MA 2073 HOMEWORK 3

3.6

(3)

SPACE	BASIS
col (A)	$\{(1,1,0),(3,4,1)\}$
row (A)	$\{(0,1,2,3,4),(0,0,0,1,2)\}$ (or R1 & R2 of A)
nul (A)	$\{(1,0,0,0,0),(0,2,-1,0,0),(0,2,0,-2,1)\}$
L nul (A)	$\{(\cancel{0},-1,1)\}$

(4) a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) if nullspace has basis, $\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}$, then A must be 3×3, which is impossible.
we need $n = \#$ of columns $= \dim(\text{col}(A)) + \dim(\text{nul}(A))$.

c) $[1 \; 1]$ d) $\begin{bmatrix} 3 & 1 \\ -1 & -1/3 \end{bmatrix}$ e) impossible.

(6) ~~AB~~

space	A	B
col A	2	1
row A	2	1
nul A	2	0
L nul A	1	2

(9) a) rowspace
null space

b) column space
L nullspace.

(11) a) no sol$^{\underline{n}}$ $\Rightarrow$ $r < m$ (since we have rows of zeros).

we know $r \leq n$.

don't know about n & m

b) Since $m - r > 0$, there is something non-triv. in L nullspace.

(13) a) F $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ b) T c) Any two invertible $n\times n$ matrices have row space = colspace = $\mathbb{R}^n$
nullspace = L. null = $\{\vec{0}\}$.

(14) rowspace basis (of A): the rows of U

null space basis of A : let $x_4 = 1$

$x_3 = -2$

$x_2 = -2x_3 - 3x_4 = 4 - 3 = 1$

$x_1 = -2x_2 - 3x_3 - 4x_4 = -2 + 6 - 4 = 0$

so $\{(0, 1, -2, 1)\}$ is a basis.

colspace basis has dimension 3, same as rowspace. And it is a subspace of $\mathbb{R}^3$, it could be $\{(1,0,0), (0,1,0), (0,0,1)\}$.

L nullspace has empty set.

(24) the colspace of A^T, aka the rowspace of A.

$\vec{y}$ is unique when the L. nullspace contains only the zero vect.

(25) a) T (rank is the same.)

b) F e.g. $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$

c) F e.g $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

d) T the subspaces are the same for A & -A.

(32) The rowspaces are the same.

Therefore, each row of B can be expressed as a lin. combo of the rows of A.

consider row 1 of B. $[\underbrace{1\ 0 \cdots 0}\ G_{11}\ G_{12} \cdots G_{1n}]$

but b/c of the first columns, this can only be a multiple of the first row of A, and in fact must be equal.

repeat for each row to see that A = B.

8.2

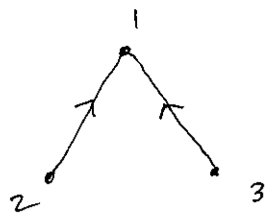
① $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ nul $A = \{k(1,1,1) : k \in \mathbb{R}\}$.

$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

② $A^T = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ left nul $A = \{k(1,-1,1) : k \in \mathbb{R}\}$.

③ $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = u$

corresponds to



④ Any vector (b_1, b_1+b_3, b_3) can be solved.

e.g $(1, 2, 1)$

but e.g $(1, 0, 0)$ has no sol^n.

The vect. $(1,-1,1)$ is in the L. nullspace of those $\vec{b}$ s.

⑤ $A^T\vec{y} = \vec{f}$ solvable for $\vec{f} = (1,-1,0)$

not " " $\vec{f} = (1,0,0)$.

This is Kirchhoff's current law.

⑮ 7 nodes ⇒ spanning tree has 6 edges.

The non-tree edges comprise the cycles ("loops")

There is only 1 loop.

⑰ a) 9 nodes ⇒ max independent rows (which gives # of independent columns) is 8.

b) $\vec{f}$ has the property that $\vec{f}^T\vec{x} = \vec{0}$ for $\vec{x} \in$ nulspace.

c) 24. Each edge goes into 2 nodes. There are 12 nodes in the graph. So diags must sum to 24.

(13) let V, W be subspaces ~~for~~ of $\mathbb{R}^n$

$$V = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_m \end{bmatrix}_{n \times m} \qquad W = \begin{bmatrix} \vec{w}_1 \cdots \vec{w}_p \end{bmatrix}_{n \times p}$$

now $V^T W$ is the $m \times p$ matrix whose entries are the dot product of $\vec{v}_i$ and $\vec{w}_j$.

If $V^T W =$ zero matrix, then all such products are zero.

$\Rightarrow$ every basis element of V is $\perp$ to every basis element of W.

Any other element of V can be written in terms of these basis elements. Therefore, $V \perp W$.

