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Fat-Tailed Distributions in the Stock Market and their Impact on Option Pricing

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Abstract

The Black-Scholes-Merton option pricing model is widely used in finance. A core assumption of the model is that stock price movements follow a geometric Brownian motion and that stock prices are log-normally distributed. However, in real financial markets here is excessive kurtosis, 'fat-tails', in the stock market that results from investor interaction such that stock prices are better fitted by a stable distribution. These fat tails cause the Black-Scholes-Merton model to under-price options that are deep out of the money. Investors account for this with an increased implied volatility for these options resulting in a volatility smile.

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I would like to give a massive thank you to my supervisor, Professor Vvedensky, for prodiving me guidance and motivation when writing this essay. I could not have done it without his insights and resources.

Contents

1 Introduction

Physics concerns itself with producing mathematical models to explain physical phenomena. Is it possible to create a model that accurately analyses the movement of stock prices? It is assumed that stock price movements follow a geometric Brownian motion – this is same branch of mathematics that models the movement of large particles suspended in fluid. The Black-Scholes-Merton model is the most commonly used method of pricing European options, and it was derived from the geometric Brownian motion model of stock prices. There is a strong correspondence between the Black-Scholes-Merton partial differential equation and the one-dimensional heat equation, and the former can be transformed into the latter as a solution method.
A consequence of modelling stock prices with geometric Brownian motion is that stock prices are log-normally distributed. However, large price fluctuations are more frequent than implied by a normal distribution and they are characterised by fat tails in stock price movements. The existence of these fat tails challenge the validity of the Black-Scholes-Merton pricing model. This essay explains the causes and consequences of fat-tailed markets on option pricing.

2 Modelling the Movement of Stock Prices

Investors and traders all ask the same question: is there any underlying pattern that governs the behaviour of stock prices? There are several methods that investors use when trying to analyse the stock market. Fundamental analysis measures the value of a stock by examining related economic, political, and geographical factors: during the COVID-19 pandemic, the stock price of hospitality and leisure companies fell because travel regulations limited the revenue of companies in that sector. Technical analysts, on the other hand, try to trade based on market sentiment and psychology. They believe that historical patterns in stock price movements tend to repeat themselves and they look for specific indicators which could suggest future stock price movements.
Stock price movements could be modelled by geometric Brownian motion. Geometric Brownian motion is when the logarithm of a stock follows Brownian motion. Fundamental analysis and technical analysis are both directional methods, which means that they try to predict whether the stock price is due to go up or down – the direction in which the stock price is moving. Geometric Brownian motion, however, is not a directional analysis method. Instead, it analyses the probabilities that a stock price moves by certain amounts in certain directions over a period.
The equation modelling the Brownian motion of a stock price is

$$dS = \mu S\,dt + \sigma S\,dz \tag{1}$$

where S is the stock price, dS is the instantaneous change in the stock price, μ is the expected return per unit time of the stock, and σ is the volatility. The expected rate of return is the profit or loss on an investment that investors expect to receive. For example, if an investor bought Apple shares for \$130 and sold them later on for \$135, they made a return of 3.8%. Volatility is a measure of the variability of returns on stock. The dz term is the change in Brownian motion over infinitesimal time dt. It is the stochastic term: $dz = \varepsilon\sqrt{dt}$ where ε has a standard normal distribution (mean of 0 and standard deviation of 1). This equation can be split a deterministic part and a probabilistic part. The expected

return over the period dt is μdt, which is deterministic. After an infinitesimal duration of time, the stock price is due to increase by μSdt. How much the actual return deviates from the expected return is probabilistic, and is determined by $\sigma\varepsilon\sqrt{t}$.
Brownian motion is a term describing random movements. It was named after the botanist Robert Brown who, in the early 19th century, noticed the phenomenon in the movement of pollen grains suspended in water: the pollen grains moved around erratically and unpredictably because of the constant bombardment of the water molecules surrounding it [1]. By modelling the movement of stock prices with stochastic differential equations, an expression can be derived for a geometric Brownian motion of share price movements. Brownian motion is also known as a generalised Wiener process, and the two will be used interchangeably. Stochastic calculus is a branch of mathematics that allows randomness to be introduced to dynamical systems. The value of a variable following a stochastic process changes in an uncertain way as stochastic processes involve probability and chance. Stock prices are modelled as continuous-time, continuous-variable stochastic processes. Continuous-time means that the variable's value can change at any given time. Continuous-variable means that the variable can take any value in a given range. However, shares can only be traded when the market is open, so they are not precisely continuous-time variables. They can also only take discrete values, perhaps multiples of a cent, so are not exactly continuous variables. Another assumption required for this model is that stock prices are Markovian variables, which means that the historical values of the stock make no impact on the future values that the stock will take [2].
The result of modelling stock price movements with geometric Brownian motion is that the percentage change in stock price, $\frac{dS}{S}$, follows a normal distribution. This can be shown by dividing Equation 1 by S:

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{2}$$

because dz is sampled from a standard normal distribution, then $\frac{dS}{S}$ is normally distributed with a mean of μdt and variance of $\sigma^2 dt$. This result is important and will be used for the statistical tests later.
By integrating both sides of the equation between the present time and some time T in the future, it can be shown that

$$S_T = S_0 e^{\mu T + \mathrm{N}(0,\sigma^2 T)} \tag{3}$$

where S_T is the stock price at time T, S_0 is the stock price currently, and $\mathrm{N}(0,\sigma^2 T)$ is a normal distribution with a mean of 0 and a variance of $\sigma^2 T$. As the logarithm of $\frac{S_T}{S_0}$ follows a Brownian motion, it can be said to follow a geometric Brownian motion.
It can also be shown that the natural logarithm of the stock price has a normal distribution:

$$\ln(S_T) - \ln(S_0) \sim \mathrm{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right] \tag{4}$$

where S_T is the stock price at some future time T, and S_0 is the current stock price. The natural logarithm of the variable follows a normal distribution, so the variable is said to have a log-normal distribution. A full derivation of this result, and a clearer definition of the Markov Property, can be found in Appendix A.
Modelling stock market returns with a normal distribution means that small price changes around the mean are much more common than large price changes. This model is widely used for risk analysis and is also incorporated into option pricing.

3 Options

An option is a financial product that allows the holder the right, but not the obligation, to buy or sell a specific quantity of an asset at some fixed time in the future. The Chicago Board Options Exchange opened in 1973. This was the first options exchange and since then, the buying and selling of options have been an integral part to the financial system. Today, the global trade volume of options was $22 trillion (this statistic was true as of 20th December 2022 according to CBOE [3]). A put option is the option to buy an asset whereas a call option is the option to sell an asset. To make option selling worthwhile, the seller will charge a fee on every option. This fee is called a premium. Figure 1 shows the potential profit and loss of an option holder as the price of the underlying asset varies. The potential losses are capped at the price of the option premium [4].

