On the Complexity of Elementary Modal Logics

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Abstract. It is well-known that modal satisfiability is PSPACE-complete [Lad77]. However, for semantically restricted classes of modal satisfiability, the complexity differs. There are many individual results on the complexity of modal logics where the allowed models are restricted. In this paper, we begin a systematic study of all possible restrictions, generalizing many previously known results. For an (infinite) subclass of these restrictions, we achieve a full complexity classification into the cases solvable in NP and PSPACE-hard.

1 Introduction

Modal logics have proven to be a valuable tool in mathmatics and computer science, to reason about things like necessity and belief. The traditional uni-modal logic enriches the propositional language with two operators $\Box$ and $\Diamond$, where $\Box\varphi$ is interpreted as φ *necessarily holds*, and $\Diamond\varphi$ as φ *possible holds*.

In the 60's of the 20th century, Saul Kripke proved a formal semantic model for modal logics: while propositional logic talks about one possible truth assignment to variables in a given formula, modal logic can be interpreted as talking about a set of related assignments, forming the set of "worlds" which are considered possible [Kri63a,Kri63b]. The connections between these possible worlds define the meaning of the operators $\Box$ and $\Diamond$ in a given structure. These structures are called Kripke structures, giving the most general modal logic in this context ist name K.

In addition to their mathematical and philisophical interest, modal logics today are widely used in practical applications: in artificial intelligence, modal logic is used to model the knowledge and beliefs of an agent, see e.g. [BZ05]. For many specific applications, there exist tailor-made variants of modal logics [BG04]. Modal logics also can be applied in cryptographic and other protocols [HMT88,FI86,LR86].

It is evident that for practical applications, the logics in question must have certain "nice" properties. In particular, the question what resources a computer needs to solve problems related to the logic is of interest. In other words, we are interested in the computational complexity of these problems. The most prominent computational problem for logics is the satisfiability problem, which is the question if some formula has a model. Many other problems like provability, implication, and equivalence, are closely related to the satisfiability problem, and often complexity results for the latter can easily be derived from results about the former.

It is also obvious that there is no single modal logic which is suitable for every possible practical application. Hence, each possible modal logic that we might consider gives rise to a new set of questions regarding the complexity of problems related to this logic.

A groundbreaking work in this context was the paper [Lad77] by Richard Ladner. He proved that the satisfiablity problem for the most general modal logic K is PSPACE-complete, and considered a number of restrictions of K. These restrictions are defined as follows: while the general modal logic K can be seen as dealing with arbitrary graphs, restrictions like $\mathsf{K4}$ and $\mathsf{S5}$ are logics considering graphs which are transitive, resp. equivalence relations. Logics dealing with these restricted classes of graphs play important roles in the application of modal logics. For example, when modal logic is used to model the belief of an agent in artificial intelligence, then we would want the beliefs of the agent to be consistent. If we interpret the modal operator $\Diamond$ as "possibly," then this boils down to demanding that φ implies $\Diamond\varphi$ for all formulas φ. It turns out that requiring this is essentially the same as restricting the study of Kripke-models to those which are reflexive (see e.g. [BdRV01], Chapter 4).

Following Ladner, many complexity results concerning restrictions of modal logics to certain graph classes were studied. However, a unified generalization of all of these cases has been missing so far. This is where our research starts: we study this problem is a general setting, where the restriction of the graphs is given by a first-order formula. For example, if we consider the relation R as an edge relation in a graph, then the first-order formula $\forall x(xRx)$ is satisfied be a graph if and only if it is reflexive.

The goal of our research is to be able to answer the following question: Given a first-order formula φ, what is the complexity of the satisfiability problem for the modal logic over the class of graphs satisfying the formula φ?

This preliminary report classifies the complexity of the arising modal satisfiability problem for the case where the formula is a universal Horn clause into those cases which can be solved in NP, and those which are PSPACE-hard.

The structure of this paper is as follows: In Section 2, we introduce the basic terminology and recall results from the literature, in particular the complexity results obtained by Ladner in [Lad77]. In Section 3, we investigate techniques which allow us to modify models for a given formula, and ensure that they keep all properties we are interested in. Section 4 then contains our results for the case where the first-order formula describing the graph class is of a special syntactically restricted kind, namely in Horn clause form. The report closes with some remarks on the next questions to consider in Section 5.

2 Preliminaries

2.1 Basic concepts and notation

Modal logic is an extension of classical propositional logic talking about "possible worlds." A modal formula is a well-formed propositional sentence using propositional variables, the usual logical symobls $\wedge, \vee, \neg$, and an additional unary operator $\Diamond$. (A further operator $\Box$ is often considered as well, this can be regarded as an abbreviation for $\neg\Diamond\neg$.) For a modal formula ϕ, $\mathsf{sf}\,(\phi)$ denotes the set of its subformulas. $\mathsf{md}\,(\phi)$ denotes the modal depth of a formula, i.e., the maximal nesting degree of the modal operator $\Diamond$.

While in propositional logic, a model for a formula is simple a propositional assignment, a model for a modal formula is a set of connected "worlds" with individual assignments. To be precise, a *frame* consists of a directed graph $G = (W, R)$, where the vertices in W are called "worlds," and an edge $(w_1, w_2) \in R$ is interpreted as w_2 is "visible" or "considered possible" from w_1. A *model* $M = (G, \pi)$ consists of a frame $G = (W, R)$, a set X of propositional variables and a function $\pi \colon X \to \mathcal{P}(W)$. We say that the model M is *based* on the graph (W, R). The function π represents truth asignments: the set $\pi(x)$ denotes the set of worlds in which the variable x is true. The propositional variables and operators behave as usual, $\Diamond\phi$ is meant to express "there is a successor world in which ϕ holds."

If $\mathcal{G}$ is a set of frames, then we say that a model $M = (G, \pi)$ is a $\mathcal{G}$-model if $G \in \mathcal{G}$.

For a formula ϕ built over variables from X, the usual propositional operators $\wedge, \vee$, and $\neg$, and the unary modal operator $\Diamond$, we define what "ϕ holds at world w in M" means for a model M. We also say that ϕ is *satisfied* at w in M, and write $M, w \models \phi$. The definition is as usual inductive:

If ϕ is a variable x, then $M, w \models \phi$ if and only if $w \in \pi(x)$. As usual, $M, w \models \phi_1 \wedge \phi_2$ if and only if ($M, w \models \phi_1$ and $M, w \models \phi_2$) and $M, w \models \neg\phi$ iff $M, w \not\models \phi$. For the modal operator $\Diamond$, we define that $M, w \models \Diamond\phi$ if and only if there is a world $w' \in W$ such that $(w, w') \in R$ and $M, w' \models \phi$.

For a class $\mathcal{G}$ of frames, a formula ϕ is $\mathcal{G}$-satisfiable if there exists a $\mathcal{G}$-model $M = (G, \pi)$ and some world $w \in W$ such that $M, w \models \phi$. Note that an analog of the Löwenheim-Skolen theorem holds for modal logic: if a modal formula is satisfiable, then it has a model which is countable. Hence, we can assume that all our graphs have a subset of the natural numbers as vertex set, in particular, we can assume that all objects arising here are sets in the sense of mathematical set theory.

As mentioned in the introduction, there are several classes of graphs which have been considered. Often, we identify a class $\mathcal{G}$ of graphs with the logic consisting of all modal formulas which are true in all $\mathcal{G}$-models. Phrased in this way, a formula ϕ is $\mathcal{G}$-satisfiable if and only if its negation $\neg\phi$ does not belong to the modal logic $\mathcal{G}$. It is possible to define modal logics with a set of axioms and a proof system. For all of the standard logics we mention in this paper, there also exists a modal axiom system. As mentioned before, the logic over all reflexive graphs can also be defined as the logic derived from the axiom $x \implies \Diamond x$ and a standard set of inference rules for modal logic. It can be shown that the set of reflexive frames is sound and complete for this logic, and therefore, these two ways of defining the logic are equivalent. Since our techniques work with the models rather than the axioms and proof systems, for our work the definition of logics as the logic over a class $\mathcal{G}$ of graphs is more natural. In Table 1, we list some well-known modal logics, and define them using the properties of the corresponding graph class.

logic name	**graph property**
K	All graphs
T	Reflexive graphs
B	Symmetric graphs
K4	Transitive graphs
S4	Graphs which are both transitive and reflexive
S5	Equivalence relation

Table 1. A few common modal logics

The purpose of this work is to generalize various results on the complexity of modal logics, and to consider restrictions of graphs in a uniform framework. We therefore introduce a way to describe graph properties by propositional formulas. The *frame language* is the first-order language containing the propositional operators $\wedge, \vee$, and $\neg$, and the binary relation R. The relation R is interpreted as the accessibility relation in a graph. For example, consider the following formula:

$$\varphi_{\text{trans}} = \forall x, y, z (xRy) \wedge (yRz) \implies (xRz).$$

It is obvious that a graph G satisfies φ_{trans} if and only if G is transitive. This leads to a general way to describe classes of graphs:

Definition 2.1 ([BdRV01]). *A class $\mathcal{G}$ of graphs is* defined *by a first-order formula φ over the frame language, if every graph is in $\mathcal{G}$ if and only if it is a model for φ. A class of graphs is* (universal) elementary *if it is defined by a first-order sentence over the frame language (where every variable in the formula is universally quantified).*

We only consider logics KL for which there exists a class of graphs which is sound and complete for this logic. This means that there is a class $\mathcal{G}$ of graphs such that a formula is valid in the logic

KL if and only if it is true for all worlds in all models based on a frame in G. We then say that a model is a KL-model if and only if it is based on a graph in $\mathcal{G}$.

Definition 2.2. – *A modal logic* KL *is* (universal) elementary *if there is a class* $\mathcal{G}$ *of graphs which is (universal) elementary and a model* M *is a* KL*-model if and only if the frame on which* M *is based is in* $\mathcal{G}$*.*
– *Let* φ *be a first-order formula over the frame language. Then* $\mathsf{K}(\varphi)$ *denotes the modal logic over the frames contained in the class of graphs defined by* φ*.*

As an example, we show how previously mentioned logics can be expressed in this formalism. In addition to the formula φ_{trans} defined above, let $\varphi_{\text{refl}} := \forall w(wRw)$, and let $\varphi_{\text{symm}} := \forall x, y(xRy) \implies (yRx)$. Finally, let φ_{taut} be some tautology over the frame language, for example let $\varphi_{\text{taut}} := \forall x(xRx) \implies (xRx)$.

logic name	formula definition
K	$\mathsf{K}(\varphi_{\text{taut}})$
T	$\mathsf{K}(\varphi_{\text{refl}})$
B	$\mathsf{K}(\varphi_{\text{symm}})$
$\mathsf{K4}$	$\mathsf{K}(\varphi_{\text{trans}})$
$\mathsf{S4}$	$\mathsf{K}(\varphi_{\text{trans}} \wedge \varphi_{\text{refl}})$
$\mathsf{S5}$	$\mathsf{K}(\varphi_{\text{trans}} \wedge \varphi_{\text{refl}} \wedge \varphi_{\text{symm}})$

Table 2. formula definitions

For a modal logic KL, a logic *extending* KL is a modal logic KL' where every formula which is true in KL is also true in KL', or equivalently every formula that is KL'-satisfiable is also KL-satisfiable. In the case of elementary logics, this is equivalent to the condition that the formula defining the logic KL' implies the formula defining the logic KL. As an example, every logic is an extension of K, and $\mathsf{S4}$ is an extension of $\mathsf{K4}$.

Every logic KL gives rise to a satisfiability problem. As mentioned before, a modal formula ϕ is KL-satisfiable if and only if its negation $\neg\phi$ is no true sentence in KL. For logics of the form $\mathsf{K}(\varphi)$ for a first-order formula φ, a modal formula ϕ is satisfiable if and only if there is a model M such that the graph on which M is based satisfies the formula φ, and there is a world $w \in M$ such that $M, w \models \phi$.

PROBLEM: KL-SAT
INPUT: A modal formula ϕ
QUESTION: Is ϕ KL-satisfiable?

As an example, the problem $\mathsf{K}(\varphi_{\text{trans}})-\mathsf{SAT}$ is the problem to decide if a given modal formula can be satisfied in a transitive graph. The aim of our research is to classify the complexity of the problem KL-SAT for every logic KL which is of the form $\mathsf{K}(\varphi)$ for a first-order formula φ over the frame language. It should be noted that when considering the problem $\mathsf{K}(\varphi)$-SAT, then we regard the formula φ as fixed. It is also interesting to study the *uniform* version of the problem, where we are given a first-order formula φ over the frame language and a modal formula ψ, and the goal is to determine whether there exists a graph satisfying both. However, this problem obviously is PSPACE-hard (this easily follows from Ladner's Theorem 2.7).

In order to talk about the relationships between formulas and the graphs they define, we introduce some notation on graphs. As usual, a *homomorphism* from a graph G_1 to a graph G_2 is a function preserving the edge relation. A *strict tree* is a tree in the usual sense, i.e., a directed, acyclic, connected graph which has a root w from which all other vertices can be reached. A tree is a srict tree, or a reflexive or transitive (or both) closure of a strict tree. We now define notation to describe paths in graphs. Note that we often identify modal models and their frames, when the propositional assignments are not important for our arguments.

Definition 2.3. *Let G be a graph, $w, v \in G$ vertices, and $i \in \mathbb{N}$. We write $G \models w \xrightarrow{i} v$ if in G, there is a path of length i from w to v. Formally, this is defined as follows:*

- *For any node $w \in G$, $G \models w \xrightarrow{0} w$,*
- *if $G \models w \xrightarrow{i-1} u$ and (u, v) is an edge in G, then $G \models w \xrightarrow{i} v$.*
- *$maxdepth^G(w) := \max\left\{i \mid \exists w' \in G, G \models w' \xrightarrow{i} w\right\}$ is the* maximal depth of w in G,
- *$maxheight^G(w) := \max\left\{i \mid \exists w' \in G, G \models w \xrightarrow{i} w'\right\}$ is the* maximal height of w in G.

