Classical Mechanics Notes

Based on Taylor's Classical Mechanics

November 25, 2024

Contents

Chapter 1

Newton's Laws of Motion

Newton's Three Laws

Newton's Laws of Motion describe the relationship between the motion of an object and the forces acting on it. Key concepts include:

- **Inertial Reference Frame**: A frame of reference in which Newton's first law (the law of inertia) holds true. Objects in such a frame either remain at rest or move at constant velocity unless acted upon by a force.
- **Newton's First Law (Law of Inertia)**:

$$\sum \vec{F} = 0 \Longrightarrow$$

constant velocity

- **Newton's Second Law**: The net force acting on an object is equal to the mass of the object multiplied by its acceleration.

$$\sum \vec{F} = m\vec{a}$$

- **Newton's Third Law**: For every action, there is an equal and opposite reaction.

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

1.1 Newton's Laws in Polar Coordinates

We start with Newton's second law in vector form:

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta}$$

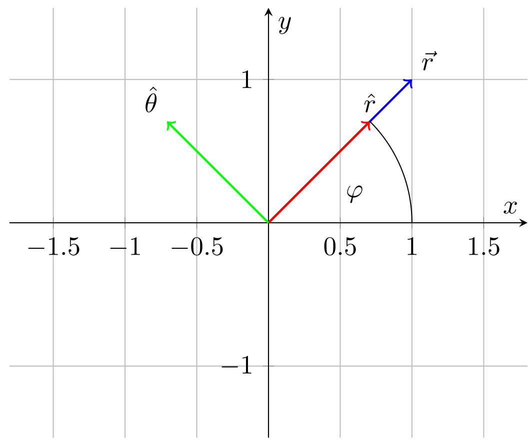

Figure 1.1: Illustration of the radial unit vector $\hat{r}$ and angular unit vector $\hat{\theta}$ in polar coordinates.

where F_r is the radial component of the force, and F_θ is the tangential component.

Express $\vec{r}$ and $\Delta\hat{r}$ in Polar Coordinates

In polar coordinates, the position vector is written as:

$$\vec{r} = r\hat{r}$$

where r is the radial distance and $\hat{r}$ is the unit vector pointing in the radial direction. To find the change in $\hat{r}$, consider a small change $\Delta\varphi$ in the angular position:

$$\Delta\hat{r} = \hat{r}_f - \hat{r}_i \approx \Delta\varphi\,\hat{\theta}$$

As $\Delta\varphi \to 0$, this approximation becomes exact:

$$\frac{\Delta\hat{r}}{\Delta t} = \frac{\Delta\varphi}{\Delta t}\hat{\theta}$$

Taking the limit as $\Delta t \to 0$, we get:

$$\dot{\hat{r}} = \dot{\varphi}\hat{\theta}$$

where $\dot{\varphi}$ is the time derivative of the angle φ.

Now, we Differentiate $\vec{r} = r\hat{r}$ to Obtain $\dot{\vec{r}}$

Now, differentiate $\vec{r} = r\hat{r}$ with respect to time:

$$\dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

Substituting $\dot{\hat{r}} = \dot{\varphi}\hat{\theta}$:

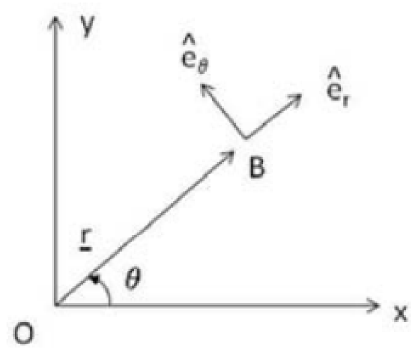

Figure 1.2: Polar Coordinates

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\theta}$$

Next we can express the Components of Velocity

From the expression for $\dot{\vec{r}}$, we can read off the components of velocity in polar coordinates:

$$v_r = \dot{r} \quad \text{and} \quad v_\theta = r\dot{\varphi} = r\omega$$

where v_r is the radial velocity and v_θ (or v_φ) is the tangential velocity, with $\omega = \dot{\varphi}$ being the angular velocity.

Now, we differentiate $\dot{\vec{r}}$ to Obtain $\ddot{\vec{r}}$

Next, differentiate $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\theta}$ with respect to time to find the acceleration:

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\varphi}\hat{\theta})$$

Applying the product rule, we get:

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\varphi}\hat{\theta} + r\ddot{\varphi}\hat{\theta} + r\dot{\varphi}\dot{\hat{\theta}}$$

Substituting $\dot{\hat{r}} = \dot{\varphi}\hat{\theta}$ and $\dot{\hat{\theta}} = -\dot{\varphi}\hat{r}$:

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\theta}$$

Finally, we arrive at Newton's Second Law in Polar Coordinates:

Now substitute the acceleration $\ddot{\vec{r}}$ into Newton's second law $\vec{F} = m\ddot{\vec{r}}$:

$$F_r\hat{r} + F_\theta\hat{\theta} = m\left[(\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\theta}\right]$$

Equating the coefficients of $\hat{r}$ and $\hat{\theta}$, we obtain the radial and angular components of Newton's second law:

$$F_r = m(\ddot{r} - r\dot{\varphi}^2) \qquad \text{(Radial Component)}$$
$$F_\theta = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \qquad \text{(Angular Component)}$$

These are the equations of motion in polar coordinates, derived from Newton's second law.

Chapter 2

Projectiles and Charged Particles

2.1 Drag

The total drag force is given by

$$\vec{f} = -f(v)\hat{v}$$

where $\hat{v}$ is the unit vector in the direction of velocity, and $f(v)$ is the magnitude of $\vec{f}$. Thsi drag force is made up of two components; linear, and quadratic.

Linear drag comes from viscosity and is proportional to the size of the object:

$$f_{\text{lin}} = bv \quad \text{where } b = \beta D$$

Quadratic drag is proportional to the cross-section and arises from the particles that the object 'moves' or carries along with it as it travels through the medium:

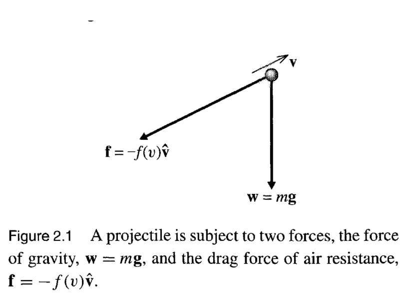

Figure 2.1: Drag

$$f_{\text{quad}} = cv^2 \quad \text{where } c = \gamma D^2$$

Thus, the total drag force is the sum of the linear and quadratic drag forces:

$$f(v) = f_{\text{lin}} + f_{\text{quad}} = bv + cv^2$$

In our cases, we can usually cancel one of these out by looking at the ratio of linear to quadratic drag. This is because in most objects, either one or the other will 'win out' depending on the relationship between the diameter and velocity and the given constants.

