THINKING IS HARD.

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CLUSTER ALGEBRAS

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Yippee!

Introduction

Math...something clever...

Seeds

Most of the content in this chapter is based on [1], with language changed for clarity.

[1] Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023

Set-up

Let $I := \{1, 2, \ldots, n\}$. Let $\mathcal{F} := \mathbb{Q}(X_i^0 : i \in I)$ be the field of rational functions on the variables X_i^0 for $i \in I$.

Labeled seeds and seed mutations

<u>**Def**</u> **1.** A *labeled seed* is a pair $(\epsilon, \mathbf{X})$, where

i. $\epsilon = (\epsilon_{ij})$ where $\epsilon_{ij} \in \mathbb{Z}$ for all $i, j \in I$ [2] and ϵ is a skew-symmetrizable[3] matrix. This means that there exists positive integers d_i for $i \in I$ such that $\epsilon_{ij} d_j = -\epsilon_{ji} d_j$ for $\epsilon_{ij} \in \epsilon$.

ii. $\mathbf{X} = (X_i)$ for $i \in I$ is a transcendence basis[4] of $\mathcal{F}_X$. $\mathcal{F}_X \cong \mathbb{Q}(X_i : i \in I)$.

[2] ϵ is an $n \times n$ matrix.

[3] Its transpose is its negative.

[4] Not sure what this means right now.

The matrix ϵ is the *exchange matrix*, and the variables X_i are called the *$\mathcal{X}$-variables*.

Let $\mathcal{S}$ denote the set of all labeled seeds in $\mathcal{F}_X$. Let $\mathrm{sgn}(x) \in \{-1, 0, +1\}$ denote the sign of a real number x.[5]

[5] -1 if $x < 0$, 0 if $x = 0$, $+1$ if $x > 0$

<u>**Def**</u> **2.** The *seed mutation* $\mu_k : \mathcal{S} \to \mathcal{S}, ((\epsilon, \mathbf{X}) \mapsto (\epsilon', \mathbf{X}')$ at some $k \in I$ is given by:

$$\epsilon'_{ij} = \begin{cases} -\epsilon_{ij} & \text{if } i = k \text{ or} j = k, \\ \epsilon_{ij} + \frac{|\epsilon_{ik}|e_{kj} + \epsilon_{ik}|\epsilon_{kj}}{2} & \text{otherwise} \end{cases} \tag{1}$$

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\mathrm{sgn}(\epsilon_{ik})})^{-\epsilon_{ik}} & \text{otherwise} \end{cases} \tag{2}$$

The transformation $\mathbf{X} \mapsto bX'$ is called the *cluster $\mathcal{X}$[6]-transformation* at k.

[6] read "x"

The seed mutation μ_k is involutive[7].

[7] Applying it twice returns you to the original seed.

Def 3. Two labeled seeds $(\epsilon, \mathbf{X}), (\epsilon', \mathbf{X}')$ are *mutation-equivalent* if there is a finite composition of seed mutations and permutations that maps $(\epsilon, \mathbf{X})$ to $(\epsilon', \mathbf{X}')$.

Equivalence classes based on mutation-equivalence are called *mutation classes* and usually labeled $\mathbf{s}$.

> Mutation classes of labeled seeds are the basic subjects in the research field of cluster algebra.[8,9]

Def 4. The relationships between labeled seeds in a mutation class $\mathbf{s}$ can be encoded in the *(labeled) exchange graph* $\mathbb{E}\mathrm{xch}_{\mathbf{s}}$. Each vertex v corresponds to a labeled seed $\mathbf{s}^{(v)} \in \mathbf{s}$. There are two kinds of labeled edges:

- labeled edges of the form

[8] Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023

[9] Approved by Dr. Dylan.

Bibliography

Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023.

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