Where the Wild Solutions Are

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Section 1 - Wild Solutions

Projective algebraic geometry seeks to capture oddities that arise from the primitiveness of affine space. To motivate our forthcoming journey, let's dive into one of those oddities.

Example 1.1. Consider the following system of equations in $\mathbb{C}[t_1, t_2, t_3, x_1, \ldots, x_4]$:

$$\begin{aligned} f_1 &= x_1 - t_1^2 \\ f_2 &= x_2 - t_1 t_2^2 \\ f_3 &= x_3 - t_1^3 t_3 \\ f_4 &= x_4 + t_1. \end{aligned}$$

Let $I = \langle f_1, \ldots, f_4 \rangle$. Notice that $\pi_3(\mathbf{V}(I)) = F(\mathbb{C}^3) \subseteq \mathbb{C}^4$, where $F(t_1, t_2, t_3) = (t_1^2, t_1 t_2^2, t_1^3 t_3, -t_1)$. By The Closure Theorem, $\mathbf{V}(I_3)$ is the Zariski closure of $F(\mathbb{C}^3)$. We now want to compute $\mathbf{V}(I_3)$. In order to do so, we determine the Gröbner basis of I (with respect to lex order) using Sage:

$$\begin{aligned} g_1 &= t_1 + x_4 \\ g_2 &= x_3 t_2^2 - t_3 x_2 x_4^2 \\ g_3 &= x_4 t_2^2 + x_2 \\ g_4 &= x_4^3 t_3 + x_3 \\ g_5 &= x_1 - x_4^2. \end{aligned}$$

By The Elimination Theorem, $I_3 = \langle x_1 - x_4^2 \rangle$. So, $\mathbf{V}(I_3) = \{(a_1, a_2, a_3, a_4) \in \mathbb{C}^4 : a_1 = a_4^2\}$. The wildness comes to the fore when we consider $(0, 0, i, 0) \in \mathbf{V}(I_3)$. Let $(u_1, u_2, u_3) \in \mathbb{C}^3$ be arbitrary. Notice that $g_4(u_1, u_2, u_3, 0, 0, i, 0) = i \neq 0$. Therefore, $(0, 0, i, 0) \notin \pi_3(\mathbf{V}(I)) = F(\mathbb{C}^3)$.
Drawing inspiration from Homework 6 Problem 3, we want to characterize all elements in $\mathbf{V}(I_3)$ that don't extend into full solutions. Let $S = \{(a_1, \ldots, a_4) \in \mathbf{V}(I_3) : (a_1, \ldots, a_4) \notin F(\mathbb{C}^3)\}$ and $W = \{(0, a, b, 0) \in \mathbb{C}^4 : a \neq 0 \text{ or } b \neq 0\}$.
We first show that $W \subseteq S$. Let $(0, a_*, b_*, 0) \in W$ be arbitrary. Notice that $(0, a_*, b_*, 0) \in \mathbf{V}(I_3)$. Now, we have two cases:

- Suppose $a_* \neq 0$. Let $(u_1, u_2, u_3) \in \mathbb{C}^3$ be arbitrary. Notice that $g_3(0, a_*, b_*, 0, u_1, u_2, u_3) = a_* \neq 0$. It follows that $(0, a_*, b_*, 0) \in S$.

- Suppose $b_* \neq 0$. Let $(u_1, u_2, u_3) \in \mathbb{C}^3$ be arbitrary. Notice that $g_4(0, a_*, b_*, 0, u_1, u_2, u_3) = b_* \neq 0$. It follows that $(0, a_*, b_*, 0) \in S$.

We now show that $S \subseteq W$. Let $(a_1, a_2, a_3, a_4) \in S$ be arbitrary. Notice that $a_1 = a_4^2$. So, $(a_1, a_2, a_3, a_4) = (a_4^2, a_2, a_3, a_4)$. Suppose for a contradiction that $a_4 \neq 0$. By The Extension Theorem, we can fix $(b_1, b_2, b_3) \in \mathbb{C}^3$ such that $(b_1, b_2, b_3, a_1, \ldots, a_4) \in \mathbf{V}(I)$. But then $(a_1, \ldots, a_4) \in F(\mathbb{C}^3)$, a contradiction. Therefore, $a_4 = 0$, and $(a_1, a_2, a_3, a_4) = (0, a_2, a_3, 0)$. Notice that $F(0, 0, 0) = (0, 0, 0, 0)$, which means that $a_2 \neq 0$ or $a_3 \neq 0$. It follows that $(a_1, a_2, a_3, a_4) \in W$.

To where did W disappear when we moved up from $\mathbf{V}(I_3)$ to $\mathbf{V}(I)$? Is there a way to capture these deviant 4-tuples? As is customary in mathematical journeys, we begin in an unexpected manner.

Section 2 - Homogeneity

Before commencing our mathematical journey, here's a logistical note: Throughout this exposition, we denote an arbitrary field by k and will explicitly specify when k is infinite and/or algebraically closed.

The heart of projective space—which is what we need to capture W—lies in tactfully expanding k^n. This expansion consists of increasing the dimension of k^n by 1 *and* increasing the dimension of points on k^n by 1, which leaves us with a geometric construction that mirrors k^n and includes just a bit more.

To engage in algebraic geometry within projective space, it is necessary to find a class of polynomials that "fit" into projective space. Throughout this section, we uncover such a class of polynomials: homogeneous polynomials. Investigating homogeneous polynomials will reveal that these objects occupy a world of their own within the polynomial ring over a field.

Definition 2.1. *Let $f \in k[x_1, \ldots, x_n]$ be nonzero, and suppose deg$(f) = \alpha$. Write*

$$f = \sum_{i=0}^{\alpha} f_i(x_1, \ldots, x_n),$$

where each term in f_i has total degree i. We say that f is homogeneous (and, in particular, α-homogeneous) if $f = f_\alpha$. Also, we define $0 \in k[x_1, \ldots, x_n]$ to be homogeneous.

Example 2.2. Consider $xy^2 + 5z, xz + y^2 \in \mathbb{R}[x, y, z]$. It is clear that $xy^2 + 5z$ is not a homogeneous polynomial while $xz + y^2$ is a 2-homogeneous polynomial. Is there a neat way to homogenize $f = xy^2 + 5z$? We may initially think to multiply $5z$ by z^2, which would give us $f_* = xy^2 + 5z^3$, a 3-homogeneous polynomial. However, there are a couple of issues with this construction. Firstly, there's a lack of uniqueness; we could have multiplied $5z$ by x^2 or y^2 instead of z^2. More severely, there's no way to recover f from f_*. That is, suppose someone gave us f_* and asked, "Which polynomial was homogenized to get f_*?" We wouldn't be able to confidently answer the question. It is for these reasons that we define the homogenization of a polynomial as follows.

Definition 2.3. *Let $f \in k[x_1, \ldots, x_n]$ be nonzero, and suppose deg$(f) = \alpha$. Write*

$$f = \sum_{i=0}^{\alpha} f_i(x_1, \ldots, x_n),$$

where f_i is i-homogeneous or $f_i = 0$. Then

$$f^h = \sum_{i=0}^{\alpha} f_i(x_1, \ldots, x_n) x_0^{\alpha - i} \in k[x_0, \ldots, x_n]$$

is the homogenization of f. Also, we define $0^h = 0$ to be the homogenization of 0.

It is a matter of unwinding Definition 2.1 and Definition 2.3 to check that f^h is α-homogeneous and $f^h(1, x_1, \ldots, x_n) = f$. Additionally, this definition of homogenization bestows upon us other slightly less apparent (but no less crucial) properties.

Proposition 2.4. *Let $f \in k[x_1, \ldots, x_n]$ be nonzero, and suppose deg$(f) = \alpha$. Then:*

(i) $f^h = x_0^\alpha f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$.

(ii) *Let $g \in k[x_0, \ldots, x_n] \backslash \{0\}$ be m-homogeneous, and let x_0^e be the highest power of x_0 dividing g. If $g_d = g(1, x_1, \ldots, x_n)$, then $g = x_0^e \cdot g_d^h$.*

(iii) *Let $S = \{h \in k[x_0, \ldots, x_n] : x_0 \nmid h$ and h is homogeneous$\} \cup \{0\}$. The maps $\phi : k[x_1, \ldots, x_n] \to S$ defined by $g \mapsto g^h$ and $\psi : S \to k[x_1, \ldots, x_n]$ defined by $h \mapsto h(1, x_1, \ldots, x_n)$ are inverses of each other, and hence are bijections.*

Proof.

(i) Write $f = \sum_{i=0}^{\alpha} f_i(x_1, \dots, x_n)$, where each term in f_i has total degree i. Notice that

$$\begin{aligned} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) &= \sum_{i=0}^{\alpha} f_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \sum_{i=0}^{\alpha} f_i(x_1, \dots, x_n)\frac{1}{x_0^i}. \end{aligned}$$

So,

$$\begin{aligned} x_0^{\alpha} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) &= x_0^{\alpha}\sum_{i=0}^{\alpha} f_i(x_1, \dots, x_n)\frac{1}{x_0^i} \\ &= \sum_{i=0}^{\alpha} f_i(x_1, \dots, x_n)x_0^{\alpha-i} \\ &= f^h. \end{aligned}$$

(ii) Notice that $g = \sum_{i=0}^{m-e} g_i(x_1, \dots, x_n)x_0^{m-i}$, where $g_i = 0$ or g_i is i-homogeneous. Also, we have that $g_d = \sum_{i=0}^{m-e} g_i(x_1, \dots, x_n)$, and

$$\begin{aligned} g_d^h &= x_0^{m-e} \cdot g_d\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= \sum_{i=0}^{m-e} x_0^{m-e} g_i(x_1, \dots, x_n)\frac{1}{x_0^i} \\ &= \sum_{i=0}^{m-e} g_i(x_1, \dots, x_n)x_0^{m-e-i}. \end{aligned}$$

It follows easily that $g = x_0^e \cdot g_d^h$.

(iii) It is immediate from Definition 2.3 that range$(\phi) \subseteq S$.
Now, let $f \in k[x_1, \dots, x_n]$ be arbitrary. Notice that $(\psi \circ \phi)(f) = \psi(\phi(f)) = \psi(f^h) = f^h(1, x_1, \dots, x_n) = f$.
Let $f \in S$ be arbitrary. Suppose first that $f \neq 0$. Notice that $(\phi \circ \psi)(f) = \phi(\psi(f)) = \phi(f(1, x_1, \dots, x_n)) = f(1, x_1, \dots, x_n)^h$. Since $x_0 \nmid f$, we have by Part (ii) that $f(1, x_1, \dots, x_n)^h = f$. Suppose now that $f = 0$. Then $(\phi \circ \psi)(0) = \phi(\psi(0)) = 0^h = 0$. $\square$

Recall that affine algebraic geometry is marked by a correspondence between radical ideals and affine varieties. In order to exploit the power of homogeneous polynomials for the sake of doing projective algebraic geometry, we need to consider an ideal-theoretic characterization of homogeneous polynomials.

Definition 2.5. *Let $f \in k[x_1, \dots, x_n]$ be nonzero, and suppose deg$(f) = \alpha$. Write*

$$f = \textstyle\sum_{i=0}^{\alpha} f_i(x_1, \dots, x_n),$$

where f_i is i-homogeneous or $f_i = 0$. If $f_i \neq 0$, then we say that f_i is a homogeneous component of f.