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v$$

Dominantly linear objects include small drops of water, as well as objects moving through a viscous fluid. Dominantly quadratic objects include larger objects such as people or planes.

The magnitude of this results is very similar to that of the **Reynold's Number**, $R = DvQ/r$, which is often used to approximate whether to use linear or quadratic drag.

Linear Drag

$$\vec{F} = f(\vec{v}) + mg = m\vec{a}$$

For the linear case:

$$f(\vec{v}) \approx b\vec{v}$$

$$m\vec{a} = -b\vec{v} + mg$$

Now we write the equations of motion in the x and y direction:

$$\begin{cases} F_x : \quad m\dot{v}_x = -bv_x \\ F_y : \quad m\dot{v}_y = -bv_y + mg \end{cases}$$

From Newton's laws we can start by finding the velocity in the x-Direction.

$$\dot{v}_x = -\frac{b}{m} v_x = -kv_x$$

$$\int \frac{dv_x}{v_x} = \int -k\,dt$$

$$\ln(v_x) = -kt + C \Longrightarrow v_x = Ae^{-kt}$$

Using $v_x(0) = v_{x0}$, we get:

$$v_x = v_{x0} e^{-kt}$$

$$x(t) = \int v_x(t)\,dt = \int v_{x0} e^{-kt}\,dt$$

$$x(t) = \frac{v_{x0}}{k}\left(1 - e^{-kt}\right)$$

Now that we've solved motion in the x diretction, we'll follow the same process for motion in the y-Direction.

Newton's equation was:

$$m\dot{v}_y = mg - bv_y$$

$$\dot{v}_y = g - kv_y$$

When $a = 0$:

$$0 = g - kv_y \Longrightarrow v_y = \frac{mg}{b} = v_{\text{ter}} \, \text{(terminal velocity)}$$

Now, returning to the orignal eqution with our newly defined terminal velocity v_{ter}:

$$m\dot{v}_y = mg - bv_y$$

$$\dot{v}_y = -\frac{b}{m}(v_y - v_{\text{ter}})$$

$$\text{Introduce } u = v_y - v_{\text{ter}}, \text{ then } u = Ae^{-kt}$$

$$v_y = v_{\text{ter}} + (v_{y0} - v_{\text{ter}})e^{-kt}$$

Now that we have$y(t)$ we can integrate to find $y(t)$

$$y(t) = \int v_y(t)\,dt = v_{\text{ter}}t + (v_{y0} - v_{\text{ter}})\frac{1}{k}\left(1 - e^{-kt}\right)$$

Now that we have the equations of motion for $x(t)$ and $y(t)$ we can now use it to eliminate t. We will do this by solving $x(t)$ and substituting our result back into $y(t)$:

$$\boxed{y = \frac{v_{y0} + v_{\text{ter}}}{v_{x0}}x + v_{\text{ter}}\ln\left(1 - \frac{x}{v_{x0}t}\right)} \tag{2.1}$$

Quadratic Drag

$$\vec{F} = -cv^2\hat{v} - mg\hat{y}$$

$$m\ddot{\vec{r}} + cv^2\hat{v} + mg\hat{y} = 0$$

1. x**-Direction**

$$m\dot{v}_x = -\frac{cv^2}{v}v_x = -\frac{c}{m}v_x^2$$

$$\int \frac{dv_x}{v_x^2} = -\frac{c}{m}\int dt$$

$$-v_x^{-1} = -\frac{c}{m}t + C \Longrightarrow v_x^{-1} = \frac{1}{v_{x0}} + \frac{c}{m}t$$

$$v_x = \frac{v_{x0}}{1 + \frac{tcv_{x0}}{m}}$$

$$x(t) = \int v_x(t)\,dt = \int \frac{v_{x0}}{1 + \frac{tcv_{x0}}{m}}\,dt$$

$$x(t) = \frac{v_{x0}m}{c}\ln\left(1 + \frac{tc}{m}\right)$$

2. y**-Direction**

$$m\dot{v}_y = mg - cv^2$$

Let $v_t er = \sqrt{\frac{mg}{c}}$, which is the terminal velocity. Then,

$$c = \frac{mg}{v_{ter}^2}$$

We can use this to simplify our main equation by taking out a factor of g:

$$\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right)$$

$$\int \frac{dv}{1 - \frac{v^2}{v_{\text{ter}}^2}} = g\int dt$$

This integral is arctanh, where $a = \frac{g}{v_{\text{ter}}}$:

$$\frac{v_{\text{ter}}}{g}\operatorname{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) = t$$

$$v = v_{\text{ter}}\tanh\left(\frac{gt}{v_{\text{ter}}}\right)$$

$$y(t) = \frac{v_{\text{ter}}^2}{g}\ln\left(\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right)$$

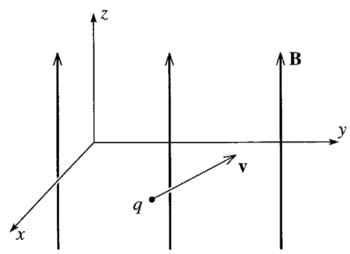

Figure 2.12 A charged particle moving in a uniform magnetic field that points in the z direction.

Figure 2.2: Uniform Magnetic Field

Motion of a Charged Particle in Uniform Electric and Magnetic Fields

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- **Charged Particles in Electric Fields**: A charged particle experiences a force proportional to the electric field.
$$\vec{F} = q\vec{E}$$
- **Charged Particles in Magnetic Fields**: A moving charged particle experiences a force perpendicular to both its velocity and the magnetic field.
$$\vec{F} = q\vec{v} \times \vec{B}$$

A charged particle of mass m and positive charge q moves in uniform electric and magnetic fields with, say, $\vec{E}$ pointing in the y-direction and $\vec{B}$ pointing in the z-direction. Suppose that, at $t = 0$, the particle is at the origin with initial velocity components $v_x(0) = v_{x0}$ and $v_y(0) = v_z(0) = 0$.

