What are Differential Forms?

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Differential forms are usually introduced at the undergraduate level by defining how they act algebraically, the algebra of the wedge product, rules for integration, etc. The purpose of this paper is to explain what kinds of objects differential forms are and their advantages over traditional vector calculus tools.

Line Integrals

Consider the classical definition for integrating a vector field F along a curve c in $\mathbb{R}^3$

$$\int_c F \cdot \mathrm{d}\vec{r} = \int_a^b F(r(t)) \cdot r'(t)\mathrm{d}t$$

where $r : [a, b] \to \mathbb{R}^3$ is a parameterization of c.

A tangent vector at a point $p \in c$ is a vector in $\mathbb{R}^3$ that is in the one dimensional subspace spanned by $r'(t)$ where $p = r(t)$. We call this vector space $T_p c$ the tangent space to c at p. At any point p along the curve c we can use F to define a function $\hat{F}_p$ that sends tangent vectors to real numbers

$$\hat{F}_p(\vec{v}) = F(p) \cdot v$$

Since dot product is linear $\hat{F}_p$ is a linear function from the vector space $\mathbb{R}^n$ to the real numbers.

Dual Spaces

We can extend the concept of a linear function from a vector space to $\mathbb{R}$ to the concept of the dual vector space, the vector space composed of all such linear functions. If V is a real vector space, let V^* be the set of all linear functions $f : V \to \mathbb{R}$. The set V^* can be made into a real vector space. Let $f, g \in V^*$ be two linear functions from V to $\mathbb{R}$, then define their sum by how it acts on vectors

$$(f + g)(v) = f(v) + g(v) \ \forall v \in V.$$

If $a \in \mathbb{R}$ is a real number, then the scalar product is defined

$$(af)(v) = af(v) \ \forall v \in V.$$

Since V^* is a vector space it would be nice to have a basis. If V has a basis, we can construct a dual basis for V^* that works nicely with the basis for V.

Definition 1 (Dual Basis)**.** Let $\{x_1, x_2, ..., x_n\}$ be a basis for V. The dual basis for V^* is $\{dx_1, dx_2, ..., dx_n\}$ defined by

$$dx_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Let W be another real vector space. If $L : V \to W$ is a linear map from V to W, is there a way to use L to create a map between the dual spaces V^* and W^*?

In fact, we can construct a linear transformation $L^* : W^* \to V^*$. If $w' \in W^*$ is a dual vector to W, $w' : W \to \mathbb{R}$ is a linear function from W to $\mathbb{R}$. Given any vector $v \in V$, $L(v)$ is a vector in W, so we can use w' to map it to $w'(L(v))$, a real number. The map $w' \circ L$ will be linear on vectors in V because both L and w' are linear, so $w' \circ L$ is an element of V^*.

Definition 2 (Dual map)**.** If $L : V \to W$ is a linear map between vector spaces V and W then the **dual map** $L^* : W^* \to V^*$ is the map defined by

$$w' \mapsto w' \circ L$$

Notice that L^* maps from W^* to V^*, which is flipped from the direction of the map L.

Dual Vector Fields and Pullbacks

So $\hat{F}_p$ is a dual vector to $\mathbb{R}^n$. Without specifying the point, $\hat{F}$ is a function that assigns a dual vector to every point: i.e. a dual vector field (often called a covector field). Dual vector fields are also differential 1-forms, so we should expect to be able to integrate $\hat{F}$.

Using $\hat{F}$ we can rewrite our line integral to be

$$\int_c \hat{F} = \int_a^b \hat{F}_{r(t)}(r'(t))\mathrm{d}t.$$

This is good, but it would be nice to define the process of using a parameterization like r in a way that generalizes to higher dimensions.

