MA 2073 HOMEWORK 5

8·5

① $$\int_0^{2\pi} 2\cos jx \cos kx \, dx = \int_0^{2\pi} \cos(j+k)x \, dx + \int_0^{2\pi} \cos(j-k)x \, dx$$

$$= \left[\frac{\sin(j+k)x}{j+k}\right]_0^{2\pi} + \left[\frac{\sin(j-k)x}{j-k}\right]_0^{2\pi}$$

$= 0 + 0$ provided $j \neq k$.

If $j = k$, then our integral becomes

$$\int_0^{2\pi} 2\cos^2 jx \, dx = \int_0^{2\pi} (1 + \cos 2jx)\, dx = 2\pi$$

⑩ Either integral is over a complete period of the function.

So $$\int_0^{2\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx$$

If f is odd, then these integrals cancel.

⑫ $$\frac{d}{dx}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x \\ \cos x \\ -2\sin 2x \\ 2\cos 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}.$$

5·1

⑱ $$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} = 1 \det\begin{bmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{bmatrix}$$

$$= (b-a)(c-a)\det\begin{bmatrix} 1 & b+a \\ 1 & c+a \end{bmatrix}$$

$$= (b-a)(c-a)(c-b)$$

(25) Row 2 is 2 times Row 1

$\Rightarrow$ determinant is zero (provided matrix has at least 2 rows).

(26) Row 3 − Row 1 = Row 2 − Row 1

So the matrix has determinant zero.

(28) a) TRUE $\det(AB) = \det(A)\det(B)$

but A not invertible $\Rightarrow \det(A) = 0$, so $\det(AB) = 0$ too.

b) FALSE row exchange multiplies by -1

c) FALSE e.g. $A = 2I$ $B = I$

$A - B = I$

but $\det(A-B) = 1$ and $\det(A) - \det(B) = 2^n - 1$.

d) TRUE $\det(AB) = \det A \det B = \det B \det A = \det(BA)$.

(29) Recall that A may not be square.

So $\det(A^T A) \neq \det(A^T)\det(A)$ since these ~~may~~ are not defined unless A is square.

5.2

(16) Note that the $(1,1)$-cofactor of the $n \times n$ matrix is F_{n-1}. The $(1,2)$-cofactor has only a single one in the first column, so it is really just a determinant of a smaller matrix.

e.g. $$\det \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} + (-1)^{2+1} \det \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} + (-1)^4 \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$F_{k+2} = F_{k+1} + F_k.$$

(21) $S_1 = 3 \qquad S_2 = 7 \qquad S_3 = 3(8) - 1(3) = 21$

$S_4 = 55$ (every 2^{nd} Fibbonacci)

(22) This reduces the determinant F_{2n+2} by ~~the~~ 1 times the cofactor of that corner entry.

This cofactor is the determinant of S_{n-1}, which is F_{2n}.

$$F_{2n+1} = F_{2n+2} - F_{2n}$$

$S_1 = 2 \qquad S_2 = 5 \qquad S_3 = 13$

(32) We want to show $F_{2n+2} = 3F_{2n} - F_{2n-2}$

$$\begin{aligned} F_{2n+2} &= F_{2n+1} + F_{2n} \\ &= F_{2n} + F_{2n-1} + F_{2n} \\ &= 2F_{2n} + (F_{2n} - F_{2n-2}) \\ &= 3F_{2n} - F_{2n-2} \end{aligned}$$

6.1 (15) first matrix

$$\det(P - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda(\lambda^2) + 1 = 1 - \lambda^3$$

the other e-vals are $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$

second

$$\det(P - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = \lambda^2(1-\lambda) + (\lambda - 1)$$
$$= (\lambda^2 - 1)(1-\lambda)$$
$$= (\lambda-1)(\lambda+1)(1-\lambda)$$

so $\lambda = 1, -1$ and 1.

