MA 2073 HOMEWORK 7 SOLUTIONS

6.7

① e-vals of A^TA:

$$\det \begin{bmatrix} \cancel{10-x} & 20 \\ 20 & 40-x \end{bmatrix} = (10-x)(40-x) - 400$$
$$= 400 - 50x + x^2 - 400$$
$$= x(x-50)$$

so $\lambda_1 = 0 \qquad \lambda_2 = 50$

$\vec{v}_1$ is $\in$ null space of $\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ so $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\vec{v}_2$ " " " " $\begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix}$ so $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

note $\lambda_2 = 50 > 0 = \lambda_1$

so let $\sigma_1 = \sqrt{50}$ and $\sigma_2 = 0$.

Then $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

put $\vec{x}_2$ here since it corresponds to first singular value.

Now $\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sqrt{50}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \frac{1}{\sqrt{50}} \frac{1}{\sqrt{5}} \begin{bmatrix} 50 \\ 100 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{50} \\ 2\sqrt{50} \end{bmatrix} = \begin{bmatrix} \sqrt{10} \\ 2\sqrt{10} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Is $\vec{u}_1$ an e-vect of AA^T?

$$\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} \sqrt{50} \\ 2\sqrt{50} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 35\sqrt{50} \\ 105\sqrt{50} \end{bmatrix} = \begin{bmatrix} 35\sqrt{10} \\ 105\sqrt{10} \end{bmatrix} = \frac{1}{\sqrt{10}} 35 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

so $\vec{u}_1$ is an evect of AA^T $\bar{w}$ eval 35.

find $\vec{u}_2$ similarly.

$$\begin{bmatrix}1&2\\3&6\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}1&3\\3&-1\end{bmatrix}\begin{bmatrix}\sqrt{50}&0\\0&0\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1&2\\2&-1\end{bmatrix}.$$

③ a) The diagonal entries of A^TA are the sum of the squares of the entries of that column.

e.g let $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ $A^TA = \begin{bmatrix}a^2+c^2&\cdots\\\cdots&b^2+d^2\end{bmatrix}$

etc.

so the i^{th} diagonal entry is

$$\sum_{j=1}^{n} a_{ji}^2$$

The trace is the sum of all such sums.

b) If A has rank 1, then so does A^TA.

Then the only nonzero e-val of A is its trace, which is the sum of all a_{ij}^2.

Then σ_1 = square root of this sum, and

σ_1^2 = the sum of all a_{ij}^2.

④ $A^TA = \begin{bmatrix}2&1\\1&1\end{bmatrix} = AA^T$.

evals: $\det\begin{bmatrix}2-x&1\\1&1-x\end{bmatrix} = (2-x)(1-x)-1$

$2-3x+x^2-1$

x^2-3x+1

$$\lambda = \frac{3\pm\sqrt{9-4}}{2} = \frac{3\pm\sqrt{5}}{2}$$

let $\sigma_1^2 = \frac{3+\sqrt{5}}{2}$ $\sigma_2^2 = \frac{3-\sqrt{5}}{2}$.

now compute e-vects.

for $\vec{v}_1$, $\begin{bmatrix}2-\left(\frac{3+\sqrt{5}}{2}\right)&1\\1&1-\left(\frac{3+\sqrt{5}}{2}\right)\end{bmatrix} \to \begin{bmatrix}\frac{1+\sqrt{5}}{2}&1\\0&0\end{bmatrix}$ so e.g. $\frac{1}{\sqrt{1+\Phi^2}}\begin{bmatrix}1\\\frac{1+\sqrt{5}}{2}\end{bmatrix}$

$\Phi = \frac{1+\sqrt{5}}{2}$.

compute $\vec{v}_2$ similarly.

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad \frac{1}{\sqrt{1+(1/\Phi)^2}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix} = \vec{v}_2 .$$

use σ_1, σ_2 to see

$\vec{u}_1 = \vec{v}_1 \qquad \vec{u}_2 = -\vec{v}_2 .$ by $A\vec{u}_1 = \sigma_1 \vec{v}_1$
$A\vec{u}_2 = \sigma_2 \vec{v}_2 .$

(11) A^TA will be a diagonal matrix $\bar{w}$ entries $\sigma_1^2, \dots, \sigma_n^2$.

