MA 2073 HOMEWORK 1 SOLUTIONS

2.1 (5) If x, y, z satisfy the 1st two eqns, then they also satisfy the 3rd eqn, since it is a linear combination of the first two.

To find sol$^{\underline{n}}$s, use elimination

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \end{aligned} \quad \xrightarrow{R2-R1} \quad \begin{aligned} x + y + z &= 2 \\ y &= 1 \end{aligned}$$

Back substituting $y = 1$ into eqn 1, we see the line of sol$^{\underline{n}}$s is given by $x + z = 1$.

3 pts on the line, for example:

$(1, 1, 0)$, $(0, 1, 1)$, $(\frac{1}{2}, 1, \frac{1}{2})$ etc.

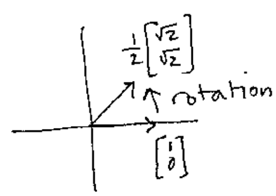
any point of the form $(x, 1, 1-x)$.

(21) Let $R = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$.

$$R\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$R\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad " \quad \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_2 \\ r_4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

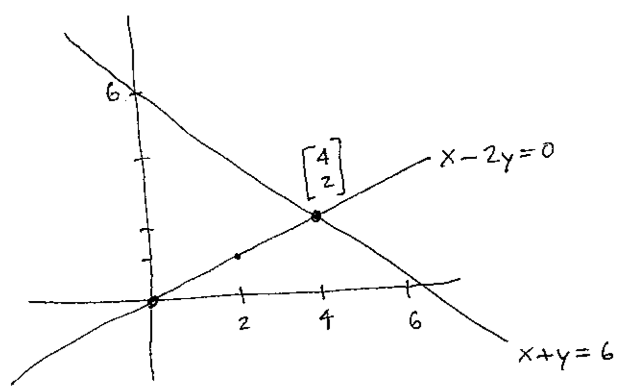
So $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$.



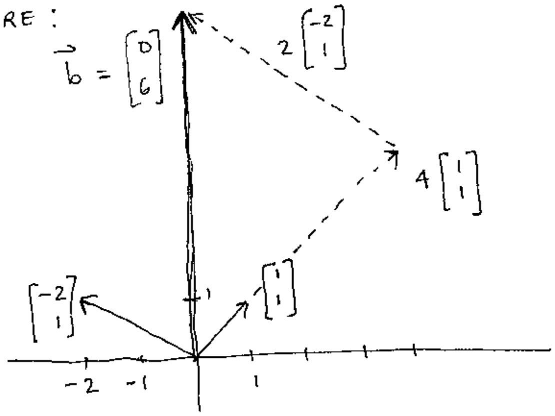
(26) $x - 2y = 0$

$x + y = 6$

ROW PICTURE:



COLUMN PICTURE :



(27) For two eqns in 3 unknowns, x, y, z, the row picture will show TWO planes in 3-dimensional space. The column picture is in 2-dimensional space. The sol$^{\underline{n}}$s normally lie on a line. (If one eqns is a scalar multiple of the other, the sol$^{\underline{n}}$s will lie on a plane).

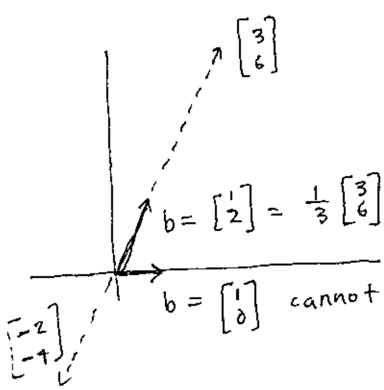
2.2 (7) when $a = 2$, the two lines are parallel $\Rightarrow$ NO SOL$^{\underline{n}}$.

when $a = 0$, elimination fails, but can be resolved by reordering the rows:

$$4x + 6y = 6$$
$$3y = -3$$

so back substitution gives $y = -1$, $x = 3$.

(9) Note that, on the left side, $6x - 4y$ is 2 times $3x - 2y$. When $b_2 = 2b_1$, then both equations describe the same line (the line of solutions, there are ∞ly many sol$^{\underline{n}}$s on this line). When $b_2 \neq 2b_1$, the equations describe parallel lines $\Rightarrow$ no sol$^{\underline{n}}$ exists.



