MA 2073 HOMEWORK 4

4.2 (13) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad P = A(A^TA)^{-1}A^T$.

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so} \quad (A^TA)^{-1} = I$$

Then $P = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4 x 4 matrix.

Then $P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

(16) Project $\vec{b}$ onto col A, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

In this case $\vec{b} \in \text{col} A$, so $P\vec{b} = \vec{b}$.

$$\vec{b} = \frac{1}{2}(1, 2, -1) + \frac{3}{2}(1, 0, 1)$$

(17) Suppose $P^2 = P$

Consider $(I - P)^2 = (I-P)(I-P)$

$$= I^2 - IP - PI + P^2$$

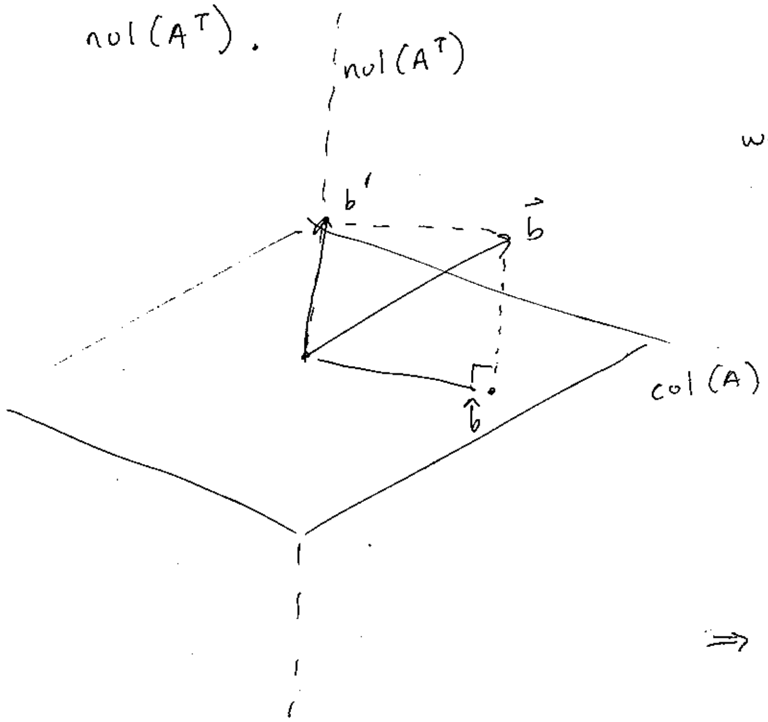
Note $IP = PI = P$ and $I^2 = I$. So our expression becomes

$$(I-P)^2 = I - P - P + P$$
$$= I - P. \quad \text{(i.e } I-P \text{ is also a projection matrix!)}$$

If P projects onto col A, then $I-P$ projects onto the left nullspace of A. Why? $\longrightarrow$

recall col A and nul A^T are orthogonal complements.

Suppose $\vec{b}$ is a vector, project $\vec{b}$ into col(A) and nul(A^T).



we write $\vec{b} = \underbrace{\hat{b}}_{\in \text{col}A} + \underbrace{\vec{b}'}_{\in \text{nul}(A^T)}$

but $\hat{b} = P\vec{b}$, so

$$\vec{b} = P\vec{b} + \vec{b}'$$

$$\Rightarrow \vec{b}' = (I - P)\vec{b}$$

$\vec{b}'$ in L nullspace is the projection of $\vec{b}$ under $(I-P)$.

(22) CLAIM: P is symmetric.

PF: consider $P^T = \left(A(A^TA)^{-1}A^T\right)^T$

note A^TA is a symmetric matrix, so $(A^TA)^{-1}$ is also symmetric $\left(\left[(A^TA)^{-1}\right]^T = (A^TA)^{-1} \right)$.

