Asymptotic Decomposition of a Scalar Field on de Sitter Space

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1 Introduction

Our modern understanding of the universe is derived from Einstein's theory of general relativity, which describes gravity as a geometric property of spacetime. The theory is underpinned Einstein's field equations,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab},$$

which relate the curvature of spacetime on the left-hand side, to the stress-energy-momentum of spacetime on the right-hand side. In contrast to other physical theories, general relativity does not have a fixed background on which a well-defined dynamical variable evolves, so we are required to construct the evolution of the metric and of the spacetime on which the metric is defined at the same time. This makes Einstein's equations extremely difficult to solve.

This severe difficulty means that in order to make progress in our understanding of the theory, it is often necessary to choose a fixed background spacetime and study the evolution of other dynamical variables on this background. Some particularly simple examples of spacetimes that are solutions of Einstein's equations are the *maximally symmetric spacetimes*, of which there are three: *Minkowski space* has a vanishing cosmological constant ($\Lambda = 0$), *de Sitter space* has a positive cosmological constant ($\Lambda > 0$), and *anti-de Sitter space* has a negative cosmological constant ($\Lambda < 0$). Of course, we are particularly interested in spacetimes that share similarities with the real universe we inhabit. As recent astronomical observations have indicated that the cosmological constant of our universe is in fact positive, we therefore have a particular interest in studying de Sitter space.

One area of great interest in general relativity is the study of *metric scattering*, which is the effect of curved spacetime on the asymptotic behaviour of fields. Much of the work in this field has been enabled by the use of the *conformal method*. This method is based on Roger Penrose's discovery in the 1960s of a way of compactfying certain spacetimes by performing a conformal transformation of the metric and attaching a boundary $\mathscr{I}$ called *null infinity* [1, 2]. This enables us to study the behaviour of the field on $\mathscr{I}$ using local differential geometry in the conformally compactified spacetime, and then translate this into conditions on the asymptotic behaviour of the field on the physical spacetime.

In this report, we study the asymptotic behaviour of a scalar field on four-dimensional de Sitter space from several different angles. We begin by giving an overview of de Sitter space and the conformal wave equation in Section 2. Once this necessary background has been introduced, we present our main results in Section 3. We start by performing calculations using the conformal method, which is used in [3] to prove that the scalar field decays exponentially in proper time along timelike geodesics approaching $\mathscr{I}$. In this work we use the method to investigate the existence of an asymptotic decomposition of the scalar field into exponentially-decaying components. The first coefficient in this decomposition was derived in [3], and here we derive the second and third coefficients in the expansion. From the pattern observed, we make a conjecture for the general form of an arbitrary coefficient in the expansion.

We also study the same problem using a different approach, by developing the method from [4] involving *quasinormal modes* – objects that describe the behaviour of fields that decay in time a via series expansion of resonant terms. This method has some similarities to the conformal method, as it again uses a compactification to enable the use of tools such as Taylor expansions at the cosmological horizon. However, it is a more direct approach, and one that gives a full asymptotic expansion in a way that allows us to easily read of the quasinormal modes and the corresponding mode solutions. We ultimately find in Section 4 that the expression obtained via this method agrees with our conjecture for the coefficients of the asymptotic expansion of the scalar field obtained via the conformal method.

2 Background

Before discussing the main results of this work, we introduce the necessary background, closely following [3, 5]. This includes a discussion of the conformal wave equation which governs the evolution of our scalar field, and an introduction to de Sitter space and its conformally compactified counterpart. Finally, we discuss the energy estimates which form the basis of our asymptotic expansion of the scalar field in the following chapter.

2.1 Conventions and Notation

We use the spacetime signature $(+,-,-,-)$. The energy estimates are calculated on the Einstein cylinder $\mathfrak{E} = \mathbb{R} \times \mathbb{S}^3$ with metric $\mathfrak{e} = g_{\mathbb{R}} \oplus \mathfrak{s}_3$, where we write $\mathfrak{s}_n$ for $g_{\mathbb{S}^n}$, the metric on the n-sphere. We use ∇ to denote the Levi-Civita connecition of the full spacetime metric $\mathfrak{e}$ or a general metric g, $\not\nabla$ to denote the connection of $\mathbb{S}^3$ and $\Box = \nabla^a \nabla_a$ to denote the corresponding d'Alembert wave operator. We use dv to denote the volume form of the metric $\mathfrak{e}$ or g. We sometimes use the notation $\lesssim$ to denote inequality up to a constant, $\simeq$ to denote equality up to a constant and $\approx$ to denote equality at $t = \infty$. We will use Penrose's sign convention for the curvature tensors, so the Riemann tensor $R^c{}_{dab}$ satisfies

