$$

The Restricted Hitchin Component

Observation

For any generator g of $T(p,q,r)$, $g_0 = \rho_0(g)$ satisfies $g_0^T F g_0 = F$,

$$F = \begin{pmatrix} 0 & & & & & \binom{n-1}{0} \\ & & & & -\binom{n-1}{1} & \\ & & & \binom{n-1}{2} & & \\ & & -\binom{n-1}{3} & & & \\ & \iddots & & & & \\ \pm\binom{n-1}{n-1} & & & & & 0 \end{pmatrix}.$$

So $g_0 \in \begin{cases} SO(m, m+1) & \textit{for } n = 2m+1 \textit{ odd} \\ Sp(2m) & \textit{for } n = 2m \textit{ even} \end{cases}.$

Which deformations also lie in these subgroups of $SL(n, \mathbb{R})$?

The Restricted Hitchin Component

Definition

The *restricted Hitchin component* $\mathcal{H}_G$ for a group G is the set of all representations in the Hitchin component whose images are contained within the group G.

Definition

For positive integers n, k, write $n = Qk + R$.

If $n_\mathcal{E} := n \pmod 2$, $k_\mathcal{E} := k \pmod 2$, and $Q_\mathcal{E} := Q \pmod 2$, then

$$\sigma_G(n,k) := \frac{1}{2}\left(kQ^2 + 2QR + R + k_\mathcal{E}\left(Q + Q_\mathcal{E}\right) - n_\mathcal{E}\left(2Q+1\right)\right).$$

Main Theorem

Theorem (W.)

If $n = 2m$ or $2m+1$, let $G = Sp(2m)$ or $SO(m, m+1)$ respectively, and let $\mathcal{H}_G$ be the restricted Hitchin component for the representation ρ_0 of the triangle group $T(p, q, r)$ into G. Then

$$dim\mathcal{H}_G = \dim G - (\sigma_G(n, p) + \sigma_G(n, q) + \sigma_G(n, r)) .$$

Projective Considerations

Proof

- *We prefer to work with matrix representatives; however, for*

$$U(p,q,r) := \left\langle \alpha, \beta, \gamma \,\middle|\, \alpha^{2p} = \beta^{2q} = \gamma^{2r} = \alpha\beta\gamma = z, z^2 = 1 \right\rangle,$$

  the commutative diagram does not admit the lift below for certain n, p, q, r.

$$\begin{tikzcd}
U(p,q,r) \arrow[r, "\psi"'] \arrow[d, "\pi_0"'] & SL(2,\mathbb{R}) \arrow[r, "\psi_n"'] \arrow[d, "\pi_2"'] & SL(n,\mathbb{R}) \arrow[d, "\pi_n"'] \\
T(p,q,r) \arrow[r, "\phi"'] \arrow[ur, dashed] & PSL(2,\mathbb{R}) \arrow[r, "\phi_n"'] & PSL(n,\mathbb{R})
\end{tikzcd}$$

- *However, it can be shown that the deformation spaces of the top and bottom row are diffeomorphic.*

Centralizers

For a generator $g \in \{\alpha, \beta, \gamma\}$ of order $2k$ in $U(p, q, r)$, consider deformations $\mathcal{D}$ of ρ_0 restricted to the cyclic subgroup $\langle g \rangle$.

- Deformations of $\rho_0(g)$ stay in its conjugacy class.
- G is connected, so any conjugate can be obtained this way.
- So $\dim \mathcal{D}$ is the dimension of the conjugacy class of $\rho_0(g)$.

Equivalently, for $C(\rho_0(g))$ the centralizer of $\rho_0(g)$ in G,

$$\dim \mathcal{D} = \dim G - \dim C(\rho_0(g)).$$

Centralizers

For Long and Thistlethwaite's work on $SL(n, \mathbb{R})$, counting repeated eigenvalues suffices for finding the dimension of the centralizer:

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \lambda_s & & \\ & \ddots & \\ & & \lambda_s \end{matrix}} \end{pmatrix}$$

But for G, $g_0^T F g_0 = F$ imposes additional restrictions.

Compatibility with Bilinear Forms

- For a generator g of order $2k$ in $SL(2, \mathbb{R})$, we can conjugate to $diag(\zeta, \overline{\zeta})$, where $\zeta := -e^{\pi i/k}$.
- If D is its image in $SL(n, \mathbb{R})$,

$$D = (-1)^{n-1} diag\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right),$$

and $D_i = \overline{D_{n-i+1}}$.

- $n = 7$, $k = 4$:

$$\begin{array}{rclcrrrrrrrl} D & = & diag & \{i, & -1, & -i, & 1, & i, & -1, & -i\} \\ F & = & adiag & \{1, & -6, & 15, & -20, & 15, & -6, & 1\} \end{array}$$

Compatibility with Bilinear Forms

- Write D as a direct sum of up to one of I and -I each, and some number of matrices of the form

$$D_\omega = \mathit{diag}\left(\omega, \ldots, \omega, \overline{\omega}, \ldots, \overline{\omega}\right), \ \omega \in \mathbb{C} \setminus \mathbb{R}.$$

- This decomposition is compatible with breaking F into a direct sum of (anti-)symmetric matrices.
- $n = 7$, $k = 4$:

$$\begin{pmatrix} \boxed{1} \ \leftarrow \ \text{dim}\,0 & & \\ & \boxed{\begin{matrix} -1 & \\ & -1 \end{matrix}} \ \leftarrow \ \text{dim}\,1 & \\ & \text{dim}\,2^2 \ \rightarrow & \boxed{\begin{matrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{matrix}} \end{pmatrix}$$

Compatibility with Bilinear Forms

The problem of finding the dimension of the centralizer of D is reduced to counting multiplicities of 1, -1, and pairs of complex conjugates.

$$D = (-1)^{n-1} diag\left(\zeta^{n-1}, \zeta^{n-3}, \dots, \zeta^{n-2i+1}, \dots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right).$$

The number of real entries depends on the parity of

- n: must the middle entry be 1?
- k: can the k in $\zeta := -e^{\pi i/k}$ divide any power(s) above?
- Q: multiplicity of ± 1 entries.