Then
$$\begin{aligned} P^T &= \left(A(A^TA)^{-1}A^T\right)^T \\ &= (A^T)^T\left[(A^TA)^{-1}\right]^T(A)^T \quad \text{(order reverses!)}. \\ &= A(A^TA)^{-1}A^T, \quad \text{as desired.} \\ &= P. \end{aligned}$$

4.3

① $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ $\cancel{y = \beta_0 + \beta_1 x \text{ or}}$

$b = C + Dt$

$A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ $A^T \vec{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$

Solve $A^T A \hat{x} = A^T \vec{b}$ for $\hat{x}$

Find $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Then $A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \vec{p}$, the projection of $\vec{b}$ onto col A.

$\vec{e} = \vec{b} - A\hat{x} = \vec{b} - \vec{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$.

Since the method of least squares minimizes the error, this is the minimal error: $\|\vec{e}\| = \sqrt{1+9+25+9} = \sqrt{44}$

② With same points the system is

$$\left.\begin{aligned} C &= 0 \\ C + D &= 8 \\ C + 3D &= 8 \\ C + 4D &= 20 \end{aligned}\right\} \text{ no sol}^{\underline{n}} \text{ exists.}$$

when our RHS is $p = (1, 5, 13, 17)$ we have:

$$\begin{aligned} C &= 1 \\ C + D &= 5 \\ C + 3D &= 13 \\ C + 4D &= 17 \end{aligned}$$

so $C = 1$, $D = 4$

and the sol$^{\underline{n}}$ is the line

$b = 1 + 4t$.

i.e $\hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(3) dot prod of $e = (-1, 3, -5, 3)$ with both columns of A is zero.

Shortest distance from $\vec{b}$ to col(A) is $\|e\| = \sqrt{44}$.

(9) We want to find the parabola

$$b = C + Dt + Et^2.$$

equations:

$$C + D \cdot 0 + E\,0^2 = 0$$
$$C + D \cdot 1 + E\,1^2 = 8$$
$$C + D \cdot 3 + E\,3^2 = 8$$
$$C + D \cdot 4 + E\,4^2 = 20$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

So we want to solve $A^T A \hat{x} = A^T \vec{b}$

For the figures, we are now projecting $\vec{b}$ onto the 3D space spanned by the cols of A. In the picture in the text, we are projecting (from $\mathbb{R}^4$) down to a (2D) plane.

(10) $b = C + Dt + Et^2 + Ft^3$

$Ax = b$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$ we can solve this!

we get $b = \frac{1}{3}\left(47t - 28t^2 + 5t^3\right)$.

So $\vec{p} = \vec{b}$ and $\vec{e} = \vec{0}$.

4.4

(15) let $\vec{q}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ (just the first column, made into a unit vect.).

let $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \left\langle \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \vec{q}_1 \right\rangle \vec{q}_1$.

Then $\vec{q}_2 = \dfrac{\vec{w}}{\|\vec{w}\|}$

so $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{3}(-9)\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

and $\vec{q}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

For q_3 use G-S on $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\vec{q}_3 = \dfrac{\vec{w}}{\|\vec{w}\|}$ where $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{q}_1 \right\rangle \vec{q}_1 - \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{q}_2 \right\rangle \vec{q}_2$

$\ldots = \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ so $\|w\| = 3$ and $\vec{q}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

b) $\vec{q}_3$ is $\perp$ to the columns of $A \Rightarrow$ it is in the left nullspace of A.

c) To solve $A\vec{x} = \begin{bmatrix}1\\2\\7\end{bmatrix}$ use least squares.

Solve $A^TA\hat{x} = A^T\begin{bmatrix}1\\2\\7\end{bmatrix}$.

we get $\hat{x} = (1,2)$.

(20) a) TRUE: Q orthogonal $\Rightarrow QQ^T = I = Q^TQ$ so Q^T is Q^{-1}, and hence Q^{-1} is orthogonal (since Q^T is!).

b) TRUE. Let $Q = \begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix}$. Then $Q\vec{x} = \vec{q}_1x_1 + \vec{q}_2x_2$, if $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.