$$[\nabla_a, \nabla_b] X^c = -R^c{}_{dab} X^d.$$

Then the Ricci tensor and scalar curvature are defined as usual as

$$R_{ab} := R^c{}_{acb} \qquad R := R^a{}_a.$$

2.2 The Conformal Wave Equation

We begin our discussion of the conformal wave equation by considering a generic spacetime $(\mathscr{M}, g)$, and studying the Lagrangian

$$\mathcal{L} = \frac{1}{2} \nabla_a \phi \nabla^a \phi - \frac{1}{12} R \phi^2,$$

where ϕ is a real scalar field on $\mathscr{M}$. The Euler-Lagrange equation obtained from this Lagrangian is

$$\Box \phi + \frac{1}{6} R \phi = 0. \tag{2.1}$$

Now consider the conformal transformation $\hat{g}_{ab} = \Omega^2 g_{ab}$, and choose the scalar field to have weight -1 under this transformation,

$$\hat{\phi} := \Omega^{-1} \phi.$$

Then the Lagrangian transforms as

$$\hat{\mathcal{L}} = \Omega^4 \hat{\mathcal{L}} + \frac{1}{2}\Omega^4 \hat{\nabla}^a (\hat{\phi}^2 \partial_a \log \Omega),$$

where $\hat{\mathcal{L}} = \frac{1}{2}\hat{\nabla}_a \hat{\phi} \hat{\nabla}^a \hat{\phi} - \frac{1}{12}\hat{R}\hat{\phi}^2$, so that the action transforms as

$$S = \hat{S} - \frac{1}{2}\int_{\mathscr{M}} \hat{\nabla}^a (\hat{\phi}^2 \partial_a \log \Omega)\, \widehat{\mathrm{dv}}\,.$$

For compactly-supported scalar fields $\phi \in \mathcal{C}_c^\infty(\mathscr{M})$, the second term vanishes upon integrating by parts, and hence the action is invariant under the conformal transformation. It then follows that the equation (2.1) is also invariant under the transformation; this equation is known as the *conformal wave equation.*

2.3 De Sitter Space

In this work we will be considering four-dimensional de Sitter space dS_4. This is the maximally symmetric solution to Einstein's equations in vacuum with positive scalar curvature, and may be defined as the hyperboloid described by

$$|x|^2 - x_0^2 = \frac{1}{H^2}$$

in $(4+1)$-dimensional Minkowski space

$$\eta_5 = \mathrm{d}x_0^2 - \mathrm{d}|x|^2 - |x|^2 \mathfrak{s}_3$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$. We now introduce the coordinate α on dS_4, defined such that

$$x_0 = \frac{1}{H}\sinh(H\alpha), \qquad |x| = \frac{1}{H}\cosh(H\alpha)$$

This causes the metric η_5 to descend to the metric $\tilde{g}$ on dS_4,

$$\tilde{g} = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha)\mathfrak{s}_3. \tag{2.2}$$

This metric is known as the *closed slicing of de Sitter space*, and provides a global coordinate system on dS_4; we will also refer to these coordinates simply as the *global coordinates.*

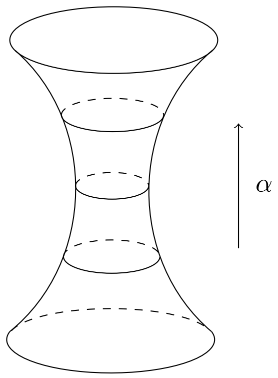

Figure 2.1: The closed slicing of de Sitter space.

