The Dimension of the Restricted Hitchin Component for Hyperbolic Triangle Groups

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Generalizing Teichmüller Space

For a compact hyperbolic surface S of genus $g > 1$, **Teichmüller Space** $\mathcal{T}(S)$ is:

- the space of conformal structures on S up to isotopy.
- a ball of dimension $6g - 6$.
- in bijection with discrete, faithful $\pi_1(S) \to PSL(2, \mathbb{R})$.

The **Hitchin component** $\mathcal{H}$ for a compact orientable hyperbolic surface (or 2-orbifold) Σ is the connected component of $\pi_1(\Sigma) \to PSL(n, \mathbb{R})$ containing a base representation

$$\rho_0 : \pi_1(\Sigma) \xrightarrow{\rho} PSL(2, \mathbb{R}) \xrightarrow{\rho_n} PSL(n, \mathbb{R}),$$

where ρ is the coefficient matrix of the Möbius transformation corresponding to an element of $\pi_1(\Sigma)$,

and ρ_n is the unique irreducible representation.

Geometry of the Hitchin Component

- **(S. Choi and W. Goldman)**: The Hitchin component in $PSL(3,\mathbb{R})$ parametrizes convex real projective structures on an orbifold.
- **(O. Guichard and A. Wienhard)**: The Hitchin component for $PSL(4,\mathbb{R})$ corresponds to certain projective structures on the unit tangent bundle of a surface; a similar characterization holds for $PSp(4,\mathbb{R})$.

Some progress has been made towards geometric interpretations of the Hitchin component for large dimensions, explicit characterizations like the above generally remain elusive.

Brief History of the Hitchin Component

- **(N.J. Hitchin, 1992)**: For a compact orientable surface S of genus $g > 1$ in $PSL(n, \mathbb{R})$,

$$\dim \mathcal{H} = -\chi(S) \dim PSL(n, \mathbb{R}).$$

- **(S. Choi and W. Goldman, 2005)**: for a compact 2-orbifold Σ in $PSL(3, \mathbb{R})$ with $\chi(\Sigma) < 0$ and without boundary,

$$\dim \mathcal{H} = -8(X_\Sigma) + 6k_c - 2b_c + 3k_r - b_r.$$

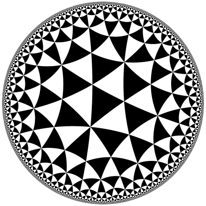
- **(F. Labourie, 2006)**: Every representation in the Hitchin component is discrete and faithful.
- **(D.D. Long and M.B. Thistlethwaite, 2018)**: For a compact 2-orbifold Σ in $PSL(n, \mathbb{R})$ with $\pi_1(\Sigma)$ a hyperbolic triangle group $T(p, q, r)$,

$$dim \mathcal{H} = (2n^2 + 1) - (\sigma(n, p) + \sigma(n, q) + \sigma(n, r)).$$

Triangle Groups

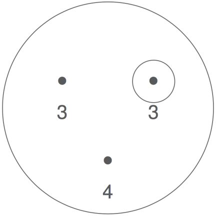
The dimension of the Hitchin component for an arbitrary 2-orbifold Σ remains an open question; we consider when $\pi_1(\Sigma) = T(p, q, r)$.

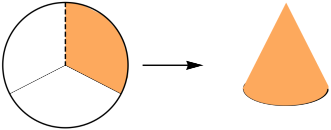
A **hyperbolic triangle group** $T(p, q, r)$ is:



$T(3, 3, 4)$ *tiling of* $\mathbb{H}^2$

- the rotational symmetries of a tiling of $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, π/r.
- $\langle a, b, c \,|\, a^p = b^q = c^r = abc = 1 \rangle$.
- the orbifold fundamental group of a sphere with cone points of order p, q, r.





An Algebraic Approach to the Hitchin Component

To find the dimension of the Hitchin component in $PSL(n,\mathbb{R})$ for triangle groups, Long and Thistlethwaite first consider deformations of cyclic generators under

$$\rho_0 : \pi_1(\Sigma) \xrightarrow{\rho} PSL(2,\mathbb{R}) \xrightarrow{\rho_n} PSL(n,\mathbb{R})$$

for

$$T(p,q,r) = \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle \, .$$

- Deformations of $\rho_0(g)$ stay in its conjugacy class.
- $SL(n,\mathbb{R})$ is connected, so any conjugate can be obtained this way.
- So $\dim \mathcal{D}$ is the dimension of the conjugacy class of $\rho_0(g)$.

For $C(g)$ the centralizer of $\rho_0(g)$ in G, $\dim \mathcal{D} = \dim G - \dim C(g)$.

