Pseudomodular Surfaces

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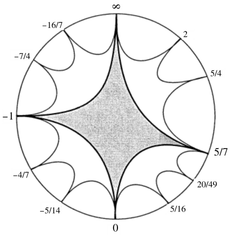

References

[1] K. Conrad, $SL_2(\mathbb{Z})$, http://www.math.uconn.edu/ kconrad/blurbs/grouptheory/SL(2,Z).pdf.

[2] D.D. Long and A.W. Reid, *Pseudomodular surfaces*, J. Reine Angew. Math. vol. 552 (2002) pp. 77-100.

[3] H.M. Hilden, M.-T. Lozano, and J.-M. Montesinos, *A characterization of arithmetic subgroups of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$*, Math. Nachr. vol. 159 (1992) pp. 245-270.

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$PSL(2,\mathbb{Z})$

Definition.

The **modular group** $PSL(2,\mathbb{Z})$ *is the quotient of*

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}$$

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by its center $\{\pm I\}$.

The modular group acts on $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ by fractional linear transformations:

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z = \frac{az + b}{cz + d}$$

Cusps of $PSL(2, \mathbb{Z})$

Proposition 1.

For every element of $\mathbb{Q} \cup \infty$, there is a parabolic element of $PSL(2, \mathbb{Z})$ fixing it.

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Proof Idea

Use the following elements of the modular group to reduce denominators:

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, translation by 1 to the right, and

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, negative inversion.

Generators of $PSL(2, \mathbb{Z})$

Theorem 2.

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $PSL(2, \mathbb{Z})$.

Generators of $PSL(2, \mathbb{Z})$

Theorem 2.

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *generate* $PSL(2, \mathbb{Z})$.

Lemma 3.

Every $z \in \mathbb{H}$ *has an element of its* $PSL(2, \mathbb{Z})$*-orbit in*

$$\mathcal{F} = \{z \in \mathbb{H} | |Re(z)| \leq 1/2, |z| \geq 1\}.$$

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Proof Idea

For $\gamma \in PSL(2,\mathbb{Z})$, move $\gamma(2i)$ back into $\mathcal{F}$ using some $g \in G$.

Constructing the Farey Tesselation

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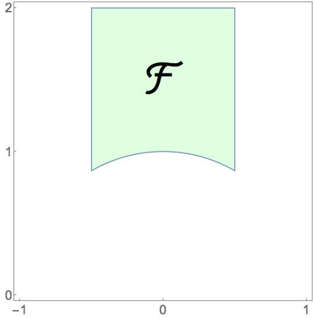

Constructing the Farey Tesselation

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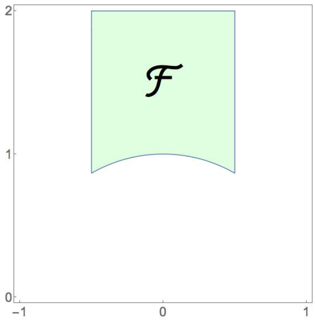
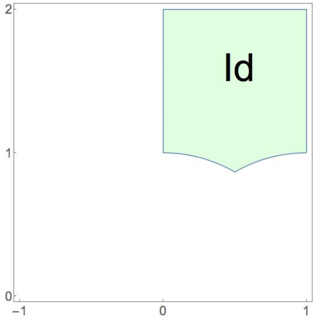


We'll start by transforming the fundamental region $\mathcal{F}$ into an ideal triangle with vertices $\{0, 1, \infty\}$.