Equations of Motion

We have $\vec{E} = (0, E, 0)$ and $\vec{B} = (0, 0, B)$. Thus, the Lorentz force is given by:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

Explicitly, this becomes:

$$\vec{F} = q\left(v_y B, E - v_x B, 0\right). \tag{1}$$

The equation of motion, $\dot{\vec{v}} = \frac{\vec{F}}{m}$, can be written as:

$$\dot{v}_x = \omega v_y, \tag{2}$$

$$\dot{v}_y = -\omega(v_x - E/B), \tag{3}$$

$$\dot{v}_z = 0, \tag{4}$$

where the **cyclotron frequency** (angular velocity) is defined as $\omega = \frac{qB}{m}$.

Confined Motion in the Plane $z = 0$

With the initial conditions $v_z(0) = 0$ and $z(0) = 0$, we can solve Eq. (4):

$$v_z(t) = 0 \Longrightarrow z(t) = 0.$$

This shows that the particle remains confined to the plane $z = 0$ at all times.

Solution in the Transverse Plane

To solve the equations for the motion in the (x, y)-plane, we differentiate Eq. (2) with respect to t and use Eq. (3):

$$\ddot{v}_x = -\omega^2 v_x + \omega E/B.$$

This is a second-order differential equation for $v_x(t)$, with the general solution:

$$v_x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{E}{B}.$$

Similarly, since $v_y = \frac{\dot{v}_x}{\omega}$, we can write:

$$v_y(t) = -A\sin(\omega t) + B\cos(\omega t).$$

Determining Constants from Initial Conditions

Using the initial conditions $v_x(0) = v_{x0}$ and $v_y(0) = 0$:

$$v_x(0) = A + \frac{E}{B} = v_{x0} \Longrightarrow A = v_{x0} - \frac{E}{B},$$

$$v_y(0) = B = 0 \Longrightarrow B = 0.$$

Thus, the solutions become:

$$v_x(t) = \left(v_{x0} - \frac{E}{B}\right)\cos(\omega t) + \frac{E}{B}, \tag{5}$$

$$v_y(t) = -\left(v_{x0} - \frac{E}{B}\right)\sin(\omega t). \tag{6}$$

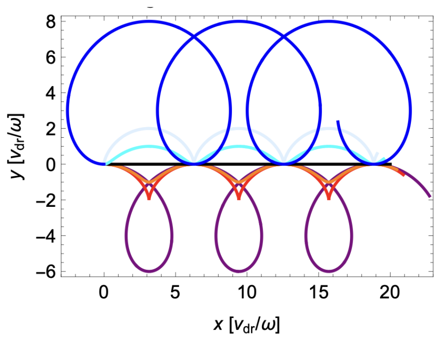

Figure 2.3: Helical Motion

Discussion

The motion of the charged particle in the presence of uniform electric and magnetic fields results in a circular trajectory in the (x, y)-plane, with a drift velocity due to the electric field. The particle's velocity components oscillate at the cyclotron frequency ω, and the electric field shifts the center of this circular motion. This behavior is characteristic of a charged particle in a crossed $\vec{E}$ and $\vec{B}$ field configuration.

If we were to interoduce the z-axis we would get motion that is **helical**, meaning that this describes a uniform field with an axis parallel to the z-axis

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Chapter 3

Momentum and Angular Momentum

Notes

Momentum and angular momentum are conserved in isolated systems. Key concepts include:

- **Linear Momentum**: The momentum of an object is the product of its mass and velocity.
$$\vec{p} = m\vec{v}$$
- **Conservation of Momentum**: In an isolated system, the total momentum is conserved.
$$\sum \vec{p}_{\text{initial}} = \sum \vec{p}_{\text{final}}$$
- **Angular Momentum**: For a rotating object or a particle moving in a circular path, the angular momentum is given by:
$$\vec{L} = \vec{r} \times \vec{p}$$
- **Conservation of Angular Momentum**: In the absence of external torques, the total angular momentum remains constant.
$$\sum \vec{L}_{\text{initial}} = \sum \vec{L}_{\text{final}}$$

Practice Problems

Chapter 4

Energy

4.1 Overview

Energy is a scalar quantity and is conserved in closed systems. Key concepts include:

- **Kinetic Energy**: The energy associated with motion.

$$T = \frac{1}{2}mv^2$$

- **Potential Energy**: Energy stored in a system due to its position in a force field, e.g., gravitational potential energy.

$$U = mgh \quad \text{(near the Earth's surface)}$$

- **Conservation of Energy**: The total energy (kinetic + potential) in an isolated system remains constant.

$$E_{\text{total}} = T + U$$

4.2 Conservative Forces

Conservative forces are forces that have a specific potential energy. There are two conditions for a force to be considered conservative:

1. F depends only on the position $\mathbf{r}$ of the object on which it acts; it must not depend on the velocity, the time, or any variables other than $\mathbf{r}$. For example, the gravitational force:

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{r^2}\mathbf{r}$$

depends only on the variable r.

2. For any two points 1 and 2, the work done by F is the same for all paths between 1 and 2. In other words, the work done is **path-independent**. This allows us to define the unique **potential energy**:

$$U(\mathbf{r}) = W(\mathbf{r}_0 \to \mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

   As we can see, U is dependent on position, and it is unique because it depends only on $\mathbf{r}_0$ and $\mathbf{r}$.

If the force on a particle is conservative, then the particle's **mechanical energy** never changes; that is, the particle's energy is conserved, which explains the use of the adjective "conservative."

If the n forces $\mathbf{F}_i$ $(i = 1, 2, \dots, n)$ acting on a particle are conservative, each with its corresponding potential energy $U_i(\mathbf{r})$, the mechanical energy is defined as

$$E \equiv U_1 + T_1 + U_2 + T_2 + \ldots + U_i + T_i \tag{4.22}$$

and is **constant in time**.

o test whether a force is conservative, you can show that work is path independent because work is path independent if and only if

$$\nabla \times \mathbf{F} = 0 \tag{4.1}$$

everywhere. This quantity is known as the **curl**. We can prove this by taking the gradient crossed with $\mathbf{F}$ and showing that for a conservative vector field, this must be true.