We also say that w is a i-step predecessor of v, and v is a i-step successor of w, if $G \models w \xrightarrow{i} v$.

Note that the maximal depth and maximal height of nodes can be (countably) infinite, even in finite graphs. The next definition is a restriction on graphs which is very natural for modal logics: for deciding whether $M, w \models \phi$ holds for some modal M, a world $w \in M$, and a modal formula ϕ, it is obvious that only the worlds which are reachable from w are important.

Definition 2.4. *Let G be a graph, and let $w \in G$. The graph G_w is obtained from G by restricting G to the worlds which can be reached from w. We say that such a graph is* rooted at w.

When interested in complexity results for modal logic, the property of having "small models" is often crucial, as these lead to a relatively "easy" satisfiability problem.

Definition 2.5. *A modal logic KL has the* polynomial size model property*, if there is a polynomial p, such that for every KL-satisfiable formula ϕ, there is a KL-model M and a world $w \in M$ such that $M, w \models \phi$, and $|M| \leq p(|\phi|)$.*

The following theorem is the key to our NP-containment results:

Theorem 2.6. *Let φ be a first-order sentence over the frame language, such that $\mathsf{K}(\varphi)$ has the polynomial size property. Then $\mathsf{K}(\varphi)$-$\mathsf{SAT} \in$ NP.*

Proof. This easily follows from the literature, since for a given graph and a fixed first-order sentence φ, it can be checked in polynomial time if the graph satisfies φ. Also, it can be verified in polynomial time if a model satisfies a modal formula. Hence, the obvious guess-and-verify approach works for NP-containment. □

It should be noted that since modal logic is an extension of propositional logic, the satisfiability problem for every non-trivial modal logic is NP-hard. Therefore, proving the polynomial size model property is the best upper complexity bound which we can hope for in this context.

2.2 Ladner's theorem and consequences

In the seminal paper [Lad77], Ladner showed PSPACE-containment and PSPACE-hardness for a variety of modal logics. In particular, he proved that the satisfiability problem for any logic between K and $\mathsf{S4}$ is PSPACE-hard. His result can be stated as follows:

Theorem 2.7. *1.* K-SAT *is* PSPACE-*complete.*

2. K4-SAT *is* PSPACE-*complete.*
3. S4-SAT *is* PSPACE-*complete.*
4. S5-SAT *is* NP-*complete.*
5. *Let* KL *be a modal logic such that* S4 *is an extension of* KL. *Then* KL-SAT *is* PSPACE-*hard.*

Ladner proves the final part with a reduction from QBF to the modal satisfiability problem. The reduction has the following property: Given a QBF-instance, a modal formula is constructed such that if the QBF-formula is true, then the modal formula can be satisfied in a reflexive, transitive tree (or in a standard tree). Otherwise, the modal formula cannot be satisfied in any model. This immediately gives the following corollary:

Corollary 2.8. *Let φ be a formula over the frame language which is is satisfied in every reflexive and transitive tree, or in every strict tree. Then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE-*hard.*

The construction from Ladner's proof can easily be modified in such a way that the resulting formula is also satisfiable in a symmetric tree, if the original QBF-instance is true. Therefore, his proof reveals the following corollary:

Corollary 2.9. *Let φ be a formula over the frame language which is satisfied in every symmetric tree. Then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE-*hard.*

Note that while the properties of reflexivity, transitivity and symmetry do not lead to a satisfiability problem for the corresponding logic which is in NP, the conjunction of all of these properties does, since S5-SAT is NP-complete due to Theorem 2.7. In contrast to the previous results, the following lemma also gives the PSPACE upper bound, and not just the hardness result.

Lemma 2.10. *Let φ be a first-order formula in the frame language, such that φ is satisfied in every tree. Then the satisfiability problem for $\mathsf{K}(\varphi)$ is* PSPACE-*complete.*

Proof. We show that any modal formula ψ is K-satisfiable if and only if it is $\mathsf{K}(\varphi)$-satisfiable. The complexity result then follows from Theorem 2.7. Obviously, a $\mathsf{K}(\varphi)$-satisfiable formula is K-satisfiable as well. Now let ψ be K-satisfiable. Then, by "unrolling" an arbitrary model for ψ, it can be shown that ψ can be satisfied in a model which is a tree. Hence ψ is $\mathsf{K}(\varphi)$-satisfiable. □

As this section shows, trees are an important subclass of modal models. And in fact, a good intuition to read this paper is to always consider the graphs we deal with as "near-trees," i.e., trees with additional edges.

3 About Modal Models

3.1 Invariants

As mentioned above, the search for logics exhibiting the polynomial size model property is an important aspect of this work. Results in this way are usually obtained by taking an arbitrary model for a given modal formula and building a smaller model out of it, which still satisfies the modal formula. Therefore, the question what kind of restrictions or modifications we can apply to our models and ensure that they still satisfy the same conditions is important. The first result in this way concerns the first-order aspect of the frames for universal formulas:

Theorem 3.1. *Let G and G' be graphs. The following conditions are equivalent:*

1. *Every universal first-order formula (with at most n variables) satisfied by G' is also satisfied by G,*

2. *Every existential first-order formula (with at most n variables) satisfied by G is also satisfied by G',*
3. *For each finite $V \subseteq G$ (such that $|V| \leq n$), there exists a functions $f\colon V \to G'$ such that for $u, v \in V$, it holds that uRv iff $f(u)R'f(v)$.*

Proof. $1 \leftrightarrow 2$ This is obvious, since negated universal first-order formulas are exactly the first-order existential formulas, and the numbers of variables of a formula and its negation are identical.

$2 \to 3$ We construct a function $f\colon V \to G'$ with the desired properties. Let $|V| = n$, and let $V = \{x_1, \ldots, x_n\}$. We construct an existential first-order formula

$$\psi = \exists x_1, \ldots, \exists x_n \bigwedge_{(x_i,x_j)\in R} x_i R x_j \wedge \bigwedge_{(x_i,x_j)\notin R} \overline{x_i R x_j}.$$

Obviously, ψ has n variables, and G obviously is a model for ψ, it follows that $G' \models \psi$. Therefore, there are $x'_1, \ldots, x'_n$ such that $(x_i, x_j) \in R$ if and only if $(x'_i, x'_j) \in R'$. Define $f(x_i) := x'_i$. This function obviously meets the criteria: Let $(x_i, x_j) \in R$. Then $x_i R x_j$ is a clause in ψ. Therefore for the values x'_i and x'_j chosen by the existential quantifiers, $(x'_i, x'_j) \in R$ must hold. Since $f(x_i) = x'_i$ and $f(x_j) = x'_j$, the claim follows. For the condition $\overline{R(x_i, x_j)}$, the proof is the same.

$3 \to 2$ Let such a function f exist for every $V \subseteq G$ with $|V| \leq n$. Let $\psi := \exists x_1, \ldots, \exists x_n \varphi$ be a first-order existential formula, let n be the number of variables in ψ, and let $G \models \psi$. We show that $G' \models \psi$. Since $G \models \psi$, there are $x_1, \ldots, x_n \in G$ such that $\varphi(x_1, \ldots, x_n)$ holds in G. This implies that $\varphi(f(x_1), \ldots, f(x_n))$ holds in G'. Therefore, $G' \models \psi$.

□

Note that while the function f required to exists in the conditions shares some properties with an isomorphism, it is not required to be injective. We would need this additional property if we wanted to allow equality clauses in the frame language.

The following is an important special case, which follows from Theorem 3.1.

Proposition 3.2. *Let G be a graph, φ a universal first-order formula over the frame language such that $G \models \varphi$. Then for every subgraph G' of G, it holds that $G' \models \varphi$. In other words, any universal elementary class of graphs is invariant under restriction.*

For a modal model, a restriction of the model is a restriction of the graph, where the propositional assignment for the remaining worlds is unchanged. We now consider restrictions which are "compatible" with the modal properties of the formulas in question. The following lemma describes a standard way to reduce the number "relevant" of successors to worlds in models. This is an application of the more general idea of modal bisimulations.

Lemma 3.3. *Let φ be a modal formula, let KL be a universal elementary logic, and let $M, w \models \varphi$, where M is a KL-model. Let M' be a restriction of M such that the following holds:*

1. *$w \in M'$,*
2. *for all $u \in M'$, and all $\psi \in \mathsf{sf}(\varphi)$ such that $M, u \models \Diamond\psi$, there is some $v \in M'$ such that (u, v) is an edge in M, and $M, v \models \psi$.*

Then M' is a KL-model, and $M', w \models \varphi$.

Lemma 3.3 immediately follows from the following Lemma, but the version stated in Lemma 3.3 is the one we almost exclusively use, except for the proof of Theorem 4.16, hence we stated this simpler version explicitly. We now give the more general result:

Lemma 3.4. *Let φ be a modal formula, let* KL *be a universal elementary logic, and let $M, w \models \varphi$, where M is a* KL*-model. Then M' be a restriction of M such that the following holds:*

1. *$w \in M'$,*
2. *for all $u \in M'$, and all $\psi \in \varphi$, such that $M, u \models \Diamond\psi$ and there exists an $i \in \mathbb{N}$ such that $M \models w \xrightarrow{i} u$ and $1 + i + \mathsf{md}(\psi) \leq \mathsf{md}(\varphi)$, there is some $v \in M'$ such that (u, v) is an edge in M, and $M, v \models \psi$.*

Then M' is a KL*-model, and $M', w \models \varphi$.*

Proof. By Proposition 3.2, M' is a KL-model. We show the following claim: Let $\chi \in \mathsf{sf}(\varphi), i \in \mathbb{N}$, and $u \in M'$ such that $M \models w \xrightarrow{i} u$, and $i + \mathsf{md}(\chi) \leq \mathsf{md}(\varphi)$, then $M, u \models \chi$ if and only if $M', u \models \chi$. For $\chi = \varphi$, $u = w$, and $i = 0$, this implies the Lemma, since w is an element of M' by definition.

We show the claim by induction on χ. If χ is a variable, then this holds trivially, since M' is a restriction of M and therefore, propositional assignments are not changed. The induction step for propositional operators is trivial. Therefore, assume that $\chi = \Diamond\psi$ for some $\psi \in \mathsf{sf}(\varphi)$, such that the claim holds for ψ. Now let u, i meet the prerequisites of the claim, i.e., let $i + \mathsf{md}(\chi) \leq \mathsf{md}(\varphi)$, and let $M \models w \xrightarrow{i} u$.

First assume that $M, u \models \chi$. Since $\chi = \Diamond\psi$, it follows that $\mathsf{md}(\chi) = \mathsf{md}(\psi) + 1$, and hence $i + 1 + \mathsf{md}(\psi) \leq \mathsf{md}(\varphi)$. Since $M, u \models \Diamond\psi$, the prerequisites of the Lemma therefore imply that there is a world $v \in M'$ such that (u, v) is an edge in M, and $M, v \models \psi$. Since $M \models w \xrightarrow{i} u$, it follows that $M \models w \xrightarrow{i+1} v$. By the induction hypothesis, we know that $M', v \models \psi$. Since M' is a restriction of M, (u, v) is an edge in M' as well, and therefore we conclude that $M', u \models \Diamond\psi$, i.e., $M', u \models \chi$.

For the other direction, assume that $M', u \models \chi$. Therefore, there is a node $v \in M'$ such that $M', v \models \psi$, and (u, v) is an edge in M'. Since $M \models w \xrightarrow{i} u$, we know that $M \models w \xrightarrow{i+1} v$ holds as well, and since $\mathsf{md}(\psi) = \mathsf{md}(\chi) - 1$, from the induction hypothesis we conclude that $M, v \models \psi$. Since (u, v) is an edge in M as well, it therefore follows that $M, u \models \Diamond\psi$, i.e., $M, u \models \chi$, concluding the proof. □

The invariance results we have so far obviously give the following easy proposition:

Proposition 3.5. *Let* KL *be a universal elementary logic, let φ be a modal formula, and let $M, w \models \varphi$, where M is a* KL*-model. Then there is a* KL*-model M' which is a restriction of M, which is rooted at w, and where every world can be reached from w in at most* $\mathsf{md}(\varphi)$ *steps, and $M', w \models \varphi$.*

Proof. The model M' is obtained by simply removing all worlds from M which cannot be reached from w in at most $\mathsf{md}(\varphi)$ steps. Due to Proposition 3.2, M' is still a KL-model. It is obvious that $M', w \models \varphi$, since any removed world cannot be relevant for the validity of φ at w. Formally, this follows from Lemma 3.4. □

3.2 Restrictions

As mentioned, our NP-containment results are obtained by proving the polysize model property and applying Theorem 2.6. The polynomial models are obtained by restricting arbitrary models to polynomial size. We will now show some restrictions which we can make in any model, showing that we can assume that certain parts of the modal are only polynomial in size.

Lemma 3.6. *Let $c \in \mathbb{N}$ be a constant. Then there exists a polynomial p, such that for every universal elementary modal logic KL, and every KL-satisfiable modal formula φ, there is a KL-model M with root w such that $M, w \models \varphi$, and the following holds:*

$$\left|\left\{v \in M \mid \mathit{maxdepth}^M(v) < c\right\}\right| \leq p(|\varphi|).$$

Moreover, every KL-model for φ contains a submodel with the required property.