Conformal Compactification of de Sitter Space

Four-dimensional de Sitter space may be conformally compactified by making the further change of coordinates

$$\tan\left(\frac{\tau}{2}\right) = \tanh\left(\frac{H\alpha}{2}\right),$$

whereupon the metric becomes

$$\tilde{g} = \frac{1}{H^2\cos^2\tau}(\mathrm{d}\tau^2 - \mathfrak{s}_3)$$

where $\tau \in (-\pi/2, \pi/2)$. We thus see that dS_4 may be conformally embedded in the Einstein cylinder $(\mathfrak{E}, \mathfrak{e})$,

$$\Omega^2 \tilde{g} = \mathrm{d}\tau^2 - \mathfrak{s}_3 = \mathfrak{e}, \tag{2.3}$$

with the conformal factor

$$\Omega = H \cos\tau. \tag{2.4}$$

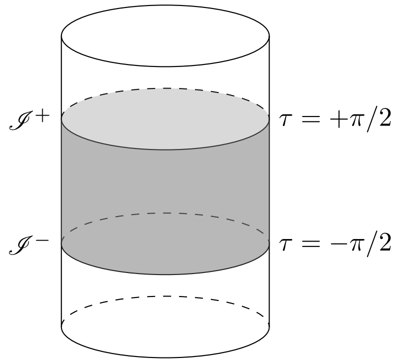

Figure 2.2: Compactified de Sitter space embedded in the Einstein cylinder.

In contrast to the physical metric (2.2), the hypersurfaces $\{\tau = \pm\pi/2\}$ are regular in this conformal scale. We thus identify compactified de Sitter space $\widehat{\mathrm{dS}}_4$ with the subset $[-\pi/2, \pi/2] \times \mathbb{S}^3$ of the Einstein cylinder $\mathfrak{E}$ by attaching to $\big((-\pi/2, \pi/2) \times \mathbb{S}^3, \mathfrak{e}\big)$ the boundary $\mathscr{I} := \{\Omega = 0\} = \{|\tau| = \pi/2\}$. The boundary is therefore the union of the spacelike hypersurfaces

$$\mathscr{I}^+ = \Big\{\tau = \frac{\pi}{2}\Big\}, \qquad \mathscr{I}^- = \Big\{\tau = -\frac{\pi}{2}\Big\},$$

called *future null infinity* and *past null infinity*, respectively.

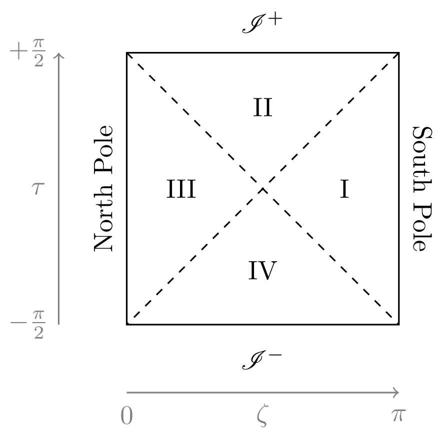

Figure 2.3: Penrose diagram for de Sitter space.

If we write the three-sphere metric as $\mathfrak{s}_3 = \mathrm{d}\zeta^2 + (\sin^2\zeta)\mathfrak{s}_2$ for $\zeta \in [0, \pi]$ and take the quotient by the SO(3) symmetry group of $\mathfrak{s}_2$, we obtain the Penrose diagram for dS_4, as shown in Figure 2.3.

Static Coordinates

Another useful coordinate system on dS_4 may be constructed by defining

$$r = \frac{\sin\zeta}{H\cos\tau}, \qquad \tanh(Ht) = \frac{\sin\tau}{\cos\zeta} \tag{2.5}$$

for $\tau \in (-\pi/2, \pi/2)$ and $\zeta \in (0, \pi)$. We note that these coordinates are only appropriate in region I of Figure 2.3, where they define a timelike, future-pointing Killing vector field ∂_t. Otherwise, the coordinates become singular on the horizons $r = 1/H$, while ∂_t is spacelike in regions II and III, and past-pointing in region III. The flow of the vector field ∂_t is represented by the arrows shown on the Penrose diagram in Figure 2.4.

In these coordinates, the dS$_4$ metric (2.2) takes the form

$$\tilde{g} = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2\mathfrak{s}_2, \tag{2.6}$$

where $F(r) = 1 - H^2r^2$, while the conformal factor is

$$\Omega = \frac{H}{\cosh(Ht)}\frac{1}{\sqrt{F_t(r)}}. \tag{2.7}$$

In region I, this metric is *static*: it has a timelike Killing vector ∂_t which is orthogonal to the family of hypersurfaces described by $t = \text{const}$. For this reason, these coordinates are known as the *static coordinates* on dS$_4$.