Definition 2.6. *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. We say that I is homogeneous if for all $f \in I\backslash\{0\}$, the homogeneous components of f are in I.*

Example 2.7. Consider $I = \langle x, y\rangle \subseteq \mathbb{R}[x, y]$. We want to show that I is homogeneous. Let $f \in I\backslash\{0\}$ be arbitrary. Fix $d \in \mathbb{N}$ such that $\deg(f) = d$. Since we can fix $g, q \in \mathbb{R}[x, y]$ such that $f = g \cdot x + q \cdot y$, it follows that $f = \sum_{i=1}^{d} f_i(x, y)$, where $f_i = 0$ or f_i is i-homogeneous. Let f_i be an arbitrary homogeneous component of f. Notice that $\deg(f_i) \geq 1$. Now, let $a_\alpha x^{\alpha_1} y^{\alpha_2}$ be an arbitrary term of f_i. Since $\alpha_1 \geq 1$ or $\alpha_2 \geq 1$, we can conclude that $a_\alpha x^{\alpha_1} y^{\alpha_2} \in I$. Therefore, $f_i \in I$. It follows that I is a homogeneous polynomial.

To state the obvious, it'd be a damn shame if we had to do a variation of this proof each time we wanted to show that an ideal is (or is not) homogeneous. Luckily, there's an algorithmic method to determine whether an ideal is homogeneous.

The following proofs involve the Division Algorithm in $k[x_1, \ldots, x_n]$ and Buchberger's Algorithm. To spare myself and the reader suffering, I avoid digging into the details of these algorithms. Thus, I urge the reader to re-investigate the Division Algorithm in $k[x_1, \ldots, x_n]$ and Buchberger's Algorithm before tackling these proofs.

It is also worth mentioning that these proofs take as given that for all homogeneous polynomials $f, g \in k[x_1, \ldots, x_n]$, we have that $f \cdot g$ is homogeneous. This claim is indeed true. I pinkie swear.

Proposition 2.8. *If $f, g \in k[x_1, \ldots, x_n] \backslash \{0\}$ are homogeneous, then $S(f, g)$ (with respect to any monomial order) is homogeneous.*

Proof. Let $>$ be an arbitrary monomial order on $\mathbb{N}^n$. Let $f, g \in k[x_1, \ldots, x_n] \backslash \{0\}$ be arbitrary such that f and g are homogeneous. Fix $m, n \in \mathbb{N}$ such that f is m-homogeneous and g is n-homogeneous. Let $x^\gamma = \text{lcm}(\text{LM}(f), \text{LM}(g))$, and let $c = \deg(x^\gamma)$. Notice that x^γ is c-homogeneous. Now, notice that $\deg\left(\frac{x^\gamma}{\text{LT}(f)}\right) = c - m$ and $\deg\left(\frac{x^\gamma}{\text{LT}(g)}\right) = c - n$. So, $\deg\left(\frac{x^\gamma}{\text{LT}(f)} \cdot f\right) = c$ and $\deg\left(\frac{x^\gamma}{\text{LT}(g)} \cdot g\right) = c$. Since $\frac{x^\gamma}{\text{LT}(f)} \cdot f$ and $\frac{x^\gamma}{\text{LT}(g)} \cdot g$ are c-homogeneous polynomials, it follows easily that $S(f, g) = \frac{x^\gamma}{\text{LT}(f)} \cdot f + \frac{x^\gamma}{\text{LT}(g)} \cdot g$ is a c-homogeneous polynomial. ☐

Proposition 2.9. *If $f, g_1, \ldots, g_t \in k[x_1, \ldots, x_n] \backslash \{0\}$ are homogeneous and we use the Division Algorithm in $k[x_1, \ldots, x_n]$ (with respect to any monomial order) to get $q_1, \ldots, q_t, r \in k[x_1, \ldots, x_n]$ with $f = q_1 g_1 + \cdots + q_t g_t + r$, then $q_1, \ldots, q_t, r$ are homogeneous.*

Proof. Let $>$ be an arbitrary monomial order on $\mathbb{N}^n$. Let $f, g_1, \ldots, g_t \in k[x_1, \ldots, x_n] \backslash \{0\}$ be arbitrary homogeneous polynomials. Fix $n, m_1, \ldots, m_t \in \mathbb{N}$ such that $\deg(f) = n$ and $\deg(g_i) = m_i$. By the Division Algorithm, we get $f = q_1 g_1 + \cdots + q_t g_t + r$, where $q_1, \ldots, q_t, r \in k[x_1, \ldots, x_n]$.
Let $i \in \{1, \ldots, t\}$ be arbitrary. If $q_i = 0$, then q_i is homogeneous. Suppose $q_i \neq 0$. Now, let $p_* \in k[x_1, \ldots, x_n]$ be an arbitrary n-homogeneous polynomial with $\text{LT}(g_i) \mid \text{LT}(p_*)$. Consider $p_0 = p_* - \left(\frac{\text{LT}(p_*)}{\text{LT}(g_i)}\right) g_i$. Notice that $\deg\left(\frac{\text{LT}(p_*)}{\text{LT}(g_i)}\right) = n - m_i$. So, $\deg\left(\left(\frac{\text{LT}(p_*)}{\text{LT}(g_i)}\right) g_i\right) = n$. Since $\frac{\text{LT}(p_*)}{\text{LT}(g_i)}$ and g_i are homogeneous polynomials, it follows that $p_0 = 0$ or p_0 is an n-homogeneous polynomial. Also, consider $p_1 = p_* - \text{LT}(p_*)$. By Definition 2.1, $p_1 = 0$ or p_1 is an n-homogeneous polynomial. It follows from these results and the construction of q_i in the Division Algorithm that q_i is an $(n - m_i)$-homogeneous polynomial.
Now, notice that either $q_i g_i = 0$ or $q_i g_i$ is an n-homogeneous polynomial. Thus, $r = f - q_1 g_1 - \cdots - q_t g_t$ equals zero or is an n-homogeneous polynomial. ☐

Theorem 2.10. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. The following are equivalent:*

(i) *I is a homogeneous ideal.*

(ii) *$I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \backslash \{0\}$ are homogeneous polynomials.*

(iii) *Let $G = \{g_1, \ldots, g_t\}$ be a reduced Gröbner basis of I with respect to any monomial order. Then $g_1, \ldots, g_t \in k[x_1, \ldots, x_n]$ are homogeneous polynomials.*

Proof. For the proof of (i) $\Leftrightarrow$ (ii), refer to p. 407-408 of *Ideals, Varieties, and Algorithms* by Cox et al.
Now, we prove (ii) $\Rightarrow$ (iii). Suppose $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s \in k[x_1, \ldots, x_n] \backslash \{0\}$ are homogeneous polynomials. Let $>$ be an arbitrary monomial order on $\mathbb{N}^n$. If $I = \{0\}$, then $G = \{0\}$ is a reduced Gröbner

basis of I with respect to $>$. Suppose $I \neq \{0\}$, and let $G = \{g_1, \dots, g_t\}$ be a reduced Gröbner basis of I with respect to $>$. We use Buchberger's Algorithm to get $f_{s+1}, \dots, f_{s+a} \in k[x_1, \dots, x_n] \backslash \{0\}$ such that $G' = \{f_1, \dots, f_s, f_{s+1}, \dots, f_{s+a}\}$ is a Gröbner basis for I. By Proposition 2.8 and Proposition 2.9, and by the construction of $f_{s+1}, \dots, f_{s+a}$ in Buchberger's Algorithm, we have that $f_{s+1}, \dots, f_{s+a}$ are homogeneous polynomials.
To get a minimal Gröbner basis G_0' of I, we scale polynomials in G' by a constant and selectively remove polynomials from G'. To get a reduced Gröbner basis G_*' of I, we selectively remove monomials from polynomials in G_0'. Thus, we have that each polynomial in G_*' is homogeneous. By Theorem 2.7.5, G_*' is the unique reduced Gröbner basis of I with respect to $>$, so $G_*' = G$. Therefore, $g_1, \dots, g_t \in k[x_1, \dots, x_n]$ are homogeneous polynomials.
The proof of (ii) $\Leftarrow$ (iii) is trivial. $\square$

This theorem is especially useful for proving that an ideal is not homogeneous.

Example 2.11. Consider $\langle x+y^2+3, xy^2+y^3\rangle \subseteq \mathbb{C}[x,y]$. By The Strong Nullstellensatz, $\mathbf{I}(\mathbf{V}(\langle x+y^2+3, xy^2+y^3\rangle)) = \sqrt{\langle x+y^2+3, xy^2+y^3\rangle}$. Using Sage, we get that $\sqrt{\langle x+y^2+3, xy^2+y^3\rangle} = \langle x+y^2+3, xy-x-3\rangle$. We also use Sage to compute that the reduced Gröbner basis of $\langle x+y^2+3, xy-x-3\rangle$ with respect to lex order is $\{x+y^2+3, y^3-y^2+3y\}$. It follows by Theorem 2.10 that $\mathbf{I}(\mathbf{V}(\langle x+y^2+3, xy^2+y^3\rangle))$ is not a homogeneous ideal.

A theme of homogeneity is its wholeness. That is, if one only considers homogeneous polynomials or homogeneous ideals, such a restriction is not much of a hindrance to understanding given homogeneous polynomials or homogeneous ideals. Hopefully, it is now clear that homogeneity indeed carves out its own world within the polynomial ring. The next two results grant us more knowledge of this lovely world.

Proposition 2.12. *Let $I_1, \dots, I_\ell \in k[x_1, \dots, x_n]$ be homogeneous ideals. Then $I_1 + \dots + I_\ell, I_1 \cap \dots \cap I_\ell$, and $I_1 \cdots I_\ell$ are homogeneous.*

Proof. We first consider $I_1 + \dots + I_\ell$. By Theorem 2.8, we fix homogeneous polynomials $f_1, \dots, f_m, g_1, \dots, g_t \in k[x_1, \dots, x_n]$ such that $I_1 = \langle f_1, \dots, f_m\rangle$ and $I_2 = \langle g_1, \dots, g_t\rangle$. So, by Proposition 4.3.2, $I_1 + I_2 = \langle f_1, \dots, f_m, g_1, \dots, g_t\rangle$. Again by Theorem 2.8, we have that $I_1 + I_2$ is homogeneous. Inductively, it follows that $I_1 + \dots + I_\ell$ is homogeneous.
Now, consider $I_1 \cap \dots \cap I_\ell$. Let $f \in (I_1 \cap \dots \cap I_\ell) \backslash \{0\}$ be arbitrary, and let $i \in \{1, \dots, \ell\}$ be arbitrary. Since $f \in I_i$, each homogeneous component of f is in I_i. It follows that each homogeneous component of f is in $I_1 \cap \dots \cap I_\ell$. So, $I_1 \cap \dots \cap I_\ell$ is homogeneous.
Finally, consider $I_1 \cdots I_\ell$. By Theorem 2.8, we can fix homogeneous polynomials $f_1, \dots, f_m, g_1, \dots, g_t \in k[x_1, \dots, x_n]$ such that $I_1 = \langle f_1, \dots, f_m\rangle$ and $I_2 = \langle g_1, \dots, g_t\rangle$. By Proposition 4.3.6, $I \cdot J = \langle f_i g_j : 1 \leq i \leq m, 1 \leq j \leq t\rangle$. Let $i \in \{1, \dots, m\}$ be arbitrary, and let $j \in \{1, \dots, t\}$ be arbitrary. Since f_i and g_j are homogeneous, it follows that $f_i g_j$ is homogeneous. Therefore, we have by Theorem 2.8 that $I_1 \cdot I_2$ is homogeneous. Inductively, it follows that $I_1 \cdots I_\ell$ is homogeneous. $\square$

Proposition 2.13. *Let $I \subsetneq k[x_1, \dots, x_n]$ be a homogeneous ideal. Then I is prime if and only if for all homogeneous polynomials $f, g \in k[x_1, \dots, x_n]$, if $fg \in I$, then $f \in I$ or $g \in I$.*

Proof. Suppose I is prime. Let $f, g \in k[x_1, \dots, x_n]$ be arbitrary homogeneous polynomials such that $fg \in I$. Then either $f \in I$ or $g \in I$.
Suppose for all homogeneous polynomials $f, g \in k[x_1, \dots, x_n]$, either $f \in I$ or $g \in I$ whenever $fg \in I$. Let $h, p \in k[x_1, \dots, x_n]$ be arbitrary such that $hp \in I$. If $h = 0$ or $p = 0$, then either $h \in I$ or $p \in I$. Suppose $h \neq 0$ and $p \neq 0$. If $\deg(hp) = 0$, then $hp \in k$, which means that $I = k[x_1, \dots, x_n]$, a contradiction. Thus, $\deg(hp) \geq 1$. We use strong induction on $\deg(hp)$. Suppose $\deg(hp) = 1$. Then either $\deg(h) = 1$ and $\deg(p) = 0$ or $\deg(h) = 0$ and $\deg(p) = 1$. Suppose without loss of generality that $\deg(h) = 1$ and $\deg(p) = 0$. Write

$$h = \sum_{i=0}^{1} h_i(x_1, \ldots, x_n)$$

where $h_i = 0$ or h_i is a homogeneous component of h with $\deg(h_i) = i$. Suppose for a contradiction that $h_0 \neq 0$. Then $h_0 p \in k$ is a homogeneous component of hp, which means that $h_0 p \in I$. So, $I = k[x_1, \ldots, x_n]$, a contradiction. We therefore conclude that $h_0 = 0$. Thus, h is homogeneous. Since h and p are homogeneous, it follows that either $h \in I$ or $p \in I$.