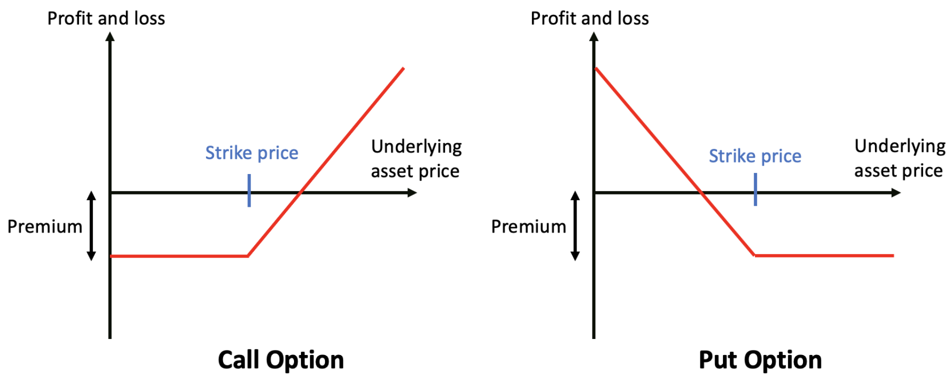

Figure 1: The profit and loss of a call and put option. For a call option holder, the profit to be gained is theoretically infinite because the share price could increase boundlessly. The losses can are capped at the option premium. For a put option holder, the profit gains can be great but are capped because an asset cannot take a negative value. But the losses are also capped at the option premium.

There are many different scenarios where options are useful. Companies may want to use options to reduce their risk to directional moves in the market. For example, if the price of jet fuel increases, the profit margin of airline companies will be impacted. So, supposing the current price of the asset, which is also called the spot price, is $2 per gallon, an airline company could buy a call option for jet fuel at this price. This will give the company the right, but not obligation, to buy fuel at $2 per gallon in a year's time. The date after which an option is no longer viable is called the expiry date, and price at which the option can be exercised is called the strike price. In this example, the strike price is $2, and the expiry date is in a year's time. If the price of jet fuel is less than $2 per gallon on the expiry date, then the company could let the option expire and the only loss in this scenario is the premium of the option. But should the price of jet fuel be more than the strike price, then the company can exercise the option and buy jet fuel from the option seller for cheaper than market price. In this example, buying an option can act as insurance. There are options available for different underlying assets ranging from cereal to stocks to cows.
If an option is out of the money, then the owner of the option will either overpay or be underpaid if they exercise their option. A call option is out of the money if the spot price

is lower than the strike price of the option – the option holder is better off buying the asset on the market instead of exercising their option because it is cheaper. A put option is out of the money when the spot price is higher than the strike price – the option holder would make a higher profit selling their asset on the market. Figure 2 shows the 'moneyness' of the above example.

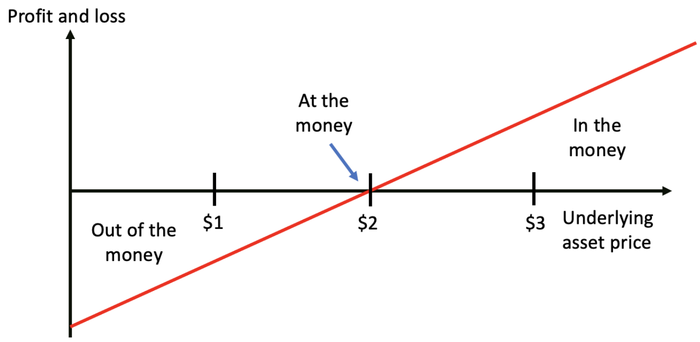

Figure 2: The 'moneyness' of a call option with a $2 strike price.

4 Black-Scholes-Merton Model

The Black-Scholes-Merton model, or just the Black-Scholes model, was developed to price options. For their work on this model, Robert C. Merton and Myron S. Scholes were awarded the Nobel prize in 1997. Unfortunately, Fischer Black died before he could receive the prize [5]. The Black-Scholes model only prices European options, where the option can only be exercised on the expiry date. Other types of options include American options which can be exercised any time between buying the option and the expiry date.

4.1 Assumptions

The model is based on several assumptions [2]:

- Stock price movements follow a geometric Brownian motion with a constant volatility and expected rate of return.
- There are no transaction fees in the process of buying and selling assets.
- During the lifetime of the option, the asset provides no dividends. Although this assumption is true for the original Black-Scholes model, the model has since been updated to include dividends.
- There are no arbitrage opportunities.
- The risk-free interest rate is constant.

A corollary of the geometric Brownian motion assumption is that stock price changes are log-normally distributed. Hence, the probability of the underlying stock taking any value can be calculated, provided the volatility, and expected rate of return is known [6].
Another important assumption is that there is no risk-free profit to be made on exchanges – it is impossible to buy the same asset on one market and sell it on another for a higher price

simultaneously. This is called arbitrage, and the Black-Scholes model assumes that there are no arbitrage opportunities available. In truth, there are small market inefficiencies that result in arbitrage opportunities. Advanced algorithms have been created to spot and utilise these opportunities which make it very difficult for regular traders to profit from arbitrage [7]. When a computer system spots an asset mispricing, it will then buy the asset cheaper on one market and sell it for a higher price on the other within a matter of seconds. This pushes the price up on the exchange where the asset is cheaper and pulls the price down on the other exchange until the prices are equal, thus removing the arbitrage opportunity.
It is possible to remove the stochastic term in an investor's portfolio by buying a stock and selling and a specific quantity of an option, thus producing a risk-less portfolio. Because of the no arbitrage assumption, the interest gained on this portfolio is equal to risk-free interest such as the interest on money in a savings account. Profit gained due to risk-free interest is not an arbitrage opportunity. This is because it is assumed that an investor starts out with no money. They would have to borrow money to leave in a savings account should they want to gain interest. However, the interest on their borrowed money would be the same as the interest on the savings account – zero profit is accrued.

4.2 Financial Interpretation

The Black-Scholes partial differential equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S} \tag{5}$$

where V is the option price, σ is the volatility of the underlying stock, S is the spot price and r is the risk-free rate. The right-hand side of this equation is the risk-free and deterministic part of the equation. It is the risk-free rate from a long position on an option and a short position in $\frac{\partial V}{\partial S}$ shares - a perfectly hedged portfolio.
The $\frac{\partial V}{\partial S}$ term is a measure of the sensitivity of the option price to changes in the spot price. For call options, the term is positive because the option gains value if the stock price increases, whereas it is negative for put options. The $\frac{\partial V}{\partial S}$ term is higher for options that are in the money and lower for options that are out of the money, because options that are in the money are more likely to expire in the money.
The time decay term, $\frac{\partial V}{\partial t}$ measures how the option price varies with time. This term is always negative for option holders as the value of the option decreases the closer it gets to expiry. It is higher for options that are at the money because it is less certain whether the option will be in or out of the money at expiry, so time plays a more important role. The $\frac{\partial^2 V}{\partial S^2}$ term is also higher for options that are at the money. This term can be useful for financial analysis of options as $\frac{\partial V}{\partial S}$ can be variable as the spot price and option price change [8].

