# Conway's Tiling Groups

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December 12, 2004

**Abstract**

In this paper I discuss a method of John Conway for determining whether a given region in the plane can be tiled with certain tiles. This technique uses infinite, finitely presented groups, with the tiles interpreted as describing relators in the group. Conway finds that a plane region can be tiled if the group element describing the boundary of the region is the trivial element of the group.

Keywords: Tiling groups, tiling by tribones, tiling by diamonds, Cayley graph

## 1 Introduction

The problem of whether a given region in the plane can be tessellated using certain shapes is a well established tiling problem. John Conway developed a method that gives a group theoretic technique for solving this problem in certain cases. In this paper I will give an overview of this method as discussed by William P. Thurston in his article *Conway's Tiling Groups*. I will consider specifically the tiling analogue of the problem involving a equilateral triangle of dots.

## 2 Definitions and Background Results

### 2.1 The Cayley Graph of a Group

Given a finitely presented group $G$ with generators $g_1, g_2, ..., g_n$ we can construct a graph $\Gamma(G)$. Let each element of $G$ be a vertex of $\Gamma(G)$, and let the generators represent the directed edges of the graph. Then each vertex will have $n$ outgoing edges and $n$ incoming edges. Note that if a generator $g_j$ has order 2, then we replace the two directed edges connecting $v$ and $vg_j$ with a single undirected edge.

Note that for every element $g$ of $G$, the map $v \to gv$ is an automorphism of the group. Furthermore, this property characterises the graphs of groups: a graph whose edges are labeled by the set $L$ such that every vertex has exactly one incoming and one outgoing edge with each label is the graph of a group if and only if the graph admits an automorphism taking any vertex to any other.

Suppose now that our group $G$ is given by the finite presentation,

$$G =< g_1, g_2, ..., g_n | R_1 = R_2 = ... = R_k = 1 >$$

where $R_i$ is a relator for the group. Then in the graph $\Gamma(G)$, if we start at the vertex $v$ and trace out the relator $R_i$, we will follow a loop that ends back at the starting vertex, $v$. In this way, $\Gamma(G)$ can be extended to a 2-complex $\Gamma^2(G)$ be sewing $k$ disks to each vertex of the graph so that the boundary of each disk traces out the relator $R_i$. This 2-complex is simply connected, meaning that every loop in $\Gamma^2(G)$ is contractible to a point.

### 2.2 Example: $S_4$

We can write the familiar group $S_4$ as follows:

$$S_4 =< a, b, c | a^2 = b^2 = c^2 = 1, (ab)^3 = (bc)^3 = (ac)^2 = 1 >$$

where $a = (12)$, $b = (23)$, and $c = (34)$. We can view the graph of this group as an assembly of elements generated from its relators. That is, consider the shapes generated by the $(ab)^3 = 1$, $(bc)^3 = 1$, and $(ac)^2 = 1$. Notice that the hexagons generated by $(ab)^3 = 1$ and $(bc)^3 = 1$ represent the $S_3$ subgroups of $S_4$, while the square generated by the relator $(ac)^2 = 1$ yields the subgroup of $S_4$ given by $\mathbb{Z}_2 \times \mathbb{Z}_2$. These polygons are shown in Figure 1. We can begin the construction of the Cayley graph of the group by attaching one copy of each shape to a single vertex, matching the $a$, $b$ and $c$ edges. We then proceed by attaching copies of the three polygons along the resulting shape, to ultimately yield a truncated octahedron. (See Figure 2)

## 3 Tiling Groups

### 3.1 Tiling by Diamonds

To discuss Conway's conception of tiling groups, consider the example of a tiling by diamonds (lozenges). First take the triangulation of the plane by

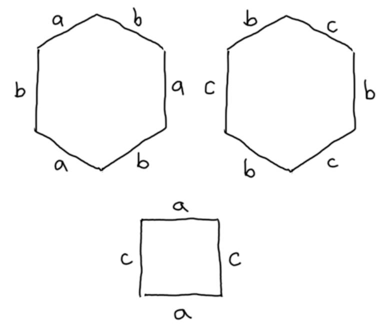


Figure 1: Relators of $S_4$: $(ab)^3 = 1$, $(bc)^3 = 1$, and $(ac)^2 = 1$

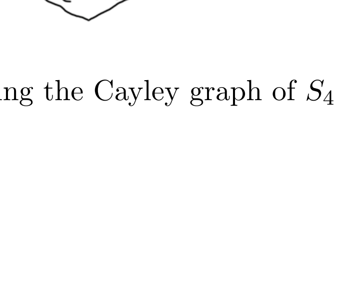

Figure 2: Building the Cayley graph of $S_4$ using the relators

equilateral triangles, and suppose that we are given a region $R$ bounded by the polygon $\pi$ with edges along sides of some triangles in the plane. Define two adjacent triangles to be a diamond. In addition, establish the labeling convention illustrated below:

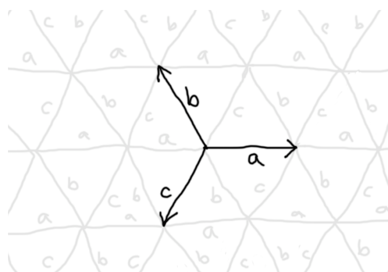


Figure 3: triangulation of the plane with labels

The triangulation of the plane with these labels defines the graph of a group, $A$. This group can be written $A =< a, b, c \mid abc = bca = 1 >$, or assembling these relators, $A =< a, b, c \mid ab = ba >$. From this presentation, it is evident that $A = \mathbb{Z} + \mathbb{Z}$, which is also clear from it's action on the plane. Now we can express the polygon $\pi$ as a word in the generators of A. Thurston suggests, however, that we instead consider $\pi$ as an element $\alpha(\pi)$ in the free group $F$ with generators $a, b, c$. If the polygon $\pi$ is closed, then we know that the homomorphism $h : F \to A$ must map $\alpha(\pi)$ to the 1, the identity of $A$. (And clearly $\pi$ must be closed for the region to be potentially tilable). Given the above labeling of the plane, we see that a diamond will be defined by one of the following:

$$D_1 = aba^{-1}b^{-1}, \quad D_2 = bcb^{-1}c^{-1}, \quad D_3 = cac^{-1}a^{-1}$$

We can thus define the diamond group: $D =< a, b, c \mid D_1 = D_2 = D_3 = 1 >$. Note that the three generators commute, hence $D = \mathbb{Z}^3$. We have the following claim:

**Proposition 1** *If a region $R$ can be tiled by diamonds, then the image $I(\pi)$ of $\alpha(\pi)$ in $D$ must be trivial*

**Proof.** Suppose we have such a region $R$. If $R$ consists of a single tile then we are done. Otherwise, take a simple arc in $R$ which cuts the region into 2 tiled subregions, $R_1$ and $R_2$ with boundaries $\pi_1$ and $\pi_2$ respectively. As an inductive hypothesis, assume $I(\pi_1) = I(\pi_2) = 1$. But then $I(\pi) = I(\pi_1) * I(\pi_2)$, so $I(\pi) = 1$, as desired. ∎

### 3.2 Tiling by Tribones

The theory of tiling groups developed by Conway has some interesting applications. In particular, it can be used to solve the problem of whether a equilateral triangular array of dots admits a covering by lines which cover three dots linearly. (Figure 4)

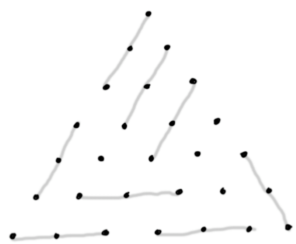

Figure 4: triangular array of dots

This is equivalent to the problem of whether it is possible to tessellate the triangular array of hexagons by tribones: three hexagons hooked together linearly. (Figure 5)

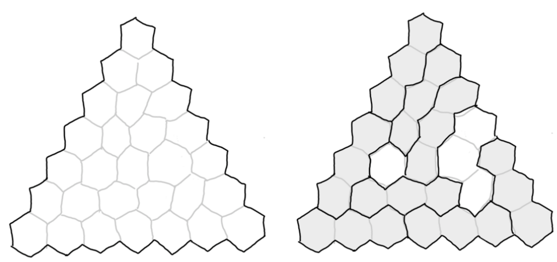

Figure 5: triangular array of hexagons, and partial tiling by tribones

Conway used his notion of tiling groups to demonstrate that this problem has no solution. Suppose the triangular region has side length $n$. Then clearly the total number of hexagons in the region will be $n(n+1)/2$. For a tiling by tribones, we require that $n$ or $n+1$ be congruent to $0(mod 3)$. Hence we require that $n \equiv 0(mod 3)$ or $n \equiv 2(mod 3)$. Now by labeling the 1-skeleton of the grid, we can create the structure of the group graph. Label as below:

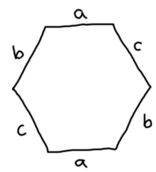


Figure 6: labels

We can now describe the group presentation of this group, say $A$:

$$A =< a, b, c \mid a^2 = b^2 = c^2 = (abc)^2 = 1 >$$

$A$ is the group of isometries of the plane given by 180° revolutions about the centres of the edges, and 180° revolutions about the centres of the hexagons themselves. Suppose $\pi$ is a path in the grid. We can then describe it as a word in the generators of $A$, or an element $\alpha(\pi)$ in the free product $F = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. There are three possible orientations for tribones in our grid structure:

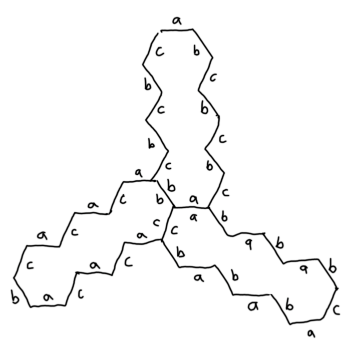

Figure 7: Possible orientations for tribones

Let

$$T_1 = (ab)^3 c (ab)^3 c$$
$$T_2 = (bc)^3 a (bc)^3 a$$
$$T_3 = (ca)^3 b (ca)^3 b$$

and we obtain the group presentation of the tribone group

$$T =< a, b, c \mid a^2 = b^2 = c^2 = T_1 = T_2 = T_3 = 1 >$$

**Proposition 2** *Let $\pi$ be a simple closed circuit in the plane such that the region $R$ bounded by $\pi$ can be tiled by tribones if the image $I(\pi)$ of $\alpha(\pi)$ in the tribone group $T$ is be trivial.*

Notice that if $T_1 = 1$, then $1 = (ab)^3c(ab)^3c$, which implies that $[(ab)^3]^{-1} = c(ab)^3c$. That is, $c$ conjugates $(ab)^3$ to its inverse. Similarly, $a$ and $b$ also conjugate $(ab)^3$ to its inverse, since $a(ab)^3a = a(ababab)a = bababa = (ba)^3 = [(ab)^3]^{-1}$. Hence $(ab)^3$ generates a normal subgroup that commutes with every word of even length. Similarly, $(bc)^3$ and $(ac)^3$ also generate normal subgroups. So then

$$J =< (ab)^3, (bc)^3, (ac)^3 >$$

defines a normal abelian subgroup of $T$. Consider the quotient group,

$$T_0 = T/J =< a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^3 = 1 >$$

Part of the Cayley Graph of $T_0$, $\Gamma(T_0)$is shown below.

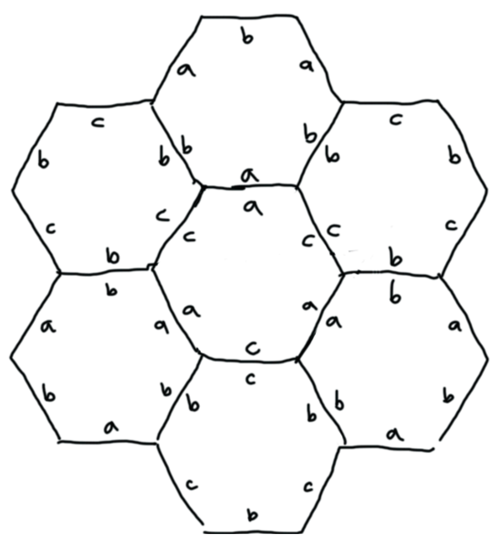

Figure 8: Part of the hexagonal grid that is the Cayley Graph of $T_0$

$\Gamma(T_0)$ is the group of isometries of the plane generated by reflections in the edges of the hexagonal tiling. Interestingly, the Cayley graph of $T_0$ is isomorphic to the graph of $A$ modulo labels. According to our claim, a region $R$ can be tiled by tribones if $\alpha(\pi)$ maps to the trivial element of $T$, hence it must also map to the trivial element of $T_0$. Recall that our region in $A$ can be described as $\alpha(\pi) = (ab)^n(ca)^n(bc)^n$. Clearly, if $n$ is a multiple of 3, then its image $I(\pi)/J$ in $T_0$ is trivial. In the other case, that $n \equiv 2(mod 3)$, we can show that its image is also trivial. Consider the triangular array of hexagons with side length $3k + 2$. We can add a row of tribones to one side of the triangle to yield a triangular array of side length $3(k + 1)$. So this case reduces to the previous case, and $I(\pi)$ is the same for both cases. So $T_0$ is not sufficient to detect the nontriviality of $I(\pi)$.