Let V be all the vectors tangent to $[a, b]$ based at the point p and W be all the vectors tangent to the curve C based at the point $r(p)$ along the curve. We will define a linear transformation dr_p that takes vectors tangent to $[a, b]$ at a point p (i.e. in V) to vectors tangent to c at the point $r(p)$ (i.e. in W). The idea of taking vectors tangent along $[a, b]$ seems a little silly, but it will generalize nicely to higher dimensions. Let $e = 1$ be the unit vector based at p in the increasing direction along $[a, b]$ and define $dr_p(e) = r'(t)$. By extending linearly, this completely defines a linear map.

From our section on dual spaces, this gives us a linear map $dr_p^*: W^* \to V^*$, so $dr_p^*(\hat{F}_{r(p)})$ is a dual vector acting on V.

Definition 3 (Pullback of a dual vector field)**.** The **pullback** $r^*\hat{F}$ of $\hat{F}$ by r is the dual vector field on $[a, b]$ defined by

$$(r^*\hat{F})_p = dr_p^*(\hat{F}_{r(p)}).$$

Consider that

$$(r^*\hat{F})_p(e) = dr_p^*(\hat{F}_{r(p)})(e)$$

and by the definition of the dual map

$$dr_p^*(\hat{F}_{r(p)})(e) = \hat{F}_{r(p)}(dr_p(e)) = \hat{F}_{r(p)}(r'(t))$$

which is exactly the term we are integrating. This "pulled back" dual vector field acts exactly how we would like a parameterizaion of the dual vector field to act!

So our final form for the line integral is

$$\int_c \hat{F} = \int_{[a,b]} r^*\hat{F} = \int_a^b (r^*\hat{F})_t(e)\mathrm{d}t.$$

Using the dot product is not the only way to define dual vector fields. The vector $x_t = r'(t)$ is a basis for the tangent space along the curve c at each point along the curve. Then the dual basis is just dx_t and any dual vector at $c(t)$ can be written as adx_t where a is a real number. If x is the vector field along the curve define to be x_t at $c(t)$, then dx is a dual vector field along the curve that gives a basis for the dual space at each point. Using this dual vector field, we can write any dual vector field as

$$\omega = f dx$$

where $f : c \to \mathbb{R}$ is some (differentiable) real valued function on the curve c. If you have seen differential 1-forms ω looks just like one. In fact, differential 1-forms and dual vector fields are exactly the same thing.

You can imagine that given just the 1-form ω on the curve you could integrate ω using the parameterization r of the curve by "pulling back" ω to r's domain,

$$\int_c \omega = \int_{[a,b]} r^*\omega = \int_a^b (r^*\omega)_t(e)\mathrm{d}t.$$

We write the left side with no reference to the parameterization r because the value of the integral is independent of the parameterization (with some loose conditions on the parameterization). A differential 1-form ω is a geometric construct. Another benefit of 1-forms is that they can be defined without any reference to a dot product. If our curve was instead in some generic smooth manifold there is no inner product on the space of tangent vectors at each point on the manifold. Defining a (smooth) inner product at every point would define a Riemannian metric on the manifold. Differential forms allow integration on smooth manifolds without dependence on additional Riemannian structure.

Surface Integrals

To motivate differential 2-forms we turn to surface integrals of vector fields from classical vector calculus. Let $\mathbf{v}(p)$ be a vector field on a surface $S \subset \mathbb{R}^3$ and let $\mathbf{n}(p)$ be a unit normal vector field along the surface. Then the surface integral of $\mathbf{v}$ over S is

$$\iint_S \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iint_S (\mathbf{v} \cdot \mathbf{n})\ \mathrm{d}S$$

Let $U \subseteq \mathbb{R}^2$ be an open subset of $\mathbb{R}^2$ and $\boldsymbol{x} : U \to S$ be a parameterization of S. Then the integral becomes

$$\iint_U \mathbf{v}(\boldsymbol{x}(s,t)) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t} \right) \mathrm{d}s\ \mathrm{d}t$$

Similarly to the case of the line integral, we can define a new function at each point, ω_p which will replace much of the formula for the surface integral.

