MA 2073 Midterm 1

February 3, 2014

Elissa Ross, WPI

Name: SOLUTIONS !

Student Number: ____________________

You have 50 minutes to complete this exam.

Examination rules and instructions:

- closed book
- no calculators or cell phones are permitted

Question	Mark
1	
2	
3	
4	
5	
6	
7	
Total	/40

Question 1 (8 marks). TRUE FALSE: Circle T for true, or F for false. Each correct answer is worth 1 mark.

a) **T / F** Every matrix with 1's down its diagonal is invertible.

b) **T / F** If A is invertible, then A^{-1} and A^2 are invertible.

c) **T / F** Every set of four non-zero vectors in $\mathbb{R}^4$ is a basis.

d) **T / F** If the rows of A are independent, and the columns of A are independent, then A is a square matrix.

e) **T / F** The column space of A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

f) **T / F** The vectors $\mathbf{b}$ that are *not* in the column space of A form a subspace.

g) **T / F** An $m \times n$ matrix has no more than n pivot variables.

h) **T / F** If A is not a square matrix, then the row vectors of A must be dependent.

Question 2 (6 marks).

a) Let A be an $n \times n$ matrix. Define A^{-1}.

b) Use the definition of the inverse of a matrix to prove that an $n \times n$ matrix with a column of zeros cannot have an inverse.

a) A^{-1} is an $n \times n$ matrix B such that $AB = BA = I$

b) Let A be a matrix with a column of zeros.

consider $BA = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{0} & a_{j+1} & \cdots & \vec{a}_n \end{bmatrix}$ The j^{th} column of BA is $\begin{bmatrix} \vec{b}_1 \vec{0} \\ \vec{b}_2 \vec{0} \\ \vdots \\ \vec{b}_n \vec{0} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vdots \\ \vec{0} \end{bmatrix}$

So BA has a column of zeros, (for any B) and hence cannot be equal to the identity matrix. Therefore, no inverse exists.

Question 3 (6 marks). Find the complete solution to the following system of equations:

$$x + y + z = 4$$

$$x - y + z = 4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

z is free.

For $\vec{x}_p$ let $z = 0$. We know $y = 0$ (from eqn 2).

$x = 4 - z = 4$

so $\vec{x}_p = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

For $\vec{x}_n$, we have $\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}}_{\text{from eqn 1}} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Hence the complete sol$^{\underline{n}}$ is $\vec{x} = \vec{x}_p + \vec{x}_n$

$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Question 4 (4 marks). What matrix E puts A into upper triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 0 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad E_{31}A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} = U, \text{ which is upper } \Delta\text{lar.}$$

So $E = E_{31}$, and $L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

So $$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 0 \end{bmatrix}$$

Question 5 (6 marks).

a) Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.

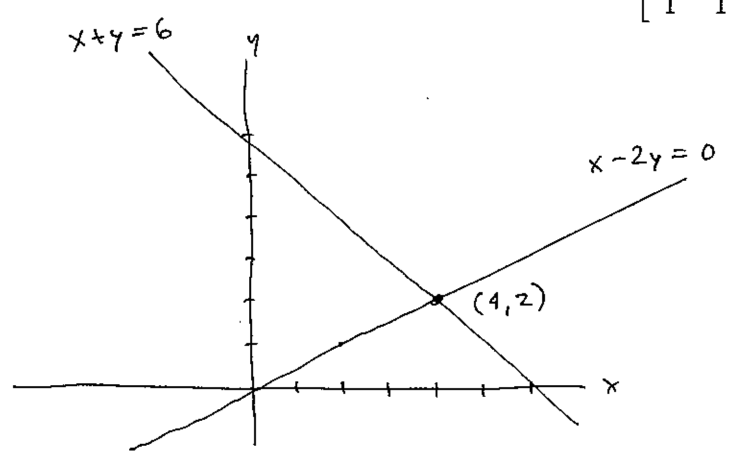
b) What is the nullspace of the matrix A?

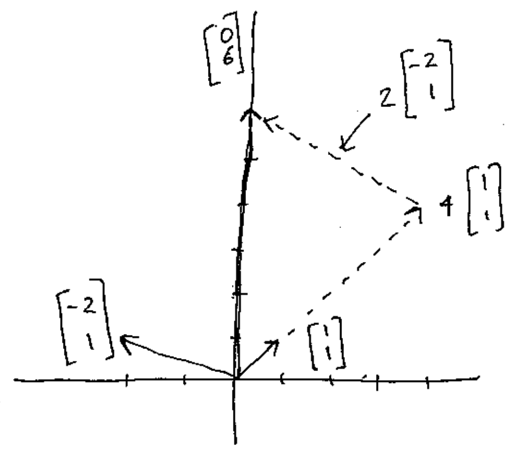
ROW PICTURE

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

COLUMN PICTURE

a)





b) nullspace is just $Z = \{\vec{0}\}$ (no free variables).

Question 6 (5 marks).

a) Is $\mathbb{R}^k$ a subspace of $\mathbb{R}^n$ when $k < n$? Why or why not?

b) Describe geometrically all subspaces of $\mathbb{R}^3$, and list their dimensions.

a) No, $\mathbb{R}^k$ is not even a subset of $\mathbb{R}^n$. A vector in $\mathbb{R}^k$ has k entries, while a vector in $\mathbb{R}^n$ has n entries.

b) $\mathbb{R}^3$ itself, dim 3.
through origin, dim 2
lines " ", dim 1
$Z = \{\vec{0}\}$, dim 0

Question 7 (5 marks).

a) Find a basis for the subspace of $\mathbb{R}^4$ consisting of all vectors whose components are equal.

b) The cosine space $\mathbf{F_3}$ contains all combinations

$$y(x) = A\cos x + B\cos 2x + C\cos 3x.$$

Find a basis for the subspace of $\mathbf{F_3}$ with $y(0) = 0$ (recall that $\cos(0) = 1$).

a) a basis is just $(1,1,1,1)$. It is independent, and we may write any vector $(\ell,\ell,\ell,\ell) = \ell(1,1,1,1)$.

b) If $y(0) = 0$, then $A + B + C = 0$.

So $A = -B - C$.

Then $A\cos x + B\cos 2x + C\cos 3x = (-B-C)\cos x + B\cos 2x + C\cos 3x$
$= B(\cos 2x - \cos x) + C(\cos 3x - \cos x)$

So $\{\cos 2x - \cos x, \cos 3x - \cos x\}$ spans the subspace. And this set is independent, since there is no linear relationship between $\cos 2x - \cos x$ and $\cos 3x - \cos x$. It is therefore a basis.

This exam has 5 pages (including the cover sheet) and 7 problems, worth a total of 40 marks.