MA 2073 HOMEWORK 2 SOLUTIONS.

3.2 (7) Row reduction on A gives $\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

The nullspace is the plane $-x+3y+5z=0$ in $\mathbb{R}^3$.

Equivalently, the null space consists of all vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+5z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Row reduction on B gives $\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

The nullspace is the intersection of the planes $-x+3y+5z=0$ and $-2x+6y+7z=0$.

This intersection is a line. given by all points of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is a point on the line.

(9) a) F, if row reduction gives rows of zeros, we have free vars (e.g. any singular matrix)

b) T, invertible $n \times n$ matrix has n pivots.

c) T, can't have more pivots than columns

d) T, can't have more pivots than rows.

(21) see also worked example 3.2A, page 139.
A could be a 2×4 matrix. Let's construct in reduced form:
The pivot cols are cols 1 & 2, the free cols are 3 & 4
(since free cols always have 0's & 1's in special sol^{n}s).

So $R = \begin{bmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & a_3 & a_4 \end{bmatrix}$ Solving $R\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} = 0$ and $R\begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$

→

gives $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$, which ~~satisfies~~ has the desired nullspace.

(so does AR, where A is any invertible 2×2 mat).

(23) A must be 3×3, Let A be

$$A = \begin{bmatrix} 1 & 0 & a_1 \\ 1 & 3 & a_2 \\ 5 & 1 & a_3 \end{bmatrix} \quad \text{and solve} \quad A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{to find } a_1, a_2, a_3.$$

$\downarrow$

row reduce $\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 3 & a_2 - a_1 \\ 0 & 1 & a_3 - 5a_1 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 0 & a_1 \\ 0 & 3 & a_2 - a_1 \\ 0 & 0 & 3a_3 - a_2 - 14a_1 \end{bmatrix}}_{R}$

$R\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $a_1 = -\frac{1}{2}$ $\quad a_2 = -2$ $\quad a_3 = -3$

So $A = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ is the desired matrix.

(27) The dimension of the null space is given by the # of free vars, but the dim. of the column space is given by the number of pivots. With 3 cols there is no way for $n - r = r$.

(30) a) $\text{nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, x \in \mathbb{R} \right\}$.

$\text{nul}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix}, y \in \mathbb{R} \right\}$.

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has x free, A^T has y free

c) A is in rref, $A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has rref $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq R^T$.

(35) Since A is invertible, the only sol$^{\underline{n}}$ to $A\vec{x} = \vec{0}$ is $\vec{x} = 0$. However, note that

$$[A \ \ A]\begin{bmatrix} \vec{y} \\ -\vec{y} \end{bmatrix} = \vec{0},$$ so the nullspace of

$[A \ \ A]$ is given by $\left\{ \begin{bmatrix} \vec{y} \\ -\vec{y} \end{bmatrix}, \vec{y} \in \mathbb{R}^4 \right\}$

(37)

① $-y_1 + y_3 - y_4 = 0$

② $y_1 - y_2 - y_5 = 0$

③ $y_2 - y_3 - y_6 = 0$

④ $y_4 + y_5 + y_6 = 0$

So $A = \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}}$ ∴

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So free vars are x_3, x_5, x_6.

special sol$^{\underline{n}}$s are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 + x_6 \\ x_3 + x_6 \\ x_3 \\ -x_5 - x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

special sol$^{\underline{n}}$s.

3.3

(8) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$ $M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}$

(10) $A = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

(17) a) recall $AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}$

so if column $\vec{b}_j = a_1 \vec{b}_1 + \cdots + a_{j-1} \vec{b}_{j-1}$ then

$A\vec{b}_j = a_1 A\vec{b}_1 + \cdots + a_{j-1} A\vec{b}_{j-1}$ is

still a combo of prev. cols of AB.

so $\text{rank}(AB) \leq \text{rank}(B)$, since we can't get new pivot columns this way.

b) $A_1 = I$, $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

(19) If $AB = I$, then $\text{rank}(AB) = \text{rank}(I) = n$.

we know $\text{rank}(A) \leq n$, since A can't have more pivots than columns. But $\text{rank}(A) \geq \text{rank}(AB) = n$

$\Rightarrow \text{rank}(A) = n$. and therefore A is invertible.

B must be its inverse, since A^{-1} is unique.

(28) $R = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ (will have cols of zeros in addition to rows of zeros).

I is $r \times r$, $r = \#$ of pivots.

note the # of pivots is unchanged.

3.4

① Augmented matrix $[A\ \vec{b}]$ reduces to

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-2b_1 \end{bmatrix}$$

$A\vec{x}=\vec{b}$ has a sol$^{\underline{n}}$ when $b_3+b_2-2b_1=0$.

col space of A is all lin combs. of cols 1 & 2, $(2,2,2)$ and $(4,5,3)$.