Now, let $m \in \mathbb{N}^+ \setminus \{1\}$ be arbitrary. Suppose $\deg(hp) = m$. Also, suppose that for all $f_*, g_* \in k[x_1, \ldots, x_n]$ with $\deg(f_* g_*) < m$, we have that $f_* g_* \in I$ implies either $f_* \in I$ or $g_* \in I$. Fix $a, b \in \mathbb{N}$ such that $\deg(h) = a$ and $\deg(p) = b$. Let h_a be a homogeneous component of h with $\deg(h_a) = a$, and let p_b be a homogeneous component of p with $\deg(p_b) = b$. Notice that $h_a p_b$ is a homogeneous component of hp. So, $h_a p_b \in I$. Since h_a and p_b are homogeneous polynomials, either $h_a \in I$ or $p_b \in I$. Suppose without loss of generality that $h_a \in I$. Now, consider $(h - h_a)(p) = hp - h_a p \in I$. If $h - h_a = 0$, then it follows that $h = h_a \in I$. Suppose $h - h_a \neq 0$. Notice that $\deg((h - h_a)(p)) < a + b = m$. By induction, either $h - h_a \in I$ or $p \in I$. Suppose $p \notin I$. Then $h - h_a + h_a = h \in I$. Therefore, either $h \in I$ or $p \in I$. So, I is prime. $\square$

Now that we've fawned over the remarkable properties of homogeneous ideals, we want a ticket into their club. Recall that for a lowly non-homogeneous polynomial, its ticket into homogeneity was homogenization. We can similarly homogenize an ideal. And in fact, ideal homogenization is heavily reliant upon polynomial homogenization.

Definition 2.14. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. We define*

$$I^h = \langle f^h : f \in I \rangle \subseteq k[x_0, \ldots, x_n]$$

to be the homogenization of I.

Proposition 2.15. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $I^h \subseteq k[x_0, \ldots, x_n]$ is homogeneous.*

Proof. By the Hilbert Basis Theorem, we can fix $g_1, \ldots, g_t \in I^h$ such that $I^h = \langle g_1, \ldots, g_t \rangle$. Let $i \in \{1, \ldots, t\}$ be arbitrary. Fix $f_{(1,i)}, \ldots, f_{(m_i,i)} \in I$ and $q_1, \ldots, q_{m_i} \in k[x_0, \ldots, x_n]$ such that $g_i = q_1 f^h_{(1,i)} + \cdots + q_{m_i} f^h_{(m_i,i)}$. Consider $I^h_* = \langle f^h_{(1,i)}, \ldots, f^h_{(m_i,i)} : 1 \leq i \leq t \rangle$. We have that $I^h_* \subseteq I^h$. Now, let $g \in \langle g_1, \ldots, g_t \rangle$ be arbitrary. Fix $r_1, \ldots, r_t \in k[x_0, \ldots, x_n]$ such that $g = r_1 g_1 + \cdots + r_t g_t$. Let $i \in \{1, \ldots, t\}$ be arbitrary. Since $g_i \in I^h_*$, it follows that $r_i g_i \in I^h_*$. So, $g \in I^h_*$.

We hence can conclude that $I^h_* = I^h$. By Theorem 2.10, I^h is a homogeneous ideal. $\square$

We now take a moment to pray that there's an easily attainable finite basis for I^h. Thankfully, the mathematical deities have answered our prayers.

Theorem 2.16. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, and let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for I with respect to a graded monomial order on $k[x_1, \ldots, x_n]$. Then $I^h = \langle g_1^h, \ldots, g_t^h \rangle \subseteq k[x_0, \ldots, x_n]$.*

Proof. Refer to p. 416-417 of *Ideals, Varieties, and Algorithms* by Cox et al. $\square$

We have thoroughly developed the algebraic side of projective algebraic geometry. It is now time to project ourselves into geometry land.

Section 3 - Projective Space

Projective space rests upon a familiar equivalence relation.

Proposition 3.1. *Let $\sim$ be a relation on $k^{n+1}\backslash\{(0,\dots,0)\}$, where $(a_0,\dots,a_n)\sim(b_0,\dots,b_n)$ if there exists $\lambda\in k\backslash\{0\}$ such that $(b_0,\dots,b_n)=\lambda(a_0,\dots,a_n)$. Then $\sim$ is an equivalence relation on $k^{n+1}\backslash\{(0,\dots,0)\}$.*

Proof. Let $(a_0,\dots,a_n)\in k^{n+1}\backslash\{(0,\dots,0)\}$ be arbitrary. We have that $(a_0,\dots,a_n)=1\cdot(a_0,\dots,a_n)$. Therefore, $(a_0,\dots,a_n)\sim(a_0,\dots,a_n)$.
Now, let $(a_0,\dots,a_n),(b_0,\dots,b_n)\in k^{n+1}\backslash\{(0,\dots,0)\}$ be arbitrary such that $(a_0,\dots,a_n)\sim(b_0,\dots,b_n)$. Fix $\lambda\in k\backslash\{0\}$ such that $(b_0,\dots,b_n)=\lambda(a_0,\dots,a_n)$. So, we have that $(a_0,\dots,a_n)=\frac{1}{\lambda}(b_0,\dots,b_n)$, which means that $(b_0,\dots,b_n)\sim(a_0,\dots,a_n)$.
Let $(a_0,\dots,a_n),(b_0,\dots,b_n),(c_0,\dots,c_n)\in k^{n+1}\backslash\{(0,\dots,0)\}$ be arbitrary such that $(a_0,\dots,a_n)\sim(b_0,\dots,b_n)$ and $(b_0,\dots,b_n)\sim(c_0,\dots,c_n)$. Fix $\lambda_1,\lambda_2\in k\backslash\{0\}$ such that $(b_0,\dots,b_n)=\lambda_1(a_0,\dots,a_n)$ and $(c_0,\dots,c_n)=\lambda_2(b_0,\dots,b_n)$. Therefore, $(c_0,\dots,c_n)=\lambda_2\lambda_1(a_0,\dots,a_n)$, which means that $(a_0,\dots,a_n)\sim(c_0,\dots,c_n)$. $\square$

It is important to recall that when $k=\mathbb{R}$ and $n=1$, this equivalence relation represents lines that go through (but do not intersect) the origin. We will soon see that by partitioning $\mathbb{R}^2\backslash\{(0,0)\}$ into these line-like sets, this equivalence relation collapses $\mathbb{R}^2\backslash\{(0,0)\}$ into a geometric space that mirrors—and includes a little bit more than—$\mathbb{R}$.

Although the case when $k=\mathbb{R}$ and $n=1$ assists in intuitively conceptualizing projective space, this beautiful construction works over any field and any dimension.

Definition 3.2. *We define n-dimensional projective space over a field k to be the set of equivalence classes of $\sim$ on $k^{n+1}\backslash\{(0,\dots,0)\}$. We write*

$$\mathbb{P}^n(k)=(k^{n+1}\backslash\{(0,\dots,0)\})/\sim.$$

Definition 3.3. *Let $(x_0,\dots,x_n)\in k^{n+1}\backslash\{(0,\dots,0)\}$, and let $p\in\mathbb{P}^n(k)$ with $(x_0,\dots,x_n)\in p$. Then we write $p=(x_0:\dots:x_n)$, and we call $(x_0:\dots:x_n)$ the homogeneous coordinates of p.*

Now that we've established a rigorous definition of projective space, it's time to prove its central characteristic.

Proposition 3.4. *For each $i\in\{0,\dots,n\}$, let*

$$U_i=\{(x_0:\dots:x_n)\in\mathbb{P}^n(k):x_i\neq 0\}.$$

(i) *Define $\phi:k^n\to U_i$ by letting $\phi((x_0,\dots,x_{i-1},x_{i+1},\dots,x_n))=(x_0:\dots:x_{i-1}:1:x_{i+1}:\dots:x_n)$. Then ϕ is a bijection.*

(ii) *Define $\varphi:\mathbb{P}^n(k)\backslash U_i\to\mathbb{P}^{n-1}(k)$ by letting $\varphi((x_0:\dots:x_{i-1}:0:x_{i+1}:\dots:x_n))=(x_0:\dots:x_{i-1}:x_{i+1}:\dots:x_n)$. Then φ is a bijection.*

(iii) $\mathbb{P}^n(k)=\bigcup_{i=1}^n U_i$.

Proof.

(i) Let $i\in\{0,\dots,n\}$ be arbitrary. Define $\psi:U_i\to k^n$ by letting $\psi((x_0:\dots:x_n))=\left(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right)$. We first show that ψ is well-defined. Let $(x_0:\dots:x_n),(x_0':\dots:x_n')\in U_i$ be arbitrary such that $(x_0:\dots:x_n)=(x_0':\dots:x_n')$. Then we can fix $\lambda\in k\backslash\{0\}$ with $(x_0',\dots,x_n')=\lambda(x_0,\dots,x_n)$. Notice that

$$\begin{aligned}\psi((x_0 : \cdots : x_n)) &= \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)\\ &= \left(\frac{\lambda x_0}{\lambda x_i}, \dots, \frac{\lambda x_{i-1}}{\lambda x_i}, \frac{\lambda x_{i+1}}{\lambda x_i}, \dots, \frac{\lambda x_n}{\lambda x_i}\right)\\ &= \left(\frac{x'_0}{x'_i}, \dots, \frac{x'_{i-1}}{x'_i}, \frac{x'_{i+1}}{x'_i}, \dots, \frac{x'_n}{x'_i}\right)\\ &= \psi((x'_0 : \cdots : x'_n)).\end{aligned}$$

We now show that ϕ and ψ are inverses. Let $(x_0 : \cdots : x_n) \in U_i$ be arbitrary. Notice that

$$\begin{aligned}(\phi \circ \psi)((x_0 : \cdots : x_n)) &= \phi\left(\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)\right)\\ &= \left(\frac{x_0}{x_i} : \cdots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : \frac{x_n}{x_i}\right)\\ &= (x_0 : \cdots : x_n).\end{aligned}$$

Now, let $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in k^n$ be arbitrary. Notice that

$$\begin{aligned}(\psi \circ \phi)(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) &= \psi((x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n))\\ &= (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n).\end{aligned}$$

We therefore conclude that there is a bijection between U_i and k^n.