Compatibility with Bilinear Forms

- Breaking the argument into cases based on the parity of n, k, and Q, we obtain a series of equations for the dimension of the centralizer of D, which can be combined into

$$\dim C_G(g) = \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right).$$

- So the deformation space of D has dimension $\dim G - \dim C_G(g)$.

- This takes care of cyclic subgroups. What about the contribution of the relation $\alpha\beta\gamma = z$ to the dimension of the Hitchin component?

Contribution of $\alpha\beta\gamma = z$

- This portion is very similar to the approach used by Long and Thistlethwaite, with minor alterations to accomodate differences in G and $SL(n, \mathbb{R})$.

- Denote images of generators of $U(p, q, r)$ under ρ_0 by α, β, γ and conjugacy classes by [ ].

- Define

$$\mathcal{S} := \left\{ \alpha'\beta'\gamma' \,\middle|\, \alpha' \in [\alpha], \beta' \in [\beta], \gamma' \in [\gamma] \right\}.$$

- If

$$\Pi : [\alpha] \times [\beta] \times [\gamma] \to \mathcal{S}$$

is the product map, then $\mathcal{H}$ is diffeomorphic to $\Pi^{-1}(\alpha\beta\gamma)$ and

$$dim\mathcal{H} = \dim[\alpha] + \dim[\beta] + \dim[\gamma] - \dim\mathcal{S}.$$

Contribution of $\alpha\beta\gamma = z$

- To complete the proof, we can show that $\mathcal{S}$ contains a neighborhood of the identity in G.
- Indeed, we can leave γ fixed, and show $S_\alpha := \{\alpha'\beta \,|\alpha' \in [\alpha]\}$ and $S_\beta := \{\alpha\beta' \,|\beta' \in [\beta]\}$ generate the tangent space of G at $\alpha\beta$.
- For $g, h \in G$ close to the identity, we can write $g = \exp(\xi)$ and $h = \exp(\eta)$ for some ξ, η in the Lie algebra $\mathfrak{g}$, so

$$g\alpha g^{-1}\beta = \exp\left((1 - Ad_\alpha)\,\xi\right)\alpha\beta$$
$$\alpha h\beta h^{-1} = \exp\left(Ad_\alpha\left(I - Ad_\beta\right)\eta\right)\alpha\beta.$$

- So every element of S_1 and S_2 can be written as $\exp(v)\alpha\beta$ for some $v \in \mathfrak{g}$. Finally, we need to show that

$$Im\left(I - Ad_\alpha\right) + Ad_\alpha Im\left(I - Ad_\beta\right) = \mathfrak{g}.$$

Contribution of $\alpha\beta\gamma = z$

- Does $Im\,(I - Ad_\alpha) + Ad_\alpha Im\,(I - Ad_\beta) = \mathfrak{g}$?

- Suppose not, and pick some $\xi \neq 0$ in the orthogonal complement of the left hand side. Then $\xi \in \ker\,(I - Ad_\alpha)$, so $\xi = Ad_\alpha \xi$.

- ξ is also in the orthogonal complement of $Ad_\alpha Im\,(I - Ad_\beta)$. Ad_α preserves $\langle\ ,\ \rangle$, so $\xi \in Ad_\alpha \ker\,(I - Ad_\beta)$. Therefore, $Ad_\alpha \xi \in Ad_\alpha \ker\,(I - Ad_\beta)$, and thus $\xi \in \ker\,(I - Ad_\beta)$.

- Since ψ_n is irreducible, Schur's Lemma gives us that

$$\ker\,(I - Ad_\alpha) \cap \ker\,(I - Ad_\beta) = \{0\}\,,$$

so $\xi = 0$, which is a contradiction.

Future Directions

- Can the results of Choi and Goldman on 3-dimensional representations in the Hitchin component for general orbifolds be generalized to higher-dimensional representations for other classes of orbifolds?
- $SL(n, \mathbb{R})$ is a real form of $SL(n, \mathbb{C})$ with respect to the anti-holomorphic involution taking a matrix to its conjugate. Can we take a description of the representation variety of a triangle group in $SL(n, \mathbb{R})$ and translate it to a description of the variety of other real forms, e.g. for an indefinite special unitary group?

References

[1] S. Choi and W.M. Goldman, *The deformation spaces of convex* $\mathbb{R}P^2$*-structures on* 2*-orbifolds*, Amer. J. Math. **127** (2005) no.5, 1019-1102.

[2] N.J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449 - 473.

[3] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Inventiones Math. **165** (2006), 51 - 114.

[4] D.D. Long and M.B. Thistlethwaite, *The Dimension of the Hitchin Component for Triangle Groups*, (2018).