4.3 Mathematical Interpretation

Equation 5 is a second-order linear differential equation of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \tag{6}$$

These PDEs can be grouped into different forms depending on the coefficients of the highest order terms [9]:

- the PDE is hyperbolic if $b^2 - 4ac > 0$
- the PDE is parabolic if $b^2 - 4ac = 0$
- the PDE is elliptic if $b^2 - 4ac < 0$

The Black-Scholes PDE is elliptic. Through substitution of variables, all elliptic equations can be reduced into an equation of the form

$$u_{xx} + ... = 0 \tag{7}$$

where ... represents lower order terms. A common elliptic equation is the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{8}$$

where u is the heat energy and k is a constant dependent on the material that the heat energy is flowing through. The $\frac{\partial u}{\partial t}$ term corresponds to the heat energy flowing into or out of a point and $\frac{\partial^2 u}{\partial x^2}$ is the diffusion term [10]. The signs on the partial derivative with respect to t are different for the Equation 5 and Equation 8 - the heat equation is a forward elliptic equation and requires initial conditions to be solved whereas the Black-Scholes is a backwards elliptic equation and requires terminal boundary conditions. These conditions are consistent with the physical nature of either problem [9]. For the Black-Scholes PDE, the value of an option at expiry is known (terminal boundary condition). The heat equation, on the other hand, requires the heat distribution at $t = 0$ (initial boundary condition). The Black-Scholes PDE can be transformed into the heat equation then solved. A full derivation of the Black-Scholes PDE from the geometric Brownian motion of share prices and the solution to the PDE can be found in Appendix B.
The solution to Equation 5 is

$$C = \Phi(d_1)S_t - \Phi(d_2)Ke^{rT} \tag{9}$$

for a call option and

$$P = Ke^{rT}\Phi(-d_2) - S_0\Phi(-d_1) \tag{10}$$

for a put option, where

$$d_1 = \frac{\ln\frac{S_t}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{11}$$

$$d_1 = d_1 - \sigma\sqrt{T} \tag{12}$$

and $\Phi(x)$ is the cumulative normal distribution function. To use the Black-Scholes equation, the spot price (S_t), the strike price (K), the risk-free interest rate (r), the volatility (σ) and the time to maturity (T) must be known. All of these parameters are market observable except the volatility, which must be estimated.
Because the Black-Scholes model is widely used in the financial sector, it is important to test its accuracy. The model assumes that stock prices are log-normally distributed. By collecting historical data, it is possible to test how well a normal distribution model describes stock price movements.

5 Market Analysis and Stable Distributions

5.1 Market Analysis

The geometric Brownian motion of stock price movements imply that the percentage change in stock prices should be normally distributed as in Equation 2. Figure 3 displays the daily price change of the S&P 500 index with two statistical distributions fitted to it: a normal distribution and a stable distribution. Different texts may refer to stable distributions as Lévy-stable distributions or Lévy distributions. (This essay uses the Nolan terminology and notation from the text *Univariate Stable Distributions* [11]).

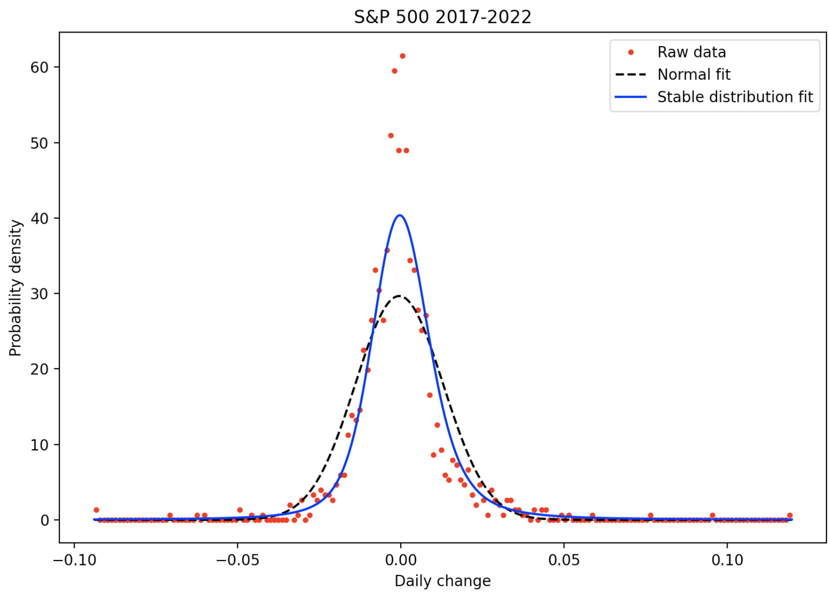

Figure 3: The daily price change of the S&P 500 index with a normal and stable distribution fitted to the data. The normal distribution has the parameters $\mu = 0$ and $\sigma = 0.0130$ and the stable distribution has the parameters $\alpha = 1.52$, $\beta = 0.190$, $\gamma = 0.0100$, and $\delta = -0.0100$ [12].

The stable distribution seems to fit the data better: it has a sharper peak and fatter tails. Figure 4 shows the same data as Figure 3 but on a semi-log axis to show the prominence of the fat tails more clearly.

5.2 Stable Distributions

It has been long known that stock prices tend not to follow a normal distribution on shorter time frames. Benoit Mandelbrot, the mathematician often hailed as the 'father of fractal geometry', was the first to suggest that stock price movements follow a stable distribution instead of a normal distribution. In 1963, he examined that the behaviour historical of cotton prices exhibited fatter tails than expected from a normal distribution [13]. Stable distributions were first characterised by Paul Lévy, Mandelbrot's supervisor.
A stable distribution can be parameterised with four parameters: α, β, γ, and δ. The form of the distribution is determined by α and β, γ determines scale, and δ is the location parameter (some sources may use σ and μ for the scale and location parameter

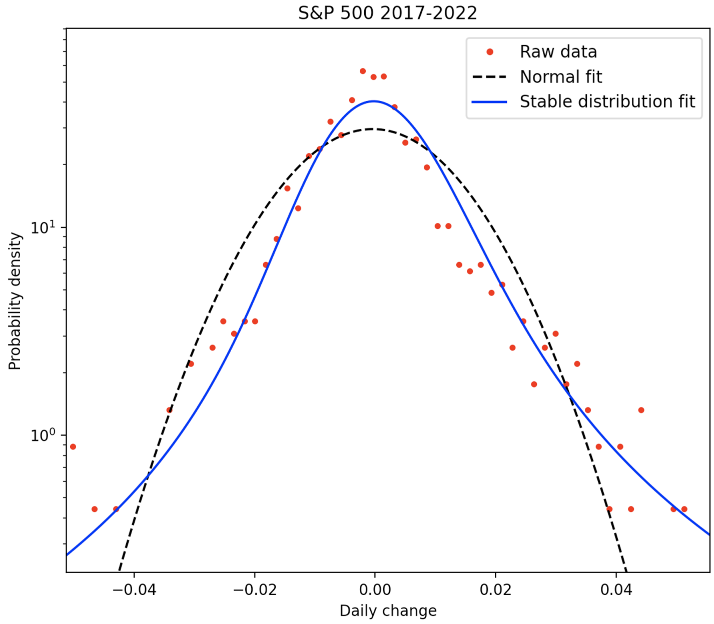

Figure 4: A semi-log plot of the daily change of the S&P 500 index plotted using the same data as Figure 3. The logarithmic y-axis shows the prominence of the fat tails more clearly.