$b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ cannot be expressed as a linear combo of $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$.

(11) a) any linear combination of (x, y, z) and (X, Y, Z) is also a $sol^{\underline{n}}$.

b) Along the line containing those 2 points.

(13) subtract 2 × row 1 from row 2

" 1 × row 1 " " 3

" 2 × row 2 " " 3

solution $(x, y, z) = (3, 1, 0)$

(24) $a = 2$ or $a = 0$

(25) $a = 0$, (zero column)

$a = 4$, (equal rows)

$a = 2$, (equal columns)

2.3

(3) $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$M = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

(4) Elimination of column 4:

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$$

back subs gives $\left(\frac{1}{2}, \frac{1}{2}, -5\right)$.

(16) a) The ages of X and Y are x and y : $x - 2y = 0$
$x + y = 33$

$sol^{\underline{n}}$ $x = 22$ and $y = 11$.

b) The line $y = mx + c$ contains $x = 2, y = 5$ and $x = 3, y = 7$ when

$$\begin{aligned} 2m + c &= 5 \\ 3m + c &= 7 \end{aligned} \Rightarrow m = 2, \ c = 1.$$

(19) $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

(25) no sol$^{\underline{n}}$ because last eqn becomes $0 = 3$

change 6 to 3. Then $R1 + R2 = R3$. ∞ly many sol$^{\underline{n}}$s.

(26) $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$

so $\vec{x} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ and $x^* = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

(28) $A = AI = \underbrace{A(BC) = (AB)C}_{\text{associative.}} = IC = C.$

2.4 (7) a) T b) F e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ a & b & c \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \end{bmatrix}$

c) T (helpful to look at the product $\begin{bmatrix} -r_1- \\ -r_2- \\ -r_3- \end{bmatrix}\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$

d) F: $(AB)^2 = ABAB$
not in general the same as $AABB = A^2B^2$.

(8) $DA = \begin{bmatrix} 3a & 3b \\ 5c & 5d \end{bmatrix}$ so $3 \times R1$, $5 \times R2$

$EA = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$ both rows are R2 of A.

The columns of AD are 3(column 1 of A) and 5(column 2 of A)

The columns of EA are zero and the sum of colums 1 & 2 of A.

(11) a) $B = 4I$ b) $B = 0$ c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ d) $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

2.5 (6) a) $AB = AC$ multiply both sides by A^{-1}, since A invertible

$A^{-1}AB = A^{-1}AC$

$B = C.$

b) $AB = AC$

$A(B-C) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ let $B-C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A(B-C) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a = -c$
$b = -d$

so $B - C = \begin{bmatrix} a & b \\ -a & -b \end{bmatrix}$

any pair of matrices satisfying this relationship will do.

(7) a) In $A\vec{x} = (1,0,0)$, equation 1 + eqn 2 − eqn 3 = 0 (LHS) but the RHS = 1. $0 = 1$ ※

b) The RHS must also satisfy $b_1 + b_2 = b_3$

c) row of zeros : no 3rd pivot.

(8) a) $\vec{x} = (1,1,-1)$ satisfies $A\vec{x} = \vec{0}$.

b) after elim. cols 1 and 2 end in zeros. Then since col 3 = col 1 + col 2, so does col 3 $\Rightarrow$ no 3rd pivot.

(11) a) If $A = -B$, the $A + B =$ zero matrix, which is NOT invertible.

b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. $A + B = I$, which is invertible ($I^{-1} = I$).

(15) CLAIM: A matrix with a column of zeros cannot have an inverse.

PROOF: If A has column of zeros, then so does the product BA. Then $BA = I$ is impossible, therefore there is no A^{-1} ∎

(18) CLAIM: If B is the inverse of A^2, then AB is the inverse of A.

PROOF: $A^2 B = I$, since B is the inverse of A

$$A(AB) = I$$

$$\Rightarrow A = (AB)^{-1}.$$

2.6 (5) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $\quad EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$

$E^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ $\quad A = LU.$

(13)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{U}$$

and $L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $A = LU$ with 4 pivots if

$a \neq 0$
$a \neq b$
$c \neq b$
$d \neq c$.