Proof. Let φ be KL-satisfiable, and let M_2 be a KL-model such that $M_2 \models \varphi$. By Proposition 3.5, there is a KL-model M_1 which is a submodel of M_2 and a world $w \in M_1$ such that $M_1, w \models \varphi$ and every world in M_1 is reachable from w in at most $\mathsf{md}(\varphi)$ steps. Now let A contain the nodes with "few" predecessors in M_1 which we want to restrict, i.e. we define

$$A := \left\{v \in M_1 \mid \mathrm{maxdepth}^{M_1}(v) < c\right\}.$$

For a vertex u in M_1, let $s_u := \{\chi \in \mathsf{sf}(\varphi) \,|M_1, u \models \Diamond\chi\}$. Obviously, $|s_u| \leq |\mathsf{sf}(\varphi)|$. Let W_u be a subset of the successors of u chosen in an arbitrary way such that W_u contains at least one vertex v such that $M_1, v \models \chi$ for each $\chi \in s_u$, and $|W_u| \leq s_u$.

We now obtain from M_1 a finite model M, where every vertex has at most $|\mathsf{sf}(\varphi)|$ "relevant" successors. We define inductively: $W_0 := \{w\}, W_{i+1} := \cup_{v \in W_i} W_v$. Let M be the model obtained from M_1 by deleting every vertex except those in $\cup_{i=0}^{\mathsf{md}(\varphi)} W_i$. This model again is a KL-model, and $M, w \models \varphi$, due to Lemma 3.3.

We now show that the model M satisfies the desired property, i.e. we show that $|M \cap A| \leq p(|\varphi|)$, where $p(n) = (c+1) \cdot n^c$. By construction, it holds that $M \cap A \subseteq \cup_{i=0}^{c} W_i$. Obviously, $|W_0| = 1$, and $|W_{i+1}| \leq |W_i| \cdot |\mathsf{sf}(\varphi)|$. Therefore, $|W_i| \leq |\mathsf{sf}(\varphi)|^i$ for all $i \in \{0, \ldots, \mathsf{md}(\varphi)\}$. Now $|M \cap A| \leq |\cup_{i=0}^{c} W_i| \leq (c+1) \cdot \mathsf{sf}(\varphi)^c$. Since $|\mathsf{sf}(\varphi)| \leq |\varphi|$, the claim follows. □

The proof of the above lemma reveals an important corollary:

Corollary 3.7. *Let KL be a universal elementary logic. Then KL has the finite model property, and $\mathsf{KL}\text{-}\mathsf{SAT} \in \mathrm{NEXPTIME}$.*

Proof. The proof of Lemma 3.6 clearly gives an exponential upper bound on the size of the constructed model. Since testing if a given model is a KL-model can be done in polynomial time, the claim follows with the same proof as Theorem 2.6. □

The following lemma shows that it is sufficient to restrict the number of those vertices in the model which have a certain minimal number of successors. In combination with Lemma 3.6, this shows that we only need to be concerned about vertices which have both a certain number of predecessors, and a certain number of successors. This is useful, because near the "top" and the "bottom" of a graph, often special cases can occur. These lemmas show that we do not need to look too closely at these exceptions.

Lemma 3.8. *Let $c \in \mathbb{N}$, and let KL be a universal elementary logic. Then for any modal formula φ and any $M, w \models \varphi$ such that M is a KL-model, there is a KL-model M' which is a restriction of M, such that $M', w \models \varphi$, and for which the following holds:*

$$|M'| \leq p\left(\left|\left\{w \mid w \in M \text{ and } w \text{ has a } c\text{-step successor}\right\}\right|\right),$$

where $p(n) = (2 \cdot |\varphi|)^c \cdot n$.

Proof. Define

$$C := \{w \mid w \in M \text{ and } w \text{ has a } c\text{-step successor}\}.$$

We construct M' from M as follows: Let $W_0 := C \cup \{w\}$, and for $i > 0$, let W_i be defined as follows:

- $W_{i-1} \subseteq W_i$,
- for each u added in the step W_{i-1} and each $\psi \in \mathsf{sf}(\varphi)$ such that $M, u \models \Diamond\psi$, and there is a world $s \in S$ such that (u, s) is an edge in M, and $M, s \models \psi$, there is one of these worlds in W_{i-1}.

Now let $M' := W_c$. Since M' is a restriction of the KL-model M, M' is a KL-model as well. We now show that $M', w \models \varphi$, by showing that M' satisfies the conditions of Lemma 3.3. The condition $w \in M'$ holds by construction. Now let $u \in M'$, let ψ be a subformula of φ, and let $M, v \models \varphi$ hold. Since $u \in M'$, there is a minimal i such that $u \in M_i$. If $i < c$, then a successor v fulfilling the condition of Lemma 3.3 is added in the step from W_i to W_{i+1}. Now assume that $i = c$. Then, by construction, u is a $c-1$-step successor of a node x added in the step from W_0 to W_1. Hence the node u cannot have a successor, since otherwise, x would have a c-step successor, a contradiction, since $x \notin C$.

Therefore, the model M' still satisfies the formula φ.

We now consider the cardinality of M' : By definition, for $i \geq 1$, it holds that

$$|W_i| \leq |W_{i-1}| + |W_{i-1}| \cdot |\mathsf{sf}(\varphi)| \leq |W_{i-1}| + |W_{i-1}| \cdot |\varphi| \leq 2 \cdot |W_{i-1}| \cdot |\varphi|,$$

since $|\varphi| \geq 1$ holds for any formula φ. Hence, it follows that $|M'| = |W_c| \leq 2^c \cdot |\varphi|^c \cdot |C|$, and the lemma follows. □

The combination of these lemmas gives the following, which generalizes the well-known result that the satisfiability problem for S5 (the logic defined by the class of graphs whose accessibility relation is an equivalence relation) is in NP.

Lemma 3.9. *Let* KL *be a universal elementary logic, such that the following holds: there are constants* c_1 *and* c_2*, such that for every* KL*-satisfiable formula* φ*, and every* KL*-model* M *such that* $M \models \varphi$*, it holds that every pair of vertices* $s, t \in M$ *such that* $\mathit{maxheight}^M(s), \mathit{maxheight}^M(t) \geq c_1$ *and* $\mathit{maxdepth}^M(s), \mathit{maxheight}^M(t) \geq c_2$*,* (s, t) *is an edge in* M*. Then* $\mathsf{KL}\text{-}\mathsf{SAT} \in \mathrm{NP}$*.*

Proof. Due to Lemma 3.6 and Lemma 3.8, by taking the maximum of the polynomials and constants given in those results, we can assume that there is a polynomial p and a constant c, such that $c \geq c_1, c_2$, and for every KL-satisfiable formula ϕ, there is a KL-model $M, w \models \phi$ such that the following holds:

1. $\left|\left\{v \in M \mid \mathrm{maxdepth}^M(v) < c\right\}\right| \leq p(|\varphi|)$.
2. $|M| \leq p(\left|\left\{v \in M \mid \mathrm{maxheight}^M(v) \geq c\right\}\right|)$.

Now let ϕ be a KL-satisfiable formula, and let M be a model with the above properties. We show how the model M can be restricted to polynomial size. The complexity result then follows from Theorem 2.6.

Let C denote the set of vertices v in M such that v both has a c-step successor and a c-step predecessor. By the prerequisites and the choice of c, it is obvious that the set C is a complete graph, and restricting C to polynomial size also restricts M to polynomial size (since Lemmas 3.6 and 3.8 guarantee the existence of a submodel with the required properties for every given model). As in the lemmas before, we can restrict the nodes inside C with an application of Lemma 3.3.

The basic idea is that inside the universal "cloud," only one world for each subformula of ϕ must be kept (with some additional nodes as neighbors for the worlds not having "enough" predecessors and successors). In this way, we can prove the polynomial size property, and Theorem 2.6 then implies that the satisfiability problem is in NP.

The result also follows from the much more difficult later Theorem 4.16: with the definitions from Section 4, it is obvious that any graph G satisfying the prerequisites is a model of the formula $\varphi^{c_1 \to c_1+1}_{w \geq c_1, x \geq c_2, y \geq c_2}$. □

4 Universal Horn Clauses

We now restrict our studies to a syntactically restricted case of universal first order formulas, namely Horn clauses. Many well-known logics can be expressed in this way.

4.1 Definitions and basic Results

Usually, a Horn clause is defined as a disjunction of literals of which at most one is positive. If a positive literal occurs, then the clause can be written as an implication, since $(\overline{x_1} \vee \cdots \vee \overline{x_n} \vee y)$ is equivalent to $(x_1 \wedge \cdots \wedge x_n \implies y)$. In the following, we make a case distinction whether a positive literal occurs, and the clause is a proper implication, or a positive clause does not appear. We will shortly see that the latter case is not very interesting in our context. We call those clauses *purely negative Horn clauses*, since no positive literal is present. Since in the context of the frame language, an atomic proposition is of the form (xRy), the following is the natural version of Horn clauses for our purposes:

Definition 4.1. – *A* universal Horn implication *over the frame language is a formula of the form*

$$(x_1 R x_2) \wedge \cdots \wedge (x_{k-1} R x_k) \implies (x_i R x_j),$$

where all (not necessarily distinct) variables are implicitly universally quantified.

– *A* universal purely negative Horn clause *over the frame language is a formula of the form*

$$\overline{(x_1 R x_2)} \vee \cdots \vee \overline{(x_{n-1} R x_n)},$$

where all (not necessarily distinct) variables are implicitly universally quantified.

– *A* universal Horn clause *is a universal Horn implication or a universal purely negative Horn clause. A* universal Horn formula *is a finite conjunction of universal Horn clauses.*

With formulas like these, many of the usually considered graph properties can be expressed, like transitivity, symmetry, euclidicity, etc. Observe that the formulas φ_{trans} and φ_{symm} defined earlier are of this form.

Definition 4.2. *Let φ be a universal Horn implication.*

– *The* prerequisite graph of *φ, denoted with* $\mathsf{prereq}\,(\varphi)$*, consists of the variables appearing on the left side of the implication φ, where (x_1, x_2) is an edge if the clause $(x_1 R x_2)$ appears.*
– *The* conclusion edge of *φ, denoted with* $\mathsf{conc}\,(\varphi)$*, is the edge (x_1, x_2), where $(x_1 R x_2)$ is the right side of the implication φ.*
– *The implication φ is called* independent*, if the vertices appearing in the conclusion edge of φ do not both appear or are not connected in the prerequisite graph of φ.*
– *The implication φ is called* reflexive*, if the vertices appearing in the conclusion edge are the same.*

– *The implication φ is called* proper*, if it is not independant and not reflexive, i.e., if both variables from the conclusion edge are different, appear in the prerequisite graph, and are connected.*

In the above definition, "connected" does not mean that there is a directed path between the respective nodes, but that there is a path in the underlying undirected graph. The name "independant" is motivated by the observation that such clauses are no "real implication," as there is no close relationship between the left and the right side of the implication. A "reflexive" Horn implication obviously is satisfied in every reflexive graph.

It is obvious that there is a one-to-one correspondence between universal Horn implications and their representation as prerequisite graph and conclusion edge. We will therefore often work with this representation. The following proposition shows that the definitions above are natural if we think about graph properties described by universal first order formulas as "forbidden subgraph properties."

Proposition 4.3. *Let φ be a universal Horn implication such that the variables of $\mathsf{conc}\,(\varphi) = (s,t)$ also appear in $\mathsf{prereq}\,(\varphi)$. A graph $G = (V,E)$ satisfies φ if and only the following holds: For every homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi) \rightarrow V$, it holds that $(\alpha(s),\alpha(t))$ is an edge in G. Equivalently, every homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi) \rightarrow G$ can be restricted to a homomorphism $\alpha'\colon \mathsf{conc}\,(\varphi) \rightarrow G$.*

Proof. Let $\varphi = R(x_1,x_2) \wedge \cdots \wedge R(x_{n-1},x_n)$, where the $x_1,\ldots,x_n$ are (not necessarily distinct) variables such that $s,t \in \{x_1,\ldots,x_n\}$, and all variables are implicitly universally quantified. First assume that $G \models \varphi$, and let $\alpha\colon \mathsf{prereq}\,(\varphi) \rightarrow G$ be a homomorphism. Due to the definition of $\mathsf{prereq}\,(\varphi)$, there are edges $(x_1,x_2),\ldots,(x_{n-1},x_n)$ is $\mathsf{prereq}\,(\varphi)$. Since α is a homomorphism, this implies that $(\alpha(x_1),\alpha(x_2)),\ldots,(\alpha(x_{n-1}),\alpha(x_n))$ are edges in G. Hence the nodes $\alpha(x_1),\ldots,\alpha(x_n)$ satisfy the formula $R(\alpha(x_1),\alpha(x_2)) \wedge \cdots \wedge R(\alpha(x_{n-1}),\alpha(x_n))$. Since $G \models \varphi$, and $\alpha(s),\alpha(t)$ appear among the nodes $\alpha(x_1),\ldots,\alpha(x_n)$, this implies that $(\alpha(s),\alpha(t))$ is an edge in G.

Now for the other direction, assume that G fulfills the homomorphism property, and let $a_1,\ldots,a_n$ be (not necessarily distinct) nodes in G such that they fulfill the prerequisite clause of φ, i.e. $(a_1,a_2),\ldots,(a_{n-1},a_n)$ are edges in G. Then obviously the function α mapping, for $i \in \{1,\ldots,n\}$, the variable x_i to the node a_i, is a homomorphism from $\mathsf{prereq}\,(\varphi)$ to G. Since s,t appear in $x_1,\ldots,x_n$, $\alpha(s)$ and $\alpha(t)$ is well-defined. By the prerequisites, we know that $(\alpha(s),\alpha(t))$ is an edge in G. Hence, G satisfies the formula φ. □

There is a natural correspondence between implication of these formulas and graph homomorphisms, which shows that considering Horn implications as graphs is a helpful concept.