respectively). The full functional form of a stable distribution and more information can be found in Appendix C. A stable distribution with $\alpha = 2$ and $\beta = 0$ is a normal distribution, $\alpha = 1.5$ and $\beta = 1$ gives a Lévy distribution and $\alpha = 1$, and $\beta = 0$ gives a Cauchy-Lorentz distribution [11]. These three specific forms of stable distributions are the only ones that have a closed formula, which made it difficult for stable distributions to be used practically before the arrival of computing. The most important difference between the shape of a normal distribution and the shape of a stable distribution is that the stable distribution can be more peaked around the mean and the tails are fatter: for stable distributions with $\alpha < 2$ (so not a normal distribution) and $-1 < \beta < 1$, the tail probabilities are asymptotically power laws [14]. The tails in a normal distribution decrease proportional to e^{-x^2}, thus stable distributions have the capability of producing fatter tails than a normal distribution. Table 1 shows the probability densities of a standard normal, Lévy and Cauchy distribution at the tails. Kurtosis is the property measuring whether a set of data is heavy- or light-tailed compared to a normal distribution. The fat-tailed (or heavy-tailed) property is called leptokurtosis.

c	P(X>c)		
	Normal	Cauchy	Lévy
0	0.5000	0.5000	1.0000
1	0.1587	0.2500	0.6827
2	0.0228	0.1476	0.5205
3	0.001347	0.1024	0.4363
4	0.00003167	0.0780	0.3829
5	0.0000002866	0.0628	0.3453

Table 1: The tail densities of a standard normal, Lévy and Cauchy distribution. Source: John P. Nolan [9].

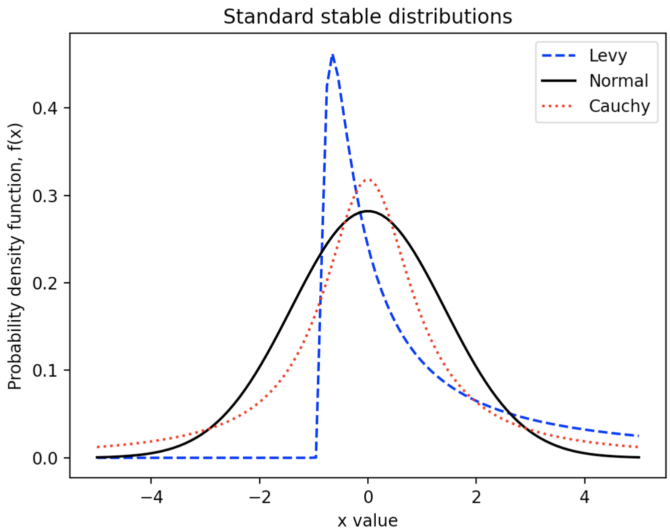

Figure 5: A standard normal, Lévy and Cauchy distribution where $\gamma = 1$ and $\delta = 0$. The Lévy distribution is skewed, whereas the others are not. The tail thickness varies with the Lévy having the heaviest tail, followed by the Cauchy then normal distribution.

5.3 Chi-Squared Test

A chi-squared test was performed on the 2017-2022 S&P 500 data for both the normal and stable distribution. Chi-squared tests are used to determine how well a set of data fits a specific statistical distribution. The test statistic of the data needs to be lower than the critical value for the distribution to be a good statistical fit to the data. The critical value is dependant on the number of free parameters of the distribution and the chosen confidence interval. The test statistic for the normal distribution is 220 and that of the stable distribution is 63. Although the test statistic for the stable distribution was much better than the normal, the stable distribution is still not a good statistical fit according to this test: in the 5% confidence interval, the critical value for the normal and stable distributions are 11 and 15 respectively (the critical values are different because the stable distribution has more free parameters). It is not uncommon for financial models to fail statistical tests, however, because of the unpredictable nature of markets. Even though the normal and stable distribution fit both failed the chi-squared test for the S&P 500 data, the stable distribution is still, objectively, a much better fit to the data.
Financial professionals have known for many decades that the stable distribution is a better model of stock price movements, but there are a few features of stable distributions that make it unpopular. All stable distributions, except the normal distribution, have infinite variance as a consequence of heavy tails [15]. However, when observing empirical data, it is apparent that the variance of a sample does not increase boundlessly as the sample size increases, instead it converges to some finite value. Stable distributions also have the property of stability, thus its name. The sum of two stable distributions is also a stable distribution. Hence, the long-term returns on a stock should follow a stable distribution because they are just the sum of the short-term returns. Empirically, however, this is not the case – the changes in stock price converge to a normal distribution over longer time frames [16]. Figure 6 shows the monthly price movements in the S&P 500 over 25 years. Compared to the S&P 500 data over 5 years, the data is more similar to a normal distribution. Because of the infinite variance of the Levy distribution, the central limit theorem cannot be applied in this case [15], [14].

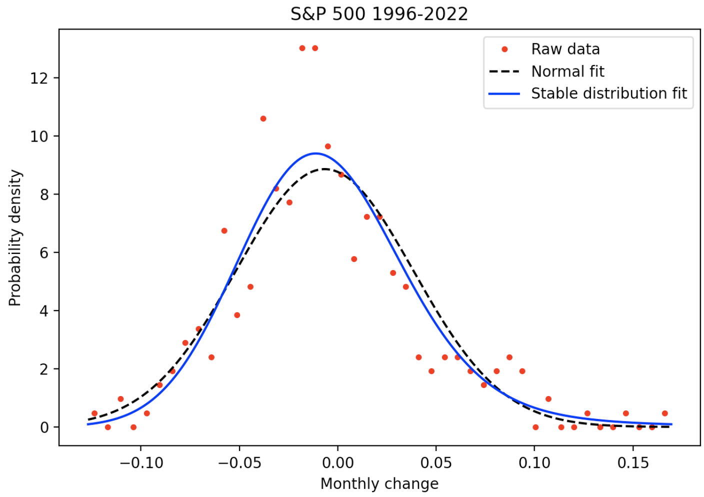

Figure 6: The monthly price change of the S&P 500 between the years 1996 and 2022 with a normal and stable distribution fitted to the data. The normal distribution has parameters $\mu = -0.000665$ and $\sigma = 0.0450$ and the stable distribution has parameters $\alpha = 1.84$, $\beta = 1.00$, $\gamma = 0.0300$ and $\delta = -0.0100$.

