MA 1024 D07 Midterm

April 8, 2014

Elissa Ross, WPI

Name: SOLUTIONS !

Student Number: ______________________

You have 50 minutes to complete this exam.

Examination rules and instructions:

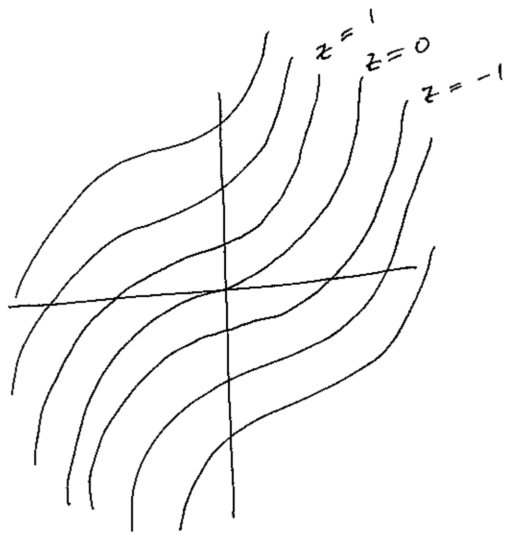
- closed book
- no calculators or cell phones are permitted

Question	Mark
1	
2	
3	
4	
5	
6	
Total	/40

Question 1 (5 marks). Describe the domain, and sketch some typical level curves of the surface described by:

$$f(x, y) = y - x^3.$$

domain: $\mathbb{R}^2$



Question 2 (5 marks). Find $\partial z/\partial x$ if $z = f(x, y)$ satisfies:

$$x^5 + xy^2 + yz = 5.$$

let $F(x, y, z) = x^5 + xy^2 + yz - 5 = 0$

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} \qquad \frac{\partial F}{\partial x} = 5x^4 + y^2$$

$$\frac{\partial F}{\partial z} = y.$$

so. $\dfrac{\partial z}{\partial x} = \dfrac{-(5x^4 + y^2)}{y}$.

Question 3 (10 marks). Find the maximum and minimum values attained by $f(x, y) = xy^2$ on the plane region R, where R is the circular disk $x^2 + y^2 \leq 3$.

$f_x(x,y) = y^2$ so $f_x = 0 \iff y = 0$

$f_y(x,y) = 2xy.$ and $f_y = 0 \iff$ either $x = 0$ or $y = 0$.

There is a line of critical points on the x-axis. (where $y=0$).

when $\boxed{y=0 \quad f(x,0) = 0}$

Now consider behaviour on boundary:

let $y^2 = 3 - x^2$

$g(x) = x(3-x^2)$ for $-\sqrt{3} \leq x \leq \sqrt{3}.$

find max/min of this function

$g'(x) = 3 - 3x^2$

$g'(x) = 0 \iff x = \pm 1$

when $x = 1$, $g(x) = 1(2) = 2$

$x = -1$, $g(x) = -1(2) = -2.$

finally, at the endpoints:

$g(-\sqrt{3}) = -\sqrt{3}(0) = 0$

$g(\sqrt{3}) = \sqrt{3}(0) = 0.$

Comparing boxed values, we find max occurs at $x=1$ $(y = \pm\sqrt{2})$
min occurs at $x=-1$ $(y=\pm\sqrt{2})$.

Question 4 (6 marks). Use differentials to approximate the number $(\sqrt{15}+\sqrt{99})^2$. *Hint:* what square integers are 15 and 99 "close to"?

$$f(x+\Delta x,\ y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

let $f(x,y) = \left(\sqrt{x}+\sqrt{y}\right)^2$

$(x,y) = (16, 100)$

$\Delta x = -1$

$\Delta y = -1.$

$$\frac{\partial f}{\partial x} = 2\left(\sqrt{x}+\sqrt{y}\right)\frac{1}{2}x^{-\frac{1}{2}} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$$

$$\frac{\partial f}{\partial y} = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{y}}.$$

$$f(15,99) \approx f(16,100) + \frac{\partial f}{\partial x}(16,100)\,\Delta x + \frac{\partial f}{\partial y}(16,100)\,\Delta y$$

$$f(16,100) = (4+10)^2 = 14^2$$

so $f(15,99) = 14^2 + \frac{14}{4}(-1) + \frac{14}{10}(-1)$

$$= 196 - \frac{140}{40} - \frac{56}{40}$$

$$= 196 - \frac{196}{40}$$

$$= \frac{1911}{10}.$$

Question 5 (7 marks). Find the directional derivative of f at P in the direction of $\mathbf{v}$:

$$f(x,y) = x^3 - x^2y + xy^2 + y^3, \quad P(1,-1), \quad \mathbf{v} = \langle 2,3 \rangle.$$

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \qquad |\vec{v}| = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle 3x^2 - 2xy + y^2, -x^2 + 2xy + 3y^2 \right\rangle$$

$$\nabla f(1,-1) = \langle 3+2+1, -1-2+3 \rangle = \langle 6, 0 \rangle$$

so $$D_{\vec{u}} f(x,y) = \langle 6, 0 \rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$$= \frac{12}{\sqrt{13}}$$

Question 6 (7 marks). Find and classify the critical points of the following function using the second derivative test.

$$f(x, y) = 3xy - x^3 - y^3.$$

~~See D03 midterm~~

$f_x(x,y) = 3y - 3x^2$

$f_y(x,y) = 3x - 3y^2$

$A = f_{xx} = -6x$

$B = f_{xy} = 3$

$C = f_{yy} = -6y$

$f_x = 0 \Rightarrow 3y = 3x^2$

$y = x^2$

then $f_y = 0 \Rightarrow 3x - 3(x^2)^2 = 0$

$3x(1 - x^3) = 0$

$\Rightarrow x = 0$ or $x = 1$

$y = 0$ $\quad$ $y = 1$

	A	B	C	$\triangle$	TYPE
(0, 0)	0	3	0	−9	saddle
(1, 1)	−6	3	−6	27	local max

This exam has 6 pages (including the cover sheet) and 6 problems, worth a total of 40 marks.