Proposition 4.4. *Let φ_1 and φ_2 be universal Horn implications where the variables from the conclusion edge also appear in the prerequisite graph such that there exists a homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi_1) \rightarrow \mathsf{prereq}\,(\varphi_2)$, which maps the conclusion edge of φ_1 to the conclusion edge of φ_2. Then φ_1 implies φ_2.*

Proof. Let G be a graph satisfying φ_1. To show that $G \models \varphi_2$, let $\beta\colon \mathsf{prereq}\,(\varphi_2) \rightarrow G$ be a homomorphism. Then $\gamma := \beta \circ \alpha$ is a homomorphism from $\mathsf{prereq}\,(\varphi_1)$ to G. Hence, since G satisfies φ_1, by Proposition 4.3, the function γ' defined as the restriction of γ to the conclusion edge of φ_1, is a homomorphism from $\mathsf{conc}\,(\varphi_1)$ to G. If $\mathsf{conc}\,(\varphi_1) = (s_1,t_1)$ and $\mathsf{conc}\,(\varphi_2) = (s_2,t_2,)$ this means the following: By the prerequisites, $\alpha(s_1) = s_2$, and $\alpha(t_1) = t_2$. Since γ' is a homomorphism from $\mathsf{conc}\,(\varphi_1)$ to G, we know that $(\gamma(s_1),\gamma(t_1))$ is an edge in G. By definition of γ, this implies that $(\beta(s_2),\beta(t_2))$ is an edge in G, and thus the restriction of β to $\mathsf{conc}\,(\varphi_2)$ is a homomorphism from $\mathsf{conc}\,(\varphi_2)$ to G. Hence, due to Proposition 4.3, $G \models \varphi_2$. □

An easy observation is that if we cannot find the prerequisite graph of our Horn clause in a tree, then, since every satisfiable modal formula can be satisfied in a tree, the Horn formula does not contribute to the complexity of the satisfiability problem.

Proposition 4.5. *1. Let φ be a conjunction of universal Horn implications such that for every clause ψ appearing there is no homomorphism from* $\mathsf{prereq}(\psi)$ *into a reflexive, transitive tree of any height. Then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE-*hard.*

2. Let φ be a conjunction of universal Horn implications such that for every clause ψ appearing there is no homomorphism from $\mathsf{prereq}(\psi)$ *into a line of any length. Then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE-*hard.*

Proof. 1. If the prerequisite graph of ψ cannot be homomorphically mapped into a reflexive, transitive tree, then by Proposition 4.3, the reflexive, transitive tree satisfies all of the ψ, and therefore satisfies φ. Hence the proposition follows from Corollary 2.8.

2. Since any reflexive, transitive tree can be homomorphically mapped to a line, and the composition of homomorphisms is again a homomorphism, this follows from the previous part.

□

4.2 Non-proper Horn Clauses

As mentioned before, Horn clauses which are not "proper" are special cases, and easy to deal with. We will give complexity classifications of these cases before turning to more interesting logics in the remainder of this paper. Many results in this section are directly implied by Ladner's Theorem 2.7 and its corollaries. First, we consider the purely negative case.

Theorem 4.6 (Purely negative Horn clause dichotomy). *Let φ be a universal purely negative Horn clause. Consider the variables as a graph, where there is an edge from x to y if and only if the clause $\overline{(xRy)}$ appears in the disjunction.*

1. *If this graph cannot be mapped to a line homomorphically, then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE-*complete.*
2. *Otherwise,* $\mathsf{K}(\varphi)$-SAT *is* NP-*complete.*

Proof. Very analogously to Proposition 4.3, it can be shown that a graph G satisfies the formula φ if and only if the variable-graph cannot be mapped homomorphically into G. Therefore, the following holds:

1. If the graph cannot be homomorphically mapped to a line, then obviously, the formula is satisfied in any tree. Therefore, the complexity result follows with Lemma 2.10.
2. If the graph can be homomorphically mapped to a line, then any graph which has a line of more then this length does not satisfy the clause φ. Therefore, models satisfying φ can only be of constant depth, and hence the logic $\mathsf{K}(\varphi)$ has the polynomial size model property due to Lemma 3.6. Membership in NP therefore follows with Theorem 2.6. The satisfiability problem is NP-hard, since the irreflexive singleton satisfies φ, and therefore the class of models is not empty. Thus, the propositional satisfiability problem trivially reduces to $\mathsf{K}(\varphi)$-SAT.

□

It should be noted that while the purely negative case is not difficult to deal with, there are some practical examples that fall into this category: for example, the formula $\varphi_{\text{irref}} := \overline{(xRx)}$ is true in a graph if and only if it is irreflexive. From Theorem 4.6, it follows that the satisfiability problem for the logic over the class of irreflexive graphs is PSPACE-complete: the variable-graph arising from this formula consists of a single reflexive node, and since a line does not have a reflexive node, this cannot be mapped to a line homomorphically.

We now show that for the other combinations which are not proper Horn implications, complexity dichotomies are also very easy to obtain.

Theorem 4.7 (Independant Horn clause dichotomy). *Let φ be an independent universal Horn implication. If the prerequisite graph can be homomorphically mapped to a line, then the satisfiability problem for any universal elementary logic extending* $\mathsf{K}(\varphi)$ *is in* NP. *Otherwise, the satisfiability problem for* $\mathsf{K}(\varphi)$ *is* PSPACE*-complete.*

Proof. First assume that the prerequisite graph can be mapped to a line, and let the conclusion edge be (x, y). Since the clause is independent, the left side of the implication is satisfied by any pair of nodes (x, y) where x and y have "enough" predecessors and successors: as soon as one vertex with sufficient depth and height exists, a graph satisfying φ must be universal. In particular, the logic $\mathsf{K}(\varphi)$ satisfies the prerequisites of Lemma 3.9, and hence the satisfiability problem is in NP.

Now if the graph cannot be mapped to a line, then the result follows directly from Lemma 2.10, since such a formula is satisfied in every strict tree. □

And finally, it is obvious that reflexive Horn clauses are of no interest either.

Lemma 4.8 (Reflexive Horn clause classification). *Let φ be a reflexive Horn clause. Then* $\mathsf{K}(\varphi)$-SAT *is* PSPACE*-hard.*

Proof. This follows directly from Corollary 2.8, since reflexive Horn clauses are satisfied in every reflexive graph. □

Now that we have cleared the uninteresting cases for our work, we concentrate on proper Horn implications for the rest of this section.

4.3 An important Horn implication

The following logics are of central interest to our work: in our Horn clauses, the following definition captures the case where the variables in the conclusion edge have a common predecessor in the prerequisite graph, but there is not necessarily a direct path between them. It turns out that this is not as special a case as it first seems: using results about graphs satisfying this formula, we will be able to show most of the NP-containment results for universal Horn implications.

Definition 4.9. *Let $\varphi^{l\to k}$ be the formula*

$$(wRx_1) \wedge (x_1Rx_2) \wedge (x_{l-1}Rx_l) \wedge (wRy_1) \wedge (y_1Ry_2) \wedge \cdots \wedge (y_{k-1}Ry_k) \implies (x_lRy_k),$$

where all variables are universally quantified (and in the case that x_0 or y_0 appear in the formula, we replace them with w).

In Figure 5, we show how the graph representation of the formula $\varphi^{2\to 4}$ looks like. The graph property described by these formulas is easy to see:

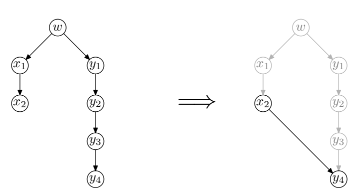

Fig. 1. Example implication $\varphi^{2\to 4}$

Proposition 4.10. *Let G be a graph, and let $k, l \in \mathbb{N}$. Then $G \models \varphi^{k \to l}$ if and only if for any nodes $w, x_k, y_l \in G$, if $G \models w \xrightarrow{k} x_k$ and $G \models w \xrightarrow{l} y_l$, then (x_k, y_l) is an edge in G.*

This definition generalizes several well-known examples:

- $\mathsf{K}\left(\varphi^{0\to 1}\right)$ is the logic over all frames,
- $\mathsf{K}\left(\varphi^{0\to 2}\right)$ is the logic over transitive frames,
- $\mathsf{K}\left(\varphi^{1\to 0}\right)$ is the logic over symmetric frames,
- $\mathsf{K}\left(\varphi^{0\to 0}\right)$ is the logic over reflexive frames,
- $\mathsf{K}\left(\varphi^{1\to 1}\right)$ is the logic over Euclidean frames.

Graphs fulfilling these formulas play a central role in our NP cases. Therefore we will prove some connectivity results for these graphs.

Lemma 4.11. *Let $1 \leq k, l \in \mathbb{N}$, and let G be a graph such that $G \models \varphi^{k \to l}$.*

1. *If $k < l$, then $G \models \varphi^{l+k-1 \to l+k}$.*
2. *If $l < k$, then there exist $k', l' \in \mathbb{N}$, such that $1 \leq k' \leq l' - 2$, and $G \models \varphi^{k' \to l'}$.*
3. *If $l = k+1$, then for any $i \geq k$, $G \models \varphi^{i \to i+1}$.*

Proof. 1. Let w be some node in G, such that $G \models w \xrightarrow{l+k-1} x_{l+k-1}$, and $G \models w \xrightarrow{l+k} y_{l+k}$, and let the (not necessarily distinct) intermediate vertices be denoted with x_i, y_i . Since $\varphi^{k\to l}$ holds in G, this implies that there is an edge (y_k, x_l). By definition, $G \models x_l \xrightarrow{k-1} x_{l+k-1}$. Combining these, we obtain a path of length k from y_k to x_{l+k-1}. On the other hand, $G \models y_k \xrightarrow{l} y_{k+l}$. Since $\varphi^{k\to l}$ holds in G, it follows that there is an edge from x_{l+k-1} to y_{l+k}, proving that $\varphi^{l+k-1\to l+k}$ holds in G.

2. We show that the claim holds for $k' := l$, and $l' := k + (k-1)(k-l) + l$. Again, let $w = x_0, x_1, \ldots, x_k$ and $w = y_0, y_1, \ldots, y_{l'}$ be nodes in G such that (x_i, x_{i+1}) and (y_j, y_{j+1}) are edges in G for all relevant i, j. It obviously follows that for all $m \geq l$, there is an edge $(y_{m+(k-l)}, y_m)$, since $G \models y_{m-l} \xrightarrow{k} y_{m+(k-l)}$ and $G \models y_{m-l} \xrightarrow{l} y_m$. Therefore, the following paths exist:
   (a) There is a $k-1$-step path from $y_{k+(k-1)(k-l)}$ to y_k,
   (b) there is, directly due to the $\varphi^{k\to l}$-property, an edge (y_k, x_l) in G,
   (c) the preceding two imply $G \models y_{k+(k-1)(k-l)} \xrightarrow{k} x_l$.
   (d) there is, by definition, an l-step path from $y_{k+(k-1)(k-l)}$ to $y_{k+(k-1)(k-l)+l}$.

   Therefore, by the $\varphi^{k\to l}$-property, there is an edge $(x_l, y_{k+(k-1)(k-l)+l}) = (x_{k'}, y_{l'})$. It remains to prove that $1 \leq k' \leq l' - 2$. Since $k' = l$, it follows that $1 \leq k'$. Now, due to the prerequisites, it follows that $k \geq l+1, l \geq 1$. Therefore, it holds that $k + (k-1)(k-l) + l \leq k + 1 \cdot 1 + 1$. Since $k' = l$, and $l < k$, it follows easily that $k' \leq k + 2 - 2$, as required.

3. Clearly it suffices to show $G \models \varphi^{k+1\to k+2}$, the claim for i follows inductively. Let w, x_j, y_j be defined as usual: $w = x_0 = y_0$, and there are edges (x_j, x_{j+1}) and (y_j, y_{j+1}). We need to show that there is an edge (x_{k+1}, y_{k+2}).
   Since $G \models \varphi^{k\to k+1}$, it follows that (y_k, x_{k+1}) is an edge in G. Since there obviously is a path of length $k-1$ from y_1 to y_k, it follows that there is a path of length k from y_1 to x_{k+1}. Since there also is a path of length $k+1$ from y_1 to y_{k+2}, it follows that there is an edge (x_{k+1}, y_{k+2}) in G, which concludes the proof.

□

4.4 Generalizations of $\varphi^{k\to l}$

As mentioned earlier, the formula $\varphi^{k\to l}$ is supposed to capture the case where the variables in the conclusion edge of a universal Horn implication have a common predecessor. But it is easy to see that not all of these cases are covered with this formula. The above Figure 5 is a graphical representation of what the implication in $\varphi^{2\to 4}$ does. But what if this is only a subgraph of the prerequisite graph? In a more general case, the node w and the nodes x_k, y_l will have more predecessors and successors. Figure 2 gives an example of a more general formula.

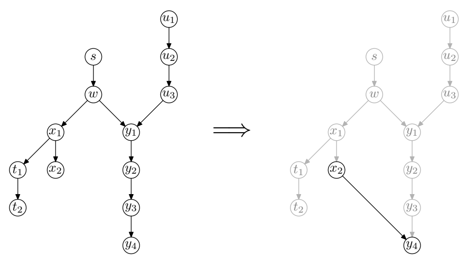

Fig. 2. More general formula

We will now see that this formula can be "simplified." This simplification is not an equivalent transformation of the formula, but we construct a new formula which is implied by the original one. The one-sided implication suffices to show many of the results we need. The simpler formula is presented in Figure 3.