(ii) Notice that $\mathbb{P}^n(k) \backslash U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) : x_i = 0\}$.
We first check that φ is well-defined. Let $(x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n), (x'_0 : \cdots : x'_{i-1} : 0 : x'_{i+1} : \cdots : x'_n) \in \mathbb{P}^n(k) \backslash U_i$ be arbitrary such that $(x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n) = (x'_0 : \cdots : x'_{i-1} : 0 : x'_{i+1} : \cdots : x'_n)$. Thus, we can fix $\lambda \in k \backslash \{0\}$ such that $(x'_0, \dots, x'_{i-1}, 0, x'_{i+1}, \dots, x'_n) = \lambda(x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$. We therefore have that

$$\begin{aligned}\varphi((x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n)) &= (x_0 : \cdots : x_{i-1} : x_{i+1} : \cdots : x_n)\\ &= (\lambda x_0 : \cdots : \lambda x_{i-1} : \lambda x_{i+1} : \cdots : \lambda x_n)\\ &= (x'_0 : \cdots : x'_{i-1} : x'_{i+1} : \cdots : x'_n)\\ &= \varphi((x'_0 : \cdots : x'_{i-1} : 0 : x'_{i+1} : \cdots : x'_n)).\end{aligned}$$

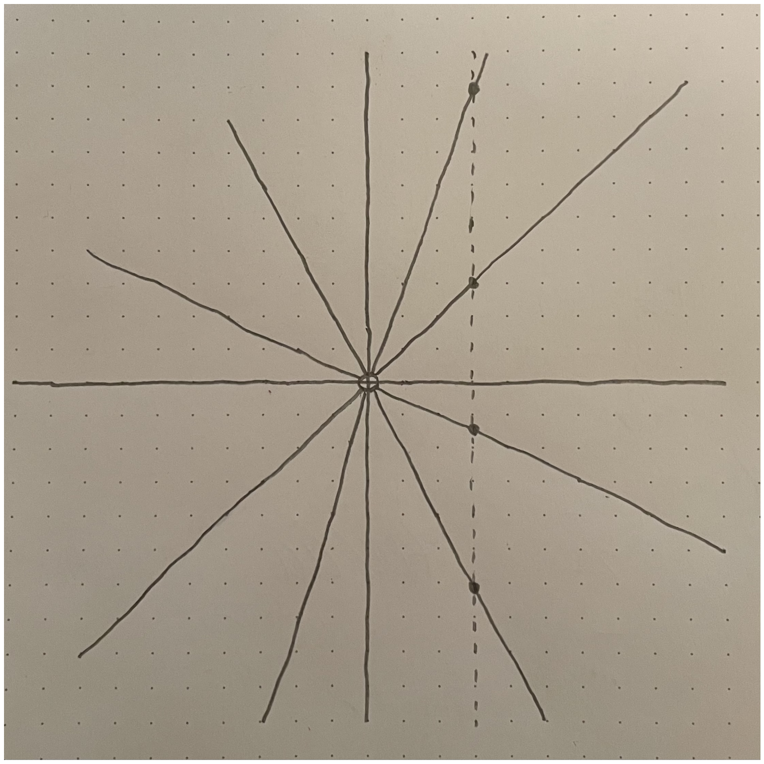
It is straightforward and annoying to prove that φ is injective and surjective, so I will spare the reader the details.

(iii) Let $(x_0 : \cdots : x_n) \in \mathbb{P}^n(k)$ be arbitrary. Since $\mathbb{P}^n(k) = (k^{n+1} \backslash \{(0, \dots, 0)\})/\sim$, it follows that $(x_0 : \cdots : x_n) \neq (0 : \cdots : 0)$. Thus, we can fix $i \in \{0, \dots, n\}$ such that $x_i \neq 0$, which means that $(x_0 : \cdots : x_n) \in \bigcup_{i=1}^n U_i$. Now, let $(y_0 : \cdots : y_n) \in \bigcup_{i=1}^n U_i$ be arbitrary. Trivially, $(y_0 : \cdots : y_n) \in \mathbb{P}^n(k)$. $\square$

As we will later uncover, the majesty of projective space is contained in $\mathbb{P}^n(k) \backslash U_i$—i.e., the little bit more that projective space gifts us. In fact, this little bit more is so special that we will call it LBM_i. Most often, LBM_0 will be of primary interest.

The next two examples showcase the diversity of projective space. Example 3.5 takes readers on a visual tour of the projective real line, and Example 3.6 demonstrates just how abstract projective space can get.

Example 3.5. Consider $\mathbb{P}^1(\mathbb{R})$, which is known as the projective real line. By Proposition 3.4, there's a bijection between $\{(x : y) \in \mathbb{P}^1(\mathbb{R}) : x \neq 0\}$ and $\mathbb{R}$. We give a visual representation of this bijection.



Notice that $(0:1)$, which may be identified with the projective real point, is the only equivalence class that doesn't intersect $x=3$.
For a more thorough discussion of the projective real line, refer to Example 3.2.8 of *Abstract Algebra* by Joe Mileti.

Example 3.6. Notice that

$$\mathbb{P}^2(\mathbb{Z}/2\mathbb{Z}) = \{(\overline{0}:\overline{1}:\overline{1}),(\overline{0}:\overline{0}:\overline{1}),(\overline{0}:\overline{1}:\overline{0}),(\overline{1}:\overline{0}:\overline{0}),(\overline{1}:\overline{0}:\overline{1}),(\overline{1}:\overline{1}:\overline{0}),(\overline{1}:\overline{1}:\overline{1})\}.$$

Let $a,b,c \in \{0,1\}$ be arbitrary with $a+b+c \geq 1$. We have that $(\overline{a}:\overline{b}:\overline{c}) = \{(\overline{a},\overline{b},\overline{c})\}$. Therefore, $\mathbb{P}^2(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3\backslash\{(\overline{0},\overline{0},\overline{0})\}$.
This weird result comports with Proposition 3.4. We have that there's a bijection from $U_0 = \{(\overline{1}:\overline{0}:\overline{0}),(\overline{1}:\overline{0}:\overline{1}),(\overline{1}:\overline{1}:\overline{0}),(\overline{1}:\overline{1}:\overline{1})\}$ to $\mathbb{Z}/2\mathbb{Z}$ defined by $(\overline{a}:\overline{b}:\overline{c}) \mapsto (\overline{b},\overline{c})$. Also, there's a bijection from LBM_0 to $(\mathbb{Z}/2\mathbb{Z})^2\backslash\{(\overline{0},\overline{0})\} = \mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$ similarly defined by $(\overline{a}:\overline{b}:\overline{c}) \mapsto (\overline{b},\overline{c})$.
Although that exploration was fun, we're going to depart from $\mathbb{P}^2(\mathbb{Z}/2\mathbb{Z})$ for the remainder of this exposition. So long, weird pal.

Now that we've surveyed projective space and homogeneous polynomials, we commence with melding geometry and algebra together. The first order of business is defining varieties in projective space.

Proposition 3.7. *Let $f \in k[x_0,\dots,x_n]$ be homogeneous. We have that $\boldsymbol{V}(\{f\}) = \{p \in \mathbb{P}^n(k) : f(p) = 0\}$ is a well-defined subset of $\mathbb{P}^n(k)$.*

Proof. Refer to p. 398 of *Ideals, Varieties, and Algorithms* by Cox et al. ☐

Corollary 3.8. *Let $I \subseteq k[x_0,\dots,x_n]$ be a homogeneous ideal. Suppose $I = \langle f_1,\dots,f_s\rangle$, where $f_1,\dots,f_s \in k[x_0,\dots,x_n]$ are homogeneous. Then*

$$\begin{aligned}\mathbf{V}(I) &= \mathbf{V}(\{f_1,\dots,f_s\})\\ &= \{(a_0:\dots:a_n) \in \mathbb{P}^n(k) : f_i(a_0,\dots,a_n) = 0 \text{ for all } 1 \leq i \leq s\}.\end{aligned}$$

We call $\boldsymbol{V}(I)$ *the projective variety defined by* I.

Proof. Immediate from Proposition 2.5.9 and Proposition 3.7. $\square$

A quick logistical aside: Hereafter, we will use $\mathbf{V}$ to refer to projective varieties and $\mathbf{V}_a$ to refer to affine varieties. Moreover, we will use $\mathbf{I}$ to denote the ideal of a projective variety (which we will define shortly) and $\mathbf{I}_a$ to denote the ideal of an affine variety.

Section 4 - Projective Algebraic Geometry

As we stroll through projective algebraic geometry, our discoveries may be trippy, as the youth say. Many results from affine space carry over completely unencumbered (e.g., Proposition 4.4), but some good friends from affine space have morphed a bit (e.g., The Projective Weak Nullstellensatz). Fundamentally, the abnormalities of this universe stem from the remarkable qualities of homogeneous polynomials combined with the stubbornness of projective space.

Definition 4.1. *Let $V \subseteq \mathbb{P}^n(k)$ be a projective variety. Then we define*

$$\mathbf{I}(V) = \{f \in k[x_0, \dots, x_n] : f(a_0, \dots, a_n) = 0 \text{ for all } (a_0 : \dots : a_n) \in V\}$$

to be the ideal of V.

Proposition 4.2. *Let $V \subseteq \mathbb{P}^n(k)$ be a projective variety. If k is infinite, then $\boldsymbol{I}(V) \subseteq k[x_0, \dots, x_n]$ is a homogeneous ideal.*

Proof. On p. 408 of *Ideals, Varieties, and Algorithms*, Cox et al. prove that $\mathbf{I}(V)$ is an ideal.
We now prove that $\mathbf{I}(V)$ is homogeneous. If $V = \emptyset$, then $\mathbf{I}(V) = k[x_0, \dots, x_n]$, which is a homogeneous ideal. Suppose $V \neq \emptyset$. Let $f \in \mathbf{I}(V) \backslash \{0\}$ be arbitrary, and fix $d \in \mathbb{N}$ such that $\deg(f) = d$. Write

$$f = \textstyle\sum_{i=0}^{d} f_i,$$

where $f_i = 0$ or f_i is i-homogeneous.
Now, let $(a_0 : \dots : a_n) \in V$ be arbitrary. Consider

$$\begin{aligned} g(x) &= f(xa_0, \dots, xa_n) \\ &= \sum_{i=0}^{d} f_i(xa_0, \dots, xa_n) \\ &= \sum_{i=0}^{d} x^i f_i(a_0, \dots, a_n) \in k[x]. \end{aligned}$$

Since $f \in \mathbf{I}(V)$, $\mathbf{V}_a(\{g(x)\}) = k \backslash \{0\}$. Since k is infinite, it follows that $g(x)$ has infinitely many roots. By Proposition 10.3.3 of *Abstract Algebra* by Joe Mileti, $g(x) = 0$. Let $i \in \{0, \dots, d\}$ be arbitrary. Since $g(x) = 0$, we have that $f_i(a_0, \dots, a_n) = 0$. Now, let $\lambda \in k \backslash \{0\}$ be arbitrary. Since $f_i(\lambda a_0, \dots, \lambda a_n) = \lambda^i f_i(a_0, \dots, a_n)$, it follows that $f_i(\lambda a_0, \dots, \lambda a_n) = 0$. Therefore, $f_i \in \mathbf{I}(V)$, and it follows that $\mathbf{I}(V)$ is homogeneous. ☐

For an infinite field k, we get that $\mathbf{I}$ and $\mathbf{V}$ are inclusion-reversing maps between projective varieties and homogeneous ideals. Also, for any projective variety $V \subseteq \mathbb{P}^n(k)$, $\mathbf{V}(\mathbf{I}(V)) = V$.

We continue along to the crux of projective algebraic geometry: a bijection between (a subset of) projective varieties and (a subset of) homogeneous ideals. In the pursuit of a bijection, we draw lessons from affine space but take care not to upset projective space.