The Restricted Hitchin Component

Observation

Each generator g in $T(p,q,r)$ satisfies $g^T F g = F$, where

$$F = \begin{pmatrix} 0 & & & & & \binom{n-1}{0} \\ & & & & -\binom{n-1}{1} & \\ & & & \binom{n-1}{2} & & \\ & & -\binom{n-1}{3} & & & \\ & \iddots & & & & \\ \pm\binom{n-1}{n-1} & & & & & 0 \end{pmatrix}.$$

$$\text{So } g \in \begin{cases} SO(m,m+1) & \text{for odd} \quad n = 2m+1 \\ Sp(2m) & \text{for even} \quad n = 2m \end{cases}.$$

The definition of the Hitchin component allows for representations $\pi_1(\Sigma) \to G$, where G is a split real simple Lie group, including $PSL(n,\mathbb{R})$, but also $SO(m,m+1)$ and $Sp(2m)$. We refer to the latter two cases as the **restricted Hitchin component** $\mathcal{H}_G$.

The Restricted Hitchin Component

Definition

Let $n, k \geq 2$ be integers, and Q, R the quotient and remainder of integer division $n \backslash k$. Let a subscript $\mathcal{E}$ denote parity.
Then we define

$$\sigma_G(n,k) := 1/2\left((n+R)Q + R + k_{\mathcal{E}}(Q + Q_{\mathcal{E}}) - n_{\mathcal{E}}(2Q+1)\right).$$

Theorem (W.)

Let $\mathcal{H}_G$ be the restricted Hitchin component of $T(p,q,r) \to G$. Then for $G = SO(m, m+1)$ or $Sp(2m)$,

$$dim\mathcal{H}_G = \dim G - \left(\sigma_G(n,p) + \sigma_G(n,q) + \sigma_G(n,r)\right),$$

where $\sigma_G(n,k)$ is the dimension of the centralizer of a cyclic generator of $T(p,q,r)$.

Cyclic subgroups of $T(p, q, r)$: Eigenvalues

For Long and Thistlethwaite's work on $PSL(n, \mathbb{R})$, counting repeated eigenvalues suffices for finding the dimension of the centralizer:

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \lambda_s & & \\ & \ddots & \\ & & \lambda_s \end{matrix}} \end{pmatrix}$$

But rearranging eigenvalues in this manner does not preserve the group relation $g^T F g = F$ for $SO(m, m+1)$ and $Sp(2m)$.

Compatibility with the Bilinear Form F

- The diagonalized image D of a generator of order k can be written so that opposite entries are complex conjugate.
- $n = 7$, $k = 4$:
$$\begin{array}{rcllllllll} D & = & diag & (i, & -1, & -i, & 1, & i, & -1, & -i) \\ F & = & adiag & (1, & -6, & 15, & -20, & 15, & -6, & 1) \end{array}$$
- Write D as a direct sum of $\pm I$ and pairs of conjugates, compatible with F as a direct sum of (anti-)symmetric matrices.

$$\begin{array}{cc} D & F \\ \begin{pmatrix} \boxed{1} & \leftarrow \quad \dim 0 & & \\ & \boxed{\begin{matrix} -1 & \\ & -1 \end{matrix}} & \leftarrow \quad \dim 1 & \\ \dim 2^2 \quad \rightarrow & & \boxed{\begin{matrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{matrix}} \end{pmatrix} & \begin{pmatrix} \boxed{-20} & & \\ & \boxed{\begin{matrix} & -6 \\ -6 & \end{matrix}} & \\ & & \boxed{\begin{matrix} & & & 1 \\ & & 15 & \\ & 15 & & \\ 1 & & & \end{matrix}} \end{pmatrix} \end{array}$$

Compatibility with the Bilinear Form F

The cost of this approach is that counting multiplicities of eigenvalues (particularly real ones) becomes decidedly more difficult.

This can be broken into cases based on the parity of the representation degree n, cyclic order k, and the quotient $Q = n \backslash k$ to obtain

$$\sigma_G(n,k) := 1/2\left((n+R)Q + R + k_{\mathcal{E}}(Q + Q_{\mathcal{E}}) - n_{\mathcal{E}}(2Q+1)\right).$$

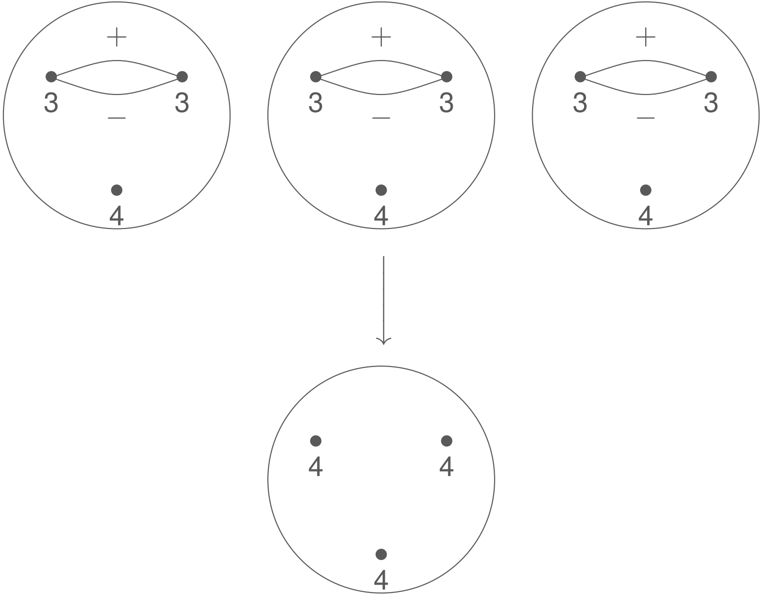
Along with finding a diagonal basis for the adjoint operator, this presents the primary difficulty in finding the dimension of the restricted Hitchin component for $T(p,q,r)$ as compared to the usual Hitchin component.

