MA 2073 Midterm 2

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Elissa Ross, WPI

Name: SOLUTIONS

Student Number: ____________________

You have 50 minutes to complete this exam.

Examination rules and instructions:

- closed book
- no calculators or cell phones are permitted

Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
Total	/50

Question 1 (8 marks). TRUE FALSE: Circle T for true, or F for false. Each correct answer is worth 1 mark.

a) **T** / **F** The matrices A and $-A$ share the same four subspaces: row space, column space, null space and left null space.

b) **T** / **F** The matrices A and A^T have the same number of pivots.

c) **T** / **F** If the row space equals the column space then $A^T = A$.

d) **T** / **F** Two subspaces that meet only in the zero vector are orthogonal.

e) **T** / **F** $(1, 1, 1)$ is orthogonal to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.

f) **T** / **F** Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.

g) **T** / **F** If A is not invertible, then AB is not invertible.

h) **T** / **F** Let S be the eigenvector matrix of A. If the columns of S are linearly independent, then A is invertible.

Question 2 (5 marks). Let A be an $m \times n$ matrix. Prove that if the columns of A^TA are linearly independent, then the columns of A are linearly independent too.

cols of A^TA lin. indep $\Longrightarrow$

$$A^TA\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$$

$$\vec{x}^T A^T A \vec{x} = \vec{0}$$ (take ~~transpose~~ product of both sides with $\vec{x}^T$)

$$(A\vec{x})^T(A\vec{x}) = \vec{0}$$ (recognize as the dot product of $A\vec{x}$ with itself)

$$\|A\vec{x}\|^2 = \vec{0} \iff A\vec{x} = \vec{0}$$

so the columns of A are linearly independent too.

Question 3 (4 marks). Let P be the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$.

a) Record a basis for $P^\perp$. Justify your answer.

b) Construct a matrix that has P as its nullspace.

a) $P^\perp$ is the set of all vectors that are $\perp$ to the vectors in P.

$$P = \{(x_1, x_2, x_3, -x_1 - x_2 - x_3)\}$$, which has dimension 3.

Note $(1,1,1,1)$ is $\perp$ to every vector in P. The set $\{(1,1,1,1)\}$ is linearly independent, and it spans $P^\perp$ (note the dimension of $P^\perp$ must be 1, since $\dim(P^\perp) = \dim(\mathbb{R}^4) - \dim(P)$).

$\Rightarrow \{(1,1,1,1)\}$ is a basis.

b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

Question 4 (6 marks). Let $A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find the least squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$.

Least squares solution: $A^T A \vec{x} = A^T \vec{b}$

$A^T A = I$

so it is just

$$\vec{x} = A^T \vec{b} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}$$

$\cancel{\vec{e} = \vec{b} - A\hat{x}}$

Question 5 (5 marks). The first three *Legendre polynomials* are $1, x$, and $x^2 - 1/3$. Use the usual inner product on $C[-1,1]$ to find c so that the polynomial $x^3 - cx$ is orthogonal to the first three.

~~note $\langle x^3 - cx, 1\rangle = \langle x^3 - cx, x^2 - 1/3\rangle = 0$~~

$\langle x^3 - cx, 1\rangle = \int_{-1}^{1} x^3 - cx \, dx = x^4 - \frac{c}{2}x^2 \Big]_{-1}^{1} = 0.$

$\langle x^3 - cx, x^2 - 1/3\rangle = 0$ too, as the integral of an odd fcn.

$\langle x^3 - cx, x\rangle = \int_{-1}^{1} x^4 - cx^2 \, dx = \left[\frac{x^5}{5} - \frac{cx^3}{3}\right]_{-1}^{1} = \left(\frac{1}{5} - \frac{c}{3}\right) - \left(\frac{-1}{5} + \frac{c}{3}\right)$

$= \frac{2}{5} - \frac{2c}{3}$

we want this to be zero $\Rightarrow$ $0 = 6 - 10c$

$c = \frac{6}{10} = \frac{3}{5}.$

Question 6 (4 marks). A 3×3 matrix B is known to have eigenvalues $0, 1, 2$. The information is enough to find three of these (give the answers where possible):

a) the rank of B — B has two lin. indep e-vects $\Rightarrow$ rank 2.