5.4 Truncated Stable Distribution

From their own analysis of the changes in stock price of the S&P 500 index, Mantegna and Stanley (1995) found that the tails of their data are fitted well by a stable distribution until six standard deviations away from the mean of their data, then the tails more accurately fit an exponential decay [17]. Similar observations have been found in the Milan stock exchange, individual French stocks, the CAC40 index and foreign exchange markets [16]. This result has encouraged the creation of the truncated stable distribution (TSD). Other sources may call the TSD the truncated Lévy distribution (TLD) instead. The TSD follows a stable distribution near the mean, but the tails fall off exponentially at extreme values on either ends of the distribution. This form of distribution is pseudo-stable - it has the property of a stable distribution at short-time scales but at longer time scales, the distribution will converge to a normal [16]. The TSD also has a finite variance which better fits empirical observation [15].

6 Fat Tails

A lot of risk management methodologies in the financial sector are based on the geometric Brownian motion model of stock price movements. However, from the analysis of the S&P 500 returns, it is clear that modelling returns with a normal distribution underestimates the leptokurtosis of asset – large price movements occur more frequently than expected, resulting in these ‘fat-tailed’ stable distributions. What are the origins of fat-tailed distributions?

The Brownian motion model of stock price movements assumes the Efficient Market Theory: investors act independently from each other and any new information influencing the stock market arrives randomly and is assimilated by all investors at the same time [18]. However, investors and traders can affect each other’s decisions.

6.1 Interaction of Investors

Investor interaction can produce positive feedback loops that amplify price movements. A good example of this is a short squeeze. Consider a public company that has been doing badly recently – they could perhaps be on the verge of bankruptcy. Traders could decide to short the asset by borrowing some stocks on an exchange and selling them at the current market price betting that the price is going to fall. They do this in the hope to then buy the stock at a lower price and return it to the person or institution that they borrowed the stock from, thus making profit. However, should the stock go up in price – the company released better than expected earnings or perhaps found new funding – the traders shorting the stock will encounter losses as they need to now buy the stock back at a higher price. These traders want to cover their losses before the stock price further increases, so they buy the stock as soon as possible. The sudden increased demand of the stock increases the stock price even more! This could then result in more traders being forced out of their short positions – thus the term a short squeeze. The cumulative effect of investors buying back this stock can produce very large price movements. A famous example of a short squeeze is the GameStop crisis of 2021 when the share price inflated from \$17.25 at the start of January to a peak of almost \$500 on 28th January [19]. Higher than expected frequencies of extreme events lead to fat tails.

Another common example of investor interaction is herd behaviour. This is when an investor imitates the decisions made by other investors - for example, research may indicate that an investor should sell an asset they are holding but the investor may choose not to do so because they know that another investor is holding or buying that asset. There are a few reasons why investors exhibit herd behaviour. An investor could believe that others have information about the return on an asset, thus copying other investors' decisions could increase returns. There may also be a psychological preference for conformity among investors. In some cases, asset managers may get a higher bonus by imitating other investors' decisions which encourages herding. Policymakers have stated that herd behaviour increases the volatility of markets and fragility of the financial system [20].

The mathematician Rama Cont and physicist Jean-Philippe Bouchaud quantified herd behaviour and investor interaction (1998). They modelled investors as agents that can form random clusters with other agents – these clusters can be thought of as mutual funds, perhaps. All agents in a cluster make the same investment decision. Clusters can buy, sell, or hold an asset and this directly impacts the price of the asset. In their model, price movements exhibited fat tails, thus showing that herd behaviour increases the frequency of extreme price movements [21].

6.2 Impact on Option Pricing

The Black-Scholes pricing model is based on the geometric Brownian motion model of stock prices and thus assumes that stock prices are log-normally distributed. So large price movements are rare. Options that are deep out of the money are priced lower by the Black-Scholes model because there is a low chance that the spot price will move such that the option is in the money. For example, consider an option's spot price to be \$100, the time till expiry is 1 year, volatility is 10% and the risk-free return is 3%. For a call option with strike price \$130, which is much out of the money, the option premium is \$0.04. But should the strike price be \$105, which is still out of the money but closer to the spot price, the option premium would be \$3.15. But the appearance of fat tails in the stock market mean that large price movements occur more frequently than the model expects, so the Black-Scholes model would under-price options that are much out of the money. The existence of volatility smiles is an example of this shortcoming.

7 Volatility

Volatility is an important topic in quantitative finance because of all the parameters required for the Black-Scholes equation, the only parameter that is not easily obtainable is volatility. It needs to be estimated and there are different methods to do so. There are two different measures of volatility: historical (realised) volatility and implied volatility.

7.1 Historical Volatility

Historical volatility uses past stock prices to calculate an estimate of the current volatility. Investors use different methods to determine historical volatility, but one of the most common is the close-close volatility estimator [22]:

$$u_i = \ln \frac{S_i}{S_{i-1}} \tag{13}$$

$$\sigma = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2} \tag{14}$$

where S is the stock's closing price and $i = 1, 2, ..., n$.
This method only considers the closing prices of the stock and ignores the intra-day volatility. There are other methods like the high-low volatility estimator that factors in the most extreme stock price into the calculation of volatility. Investors must also choose an appropriate number of samples, n, when estimating volatility. More data could produce a more accurate estimate, but as volatility changes over time, data that is too old may no longer be relevant.
Further methods of volatility estimation are also used including the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) approach, which is popular because it assumes that volatility is not constant and that the current volatility depends on the past measures - the price of an asset tends to be more volatile after a big price movement, such as a crash [23]. More information on the GARCH model can be found in Engle (2001) [24].

7.2 Implied Volatility

Historical volatility is a backwards-looking measure of volatility whereas implied volatility is future-looking: it reflects the volatility of the underlying asset given its option price. For highly liquid (frequently traded) underlying assets, the option price should be readily obtainable. It is not possible to invert the Black-Scholes equation to determine volatility as a function of the other parameters, but an iterative search procedure can be used instead [2]. For assets with high volatilities, the option premium is higher as well because there is an increased probability of the option expiring in the money.

The 30-day implied volatility of the U.S. stock market can easily be obtained from the CBOE volatility index (VIX). This index measures implied volatility by analysing the price of call and put options on the S&P 500 whose expiry is between 27 to 37 days [25]. If the VIX index has a value of 9, for example, this means that the volatility of S&P 500 30-day options are 9%.