2.7 (4) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$.

The diagonal of $A^T A$ has dot products of columns with themselves. If $A^T A = 0$, zero diagonals $\Rightarrow$ zero dot products

$\Rightarrow$ zero columns

$\Rightarrow A = 0$.

(7) a) F. $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^T$ (i.e A is symmetric).

b) F. The transpose of AB is $B^T A^T = BA$, since A, B symmetric.
So $(AB)^T = AB$ implies $AB = BA$, which is not true in general.

c) T. Consider the transpose of $AA^{-1} = I$:
$(AA^{-1})^T = (A^{-1})^T A^T = (A^{-1})^T A = I.$ Mult. both sides on R by A^{-1} →

$(A^{-1})^T A A^{-1} = A^{-1} \Rightarrow (A^{-1})^T = A^{-1}$, so A^{-1} is symmetric

d) T. $(ABC)^T = C^T (AB)^T = C^T B^T A^T = CBA$.

3-1 (4) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. cA is the smallest subspace containing A, for $c \in \mathbb{R}$.

(7) rule 8: Note $(c_1+c_2)f(x) = f((c_1+c_2)x)$, which is not in general the same as $f(c_1x) + f(c_2x). = c_1 f(x) + c_2 f(x)$.

(10) a, d, e

(11) a) all matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ b) all matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$, $a \in \mathbb{R}$.

$a, b \in \mathbb{R}$

c) all matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (all diagonal matrices)

(14) a) all of $\mathbb{R}^2$, lines through the origin, $\{(0,0)\}$.

b) all of D, all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, all matrices of the form $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$

all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, and $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$.

(15) a) line, plane

b) point, line

c) if $\vec{x}, \vec{y}$ are in both S and T, then since S, T are subspaces, $\vec{x}+\vec{y} \in S$ and T. So $\vec{x}+\vec{y} \in S \cap T$.

Similarly $c\vec{x}$ ~~and~~ is in both S and T, so $c\vec{x} \in S \cap T$ too.

(22) a) sol$^{\underline{n}}$s for every $\vec{b}$, b) solvable only if $b_3 = 0$, c) solvable only if $b_2 = b_3$

(23) unless $\vec{b}$ is already in the col space.

$[A \ \vec{b}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ larger col. space no sol$^{\underline{n}}$ to $A\vec{x} = \vec{b}$.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\vec{b}$ is in col space
$A\vec{x} = \vec{b}$ has a sol^{n}.

(24) $B = 0$, $A \neq 0$.

Then the col space of AB is smaller than the col space of A.

(27) a) F. $(0, \dots, 0)$ is not in this set.

b) T

c) T

d) F. Let $A = I$. Then $\text{col}\{A - I\} = 0$, but $\text{col}\{A\} = \mathbb{R}^2$.

Additional probs:

① First note $\vec{0} \in H$, since $0\vec{v} \in \text{span}\{\vec{v}, \vec{w}\}$.

Consider two elements of $\text{span}\{\vec{v}, \vec{w}\}$, $c_1\vec{v} + c_2\vec{w}$ and $d_1\vec{v} + d_2\vec{w}$.

Then $(c_1\vec{v} + c_2\vec{w}) + (d_1\vec{v} + d_2\vec{w}) = (c_1 + d_1)\vec{v} + (c_2 + d_2)\vec{w}$, which is also in the span of $\vec{v}$ and $\vec{w}$.

Finally, consider $c(c_1\vec{v} + c_2\vec{w}) = cc_1\vec{v} + cc_2\vec{w}$, which is also in $\text{span}\{\vec{v}, \vec{w}\}$.

Hence $\text{span}\{\vec{v}, \vec{w}\}$ meets the requirements of a subspace.

② Since K contains $\vec{v}$ and $\vec{w}$, by definition of a subspace K contains all linear combinations $c_1\vec{v} + c_2\vec{w}$, which is precisely the definition of the span of $\vec{v}$ and $\vec{w}$. So K contains the span of $\vec{v}$ and $\vec{w}$.