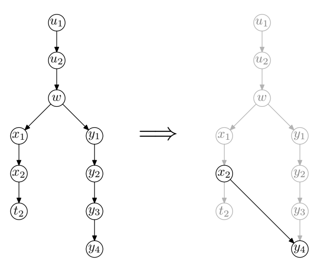

Fig. 3. Simplified formula

It is easy to see that every graph which satisfies the formula displayed in Figure 2 also satisfies the formula from Figure 3. This follows directly from Proposition 4.4, since the prerequisite graph from Figure 2 can obviously be mapped homomorphically to the prerequisite graph from Figure 3 (the homomorphism h is defined as $h(t_1) := x_2, h(s) := u_2, h(u_3) := w$, and maps the other nodes to the ones with the same labels). Hence, if we can show NP-containment for all modal logics extending the one defined by the later formula, this puts the logic defined by the original formula into NP as well.

Any universal Horn implication which can be mapped onto a tree can be embedded in a graph with certain properties, namely the properties of the formula we now define. These formulas capture the generalizations of $\varphi^{k\to l}$ mentioned above, where we demand that the nodes w, x_k, y_l have a sufficient number of predecessors or successors. We again use the representation of Horn clauses as graphs.

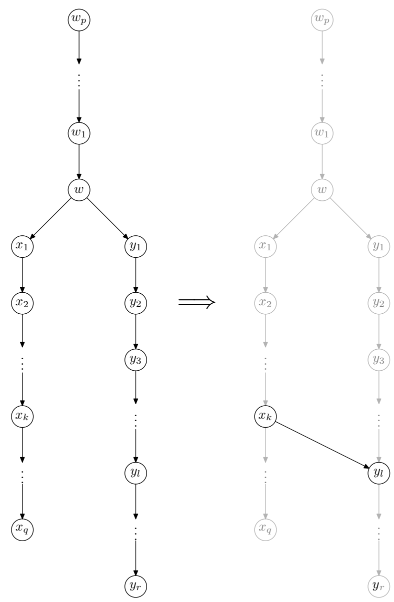

Fig. 4. The formula $\varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$

Definition 4.12. *For $k, l, p, q, r \in \mathbb{N}$, the formula $\varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$ is defined as the universal Horn clause displayed in Figure 4.*

It should be noted that the notation $\varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$ suggests that w, y, x can be compared to natural numbers, but what is meant in that notation is simply that the number of predecessors (successors, resp.) of w, x_k, y_l can be compared to p, q, and r, respectively. Hence, if we use "natural names" for the vertices, i.e. we have vertices $w = x_0, x_1, \dots,$ and $w = y_0, y_1, \dots$, then this ensures that the vertices up to x_q and y_r exist. When proving that this formula holds in a graph, we will usually rely on the notation provided in Figure 4, and assume that there are nodes $w_{-p}, \dots, w_0 = w = x_0 = y_0, x_1, \dots, x_q, x_1, \dots, y_r$ with edges as seen in Figure 4, i.e., most of the time we do not mention the homomorphism explicitly.

The formula $\varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$ can be seen to be only a slight generalization of the formulas $\varphi^{k\to l}$ we already considered, as exhibited by the following proposition:

Proposition 4.13. *Let G be a graph, and let $k, l, p, q, r \in \mathbb{N}$. Then $G \models \varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$ if and only if the following condition holds: For any nodes $w, x_k, y_l \in G$, such that w has a p-step predecessor, x has a $q-k$-step successor and y has an $r-l$-step successor, $G \models w \xrightarrow{k} x_k$ and $G \models w \xrightarrow{l} y_l$, it follows that (x_k, y_l) is an edge in G.*

The above proposition immediately implies the following:

Proposition 4.14. *Let $G \models \varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$, and let G' be the restriction of G to the set*

$$C := \left\{ w \in G \mid \mathit{maxdepth}^G(w) \geq p \text{ and } \mathit{maxheight}^G(w) \geq \max(q-k, r-l) \right\},$$

then $G' \models \varphi^{k\to l}$.

Hence, we can use Lemmas 3.6 and 3.8 together with Proposition 4.14 to restrict our attention to the vertices in the "middle" of the graph, and then talk about the formula $\varphi^{k\to l}$ instead of $\varphi^{k\to l}_{w\geq p, x\geq q, y\geq r}$.

There obviously is a close relationship between formulas of the form $\varphi^{k\to l}$ and formulas of the form $\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}$. In particular, this relationship allows for the "lifting" of implications, as is shown in the following easy lemma.

Lemma 4.15. *Let $p,q,r,k,l,k',l'\in\mathbb{N}$, and let $\varphi^{k\to l}$ imply $\varphi^{k'\to l'}$. Then $\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}$ implies $\varphi^{k'\to l'}_{w\geq p',x\geq q',y\geq r'}$, where*

$$\begin{aligned}p' &:= p,\\ q' &:= k'+\max(q-k,r-l),\\ r' &:= l'+\max(q-k,r-l).\end{aligned}$$

Proof. Let G be a graph satisfying $\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}$. Let w,x_i,y_i be the nodes in the graph connected as the nodes in the prerequisite graph of $\varphi^{k'\to l'}_{w\geq p',x\geq q',y\geq r'}$. Let G' be the graph G restricted to the set of vertices which have a p-step predecessor, and $\max(q-k,r-l)$-step successor in G. Then, by choice of nodes, and Propositions 4.13 and 4.10, it follows that $G'\models\varphi^{k\to l}$. Hence, due to the prerequisites, we know that $G\models\varphi^{k'\to l'}$. In particular, since the nodes w,x_i,y_i satisfy the prerequisite graph of $\varphi^{k'\to l'}_{w\geq p',x\geq q',y\geq r'}$, we know that $w,x_k,y_l\in G'$. Hence, it follows that (x_k,y_l) is an edge in G', and therefore it is an edge in G, as claimed. ☐

4.5 NP results

In this section, we prove NP-containment results. Using connectivity results from the previous section, we show that "most" of the logics of the form $\mathsf{K}\left(\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}\right)$ give rise to a satisfiability problem in NP. In the remainder of the section, we obtain further NP-containment results for cases not covered directly by this theorem.

The relationship between formulas of the form $\varphi^{k\to l}$ for arbitrary $k<l$ and formulas $\varphi^{k\to k+1}$ is interesting. As seen in Lemma 4.11, a formula of the former type always implies one of the latter type. Therefore, it is sufficient to prove results for the latter. However, in the proof for Theorem 4.16 shows that if we consider graphs satisfying a formula $\varphi^{k\to k+1}$, then in many cases, sub-graphs which satisfy formulas of the more general kind $\varphi^{k\to l}$ for some $k\leq l-2$ arise. Hence, both types of formulas are relevant for us.

It is comparably easy to show that a logic of the form $\mathsf{K}\left(\varphi k\to k+1\right)$ leads to a satisfiability problem in NP, by carefully "copying" vertices and adding the right neighbors. However, while this process leads to a model which still satisfies $\varphi k\to k+1$, it does not give the desired result that the problem can be solved in NP for all logics above $\mathsf{K}\left(\varphi k\to k+1\right)$. In order to prove this, our model-manipulations must be consistent with the conditions of Theorem 3.1. In the proof of the following theorem, we can again construct a small model using only restriction.

Theorem 4.16. *Let KL be a universal elementary modal logic which is an extension of $\varphi^{k\to k+1}_{w\geq p,x\geq q,y\geq r}$ for some $k,p,q,r\in\mathbb{N},k\geq 1$. Then $\mathsf{KL}\text{-}\mathsf{SAT}\in\mathrm{NP}$.*

Proof. We show the claim with proving that the logic KL has the polynomial size model property. The result then follows from Theorem 2.6. Hence, let ϕ be a modal formula, and let M be a KL-model with world w such that $M,w\models\phi$. If ϕ is KL-satisfiable, then such a model always exists due to Proposition 3.5. Now, define

$$\begin{aligned}A &:= \left\{v\in M \mid \mathrm{maxdepth}^M(v)\leq p+k+2\right\}\cup\{w\},\\ S &:= \left\{v\in M \mid \mathrm{maxheight}^M(v)\geq \max(q-k,r-k-1)\right\},\\ C &:= (M\setminus A)\cap S.\end{aligned}$$

The set C contains all the nodes in M which have “enough” predecessors and successors to ensure that the formula $\varphi^{k\to k+1}_{w\geq p,x\geq q,y\geq r}$ gives us all the necessary edges that we are interested in. To be more precise, we show the following fact:

Fact 1 $C \models \varphi^{1\to 2}$.

Proof. Let w', x_1, y_1, y_2 be vertices from C, such that they satisfy the prerequisites of $\varphi^{1\to 2}$, i.e., let (w', x_1), (w', y_1), and (y_1, y_2) be edges in M. Since $w' \in C$, we know that w' has a $k-1$-step predecessor w', which in turn has a p-step predecessor. Due to the edges defined above, it obviously holds that $M \models w'' \xrightarrow{k} x_1$, and $M \models w'' \xrightarrow{k+1} y_2$. Since both x_1 and x_2 are elements of C, they have the required number of successors, and therefore Proposition 4.13 implies that (x_1, y_2) is an edge in M, as claimed. □

This immediately gives the following corollary:

Fact 2 *For all $l \geq 0$, it holds that $C \models \varphi^{l\to l+1}$.*

Proof. This follows directly from Fact 1 and Lemma 4.11, and the observation that $\varphi^{0\to 1}$ is true in any graph. □

Due to Lemma 3.6, we can assume, without loss of generality, that $|A|$ is polynomially bounded (in the length of ϕ). Due to Proposition 3.5, we can also assume that every world in M can be reached from w in $\mathsf{md}(\phi)$ steps. Therefore, we can choose a set $G \subseteq M$ with the following properties:

- G is polynomially bounded,
- $G \subseteq C$,
- For every node $u \in A$ and every subformula ψ of ϕ such that $M, u \models \Diamond\psi$, there is a node $v \in G$ such that (u, v) is an edge in M, and $M, v \models \psi$.
- every node in $M \setminus A$ can be reached on a path of length at most $\mathsf{md}(\phi)$ from some node in G.

Such a set can be chosen with a method which is identical to the one in the proof of Lemma 3.6 (we can choose the set G to be the set W_{p+k+2} from the construction in the proof of Lemma 3.6).

For every $g \in M$, we define M_g^C to be the set $M_g \cap C$. Obviously, M_g^C is a graph with root g, and if $g \in G$, then M_g^C is a restriction of C. Hence, Proposition 3.2 and Fact 2 immediately imply the following:

Fact 3 *Let $g \in G$. Then M_g^C is a graph with root g such that $M_g^C \models \varphi^{l\to l+1}$ for all $l \geq 0$.*

For any $g \in M$ and $i \in \mathbb{N}$, we define

$$L_i^g := \left\{ v \in M_g^C \mid M \models g \xrightarrow{i} v \right\}.$$

We observe the following fact:

Fact 4 *Let $g \in G$, $i \in \mathbb{N}$, $x \in L_i^g$, and $y \in L_{i+1}^g$. Then (x, y) is an edge in M.*

Proof. This immediately follows from Fact 3 and the definition of L_i^g. □

Graphs fulfilling the formula $\varphi^{1\rightarrow 2}$ are "layered:" whenever there are nodes x and y such that there is a path of length i from the root to i, and of length $i+1$ to the node y, then there is an edge between x and y. We say that such a graph is *canonical*, if there are no other edges than the ones demanded by this property, i.e., if the "layers" given by these edges are disjoint. To be more precise, for the graphs M_g^C appearing in our construction, we say that for some $g \in M$, the graph M_g^C is *canonical*, if for every pair of natural numbers $i \neq j$, it follows that $L_i^g \cap L_j^g = \emptyset$. Intuitively, this means that every node in this graph has a uniquely determined depth. We make a distinction between those elements in G which lead to a canonical graph, and those which do not. In light of Lemma 3.4, it is obvious that we do not need to look at the entire graph, but can ignore nodes which do not have "short" paths from the root of the graph leading to it. For a natural number b, we say that a graph G is *b-canonical*, if for all $0 \leq i < j \leq b$ it holds that $L_i^g \cap L_j^g = \emptyset$. It is obvious that if G is canonical, then G also is b-canonical for any b.

We define the following:

$$G_{\text{can}} := \left\{g \in G \mid M_g^C \text{ is } \mathsf{md}\,(\phi)\text{-canonical}\right\},$$
$$G_{\text{non-can}} := \left\{g \in G \mid M_g^C \text{ is not } \mathsf{md}\,(\phi)\text{-canonical}\right\},$$

It is obvious that $G = G_{\text{can}} + G_{\text{non-can}}$. For each $g \in G_{\text{non-can}}$, let $i(g), j(g)$ denote natural numbers, and let $n(g)$ denote some node such that $n(g) \in L_{i(g)}^g \cap L_{j(g)}^g$, with $0 \leq i(g) < j(g) \leq \mathsf{md}\,(\phi)$, and $i(g)$ is minimal with these properties. The subgraphs of these "minimal non-canonical points" have an interesting property:

Fact 5 *Let $g \in G_{\text{non-can}}$. Then $M_{n(g)}^C \models \varphi^{1\rightarrow 2+j(g)-i(g)}$.*[3]

Proof. Let $w', x_1, y_{2+j(g)-i(g)} \in M_{n(g)}^C$ such that $M \models w' \xrightarrow{1} x_1$ and $M \models w' \xrightarrow{2+j(g)-i(g)} y_{2+j(g)-i(g)}$. Due to Proposition 4.10, it suffices to show that $(x_1, y_{2+j(g)-i(g)})$ is an edge in M.

We know that the following holds: $M \models n(g) \xrightarrow{j(g)-i(g)} n(g)$. This is because since due to induction on Fact 4, there is a path of length $j(g)-i(g)$ between any pair of vertices from $L_{i(g)}^g \times L_{j(g)}^g$, and since $n(g)$ is a member of both, this implies that there is a path of this length from $n(g)$ to itself. Now, since $w' \in M_{n(g)}^C$, we know that there is a path from $n(g)$ to w' of some length s. In combination with the self-loop at $n(g)$, this implies the following:

- $M \models n(g) \xrightarrow{s} w'$,
- $M \models n(g) \xrightarrow{s+j(g)-i(g)} w'$.