Example: Electric and Magnetic fields

$$\gamma^2 + \theta^2 = \omega^2 \tag{4.2}$$

"Maxwell's equations" are named for James Clark Maxwell and are as follow:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's Law of Induction} \tag{4.3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right) \qquad \text{Ampere's Circuital Law} \tag{4.4}$$

In electrostatics, the electric field is a conservative, central force, as B is not moving, so $\frac{\partial \vec{B}}{\partial t} = 0$. However, the magnetic field is not. 11

4.3 Force as the Gradient of Potential Energy

Consider a particle described by $\mathbf{F}(\mathbf{r})$ with potential energy $U(\mathbf{r})$. The work done by a small displacement from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$ is:

$$W(\mathbf{r} \to \mathbf{r}_1) = W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = F_x\, dx + F_y\, dy + F_z\, dz$$

Now, we know that work is the same as the change in U (potential energy):

$$W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = -dU = -\left(U(\mathbf{r} + d\mathbf{r}) - U(\mathbf{r})\right)$$

Let's just look at dU:

$$\begin{aligned} dU &= U(x + dx, y + dy, z + dz) - U(x, y, z) \\ &= \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz \end{aligned}$$

Thus:

$$W(\mathbf{r} \to \mathbf{r} + d\mathbf{r}) = dW$$

$$dW = F_x\,dx + F_y\,dy + F_z\,dz$$

But we know that:

$$dW = -dU$$

So:

$$F_x\,dx + F_y\,dy + F_z\,dz = -\left(\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz\right)$$

Thus:

$$\boxed{\mathbf{F} = -\nabla U}$$

Thus, we can say that **any conservative force must be derivable from potential energy**.

4.4 Central Forces

Central forces are described by this form:

$$\mathbf{F}(\mathbf{r}) = f(r)$$

That is, $\mathbf{F}(\mathbf{r})$ is **spherically symmetric** and **rotationally invariant**, meaning the magnitude function $f(r)$ is independent of the direction of $\mathbf{r}$ and, hence, has the same value at all points the same distance from the origin.

We can prove this using spherical polar coordinates.

Recall that for spherical coordinates:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad \text{and} \quad z = r\cos\theta. \tag{4.68}$$

$\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are all mutually perpendicular, so dot products behave in the same way.

To find the gradient of a scalar field in spherical coordinates:

We have seen that the components of ∇f are precisely the partial derivatives of f with respect to x, y, and z,

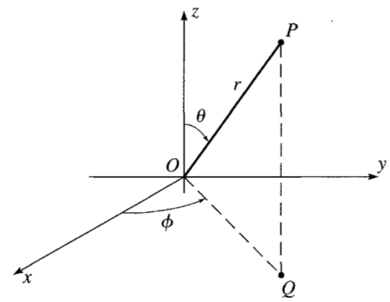


Figure 4.1: Spherical Coordinates

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}. \tag{4.70}$$

The corresponding expression for ∇f in spherical coordinates is not so straightforward. To find it, recall that, in a small displacement $\mathrm{d}\mathbf{r}$, the change in any function $f(r)$ is:

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{r}. \tag{4.71}$$

To evaluate the small vector $\mathrm{d}\mathbf{r}$ in spherical coordinates, we must examine carefully what happens to the point $\mathbf{r}$ when we change r, θ, and ϕ:

- A small change $\mathrm{d}r$ in r moves the point a distance $\mathrm{d}r$ radially out, in the direction of $\hat{\mathbf{r}}$. - A small change $\mathrm{d}\theta$ in θ moves the point around a circle of longitude (radius r) through a distance $r\,\mathrm{d}\theta$ in the direction of $\hat{\boldsymbol{\theta}}$. - A small change $\mathrm{d}\phi$ in ϕ moves the point around a circle of latitude (radius $r\sin\theta$) through a distance $r\sin\theta\,\mathrm{d}\phi$.

Putting all this together, we see that:

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\mathbf{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{\phi}}.$$

Knowing the components of $\mathrm{d}\mathbf{r}$, we can now evaluate the dot product in (4.71) in terms of the unknown components of ∇f:

$$\mathrm{d}f = (\nabla f)_r\,\mathrm{d}r + (\nabla f)_\theta\,r\,\mathrm{d}\theta + (\nabla f)_\phi\,r\sin\theta\,\mathrm{d}\phi. \tag{4.72}$$

Meanwhile, since f is a function of the three variables r, θ, and ϕ, the change in f is, of course:

$$\mathrm{d}f = \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\mathrm{d}\phi. \tag{4.73}$$

Comparing (4.72) and (4.73), we conclude that the components of ∇f in spherical coordinates are:

$$(\nabla f)_r = \frac{\partial f}{\partial r}, \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}, \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{4.74}$$

Or, a little more compactly:

$$\boxed{\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}.} \tag{4.75}$$

Chapter 5

Oscillations

Notes

Oscillatory motion is periodic and is described by differential equations. Key concepts include:

5.1 Deriving the Equations of Motion for a Simple Harmonic Oscillator

Starting from Hooke's law, we have the force acting on the cart as:

$$F = -kx,$$

where k is the spring constant and x is the displacement from the equilibrium position. According to Newton's second law, we can relate the force to the mass and acceleration of the cart:

$$F = ma = m\frac{d^2x}{dt^2}.$$

Setting the two expressions for force equal gives:

$$m\frac{d^2x}{dt^2} = -kx.$$

Rearranging this, we find the equation of motion:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

Here, we introduce the constant $\omega = \sqrt{\frac{k}{m}}$, which we will see later is the **frequency.** leading to the familiar form of the differential equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

This is a second-order, linear, homogeneous differential equation, which has two independent solutions. We can choose these solutions as follows:

$$x(t) = e^{i\omega t} \quad \text{and} \quad x(t) = e^{-i\omega t}.$$

By the **principle of superposition**, we can combine these solutions with arbitrary constants C_1 and C_2:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}.$$

Using **Euler's formula**, we can express the exponential functions in terms of sine and cosine:

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t),$$
$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t).$$

Thus, we have:

$$x(t) = (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)$$

Letting $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$, we can rewrite the solution as:

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t).$$

This shows that the motion described by $x(t)$ is **simple harmonic motion**, as it can be represented as a combination of sine and cosine functions, both of which are periodic. We can see that this is the very definition of simple harmonic motion.

5.2 Simple Harmonic Oscillator

5.2.1 Equation of Motion

Starting from Hooke's Law:

$$F = -kx \tag{5.1}$$

Using Newton's second law:

$$F = ma = m\ddot{x} \tag{5.2}$$

We get:

$$m\ddot{x} + kx = 0 \tag{5.3}$$

Rewriting it as:

$$\ddot{x} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}} \tag{5.4}$$

We will see later that ω is the frequency of oscillation.

