



THINKING IS HARD.

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# CLUSTER ALGEBRAS

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*Yippee!*



# *Introduction*

Math...something clever...



# Seeds

Most of the content in this chapter is based on <sup>1</sup>, with language changed for clarity.

<sup>1</sup> Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023

## Set-up

Let  $I := \{1, 2, \dots, n\}$ . Let  $\mathcal{F} := \mathbb{Q}(X_i^0 : i \in I)$  be the field of rational functions on the variables  $X_i^0$  for  $i \in I$ .

## Labeled seeds and seed mutations

**Def 1.** A labeled seed is a pair  $(\epsilon, \mathbf{X})$ , where

- $\epsilon = (\epsilon_{ij})$  where  $\epsilon_{ij} \in \mathbb{Z}$  for all  $i, j \in I$ <sup>2</sup> and  $\epsilon$  is a skew-symmetrizable<sup>3</sup> matrix. This means that there exists positive integers  $d_i$  for  $i \in I$  such that  $\epsilon_{ij}d_j = -\epsilon_{ji}d_i$  for  $\epsilon_{ij} \in \epsilon$ .
- $\mathbf{X} = (X_i)$  for  $i \in I$  is a transcendence basis<sup>4</sup> of  $\mathcal{F}_{\mathbf{X}}$ .  $\mathcal{F}_{\mathbf{X}} \cong \mathbb{Q}(X_i : i \in I)$ .

<sup>2</sup>  $\epsilon$  is an  $n \times n$  matrix.

<sup>3</sup> Its transpose is its negative.

<sup>4</sup> Not sure what this means right now.

The matrix  $\epsilon$  is the *exchange matrix*, and the variables  $X_i$  are called the  $\mathcal{X}$ -variables.

Let  $\mathcal{S}$  denote the set of all labeled seeds in  $\mathcal{F}_{\mathbf{X}}$ . Let  $\text{sgn}(x) \in \{-1, 0, +1\}$  denote the sign of a real number  $x$ .<sup>5</sup>

<sup>5</sup>  $-1$  if  $x < 0$ ,  $0$  if  $x = 0$ ,  $+1$  if  $x > 0$

**Def 2.** The seed mutation  $\mu_k : \mathcal{S} \rightarrow \mathcal{S}, ((\epsilon, \mathbf{X}) \mapsto (\epsilon', \mathbf{X}'))$  at some  $k \in I$  is given by:

$$\epsilon'_{ij} = \begin{cases} -\epsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \epsilon_{ij} + \frac{|\epsilon_{ik}|e_{kj} + \epsilon_{ik}|\epsilon_{kj}|}{2} & \text{otherwise} \end{cases} \quad (1)$$

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\text{sgn}(\epsilon_{ik})})^{-\epsilon_{ik}} & \text{otherwise} \end{cases} \quad (2)$$

The transformation  $\mathbf{X} \mapsto b\mathbf{X}'$  is called the *cluster  $\mathcal{X}^6$ -transformation* at  $k$ .

<sup>6</sup> read "x"

The seed mutation  $\mu_k$  is involutive<sup>7</sup>.

<sup>7</sup> Applying it twice returns you to the original seed.

**Def 3.** Two labeled seeds  $(\epsilon, \mathbf{X}), (\epsilon', \mathbf{X}')$  are *mutation-equivalent* if there is a finite composition of seed mutations and permutations that maps  $(\epsilon, \mathbf{X})$  to  $(\epsilon', \mathbf{X}')$ .

Equivalence classes based on mutation-equivalence are called *mutation classes* and usually labeled  $\mathbf{s}$ .

Mutation classes of labeled seeds are the basic subjects in the research field of cluster algebra.<sup>8,9</sup>

**Def 4.** The relationships between labeled seeds in a mutation class  $\mathbf{s}$  can be encoded in the (*labeled*) *exchange graph*  $\mathbb{E}\text{ch}_{\mathbf{s}}$ . Each vertex  $v$  corresponds to a labeled seed  $\mathbf{s}^{(v)} \in \mathbf{s}$ . There are two kinds of labeled edges:

- labeled edges of the form

<sup>8</sup> Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023

<sup>9</sup> Approved by Dr. Dylan.



## *Bibliography*

Takeru Asaka, Tsukasa Ishibashi, and Shunsuke Kano. Earthquake theorem for cluster algebras of finite type, 2023.



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