Combining this with the paths from w' to x_1 and $y_{2+j(g)-i(g)}$, this gives the following:

- $M \models n(g) \xrightarrow{s+j(g)-i(g)+1} x_1$,
- $M \models n(g) \xrightarrow{s+2+j(g)-i(g)} y_{2+j(g)-i(g)}$.

Hence, since due to Fact 3, $M_{n(g)}^C \models \varphi^{s+j(g)-i(g)+1\rightarrow s+j(g)-i(g)+2}$, there is an edge $(x_1, y_{2+j(g)-i(g)})$, as required. the node w' satisfies the condition of the node z as defined above, concluding the proof of Fact 5. ☐

[3] proofread

Above, we claimed that $n(g)$ was a "minimal non-canonical point" in M_g^C. The following makes this more precise:

Fact 6 *Let $g \in G_{\text{non-can}}$. Then $M_g^C \setminus M_{n(g)}^C$ is $\mathsf{md}(\phi)$-canonical.*

Proof. Assume that this is not the case. Then there exist natural numbers $i_1 < i_2 \leq \mathsf{md}(\phi)$, and some node $v \in L_{i_1}^g \cap L_{i_2}^g$. Due to the minimality of $i(g)$, it follows that $i(g) \leq i_1$, and hence $i(g) < i_2$. In particular, it follows by induction on Fact 4, that there is a path from $n(g) \in L_{i(g)}^g$ to $v \in L_{i_2}^g$, i.e., $v \in M_{n(g)}^C$. This is a contradiction. □

We therefore have the following structure of the model M : It holds that M can be written as the union of sub-graphs as follows:

$$M = A \cup \bigcup_{g \in G_{\text{can}}} \underbrace{M_g^C}_{\mathsf{md}(\phi)-\text{canonical}} \cup \bigcup_{g \in G_{\text{non-can}}} \underbrace{(M_g^C \setminus M_{n(g)}^C)}_{\mathsf{md}(\phi)-\text{canonical}} \cup \bigcup_{g \in G_{\text{non-can}}} \underbrace{M_{n(g)}^C}_{\models \varphi^{1 \to 2+j(g)-i(g)}} \cup (M \setminus S).$$

Note that the set G is polynomial in $|\phi|$. Due to Lemmas 3.6 and 3.8, it suffices to restrict the "middle part" of this equation, i.e., the components except A and $M \setminus S$, to polynomial size in order to obtain the desired polynomial model. In order to do this, we will now prove further connectivity results for the sub-models $M_{n(g)}^C$. It is important to note that due to Proposition 3.2, all of these submodels inherit all of the properties of their respective super-models which can be expressed by a universal first order formula.

The idea behind the construction is the following: for the canonical models, we know that we can stop adding nodes at depth $\mathsf{md}(\phi)$. For the other models, we cannot do this, but we also do not need to: since we will see in Fact 9 that we have "circular" edges, we know that nodes in a "low" level also are in a "high" level.

As the next connectivity result, we prove the following:

Fact 7 [4] *Let $g \in G_{\text{non-can}}$, and let $x_{i_1}, y_{i_2} \in M_{n(g)}^C$, such that $M \models n(g) \xrightarrow{i_1} x_{i_1}$, $M \models n(g) \xrightarrow{i_2} y_{i_2}$, where $i_1 \geq 1$ and $i_2 = i_1 + 1 + j(j(g) - i(g))$ for some $j \geq 0$. Then (x_{i_1}, y_{i_2}) is an edge in M.*

Proof. We show the claim by induction over j. For $j = 0$, it follows that $i_2 = i_1 + 1$, and hence the claim follows from Fact 4. Therefore, assume that the claim holds for $j - 1$. As usual, let x_i and y_i denote the vertices on the path from $n(g)$ to x_{i_1} and y_{i_2}. Since by Fact 5, we know that $M_{n(g)}^C \models \varphi^{1 \to 2+j(g)-i(g)}$, it suffices to show that there is a world $z \in M_{n(g)}^C$ such that $M \models z \xrightarrow{1} x_{i_1}$ and $M \models z \xrightarrow{2+j(g)-i(g)} y_{i_2}$. We show that the world $z = x_{i_1 - 1}$ satisfies these conditions. Since $i_1 \geq 1$, this world exists, and is a predecessor of x_{i_1}.

Let $i = i_2 - (j(g) - i(g))$. Then $i - i_1 = 1 + (j-1)(j(g) - i(g))$, and hence due to induction hypothesis, we know that there is an edge (x_{i_1}, y_i) in M, i.e., an edge $(x_{i_1}, y_{i_2 + i(g) - j(g)})$ (since $j \geq 1$, $i_2 + i(g) - j(g) \geq 0$, this node exists). We now construct a $2 + j(g) - i(g)$-step path from $x_{i_1 - 1}$ to y_{i_2} as follows:

1. $M \models x_{i_1 - 1} \xrightarrow{1} x_{i_1}$ by choice of nodes,
2. Due to the above, $M \models x_{i_1} \xrightarrow{1} y_{i_2 - (j(g) - i(g))}$,
3. By choice of nodes, $M \models y_{i_2 - (j(g) - i(g))} \xrightarrow{j(g) - i(g)} y_{i_2}$.

Hence we know that by combining the above, there is a path of length $1 + 1 + j(g) - i(g)$ from $z = x_{i_1 - 1}$ to y_{i_2}, and since there is an edge (z, x_{i_1}) be choice of nodes, this shows that the world z indeed satisfies the desired properties, concluding the proof of Fact 7. □

[4] proofread

Intuitively, Fact 7 shows that we have lots of "forward edges" from "higher" levels in $M^C_{n(g)}$ to "lower" levels. The following shows that we also have the corresponding "back edges:"

Fact 8 *Let $g \in G_{\text{non-can}}$, let $x_{i_1}, y_{i_2} \in M^C_{n(g)}$, and let $i_1, i_2 \geq 1$ such that $M \models n(g) \xrightarrow{i_1} x_{i_1}$, $M \models n(g) \xrightarrow{i_2} x_{i_2}$, where $i_2 = i_1 + 1 - j \cdot (j(g) - i(g))$ for some $j \geq 1$. Then (x_{i_1}, y_{i_2}) is an edge in M.*

Proof. As usual, let x_i and y_i denote the intermediate vertices on the paths from $w = x_0 = y_0$ to x_{i_1} and y_{i_2}. We show the claim by induction on j. First, assume that $j = 1$, i.e., $i_2 = i_1+1-(j(g)-i(g))$.

Since $n(g) \in L^g_{i(g)} \cap L^g_{j(g)}$, induction on Fact 4 shows that $M \models n(g) \xrightarrow{j(g)-i(g)} n(g)$. In particular, the following holds:

- $M \models n(g) \xrightarrow{i_1} x_{i_1}$,
- $M \models n(g) \xrightarrow{j(g)-i(g)+i_2} y_{i_2}$,

and therefore it follows that

$$M \models n(g) \xrightarrow{i_1+1-(j(g)-i(g))+j(g)-i(g)} y_{i_2},$$

hence $M \models n(g) \xrightarrow{i_1+1} y_{i_2}$. Since due to Fact 3, $M^C_{n(g} \models \varphi^{i_1 \to i_1+1}$, it follows that (x_{i_1}, y_{i_2}) is an edge as claimed.

Now assume that the claim holds for $j-1$, and for the case $j = 1$. Again, it suffices to prove that there is a node $z \in M^C_{n(g)}$, such that $M \models z \xrightarrow{1} x_{i_1}$, and $M \models z \xrightarrow{2} y_{i_2}$, since due to Fact 1, we know that $M^C_g \models \varphi^{1 \to 2}$, and since $M^C_{n(g)}$ is a submodel of M^C_g, Proposition 3.2 implies that this formula also holds in $M^C_{n(g)}$.

We choose z to be x_{i_1-1}. Since $i_1 \geq 1$, this world exists. By choice of nodes, we know that $M \models z = x_{i_1-1} \xrightarrow{1} x_{i_1}$.

Since $i_2 \geq 1$, and $i_1 = i_2 + j(j(g) - i(g)) - 1$, and $j(g) > i(g)$, it follows that $i_1 \geq j$. In particular, since the case $j = 1$ is covered in the induction hypothesis, we can conclude that $i_1 \geq 2$. Therefore we can apply the induction hypothesis for $j-1$, and conclude that $M \models x_{i_1-1} \xrightarrow{1} x_{i_1-1+1-(j-1)(j(g)-i(g))}$, i.e., $M \models x_{i_1-1} \xrightarrow{1} x_{i_1-(j-1)(j(g)-i(g))}$ (the node x_{i_1-1} exists, since $i_1 > i_2 \geq 1$). It therefore remains to show that $M \models x_{i_1-(j-1)(j(g)-i(g))} \xrightarrow{1} y_{i_2}$. By the induction hypothesis for the case $j = 1$, it follows that

$$M \models x_{i_1-(j-1)(j(g)-i(g))} \xrightarrow{1} y_{i_1-(j-1)(j(g)-i(g))+1-(j(g)-i(g))}.$$

Since $i_1 - (j-1)(j(g)-i(g)) + 1 - (j(g)-i(g)) = i_1 - j(j(g)-i(g)) + 1 = i_2$, the claim follows, and hence the proof of Fact 8 is complete. □

The preceding two facts give the following easy corollary:

Fact 9 *Let $g \in G_{\text{non-can}}$, and let $x \in L^{n(g)}_{i_1}$, $y \in L^{n(g)}_{i_2}$, where $i_1, i_2 \geq 1$. amd $i_2 \equiv i_1+1 \mod (j(g)-i(g))$. Then (x, y) is an edge in M.*

Proof. Let $i_2 = i_1 + 1 + j(j(g) - i(g))$ for some $j \in \mathbb{Z}$. If $j \geq 0$, then the claim follows directly from Fact 7, if $j < 0$, then it follows from Fact 8. □

We now need to ensure that for $g \in G_{\text{non-can}}$, the graph $M_g^C \setminus M_{n(g)}^C$ still has root g, unless it is empty.

Fact 10 *Let $g \in G_{\text{non-can}}$, such that $M_g^C \setminus M_{n(g)}^C$ is not empty. Then $M_g^C \setminus M_{n(g)}^C$ has root g.*

Proof. Assume that this is not the case, i.e., that there is some $v \in M_g^C \setminus M_{n(g)}^C$, and in this graph, there is no path from g to v. Since v is an element of M_g^C, a path from g to v exists in the original model M. Let $g \to v_1 \to \cdots \to v_i \to v$ be this path. Since the path does not exist in the graph $M_g^C \setminus M_{n(g)}^C$, it follows that one of the v_j must be an element of $M_{n(g)}^C$. Since there is a path from this v_j to v, it follows that v is an element of $M_{n(g)}^C$ as well, which is a contradiction. □

We now have all the necessary information to construct the polynomial-size model to conclude the proof of Theorem 4.16. For this, let $G_{\text{can}} = \{g_1, \ldots, g_n\}$, and let $G_{\text{non-can}} = \{g_{n+1}, \ldots, g_{n+m}\}$, For $1 \leq i \leq m$, define $g_{n+m+i} := n(g_{n+i})$ (note that in this case, by definition g_{n+i} is a member of $G_{\text{non-can}}$). For $1 \leq i \leq n + 2m$, define

$$N_i := \begin{cases} M_{g_i}^C, & \text{if } 1 \leq i \leq n, \\ M_{g_i}^C \setminus M_{n(g_i)}^C, & \text{if } n+1 \leq i \leq n+m, \\ M_{n(g_i)}^C, & \text{if } n+m+1 \leq i \leq n+2m. \end{cases}$$

Note that by definition and Fact 10 it follows that N_i is a graph which is either empty or has root g_i. It also follows from the above that for $1 \leq i \leq n+m$, the graph N_i is canonical. Since the union over all N_i is the same as the union over all M_g^C (for all $g \in G$), it follows that

$$M = A \cup \left(\bigcup_{1 \leq i \leq n+2m} N_i \right) \cup (M \setminus S).$$

Note that $n + 2m$ is polynomial in $|\phi|$, since G is. We now define a series of models M_i for $0 \leq i \leq n+2m$, which, step by step, integrate "enough" vertices of the original model M to ensure that the fomrula ϕ still holds, but restrict the size to a polynomial. As the induction start, we define $M_0 := A$. For $i \geq 1$, the construction is as follows:

- Add every world from M_{i-1} to M_i,
- add g_i to M_i,
- If $1 \leq i \leq m+n$, i.e., if N_i is $\mathsf{md}(\phi)$-canonical, then for each $0 \leq j \leq \mathsf{md}(\phi)$, and each formula $\psi \in \mathsf{sf}(\phi)$: if there is a world $v \in L_j^{g_i}$ such that $M, v \models \psi$, then add *one* of these worlds into M_i.
- If $n+m+1 \leq i \leq n+2m$, i.e., if N_i is not $\mathsf{md}(\phi)$-canonical, then for each $0 \leq j \leq j(g)-i(g)+1$, and each formula $\psi \in \mathsf{sf}(\phi)$: if there is a world $v \in L_j^{g_i}$ such that $M, v \models \psi$, then add *one* of these worlds into M_i.