Note b is in the col space when $A\vec{x}=\vec{b}$ is solvable, that is, when $b_3+b_2-2b_1=0$.

Nullspace: reduce more: (with $\vec{b}$ - will need later).

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 & x_4 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3+2x_4 \\ -x_3-2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_n}.$$

complete sol$^{\underline{n}}$: $\vec{x}_p+\vec{x}_n$.

Find $\vec{x}_p$: Set $x_3=x_4=0$, $x_2=-1$, $x_1=4$

so complete sol$^{\underline{n}}$ is

$$\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

④ $\vec{x}_p+\vec{x}_n = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right) + x_2(-3,1,0,0) + x_4(0,0,-2,1)$

⑥ a) solvable if $b_2=2b_1$, $3b_1-3b_3+b_4=0$. $\Big\}$ then $\vec{x} = \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{bmatrix} = \vec{x}_p$

b) " " ~~$b_2=2b_1$~~ " then $\vec{x} = \begin{bmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

(10) let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow a+b+c = d+e+f = 0.$$

and $A \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a+4b \\ 2d+4e \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

let $a=1$ $b=0$ $\Rightarrow c=-1$
$d=0$ $e=1$ $\Rightarrow f=-1$

check that $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix}$ satisfies these reqs, with $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

(13) a) $\vec{x}_p$ is always mult. by 1.
b) $A\vec{x} = \vec{b}$ may have a line or plane (for example) of sol$^{\underline{n}}$s. Any point on that line or plane is a sol$^{\underline{n}}$.
c) $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2a \\ 2a \end{bmatrix}$. so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are both sol$^{\underline{n}}$s.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: length $\sqrt{2}$ $\qquad$ $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$: length 2.

(recall $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$).

d) $\vec{x}_n = \vec{0}$ is in the nullspace.

(16) 3, row, always exists, $\mathbb{R}^3$, $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \end{bmatrix}$.

(22) $A\vec{x} = \vec{b}$ has ∞ly many sol$^{\underline{n}}$s $\Rightarrow$ there are free vars, these don't depend on B.

If $B \notin col(A)$, then we might have no sol$^{\underline{n}}$s.

(24) a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
one sol$^{\underline{n}}$ if $b_1 = b_2$
no sol$^{\underline{n}}$ " $b_1 \neq b_2$

b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [b]$
∞ly many sol$^{\underline{n}}$s.

c) A with $r < n$, $r < m$.

d) $A = I$ (any invertible matrix).

(34) a) rank A is 3

complete sol$^{\underline{n}}$ to $A\vec{x} = \vec{0}$ is $c\begin{bmatrix}2\\3\\1\\0\end{bmatrix}$

b) x_3 is the free var (can't be x_4, since $x_4 = 0$ in).

so R has form

$$\begin{bmatrix}1 & 0 & a & 0\\0 & 1 & b & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

just need to solve for a, b from $R\begin{bmatrix}2\\3\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $a = -2$, $b = -3$.

c) full row rank! (No rows of zeros).

(36) let $A = \begin{bmatrix}\vec{a}_1 & \cdots & \vec{a}_n\end{bmatrix}$. Let $\vec{b} = \vec{a}_1$. Then $\vec{x} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ solves

$A\vec{x} = \vec{b}$. Therefore $\vec{x}$ also solves $C\vec{x} = \vec{b}$. So

$\vec{c}_1 = \vec{a}_1$, where $\vec{c}_1$ is the first column of C. Repeat

for the other columns!

3.5 (2) Note that $(1, 1, 1, 1)\, v_i = 0$, so all six vectors lie on a plane. No more than 3 can be independent!
$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are indep.

(5) put vectors as columns of a matrix, row reduce.
a) 3 pivots $\Rightarrow$ independent
b) 2 pivots $\Rightarrow$ dependent

(7) Take $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1(\vec{w}_2 - \vec{w}_3) + c_2(\vec{w}_1 - \vec{w}_3) + c_3(\vec{w}_1 - \vec{w}_2) = 0.$

let $c_1 = c_3 = 1$, $c_2 = -1$. This is a non-trivial linear comb.

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = \begin{bmatrix}\vec{w}_1 & \vec{w}_2 & \vec{w}_3\end{bmatrix}\begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}$$

so e.g $\vec{v}_1 = \vec{w}_2 - \vec{w}_3 = a\vec{w}_1 + d\vec{w}_2 + g\vec{w}_3$ so $a = 0$, $d = 1$, $g = -1$

$A = \begin{bmatrix}0 & 1 & 1\\1 & 0 & -1\\-1 & -1 & 0\end{bmatrix}$ is singular. (row reduce to get rows of zeros).