Recall the definition of the tangent space:

Definition 4 (Tangent Space to a Surface)**.** A vector v is tangent to a point $p \in S$ if there exists a curve $r : [a,b] \to S$ such that $r(t) = p$ and $r'(t) = v$. The tangent space T_pS to S at a point p is the vector space contained every vector tangent to S at p.

Let ω_p be defined by taking two tangent vectors u, w at p and sending them to a real number,

$$(u,w) \mapsto v(p) \cdot (u \times w)$$

The function ω_p has some interesting properties. If $u, w, z \in T_pS$ are three tangent vectors to S at p and $a \in \mathbb{R}$ is a real number, then

$$\omega_p(au + w, z) = v(p) \cdot ((au + w) \times z)\,.$$

The cross product distributes over vector addition and scalar multiplication so this is

$$v(p) \cdot (au \times z + w \times z)\,,$$

and becauce the dot product also distributes it is

$$av(p) \cdot (u \times z) + v(p) \cdot (w \times z) = a\omega_p(u,z) + \omega_p(w,z).$$

For the same reasons,

$$\omega_p(u, aw + z) = a\omega_p(u,w) + \omega_p(u,z).$$

The function ω is linear over each of its arguments, or multilinear.

Additionally, the cross product is antisymmetric ($u \times w = -w \times u$), which means that since

$$\omega_p(u, w) = v(p) \cdot (u \times w) = -v(p) \cdot (w \times u) = -\omega_p(w, u),$$

ω_p is also antisymmetric. Again ignoring the point p, ω assigns to every point on the surface a multilinear, antisymmetric function on the tangent space.

With dual vectors $w' \in W^*$, we could use a linear transformation $L : V \to W$ to pull back w' to a dual vector $w' \circ L$ in V^*. We can do something similar with multilinear maps from $f : W \times W \to \mathbb{R}$. Consider that $f \circ (L \times L)$ is a function that sends

$$(v_1, v_2) \mapsto f(L(v_1), L(v_2))$$

and will be a multilinear map from $W \times W$ to $\mathbb{R}$.

Definition 5 (Pullback of a multilinear map)**.** Let $f : V \times V \to \mathbb{R}$ be a multilinear map. The **pullback** L^*f of f by L is the multilinear map

$$L^*f = f \circ (L \times L).$$

If f is antisymmetric, L^*f will also be antisymmetric since

$$L^*f(v_1, v_2) = f(L(v_1), L(v_2)) = -f(L(v_2), L(v_1)) = -L^*f(v_2, v_1)$$

Given an analog to our linear map dr_p for the parameterization $\boldsymbol{x}$, we could pull back the antisymmetric multilinear map $\omega_{\boldsymbol{x}(q)}$ to an antisymmetric multilinear map on the tangent space at $q \in \mathbb{R}^2$. In fact, there is a well defined linear transformation $d\boldsymbol{x}_p$ which sends tangent vectors at a point $p \in U$ to tangent vectors at the point $\boldsymbol{x}(p)$ in S. In particular, if ∂_s and ∂_t are the unit vectors at p in the coordinate directions of s and t respectively then $d\boldsymbol{x}_p(\partial_s) = \frac{\partial \boldsymbol{x}}{\partial s}$ and $d\boldsymbol{x}_p(\partial_t) = \frac{\partial \boldsymbol{x}}{\partial t}$.

For any point $q \in U$, using the pullback of a multilinear map we obtain a multilinear map $d\boldsymbol{x}_q^*(\omega_{\boldsymbol{x}(q)})$ on the tangent space of $\mathbb{R}^2$ at q.

Definition 6 (Pullback of a multilinear field)**.** The **pullback** $\boldsymbol{x}^*\omega$ of ω by $\boldsymbol{x}$ is the multilinear field defined by

$$(\boldsymbol{x}^*\omega)_q = d\boldsymbol{x}_q^*(\omega_{\boldsymbol{x}(q)})$$

at each point $q \in \mathbb{R}^2$.