To finish the problem, we return to the full group $T$. To begin, we wish to consider the image of the tribones in $T_0$. Consider the path in the graph of $T_0$ determined by the relator $T_1$. Starting at a vertex $*$, we traverse the

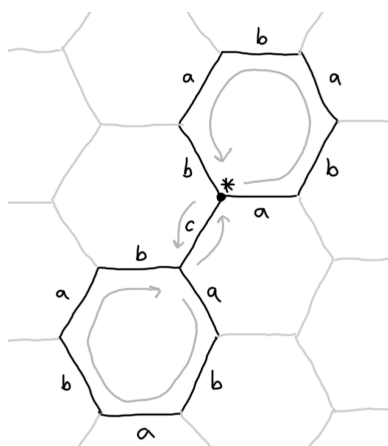


Figure 9: The image of a tribone in $T_0$

circuit *ababab* counterclockwise around a hexagon, then along the path $c$ followed by the traversal of the circuit *ababab* clockwise around the second hexagon, and finally back along the path $c$ to return to $*$ (Figure 9). It is clear that the signed total of enclosed hexagons is 0, counted according to the degree of winding around the hexagons.

Notice now that we can describe $T$ as an extension of the form, $D = \mathbb{Z}^3 \rightarrow T \rightarrow T_0$. We now may interpret an element of $T$ to be a vertex $v$ in $T_0$, with an associated path $p$ from $*$ to $v$, subject to the following equivalence relation: If $q$ is another path from $*$ to $v$, then $p \sim q$ if the signed totals of the three types of hexagons in $T_0$, $C_1, C_2, C_3$ are all 0, where $C_1 = (ab)^3$, $C_2 = (bc)^3$, and $C_3 = (ac)^3$. Given this definition, we can see that the tribone relations $T_1, T_2$ and $T_3$ are satisfied. Hence a group constructed in this fashion is at least a quotient group of T. Recall that the kernel $J$ of the map $T \to T_0$ is abelian, and generated by the relations $C_1, C_2$ and $C_3$. But in our construction we have that the kernel is the free abelian group on the generators $C_i$, so our group must in fact be $T$.

Now that we have constructed $T$, we can easily read $I(\pi)$ by inspection. We have reduced our problem to the question of $n = 3k$. Recall that the image of $\pi$ in $\Gamma(T)$ is given by $(ab)^n(bc)^n(ac)^n$. When $n = 3k$ this becomes $((ab)^3)^k((bc)^3)^k((ac)^3)^k$ which is just ${C_1}^k{C_2}^k{C_3}^k$. This is clearly not trivial in $T$. Hence the tiling is impossible, and any triangular array of hexagons cannot be tiled by tribones.

### 3.3 Further Results

One question that Thurston asks in this paper is whether this method can be used to find a lower bound on the number of holes left by partial tiling of the region by tribones. Thurston describes a method of considering this problem, and conjectures that if there is a single hole, it must be in the exact centre of the triangle.

Thurston also gives a detailed description of several other applications of Conway's tiling groups. He considers a problem similar tothe tribone tiling described above: Whether a triangular array of dots with $N$ dots on each side can be subdivided into disjoint triangular arrays of dots with $M$ dots on each side. Like the previous problem, this question can be posed in terms of tilings by hexagons. Thurston presents a theorem of Conway:

**Theorem 3** *When $N > M > 2$, the triangular array $T_N$ of hexagons cannot be tiled by $T_M$'s.*

In other words, the only possible solutions to such a problem will be a tiling by $T_2$'s, that is, blocks consisting of three hexagons (or three dots). A second theorem of Conway gives the following interesting result:

**Theorem 4** *A triangular array $T_k$ of hexagons can be tiled by $T_2$'s if and only if $k$ is congruent to $0, 2, 9$ or $11$ modulo $12$.*

## 4 Discussion and Conclusion

In conclusion, it is clear that the idea of Conway's tiling groups can be a useful tool in solving tiling problems that may be resistant to other forms of analysis.

I am interested in applying the idea of Conway's tiling groups to nonperiodic or quasiperiodic tilings, such as Penrose tiles. How would one characterize the Penrose rhombs in terms of relations in a group? Furthermore, could a Penrose tiling group determine whether a certain region was tilable? Perhaps more interestingly, could this technique shed light on the question of whether errors will occur in a given region of such tilings?

## References


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[2] William P. Thurston, Conway's Tiling Groups in *Amer. Math Monthly*, **97** (1990), 757–773.