so $\|Q\vec{x}\|^2 = \langle \vec{q}_1x_1 + \vec{q}_2x_2, \vec{q}_1x_1 + \vec{q}_2x_2\rangle$

$$= \underbrace{\langle\vec{q}_1, \vec{q}_1\rangle}_{=1}x_1^2 + 2\underbrace{\langle\vec{q}_1, \vec{q}_2\rangle}_{=0}x_1x_2 + \underbrace{\langle\vec{q}_2, \vec{q}_2\rangle}_{=1}x_2^2$$

$$= x_1^2 + x_2^2$$

$$= \|\vec{x}\|^2.$$

(21) $A = \begin{bmatrix}\vec{a}_1 & a_2\end{bmatrix} = \begin{bmatrix}1 & -2\\1 & 0\\1 & 1\\1 & 3\end{bmatrix}$

let $\vec{u}_1 = \dfrac{\vec{a}_1}{\|\vec{a}_1\|} = \dfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ this is the first orthonormal vect.

now $\vec{u}_2 = \dfrac{\vec{w}}{\|\vec{w}\|}$, where $\vec{w} = \cancel{\begin{bmatrix}2\\0\\1\\3\end{bmatrix}}\vec{a}_2 - \langle\vec{a}_2, \vec{u}_1\rangle\vec{u}_1$

$$\vec{w} = \begin{bmatrix}-2\\0\\1\\3\end{bmatrix} - \frac{1}{2}\cdot\frac{1}{2}\cdot 2\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2\\0\\1\\3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-5/2\\-1/2\\1/2\\5/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-5\\-1\\1\\5\end{bmatrix}.$$

and $\|\vec{w}\| = \sqrt{\langle \vec{w}, \vec{w} \rangle}$

$= \frac{1}{2}\sqrt{52}$

So $\vec{u}_2 = \frac{1}{\sqrt{52}} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$.

Now $Q = \begin{bmatrix} \vec{u}_1 & u_2 \end{bmatrix}$ and projection onto col(A) is

projection onto col(Q)

Projection matrix is $Q \underbrace{(Q^T Q)^{-1}}_{=I} Q^T$

so it is just QQ^T

$$QQ^T = \begin{bmatrix} 1/2 & -5/\sqrt{52} \\ 1/2 & -1/\sqrt{52} \\ 1/2 & 1/\sqrt{52} \\ 1/2 & 5/\sqrt{52} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -5/\sqrt{52} & -1/\sqrt{52} & 1/\sqrt{52} & 5/\sqrt{52} \end{bmatrix} = \cancel{\frac{1}{4}+\frac{25}{52}}$$

and the projection of $\vec{b}$ is $QQ^T\vec{b}$

$$Q^T\vec{b} = Q^T \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -4/2 - 3/2 + 3/2 \\ +20/\sqrt{52} + 3/\sqrt{52} + 3/\sqrt{52} \end{bmatrix} = \begin{bmatrix} -2 \\ 26/\sqrt{52} \end{bmatrix} = \begin{bmatrix} -2 \\ \sqrt{52}/2 \end{bmatrix} \quad \left(\text{since } 26 = \frac{52}{2}\right)$$

Then $$QQ^Tb = Q\begin{bmatrix} -2 \\ \frac{\sqrt{52}}{2} \end{bmatrix} = \begin{bmatrix} -1 - 5/2 \\ -1 - 1/2 \\ -1 + 1/2 \\ -1 + 5/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 \\ -3 \\ -1 \\ 3 \end{bmatrix}$$ is the projection of $\vec{b}$ onto col A.

(24) note $x_4 = x_1 + x_2 + x_3$ so we could find a basis:

a)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1+x_2+x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

These three vectors are independent, so they form a basis for S.

b) The vector $(1,1,1,-1)$ is $\perp$ to all three.

and so it forms a basis for $S^\perp$

c) we may write $(1,1,1,1) = \vec{b}_1 + \vec{b}_2$ where

$$\vec{b}_1 = \frac{1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}$$

and $\vec{b}_2 = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$

ADDITIONAL PROBLEMS

① a) if $A\vec{x} = \vec{0}$, then $A^TAx = A^T(A\vec{x}) = A^T(\vec{0}) = \vec{0}$.

b) Suppose $A^TA\vec{x} = \vec{0}$, and ~~consider~~ note $\vec{x}^TA^TA\vec{x} = \vec{x}^T(A^TA\vec{x})$
$= \vec{x}^T(\vec{0})$
$= \vec{0}$.