Constructing the Farey Tesselation

Alter $\mathcal{F}$ by:

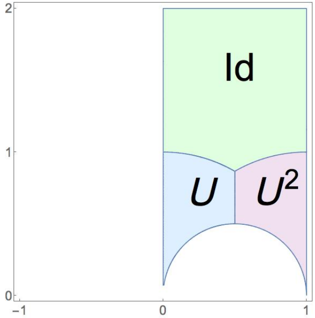
- chopping off the left half and translating it right 1,



Constructing the Farey Tesselation

Alter $\mathcal{F}$ by:

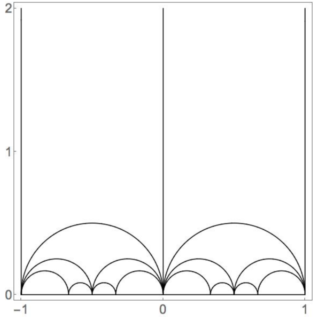
- chopping off the left half and translating it right 1, and
- taking the image under Id, $U = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = ST^{-1}$, and U^2



Constructing the Farey Tesselation

Definition 4.

The **Farey Tesselation** $\mathcal{T}$ is the images of the ideal triangle with vertices $\{0, 1, \infty\}$ under the action of $PSL(2, \mathbb{Z})$ by fractional linear transformations.



Main Theorem: Existence of Pseudomodular Groups

Theorem 5.

There is a finite coarea discrete group $\Gamma \leq PSL(2, \mathbb{Q})$ not commensurable with the modular group whose cusp set is precisely $\mathbb{Q} \cup \{\infty\}$. We call groups that satisfy these properties **pseudomodular**.

(Potentially) Pseudomodular Groups: $\Delta(u^2, 2\tau)$

Definition 6.

Let $u^2, \tau \in \mathbb{Q}$, $0 < u^2 < \tau - 1$. Then $\Delta(u^2, 2\tau) := \langle g_1, g_2 \rangle$, where

$$g_1 := \frac{1}{D} \begin{pmatrix} \tau - 1 & u^2 \\ 1 & 1 \end{pmatrix}, \; g_2 := \frac{1}{D} \begin{pmatrix} u & u \\ 1/u & \frac{\tau - u^2}{u} \end{pmatrix}$$

and $D = \sqrt{\tau - u^2 - 1}$.

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and $D = \sqrt{\tau - u^2 - 1}$.

Then $g_1 g_2^{-1} g_1^{-1} g_2 = \begin{pmatrix} -1 & -2\tau \\ 0 & -1 \end{pmatrix}$, right translation by 2τ.

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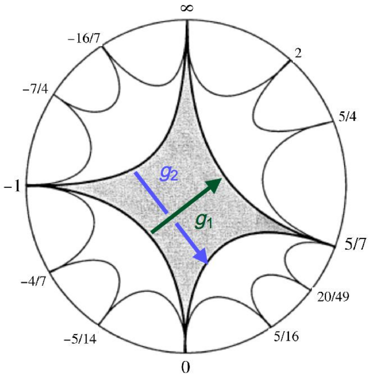

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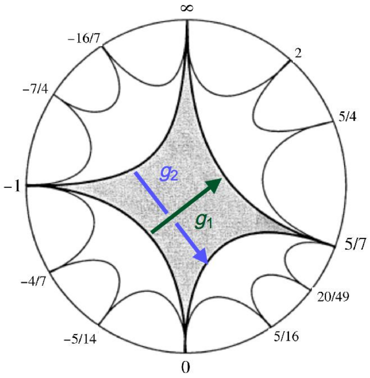

Theorem 7.

The group $\Delta(5/7, 6)$ *is pseudomodular.*

$\Delta(5/7, 6)$

Note

$\Delta(5/7, 6)$ is not commensurable with $SL(2, \mathbb{Z})$ because $tr(g_2^2) = 39/5 \notin \mathbb{Z}$. See [3].

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Proof Idea

Cover $\mathbb{Q}$ with intervals where a group element always reduces the denominator of rationals inside that interval, called **killer intervals**.

$\Delta(5/7, 6)$

Lemma 8.

If $\Delta(u^2, 2\tau)$ is such that $[0, 2\tau]$ can be covered by killer intervals, then its cusp set is $\mathbb{Q} \cup \{\infty\}$.

