The Dimension of the Restricted Hitchin Component for a Triangle Group

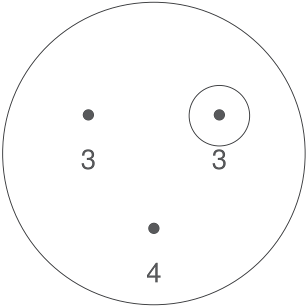
Elise A. Weir

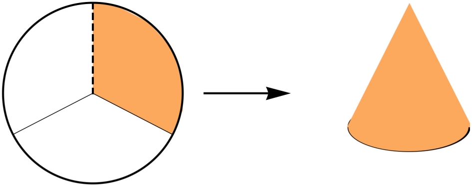
April 8th, 2018

THE UNIVERSITY OF
TENNESSEE
KNOXVILLE

Object of Interest: Orbifolds

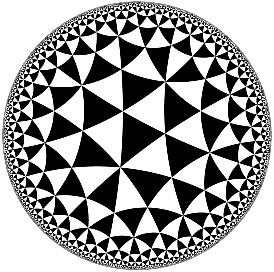
- A *surface group* is the fundamental group of a compact surface.
- An *orbifold* is a generalization of a surface that looks locally like the quotient of $\mathbb{R}^2$ by the linear action of a finite group.
- A sphere with cone points of order 3, 3, and 4:





Object of Interest: Triangle Groups

- A *hyperbolic triangle group* $T(p, q, r)$ is:

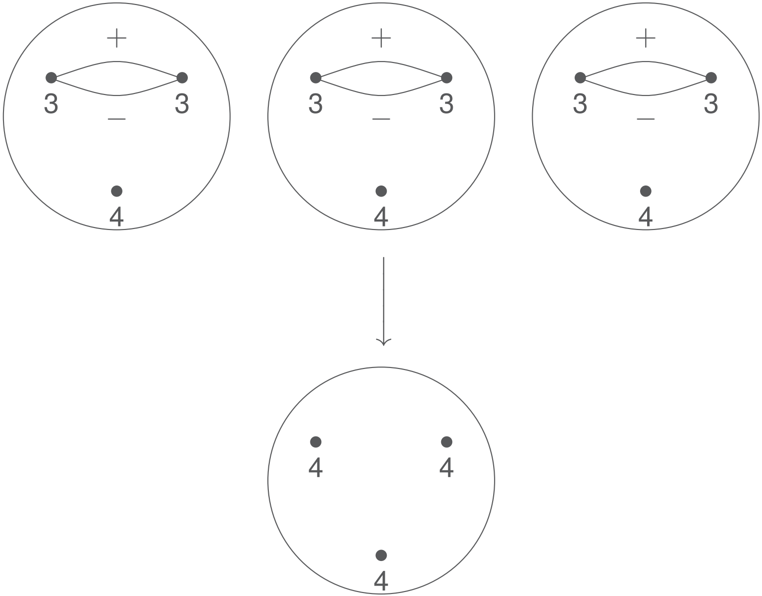


$T(3, 3, 4)$ *universal cover*

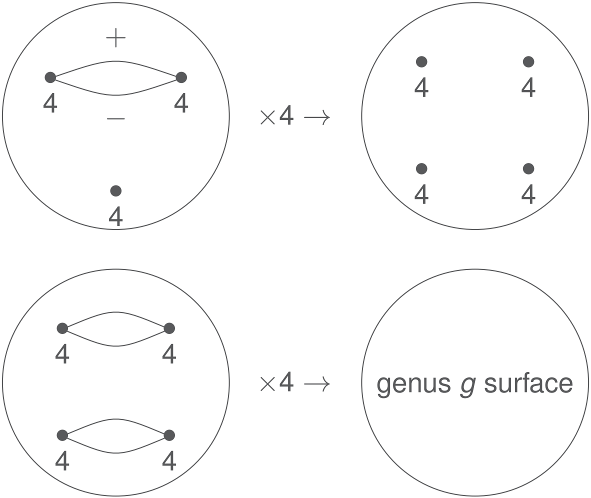
  - the rotational symmetries of a tiling of $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, and π/r.
  - $\langle a, b, c \,|\, a^p = b^q = c^r = abc = 1 \rangle$
  - the orbifold fundamental group of a sphere with cone points of order p, q, and r.