Proposition 4.3. *Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. Then $\sqrt{I}$ is a homogeneous ideal.*

Proof. Refer to p. 409-410 of *Ideals, Varieties, and Algorithms* by Cox et al. ☐

Proposition 4.4. *Let $V \subseteq \mathbb{P}^n(k)$ be a projective variety. Then $\boldsymbol{I}(V) \subseteq k[x_0, \dots, x_n]$ is radical.*

Proof. Trivially, $\mathbf{I}(V) \subseteq \sqrt{\mathbf{I}(V)}$. Now, let $f \in \sqrt{\mathbf{I}(V)}$ be arbitrary. Fix $m \in \mathbb{N}^+$ such that $f^m \in \mathbf{I}(V)$. Let $(a_0 : \dots : a_n) \in V$ be arbitrary, and let $\lambda \in k \backslash \{0\}$ be arbitrary. Since $f^m(\lambda a_0, \dots, \lambda a_n) = 0$, it follows that $f(\lambda a_0, \dots, \lambda a_n) = 0$. Therefore, $f \in \mathbf{I}(V)$. Thus, $\mathbf{I}(V)$ is radical. ☐

Proposition 4.5. *$\langle x_0, \dots, x_n \rangle \subseteq k[x_0, \dots, x_n]$ is maximal.*

Proof. Cox et al. prove a more general version of this result using the Division Algorithm (see Proposition 4.5.9). However, I posit another proof. This proof can easily be modified to prove Cox et al.'s more general statement, but the weaker claim is sufficient for our purposes.
Let $I_0 = \langle x_0, \dots, x_n \rangle$. We want to show that $k \cong k[x_0, \dots, x_n]/I_0$.
Notice that $\langle \mathrm{LT}(I_0) \rangle = \langle x_0, \dots, x_n \rangle$. Therefore, $\{x^\alpha : x^\alpha \notin \langle \mathrm{LT}(I_0) \rangle\} = \{1\}$. Consider $\phi : k \to k[x_0, \dots, x_n]/I_0$, where $\phi(a) = a + I_0 = \overline{a}$.
Let $a_1, a_2 \in k$ be arbitrary such that $\phi(a_1) = \phi(a_2)$. Then $\overline{a_1} = \overline{a_2}$. So, $\overline{(a_1 - a_2) \cdot 1} = \overline{0}$. We have by Proposition 5.3.1 that $a_1 - a_2 = 0$, which means that $a_1 = a_2$. It follows that ϕ is injective.
Let $\overline{f} \in k[x_0, \dots, x_n]/I_0$ be arbitrary. Since $\{x^\alpha : x^\alpha \notin \langle \mathrm{LT}(I_0) \rangle\} = \{1\}$, we have by Proposition 5.3.1 that we can fix $a_* \in k$ with $\overline{f} = \overline{a_*}$. Thus, $\phi(a_*) = \overline{f}$. It follows that ϕ is surjective.
Now, let $b_1, b_2 \in k$ be arbitrary. Notice that $\phi(b_1 + b_2) = \overline{b_1 + b_2} = \overline{b_1} + \overline{b_2} = \phi(b_1) + \phi(b_2)$. Similarly, $\phi(b_1 \cdot b_2) = \overline{b_1 \cdot b_2} = \overline{b_1} \cdot \overline{b_2} = \phi(b_1) \cdot \phi(b_2)$.
We have thus shown that $k \cong k[x_0, \dots, x_n]/I_0$. Since k is a field, it follows that $k[x_0, \dots, x_n]/I_0$ is a field. Therefore, I_0 is maximal. $\square$

Any hope that The Weak Nullstellensatz carries over unscathed into projective space is utterly broken after considering $\langle x_0, \dots, x_n \rangle \subseteq k[x_0, \dots, x_n]$, where k is an algebraically closed field. Notice that $\mathbf{V}(\langle x_0, \dots, x_n \rangle) = \emptyset$, which follows directly from $\mathbb{P}^n(k)$ excluding the origin. This exclusion is precisely the obstacle that The Projective Weak Nullstellensatz seeks to overcome.

Theorem 4.6 (The Projective Weak Nullstellensatz). *Let k be an algebraically closed field, and let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. The following are equivalent:*

(i) $\mathbf{V}(I) = \emptyset$.

(ii) $I : \langle x_0, \dots, x_n \rangle^\infty = k[x_0, \dots, x_n]$.

(iii) $\sqrt{I} = \langle x_0, \dots, x_n \rangle$ *or* $\sqrt{I} = k[x_0, \dots, x_n]$.

Proof. For (i) $\Leftrightarrow$ (ii), refer to p. 411 of *Ideals, Varieties, and Algorithms* by Cox et al.
We prove (i) $\Leftrightarrow$ (iii). Suppose $\mathbf{V}(I) = \emptyset$. Thus, $I : \langle x_0, \dots, x_n \rangle^\infty = k[x_0, \dots, x_n]$. By Proposition 4.4.12, $\langle x_0, \dots, x_n \rangle \subseteq \sqrt{I}$. Since $\langle x_0, \dots, x_n \rangle$ is maximal, $\sqrt{I} = \langle x_0, \dots, x_n \rangle$ or $\sqrt{I} = k[x_0, \dots, x_n]$.
Now, suppose $\sqrt{I} = \langle x_0, \dots, x_n \rangle$ or $\sqrt{I} = k[x_0, \dots, x_n]$. Suppose $\sqrt{I} = \langle x_0, \dots, x_n \rangle$. By Theorem 4.2.7, $\mathbf{V}_a(\sqrt{I}) = \mathbf{V}_a(I) = \{(0, \dots, 0)\}$. Since $(0 : \cdots : 0) \notin \mathbb{P}^n(k)$, it follows that $\mathbf{V}(I) = \emptyset$. Suppose $\sqrt{I} = k[x_0, \dots, x_n]$. Again by Theorem 4.2.7, $\mathbf{V}_a(\sqrt{I}) = \mathbf{V}_a(I) = \emptyset$. We therefore have that $\mathbf{V}(I) = \emptyset$. $\square$

The Projective Strong Nullstellensatz contains less hiccups than The Projective Weak Nullstellensatz but nonetheless differs from its affine counterpart in one key way. The Projective Strong Nullstellensatz requires that $\mathbf{V}(I) \neq \emptyset$ (where I is a homogeneous ideal of a multivariate polynomial ring over an algebraically closed field). Let's see what can go wrong when $\mathbf{V}(I) = \emptyset$.

Example 4.7. Once again, our interest and ire zeroes (or, shall I say, nulls) in on $\langle x_0, \dots, x_n \rangle \subseteq k[x_0, \dots, x_n]$, where k is an algebraically closed field. Since $\langle x_0, \dots, x_n \rangle$ is maximal, we have that $\langle x_0, \dots, x_n \rangle$ is prime, which means that $\langle x_0, \dots, x_n \rangle$ is radical. By The Projective Weak Nullstellensatz, $\mathbf{V}(\langle x_0, \dots, x_n \rangle) = \emptyset$. Thus, $\mathbf{I}(\mathbf{V}(\langle x_0, \dots, x_n \rangle)) = \mathbf{I}(\emptyset) = k[x_0, \dots, x_n]$. And therein lies the issue.

Theorem 4.8 (The Projective Strong Nullstellensatz). *Let k be an algebraically closed field, and let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. If $\mathbf{V}(I) \neq \emptyset$, then*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Proof. Refer to p. 412 of *Ideals, Varieties, and Algorithms* by Cox et al. $\square$

Theorem 4.9. *Let k be an algebraically closed field. Now, let*

$$P = \{V \subseteq \mathbb{P}^n(k) : V \text{ is a projective variety and } V \neq \emptyset\}$$

and

$$S = \{I \subseteq k[x_0, \dots, x_n] : I \text{ is a homogeneous radical ideal and } I \subsetneq \langle x_0, \dots, x_n \rangle\}.$$

Then $\boldsymbol{I}\colon P \to S$ *and* $\boldsymbol{V}\colon S \to P$ *are inverses of each other.*

Proof. We first show that $\mathbf{I} : P \to S$ and $\mathbf{V} : S \to P$ are well-defined.
Let $V \in P$ be arbitrary. Notice that $\mathbf{V}(\mathbf{I}(V)) = V \neq \emptyset$. By The Projective Weak Nullstellensatz $\sqrt{\mathbf{I}(V)} \neq \langle x_0, \dots, x_n \rangle$ and $\sqrt{\mathbf{I}(V)} \neq k[x_0, \dots, x_n]$. Since $\sqrt{\mathbf{I}(V)} = \mathbf{I}(V)$ and $\langle x_0, \dots, x_n \rangle$ is maximal, it follows that $\mathbf{I}(V) \subsetneq \langle x_0, \dots, x_n \rangle$. Thus, $\mathbf{I}(V) \in S$.
Let $I \in S$ be arbitrary. Since $I = \sqrt{I} \subsetneq \langle x_0, \dots, x_n \rangle$, we have by The Projective Weak Nullstellensatz that $\mathbf{V}(I) \neq \emptyset$. So, $\mathbf{V}(I) \in P$.
Now, notice that $\mathbf{V}(\mathbf{I}(V)) = V$. Also, by The Projective Strong Nullstellensatz, $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = I$. $\square$

We've created a projective version of many of the most salient facts about affine space. It's natural to wonder whether prime ideals and irreducible varieties remain tight-knit in projective space. Thankfully (and, in particular, thanks to Proposition 2.13), the answer is a resounding affirmative.

A quick side note: Irreducible varieties in projective space are defined completely analogously to irreducible varieties in affine space (see Definition 4.5.1).

Proposition 4.10. *Let k be an algebraically closed field, and let $I \subseteq k[x_0, \dots, x_n]$ be a radical homogeneous ideal. Then $\boldsymbol{V}(I)$ is irreducible if and only if I is prime.*

Proof. Suppose I is prime. If $I = \langle x_0, \dots, x_n \rangle$ or $I = k[x_0, \dots, x_n]$, then $\mathbf{V}(I) = \emptyset$ by The Projective Weak Nullstellensatz. Suppose $I \subsetneq \langle x_0, \dots, x_n \rangle$. Let $V_1, V_2 \subseteq \mathbb{P}^n(k)$ be arbitrary projective varieties such that $\mathbf{V}(I) = V_1 \cup V_2$. Now, suppose $\mathbf{V}(I) \neq V_1$. Notice that $V_2 \subseteq \mathbf{V}(I)$. So, $\mathbf{I}(\mathbf{V}(I)) \subseteq \mathbf{I}(V_2)$. Since $V_1 \neq \mathbf{V}(I)$, it follows that $\mathbf{I}(\mathbf{V}(I)) \subsetneq \mathbf{I}(V_1)$. We can thus fix $f \in \mathbf{I}(V_1) \backslash \mathbf{I}(\mathbf{V}(I))$. Now, let $g \in \mathbf{I}(V_2)$ be arbitrary. Notice that $fg \in \mathbf{I}(\mathbf{V}(I))$. By The Projective Strong Nullstellensatz, $\mathbf{I}(\mathbf{V}(I)) = I$. So, either $f \in I$ or $g \in I$. Since $f \notin I$, it follows that $g \in I$. Thus, $\mathbf{I}(V_2) \subseteq \mathbf{I}(\mathbf{V}(I))$. So, $\mathbf{I}(V_2) = I$, which means that $V_2 = \mathbf{V}(I)$. We therefore conclude that $\mathbf{V}(I)$ is irreducible.
Now, suppose $\mathbf{V}(I)$ is irreducible. If $\mathbf{V}(I) = \emptyset$, then $I = \langle x_0, \dots, x_n \rangle$ or $I = k[x_0, \dots, x_n]$ by The Projective Weak Nullstellensatz. Suppose $\mathbf{V}(I) \neq \emptyset$. Let $f, g \in k[x_0, \dots, x_n]$ be arbitrary homogeneous polynomials such that $fg \in I$. Since f and g are homogeneous, $\mathbf{V}(\langle f \rangle), \mathbf{V}(\langle g \rangle) \subseteq \mathbb{P}^n(k)$ are projective varieties. By Proposition 2.12, $I + \langle f \rangle$, $I + \langle g \rangle \subseteq k[x_0, \dots, x_n]$ are homogeneous ideals. So, $\mathbf{V}(I + \langle f \rangle), \mathbf{V}(I + \langle g \rangle) \subseteq \mathbb{P}^n(k)$ are projective varieties. By an argument completely analogous to the affine case (Theorem 4.3.4), $\mathbf{V}(I + \langle f \rangle) = \mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle)$ and $\mathbf{V}(I + \langle g \rangle) = \mathbf{V}(I) \cap \mathbf{V}(\langle g \rangle)$.
We now show by double containment that $\mathbf{V}(I) = \mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle) \cup \mathbf{V}(I) \cap \mathbf{V}(\langle g \rangle)$. Let $(a_0 : \cdots : a_n) \in \mathbf{V}(I)$ be arbitrary. Since $fg \in I$, we have that $(fg)(a_0, \dots, a_n) = f(a_0, \dots, a_n) \cdot g(a_0, \dots, a_n) = 0$. So, either $f(a_0, \dots, a_n) = 0$ or $g(a_0, \dots, a_n) = 0$. Assume without loss of generality that $f(a_0, \dots, a_n) = 0$. Since f is homogeneous, $f(\lambda a_0, \dots, \lambda a_n) = 0$ for all $\lambda \in k \backslash \{0\}$. Thus, $(a_0 : \cdots : a_n) \in \mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle)$. It follows that $\mathbf{V}(I) \subseteq \mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle) \cup \mathbf{V}(I) \cap \mathbf{V}(\langle g \rangle)$. Trivially, $\mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle) \cup \mathbf{V}(I) \cap \mathbf{V}(\langle g \rangle) \subseteq \mathbf{V}(I)$.
Since $\mathbf{V}(I)$ is irreducible, we can assume without loss of generality that $\mathbf{V}(I) = \mathbf{V}(I) \cap \mathbf{V}(\langle f \rangle)$. Thus, $\mathbf{V}(I) \subseteq \mathbf{V}(\langle f \rangle)$, which means that $\mathbf{I}(\mathbf{V}(\langle f \rangle)) \subseteq \mathbf{I}(\mathbf{V}(I))$. By The Projective Strong Nullstellensatz, $\mathbf{I}(\mathbf{V}(\langle f \rangle)) \subseteq I$. Since $f \in \mathbf{I}(\mathbf{V}(\langle f \rangle))$, it follows that $f \in I$. By Proposition 2.13, I is prime. $\square$

