MA 2073. HOMEWORK 4
DUE: FRIDAY FEBRUARY 14

Problems from text:

Section 4.2: 13, 16, 17, 22

Section 4.3: 1, 2, 3, 9, 10

Section 4.4: 15, 20, 21, 24

~~*Section 8.5:* 1, 10, 12~~

Additional problems:

Questions 1 – 3 will prove that for an $m \times n$ matrix A, the matrix $A^T A$ is invertible if and only if A has linearly independent columns.

1. Let A be an $m \times n$ matrix. Use the steps below to show that a vector $\mathbf{x}$ in $\mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$ if and only if $A^T A\mathbf{x} = \mathbf{0}$.
   a) Show that if $A\mathbf{x} = \mathbf{0}$, then $A^T A\mathbf{x} = \mathbf{0}$.
   b) Suppose that $A^T A\mathbf{x} = \mathbf{0}$. Explain why $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{0}$, and use this to show that $A\mathbf{x} = \mathbf{0}$.

2. Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Show that the columns of A are linearly independent. (*N.B.* You may not assume A is invertible – it may not even be square!).

3. Let A be an $m \times n$ matrix whose columns are linearly independent.
   a) Use Question 1 to show that $A^T A$ is an invertible matrix.
   b) Explain why A must have at least as many rows as columns.
   c) Determine the rank of A.

4. Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Prove that $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that
$$||\mathbf{y} - \hat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}||$$
for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.

5. Prove the following lemma about the inner product: If $||\mathbf{u}|| = ||\mathbf{v}|| = 1$, then $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq 1$.