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7.4. Killer intervals for $\Delta(5/7, 6)$.

$$\begin{aligned}
&\{(0:7), (1/7:125), (5/28:1), (5/21:1), (2/7:25), (45/154:5),\\
&(40/133:1), (5/16:7), (30/91:1),\\
&(55/161:1), (130/371:1), (55/156:7), (3190/9023:1),\\
&(255/721:25), (5735/16212:1),\\
&(635/1792:1), (5/14:25), (35/97:7), (120/329:1),\\
&(235/637:1), (115/308:1), (45/119:5), (155/406:1), (65/168:1),\\
&(20/49:1), (3/7:25), (10/21:1), (85/161:1), (15/28:5), (65/119:1),\\
&(55/98:1), (4/7:25), (5/7:1), (6/7:125),\\
&(25/28:1), (13/14:5), (125/133:1), (20/21:5), (75/77:1), (125/126:1),\\
&(1:175), (50/49:1), (270/259:1), (300/287:1), (1550/1477:1),\\
&(125/119:25), (2825/2688:1), (2525/2401:1), (325/308:1),\\
&(15/14:5), (25/23:7), (100/91:1), (125/112:1), (8/7:5),\\
&(25/21:1), (5/4:7), (9/7:5), (10/7:1), (11/7:1), (12/7:5), (13/7:1),\\
&(2:7), (15/7:1), (16/7:5), (17/7:1), (18/7:25), (37/14:1), (19/7:5),\\
&(58/21:1), (453/161:1), (2033/721:1), (79/28:25), (1917/679:1),\\
&(337/119:1), (20/7:5), (3:7)\}
\end{aligned}$$

Killer Interval Example for $\Delta(5/7, 6)$

Example 9.

We can construct a killer interval centered at $1/2$.

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- $g_1^{-1} g_2^{-1} = \frac{1}{3} \begin{pmatrix} 7\sqrt{7/5} & -4\sqrt{5/7} \\ -2\sqrt{35} & \sqrt{35} \end{pmatrix}$ takes $1/2$ to ∞.

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- For $p, q \in \mathbb{Z}$, $(g_1^{-1} g_2^{-1}) \cdot \frac{p}{q} = \frac{-49p+20q}{35(2p-q)}$.

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- Consider the interval $|35(2p-q)| < |q|$ where $g_1^{-1} g_2^{-1}$ strictly decreases denominators:

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$$\left(\frac{1}{2} - \frac{1}{2 \cdot 35}, \frac{1}{2} + \frac{1}{2 \cdot 35} \right)$$

Not-so-pseudomodular $\Delta(u^2, 2\tau)$

Definition 10.

Hyperbolic elements of a discrete subgroup of $PSL(2, \mathbb{C})$ which fix points in the rationals are **special hyperbolics**.

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$\Delta(3/4, 4)$ has a collection of killer intervals which miss only $1/2$ and $3/2$ in $[0, 2]$.

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Example 12.

$\Delta(1, 6)$ is commensurable with $PSL(2, \mathbb{Z})$.

A pseudo-Farey Tesselation of $\mathbb{H}$

Take the fundamental domain $\mathscr{Q}_{(u^2,2\tau)}$ formed by the vertex set $\{-1, 0, u^2, \infty\}$ and act by $\Delta(u^2, 2\tau)$.

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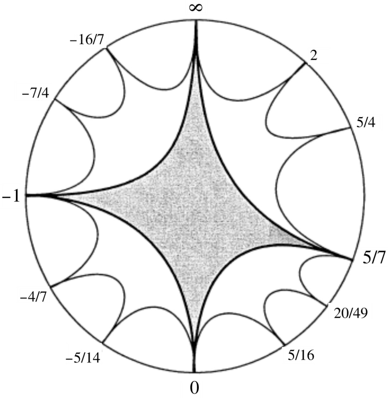

Relating the pseudo-Farey and Farey Tesselations

Since $\Delta(1,6)$ is commensurable with $PSL(2,\mathbb{Z})$, the previous construction yields the usual Farey tessellation if we include the geodesic from 0 to ∞:

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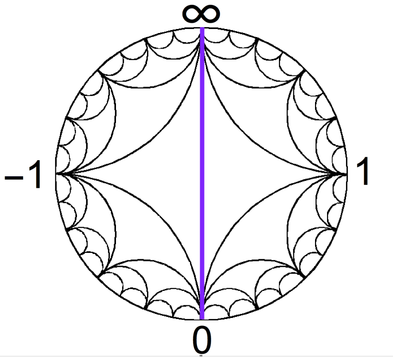

Classifying $\Delta(u^2, 2\tau)$

Table 5.1. $2\tau = 4$

$0 < u^2 \leq 1$	structure	$0 < u^2 \leq 1$	structure
1	arithmetic	1/9	special fixing 1/3
1/2	arithmetic	2/9	special fixing 1/3
1/3	arithmetic	4/9	special fixing 2/3
2/3	arithmetic	5/9	special fixing 1/3
1/4	special fixing 1/2	7/9	special fixing 1/3
3/4	special fixing 1/2	8/9	special fixing 2/3
1/5	arithmetic	1/10	special fixing 7/2
2/5	**pseudomodular**	3/10	special fixing 1/5
3/5	**pseudomodular**	7/10	special fixing 1/2
4/5	arithmetic	9/10	special fixing 6/5
1/6	special fixing 3/2	1/11	conjectural pseudomodular
5/6	special fixing 1/2	2/11	conjectural pseudomodular
1/7	conjectural pseudomodular	3/11	**pseudomodular**
2/7	conjectural pseudomodular	4/11	conjectural pseudomodular
3/7	**pseudomodular**	5/11	conjectural pseudomodular
4/7	conjectural pseudomodular	6/11	conjectural pseudomodular
5/7	conjectural pseudomodular	7/11	conjectural pseudomodular
6/7	conjectural pseudomodular	8/11	conjectural pseudomodular
1/8	special fixing 1/2	9/11	conjectural pseudomodular
3/8	special fixing 1/2	10/11	conjectural pseudomodular
5/8	special fixing 1/2		
7/8	special fixing 1/2		

Classifying $\Delta(u^2, 2\tau)$

Table 5.2. $2\tau = 6$

$0 < u^2 \leqq 1$	structure	$0 < u^2 \leqq 1$	structure
1	arithmetic	1/9	special fixing −100/117
1/2	arithmetic	2/9	special fixing 545/1521
1/3	special fixing 1	4/9	special fixing −52/9
2/3	special fixing 1/3	5/9	special fixing −5/16
1/4	special fixing −5/8	7/9	special fixing 29/9
3/4	special fixing 3/2	8/9	special fixing −205/9
1/5	arithmetic	1/10	special fixing 5/52
2/5	special fixing 1/7	3/10	special fixing 1/2
3/5	conjectural pseudomodular	7/10	special fixing 1/2
4/5	conjectural pseudomodular	9/10	special fixing 6/5
1/6	special fixing −1/35	1/11	conjectural pseudomodular
5/6	special fixing −17/24	2/11	special fixing −266/4717
1/7	special fixing −37/14	3/11	undecided
2/7	conjectural pseudomodular	4/11	special fixing 1/5
3/7	special fixing 3/4	5/11	special fixing −1778/741
4/7	special fixing 2/7	6/11	special fixing 69/11
5/7	**pseudomodular**	7/11	special fixing 149/136
6/7	special fixing 5/3	8/11	special fixing −79/93
1/8	special fixing 1/14	9/11	conjectural pseudomodular
3/8	special fixing −15/2	10/11	special fixing 1/3
5/8	special fixing 7/4		
7/8	special fixing 1/2		

The End!

Questions?

References

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Thanks!

A special thanks goes to my advisor

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and to the rest of my committee

Dr. James Conant and Dr. Marie Jameson