Theorem 4.11. *Let k be an algebraically closed field. Now, let*

$$P' = \{V \subseteq \mathbb{P}^n(k) : V \text{ is a nonempty irreducible projective variety}\}$$

and

$$S' = \{I \subseteq k[x_0, \dots, x_n] : I \text{ is a homogeneous prime ideal and } I \subsetneq \langle x_0, \dots, x_n \rangle\}.$$

Then $\boldsymbol{I}\colon P' \to S'$ *and* $\boldsymbol{V}\colon S' \to P'$ *are inverses of each other.*

Proof. We first show that $\mathbf{I} : P' \to S'$ and $\mathbf{V} : S' \to P'$ are well-defined. Let $V \in P'$ be arbitrary. By Theorem 4.9 and Proposition 4.10, $\mathbf{I}(V) \in S'$. Let $I \in S'$ be arbitrary. By Theorem 4.9 and Proposition 4.10, $\mathbf{V}(I) \in P'$.
We have that $\mathbf{V}(\mathbf{I}(V)) = V$. By The Projective Strong Nullstellensatz, $\mathbf{I}(\mathbf{V}(I)) = I$. $\square$

Section 5 - From Affine to Projective

A theme of this exposition—and perhaps of mathematics—is crafting elegant worlds and efficient transportation to those worlds. For example, in real analysis, power series comprise the elegant world and Taylor series is the way for an ordinary function to get there.

Another example is the Zariski closure of a subset S of affine space, which is the smallest affine variety containing S. We noted that the Zariski closure of S is equal to $\mathbf{V}_a(\mathbf{I}_a(S))$. Using the Zariski closure as our guiding light, we want to construct efficient tracks from affine space to projective space.

Definition 5.1. *Let $W \subseteq k^n$ be an affine variety. We define the projective closure of W to be $\overline{W}_p = \boldsymbol{V}(\boldsymbol{I}_a(W)^h) \subseteq \mathbb{P}^n(k)$.*

Example 5.2. Let $\langle y \rangle \subseteq \mathbb{C}[y,z]$. Consider $W = \mathbf{V}_a(\langle y \rangle) \subseteq \mathbb{C}^2$. Notice that $W = \{(0,a) \in \mathbb{C}^2\}$. By The Strong Nullstellensatz, $\mathbf{I}_a(W) = \sqrt{\langle y \rangle} = \langle y \rangle$. By Theorem 2.16, $\langle y \rangle^h = \langle y \rangle \subseteq \mathbb{C}[x,y,z]$. Also, $\mathbf{V}(\langle y \rangle) = \{(a:0:c) \in \mathbb{P}^2(\mathbb{C}) : a \neq 0 \text{ or } c \neq 0\}$. Therefore, the projective closure of $\{(0,a) \in \mathbb{C}^2\}$ is $\{(a:0:c) \in \mathbb{P}^2(\mathbb{C}) : a \neq 0 \text{ or } c \neq 0\}$.

If you look closely at Example 5.2, a bijection between the affine variety and the projective closure becomes clear. This bijection is completely generalizable.

Proposition 5.3. *Let $W \subseteq k^n$ be an affine variety. Then:*

(i) *Define $\psi : \overline{W}_p \cap U_0 \to W$ by letting $\psi((a_0 : \cdots : a_n)) = \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$. It follows that ψ is a bijection.*

(ii) *If $V \subseteq \mathbb{P}^n(k)$ is a projective variety such that there's a bijection between $V \cap U_0$ and W, then $\overline{W}_p \subseteq V$.*

Proof.

(i) We first show that ψ is well-defined. Let $(b_0 : \cdots : b_n) \in \overline{W}_p \cap U_0$ be arbitrary, and let $(b_0' : \cdots : b_n') \in \mathbb{P}^n(k)$ be arbitrary such that $(b_0 : \cdots : b_n) = (b_0' : \cdots : b_n')$. By an argument completely analogous to Proposition 3.4 Part (i), $\psi((b_0 : \cdots b_n)) = \psi((b_0' : \cdots : b_n'))$.
Let $>$ be a graded monomial order on $k[x_1, \ldots, x_n]$. Fix $g_1, \ldots, g_t \in k[x_1, \ldots, x_n]$ such that $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for $\mathbf{I}_a(W)$ with respect to $>$. By Theorem 2.16, $\mathbf{I}_a(W)^h = \langle g_1^h, \ldots, g_t^h \rangle$.
Let $i \in \{1, \ldots, t\}$ be arbitrary. Since $(b_0 : \cdots : b_n) \in \mathbf{V}(\mathbf{I}_a(W)^h) \cap U_0$, we have that

$$\begin{aligned} g_i^h\left(\frac{b_0}{b_0}, \ldots, \frac{b_n}{b_0}\right) &= g_i^h\left(1, \frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right) \\ &= g_i\left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right) \\ &= 0. \end{aligned}$$

Thus, $\psi((b_0 : \cdots : b_n)) \in \mathbf{V}_a(\mathbf{I}_a(W)) = W$. It follows that ψ is well-defined.
We now show that ψ is injective. Let $(b_0 : \cdots : b_n), (c_0 : \cdots : c_n) \in \overline{W}_p \cap U_0$ be arbitrary such that $\psi((b_0 : \cdots : b_n)) = \psi((c_0 : \cdots : c_n))$. Thus, $\left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right) = \left(\frac{c_1}{c_0}, \ldots, \frac{c_n}{c_0}\right)$. So, $(b_1, \ldots, b_n) = \frac{b_0}{c_0}(c_1, \ldots, c_n)$. It follows that $(b_0, b_1, \ldots, b_n) = \frac{b_0}{c_0}(c_0, c_1, \ldots, c_n)$. Thus, $(b_0 : \cdots : b_n) = (c_0 : \cdots : c_n)$.
Finally, we show that ψ is surjective. Let $(b_1, \ldots, b_n) \in W$ be arbitrary. Since $g_i(b_1, \ldots, b_n) = g_i^h(1, b_1, \ldots, b_n) = 0$, we have that $(1 : b_1 : \cdots : b_n) \in \overline{W}_p \cap U_0$. Also, $\phi(((1 : b_1 : \cdots : b_n)) = (b_1, \ldots, b_n)$. Therefore, ψ is a bijection.

(ii) Refer to p. 418 of *Ideals, Varieties, and Algorithms* by Cox et al. □

Now that we've laid efficient tracks from affine to projective, we ought to check whether these tracks are sufficiently smooth. That is, we want to determine whether properties of an affine variety like, say, irreducibility carry over to the projective closure of an affine variety. Indeed, our tracks are smooth enough to retain irreducibility, and the next three results prove this fact.

By the way, it is important to note that the following proofs make significant usage of the fact that polynomial evaluation is a ring homomorphism.

Lemma 5.4. *Let $f, g \in k[x_1, \ldots, x_n]$. We have that*

$$(fg)^h = f^h g^h.$$

Also, for all $m \in \mathbb{N}$,

$$(f^m)^h = (f^h)^m.$$

Proof. If $f = 0$ or $g = 0$, it is trivial that $(fg)^h = f^h g^h$. Also, if $f = 0$, it is trivial that $(f^m)^h = (f^h)^m$ for all $m \in \mathbb{N}$. Suppose $f \neq 0$ and $g \neq 0$. Fix $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\deg(f) = \alpha_1$ and $\deg(g) = \alpha_2$. By Proposition 2.4, $f^h = x_0^{\alpha_1} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ and $g^h = x_0^{\alpha_2} g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$. Notice that $\deg(fg) = \deg(f) + \deg(g) = \alpha_1 + \alpha_2$. So,

$$\begin{aligned}
(fg)^h &= x_0^{\alpha_1+\alpha_2}(fg)\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\
&= x_0^{\alpha_1+\alpha_2} \cdot f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \cdot g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\
&= x_0^{\alpha_1} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \cdot x_0^{\alpha_2} g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \\
&= f^h g^h.
\end{aligned}$$

Now, let $m \in \mathbb{N}$ be arbitrary. By Proposition 2.4,

$$\begin{aligned}
(f^h)^m &= \left(x_0^{\alpha_1} f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)\right)^m \\
&= x_0^{m\alpha_1} f^m\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)
\end{aligned}$$

Since $\deg(f^m) = m\alpha_1$, we have by Proposition 2.4 that $(f^m)^h = x_0^{m\alpha_1} f^m\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = (f^h)^m$. $\square$

Proposition 5.5. *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then I is a prime ideal if and only if $I^h \subseteq k[x_0, \ldots, x_n]$ is a prime ideal.*

Proof. Suppose I is a prime ideal. Now, let $f, g \in k[x_0, \ldots, x_n]$ be arbitrary homogeneous polynomials with $fg \in I^h$. If $f = 0$ or $g = 0$, then $f \in I^h$ or $g \in I^h$. Suppose $f \neq 0$ and $g \neq 0$. Fix $m_1, m_2 \in \mathbb{N}$ such that f is m_1-homogeneous and g is m_2-homogeneous. Fix $e_1, e_2 \in \mathbb{N}$ such that $x_0^{e_1}$ is the highest power of x_0 dividing f and $x_0^{e_2}$ is the highest power of x_0 dividing g. Fix $g_1, \ldots, g_t \in k[x_1, \ldots, x_n]$ such that $G = \{g_1, \ldots, g_t\}$ is a Gröbner basis for I with respect to a graded monomial order. By Theorem 2.16, we can fix $A_1, \ldots, A_t \in k[x_0, \ldots, x_n]$ such that $fg = A_1 g_1^h + \cdots + A_t g_t^h$. Now, notice that

$$\begin{aligned}
(fg)(1, x_1, \ldots, x_n) &= (A_1 g_1^h + \cdots + A_t g_t^h)(1, x_1, \ldots, x_n) \\
&= A_1(1, x_1, \ldots, x_n) g_1^h(1, x_1, \ldots, x_n) + \cdots + A_t(1, x_1, \ldots, x_n) g_t^h(1, x_1, \ldots, x_n) \\
&= A_1(1, x_1, \ldots, x_n) g_1 + \cdots + A_t(1, x_1, \ldots, x_n) g_t \in I.
\end{aligned}$$

Therefore, $f(1, x_1, \ldots, x_n) g(1, x_1, \ldots, x_n) \in I$. Since I is prime, either $f(1, x_1, \ldots, x_n) \in I$ or $g(1, x_1, \ldots, x_n) \in I$. Assume without loss of generality that $f(1, x_1, \ldots, x_n) \in I$. Thus, $f(1, x_1, \ldots, x_n)^h \in I^h$. By Proposition 2.4, $f = x_0^{e_1} f(1, x_1, \ldots, x_n)^h$. So, $f \in I^h$. Since I^h is homogeneous by Proposition 2.15, we have by Proposition 2.13 that I^h is prime.