(36)

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc$$
$$= ad - \lambda(a+d) + \lambda^2 - bc$$
$$= \lambda^2 - (a+d)\lambda + (ad - bc)$$

this isn't good $\therefore$ let's use another method $\rightarrow$

of course $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\end{bmatrix} = \begin{bmatrix}a+b\\a+b\end{bmatrix} = (a+b)\begin{bmatrix}1\\1\end{bmatrix}$

or $\begin{bmatrix}a+b\\c+d\end{bmatrix} = \begin{bmatrix}c+d\\c+d\end{bmatrix} = (c+d)\begin{bmatrix}1\\1\end{bmatrix}$

so $\begin{bmatrix}1\\1\end{bmatrix}$ is an e-vector $\bar{w}$ e-val $\lambda_1 = (a+b) = (c+d)$.

To find second e-val, check trace$(A) = a+d$. So

$a+d = (a+b) + \lambda_2$, hence $\lambda_2 = d-b$

(32) a) Since 0 is an e-val, the corr. e-vect is a basis for null(A). $(A\vec{u} = 0\vec{u} = \vec{0})$

$\{\vec{v}, \vec{w}\}$ form a basis for col A, why?

First we note A is 3×3, since it has 3 e-vals.

The dimension of nul(A) = 1, so dimension of col(A) = 2. (= dim(row(A))).

$\vec{v}, \vec{w}$ independent, and both are in col(A). They must therefore span col(A) and are a basis.

b) let $x = c\vec{v} + d\vec{w}$

then $Ax = 3c\vec{v} + 5d\vec{w}$ so if $c = \frac{1}{3}$, $d = \frac{1}{5}$ then

$\vec{x} = \frac{\vec{v}}{3} + \frac{\vec{w}}{5}$ is a partic. sol^{n}.

The complete sol^{n} is

$$\vec{x} = \frac{\vec{v}}{3} + \frac{\vec{w}}{5} + k\vec{u}, \quad k \in \mathbb{R}.$$

c) $\vec{u}$ would be in the column space.

(35) Permutation matrices have

$$P^T P = I$$

$$\Rightarrow (\det P)^2 = 1 \quad \text{hence determinant is } \pm 1$$

pivots are always 1

the trace can be 3 $(P = I)$

or 1 (row exchange, e.g $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$)

or 0 (2 row exchanges, e.g $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$)

from problem 15 we know $\lambda_1 = 1$ is always an eigenvalue. To find others use the fact that

$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace } P$ and $\lambda_1 \lambda_2 \lambda_3 = \det P$

we find possible determinants are ± 1 and

$$\frac{-1 + \sqrt{3}\, i}{2} = e^{2\pi i/3} \quad \text{and} \quad \frac{-1 - \sqrt{3}\, i}{2} = e^{-2\pi i/3}$$

(11) a) TRUE - no zero e-val

b) FALSE - we need distinct e-vals for lin. inped. e-vects

c) FALSE - it is possible for a matrix to have indep e-vects

for example I has e-val 1 with multiplicity n.

evects are the standard basis.

(15) $|\lambda| < 1$ $\quad A_1$ has e-val $\lambda = 1$ (it is a markov matrix)

thus $A_1^k \to A_1^\infty \neq 0$ matrix.

The e-vals of A_2 are all less than 1, so

$A_2^k \to 0$

(16)
$$\det(A_1 - \lambda I) = \det \begin{bmatrix} 6/10 - \lambda & 9/10 \\ 4/10 & 1/10 - \lambda \end{bmatrix} = \left(6/10 - \lambda\right)\left(1/10 - \lambda\right) - \frac{36}{100}$$

$$= \frac{6}{100} - \frac{7}{10}\lambda + \lambda^2 - \frac{36}{100}$$

$$= \lambda^2 - \frac{7}{10}\lambda - \frac{3}{10}$$

$$= (\lambda - 1)\left(\lambda + \tfrac{3}{10}\right)$$

So $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -3/10 \end{bmatrix}$

for $\lambda_1 = 1$ $\quad A_1 - \lambda_1 I = \begin{bmatrix} -4/10 & 9/10 \\ 4/10 & -9/10 \end{bmatrix}$ So $\vec{x}_1 = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ is an e-vect