For future reference, we note that the inverse transformation to (2.5) may be written as

$$\tan\tau = F(r)^{1/2}\sinh(Ht), \qquad \tan\zeta = HrF(r)^{-1/2}\,\mathrm{sech}(Ht). \tag{2.8}$$

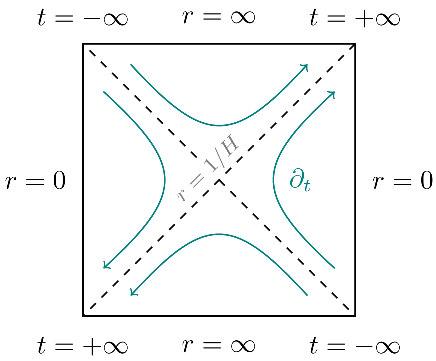

Figure 2.4: Static coordinates on de Sitter space.

The Wave Equation on de Sitter Space

The scalar curvature of dS$_4$ is $\tilde{R} = 12H^2$, so that referring to (2.1), we see that the conformal wave equation on dS$_4$ is

$$\tilde{\Box}\tilde{\phi} + 2H^2\tilde{\phi} = 0. \tag{2.9}$$

Under the rescaling

$$\hat{\phi} = \Omega^{-1}\tilde{\phi}$$

the wave equation (2.9) becomes the conformal wave equation on the Einstein cylinder $\mathfrak{E}$,

$$\hat{\Box}\hat{\phi} + \hat{\phi} = 0. \tag{2.10}$$

Initial data for (2.9) given by $(\tilde{\phi}, \partial_\alpha\tilde{\phi})|_{\tilde{\Sigma}} = (\tilde{\phi}_0, \tilde{\phi}_1)$ on the Cauchy surface $\tilde{\Sigma} \simeq \{\alpha = 0\} \times \mathbb{S}^3$ in fact also define initial data $(\hat{\phi}, \partial_\tau\hat{\phi})|_{\hat{\Sigma}} = (\hat{\phi}_0, \hat{\phi}_1)$ for (2.10) on $\hat{\Sigma} \simeq \{\tau = 0\} \times \mathbb{S}^3$, since the Cauchy surface is a three-sphere in both cases, and the conformal factor is simply a constant on this surface, $\Omega|_{\alpha=0} = H$.

2.4 Sobolev Spaces

Before we move on to make energy estimates on de Sitter space, we must introduce the notion of a Sobolev space, and state a particular case of the Sobolev embedding theorem which will be used in the following section. The definitions in this section may be found in Chapter 2 of [6].

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold of dimension n. For a real function ϕ belonging to $\mathcal{C}^k(\mathcal{M})$, with $k \geq 0$ an integer, we define

$$|\nabla^k\phi|^2 := (\nabla^{a_1}\nabla^{a_2}\dots\nabla^{a_k}\phi)(\nabla_{a_1}\nabla_{a_2}\dots\nabla_{a_k}\phi).$$

For any real number $p \geq 1$, we denote by $\mathfrak{C}^{k,p}$ the vector space of $\mathcal{C}^\infty$ functions ϕ such that $|\nabla^l \phi| \in L^p(\mathscr{M})$ for all $0 \leq l \leq k$. The Sobolev space $W^{k,p}(\mathscr{M})$ is defined to be the completion of $\mathfrak{C}^{k,p}$ with respect to the norm

$$\|\phi\|_{W^{k,p}} := \sum_{l=0}^{k} \|\nabla^l \phi\|_p.$$

We will only be working with Sobolev spaces with $p = 2$, which we denote by $H^k(\mathscr{M}) := W^{k,2}(\mathscr{M})$.

The Sobolev Embedding Theorem

As a consequence of the Sobolev embedding theorem, found in Chapter 2 of [6], we have the following result: if $k \geq 0$, $r \geq 0$ are integers satisfying

$$\frac{(k-r)}{n} \geq \frac{1}{2},$$

then

$$H^k(\mathscr{M}) \subset \mathcal{C}^r(\mathscr{M}),$$

and the identity operator is continuous (the embedding is compact), where $\mathcal{C}^r$ is the space of functions ϕ of finite $\|\phi\|_{\mathcal{C}^r} := \max_{0 \leq l \leq r} \sup |\nabla^l u|$ norm.