7.3 Volatility Smiles

The Black-Scholes model assumes that the volatility of an underlying asset is constant [26]. Given the same underlying asset but with different strike prices, the implied volatilities should be the same. However, empirical market data has shown this not to be the case. Plotting implied volatilities by different strike prices produces a volatility smile – the volatility is higher for options that are deep out of the money. The Black-Scholes model would predict the volatility curve to be flat.
Different underlying assets exhibit different patterns of the volatility smile. Equity options traded on American markets showed a volatility smile only after the market crash in 1987 [2]. Financial crashes are examples of price large movements. Investor reassessments of the frequencies of large market movements - fat tails - have led to higher prices for out of the money options, thus an increased implied volatility.

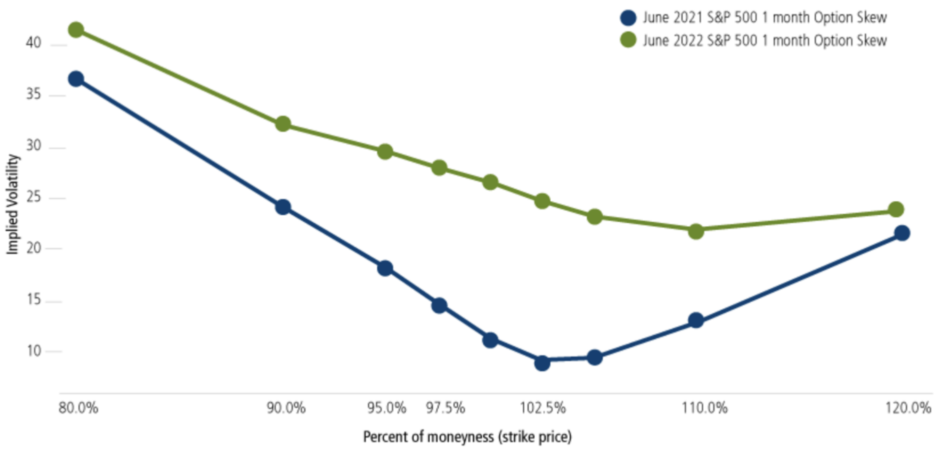

Figure 7: The volatility smile for S&P 500 30-day options from June 2021 and June 2022. Source: Calamos Investments [27].

8 Conclusion

The Black-Scholes-Merton model of pricing options is widely used in the financial industry, so it is important to test the assumptions that this model is based on. A method of solving the Black-Scholes partial differential equation is to first transform it into the heat equation before solving it by analytical techniques. There are stark similarities between both equations as they each contain a time-decay and diffusion term, and are both elliptic linear second-order partial differential equations. The Black-Scholes model is based off the assumption that stock price movements follow a geometric Brownian motion and thus are log-normally distributed. From analysis of the S&P 500 market data, it is evident that real share prices are not log-normally distributed. Instead, there are large price fluctuations that are more frequent than expected from a normal distribution. These fluctuations result in excessive kurtosis – or fat tails – in the market data. A stable distribution, or a truncated stable distribution, fits the data better because they can be more fat-tailed than a normal distribution. Investor interaction can produce positive feedback loops and amplify price movements which result in these fat tails. A consequence of large price movements is that the Black-Scholes model under-prices options that are deep out of the money – it underestimates the probability of the spot price moving such that the option is in the money. Investors have accounted for this by assuming that the implied volatility of the underlying asset for out of the money options is higher than options that are in the money, as the option price increases with volatility. This results in the implied volatility smile effect. The Black-Scholes model requires the volatility of the underlying asset to be constant. Hence, the existence of the volatility smile is a downfall of the model.

There are other models that tackle these issues. The Heston model accounts for the volatility smile by allowing volatility to be a stochastic variable. This model also accounts for the fat tails [28]. Besides this, it is possible to incorporate the truncated stable distribution into option pricing [15].

Although the Black-Scholes model has its shortcomings, it is still commonly used mainly because of its mathematical simplicity and ease of use. The downfalls of the model can be circumvented easily: the model has been modified to include dividends, and investors have accounted for fat tail risk by assuming high volatility for options deep out of the money [29].

There is no model that exists that can accurately analyse every aspect of financial markets because of their unpredictable nature. Institutions must use a wide variety of models depending on the economic situation and investor sentiment. When using the Black-Scholes model to price options, it is important to keep in mind the assumptions that this model as based on and the model's limitations.

Appendices

The derivations of geometric Brownian motion in Appendix A and the Black-Scholes partial differential equation in Appendix B were reproduced from the text *Options, Future, and Other Derivatives* by Hull (2015) [2]. The transformation of the Black-Scholes equation into the heat equation was reproduced from Fedotov (2008) [9], [30]. The specific parameterisations and notations of stable distributions were reproduced from the text *Univariate Stable Distributions: Models for Heavy Tailed Data* by Nolan (2020) [11].

A Geometric Brownian Motion Derivation

A.1 Markov Variables

Suppose a change in a variable is entirely dependent on its value at an instantaneous point in time, the variable is said to possess a Markov property [2]. A consequence of this is that the historical values of the variable make no impact on the future values that the variable will take. If stocks were modelled with this property, it would be consistent with the weak form of the Efficient Market Theory. This states that stock prices already contain all historical information - there are no superior returns to be gained from technical analysis. A Markovian variable, z, is said to follow a Wiener process or Brownian motion if the change in z, Δz, in short time period Δt, follows

$$\Delta z = \varepsilon\sqrt{\Delta t} \tag{15}$$

where ε has a standard normal distribution. The drift rate of a stochastic process is the change in the mean per unit time, and the variance rate is the change in the variance per unit time of a variable.
Consider the change in the variable z after a long time T, given by $z(T) - z(0)$. T can be split up into up into X smaller time periods Δt where the variable changes by Δz:

$$X = \frac{T}{\Delta t}$$

The change in variable after time T is then given by

$$z(T) - z(0) = \sum_{i=1}^{X} \varepsilon_i\sqrt{\Delta t} \tag{16}$$

Because of the additive property of the mean and variance of Markov variables,

$$\mu = 0$$

$$\sigma^2 = X\Delta t = T$$

$$\sigma = \sqrt{T}$$

which is consistent to the properties of Markov variables aforementioned. The change in this variable for a long time period is normally distributed.

A.2 Generalised Wiener Process and Itô Process

The variable x follows a generalised Wiener process if the rate of change can be given by

$$dx = a\,dt + b\,dz \tag{17}$$

where a and b are constats. There are two parts to this equation. The $a\,dt$ part is the corresponds to the drift of the variable: x has expected drift of a per unit time. The $b\,dz$ part of the equation gives the volatility of this drift. The expected variance rate is b^2. For discrete-time, the generalised Wiener Process can be written as

$$\Delta x = a\,\Delta t + b\,\Delta z \tag{18}$$

Figure 8 shows a plot of a discrete generalised Wiener process (18) with parameters $a = 0.3$ and $b = 1.2$ for 100 time steps.

