MA 2073 HOMEWORK 6

6.4

(4) $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$. Find e-vals:

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{bmatrix} = (-2-\lambda)(7-\lambda) - 36$$
$$= \lambda^2 - 5\lambda - 14 - 36$$
$$= \lambda^2 - 5\lambda - 50$$
$$= (\lambda - 10)(\lambda + 5)$$

so $\lambda_1 = 10$, $\lambda_2 = -5$.

$$\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$$

e-vect corr to λ_1: null space of $\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{x}_1$

λ_2: " " $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{x}_2$

normalizing, $\|\vec{x}_1\| = \sqrt{5} = \|\vec{x}_2\|$ so

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}.$$

(11) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ e-vals: $\det \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1$

$$= \lambda^2 - 6\lambda + 8$$
$$= (\lambda - 2)(\lambda - 4)$$

$\lambda_1 = 2$ $\lambda_2 = 4$

e-vects: nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, normalized $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

" " $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, normalized $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ $Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\lambda_1 \vec{x}_1 \vec{x}_1^T + \lambda_2 \vec{x}_2 \vec{x}_2^T$

$$\vec{x}_1 \vec{x}_1^T = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\vec{x}_2 \vec{x}_2^T = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

hence $A = 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} + 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$

same idea for matrix B.

$$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0 \begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25 \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$$

(14) orthogonal.

trace $M = 0$

we know $\det(M) = 1$, since M orthogonal. Hence $\lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$

and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

The only possibility is $\lambda_1 = \lambda_2 = i$
$\lambda_3 = \lambda_4 = -i$

then trace $= 0 = 2i + 2(-i)$

and $\det = 1 = (i^2)(-i)^2 = (-1)(-1) = 1.$

(16) a) $B \begin{bmatrix} y \\ -z \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \vec{y} \\ -\vec{z} \end{bmatrix} = \begin{bmatrix} -A\vec{z} \\ A^T \vec{y} \end{bmatrix} = \begin{bmatrix} -\lambda \vec{y} \\ \lambda \vec{z} \end{bmatrix} = -\lambda \begin{bmatrix} \vec{y} \\ -\vec{z} \end{bmatrix}$

b) $A^T A \vec{z} = A^T \lambda \vec{y} = \lambda A^T \vec{y} = \lambda^2 \vec{z}.$

$\Rightarrow \lambda^2$ is an e-val of $A^T A$.

c) $B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (when $A = I$, $A = A^T$).

$\det(B - \lambda I) = (\lambda^2 - 1)(\lambda^2 - 1)$

$\lambda = \pm 1$, each $\bar{w}$ multiplicity 2.

we have $I\vec{z} = \vec{y}$ and $I\vec{y} = -\vec{z}$

so combinations: $(1,0,1,0)$
$(0,1,0,1)$
$(1,0,-1,0)$
$(0,1,0,-1)$

(21) a) False $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ has real e-vals & evects

b) True. Every such matrix can be written

$A = Q\Lambda Q^T$ which is symmetric.

(since $A^T = (Q\Lambda Q^T)^T = Q\Lambda Q^T$.)

c) True. $A^{-1} = (Q\Lambda Q^T)^{-1} = (Q^T)^{-1}\Lambda^{-1}Q^{-1}$

$= Q\Lambda^{-1}Q^T$ (recall $Q^T = Q^{-1}$ and Λ has an inverse).

d) False. Q does not always equal Q^T.

(23) A = invertible, orthogonal, permutation, diagonalizable, markov.

B = projection (symmetric & $B^2 = B$), diagonalizable, markow

A allows: $S\Lambda S^{-1}$ and $Q\Lambda Q^T$

B allows: $S\Lambda S^{-1}$ and $Q\Lambda Q^T$

8.3

(4) A^T always has evect $(1,1,1,1)$.

since colums of A sum to 1.

(5) we need to find e-vect of A corresponding to e-val $\lambda = 1$.

nullspace of $\begin{bmatrix} -0.02 & .00 & 0 \\ .02 & -.03 & 0 \\ .00 & .03 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -0.02 & .00 & 0 \\ .00 & .03 & 0 \\ .00 & .00 & 0 \end{bmatrix}$

So e-vect is $(0, 0, 1)$

(EVERYONE IS DEAD!).

(9) Let M be markov.

certainly the entries of M^2 are still positive.

Note $(1, 1, \ldots, 1) M = (1, 1, \cdots 1)$. b/c col sums are 1.

multiplying on the right by M gives

$(1, 1, \ldots, 1) M^2 = (1, 1, \ldots, 1)$

hence col sums of M^2 are 1 too.

6.6. (1) $B = GCG^{-1}$

$= GF^{-1} A F G^{-1}$

hence $M = GF^{-1}$ (and then $M^{-1} = FG^{-1}$)

(4) Since A has distict e-vals it can be diagonalized.

hence it is similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Since all 2x2's with these e-vals are similar to Λ, they are all similar to each other by (1)

(12) $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$

If $JM = MK$ then

$m_{21} = m_{22} = m_{23} = m_{24} = 0$ $\rightsquigarrow$ a row of zeros $\rightarrow$

hence $\det M = 0$, not invertible.

(13) J and K from prob 12 plus

$$\begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & 1 \\ 0 & & & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 0 \\ & & 0 & 0 \end{array}\right]$$ and zero matrix.

(17) a) FALSE :

If A is a nonsymmetric diagonalizible matrix, it can be written as $S \Lambda S^{-1}$, hence is Similar to Symmetric Λ

b) True : A singular matrix has $\lambda = 0$

c) False : $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ have e-vals ± 1 and are similar.

d) TRUE : The e-vals of $A + I$ are the evals of A plus 1. $(\lambda_i + 1)$.

so they are similar to different diagonal matrices.

(18) $AB = (B^{-1}B)AB = B^{-1}(BA)B$ So BA is similar to AB.

(another way: if $AB\vec{x} = \lambda\vec{x}$ then $B(AB\vec{x}) =$ ~~[illegible]~~

$$= B\lambda\vec{x}$$

or $(BA)(B\vec{x}) = \lambda(B\vec{x})$

so λ is an e-val of BA with e-vect $B\vec{x}$.

(24) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$

both have e-vals 2 and $\frac{1}{2}$.

ADDITIONAL PROBLEMS

① Claim: M is column stochastic

$$M = (1-p)\cdot A + pB$$

note A & B are column stochastic.

hence all the entries of M are positive ($1>p>0$, $1-p>0$).

The column sum of M, for column j:

$$\sum_{i=1}^{n} M_{ij} = \sum_{i=1}^{n}\left[(1-p)A_{ij} + pB_{ij}\right]$$

$$= (1-p)\sum_{i=1}^{n} A_{ij} + p\sum_{i=1}^{n} B_{ij}$$

$$= (1-p)\cdot 1 + p\cdot 1$$

$$= 1$$

② Dangling Nodes.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$M = \begin{bmatrix} p/3 & p/3 & p/3 \\ p/3 & p/3 & p/3 \\ (1-p)+p/3 & (1-p)+p/3 & p/3 \end{bmatrix} \quad \text{use } p = 0.15 \quad M = \begin{bmatrix} .05 & .05 & .05 \\ .05 & .05 & .05 \\ .9 & .9 & .05 \end{bmatrix}$$

The eigenvector of M corresponding to the eigenvalue 1

is the nullspace of

$$M - I.$$

BUT actually, A wasn't markov, so

M isn't either $\Rightarrow$ not necessarily eval 1.

and $M^n \to 0$ matrix as $n \to \infty$.

So the problem of dangling nodes persists.