b) the determinant of B^TB — Recall $\det(B^T) = \det B$, so $\det(B^TB) = (\det(B))^2 = 0^2 = 0$.

c) the eigenvalues of B^TB — can't find.

d) the eigenvalues of $(B^2 + I)^{-1}$

$$(B^2 + I)^{-1}\vec{x} = \lambda\vec{x}$$

$$\frac{1}{\lambda}\vec{x} = (B^2 + I)\vec{x}$$

$$\frac{1}{\lambda}\vec{x} = B^2\vec{x} + I\vec{x}$$

$\left(\frac{1}{\lambda} - 1\right)\vec{x} = B^2\vec{x}$. Eigenvalues of B^2 are the squares the e-vals of B.

so $\left(\frac{1}{\lambda} - 1\right) = 0^2 \Rightarrow \lambda_1 = 1$

$\left(\frac{1}{\lambda} - 1\right) = 1^2 \Rightarrow \lambda_2 = \frac{1}{2}$

$\left(\frac{1}{\lambda} - 1\right) = 2^2 \Rightarrow \lambda_3 = \frac{1}{5}.$

Question 7 (10 marks). Suppose G_{k+2} is the *average* of the previous numbers G_{k+1} and G_k.

3 a) Write the equations

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k$$
$$G_{k+1} = G_{k+1}$$

as a matrix expression $\mathbf{u}_{k+1} = A\mathbf{u}_k$, where $\mathbf{u}_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ (i.e. find A)

3 b) Find the eigenvalues and eigenvectors of A

2 c) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

2 d) If $G_0 = 0$ and $G_1 = 1$ show that the "Gibonacci numbers" approach $\frac{2}{3}$.

a) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$

b) $\det(A - \lambda I) = \det \begin{bmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{bmatrix} = (1/2 - \lambda)(-\lambda) - \frac{1}{2}$
$= \lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}$
$= (\lambda - 1)(\lambda + \frac{1}{2})$

$\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$

$\vec{x}_1 = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{x}_1 = (1, 1)$

$\vec{x}_2 \quad \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{x}_2 = (1, -2)$

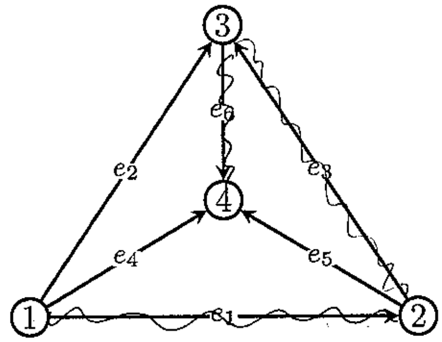
c) so $S = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ $\quad S^{-1} = \frac{-1}{3}\begin{bmatrix} -2 & -1 \\ -1 & \phi \end{bmatrix}$ $\quad \Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix}$

So $\lim_{n \to \infty} \Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $A^n = S\Lambda^n S^{-1} = S\begin{bmatrix} 2/3 & 1/3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$

and $\cancel{A}$ $\cancel{S}$ $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$ as $n \to \infty$.

Question 8 (8 marks).



2 a) Write down the adjacency matrix A for the graph shown above.

2 b) Row reduce the matrix. The non-zero rows correspond to the edges of a spanning tree. Mark these edges on the diagram above.

1 c) Find one solution to $A\mathbf{x} = \mathbf{0}$

2 d) Find three solutions to $A^T\mathbf{y} = \mathbf{0}$

1 e) List the dimensions of the four subspaces.

a)
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

b)
$$\longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c) $\vec{x} = (1, 1, 1, 1, 1, 1)$

d) $\vec{x}_1 = (1, 0, 0, -1, 1, 0)$ (edges $e_1 + e_5 - e_4$)

$\vec{x}_2 = (1, 0, 0, -1, 0, 1)$ ($e_2 + e_6 - e_4$)

$\vec{x}_3 = (0, 0, 1, 0, -1, 1)$ ($e_3 + e_6 - e_5$).

e) rowspace = col space = 3

nullspace = 1 Left nullspace = 3.

This exam has 7 pages (including the cover sheet) and 8 problems, worth a total of 50 marks.