5.2.2 Solution of the Differential Equation

This is a second-order linear differential equation with the general solution:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{5.5}$$

By applying Euler's formula, we can rewrite this as:

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) \tag{5.6}$$

Or in a simplified form:

$$x(t) = A \cos(\omega t - \delta) \tag{5.7}$$

where δ is a phase shift.

5.3 Extensions

Now, consider the initial conditions. If I start the oscillations by pulling the cart aside to $x = x_0$ and releasing it from rest ($v_0 = 0$), then $B_2 = 0$ in the general solution, and only the cosine term survives:

$$x(t) = B_1 \cos(\omega t). \tag{5.8}$$

If I launch the cart from the origin ($x_0 = 0$) by giving it a kick at $t = 0$, only the sine term survives:

$$x(t) = B_2 \sin(\omega t).$$

These two simple cases are illustrated in Figure 5.3. Notice that both solutions, like the general solution, are periodic because both the sine and cosine functions are periodic. Since the argument of both sine and cosine is ωt, the function $x(t)$ repeats itself after the time T for which $\omega t = 2\pi$. That is, the period is

$$T = \frac{2\pi}{\omega}. \tag{5.9}$$

This can also be written as a **phase-shifted cosine solution** which uses only cosine. This can help us better visualize the wave mechanics of the situation:

$$x(t) = A[-\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)]$$

$$x(t) = A[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)]$$

$$x(t) = A\cos(\omega t - \theta) \tag{5.11}$$

he cart is oscillating with amplitude A, but instead of being a simple cosine as in (5.8), it is a cosine which is shifted in phase: When t = 0 the argument of the cosine is —δ, and the oscillations lag behind the simple cosine by the **phase shift** δ.

5.4 Damped Harmonic Oscillator

5.4.1 Equation of Motion with Damping

Introducing a damping term characterized as a resistive force $-b\dot{x}$:

$$F = -kx - b\dot{x} = m\ddot{x} \tag{5.8}$$

The equation becomes:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{5.9}$$

Dividing through by m and defining $2\beta = \frac{b}{m}$ (the damping constant), we get:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{5.10}$$

5.4.2 Solution of the Differential Equation

The characteristic equation is:

$$r^2 + 2\beta r + \omega_0^2 = 0 \tag{5.11}$$

The roots are:

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{5.12}$$

The general solution is:

$$x(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_0^2}t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}t}\right) \tag{5.13}$$

5.5 Damping Conditions

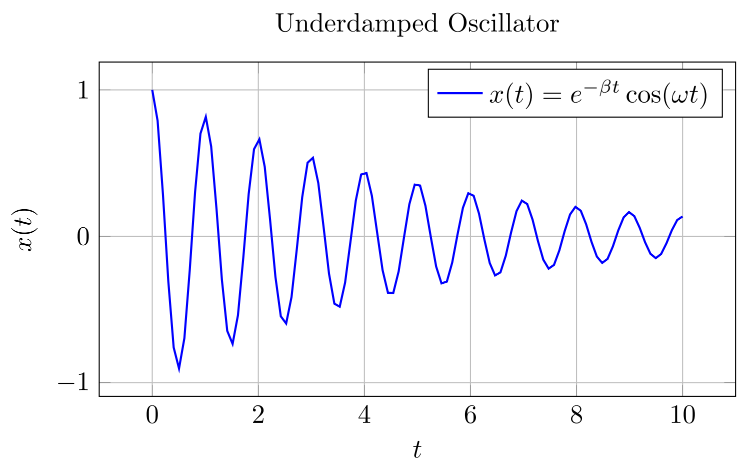
5.5.1 No Damping ($\beta = 0$)

In this case, the solution is:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{5.14}$$

(Same as simple harmonic motion, unchanged.)

Weak Damping ($\beta < \omega_0$)



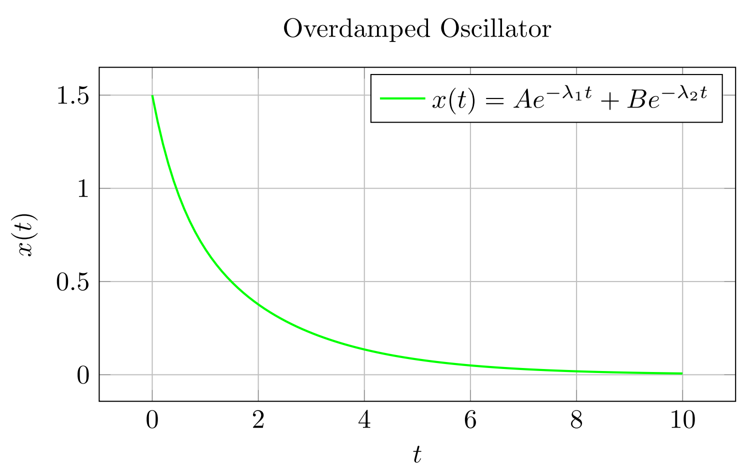
This is known as "**underdamping**" where $\beta < \omega_0$. The solution can be written as:

$$x(t) = e^{-\beta t}\left(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\right) \tag{5.15}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

This is because the perturbation is not large enough to cause a large change, so there is a frequency ω_1 that is only slightly smaller than the natural frequency ω_0.

Strong Damping ($\beta > \omega_0$)

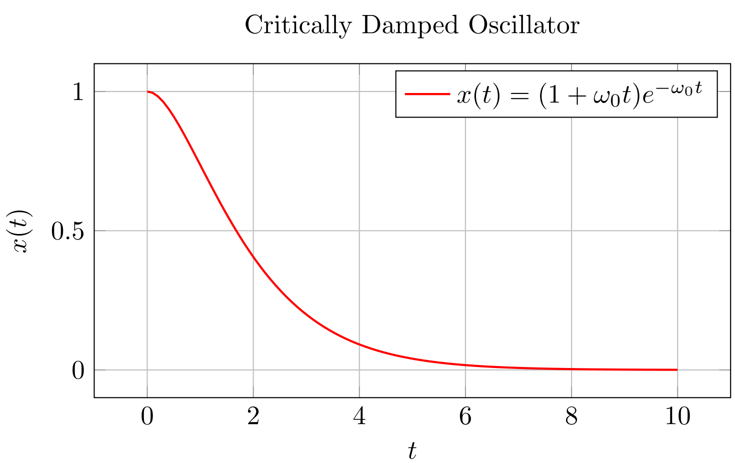


This is known as "**overdamping**." The solution becomes:

$$x(t) = C_1 e^{(-\beta + \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{(-\beta - \sqrt{\beta^2 - \omega_0^2})t} \tag{5.16}$$

Since both exponents are negative, $x(t)$ decreases over time without ever oscillating. We can see on the graph that the oscillation begins but is not able to complete.