- One motivation for studying triangle groups: each one contains a surface group as a subgroup of finite index.

Motivation: Surface Groups in $T(3, 3, 4)$



Motivation: Surface Groups in $T(3,3,4)$



Method of Study: The Hitchin Component

Base Representation

$$\rho_0 : \pi_1(\Sigma) \xrightarrow{\rho} PSL(2,\mathbb{R}) \xrightarrow{\rho_n} PSL(n,\mathbb{R})$$

- ρ_n: *unique irreducible representation* $PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$
- ρ *for* $\pi_1(\Sigma) = T(p,q,r)$*:*

  any rotation can be written as $\frac{fz+g}{hz+k}$*; map to* $\begin{pmatrix} f & g \\ h & k \end{pmatrix}$

Definition

For an orbifold group $\pi_1(\Sigma)$, the ***Hitchin component*** $\mathcal{H}$ is the connected component of the space of representations $\pi_1(\Sigma) \to PSL(n,\mathbb{R})$ containing the base representation ρ_0.

Method of Study: The Hitchin Component

- **(N.J. Hitchin, 1992)**: For a compact oriented surface S, the Hitchin component of $\pi_1(S)$ into $PSL(n, \mathbb{R})$ has dimension $\chi(S)(1-n^2)$.
- **(S. Choi and W. Goldman, 2005)**: for a compact 2-orbifold Σ with $\chi(\Sigma) < 0$ and without boundary, there is a formula for the dimension of the Hitchin component of $\pi_1(\Sigma)$ into $PSL(3, \mathbb{R})$.
- **(F. Labourie, 2006)**: Every representation in the Hitchin component is discrete and faithful.
- **(D.D. Long and M.B. Thistlethwaite, 2018)**: Let $\mathcal{H}$ be the Hitchin component of $T(p, q, r) \to PSL(n, \mathbb{R})$. Then

$$dim\mathcal{H} = (2n^2+1) - (\sigma(n,p) + \sigma(n,q) + \sigma(n,r)),$$

where $\sigma(n, k)$ arises from counting repeated eigenvalues.

The Restricted Hitchin Component

Observation

If g is the image of a generator under the base representation ρ_0, it satisfies $g^T F g = F$, where

$$F = \begin{pmatrix} 0 & & & & & \binom{n-1}{0} \\ & & & & -\binom{n-1}{1} & \\ & & & \binom{n-1}{2} & & \\ & & -\binom{n-1}{3} & & & \\ & \iddots & & & & \\ \pm\binom{n-1}{n-1} & & & & & 0 \end{pmatrix}.$$

So $g \in \begin{cases} SO(m, m+1) & \text{for odd} \quad n = 2m+1 \\ Sp(2m) & \text{for even} \quad n = 2m \end{cases}.$

Which Hitchin representations also lie in $SO(m, m+1)$ or $Sp(2m)$?

The Restricted Hitchin Component

Definition

The *restricted Hitchin component* $\mathcal{H}_G$ for a group G is the set of all representations in the Hitchin component whose images are contained within the group G.

Theorem (W.)

Let $\mathcal{H}_G$ be the restricted Hitchin component of $T(p,q,r) \to G$. Then for $G = SO(m,m+1)$ or $Sp(2m)$,

$$dim\mathcal{H}_G = \dim G - \left(\sigma_G(n,p) + \sigma_G(n,q) + \sigma_G(n,r)\right),$$

where $\sigma_G(n,k)$ arises from counting repeated real eigenvalues, and repeated pairs of complex conjugate eigenvalues.

Cyclic subgroups of $T(p, q, r)$

Recall:

$$T(p,q,r) = \langle a, b, c \,|\, a^p = b^q = c^r = abc = 1 \rangle$$

For a generator g, consider deformations $\mathcal{D}$ of the base representation ρ_0 restricted to the cyclic subgroup $\langle g \rangle$.

