The Hitchin Component for Triangle Groups in Sp(2m) & SO(m,m+1)

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Summary

For representations of hyperbolic triangle groups $T(p,q,r)$ into $PSL(n,\mathbb{R})$, we give a dimension formula for representations which are both in the Hitchin component and contained in a subgroup $SO(m,m+1)$ or $Sp(2m)$ of $SL(n,\mathbb{R})$.

Background

A ***hyperbolic triangle group*** $T(p,q,r)$ is:

- the rotational symmetries of a tiling of $\mathbb{H}^2$ by geodesic triangles with angles π/p, π/q, and π/r.
- the orbifold fundamental group of a sphere with cone points of order p, q, and r.
- $\langle a,b,c\,|\,a^p = b^q = c^r = abc = 1\rangle$

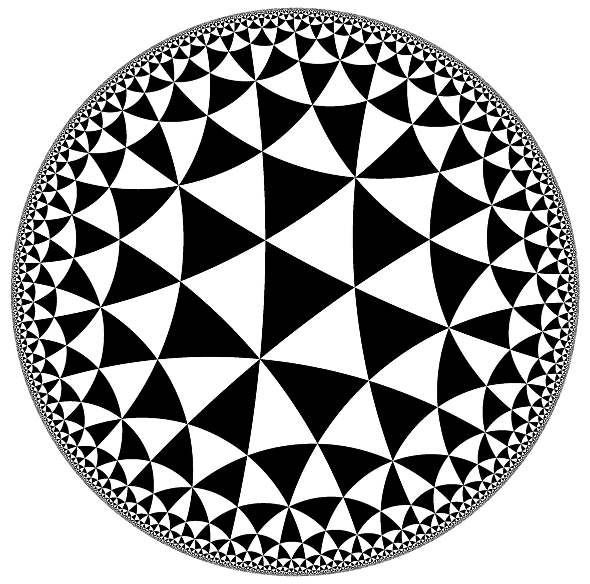

Figure 1: $T(3,3,4)$ universal cover

Each triangle group contains surface groups as subgroups of finite index; as a result, finding a representation of a triangle group provides a representation for those surface groups as well, often inheriting desirable properties like Zariski density.

For an orbifold group $\pi_1(\Sigma)$, the ***Hitchin component*** $\mathcal{H}$ is the connected component of the space of representations $\pi_1(\Sigma) \to PSL(n,\mathbb{R})$ containing the base/canonical representation

$$\rho_0 : \pi_1(\Sigma) \xrightarrow{\rho} PSL(2,\mathbb{R}) \xrightarrow{\rho_n} PSL(n,\mathbb{R}),$$

where ρ is the holonomy representation and ρ_n is the unique irreducible representation.

Our interest in the Hitchin component is, in part, due to the result of Labourie in [3] that guarantees that every representation there is discrete and faithful.

Historical Context

- **(N. Hitchin, 1992)**: For a compact oriented surface S, the Hitchin component of $\pi_1(S)$ into $PSL(n,\mathbb{R})$ has dimension $\chi(S)(1-n^2)$.
- **(S. Choi and W. Goldman, 2005)**: for a compact 2-orbifold Σ with $\chi(\Sigma) < 0$ and without boundary, there is a formula for the dimension of the Hitchin component of $\pi_1(\Sigma)$ into $PSL(3,\mathbb{R})$.
- **(F. Labourie, 2006)**: Every representation in the Hitchin component is discrete and faithful.
- **(D. Long and M. Thistlethwaite, 2018)**: Let $\mathcal{H}$ be the Hitchin component of $T(p,q,r) \to PSL(n,\mathbb{R})$. Then
$$dim\mathcal{H} = (2n^2+1) - (\sigma(n,p) + \sigma(n,q) + \sigma(n,r)),$$
where $\sigma(n,k)$ arises from counting repeated eigenvalues.