2.5 Energy Estimates

Energy Estimate on de Sitter Space

The stress-energy tensor for the conformal wave equation on $\mathfrak{E}$ is

$$\hat{\mathbf{T}}_{ab} = \hat{\nabla}_a \hat{\phi} \hat{\nabla}_b \hat{\phi} - \frac{1}{2} \mathfrak{e}_{ab} \hat{\nabla}_c \hat{\phi} \hat{\nabla}^c \hat{\phi} + \frac{1}{2} \mathfrak{e}_{ab} \hat{\phi}^2,$$

which is symmetric and satisfies

$$\hat{\nabla}^a \hat{\mathbf{T}}_{ab} = \big(\hat{\square} \hat{\phi} + \hat{\phi}\big) \hat{\nabla}_b \hat{\phi}.$$

Thus the equation (2.10) implies that $\hat{\mathbf{T}}_{ab}$ is conserved, and hence implies the existence of a corresponding conserved current

$$\hat{J}_b = \hat{T}^a \hat{\mathbf{T}}_{ab},$$

where $\hat{T}^a = \partial_\tau$ is a timelike Killing vector field with respect to $\mathfrak{e}$. Since this current is conserved, we have $\hat{\nabla}^b \hat{J}_b = 0$.

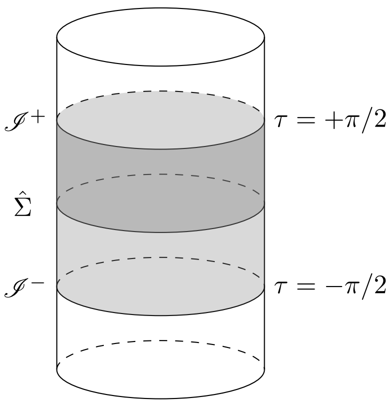

Figure 2.5: Energy estimate on de Sitter space.

We now wish to integrate this conservation equation over the region of compactified de Sitter space bounded by $\hat{\Sigma}$ and $\mathscr{I}^+$. Using the divergence theorem, we have

$$0 = \int_{D^+(\hat{\Sigma})} \hat{\nabla}^b \hat{J}_b \, \widehat{\mathrm{dv}} = \int_{\mathscr{I}^+} \hat{J}^b \partial_b \lrcorner \, \widehat{\mathrm{dv}} - \int_{\hat{\Sigma}} \hat{J}^b \partial_b \lrcorner \, \widehat{\mathrm{dv}} \,. \tag{2.11}$$

On the initial surface $\hat{\Sigma}$, we have

$$\int_{\hat{\Sigma}} \hat{J}^b \partial_b \lrcorner \widehat{\mathrm{dv}} = \int_{\hat{\Sigma}} \hat{J}^\tau \, \widehat{\mathrm{dv}}_{\hat{\Sigma}} = \int_{\hat{\Sigma}} \hat{\mathbf{T}}_{00} \, \widehat{\mathrm{dv}}_{\hat{\Sigma}} = \frac{1}{2} \int_{\hat{\Sigma}} \Big((\partial_\tau \hat{\phi})^2 + (\hat{\nabla} \hat{\phi})^2 + \hat{\phi}^2 \Big) \, \widehat{\mathrm{dv}}_{\hat{\Sigma}} = \frac{1}{2} \Big(\|\hat{\phi}_1\|^2_{L^2(\hat{\Sigma})} + \|\hat{\phi}_0\|^2_{H^1(\hat{\Sigma})} \Big),$$

and similarly on $\mathscr{I}^+$,

$$\int_{\mathscr{I}^+} \hat{J}^b \partial_b \lrcorner \widehat{\mathrm{dv}} = \frac{1}{2} \Big(\|(\hat{T}\hat{\phi})^+\|^2_{L^2(\mathscr{I}^+)} + \|\hat{\phi}^+\|^2_{H^1(\mathscr{I}^+)} \Big),$$

where $\hat{\phi}^+$ and $(\hat{T}\hat{\phi})^+$ are the restrictions of $\hat{\phi}$ and $(\hat{T}\hat{\phi}) = \partial_\tau \hat{\phi}$ to $\mathscr{I}^+$, respectively. Then by using (2.11), we can make the energy estimate

$$\|\hat{\phi}_1\|^2_{L^2(\hat{\Sigma})} + \|\hat{\phi}_0\|^2_{H^1(\hat{\Sigma})} \simeq \|(\hat{T}\hat{\phi})^+\|^2_{L^2(\mathscr{I}^+)} + \|\hat{\phi}^+\|^2_{H^1(\mathscr{I}^+)}. \tag{2.12}$$