$\lambda_2 = -3/10$ $\quad A_1 - \lambda_2 I = \begin{bmatrix} 9/10 & 9/10 \\ 4/10 & 4/10 \end{bmatrix}$ So $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

hence $S = \begin{bmatrix} 9 & 1 \\ 4 & -1 \end{bmatrix}$ $\quad S^{-1} = \frac{-1}{13}\begin{bmatrix} -1 & -1 \\ -4 & 9 \end{bmatrix}$

and $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so

$$S\Lambda^k S^{-1} \to S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$$

$$= S \begin{bmatrix} 1/13 & 1/3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9/13 & 9/13 \\ 4/13 & 4/13 \end{bmatrix}$$, the steady state.

(20) $A = S\Lambda S^{-1}$

$$\det(A) = \det(S)\det(\Lambda)\det(S^{-1})$$
$$= \det(\Lambda)\det(S)\det(S^{-1})$$
$$= \det(\Lambda)\det(SS^{-1})$$
$$= \lambda_1\lambda_2\cdots\lambda_n \det(I)$$
$$= \lambda_1\lambda_2\cdots\lambda_n.$$
$$\left(= \det(\Lambda).\right)$$

This only works when A can be diagonalized.

(26) The column space and the nullspace may have vectors in common! So $\vec{x}$ could be in both.

And there may not be r linearly independent eigenvectors in the column space.

ADDITIONAL PROBLEMS

Lay

(9) a) calculate "vectors of values"

$$\begin{array}{c|ccc} & p_0 & p_1 & p_2 \\ \hline -3 & 1 & -3 & 9 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 3 & 9 \end{array}$$

so $p_0 \perp p_1$ and $p_1 \perp p_2$.

since p_0 & p_1 are orthogonal, the projection of p_2 onto span$\{p_0, p_1\}$ is given by

$$\hat{p}_2 = \langle p_2, p_0 \rangle \frac{p_0}{\|p_0\|^2} + \langle p_2, p_1 \rangle \frac{p_1}{\|p_1\|^2}$$

but $\langle p_2, p_1 \rangle = 0$, so it is just the first term.

$$\hat{p}_2 = 20 \frac{p_0}{4} = 5 \quad \text{(the constant polynomial)}.$$

b) Use gram-schmidt. we have already done most of the work.

$$q = p_2 - \hat{p}_2 = t^2 - 5$$

[note only need orthogonal, not orthonormal].

vector of values for q:

$$\begin{array}{c|c} & q \\ \hline -3 & 4 \\ -1 & -4 \\ 1 & -4 \\ 3 & 4 \end{array}$$

so $\frac{1}{4} q$ has the desired values.

(25) $V = C[-1,1]$

let $p_0 = 1$

$p_1 = t$

$p_2 = t^2$

$$\langle p_0, p_1 \rangle = \int_{-1}^{1} t\,dt = \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0 \quad \text{so} \quad p_0 \perp p_1$$

to find q orthogonal to p_0, p_1, and $\text{span}\{p_0, p_1, q\} = \text{span}\{p_0, p_1, p_2\}$,

use G-S:

$$q = p_2 - \left[\langle p_2, p_0 \rangle \frac{p_0}{\|p_0\|^2} + \langle p_2, p_1 \rangle \frac{p_1}{\|p_1\|^2} \right]$$

let's compute the parts:

$$\langle p_2, p_0 \rangle = \int_{-1}^{1} t^2\,dt = \left[\tfrac{1}{3}t^3\right]_{-1}^{1} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

$$\langle p_2, p_1 \rangle = \int_{-1}^{1} t^3\,dt = \left[\tfrac{1}{4}t^4\right]_{-1}^{1} = 0$$

$$\|p_0\|^2 = \langle p_0, p_0 \rangle = \int_{-1}^{1} dt = [t]_{-1}^{1} = 2 \qquad \cancel{\text{so } \|p_0\| = \sqrt{2}}$$

Then $q = t^2 - \frac{2}{3}\frac{1}{2} - 0$

$= t^2 - \frac{1}{3}$

$q = t^2 - \frac{1}{3}$ (or $q = 3t^2 - 1$) is the 3rd Legendre poly.