(11) a) line b) plane spanned by $(0,1,1)$ & $(1,1,0)$

c) all of $\mathbb{R}^3$ (scalar multiples give everything)

d) all of $\mathbb{R}^3$ "

(13) a) 2 b) 2 c) 2 d)

The row spaces of A and U are the same.

(16) a) $\{(1,1,1,1)\}$ b) $\{(a,b,c,d) \mid a+b+c+d=0\}$ is the space.

To find a basis let $a=1, b=0, c=0 \Rightarrow d=-1$ similarly find 3 other combinations:

$\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1)\}$ is a basis.

c) $\perp$ to the two vectors :

$(1,1,0,0)\cdot(a,b,c,d) = a+b=0 \Rightarrow a=-b$

$(1,0,1,1)\cdot(a,b,c,d) = a+c+d=0 \Rightarrow c=b-d$

so $(a,b,c,d) = (-b, b, b-d, d) = b(-1,1,1,0) + d(0,0,-1,1)$

and $\{(-1,1,1,0), (0,0,-1,1)\}$ form a basis.

d) columns of I are a basis for $\text{col}(I)$, $\text{nul}(I) = \{\vec{0}\}$, which has the empty set as a basis.

(20) a) $x = 2y-3z$ so $(x,y,z) = (2y-3z, y, z)$

$= y(2,1,0) + z(-3,0,1)$

and $\{(2,1,0), (-3,0,1)\}$ form a basis.

b) ~~The xy-plane has basis (0,0,1).~~

Note that all pts of the form $c(2,1,0)$ lie on the plane. This is precisely the plane $z=0$ (the xy-plane).

So $\{(2,1,0)\}$ is a basis.

c) The normal vector is $(1,-2,3)$ is $\perp$ to everything in the plane. $\Rightarrow$ it is a basis.

(29) a) F : when we have $n > m$, can have indep rows & dependent cols e.g $A = \begin{bmatrix} I & B \end{bmatrix}$

b) F: consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

c) T

d) F : if we have dependent cols, they are not independent.

(30) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & -2a-b \\ d & e & -2d-e \end{bmatrix} = a \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ d \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

These 4 mats. are a basis.

(35) $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ ← general form

so $\{1, x, x^2, x^3\}$ is a basis

when $p(1) = 0$, we have $a_0 + a_1 + a_2 + a_3 = 0$.

so $a_0 + a_1 x + a_2 x^2 + (-a_0 - a_1 - a_2) x^3 = a_0 (1-x^3) + a_1 (x - x^3) + a_2 (x^2 - x^3)$

$\Rightarrow \{1-x^3, x-x^3, x^2-x^3\}$ is one basis.

(43) Consider a lin combo:

$$\underbrace{a_1 \vec{u}_1 + \cdots + a_r \vec{u}_r}_{V \cap W} + \underbrace{b_1 \vec{v}_1 + \cdots + b_s \vec{v}_s}_{V} + \underbrace{c_1 \vec{w}_1 + \cdots + c_t \vec{w}_t}_{W.} = \vec{0}. \quad (*)$$

want to prove: all coeffs must be zero.

let $\vec{x} = a_1 \vec{u}_1 + \dots + a_r \vec{u}_r + b_1 \vec{v}_1 + \dots + b_s \vec{v}_s$.

Note that $\vec{x}$ is in V.

This means that $c_1 \vec{w}_1 + \dots + c_t \vec{w}_t = -\vec{x}$, from ⊛

Therefore $\vec{x}$ is in W too.

$\vec{x} \in V, W \Rightarrow \vec{x} \in V \cap W$.

$\Rightarrow \vec{x}$ can be written using only the v_i's.

e.g ⊛ becomes

$$a_1 \vec{u}_1 + \dots + a_r \vec{u}_r + b_1 \vec{v}_1 + \dots + b_s \vec{v}_s + \underbrace{d_1 \vec{u}_1 + \dots + d_r \vec{u}_r}_{-\vec{x} \text{ expressed in terms of the } u_i\text{'s.}} = \vec{0}$$

But now this is a linear comb. of the $\vec{u}_i$'s and the $\vec{v}_j$'s, so it is an element of V, and we know the u_i's & the v_i's are indep.

$\Rightarrow$ all coeffs $= 0$

$\Rightarrow \vec{x} = \vec{0}$

$\Rightarrow -\vec{x} = \vec{0} = c_1 \vec{w}_1 + \dots + c_t \vec{w}_t$

but $\vec{w}_1, \dots, \vec{w}_t$ are lin indep $\Rightarrow c_1 = \dots = c_t = 0$ too,

which proves the claim.