Critical Damping ($\beta = \omega_0$)



2. General Solution to the Damped Oscillator

The characteristic equation for the above second-order differential equation is:

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

The solutions to this quadratic equation are:

$$r = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

This gives us three cases for the general solution based on the value of the damping ratio ζ:

- **Underdamped case** ($0 < \zeta < 1$):

$$x(t) = e^{-\zeta\omega_0 t}\left[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right]$$

where $\omega_d = \omega_0\sqrt{1 - \zeta^2}$ is the damped natural frequency.

- **Critically damped case** ($\zeta = 1$):

$$x(t) = (C_1 + C_2 t)e^{-\omega_0 t}$$

- **Overdamped case** ($\zeta > 1$):

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where r_1 and r_2 are the two real roots of the characteristic equation:

$$r_1 = -\omega_0(\zeta + \sqrt{\zeta^2 - 1}), \quad r_2 = -\omega_0(\zeta - \sqrt{\zeta^2 - 1})$$

3. Phase-Shifted Solution for the Underdamped Case

For the underdamped case, the solution can be written as a phase-shifted cosine:

$$x(t) = Ae^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)$$

where A is the amplitude and ϕ is the phase shift, both determined by initial conditions.

5.6 Driven Harmonic Oscillators

We start with the equation for a driven harmonic oscillator:

$$m\ddot{x} + b\dot{x} + kx = F(t),$$

where: - m is the mass, - b is the damping coefficient, - k is the spring constant, and - $F(t)$ is the driving force. This driving force kees the oscillator moving

Dividing through by m, just like for damped oscillation we get:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m}.$$

From here, we can define the following parameters: - $\beta = \frac{b}{2m}$, the damping constant, - $\omega_0 = \sqrt{\frac{k}{m}}$, the natural frequency, - $f(t) = \frac{F(t)}{m}$, the force per unit mass.

Substituting these into the equation, we obtain gives us the normal damped oscillation function :

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t).$$

This is a second order differential equation.

Define a Linear Differential Operator

It is often helpful to think of the left side of this equation as the result of an operator D acting on $x(t)$. This operator serves to take out x.

$$D = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2.$$

When we multiply this by **x**, we will get back the function that we previously had before. Thus when D acts on $x(t)$, it gives:

$$Dx = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x.$$

This allows us to rewrite the equation for the driven harmonic oscillator as:

$$Dx = f(t).$$

Defining D is useful because we know that this differential equation is linear. Because it's linear, we can treat D as a 'constant' and take it out of the equation when we differentiate.

We know from basic calculus that the derivative of ax is $a\dot{x}$. The same thing applies here.

The operator D is linear, meaning that it satisfies the following properties: - For any constant a, $D(ax) = aDx$. - For any functions $x_1(t)$ and $x_2(t)$, $D(x_1 + x_2) = Dx_1 + Dx_2$.

Combining these properties, we get:

where a and b are constants. This linearity is fundamental to solving differential equations.

5.6.1 Solution of the Driven Harmonic Oscillator

To solve the equation $Dx = f(t)$, we typically look for two parts of the solution:

1. The **homogeneous solution** x_h, which satisfies:

$$Dx_h = 0.$$

2. The **particular solution** x_p, which satisfies:

$$Dx_p = f(t).$$

This is also known as the **inhomogenous** soution and its much more difficult to solve.

The general solution is then a sum of these two parts:

$$x(t) = x_h(t) + x_p(t).$$

Now you can define your two equations as follows:

$$x_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$x_p = f_0 cos(\omega t)$$

We do this because most driven damped oscillators are near sinusoidal patterns, and using Fourier series, you can make sure that it is sinusoidal. The Fourier transform is a superposition of waves.

Now we know that there must be another solution with sine instead of cosine because these only differ by a time shift.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$
$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

Now we will use Euler's formula to simplify this.

Suppose now we define the complex function

$$z(t) = x(t) + iy(t),$$

with $x(t)$ as its real part and $y(t)$ as its imaginary part. If we multiply (5.58) by i and add it to (5.57), we find that

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}.$$

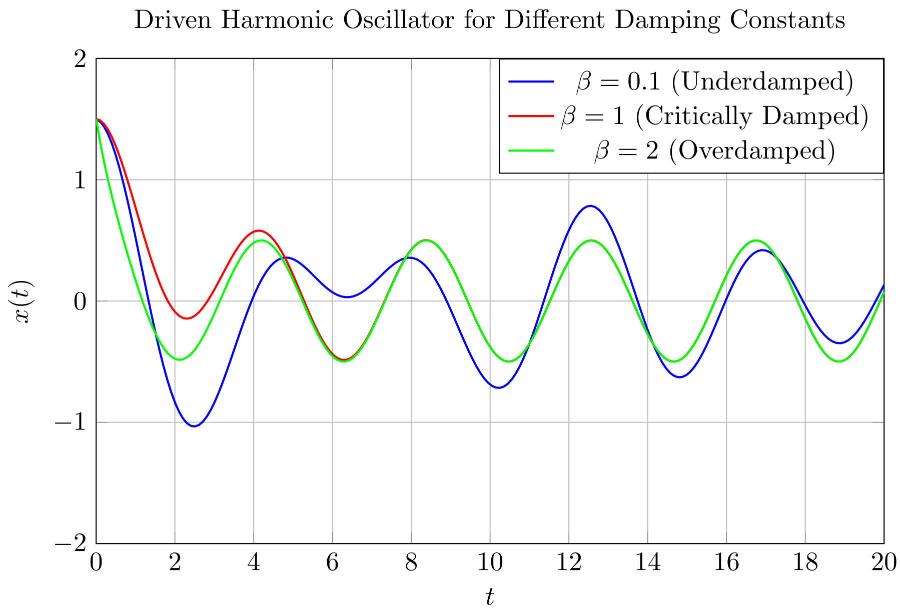
5.6.2 Analysis

- We have already found the homogeneous solution x_h, which would be the case when there is no driving force. This depends on the characteristic equation derived from D:

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0.$$

- The particular solution x_p depends on the form of $f(t)$, which is the driving term. If $f(t)$ is sinusoidal, such as $f(t) = A\cos(\omega t)$, methods like undetermined coefficients or variation of parameters can be used to find x_p.