Now using ω and the pullback we can rewrite our surface integral as

$$
\begin{aligned}
&\iint_U \mathbf{v}(\boldsymbol{x}(s,t)) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial s} \times \frac{\partial \boldsymbol{x}}{\partial t}\right) \mathrm{d}s \ \mathrm{d}t \\
&\quad = \iint_U \omega_{\boldsymbol{x}(s,t)} \left(\frac{\partial \boldsymbol{x}}{\partial s}, \frac{\partial \boldsymbol{x}}{\partial t}\right) \mathrm{d}s \ \mathrm{d}t && \text{(definition of } \omega\text{)} \\
&\quad = \iint_U \omega_{\boldsymbol{x}(s,t)} \left(d\boldsymbol{x}_{(s,t)}(\partial_s), d\boldsymbol{x}_{(s,t)}(\partial_t)\right) \mathrm{d}s \ \mathrm{d}t && \text{(using that } d\boldsymbol{x}_{(s,t)}(\partial_s) = \frac{\partial \boldsymbol{x}}{\partial s}, d\boldsymbol{x}_{(s,t)}(\partial_t) = \frac{\partial \boldsymbol{x}}{\partial t}\text{)} \\
&\quad = \iint_U \boldsymbol{x}^*\omega(\partial_s, \partial_t) \mathrm{d}s \ \mathrm{d}t && \text{(definition of the pullback } \boldsymbol{x}^*\omega\text{)} \\
&\quad = \int_U \boldsymbol{x}^*\omega = \int_S \omega && \text{(definition of integration of multilinear fields)}
\end{aligned}
$$

Differential Forms

Just as in the case with dual vector fields or 1-forms, we can define these antisymmetric multilinear without any reference to a cross product or $\mathbb{R}^3$. The vectors $\frac{\partial \boldsymbol{x}}{\partial s}$ and $\frac{\partial \boldsymbol{x}}{\partial t}$ give a basis for T_pS at every point. Recall that given this basis there is a corresponding dual basis, ds, dt such that

$$ds(\frac{\partial \boldsymbol{x}}{\partial s}) = 1, \quad ds(\frac{\partial \boldsymbol{x}}{\partial t}) = 0, \quad dt(\frac{\partial \boldsymbol{x}}{\partial t}) = 1, \text{ and } dt(\frac{\partial \boldsymbol{x}}{\partial s}) = 0.$$

It would be a nice if there was a way to also use this dual basis to define a basis for the space of alternating multilinear functions $T_pS \times T_pS \to \mathbb{R}$. In fact, there is a operator $\wedge$ called the wedge product that takes two dual vectors $T_pS \to \mathbb{R}$ and makes an alternating multilinear function from $T_pS \times T_pS \to \mathbb{R}$. The construction of the wedge product is beyond the scope of this paper, but what is important is that it allows us to write these alternating multilinear maps in the form

$$a \ ds \wedge dt$$

where a is a real number. Since this dual basis is defined at every point $p \in S$, we can express every alternating multilinear field in two variables on S as

$$f ds \wedge dt.$$

This looks just like a 2-form because it is a 2-form!

In general, if M is some space with differential structure (i.e. a curve, a surface in $\mathbb{R}^3$, or a smooth manifold of any dimension) a differential n-form ω is a field which assigns to each point $p \in M$ a multilinear function

$$\omega_p : \underbrace{T_pM \times ... \times T_pM}_{n \text{ times}} \to \mathbb{R}$$

such that swapping any two inputs adds a negative sign,

$$\omega_p(v_1, v_2, ..., v_i, ..., v_j, ..., v_n) = -\omega_p(v_1, ..., v_j, ..., v_i, ..., v_n).$$

This last property is called alternating can be thought of as a generalization of the antisymmetric property for 2-forms.

Given a parameterization $\boldsymbol{x} : U \subset \mathbb{R}^n \to M$ of our n-dimensional space M and a differential n-form ω, we can use the same machinery we created for curves and surfaces to integrate ω over M,

$$\int_M \omega = \int_U \boldsymbol{x}^* \omega.$$