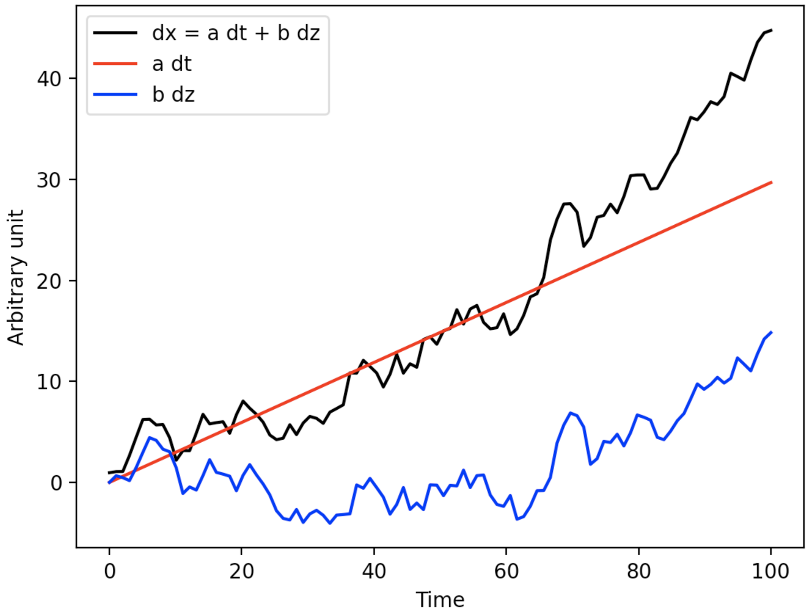

Figure 8: Discrete generalised Wiener process with parameters $a = 0.3$, $b = 1.2$ for 100 time steps.

An Itô process is a type of Wiener process where the expected drift rate, a, and the expected variance rate, b^2, changes with time. The drift rate and variance rate are also dependent on the value of the instantaneous value of the variable, but not the variable's historic values, meaning that an Itô process is a type of Markov process.

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{19}$$

The mathematics laid out in previously in this chapter can be used to model stock prices. The process required is an Itô process, because the drift and variance rate are not constant. However, the expected return rate on a stock should be kept constant.

$$\text{Expected return rate} = \frac{\text{expected drift rate}}{\text{stock price}} \tag{20}$$

The most common equation used to model the change in stock price is

$$dS = \mu\,S\,dt + \sigma\,S\,dz \tag{21}$$

where S is the stock price, μ is the expected rate of return, and σ is the volatility of the stock price. As before, $\Delta z = \varepsilon\sqrt{t}$ where ε is has a standard normal distribution. By dividing Equation 21 by S and integrating both sides between the present time and time T in the future,

$$\int_{S_0}^{S_T} \frac{dS}{S} dS = \int_0^T \mu\, dt + \int_{z0}^{z_T} \sigma\, dz' \tag{22}$$

where S_0 is the stock price currently and S_T is the stock price at time T in the future. Because dz is normally distributed, and normal distributions have an additive property, Equation 22 can be written as

$$S_T = S_0 e^{\mu T + \mathrm{N}(0,\sigma^2 T)} \tag{23}$$

which is the same as in Equation 3. This is known as geometric Brownian motion. In a risk-neutral world, μ is called the risk-free rate.

A.3 Itô's Lemma and Log-Normal property

The equation for stock price movement in Equation 21 has a variable S as the drift and variance rate. It requires a transformation to remove this S coefficient. Itô's Lemma, which won't be proved here, shows that should the variable x follow an Itô process as in 19, then a function $G = G(x,t)$ will also follow a Wiener process:

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial^2 x^2}\right) dt + \frac{\partial G}{\partial x} b\, dz \tag{24}$$

Now, if $G = G(S) = \ln(S)$, then G will follow the generalised Wiener process given by

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz \tag{25}$$

with drift rate of $\mu - \sigma^2/2$, and variance rate of σ^2. The term dG is the change in the natural logarithm of the stock price, and this follows the normal distribution given in Equation 4 .

B Black-Scholes-Merton Model Derivation and Solution

B.1 Derivation

As shown previously, stock prices can be modelled to follow a geometric Brownian motion as in Equation 21. The price of a derivative, f, is dependent on the price of the underlying stock, S, so is a function of the stock price, $f = f(S)$. By Itô's Lemma 24, we can write

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dz \tag{26}$$

Equation 26 implies that the part of the equation governing the 'randomness' is the same for both derivative price and stock price (in discrete terms, the $\Delta z = \varepsilon\sqrt{\Delta t}$ is the same for both Δf and ΔS).

Hence, a risk-free portfolio, Π can be made by buying a specific ratio of the underlying stock and selling a derivative and thus eliminating the Weiner process.

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{27}$$

The above equation shows that for an instant in time, if $\frac{\partial f}{\partial S}$ number of shares are bought, then to balance the risk, a single put-option must be bought (the negative sign indicates that f is a put option, the choice to sell shares at a specific price). This portfolio is only risk-free for an instantaneous point in time, however, and must be constantly balanced.
The change in the risk-free portfolio is

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \tag{28}$$

but ΔS is a geometric Wiener process, as shown in the discrete version of Equation 21, and Δf is the discrete version of Equation 26. So, the discrete version of Equations 21 and 26 can be substituted into Equation 28 to give

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{29}$$

which doesn't depend on Δz so must be risk-less in Δt.
The instantaneous rate of change of the risk-less portfolio is

$$\Delta\Pi = r\Pi\Delta t \tag{30}$$

where r is the expected return, which was previously called μ. This renaming has been done to show that μ must be the risk-free rate of return, r because of the assumption that there is no arbitrage - all rates of return that carry no risk are the same.
By substituting Equation 29 and Equation 27 into Equation 30, the Black-Scholes-Merton differential equation can be derived.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{dS^2} = rV \tag{31}$$

where $V = f$.

B.2 Solution

To solve these equations, the following substitutions are required,

$$S = Xe^x \tag{32}$$

$$t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \tag{33}$$

$$V = X\nu(x,t) \tag{34}$$

where X, T, andτ are constants and ν is a function of x and t. So, the partial derivatives in Equation 31 can be rewritten as

$$\frac{\partial V}{\partial t} = X\frac{\partial v}{\partial \tau}\frac{d\tau}{dt} = X\frac{\partial v}{\partial \tau}\left(-\frac{\sigma^2}{2}\right) = -\frac{X\sigma^2}{2}\frac{\partial v}{\partial \tau} \tag{35}$$

$$\frac{\partial V}{\partial S} = X\frac{\partial v}{\partial x}\frac{dx}{dS} = X\frac{\partial v}{\partial x}\frac{1}{S} = e^{-x}\frac{\partial v}{\partial x} \tag{36}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{e^{-x}}{X}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-x}}{X}\left(e^{-x}\frac{\partial^2 v}{\partial x^2} - e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-2x}}{X}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) \tag{37}$$

Then by substituting the transformed partial derivatives back into the Black-Scholes equation:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \tag{38}$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$.
The final substitution required is

$$v(x,t) = e^{\alpha x + \beta \tau}u(x,\tau) \tag{39}$$

where α and β are arbitrary constants that can be determined. Hence, the partial derivatives in Equation 38 can be written as

$$\frac{\partial v}{\partial \tau} = e^{\alpha x + \beta \tau}\left(\beta u + \frac{\partial u}{\partial \tau}\right) \tag{40}$$

$$\frac{\partial v}{\partial x} = e^{\alpha x + \beta \tau}\left(\alpha u + \frac{\partial u}{\partial x}\right) \tag{41}$$

$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x + \beta \tau}\left(\alpha^2 u + 2u\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) \tag{42}$$

Again, by substuting the transformed partial derivatives into Equation 38, the following expression is obtained.