- Deformations of $\rho_0(g)$ stay in its conjugacy class.
- G is connected, so any conjugate can be obtained this way.
- So $\dim \mathcal{D}$ is the dimension of the conjugacy class of $\rho_0(g)$.

Equivalently, for $C(g)$ the centralizer of $\rho_0(g)$ in G,

$$\dim \mathcal{D} = \dim G - \dim C(g) .$$

Cyclic subgroups of $T(p, q, r)$: Eigenvalues

For Long and Thistlethwaite's work on $PSL(n, \mathbb{R})$, counting repeated eigenvalues suffices for finding the dimension of the centralizer:

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \lambda_s & & \\ & \ddots & \\ & & \lambda_s \end{matrix}} \end{pmatrix}$$

But for $SO(m, m+1)$ and $Sp(2m)$, the group relation $g^T F g = F$ imposes additional restrictions.

Compatibility with the Bilinear Form F

- The diagonalized image D of a generator of order k can be written so that opposite entries are complex conjugate.
- $n = 7$, $k = 4$:

$$\begin{array}{rclccccccc} D & = & \mathit{diag} & (i, & -1, & -i, & 1, & i, & -1, & -i) \\ F & = & \mathit{adiag} & (1, & -6, & 15, & -20, & 15, & -6, & 1) \end{array}$$

- Write D as a direct sum of $\pm I$ and pairs of conjugates, compatible with F as a direct sum of (anti-)symmetric matrices.

$$D \qquad\qquad\qquad\qquad F$$

$$\left(\begin{array}{ccccccc} \boxed{1} & \leftarrow & \text{dim } 0 & & & & \\ & \boxed{\begin{matrix} -1 & \\ & -1 \end{matrix}} & \leftarrow \text{dim } 1 & & & & \\ & \text{dim } 2^2 & \rightarrow & \boxed{\begin{matrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{matrix}} & & & \end{array}\right) \qquad \left(\begin{array}{ccc} \boxed{-20} & & \\ & \boxed{\begin{matrix} & -6 \\ -6 & \end{matrix}} & \\ & & \boxed{\begin{matrix} & & & 1 \\ & & 15 & \\ & 15 & & \\ 1 & & & \end{matrix}} \end{array}\right)$$

Compatibility with the Bilinear Form F

So the problem is reduced to counting multiplicities of ± 1 and complex conjugate pairs. Real entries depend on the parity of

- **n (degree)**: must the middle entry be 1?
- **k (cyclic order)**: can k divide the powers of roots of unity that appear in the diagonalized image D of a generator g?
- **Q (integer division $n \backslash k$)**: multiplicity of ± 1 entries.

Break the argument into cases based on the parity of n, k, Q to get a series of equations for the dimension of the centralizer, which can be combined into

$$\sigma_G(n,k) := \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right),$$

where subscript $\mathcal{E}$ denotes parity, and R is the remainder of $n \backslash k$.

$$T(p,q,r) = \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle$$

- What about the contribution of the last relation to the Hitchin component?
- Similar to the approach used by Long and Thistlethwaite, with minor alterations to accommodate differences in $SO(m, m+1)$ and $Sp(2m)$ compared to $SL(n, \mathbb{R})$.
- Basic idea: show that a product map on conjugacy classes $\Pi : [a] \times [b] \times [c] \to G$ is a submersion using tangent spaces. Then the relation $abc = 1$ subtracts $\dim G$ from the dimension of the Hitchin component.

Future Directions

Can the results of Choi and Goldman on degree 3 *representations for general orbifolds be generalized to higher-dimensional representations for classes of orbifolds other than triangle groups?*

References

[1] S. Choi and W.M. Goldman, *The deformation spaces of convex $\mathbb{R}P^2$-structures on 2-orbifolds*, Amer. J. Math. **127** (2005) no.5, 1019-1102.

[2] N.J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449 - 473.

[3] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Inventiones Math. **165** (2006), 51 - 114.

[4] D.D. Long and M.B. Thistlethwaite, *The dimension of the Hitchin component for triangle groups*, to appear in Geometriae Dedicata (2018).

Slides available at `http://www.math.utk.edu/~weir/`