Now, suppose I^h is a prime ideal. Let $f, g \in k[x_1, \ldots, x_n]$ be arbitrary such that $fg \in I$. Notice that

$(fg)^h \in I^h$. By Proposition 5.4, $f^h g^h \in I^h$. Since I^h is prime, either $f^h \in I^h$ or $g^h \in I^h$. Assume without loss of generality that $f^h \in I^h$. By an argument completely analogous to the argument above that $(fg)(1, x_1, \ldots, x_n) \in I$, we have that $f = f^h(1, x_1, \ldots, x_n) \in I$. $\square$

Theorem 5.6. *Let $W \subseteq k^n$ be an affine variety. Then W is irreducible if and only if $\overline{W}_p$ is irreducible.*

Proof. Suppose W is irreducible. Consider $\overline{W}_p = \mathbf{V}(\mathbf{I}_a(W)^h)$. By Proposition 4.5.3, $\mathbf{I}_a(W) \subseteq k[x_1, \ldots, x_n]$ is a prime ideal. By Proposition 5.5, $\mathbf{I}_a(W)^h \subseteq k[x_0, \ldots, x_n]$ is a prime ideal. So, by Proposition 4.10, $\overline{W}_p$ is irreducible.
Suppose $\overline{W}_p$ is irreducible. By Proposition 4.10, $\mathbf{I}_a(W)^h$ is prime. So, by Proposition 5.5, $\mathbf{I}_a(W)$ is prime. It follows by Proposition 4.5.3 that W is irreducible. $\square$

We end this section with a result that renders life significantly easier in algebraically closed fields. Fundamentally, this result stems from The Strong Nullstellensatz.

Theorem 5.7. *Let k be an algebraically closed field, and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $\boldsymbol{V}(I^h) \subseteq \mathbb{P}^n(k)$ is the projective closure of $\boldsymbol{V}_a(I) \subseteq k^n$.*

Proof. Refer to p. 419 of *Ideals, Varieties, and Algorithms* by Cox et al. $\square$

Section 6 - Melding Projective and Affine

Before capturing the wild solutions, we must synthesize projective algebraic geometry and affine algebraic geometry. We start by defining (n, m)-dimensional projective-affine space. As Proposition 6.2 demonstrates, this geometric construction behaves exactly as we'd expect.

Definition 6.1. *We define (n, m)-dimensional projective-affine space over a field k to be $\mathbb{P}^n(k) \times k^m$.*

Proposition 6.2. *For each $i \in \{0, \ldots, n\}$, let*

$$U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) : x_i \neq 0\}.$$

(i) *Define $\phi : k^n \times k^m \to U_i \times k^m$ by letting $\phi((x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, \ldots, y_m)) = (x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n; y_1, \ldots, y_m)$. Then ϕ is a bijection.*

(ii) *Define $\varphi : \mathbb{P}^n(k) \backslash U_i \times k^m \to \mathbb{P}^{n-1}(k) \times k^m$ by letting $\varphi((x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n; y_1, \ldots, y_m)) = (x_0 : \cdots : x_{i-1} : x_{i+1} : \cdots : x_n; y_1, \ldots, y_m)$. Then φ is a bijection.*

(iii) $\mathbb{P}^n(k) \times k^m = \bigcup_{i=1}^n (U_i \times k^m)$.

Proof. Immediate from Proposition 3.4. $\square$

It'd be wonderful if varieties of homogeneous polynomials were well-defined in projective-affine space. However, that fantasy gets shattered when we consider $\langle xz^2 + y^2 z\rangle \subseteq k[x, y, z]$. Suppose we wanted to define $\mathbf{V}(\langle xz^2 + y^2 z\rangle) \subseteq \mathbb{P}^1(k) \times k$. Notice that $(1)(-1)^2 + (1)^2(-1) = 0$, so $(1, 1, -1) \in \mathbf{V}(\langle xz^2 + y^2 z\rangle)$, but $(2)(-1)^2 + (2)^2(-1) = -2$, so $(2, 2, -1) \notin \mathbf{V}(\langle xz^2 + y^2 z\rangle)$. Since $(1 : 1; -1) = (2 : 2; -1)$, it follows that $\mathbf{V}(\langle xz^2 + y^2 z\rangle)$ is not well-defined in $\mathbb{P}^1(k) \times k$.

We thus need to delimit a novel subset of the multivariate polynomial ring over a field that is particularly well-suited for projective-affine space. I forewarn the reader that the remainder of this section contains an avalanche of definitions and results. This avalanche, however, has a very familiar quality to it.

Definition 6.3. *Let $f \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be nonzero, and write*

$$f = \textstyle\sum_{i=0}^s g_{\beta_i}(y_1, \ldots, y_m) x^{\beta_i},$$

where $s \in \mathbb{N}$ and $x^{\beta_i} \in \{x_0^{b_0} \cdots x_n^{b_n} : (b_0, \ldots, b_n) \in \mathbb{N}^{n+1}\}$. We let $(x_0, \ldots, x_n)$-$\deg(f) = \max\{|\beta_i| : i \in \{0, \ldots, s\}\}$.

Definition 6.4. *Let $f \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be nonzero, and suppose $(x_0, \ldots, x_n)$-$\deg(f) = d$. Write*

$$f = \textstyle\sum_{i=0}^s g_{\beta_i}(y_1, \ldots, y_m) x^{\beta_i},$$

where $s \in \mathbb{N}$ and $x^{\beta_i} \in \{x_0^{b_0} \cdots x_n^{b_n} : (b_0, \ldots, b_n) \in \mathbb{N}^{n+1}\}$. If $|\beta_i| = d$ for all $i \in \{0, \ldots, s\}$, then f is $(x_0, \ldots, x_n)$-homogeneous (and, in particular, d-$(x_0, \ldots, x_n)$-homogeneous). Also, we define 0 to be $(x_0, \ldots, x_n)$-homogeneous.

Proposition 6.5. *Let $f \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be $(x_0, \ldots, x_n)$-homogeneous. Then $\boldsymbol{V}(\{f\}) = \{p \in \mathbb{P}^n(k) \times k^m : f(p) = 0\}$ is a well-defined subset of $\mathbb{P}^n(k) \times k^m$.*

Proof. If $f = 0$, then $\mathbf{V}(\{f\}) = \mathbb{P}^n(k) \times k^m$. Now, suppose $f \neq 0$, and fix $d \in \mathbb{N}$ such that f is d-$(x_0, \ldots, x_n)$-homogeneous. Let $(a_0 : \cdots : a_n; b_1, \ldots, b_m), (a_0^* : \cdots : a_n^*; b_1, \ldots, b_m) \in \mathbb{P}^n(k) \times k^m$ be arbitrary such that $(a_0 : \cdots : a_n; b_1, \ldots, b_m) = (a_0^* : \cdots : a_n^*; b_1, \ldots, b_m)$. Suppose $f(a_0, \ldots, a_n, b_1, \ldots, b_m) = 0$. We can fix

$\lambda \in k\backslash\{0\}$ such that $(a_0^*, \ldots, a_n^*, b_1, \ldots, b_m) = (\lambda a_0, \ldots, \lambda a_n, b_1, \ldots, b_m)$. Now, fix $s \in \mathbb{N}$ such that

$$\begin{aligned} f(\lambda a_0, \ldots, \lambda a_n, b_1, \ldots, b_m) &= \sum_{i=0}^{s} g_{\beta_i}(b_1, \ldots, b_m)(\lambda a)^{\beta_i} \\ &= \lambda^d \sum_{i=0}^{s} g_{\beta_i}(b_1, \ldots, b_m)(a)^{\beta_i} \\ &= \lambda^d \cdot f(a_0, \ldots, a_n, b_1, \ldots, b_m) \\ &= 0. \end{aligned}$$

It follows that $\mathbf{V}(\{f\})$ is a well-defined subset of $\mathbb{P}^n(k) \times k^m$. $\square$

Definition 6.6. *Let $I \subseteq k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal. We say that I is $(x_0, \ldots, x_n)$-homogeneous if I is generated by $(x_0, \ldots, x_n)$-homogeneous polynomials.*

Corollary 6.7. *Let $I \subseteq k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ be an $(x_0, \ldots, x_n)$-homogeneous ideal. Suppose $I = \langle f_1, \ldots, f_t \rangle$, where $f_1, \ldots, f_t \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ are $(x_0, \ldots, x_n)$-homogeneous. Then*

$$\begin{aligned} \mathbf{V}(I) &= \mathbf{V}(\{f_1, \ldots, f_t\}) \\ &= \{(a_0 : \cdots : a_n; b_1, \ldots, b_m) \in \mathbb{P}^n(k) \times k^m : f_i(a_0, \ldots, a_n, b_1, \ldots, b_m) = 0 \text{ for all } 1 \leq i \leq t\}. \end{aligned}$$

We call $\boldsymbol{V}(I)$ the $(x_0, \ldots, x_n)$-projective variety defined by I.

Proof. Immediate from Proposition 2.5.9 and Proposition 6.5. $\square$

Definition 6.8. *Let $f \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be nonzero. Suppose $(x_1, \ldots, x_n)$-$\deg(f) = d$, and write*

$$f = \textstyle\sum_{i=0}^{s} g_{\beta_i}(y_1, \ldots, y_m) x^{\beta_i},$$

where $s \in \mathbb{N}$ and $x^{\beta_i} \in \{x_1^{b_1} \cdots x_n^{b_n} : (b_1, \ldots, b_n) \in \mathbb{N}^n\}$. Then

$$f^h_{(x_0,\ldots,x_n)} = \textstyle\sum_{i=0}^{s} g_{\beta_i}(y_1, \ldots, y_m) x^{\beta_i} x_0^{d-|\beta_i|} \in k[x_0, \ldots, x_n, y_1, \ldots, y_m]$$

is the $(x_0, \ldots, x_n)$-homogenization of f. Also, we define $0^h_{(x_0,\ldots,x_n)} = 0$ to be the $(x_0, \ldots, x_n)$-homogenization of 0.

Definition 6.8 has similar benefits as Definition 2.3. For any $f \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, we have that $f^h_{(x_0,\ldots,x_n)}$ is d-$(x_0, \ldots, x_n)$-homogeneous and $f^h_{(x_0,\ldots,x_n)}(1, x_1, \ldots, x_n, y_1, \ldots, y_m) = f$.

Definition 6.9. *Let $I \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal. We define*

$$I^h_{(x_0,\ldots,x_n)} = \langle f^h_{(x_0,\ldots,x_n)} : f \in I \rangle \subseteq k[x_0, \ldots, x_n, y_1, \ldots, y_m]$$

to be the $(x_0, \ldots, x_n)$-homogenization of I.

Proposition 6.10. *Let $>$ be a monomial order on $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ such that for all monomials $x^\alpha y^\gamma, x^\beta y^\delta \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, we have*

$$|\alpha| > |\beta| \Rightarrow x^\alpha y^\gamma > x^\beta y^\delta.$$

Let $I \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal, and let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for I with respect to $>$. Then $I^h_{(x_0,\ldots,x_n)} = \langle g^h_{1(x_0,\ldots,x_n)}, \ldots, g^h_{t(x_0,\ldots,x_n)} \rangle \subseteq k[x_0, \ldots, x_n, y_1, \ldots, y_m]$.