The eigenvectors of A^TA will be just the columns of I, since A^TA is diagonal.

Hence $V = I$.

Then $\vec{u}_i = \dfrac{A\vec{v}_i}{\sigma_i} = \dfrac{\vec{w}_i}{\sigma_i}$ ← ith column of A.

hence $U = A\Sigma^{-1}$

and the SVD of A is $U\Sigma V^T$

$= (A\Sigma^{-1})\,\Sigma \cdot I.$

(15) The singular values for $A+I$ come from $(A+I)^T(A+I)$ which is different from $A^TA + I$.

7.1

(3) d) is the only non-linear transformation.
(does not map origin to itself).

(6) a) does not satisfy either

$T(\vec{v}+\vec{w}) = \dfrac{\vec{v}+\vec{w}}{\|\vec{v}+\vec{w}\|}$ and $T(\vec{v}) + T(\vec{w}) = \dfrac{\vec{v}}{\|\vec{v}\|} + \dfrac{\vec{w}}{\|\vec{w}\|}$

not the same.

$T(c\vec{v}) = \dfrac{c\vec{v}}{\|c\vec{v}\|} = T(\vec{v}) \neq cT(\vec{v})$

b) and c) are linear, hence satisfy both.

d) ~~satisfies T(c~~ does not satisfy either.

(7) a) $T(T(\vec{v})) = \vec{v}$ linear

b) $T(T(\vec{v})) = T(\vec{v} + (1,1)) = \vec{v} + (2,2)$ not linear (shift)

c) $T(T(\vec{v})) =$ 180° rotation linear.

d) $T(T(\vec{v})) = T(\vec{v}) = \left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)$ linear.

(8) a) range : line of vectors $(x, 0)$
kernel : line of vectors (x, x)

b) range : $\mathbb{R}^2$
kernel : line of vectors $(0, 0, v_3)$

c) range : $(0,0)$
kernel : $\mathbb{R}^2$

d) range : $\{(x,x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$
kernel : line of vectors $(0, x) \in \mathbb{R}^2$.

(10) a) $T(x, 0) = \vec{0}$

b) $(0, 0, x)$ is not in the range (i.e range is not all of $\mathbb{R}^3$)

c) $T(0, x) = 0$.

(14) note $\det A = -1 \neq 0$, so A invertible.

If $AM = 0$, multiplying both sides by A^{-1} gives $M = 0$ too.

Furthermore $AM = B \Rightarrow M = A^{-1}B$.

hence any matrix B can be the output of this linear transformation.

(17) a, b, c are true

d) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$

so this is false.

(18) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

So any matrix with $b \neq 0$ will not be the zero matrix under $T(m)$.

The range is the set $\left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, x \in \mathbb{R} \right\}$.

The kernel is the set $\left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, a, c, d \in \mathbb{R} \right\}$.

7.2 (1) $S\vec{v} = \dfrac{d^2\vec{v}}{dx^2}$

$S\vec{v}_1 = 0$
$S\vec{v}_2 = 0$
$S\vec{v}_3 = 2 = 2\vec{v}_1$
$S\vec{v}_4 = 6x = 6\vec{v}_2$

so $B = \begin{bmatrix} 0 & 6 & 2 & 0 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the matrix for S.

(2) The second derivative of every function $ax+b$ is zero.

The nullspace of B is $\{ (a, b, 0, 0) , a, b \in \mathbb{R} \}$

⑤ $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \cancel{0} & 1 \end{bmatrix}$

$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$T(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 2\vec{w}_1 + \vec{w}_2 + 2\vec{w}_3$

⑥ $T(\vec{v}_2) = T(\vec{v}_3) \Rightarrow T(\vec{v}_2 - \vec{v}_3) = 0$

sol$^{\underline{n}}$s to $T(\vec{v}) = 0$ are $\vec{v} = c(\vec{v}_2 - \vec{v}_3)$.

