This is a graded discussion: **50 points possible** due Feb 28, 2022 at 12:50pm

Since this is a group discussion, each group has its own conversation for this topic. Here are the ones you have access to:

- ODE Group 1 8 20
- ODE Group 2 3 24
- ODE Group 3 1 40

Math Concepts: Linear Differential Equations with Constant Coefficients

ToDo: Respond to each of the following.

1. The *Wronskian* $W(x)$of a collection of functions $f_1(x),\ldots,f_n(x)$ is the determinant of the matrix $\begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_1' & f_2' & \cdots & f_{n-1}' & f_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1^{(n-2)} & f_2^{(n-2)} & \cdots & f_{n-1}^{(n-2)} & f_n^{(n-2)} \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & f_n^{(n-1)} \end{bmatrix}$.
   A. Prove that $f_1,\ldots,f_n$ being linearly dependent implies $W(x)\equiv 0$. Hint: Use the linearity of the derivative to show that the Wronskian matrix has rank less than n.
   B. Show that the functions $e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}$ are linearly independent by computing their Wronskian. Hint: What is the contrapositive of the statement shown in part A?
   C. Fix $b\in\mathbb{R}$. Show that the functions $\cos(bx)$ and $\sin(bx)$ are linearly independent by computing their Wronskian.
2. A smooth function f is *analytic at $x=a$* if it equals its Taylor series $\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$ in a neighborhood of a (i.e. this series has a nonzero radius of convergence in which it converges to the value of the function). Define $\mathcal{A}_a(\mathbb{R})\subset\mathcal{C}^\infty(\mathbb{R})$ to be the subset of functions which are analytic at $x=a$.
   A. Recall that $\mathcal{C}^\infty(\mathbb{R})$ has the structure of a real vector space. Prove that $\mathcal{A}_a(\mathbb{R})$ is a subspace. Hint: If f and g are analytic at $x=a$ with radii of convergence R_1 and R_2, what is the radius of convergence for $f+g$?
   B. For smooth functions $f,g\in\mathcal{C}^\infty(\mathbb{R})$, prove the following generalization of the product

rule by induction on n: $\frac{d^n}{dx^n}(f \cdot g) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \cdot g^{(n-k)}$. Note: This looks remarkably similar to the binomial expansion formula.

C. Suppose f and g are analytic at $x = a$ with radii of convergence R_1 and R_2, respectively. Prove that $f \cdot g$ is also analytic and equals the product $\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k\right)\left(\sum_{\ell=0}^{\infty} \frac{g^{(\ell)}(a)}{\ell!}(x-a)^\ell\right)$ in its domain of analyticity.

D. Explain why any polynomial $p(x)$ with real coefficients can be viewed as an analytic function at any $a \in \mathbb{R}$.

E. Prove for any $\lambda \in \mathbb{R}$ that $e^{\lambda x}$ is analytic at any $a \in \mathbb{R}$.

F. Prove for any $b \in \mathbb{R} \setminus \{0\}$ that $\cos(bx)$ and $\sin(bx)$ are analytic at $x = 0$. Note: This works for any $a \in \mathbb{R}$, but the explicit expression is a little gross to write down.

G. Use parts C, D, and E above to show for any $m \geq 0$ and any $\lambda \in \mathbb{R}$ that $x^m e^{\lambda x}$ is analytic at any $a \in \mathbb{R}$.

H. Use parts C, E, and F above to show for any $\lambda \in \mathbb{R}$ and any $b \in \mathbb{R} \setminus \{0\}$ that $e^{\lambda x}\cos(bx)$ and $e^{\lambda x}\sin(bx)$ are analytic at any $a \in \mathbb{R}$.

3. Consider the homogeneous linear differential equation with constant coefficients $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$ for $a_k \in \mathbb{R}$.

   A. Show that the set of solutions to this differential equation forms a vector space. Hint: What structure must be present on the set of all possible solutions?

   B. Assume $f_0, f_1, \ldots, f_n$ are all solutions to the differential equation above. Show that their Wronskian is zero.

   C. Use part B to prove that the differential equation above can have at most n linearly independent *analytic* solutions in a neighborhood of a. Hint: Assume $n+1$ solutions $f_0, f_1, \ldots, f_n$ which are all analytic at $x = a$ and prove that they have to be linearly dependent.

   D. Use part C to give a bound on the possible dimension of the space of analytic solutions to this differential equation.

4. Find all *real* solutions to each of the following homogeneous linear differential equations in one-variable (with distinct eigenvalues). Show enough of your work that it is clear how you arrived at your answer. Note: Since the coefficients are real numbers, your solutions should be real-valued functions.

   A. $y' - 5y = 0$

   B. $y'' - 3y' - 4y = 0$

   C. $y^{(4)} - y = 0$

5. Find all *real* solutions to each of the following homogeneous linear differential equations in one-variable (with repeated eigenvalues). Show enough of your work that it is clear how you arrived at your answer. Note: Since the coefficients are real numbers, your solutions should be real-valued functions.

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A. $y'' + 4y' + 4y = 0$

B. $y^{(4)} - 4y^{(3)} + 6y^{(2)} - 4y^{(1)} + y^{(0)} = 0$

C. $y^{(6)} + 2y^{(5)} + 2y^{(4)} = 0$

6. Use your results from questions 2 and 3 to explain how you know that you have found all possible analytic solutions to the differential equations in questions 4 and 5.

ToDo: Review the **RISE Model** **(https://pasadena.instructure.com/courses/34818/files/4586595/download?wrap=1)** **(https://pasadena.instructure.com/courses/34818/files/4586595/download?download_frd=1)** (click that link) for student discussion feedback.

ToDo: Respond to two of your classmates' posts following the RISE Model, make sure to include encouraging and helpful remarks in your responses. Please respond to a post that has not yet received feedback if possible.

ToDo: Edit your original post if needed and submit responses to your peer feedback.