(14) We want $A\vec{x} = B\hat{x}$.

or $A\vec{x} - B\hat{x} = \vec{0}$

write $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \vec{x} \\ -\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

so we are looking for elements in the null space of $[A \ B]$. The nullspace must be non-trivial, since $[A \ B]$ is 3×4 (not every col can be a pivot col).

$$\begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \rightarrow$$

so x_3 free

$x_4 = 0$ let $x_3 = 1$ then $x_2 = -1$, $x_1 = 2 - 5 = -3$.

so $\vec{x} = (-3, -1)$ and $(-1, 0) = \hat{x}$.

(15) If $p+q > n$, then the subspaces overlap nontrivially.

(we can't have more than n basis vectors, so they must have common ones).

(18) $A = \cancel{\begin{bmatrix} 1 & 2 \\ 5 & 2 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

(21) Two vectors that span $S^{\perp}$ are the nullspace of

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

x_3, x_4 free

nullspace is spanned by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

so $\{(0,-1,1,0), (-5,1,0,1)\}$ is a basis

(23) If $\vec{x} \in V^{\perp}$, then $\vec{x}$ is $\perp$ to everything in V.

Since $S \subseteq V$, $\vec{x}$ is also $\perp$ to everything in S

$\Rightarrow \vec{x} \in S^{\perp}$.

(24) $A A^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

The dot prod. of the first row of A w/ the first column of A^{-1} is <u>nonzero</u>.

The dot prod of the second row of A w/ the first col of A^{-1} is <u>zero</u>

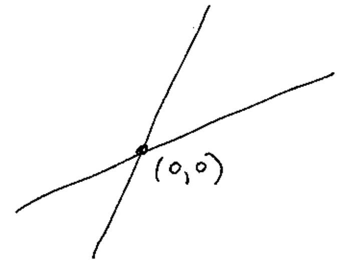
Same for $3^{rd}, \ldots, n^{th}$ rows.

(28) a) The planes are not parallel, therefore they meet in a line. Any vector on this line is not orthogonal to ~~both~~ either.

b) both are subspaces of $\mathbb{R}^5$ of dimension 2.

But a subspace of dim 2 has orthog. complement of dimension 3.

c) consider two lines in $\mathbb{R}^2$:



← not orthogonal!

(32) row space $\perp$ to nullspace $\vec{r}^{\,T}\vec{n} = 0$

col space $\perp$ to L nullspace $\vec{\ell}^{\,T}\vec{c} = \vec{0}$

The matrix $\vec{c}\,\vec{r}^{\,T}$ has this property. (As does any scalar multiple). since $\vec{\ell}^{\,T}(\vec{c}\,\vec{r}^{\,T}) = \vec{0}$ and $(\vec{c}\,\vec{r}^{\,T})\vec{n} = \vec{0}$

Additional problems.

(1) Let $\vec{x}$ be s.t $\vec{x}^{\,T}\vec{v}_j = 0$ for $1 \le j \le p$.

Any vector $\vec{w} \in W$ can be expressed as a linear combo of the $\vec{v}_j$'s.

$$\vec{w} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$$

Then
$$\begin{aligned}\vec{x}\cdot\vec{w} = \vec{x}^{\,T}\vec{w} &= \vec{x}^{\,T}(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) \\ &= c_1\vec{x}^{\,T}\vec{v}_1 + \cdots + c_p\vec{x}^{\,T}\vec{v}_p \\ &= 0 + \cdots + 0 \\ &= 0.\end{aligned}$$

hence $\vec{x}$ is orthog. to everything in W.

(2) We want to show that $W^\perp$ is a subspace of $\mathbb{R}^n$

First note $\vec{0} \in W^\perp$ since $\vec{0}\cdot\vec{x} = \vec{0}^{\,T}\vec{x} = \vec{0}$ for every vector $\vec{x} \in W$

Now let $\vec{w}_1, \vec{w}_2 \in W^\perp$. We want to show that $\vec{w}_1 + \vec{w}_2$ is also in $W^\perp$. Let $\vec{x} \in W$.

consider
$$\begin{aligned}(\vec{w}_1 + \vec{w}_2)\cdot\vec{x} = (\vec{w}_1 + \vec{w}_2)^T\cdot\vec{x} = (\vec{w}_1^{\,T} + \vec{w}_2^{\,T})\vec{x} &= \vec{w}_1^{\,T}\vec{x} + \vec{w}_2^{\,T}\vec{x} \\ &= 0 + 0\end{aligned}$$

since $\vec{w}_1, \vec{w}_2 \in W^\perp$

So $W^\perp$ is closed under vector addition.

Finally, consider $c\vec{w} \cdot \vec{x}$, where $\vec{w} \in W^{\perp}$, $\vec{x} \in W$, $c \in \mathbb{R}$.

Then $c\vec{w} \cdot \vec{x} = (c\vec{w})^T \vec{x} = c\vec{w}^T\vec{x} = c\,0 = 0$

So $W^{\perp}$ is also closed under scalar multiplication.

Since $W^{\perp}$ contains the zero vector, and is closed under vector add$^{\underline{n}}$ & scalar mult., it is a subspace of $\mathbb{R}^n$.

:)