But $\vec{x}^TA^TA\vec{x} = (A\vec{x})^TA\vec{x}$

so $\vec{0} = \vec{x}^TA^TA\vec{x} = (A\vec{x})^TA\vec{x} = \langle A\vec{x}, A\vec{x}\rangle$

and the inner product is zero $\Leftrightarrow$ $A\vec{x}$ is the zero vector.

② Suppose A is $m\times n$, and A^TA is invertible. Thus the columns of A^TA are linearly independent, and hence the only sol$^{\underline{n}}$ to $A^TA\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. By ① the only sol$^{\underline{n}}$ to $A\vec{x} = \vec{0}$ is also $\vec{x} = 0$, hence the columns of A are linearly independent.

③ a) CLAIM: A^TA is invertible when the columns of $m \times n$ matrix A are lin indep.

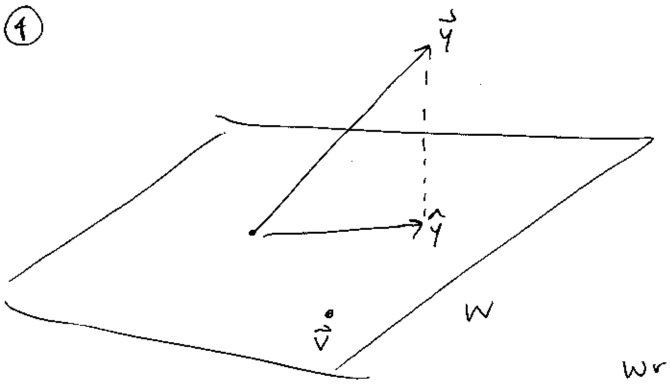
PROOF: ~~Since~~ Recall ① showed that $A\vec{x} = \vec{0} \iff A^TA\vec{x} = 0$

But since the columns of A are linearly independent, there is only ONE vector ~~$\vec{x}$~~ satisfying $A\vec{x} = \vec{0}$, namely $\vec{x} = \vec{0}$. Hence the columns of A^TA are linearly independent too. Since A^TA is square it is therefore invertible.

b) Since A lin indep. there are n pivot cols. There must be at least n rows to hold these pivots, thus $m \geq n$.

c) rank $A = n$.

④



Take $\vec{v} \in W$, $\vec{v} \neq \hat{y}$

So $\hat{y} - \vec{v} \in W$.

we know $\vec{y} - \hat{y}$ is $\perp$ to W,

and so $\vec{y} - \hat{y}$ is $\perp$ to $\hat{y} - \vec{v}$.

write $\vec{y} - \vec{v} = (y - \hat{y}) + (\hat{y} - \vec{v})$

and use pythagoras:

$$\|\vec{y} - \vec{v}\|^2 = \|y - \hat{y}\|^2 + \|\hat{y} - \vec{v}\|^2$$

$\|\hat{y} - \vec{v}\| > 0$ since $\hat{y} - \vec{v} \neq \vec{0}$, which gives the desired result. ∎

⑤ we know $\|\vec{u}-\vec{v}\| \geqslant 0$

$$\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle \geqslant 0$$

$$\langle \vec{u},\vec{u}\rangle - \langle \vec{u},\vec{v}\rangle - \langle \vec{v},\vec{u}\rangle + \langle \vec{v},\vec{v}\rangle \geqslant 0 \qquad (\text{bilinearity of } \langle\, , \rangle)$$

$$\underbrace{\langle \vec{u},\vec{u}\rangle}_{=1} + \underbrace{\langle \vec{v},\vec{v}\rangle}_{=1} - 2\langle \vec{u},\vec{v}\rangle \geqslant 0$$

$$\Rightarrow \langle \vec{u},\vec{v}\rangle \leqslant 1.$$

Similarly, since $\langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle \geqslant 0$, we obtain

$$-\langle \vec{u},\vec{v}\rangle \leqslant 1 \quad (\text{or } \langle \vec{u},\vec{v}\rangle \geqslant -1)$$

Combining, we get $|\langle \vec{u},\vec{v}\rangle| \leqslant 1$. ∎