By construction, since $j(g)$ and $i(g)$ are polynomial in $|\phi|$ for each $g \in G_{\text{non-can}}$, and $|\mathsf{sf}(\phi)|$ also is obviously polynomial in $|\phi|$, there are only polynomially many worlds in M_{n+2m}. We now define M' to be the model $M_{n+2m} \cup (M \setminus S)$. Then the set of worlds in M' which have $\max(q-k, r-k-1)$-step successor is polynomially bounded, since this is a subset of M_{n+2m}. Hence, due to Lemma 3.8, it suffices to show that $M', w \models \phi$ in order to exhibit a model of ϕ which is polynomial in size. In order to prove this, we show that M' satisfies the conditions of Lemma 3.4. Since $w \in A$ by definition and $M_0 = A$, w is an element of M' by definition. Therefore, let u be an element of M', and let ψ be a subformula of ϕ, such that $M, u \models \Diamond\psi$, and let a be a natural number such that $M \models w \xrightarrow{a} u$, and $1 + a + \mathsf{md}(\psi) \leq \mathsf{md}(\phi)$. It suffices to show that there is a world $v \in M'$ such that $M, v \models \psi$, and (u, v) is an edge in M.

Since $M, u \models \Diamond\psi$, there is a world $v' \in M$, such that (u, v') is an edge in M, and $M, v' \models \psi$.

There are several cases to consider. If $v' \in A$, then by the construction of M', v' is an element of M' as well. Hence we can choose v to be v'. If $u \in A$, and $v \notin A$, then by construction there is a world $v' \in G$, such that $M, v' \models \psi$, and (u, v') is an edge in M. Since $G \subseteq M'$, we can again choose $v = v'$. If $v' \notin S$, then $v' \in M'$ by construction, and we can again choose $v = v'$. If $u \in M \setminus S$, then $v' \in M \setminus S$ holds as well.

Therefore, it remains to consider the case $u, v' \in C$. Hence, there is some $i \in \{1, \ldots, n + 2m\}$, such that $u, v' \in N_i$. Let i be maximal with this property, i.e., if it is possible to choose this i in such a way that N_i is not canonical, then we do so.

In particular, since (u, v') is an edge in M, there is some natural number j such that $u \in L_j^{g_i}$, and $v' \in L_{j+1}^{g_i}$. If it is only possible to choose an i leading to a $\mathsf{md}(\phi)$-canonical model N_i, then choose i and j in such a way that j is minimal with this property. We make a case distinction.

Case 1: $i \in \{n + m + 1, \ldots, n + 2m\}$, **i.e.,** N_i **is not canonical.** Note that Fact 7 does only give us the forward edges for the case that i_1 is at least 1. Because of this, there are a few sub-cases to consider.

We first show that in this case, there is some natural number $j' \leq j(g) - i(g) + 1$ such that $u \in L_{j'}^{g_i}$. If $j \leq j(g) - i(i) + 1$, then this is obvious. Hence, assume that $j \geq j(g) - i(g) + 2$. In this case, we can write j as $t + s(j(g) - i(g))$ for some $t < j(g) - i(g)$ and a natural number s.

- **Case a:** $t = 0$ In this case, it follows that $s \geq 2$. Let x be some world from $L_{j(g)-i(g)-1}^{g_i}$ (note that this set cannot be empty, since u is in $L_j^{g_i}$, and hence the preceeding "levels" also cannot be empty). Then, by Fact 9, there is an edge from x to u, and hence $u \in L_{j(g)-i(g)}^{g_i}$.
- **Case b:** $t = 1$ In this case, it follows that $s \geq 2$. Choose a world $x \in L_{j(g)-i(g)}^{g_i}$ (again, this set is non-empty). Then again Fact 9 implies that (x, u) is an edge in M, and hence $u \in L_{j(g)-i(g)+1}^{g_i}$.
- **Case c:** $t \geq 2$ Let $x \in L_{t-1}^{g_i}$. Since $t - 1 \geq 1$, Fact 10 implies that (x, u) is an edge in M, and hence $u \in L_t^{g_i}$.

Since $u \in L_{j'}^{g_i}$, we also know that $v' \in L_{j'+1}^{g_i}$. Now if $j' + 1 \leq j(g) - i(g) + 1$, by construction there is an $v \in L_{j'+1}^{g_i} \cap M'$, such that $M, v \models \psi$, and due to Fact 4, there is an edge (u, v) in M. Therefore, assume that $j' + 1 > j(g) - i(g) + 1$, i.e., $j' > j(g) - i(g)$. Due to the choice of j', it then follows that $j' = j(g) - i(g) + 1$. In particular, we know that $j' \equiv 1 \bmod (j(g) - i(g))$, and $j' + 1 \equiv 2 \bmod (j(g) - i(g))$. Let x be a world in $L_1^{g_i}$, then from Fact 9, we know that there is an edge from x to v' in M. In particular, it follows that $v' \in L_2^{g_i}$. Hence, by construction, there is a world $v \in L_2^{g_i}$ such that $M, v \models \psi$. Since $u \in L_1^{g_i}$, and $v \in L_2^{g_i}$, Fact 4 implies that there is an edge (u, v) in M as claimed.

Case 2: $i \in \{1, \ldots, n + m\}$, **i.e.,** N_i **is** $\mathsf{md}(\phi)$**-canonical.** If $j' + 1 \leq \mathsf{md}(\phi)$, then, due to the construction of M', a world v from $L_{j'+1}^{g_i}$ satisfying ψ was added in the construction, and due to Fact 4, (u, v) is an edge in M.

Therefore, assume that $j' + 1 > \mathsf{md}(\phi)$, i.e., $j' \geq \mathsf{md}(\phi)$. Since N_i is $\mathsf{md}(\phi)$-canonical, we know that u cannot be reached from g_i with a path shorter than $\mathsf{md}(\phi)$ steps. Due to the choice of i and j, we also know that u does not appear in a non-$\mathsf{md}(\phi)$-canonical submodel, and that in each $\mathsf{md}(\phi)$-canonical submodel where u appears, it has depth of at least $\mathsf{md}(\phi)$. This means that from the root w of the original model, u cannot be reached on any path which is shorter than $\mathsf{md}(\phi)$. In particular, this implies that $a \geq \mathsf{md}(\phi)$, which is a contradiction to the choice of a.

Hence, we know that $M', w \models \phi$, finally concluding the proof of Theorem 4.16. □

The above Theorem 4.16 is our most general NP-containment result, covering a lot of logics. The following corollary shows that indeed most of the "interesting" cases are covered:

Corollary 4.17. *Let* KL *be a modal logic above* $\mathsf{K}\left(\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}\right)$, *where* $1\leq k<l$, *and* $p,q,r\in\mathbb{N}$. *Then* KL-SAT $\in$ NP.

Proof. From Lemmas 4.11 and 4.15, it follows that such a logic also is above $\mathsf{K}\left(\varphi^{k'\to k'+1}_{w\geq p',x\geq q',y\geq r'}\right)$ for some numbers k',p',q',r', where $1\leq k'$. Hence, the result follows directly from Theorem 4.16. $\square$

However, there are some cases where NP-membership is proven in a different way:

Theorem 4.18. *Let* KL *be a universal elementary logic above* $\mathsf{K}\left(\varphi^{k\to k}_{w\geq p,x\geq q,y\geq r}\right)$ *for some* $k,p,q,r\in\mathbb{N}$, *where* $k\geq 1$. *Then* KL-SAT $\in$ NP.

Proof. We show the polynomial-size model property for KL, the result then shows from Theorem 2.6. Let φ be a KL-satisfiable formula. Then, by proposition 3.5, there is a KL-model M, such that $M,w\models\varphi$, and M is rooted at w. Let

$$C:=\left\{u\in M \mid \mathrm{maxdepth}^M(u), \mathrm{maxheight}^M(u)\geq \max(q-k,r-k)+p+k\right\}.$$

We show that C is a universal graph. The lemma then follows with Lemma 3.9. We first show that C is reflexive. Let $u\in C$. Then, by definition of C, the pair u,u fulfills the prerequisites of $\varphi^{k\to k}_{w\geq p,x\geq q,y\geq r}$ and hence (u,u) is an edge. We now show that C is symmetric: Let $v,u\in C$ such that (v,u) is an edge. Since v and u are reflexive, it follows that $M\models v\xrightarrow{k}v$, $M\models v\xrightarrow{k}u$, and $M\models u\xrightarrow{k}u$. Hence, v is a k-step predecessor of both v and u, and hence by the $\varphi^{k\to k}_{w\geq p,x\geq q,y\geq r}$ -condition (since $u,v\in C$, the conditions about required predecessors and successors are satisfied), (u,v) is an edge.

Finally, we show that C is transitive: Let $u,v,z\in C$ such that (u,v) and (v,z) are edges in M. Since C is symmetric due to the above, it follows that (v,u) is an edge as well. Since v is also reflexive, this implies that $M\models v\xrightarrow{k}u$ and $M\models v\xrightarrow{k}z$. Hence, v is a k-step predecessor of both u and z, and the $\varphi^{k\to k}_{w\geq p,x\geq q,y\geq r}$ -property implies that (u,z) is an edge in M, concluding the proof. $\square$

So we now have cleared the cases of logics defined by formulas $\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}$, where $1\leq k\leq l-2$, or where $k=l$. The next case that we consider is where one of the values k,l is 0. While logics satisfying only the formula $\varphi^{0\to k}$ are easily seen to be PSPACE-hard (since this formula is satisfied in every reflexive, transitive graph, this follows from Corollary 2.8), the "reverse" of this formula, $\varphi^{k\to 0}$, behaves very differently. Intuitively, this is because if we want to add edges to a tree to make it satisfy this formula, then in contrast to the transitive case, we get "back edges" as well. This allows us to show a high degree of connectedness. An exception to this is the case $k=1$: the formula $\varphi^{1\to 0}$ with arbitrary lower indices holds in any symmetric graph, and therefore PSPACE-completeless for these logics follows from Corollary 2.9.

Now for the logics extending $\varphi^{k\to 0}_{w\geq p,x\geq q,y\geq r}$ for some $k\geq 2$, we can show that this formula implies one of the formulas we already studied, and therefore Corollary 4.17 can be applied to give the complexity bound.

Lemma 4.19. *Let* $p,q,r,k\in\mathbb{N},k\geq 2$, *and let* G *be a graph.*

1. *If* $G\models\varphi^{k\to 0}$, *then* $G\models\varphi^{0\to k^2}$.
2. *If* $G\models\varphi^{k\to 0}_{w\geq p,x\geq q,y\geq r}$, *then* $G\models\varphi^{k-1\to k^3}_{w\geq p',x\geq q',y\geq r'}$, *where*

$$\begin{aligned}p'&:=p,\\ q'&:=k^3+\max(q-k,r),\\ r'&:=k-1+\max(q-k,r).\end{aligned}$$

Proof. Let $w_{-p}, \ldots, w = x_0, \ldots, x_q, w = y_0, \ldots, y_k$ be nodes in the graph fulfilling the prerequisite graph of the respective formulas.

1. By the $\varphi^{k \to 0}$-property, it is clear that $(y_k, y_0), (y_{2k}, y_k), \ldots, (y_{k^2}, y_{(k-1)\cdot k})$ are edges in G. Hence, $G \models y_{k^2} \xrightarrow{k} w$. Again due to the property, it follows that (w, y_{k^2}) is an edge in G, as required.
2. Consider the subgraph $G' := G \cap \{w, x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k^3}\}$. By choice of p', q', r', every node in G' has a p-step predecessor, a $l-k$-step successor, and a $r-k$-step successor in G. Hence, $G' \models \varphi^{k \to 0}$. Due to part 1, this implies that $G' \models \varphi^{0 \to k^2}$. Hence, in G' there are edges $(w, y_{k^2}), (y_{k^2}, y_{2k^2}), \ldots, (y_{(k-1)\cdot k^2}, y_{k^3})$. Therefore, due to the $\varphi^{k \to 0}$-property, it follows that (y_{k^3}, w) is an edge in G'. Hence, it follows that $G \models y_{k^3} \xrightarrow{k} x_{k-1}$. The $\varphi^{k \to 0}$-property implies that (x_{k-1}, y_{k^3}) is an edge, as required.

□

As mentioned, this implication directly gives the desired complexity result:

Corollary 4.20. *Let $p, q, r, k \in \mathbb{N}$, and let $k \geq 2$. Let KL be a universal elementary logic extending $\mathsf{K}\left(\varphi^{k \to 0}_{w \geq p, x \geq q, y \geq r}\right)$. Then KL-$\mathsf{SAT} \in \mathrm{NP}$.*

Proof. Due to Lemma 4.19, KL is also an extension of $\mathsf{K}\left(\varphi^{k-1 \to k^3}_{w \geq p', x \geq q', y \geq r'}\right)$ for some constants p', q', r'. Due to the prerequisites, it holds that $1 \leq k-1 \leq k^3 - 2$. Hence, the result follows from Corollary 4.17. □

Lemma 4.21. *Let $k, l, p, q, r \in \mathbb{N}$, $l < k$. Let G be a graph such that $G \models \varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}$. Then $G \models \varphi^{k' \to l'}_{w \geq p', x \geq q', y \geq r'}$, where*

$$\begin{aligned} k' &:= l, \\ l' &:= k + (k-1)(k-l) + l, \\ p' &:= p, \\ q' &:= k' + \max(k-q, l-r), \\ r' &:= l' + \max(k-q, l-r). \end{aligned}$$

Proof. Let w_i, x_i, y_i be nodes satisfying the prerequisite graph of $\varphi^{k' \to l'}_{w \geq p', x \geq q', y \geq r'}$. It is easy to see that for any $m \in \{k, \ldots, l' - (k-l)\}$, there is an edge $(y_{m+(k-l)}, y_m)$ in G, since by definition, $G \models y_{m-l} \xrightarrow{k} y_{m+(k-l)}$, and $G \models y_{m-l} \xrightarrow{l} y_m$, and by choice of p', q', r', these nodes have a q, resp. r-step successor, and a p-step predecessor. Hence, the following paths exist:

- By applying the above $k-1$ times, we conclude that there is a $k-1$-step path from $y_{k+(k-1)(k-l)}$ to y_k
- Due to the $\varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}$-property, and since $p' \geq p$, $q' \geq q$, and $r' \geq r$, there is an edge (x_k, y_l) in G.
- The preceding two paths imply $G \models y_{k+(k-1)(k-l)} \xrightarrow{k} x_l$.
- By definition, it holds that $G \models y_{k+(k-1)(k-l)} \xrightarrow{l} y_{k+(k-1)(k-l)+l}$.

