MA 1024 D03 Midterm

April 8, 2014

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Name: SOLUTIONS !

Student Number: ____________________

You have 50 minutes to complete this exam.

Examination rules and instructions:

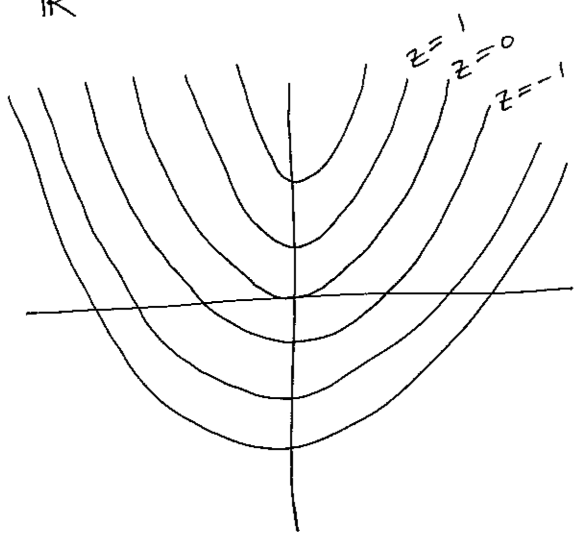
- closed book
- no calculators or cell phones are permitted

Question	Mark
1	
2	
3	
4	
5	
6	
Total	/40

Question 1 (5 marks). Describe the domain and sketch some typical level curves of the surface described by:

$$f(x, y) = y - x^2.$$

Domain: $\mathbb{R}^2$



Question 2 (5 marks). Find $\partial z/\partial x$ if $z = f(x, y)$ satisfies:

$$x^{2/3} + y^{2/3} + z^{2/3} = 1.$$

let $F(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 1 = 0$.

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} \qquad \frac{\partial F}{\partial x} = \frac{2}{3}x^{-\frac{1}{3}} \qquad \frac{\partial F}{\partial z} = \frac{2}{3}z^{-\frac{1}{3}}$$

$$\text{so} \quad \frac{\partial z}{\partial x} = \frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}z^{-\frac{1}{3}}} = -\sqrt[3]{\frac{z}{x}}$$

Question 3 (10 marks). Find the maximum and minimum values attained by $f(x, y) = xy^2$ on the plane region R, where R is the circular disk $x^2 + y^2 \leq 3$.

$f_x(x,y) = y^2$ so $f_x = 0 \iff y = 0$

$f_y(x,y) = 2xy.$ and $f_y = 0 \iff$ either $x = 0$ or $y = 0$.

There is a line of critical points on the x-axis. (where $y = 0$).

when $\boxed{y = 0 \quad f(x,0) = 0}$

Now consider behaviour on boundary:

let $y^2 = 3 - x^2$

$g(x) = x(3 - x^2)$ for $-\sqrt{3} \leq x \leq \sqrt{3}.$

find max/min of this function

$g'(x) = 3 - 3x^2$

$g'(x) = 0 \iff x = \pm 1$

when $x = 1$, $\boxed{g(x) = 1(2) = 2}$

$x = -1$, $\boxed{g(x) = -1(2) = -2.}$

finally, at the endpoints:

$\boxed{g(-\sqrt{3}) = -\sqrt{3}(0) = 0}$

$\boxed{g(\sqrt{3}) = \sqrt{3}(0) = 0.}$

Comparing boxed values, we find max occurs at $x = 1$ $(y = \pm\sqrt{2})$

min occurs at $x = -1$ $(y = \pm\sqrt{2})$.

Question 4 (6 marks). Use differentials to approximate the number $e^{0.4} = \exp(1.1^2 - 0.9^2)$.

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

let $f(x,y) = e^{x^2-y^2}$

$(x,y) = (1,1)$

$\Delta x = 0.1$

$\Delta y = -0.1.$

then $\frac{\partial f}{\partial x} = 2x\,e^{x^2-y^2}$ $\qquad \frac{\partial f}{\partial x} = -2y\,e^{x^2-y^2}.$

$$e^{0.4} \approx f(1,1) + \frac{\partial f}{\partial x}(1,1)\,\Delta x + \frac{\partial f}{\partial x}(1,1)\,\Delta y$$

$$= e^0 + 2e^0(0.1) + (-2e^0)(-0.1)$$

$$= 1 + 0.2 + 0.2$$

$$= 1.4$$

Question 5 (7 marks). Find the directional derivative of f at P in the direction of $\mathbf{v}$:

$$f(x,y) = \sin x \cos y, \quad P(\pi/3, -2\pi/3), \quad \mathbf{v} = \langle 4, -3 \rangle.$$

Hint:

$$\sin(\pi/3) = \sqrt{3}/2 \qquad \sin(-2\pi/3) = -\sqrt{3}/2$$
$$\cos(\pi/3) = 1/2 \qquad \cos(-2\pi/3) = -1/2.$$

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}.$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \qquad |\vec{v}| = \sqrt{4^2+3^2} = 5.$$

$$\vec{u} = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle.$$

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle \cos x \cos y, -\sin x \sin y \rangle$$

$$\nabla f\left(\frac{\pi}{3}, -\frac{2\pi}{3}\right) = \left\langle \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{-2\pi}{3}\right), -\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{-2\pi}{3}\right) \right\rangle$$

$$= \left\langle \frac{1}{2} \cdot \frac{-1}{2}, \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2} \right\rangle$$

$$= \left\langle \frac{-1}{4}, \frac{3}{4} \right\rangle$$

$$\text{so } D_{\vec{u}} f(x,y) = \left\langle -\frac{1}{4}, \frac{3}{4} \right\rangle \cdot \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$$

$$= \frac{-4}{20} \quad \frac{-9}{20}$$

$$= \frac{-13}{20}.$$

Question 6 (7 marks). Find and classify the critical points of the following function using the second derivative test.

$$f(x,y) = x^2 - 2xy + y^3 - y.$$

$f_x = 2x - 2y$

$f_y = -2x + 3y^2 - 1$

$f_x = 0 \iff x = y.$

$f_y = 0 \iff 3y^2 - 2y - 1 = 0$

so $y = \frac{2 \pm \sqrt{4+12}}{6} = \frac{2 \pm 4}{6} = 1$ or $-\frac{1}{3}$.

$A = f_{xx} = 2$

$B = f_{xy} = -2$

$C = f_{yy} = 6y$

CRIT PTS	A	B	C	Δ	TYPE
$(1,1)$	2	-2	6	8	local min
$(-\frac{1}{3}, -\frac{1}{3})$	2	-2	-2	-8	saddle.

This exam has 6 pages (including the cover sheet) and 6 problems, worth a total of 40 marks.