The Hitchin Component and Surface Subgroups of $SL(n, \mathbb{R})$

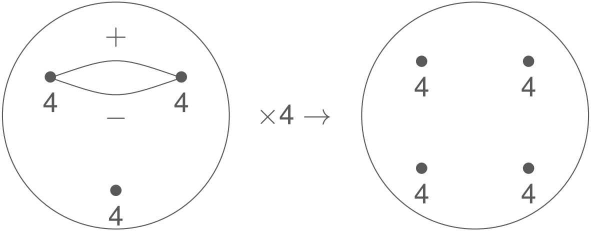
The geometric interpretation of the Hitchin component in $PSL(n, \mathbb{R})$ remains conjectural for large n, and even less is known for the restricted Hitchin component.

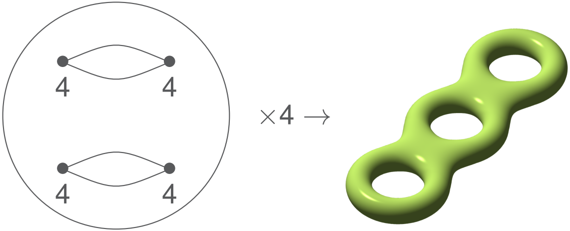
Another reason for studying Hitchin representations of triangle groups is that each $T(p, q, r)$ contains surface groups as subgroups of finite index.

Motivation: Surface Groups in $T(3,3,4)$



Motivation: Surface Groups in $T(3,3,4)$





Zariski dense surface subgroups of $SL(n, \mathbb{R})$

So one can use deformations of the base representation $\rho_0 : T(p, q, r) \to PSL(n, \mathbb{R})$ to find surface subgroups in $SL(n, \mathbb{R})$.

As is done by Long, Reid, and Thistlethwaite, one can use a result of Lubotzky to determine whether particular representations found in this manner are Zariski dense.

What can the dimension of the restricted Hitchin component for triangle groups tell us about obstructions to finding these Zariski dense surface subgroups?

Selected dimensions of the restricted Hitchin component

n	p	$\leq q$	$\leq r$	$\dim \mathcal{H}_G$
3	$\cdot$	$\cdot$	$\cdot$	0
4	2	3	$\cdot$	0
	2	≥ 4	$\cdot$	2
	3	3	$\cdot$	0
	3	≥ 4	$\cdot$	2
	≥ 4	$\cdot$	$\cdot$	4
5	2	3	$\cdot$	0
	2	≥ 3	$\cdot$	2
	3	3	$\cdot$	0
	3	≥ 4	$\cdot$	2
	≥ 4	$\cdot$	$\cdot$	4
6	2	≥ 6	$\cdot$	6
	3	≥ 6	$\cdot$	8
	4	≥ 6	$\cdot$	10
	≥ 6	$\cdot$	$\cdot$	12

- Since $\dim \mathcal{H}_G = 0$ for $T(3,3,4)$ in degrees 4 and 5, the only Hitchin representation in $SO(3,2)$ is ρ_0.
- This prevents an obstruction to ensuring images are Zariski dense.

Comparing Hitchin and Restricted Hitchin Components

n	p	$\leq q$	$\leq r$	$\dim\mathcal{H} = \dim\mathcal{H}_G$
4	2	≥ 4	$\cdot$	2
5	2	≥ 4	$\cdot$	2
6	2	3	$\cdot$	2
	2	4	5	2
	2	4	≥ 6	4
7	2	3	$\cdot$	2
	2	4	5	2
	2	4	6	4
8	2	3	7	2
	2	3	≥ 8	4
	2	4	5	4
	2	4	6	6
9	2	3	7	2
	2	3	8	4
	2	4	5	4
	2	4	6	6
10	2	3	7	2
	2	3	8	4
	2	4	5	6
	2	4	6	8

When $\dim\mathcal{H} = \dim\mathcal{H}_G$, there are no representations in the Hitchin component with images that are Zariski dense in $SL(n, \mathbb{R})$.

Future Directions

- What is the dimension of the (restricted) Hitchin component for a family of orbifolds with shared algebraic structure?
- What is the relationship between the dimension of the (restricted) Hitchin component of $T(p,q,r)$ and that of the reflection-inclusive $\tilde{T}(p,q,r)$, which has presentation

$$\left\langle a,b,c : (ab)^p = (bc)^q = (ca)^r = a^2 = b^2 = c^2 = 1 \right\rangle?$$

- What can be said about the deformation space for a triangle group (or other orbifold group) into a special unitary group?

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