Then nullspace $= \{(0, c, -c),\ c \in \mathbb{R}\}$.

$T(\vec{v}) = \vec{w}_2$ has solutions $(1,0,0) + (0,c,-c)$, $c \in \mathbb{R}$.

⑦ e.g. $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \notin$ col A. so $\vec{v}_1 + 4\vec{v}_3 \notin$ range of T.

⑩ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. $\vec{w}_1$ corresponds to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. so input $\vec{v}_1 - \vec{v}_2$ gives output $\vec{w}_1$.

⑪ To invert A:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$T^{-1}(\vec{w}_1) = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ which corresponds to $\vec{v}_1 - \vec{v}_2$.

Similarly $T^{-1}(\vec{w}_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 - \vec{v}_3$

$T^{-1}(\vec{w}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_3$.

(21) $\vec{v}_1 = 1$ $\qquad \vec{w}_1 = \frac{1}{2}(x^2+x)$

$\vec{v}_2 = x$ $\qquad \vec{w}_2 = 1 - x^2$

$\vec{v}_3 = x^2$. $\qquad \vec{w}_3 = \frac{1}{2}(x^2-x)$

$$T(\vec{v}_1) = \vec{v}_1 = 1 = \vec{w}_2 + \vec{w}_1 + \vec{w}_3$$

$$T(\vec{v}_2) = \vec{v}_2 = x = \vec{w}_1 - \vec{w}_3$$

$$T(\vec{v}_3) = \vec{v}_3 = x^2 = \vec{w}_1 + \vec{w}_3$$

so change of basis matrix from v's to w's.

$$\cancel{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & -1/2 & 1/2 \end{bmatrix}} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

The other way is even easier: $\vec{w}_1 = \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3$ etc.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

(26) The matrix is just the diagonal matrix with the eigenvalues as the diagonal elements.

(27) If we want $T(v_i)$ to be a basis, then the matrix representing T must be invertible. If it is not invertible, then $\{A\vec{v}_i\}$ may not form a basis of $\mathbb{R}^n$.

(32) FALSE: we need to know $T(\vec{w})$ for basis vectors.

ADDITIONAL PROBLEMS

① Let $A = U \Sigma V^T$.

Since A invertible, rank $A = n$, and the entries on the diagonal of Σ must be nonzero. (these are the evals of $A^T A$, which is also $n \times n$ & rank n).

$$A^{-1} = (U \Sigma V^T)^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$$
$$= V \Sigma^{-1} U^T$$

note this exists b/c the entries on the diag of Σ are nonzero.

if $\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ 0 & \cdots & \sigma_n \end{bmatrix}$ $\quad \Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & \cdots & 0 \\ 0 & \cdots & 1/\sigma_n \end{bmatrix}$

② If $A = U \Sigma V^T$

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$$
$$= V \Sigma^T U^T U \Sigma V^T$$
$$= V (\Sigma^T \Sigma) V^T$$

so in other words V diagonalizes $A^T A$.

$\Rightarrow$ * The columns of V are the e-vects of $A^T A$.

Similarly
$$A A^T = (U \Sigma V^T)(U \Sigma V^T)^T$$
$$= U \Sigma V^T V \Sigma^T U^T$$
$$= U \Sigma \Sigma^T U^T$$

and U diagonalizes $A A^T$.

③ $T: M_{n \times n} \longrightarrow \mathbb{R}$

CLAIM: $T(A) = \text{trace}(A)$ is linear

PROOF: consider $T(A+B) = \text{trace}(A+B)$

$= \text{trace}(A) + \text{trace}(B)$, by properties of trace

$= T(A) + T(B)$, as desired.

And $T(cA) = \text{trace}(cA) = c\,\text{trace}(A) = c\,T(A)$ ✓.

④ CLAIM: $D(A) = \det(A)$ is not linear.

PROOF: ~~counterexample:~~ $\det(A+B) \neq \det(A) + \det(B)$ in general.

e.g. let $A = B = I_{2\times 2}$.

then $\det A = \det B = 1$

but $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\det(A+B) = 4$.