Hence, from the $\varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}$-property, since all involved nodes are between w and $x_{k'}$ or $y_{l'}$ and therefore have successors and predecessors of the appropriate degree, it follows that $(x_l, y_{k+(k-1)(k-l)+l}) = (x_{k'}, y_{l'})$ is an edge in G, as claimed. □

Corollary 4.22. *Let $k, l, p, q, r \in \mathbb{N}$, such that $1 \leq l < k$, and let KL be a universal elementary modal logic extending $\mathsf{K}\left(\varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}\right)$. Then KL has the polysize model property. In particular, KL-$\mathsf{SAT} \in \mathrm{NP}$.*

Proof. Due to Lemma 4.21, KL is also an extension of $\mathsf{K}\left(\varphi^{k' \to l'}_{w \geq p', x \geq q', y \geq r'}\right)$, where $p', q', r' \in \mathbb{N}$, $k' = l$, and $l' = k + (k-1)(k-l) + l$. It is easy to see that $1 \leq k' \leq l' - 2$, hence the result follows from Corollary 4.17. □

4.6 A dichotomy for Horn clauses

Although the NP-results in the previous section only dealt with Horn clauses of the specific forms $\varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}$, these results allow for a complete complexity classification of the problem $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT}$ for the case where φ is a universal Horn clause over the frame language.

In Section 4.2, we covered all cases where the Horn clause is non-proper. In Proposition 4.5, we saw that among the proper clauses, we only need to consider those which can be homomorphically mapped to a tree. Therefore, we give the following definition:

Definition 4.23. *Let φ be a proper Horn implication. Then let T_φ denote the class of trees T such that there exists a homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi) \to T$.*

In the members of T_φ, there are three possibilities to consider: the vertices from the conclusion edge can

- be connected with a directed path,
- have a common predecessor,
- have a common successor.

If for a proper Horn implication φ with conclusion edge (x, y), for every homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi) \to T \in T_\varphi$, $\alpha(x)$ and $\alpha(y)$ are connected with a diretced path, we say that φ is *linear*. It is obvious that if there is a path connecting the vertices from the conclusion edge of φ in $\mathsf{prereq}\,(\varphi)$, then φ is linear. Also note that any Horn clause φ which cannot be mapped homomorphically to a tree is, by definition, linear as well.

Theorem 4.24 (Dichotomy for linear case). *Let φ be a linear Horn implication, and let (x, y) be the conclusion edge. Then the following holds:*

1. *If for every tree $T \in T_\varphi$, such that $\alpha\mathsf{prereq}\,(\varphi) : T$ is a homomorphism, there is a path from $\alpha(x)$ to $\alpha(y)$ in T, then $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT}$ is* PSPACE*-hard.*
2. *Else, if in every tree T_φ such that $\alpha\mathsf{prereq}\,(\varphi) : T$ is a homomorphism, there is a path from x to y of length* 1, *then $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT}$ is* PSPACE*-hard.*
3. *In all other cases, $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT}$ is in* NP.

Proof. 1. In this case, φ is satisfied in every transitive tree. Hence the claim follows from Corollary 2.8.
2. In this case, φ is satisfied in any graph which is symmetric. Hence PSPACE-hardness follows from Corollary 2.9.
3. In this case, it is easy to see that φ implies a formula of the form $\varphi^{k \to 0}_{w \geq p, x \geq q, y \geq r}$ for some $k \geq 2$. Hence, $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT}$ is in NP due to Corollary 4.20.

□

If the clause is proper and non-linear, then we can show that it gives enough connectivity in graphs to lead to the polynomial size model property:

Theorem 4.25 (Non-linear case). *Let φ be a proper non-linear Horn clause. Then $\mathsf{K}\,(\varphi)\text{-}\mathsf{SAT} \in$* NP.

Proof. Let $\mathsf{conc}\,(\varphi) = (x, y)$. Since φ is a non-linear, proper Horn clause, there is a tree T and a homomorphism $\alpha\colon \mathsf{prereq}\,(\varphi) \to T$, such that there is no directed path between $\alpha(x)$ and $\alpha(y)$ in T. If $\alpha(x) \neq \alpha(y)$, then $\alpha(x)$ and $\alpha(y)$ have a common predecessor w in T. Let k be the distance of x from w, and let l be the distance of y from w. It follows that we can homomorphically map T into the predecessor graph of $\varphi^{k \to l}_{w \geq p, x \geq q, y \geq r}$ for some p, q, r allowing for the height of T. Therefore,

$\mathsf{K}(\varphi)$ is a logic extending $\mathsf{K}\left(\varphi^{k\to l}_{w\geq p,x\geq q,y\geq r}\right)$, and the complexity result follows from Corollary 4.17, Theorem 4.18, or Corollary 4.22, depending on whether $k < l$, $k > l$, or $k = l$.

Therefore, assume now that $\alpha(x) = \alpha(y)$ for every such tree T, i.e., when mapping $\mathsf{prereq}(\varphi)$ into a tree T, then $\alpha(x)$ and $\alpha(y)$ are connected with a path, or are identical. In particular, there exist such a tree T and a homomorphism mapping both x and y to the same irreflexive node. By taking α to be T's homomorphic mapping to an irreflexive line $x_0, \ldots, x_n$, it follows that $\alpha(x)$ and $\alpha(y)$ are mapped to the same element x_i. Since x and y are different nodes in the predecessor graph of φ, we can modify the line as follows: introduce a new node y_i, which is connected to everything that x_i is connected to. Call this line L'. We can construct a homomorphism $\beta\colon \mathsf{prereq}(\varphi) \to L'$, by defining $\beta(z) = \alpha(z)$ for all $z \neq y$, and $\alpha(y) = y_i$. Since α is a homomorphism, the nodes x_i and y_i are irreflexive, and y_i has all connections that x_i has, β is a homomorphism. Let φ' be the universal Horn implication with prerequisite graph L', and conclusion edge (x_i, y_i). The homomorphism β and Proposition 4.4 show that φ implies φ'. We now show that any graph G with root w satisfying φ' satisfies the conditions of Lemma 3.9. For this, let C denote the set of vertices in G which have a i-step predecessor, and a $n-i$-step successor. We claim that this is a universal graph, which concludes the proof. We first show that C is reflexive. Let v be a node in G. Since v satisfies the conditions of both x_i and y_i in the graph L', v is reflexive. Now for symmetry, assume that v_1 and v_2 are elements from C, such that (v_1, v_2) is an edge in C. Since they are both reflexive, we can map L' to their predecessors and successors in such a way that x_j for $j \leq i$ is mapped to v_1, x_i is mapped to v_2, and y_1 is mapped to v_2, the nodes x_j for $j > i$ are mapped to v_2. Hence the formula φ' gives the edge (v_2, v_1). Finally, for transitivity, let (u, v) and (v, z) be edges in C. Then (v, u) and (z, v) also are edges, since we already proved symmetry. Then we can again construct a homomorphism mapping all elements of L' to v, except mapping x_i to u, and y_i to z. Hence, the φ'-property shows that (u, z) is an edge, as claimed.

□

The following is an example of a proper universal Horn implication φ which is not linear, but where the vertices x and y from the conclusion edge are mapped to the same node in every tree. The satisfiability problem for the logic $\mathsf{K}(\varphi)$ is in NP due to Theorem 4.25 (the second case in the proof applies).

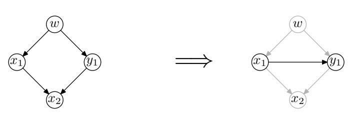

Fig. 5. Example non-linear implication

Note that this clause is not linear, since we can map the prerequisite graph to the line (l_0, l_1, l_2) with $\alpha(w) = l_0, \alpha(x_1) = \alpha(y_1) = l_1$, and $\alpha(x_2) = l_2$. It is true that every tree-like homomorphic image of φ is a line, but x_1 and y_1 are not necessarily connected with a directed path in this image. Hence, the implication is not linear.

5 Conclusion and Future Research

We have analyzed the complexity of modal logics for which a class of graphs which can be described by a first-order Horn clause gives the proper semantics, i.e., where this class of graphs is sound and complete. For these cases, we achieved a full classification, generalizing many previously known results.

There are obvious open questions to study next: Up to now, we only looked at individual Horn clauses, and proved that many of them lead to NP-complete modal satisfiability problems. But it is possible that the conjunction of Horn clauses which lead to PSPACE-hard logics individually gives a logic which is in NP.

We do not have a full classification of these cases, but we have some initial results. A well-known example for such a case is the modal logic $\mathsf{K4B}$, which is the logic over graphs which are both symmetric and transitive. On their own, both of these properties lead to PSPACE-hard logics, but their conjunction gives a logic which is in NP. In our notation, it can easily be seen that $\mathsf{K4B} = \mathsf{K}\left(\varphi^{1\to 0} \wedge \varphi^{0\to 2}\right)$. For this concrete example, the following Theorem 5.2 gives this result. We first need a technical condition:

Proposition 5.1. *Let $k \in \mathbb{N}$. Then $\varphi^{0\to k}$ implies $\varphi^{0\to k^2}$.*

Proof. Let G be a graph such that $G \models \varphi^{0\to k}$, and let $w = x_0, w = y_0, y_1, \ldots, y_{k^2}$ be nodes in G such that there are edges between y_i and y_{i+1} for all relevant i. Then, due to the $\varphi^{0\to k}$-property, there are edges $(y_0, y_k), (y_k, y_{2k}), \ldots, (y_{(k-1)k}, y_{k^2})$. Hence, it follows that $G \models w \xrightarrow{k} y_{k^2}$, and the claim follows directly from the $\varphi^{0\to k}$-property. □

We can now show the generalization of the complexity result for $\mathsf{K4B}$ as mentioned above.

Theorem 5.2. *et $k \geq 2 \in \mathbb{N}$. Then $\varphi^{1\to 0} \wedge \varphi^{0\to k}$ implies $\varphi^{1\to(k-1)k}$ if k is odd, and $\varphi^{2\to(k-1)k^2}$ if k is even. Hence, for any $p_1, q_1, r_1, p_2, q_2, r_2 \in \mathbb{N}$, the logic $\mathsf{K}\left(\varphi^{1\to 0}_{w\geq p_1, x\geq q_1, y\geq r_1} \wedge \varphi^{0\to k}_{w\geq p_2, x\geq q_2, y\geq r_2}\right)$ has an* NP-*complete satisfiability problem.*

Proof. First, let k be odd. Since $G \models \varphi^{1\to 0}$, it follows that G is symmetric. Hence, we know that $G \models w \xrightarrow{k-1} w$, and hence $G \models w \xrightarrow{k} x_1$. By applying the $\varphi^{0\to k}$-property $k-1$ times, we get a path of length $k-1$ from w to $y_{(k-1)k}$. Hence, since (x_1, w) is an edge, it follows that $G \models x_1 \xrightarrow{k} y_{(k-1)k}$, and due to the $\varphi^{0\to k}$-property, this implies that $(x_1, y_{(k-1)k})$ is an edge as required.

Now let k be even. By applying Proposition 5.1 we know that $G \models \varphi^{0\to k^2}$, hence, (w, y_{k^4}) is an edge in G. Since $k \equiv 0 \mod 2$, and $k \geq 2$, the symmetry of G ensures that there is a path of length k from w to x_2. Hence, there is an edge (w, x_2), and due to symmetry, an edge (x_2, w). By $(k-1)$ applications of the $\varphi^{0\to k^2}$-property, we know that $G \models w \xrightarrow{k-1} y_{(k-1)k^2}$. Hence, we conclude that $G \models x_2 \xrightarrow{k} y_{(k-1)k^2}$, and another application of the $\varphi^{0\to k}$-property gives the edge $(x_2, y_{(k-1)k^2})$, as required.

For the NP-containment of the more general logics, observe that $\varphi^{1\to 0}_{w\geq p_1, x\geq q_1, y\geq r_1} \wedge \varphi^{0\to k}_{w\geq p_2, x\geq q_2, y\geq r_2}$ implies $\varphi^{1\to 0}_{w\geq p, x\geq q, y\geq r} \wedge \varphi^{0\to k}_{w\geq p, x\geq q, y\geq r}$, where $p = \max(p_1, p_2)$, $q = \max(q_1, q_2)$, and $r = \max(r_1, r_2)$. Due to the above and Lemma 4.15, we know that this formula implies $\varphi^{1\to(k-1)k}_{w\geq p', x\geq q', y\geq r'}$ for some p', q', r' if k is odd, and it implies $\varphi^{2\to(k-1)k^2}_{w\geq p'', x\geq q'', y\geq r''}$ for some p'', q'', r'' if k is even. Further, if k is odd, then it follows that $k \geq 3$. Hence, $(k-1)k \geq 6$. If k is even, then, since $k \geq 2$, we know that $(k-1)k^2 \geq 4$, and in both cases the NP result follows from Corollary 4.17. □

It is not to be expected that this is the only case where the combination of first-order properties which, on their own, lead to PSPACE-hard cases give a logic with a satisfiability problem in NP. Hence, this is the next interesting question to consider in this context.

Also, it is interesting to see whether we can obtain PSPACE upper bounds for those logics we showed to be PSPACE-hard. We believe that this can be done for all the logics obtained from universal Horn formulas.

The next challenge then is to go beyond formulas in the Horn form. Our tools from Section 3 work for any universal formula over the frame language, but a full classification of all of these cases requires new techniques as well. Finally, the next major open question is to consider arbitrary first-order formulas instead of the universally quantified ones we considered up to now.

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