OS Homework 1

1.8

a. $\sum_{i=0}^{\infty} \frac{1}{4^i}$

$$\sum_{i=0}^{\infty} \frac{1}{4^i} = \underset{(4^0)}{\overset{(1)}{\frac{1}{1}}} + \underset{(4^1)}{\frac{1}{4}} + \underset{(4^2)}{\frac{1}{16}} + \underset{(4^3)}{\frac{1}{64}} \quad \underset{(4^4)}{\frac{1}{256}}$$

3 terms:

$$1 + \frac{1}{4} = \frac{5}{4}$$

$$1 + \frac{1}{4} + \frac{1}{16} = \frac{21}{16}$$

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{85}{64}$$

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} = \frac{341}{256}$$

partial sums:

$$\left(1, \frac{5}{4}, \frac{21}{16}, \frac{85}{64}, \frac{341}{256}, \ldots\right)$$

So,

$$\sum_{i=0}^{\infty} \frac{1}{4^i} = 1.\overline{33} \qquad \sum_{n=0}^{\infty} \frac{1}{4^n}$$

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6. $\sum_{i=0}^{\infty} \frac{i}{4^i}$

Sums: $\sum_{i=0}^{\infty} \frac{i}{4^i} = \underset{(4^0)}{\overset{(0)}{\frac{0}{1}}} + \underset{(4^1)}{\frac{1}{4}} + \underset{(4^2)}{\overset{(\frac{1}{8})}{\frac{2}{16}}} + \underset{(4^3)}{\frac{3}{64}} + \underset{(4^4)}{\overset{(\frac{1}{64})}{\frac{4}{256}}}$

$0 + \frac{1}{4} = \frac{1}{4}$

$0 + \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

$0 + \frac{1}{4} + \frac{1}{8} + \frac{3}{64} = \frac{27}{64}$

$0 + \frac{1}{4} + \frac{1}{8} + \frac{3}{64} + \frac{4}{64} = \frac{7}{16}$

Partial Sums:

$\left(\frac{1}{4}, \frac{3}{8}, \frac{27}{64}, \frac{7}{16}, \ldots\right)$

$\sum_{i=0}^{\infty} \frac{i}{4^i} = .\overline{4}$

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1.12 Prove the following

a) $\sum_{i=1}^{n}(2i-1)=n^2$

let $P(n)$ be $\sum_{i=1}^{n}(2i-1)=n^2$

1. Base case; $n=1$

Prove: $\sum_{i=1}^{1}(2i-1)=1^2$

Proof: $\sum_{i=1}^{1}(2(1)-1)=1$ and $1^2=1$

thus, $P(1)$.

2. Induction Step; let $n\in\mathbb{N}$, we fix arbitrary integer $n\geq 1$ to show that $P(n+1)$ is true

i) Assume $P(n)$: $\sum_{i=1}^{n}(2i-1)=n^2$ for some int $n\geq 1$

ii) Induction:

$\sum_{i=1}^{n+1}(2i-1)=(n+1)(n^2)$

Proof: Sum the left hand side

$$\sum_{i=1}^{n+1}(2i-1)=0+1+2+3+\dots+n+(n+1)$$
$$=(0+1+2+3+\dots+n)+(n+1)$$
$$=\left(\sum_{i=1}^{n}(2i-1)\right)+(n+1)$$

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$= [n^2] + (n+1)$

$= n^2 + n + 1$

$= \cancel{n(n+1)+1}$

$= n^2 + n + 1$

by induction it follows

$$\sum_{i=1}^{n} (2i-1) = n^2$$

b. Prove $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$

~~proof~~ let $P(n)$ be $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$

1. Base case; $n=1$

Prove $\sum_{i=1}^{1} i^3 = \left(\sum_{i=1}^{1} i\right)^2$

Proof $\sum_{i=1}^{1} 1^3 = 1$ and $\left(\sum_{i=1}^{1} 1\right)^2 = 1$

Thus, $P(1)$.

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2. Induction Step; let $n \in \mathbb{N}$ and we fix an arbitrary integer $n \geq 1$. Then we'll show that $P(n+1)$ is also true.

i) Assume $P(n)$: $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ for some int $n \geq 1$

ii) use induction:

$$\sum_{i=1}^{n+1} i^3 = (n+1)\left(\sum_{i=1}^{n} i\right)^2$$

proof; write sum on the lhs ~~to~~ be proven

$$\sum_{i=1}^{n+1} i^3 = 1+4+9+16+\dots+n+(n+1)$$

$$= (1+4+9+16+\dots+n)+(n+1)$$

$$= 1+8+27+64+\dots+n+(n+1)$$

$$= (1+8+27+64+\dots+n)+(n+1)$$

$$= \left(\sum_{i=1}^{n} i^3\right) + (n+1)$$

$$= \left[\left(\sum_{i=1}^{n} i\right)^2\right] + (n+1)$$

by induction it follows

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

1.7 Prove the following

a. $\log X < X$ for all $X > 0$

It is true that $0 < x \leq 1$

if $x < 1$ then $\log x < 0$

and

if $x = 1$ then $\log x = 0$

now we use induction to show $x > 1$ is true

Base case;

if $1 < x < 2$ then $\log x < 1$

and

if $x = 2$ then $\log x = 1$

$\therefore 1 < x \leq 2$

Inductive Hypothesis;

Assume true $P < x \leq 2P$

prove $2P < y \leq 4P$ this is because y is between the limits

$2(P) < y \leq 2(2P)$

or

$P < \frac{y}{2} \leq 2P$

$\log y = \log 2 + \log \frac{y}{2}$

$= \log y = 1 + \log \frac{y}{2}$

This follows that

$\log \frac{y}{2} < \frac{y}{2} \Rightarrow \log y < 1 + \frac{y}{2}$

substitute

$\frac{y}{2}$ will always be greater than 1

$\therefore \log y < \frac{y}{2} + \frac{y}{2} = \log y \leq y$

6. $\log_c(A^B) = B \log_c A$

1. let $\log_c(A^B) = x$ Reason: $\log(A^B)$ must be a number

2. $C^x = A^B$ Reason: exponentiated form of #1

3. $(C^x)^{1/B} = (A^B)^{1/B}$ Reason: exponents if $A=B$ then $A^N = B^N$

4. $C^{x/B} = A$ Reason: exponents $(A^P)^N = A^{P \cdot N}$

5. $\log_c(A) = \frac{x}{B}$ Reason: equivalent log form of #4

6. $B \cdot \log_c(A) = x$ Reason: if $A=B$ then $R \cdot A = R \cdot B$

7. $B \cdot \log_c(A) = \log_c(A^B)$ Reason: substitute from #1 for x in #6