Let's look for the sinusoidal form of the solution.



5.7 Resonance

Chapter 6

Calculus of Variations

6.1 Action and Lagrange's Equation

The action, S, for a system is defined as the integral of the Lagrangian, L, over time:

$$\boxed{S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt} \tag{6.1}$$

where $L = T - V$ is the difference between the kinetic energy, T, and the potential energy, V.

According to the principle of least action, the actual path taken by the system between two points in configuration space is the one that minimizes the action. Mathematically, this condition leads to Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{6.2}$$

6.2 Derivation of Euler-Lagrange Equation

To derive the Euler-Lagrange equation, consider a small variation $\delta q(t)$ in the path $q(t)$. The variation in the action is given by:

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt. \tag{6.3}$$

After integrating by parts and applying the boundary conditions $\delta q(t_1) = \delta q(t_2) = 0$, we obtain the Euler-Lagrange equation:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.} \tag{6.4}$$

6.3 Example: The Simple Harmonic Oscillator

Let's apply the calculus of variations to a simple harmonic oscillator. The Lagrangian for a harmonic oscillator is:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{6.5}$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}(m\dot{x}) + kx = 0, \tag{6.6}$$

which simplifies to the familiar equation of motion:

$$m\ddot{x} + kx = 0. \tag{6.7}$$

6.4 Constraints and Generalized Coordinates

In many mechanical systems, constraints limit the motion of the system. By using generalized coordinates, we can describe such systems in a simpler manner, even when the constraints are complex.

6.5 Shortest path between two points

The length of the path between two points is given by:

$$L = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

This has the standard form for the Euler-Lagrange problem with the function $f(y, y', x) = \sqrt{1 + y'^2}$.

Recalling the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

So we find:
- $\frac{\partial f}{\partial y} = 0$ because the equation is not dependent on y. - $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$
Thus,

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

So $\frac{\partial f}{\partial y'}$ must be a constant with any change in x.
Thus,

$$\frac{\partial f}{\partial y'} = C = \frac{y'}{\sqrt{1 + y'^2}}$$

$$y'^2 = C^2(1 + y'^2)$$

And y'^2 is a constant.

If we call y' as m:

$$y(x) = \int m\,dx = mx + C = mx + b$$

Thus, the shortest path is described by the equation for a line:

$$y(x) = mx + b$$

Brachistochrone problem

In the brachistochrone problem, we want to find what arrangement will allow us to reach point 2 in the shortest possible time.

Let's first think about how to solve for time. We know velocity $v(t) = \frac{\Delta d}{\Delta t}$, so we can rearrange this equation:

$$dt = \frac{ds}{v(t)}$$

$$t_{1\to 2} = \int_1^2 \frac{ds}{v(t)}$$

Where s is displacement. The height at any point y can be determined using energy constraints:

$$mgy_1 = \frac{1}{2}mv^2 + mgy_2$$

$$g\Delta y = \frac{1}{2}v^2$$

$$v = \sqrt{2gy}$$

We can substitute this back into the equation and use y as the independent variable because v is a function of y. We can simplify ds by taking the dy out of the square root:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \sqrt{x'(y)^2 + 1}\,dy$$

Thus,

$$t = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{x'(y)^2 + 1}}{\sqrt{y}}\,dy$$

Now we have a function $f(x(y), x'(y), y)$ for which the Euler-Lagrange equation can be used.

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

Since f **is independent of** x, we know that the derivative $\frac{\partial f}{\partial x}$ will be 0. Thus,

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

This means that the derivative will just be a constant.

$$\frac{x'(y)^2}{y(1 + x'(y)^2)} = \text{const} = \frac{1}{2a}$$

The constant $\frac{1}{2a}$ is named for "future convenience".
Now, simplifying this equation and solving for $x'(y)$:

$$x'(y) = \sqrt{\frac{y}{2a - y}}$$

$$x = \int \sqrt{\frac{y}{2a - y}}\, dy$$

Substitute that $\boldsymbol{y = a(1 - \cos\theta)}$. This will give:

$$x = a \int (1 - \cos\theta)\, d\theta$$

$$x = a(\theta - \sin\theta) + C$$

Since we can choose the initial conditions, we can choose that point one has $x = y = 0$. Then, $C = 0$. Thus:

$$x = a(\theta - \sin\theta) \tag{6.8}$$
$$y = a(1 - \cos\theta) \tag{6.9}$$

Chapter 7

Lagrange's Equations

7.1 Lagrange's Equations for Unconstrained Motion

The Lagrangian of a conservative system is:

$$\boxed{L = T - U} \tag{7.1}$$

where T and U are respectively the kinetic and potential energies.

To show why this is important, let's first define a function $L = T - U$. Here, T is the kinetic energy, which depends on $\dot{r}$, and U is the potential energy, which depends on the position r. So, if we take the derivative of L with respect to the position r, we get:

$$\frac{dL}{dr} = -\frac{dU}{dr} = F_x$$

by the properties of potential energy. Additionally, if we take the derivative with respect to $\dot{r}$, we get:

$$\frac{dL}{d\dot{r}} = \frac{dT}{d\dot{r}} = p_x$$

which is the momentum.

Since $p = m\dot{x}$, then $\frac{d}{dt}\left(\frac{dL}{d\dot{r}}\right) = F_x$. Thus,

$$\frac{dL}{dr} = \frac{d}{dt}\left(\frac{dL}{d\dot{r}}\right).$$

For Cartesian coordinates, this implies three Lagrange equations:

$$\frac{dL}{dx} = \frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right),$$

$$\frac{dL}{dy} = \frac{d}{dt}\left(\frac{dL}{d\dot{y}}\right),$$

$$\frac{dL}{dz} = \frac{d}{dt}\left(\frac{dL}{d\dot{z}}\right).$$

These equations exactly follow the form of the Euler-Lagrange equation. This implies that we can infer that the integral $S = \int L\, dt$ along the path of the particle is stationary.

7.1.1 Hamilton's Principle

$$S = \int_{t_1}^{t_2} L\, dt \tag{7.2}$$

if taken along the path.

Thus, a particle's path can be defined in three ways: by Newton's second law $F = ma$ (Newtonian Mechanics), by the three Lagrange equations (Lagrangian Mechanics), and by Hamilton's principle (Hamiltonian Mechanics).

7.1.2 Spherical Coordinates

When we choose r and θ for our coordinates, the θ equation turns out to be the equation for angular momentum.