Restriction to Subgroups

The ***restricted Hitchin component*** $\mathcal{H}_G$ is the space of representations in the Hitchin component which lie in a subgroup G of $SL(n,\mathbb{R})$. In the case of $G = SO(m,m+1)$ or $G = Sp(2m)$, this means that images g of these representations satisfy

$$g^T F g = F,$$

where F is an (skew-)symmetric antidiagonal matrix. This is a natural restriction to make, since images of triangle groups under the base representation ρ_0 already lie in these subgroups.

Method

Historically, approaches to understanding the Hitchin component for orbifold groups have been largely geometric in nature, typically involving Higgs bundles. This has the advantage of being effective for any orbifold group $\pi_1(\Sigma)$, but is difficult to generalize from degree 3 representations to degree n representations.

We follow the approach of [4], which seeks to understand one family of orbifold groups ($T(p,q,r)$) by analyzing their shared algebraic structure. In particular, we consider the number of repeated eigenvalues of cyclic generators.

Our approach differs from that of [4] in that we distinguish between real and complex eigenvalues, as a means of ensuring that our representations satisfy the group relation $g^T F g = F$. For example, for degree $n = 7$ and a cyclic generator D of order 4, we obtain the following direct sum decompositions for D and F:

$$D = \begin{pmatrix} \boxed{1} & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & i & & & \\ & & & & i & & \\ & & & & & -i & \\ & & & & & & -i \end{pmatrix} \qquad F = \begin{pmatrix} \boxed{-20} & & & & & & \\ & & -6 & & & & \\ & -6 & & & & & \\ & & & & & & 1 \\ & & & & & 15 & \\ & & & & 15 & & \\ & & & 1 & & & \end{pmatrix}$$

$\leftarrow$ dim 0 $\leftarrow$ dim 1   dim 2^2 $\rightarrow$

Our formulation of $\sigma_G(n,k)$ below follows from using the parity of the degree, cyclic order, and quotient $n\backslash k$ to determine the number of repeated real eigenvalues and repeated complex eigenvalue pairs for a generator, and as a result, the dimension of the centralizer for that generator.

Main Theorem

Let a subscript $\mathcal{E}$ denote parity.

For n the degree of the representation, k the order of a cyclic generator, and Q,R the quotient and remainder respectively of integer division $n\backslash k$, we define

$$\sigma_G(n,k) := \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\,(Q + Q_{\mathcal{E}}) - n_{\mathcal{E}}\,(2Q+1)\right).$$

Let $\mathcal{H}_G$ be the restricted Hitchin component of $T(p,q,r) \to G$. Then for $G = SO(m,m+1)$ or $Sp(2m)$,

$$dim\mathcal{H}_G = \dim G - (\sigma_G(n,p) + \sigma_G(n,q) + \sigma_G(n,r)),$$

where $\sigma_G(n,k)$ arises from counting repeated real eigenvalues, and repeated pairs of complex conjugate eigenvalues.

Future Directions

- While these results only pertain to one class of orbifold groups, it is likely that other classes of orbifold groups with shared algebraic structure can be analyzed in a similar manner. In fact, since orbifolds can be characterized by the order and number of cone points and corner reflectors, an algebraic approach might be generalizable to orbifold groups as a whole.
- Knowing the dimension of the (restricted) Hitchin component gives us a foothold in the pursuit of finding parameterized infinite families of representations for particular triangle groups. Though our methods do not specify how to find these families, the results can tell us, in some sense, when such a search is "complete".

References

[1] S. Choi and W. Goldman, *The deformation spaces of convex $\mathbb{R}P^2$-structures on 2-orbifolds*, Amer. J. Math, 127 (2005), pp. 1019–1102.

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[4] D. Long and M. Thistlethwaite, *The dimension of the hitchin component for triangle groups*, Geometriae Dedicata (to appear), (2018). `http://web.math.ucsb.edu/~long/pubpdf/slnr_submitted.pdf`.

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