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku \tag{43}$$

To simplify further, the α and β terms can be defined as

$$\alpha = -\frac{1}{2}(k-1) \tag{44}$$

$$\beta = -\frac{1}{4}(k+1)^2 \tag{45}$$

and by substituting these expressions into Equation 43, the $\frac{\partial u}{\partial x}$ and u terms can be removed. The final expression is the heat equation,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{46}$$

where the range of the variables are $-\infty < x < \infty$ and $\tau > 0$. Hence,

$$V(S,t) = Xv(x,t) = Xe^{\alpha x + \beta \tau}u(x,\tau) = Xe^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau) \tag{47}$$

To determine the final boundary conditions, the conditions for a call option can be transformed. For a call option, the boundary condition on the option price is

$$V(S,T) = \max(S-K,0) \tag{48}$$

as the price of a call option at its expiry date, T, is the price of the underlying stock minus the strike price, K, if the option is in the money, or is worth nothing if the option is out of the money. By allowing $K = X$, and using using the definitions S, τ, and ν in Equations 32, 33 and 39, the boundary conditions can be written as

$$u(x,0) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right] \tag{49}$$

and a similar procedure can be done for a put option which gives the boundary conditions

$$u(x,0) = \max\left[e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right] \tag{50}$$

Solving the initial boundary condition for the heat equation will solve the Black-Scholes PDE for a call or put option.
The general solution to the initial value problem of the heat equation for $\tau > 0$ and $-\infty < x < \infty$ is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}[\int_{\infty}^{-\infty} u_0(s)e^{\frac{(-x-s)^2}{4\tau}}\,ds] \tag{51}$$

where $u_0 = u(x,0)$. The derivation of the above general solution and why it works can be found in Fedotov (2008) [30]. Hence, the initial boundary conditions can be substituted into the general solution. But because $u_0(x) = 0$ for all values of x below 0, the integral can further be simplified to

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}[\int_{0}^{\infty} \left[e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s}\right] e^{\frac{(-x-s)^2}{4\tau}}\,ds] \tag{52}$$

and by making the substitution

$$x' = \frac{s-x}{\sqrt{2\tau}} \tag{53}$$

the following expression is obtained.

$$\begin{aligned} u(x,\tau) =& \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(x'-\frac{1}{2}(k+1)\sqrt{2\tau}\right)^2} dx' \\ =& -\frac{e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(x'-\frac{1}{2}(k-1)\sqrt{2\tau}\right)^2} dx' \\ =& I_1 - I_2 \end{aligned} \tag{54}$$

The cumulative normal distribution has the expression

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2}\,ds \tag{55}$$

By making the substitutions

$$x_1 = x' - \frac{1}{2}(k+1)\sqrt{2\tau} \tag{56}$$

$$x_2 = x' - \frac{1}{2}(k-1)\sqrt{2\tau} \tag{57}$$

for I_1 and I_2 respectively, then u can be rewritten as

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(d_2) \tag{58}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \tag{59}$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau} \tag{60}$$

Now, the variables V, x, τ and k must be transformed back to their usual definitions using Equations 34, 32, 33 and $k = \frac{r}{\frac{1}{2}\sigma^2}$. This gives the following Equation for the price of a European call option.

$$C(S,t) = V(S,t) = S\Phi(d_1) - Xe^{r(T-t)}\Phi(d_2) \tag{61}$$

By the same method, the price of a put option is

$$P(S,t) = Xe^{r(T-t)}\Phi(-d_2) - S\Phi(-d_1) \tag{62}$$

where, in both cases,

$$d_1 = \frac{\ln\frac{S_t}{X} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{63}$$

$$d_1 = d_1 - \sigma\sqrt{T-t} \tag{64}$$

These expressions are the same as in Equations 9 and 10 except the definition of T differs: in Equations 9 and 10, T is defined as the time until expiry, whereas in the derivation, T is the expiry date and t is the present time.

C Stable Distributions

Stable distributions have the notation $\mathbf{S}(\alpha,\beta,\gamma,\delta;k)$. The parameter k helps distinguish between the two different parameterisations used to describe stable distributions:
A random variable X is $\mathbf{S}(\alpha,\beta,\gamma,\delta;0)$ if

$$X = \begin{cases} \gamma(Z-\beta\tan\frac{\pi a}{2})+\delta & \alpha\neq 1 \\ \gamma Z+\delta & \alpha = 1 \end{cases} \tag{65}$$

where $Z = Z(\alpha,\beta)$ is a characteristic function given by

$$E\exp(iuZ) = \begin{cases} \exp(-|u|^{\alpha}[1-i\beta\tan\frac{\pi\alpha}{2}(\text{sign}u)] & \alpha\neq 1 \\ \exp(-|u|[1+i\beta\frac{2}{\pi}(\text{sign}u)\log|u|]) & \alpha = 1 \end{cases} \tag{66}$$

and X has the characteristic function given by

$$E\,exp(iuX) = \begin{cases} \exp(-\gamma^{\alpha}|u|^{\alpha}[1+i\beta(\tan\frac{\pi a}{2})(\text{sign}u(|\gamma u|^{1-\alpha}-1]+i\delta u & \alpha\neq 1 \\ \exp(-\gamma|u|[1+i\beta\frac{2}{\pi}(\text{sign}u)\log(\gamma|u|)]+i\delta u & \alpha = 1 \end{cases} \tag{67}$$

Alternatively, the parameterisation $\mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ can be used instead:
A random variable X is $\mathbf{S}(\alpha,\beta,\gamma,\delta;1)$ if

$$X = \begin{cases} \gamma Z + \delta & \alpha \neq 1 \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \gamma \log \gamma & \alpha = 1 \end{cases} \tag{68}$$

where X is the characteristic function given by

$$E\exp(iuX) = \begin{cases} \exp(-\gamma^{\alpha}|u|^{\alpha}[1 - i\beta(\tan\frac{\pi\alpha}{2}(\mathrm{sign} u)] + i\delta u) & \alpha \neq 1 \\ \exp(-\gamma|u|[1 + i\beta\frac{2}{\pi}(\mathrm{sign} u)\log|u|] + i\delta u & \alpha = 1 \end{cases} \tag{69}$$

The parameterisation 0 is the choice used throughout the essay when fitting the shape of the stable distributions.

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