In this case, the θ component of the generalized force is just the torque:

$$(\theta \text{ component of generalized force}) = \tau = \gamma(\text{torque})$$

and the corresponding component of the generalized momentum is:

$$\theta \text{ component of generalized momentum} = \frac{dL}{d\dot{\phi}} = L(\text{angular momentum}).$$

This shows that the generalized force and momentum components aren't necessarily the same as actual force and momentum. For example, here the generalized force is a torque (force $\times$ distance) and the generalized momentum is angular momentum (momentum $\times$ distance).

7.2 Constrained Systems

7.2.1 Pendulum Example

The kinetic energy of the system is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\phi}^2$$

The potential energy is:

$$U = mgh$$

where height h can be expressed in terms of ϕ as $h = l(1 - \cos\phi)$. Thus,

$$U = mgl(1 - \cos\phi).$$

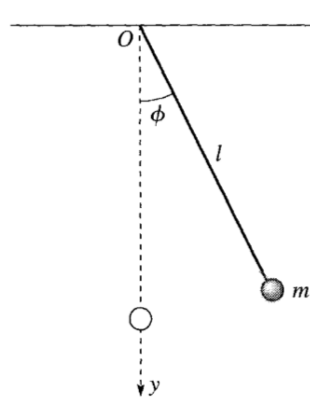


Figure 7.1: Pendulum

So,

$$L = T - U = \frac{1}{2} ml^2 \dot{\phi}^2 - mgl(1 - \cos\phi).$$

Now, we have expressed the Lagrangian in terms of a generalized coordinate q and its time derivative $\dot{q}$, and we can proceed by defining the Lagrange equation for this system as:

$$\frac{dL}{d\phi} = \frac{d}{dt}\left(\frac{dL}{d\dot{\phi}}\right).$$

Then the Lagrange equation is:

$$-mgl\sin(\phi) = \frac{d}{dt}(ml^2\dot{\phi}) = ml^2\ddot{\phi}.$$

We can see that the left-hand side of this equation is torque, τ, and the right-hand side ml^2 is the moment of inertia I times the angular acceleration α. So the Lagrange equation shows that:

$$\tau = I\alpha,$$

which is a familiar result from Newtonian mechanics.

7.3 Generalized Coordinates

The general definition applies to generalized coordinates.

The general definition of a Lagrangian is a function that acts on a set of generalized coordinates $q = (q_1, q_2, \dots, q_n)$, and the velocities such that $L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$ satisfies the following equation:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

This is more general and shows that the Lagrangian does not necessarily have to be $L = T - U$.

Chapter 8

Two-Body Central-Force Problems

In this section we will essentially just go over how to solve a two body problem with central forces, in this case, the force of gravity.

Chapter 9

Non-Inertial Reference Frames

Newtonian classical mechanics are valid under inertial reference frames. However, Lagrangian mechanics are global, giving us the tools to evaluate **non-inertial reference frames**. In this case, we will consider linearly accelerating non-inertial reference frames.

9.0.1 Acceleration without Rotation

Let S_0 be an inertial frame, S is a no-inertial reference frame with acceleration a=v

Rotation through velocity $\mathbf{w} = \mathbf{w} * \mathbf{r}$, right hand rule

Relative velocity:

$$\mathbf{v_{m1}} = \mathbf{v_{m2}} + \mathbf{v_{21}}$$

We can use this to evaluate problems using Newton's laws in spherical polar coordinates

9.0.2 Time Derivatives in a Rotating Reference Frame

Our inertial frame S is rotating with ω relative to inertial reference frame S_0.

We will use the same origin σ for both S_0, S.

Consider some vector $\mathbf{Q}$. We can decompose this in our reference frame S, and express it as a summation:

$$Q = \sum_i Q_i \hat{i}$$

$$\frac{dQ}{dt}|_S = \sum_i \dot{Q} \hat{i}$$

$$\frac{dQ}{dt}|_{S_0} = \sum_i \dot{Q}\hat{i} + \sum_i Q_i\dot{\hat{i}}|_{S_0}$$

Nowe ew can find Newton's second law in rotational framed by just taking two derivatives. If we hvve acceleration A we only just have the inertial forces to add. We can find the inertial forces now based on rotational motion, sicne we know that the inertial frame is rotating.

9.0.3 Newton's second law when Ω is constant

This is the simplest case of rotating reference frames. if we assume ω is constant in S_0, then it will be constant in all inertial reference frames S since $\omega x \omega = 0$. Also, the time derivative of ω will always be 0.

Newton's second law in our original inertial frame:

$$m\ddot{\vec{r}}|_{S_0} = \mathbf{F}$$

$$\ddot{\vec{r}}|_{S_0} = \frac{d}{dt}|_{S_0}[\dot{\vec{r}}|_S + [\vec{\Omega} \times \vec{r}]$$

Generalizing to a non-inertial reference frame S

$$\frac{d}{dt}|_S = [\dot{\vec{r}}|_S + \Omega \vec{\times} r] + \dot{\vec{\Omega}} \times [\dot{\vec{r}}|_S + \Omega \vec{\times} r]$$

From now on, we can set $\dot{\mathbf{r}}|_S = \dot{\vec{r}}$ for simplicity.

We can simplify these results and plug them into newton's second law, understanding that our result is in a rotating reference frame.

$$m\ddot{\mathbf{r}}) = \vec{F} + 2m(\vec{r} \times \Omega) + m[\Omega \times \vec{r}] \times \Omega$$

9.0.4 Time derivatives

$$\vec{L} = \vec{R} \times \vec{P} + \sum_\alpha P\vec{r_\alpha} \times \vec{p_\alpha} = \vec{L_{CM}} \tag{9.1}$$

This relies on us having chose an ceter of mas position

We can associate the momentum with two aspects, one of which does have to do with the centere of mass and one which doesnt

We can think of earth like this, where $\vec{L}_Earth = \vec{L}_{orb} + \vec{L}_{spin}$

We have another component that has to do witht he spinning motion which we do not usually consider in our normal calculations

$$\vec{L}_{orb} = \vec{R} \times \vec{P}$$
$$\dot{\vec{L}}_{orb} = \dot{\vec{R}} \times \vec{P} + \vec{R} \times \dot{\vec{P}}$$
$$\vec{L}_{spin} = \dot{L} - \dot{L}_{orb} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = \Gamma^{\vec{ext}}$$

Zero if it's a central force!