Proof. Refer to p. 432 of *Ideals, Varieties, and Algorithms* by Cox et al. $\square$

Section 7 - Wild Solutions Found

Let's craft a monomial order on $\mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$.

Example 7.1. Let $\alpha, \beta \in \mathbb{N}^7$. We say that $\alpha > \beta$ if

$$(\alpha_1, \alpha_2, \alpha_3) >_{grlex} (\beta_1, \beta_2, \beta_3) \text{ or } (\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3) \text{ and } (\alpha_4, \dots, \alpha_7) >_{lex} (\beta_4, \dots, \beta_7).$$

We want to prove that $>$ is a monomial order on $\mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$. Also, we want to show that for all $t^\alpha x^\gamma, t^\beta x^\delta \in \mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$, $|\alpha| > |\beta|$ implies $t^\alpha x^\gamma > t^\beta x^\delta$.
Let $\alpha, \beta \in \mathbb{N}^7$ be arbitrary. If $(\alpha_1, \alpha_2, \alpha_3) \neq (\beta_1, \beta_2, \beta_3)$, then either $\alpha > \beta$ or $\beta > \alpha$, since $>_{grlex}$ is a monomial order. Now, suppose $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$. If $(\alpha_4, \dots, \alpha_7) \neq (\beta_4, \dots, \beta_7)$, then either $\alpha > \beta$ or $\beta > \alpha$, since $>_{lex}$ is a monomial order. Suppose $(\alpha_4, \dots, \alpha_7) = (\beta_4, \dots, \beta_7)$. Then $\alpha = \beta$.
Now, let $\alpha, \beta, \gamma \in \mathbb{N}^7$ be arbitrary such that $\alpha > \beta$ and $\beta > \gamma$. We three cases:

- Suppose $(\alpha_1, \alpha_2, \alpha_3) >_{grlex} (\beta_1, \beta_2, \beta_3)$. Since $(\beta_1, \beta_2, \beta_3) \geq_{grlex} (\gamma_1, \gamma_2, \gamma_3)$ and $>_{grlex}$ is a monomial order, $(\alpha_1, \alpha_2, \alpha_3) >_{grlex} (\gamma_1, \gamma_2, \gamma_3)$. So, $\alpha > \gamma$.
- Suppose $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$ and $(\beta_1, \beta_2, \beta_3) >_{grlex} (\gamma_1, \gamma_2, \gamma_3)$. Trivially, $\alpha > \gamma$.
- Suppose $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3) = (\gamma_1, \gamma_2, \gamma_3)$. Notice that $(\alpha_4, \dots, \alpha_7) >_{lex} (\beta_4, \dots, \beta_7)$ and $(\beta_4, \dots, \beta_7) >_{lex} (\gamma_4, \dots, \gamma_7)$. Since $>_{lex}$ is a monomial order, we have that $(\alpha_4, \dots, \alpha_7) >_{lex} (\gamma_4, \dots, \gamma_7)$. So, $\alpha > \gamma$.

Since these cases exhaust all possibilities, the result follows that $>$ is transitive.
Let $\alpha, \beta, \gamma \in \mathbb{N}^7$ be arbitrary such that $\alpha > \beta$. We have two cases:

- Suppose $(\alpha_1, \alpha_2, \alpha_3) >_{grlex} (\beta_1, \beta_2, \beta_3)$. Since $>_{grlex}$ is a monomial order, $(\alpha_1, \alpha_2, \alpha_3) + (\gamma_1, \gamma_2, \gamma_3) >_{grlex} (\beta_1, \beta_2, \beta_3) + (\gamma_1, \gamma_2, \gamma_3)$. Therefore, $\alpha + \gamma > \beta + \gamma$.
- Suppose $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3)$. Then $(\alpha_4, \dots, \alpha_7) >_{lex} (\beta_4, \dots, \beta_7)$. Since $>_{lex}$ is a monomial order, $(\alpha_4, \dots, \alpha_7) + (\gamma_4, \dots, \gamma_7) >_{lex} (\beta_4, \dots, \beta_7) + (\gamma_4, \dots, \gamma_7)$. Therefore, $\alpha + \gamma > \beta + \gamma$.

Now, let $\alpha \in \mathbb{N}^7$ be arbitrary. Since $>_{grlex}$ is a monomial order, we have by Corollary 2.4.6 that $(\alpha_1, \alpha_2, \alpha_3) \geq_{grlex} (0, 0, 0)$. Similarly, since $>_{lex}$ is a monomial order, it follows that $(\alpha_4, \dots, \alpha_7) \geq_{lex} (0, \dots, 0)$. Therefore, $\alpha \geq (0, \dots, 0)$, and we have by Corollary 2.4.6 that $>$ is a well-ordering.
Therefore, $>$ is a monomial order on $\mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$.
Let $t^\alpha x^\gamma, t^\beta x^\delta \in \mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$ be arbitrary such that $|\alpha| > |\beta|$. Thus, $\alpha >_{grlex} \beta$, which means that $t^\alpha x^\gamma > t^\beta x^\delta$.

We use this monomial order, along with Proposition 6.10, to make serious headway towards our mission.

Example 7.2. Recall our system of equations in $\mathbb{C}[t_1, t_2, t_3, x_1, \dots, x_4]$:

$$\begin{aligned} f_1 &= x_1 - t_1^2 \\ f_2 &= x_2 - t_1 t_2^2 \\ f_3 &= x_3 - t_1^3 t_3 \\ f_4 &= x_4 + t_1. \end{aligned}$$

We use Sage to calculate a Gröbner basis for $I = \langle f_1, f_2, f_3, f_4 \rangle$ with respect to $>$:

$$\begin{aligned} g_1 &= t_2^2 x_3 - t_3 x_2 x_4^2 \\ g_2 &= t_2^2 x_4 + x_2 \\ g_3 &= t_1 + x_4 \\ g_4 &= t_3 x_4^3 + x_3 \\ g_5 &= x_1 - x_4^2. \end{aligned}$$

We now calculate the (t_0, t_1, t_2, t_3)-homogenization of each g_i:

$$\begin{aligned}
g_1^* &= g^h_{1(t_0,\ldots,t_3)} = t_2^2 x_3 - t_0 t_3 x_2 x_4^2 \\
g_2^* &= g^h_{2(t_0,\ldots,t_3)} = t_2^2 x_4 + t_0^2 x_2 \\
g_3^* &= g^h_{3(t_0,\ldots,t_3)} = t_1 + t_0 x_4 \\
g_4^* &= g^h_{4(t_0,\ldots,t_3)} = t_3 x_4^3 + t_0 x_3 \\
g_5^* &= g^h_{5(t_0,\ldots,t_3)} = x_1 - x_4^2.
\end{aligned}$$

By Proposition 6.10, $I^* = I^h_{(t_0,\ldots,t_3)} = \langle g_1^*, g_2^*, g_3^*, g_4^*, g_5^* \rangle \subseteq \mathbb{C}[t_0, \ldots, t_3, x_1, \ldots x_4]$.

This discovery would be utterly useless without the following extraordinary and extraordinarily crucial result.

Proposition 7.3. *Let k be an algebraically closed field, and let $I \subseteq k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be an ideal. Then*

$$\mathbf{V}_a(I_n) = \pi_{n+1}\left(\mathbf{V}\left(I^h_{(x_0,\ldots,x_n)}\right)\right).$$

Proof. Refer to p. 431 of *Ideals, Varieties, and Algorithms* by Cox et al. □

This result illuminates the system of equations that our wild solutions inhabit. Each deviant 4-tuple—i.e., each element in $W = \{(0, a, b, 0) \in \mathbb{C}^4 : a \neq 0 \text{ or } b \neq 0\}$—extends to at least one complete solution in $\mathbf{V}(I^*) \subseteq \mathbb{P}^3(\mathbb{C}) \times \mathbb{C}^4$.

We now determine whether there's anything peculiar about the way in which the deviant 4-tuples extend to complete solutions in $\mathbf{V}(I^*)$.

Example 7.4. We want to show that $W = \pi_4(\mathbf{V}(I^*) \backslash (U_0 \times \mathbb{C}^4)) \backslash \pi_4(\mathbf{V}(I^*) \cap (U_0 \times \mathbb{C}^4))$. Let $S = \pi_4(\mathbf{V}(I^*) \backslash (U_0 \times \mathbb{C}^4)) \backslash \pi_4(\mathbf{V}(I^*) \cap (U_0 \times \mathbb{C}^4))$.
Let $(0, a_*, b_*, 0) \in W$ be arbitrary. Notice that $(0 : 0 : 0 : 1; 0, a_*, b_*, 0) \in \mathbf{V}(I^*) \backslash (U_0 \times \mathbb{C}^4)$. Let $(u_0 : u_1 : u_2 : u_3) \in U_0$ be arbitrary. Now, suppose $a_* \neq 0$. Notice that $g_2^*(u_0, u_1, u_2, u_3, 0, a_*, b_*, 0) = u_0^2 a_*$. Since $u_0 \neq 0$ and $a_* \neq 0$, it follows that $(u_0 : u_1 : u_2 : u_3; 0, a_*, b_*, 0) \notin \mathbf{V}(I^*)$. Suppose $b_* \neq 0$. Then $g_4^*(u_0, u_1, u_2, u_3, 0, a_*, b_*, 0) = u_0 b_*$. Since $u_0 \neq 0$ and $b_* \neq 0$, it follows that $(u_0 : u_1 : u_2 : u_3; 0, a_*, b_*, 0) \notin \mathbf{V}(I^*)$. So, $(0, a_*, b_*, 0) \notin \pi_4(\mathbf{V}(I^*) \cap (U_0 \times \mathbb{C}^4))$. It follows that $(0, a_*, b_*, 0) \in S$.
Let $(a_1, a_2, a_3, a_4) \in S$ be arbitrary. Fix $(0 : u_1 : u_2 : u_3) \in \mathbb{P}^3(\mathbb{C}) \backslash U_0$ such that $(0 : u_1 : u_2 : u_3; a_1, a_2, a_3, a_4) \in \mathbf{V}(I^*) \backslash (U_0 \times \mathbb{C}^4)$. Since $g_5^*(0, u_1, u_2, u_3, a_1, a_2, a_3, a_4) = a_1 - a_4^2 = 0$, we have that $a_1 = a_4^2$. Suppose for a contradiction that $a_4 \neq 0$. Since $g_4^*(0, u_1, u_2, u_3, a_1, a_2, a_3, a_4) = u_3 a_4^3 = 0$, it follows that $u_3 = 0$. Also, since $g_2^*(0, u_1, u_2, u_3, a_1, a_2, a_3, a_4) = u_2 a_4 = 0$, we have that $u_2 = 0$. Furthermore, $g_3^*(0, u_1, u_2, u_3, a_1, a_2, a_3, a_4) = u_1 = 0$. Therefore, $(0 : u_1 : u_2 : u_3) = (0 : 0 : 0 : 0)$, a contradiction. It follows that $a_1 = 0 = a_4$. Since $(1 : 0 : 0 : 0; 0, 0, 0, 0) \in \mathbf{V}(I^*) \cap (U_0 \times \mathbb{C}^4)$, it follows that $(0, 0, 0, 0) \notin S$. So, either $a_2 \neq 0$ or $a_3 \neq 0$. Therefore, $(a_1, a_2, a_3, a_4) \in W$.
Behold, $W = S$.

It follows that W consists of all the partial solutions in $\mathbb{C}^4$ of $\mathbf{V}(I^*)$ that extend to complete solutions exclusively in $\mathrm{LBM}_0 \times \mathbb{C}^4$. In other words, the little bit more that projective space gifts us is where the wild solutions are.

Bibliography

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