Higher Order Energy Estimate

We now claim that for all $m \geq 0$, we can make the higher-order energy estimate

$$\|\hat{T}\hat{\phi}\|^2_{H^m(\hat{\Sigma})} + \|\hat{\phi}\|^2_{H^{m+1}(\hat{\Sigma})} \simeq \|\hat{T}\hat{\phi}\|^2_{H^m(\mathscr{I}^+)} + \|\hat{\phi}\|^2_{H^{m+1}(\mathscr{I}^+)}. \tag{2.13}$$

The proof proceeds by an induction argument. Recalling that $H^0 = L^2$, we see that we have already proven the statement for $m = 0$ in our basic energy estimate (2.12). For the case $m = 1$, we begin by considering the conformal wave equation (2.10),

$$\partial_\tau^2 \hat{\phi} - \hat{\Delta} \hat{\phi} + \hat{\phi} = 0. \tag{2.14}$$

For sufficiently regular initial data, we can commute $\hat{T} = \partial_\tau$ into this equation to obtain

$$\partial_\tau^2 (\hat{T}\hat{\phi}) - \hat{\Delta}(\hat{T}\hat{\phi}) + \hat{T}\hat{\phi} = 0.$$

This shows that $\hat{T}\hat{\phi}$ is itself a solution of the conformal wave equation (2.10), and hence we can apply our energy estimate (2.12) to $\hat{T}\hat{\phi}$ to obtain

$$\|\hat{T}^2\hat{\phi}\|^2_{L^2(\hat{\Sigma})} + \|\hat{T}\hat{\phi}\|^2_{H^1(\hat{\Sigma})} \simeq \|\hat{T}^2\hat{\phi}\|^2_{L^2(\mathscr{I}^+)} + \|\hat{T}\hat{\phi}\|^2_{H^1(\mathscr{I}^+)}. \tag{2.15}$$

Referring again to (2.14), we see that $\hat{T}^2\hat{\phi} = (\hat{\Delta} - 1)\hat{\phi}$, so that

$$\begin{aligned} \|\hat{T}^2\hat{\phi}\|^2_{L^2(\hat{\Sigma})} = \|(\hat{\Delta} - 1)\hat{\phi}\|^2_{L^2(\hat{\Sigma})} &= \int_{\hat{\Sigma}} \Big(|\hat{\Delta}\hat{\phi}|^2 - 2\hat{\phi}\hat{\Delta}\hat{\phi} + \hat{\phi}^2 \Big) \, \widehat{\mathrm{dv}}_{\hat{\Sigma}} \\ &= \int_{\hat{\Sigma}} \Big(|\hat{\nabla}^2 \hat{\phi}|^2 + |\hat{\nabla}\hat{\phi}|^2 + \hat{\phi}^2 \Big) \, \widehat{\mathrm{dv}}_{\hat{\Sigma}} = \|\hat{\phi}\|^2_{H^2(\hat{\Sigma})}, \end{aligned}$$

where in the second line the second term has been integrated by parts. A similar result holds for the integral on $\mathscr{I}^+$, and upon substituting these results into (2.15), we obtain the claim for $m = 1$. The result for general $m \geq 0$ can be proven in an analogous manner by commuting m powers of $\hat{T}$ into the conformal wave equation.

3 Results

3.1 Asymptotic Decomposition via the Conformal Method

Decay Estimate

Invoking the Sobolev embedding theorem for the case where the dimension of the manifold is $n = 3$, we see that since $(2 - 0)/3 \geq 1/2$, we have

$$H^2(\mathbb{S}^3) \subset \mathcal{C}^0(\mathbb{S}^3).$$

Thus for initial data $(\hat{\phi}_0, \hat{\phi}_1) \in H^2(\mathbb{S}^3) \times H^1(\mathbb{S}^3)$, we have $\hat{\phi}^+ \in \mathcal{C}^0(\mathbb{S}^3)$, and hence $\hat{\phi} \in \mathcal{C}^0(\mathbb{R} \times \mathbb{S}^3)$. This shows that $|\hat{\phi}| \leq C$ on $[-\pi/2, \pi/2] \times \mathbb{S}^3$, and in particular as $\tau \to \pi/2$, for some constant C. Then using the fact that $\hat{\phi} = \Omega^{-1}\tilde{\phi}$, we see that in the global coordinates,

$$|\tilde{\phi}| \lesssim \Omega \lesssim \frac{1}{\cosh(H\alpha)} \lesssim \mathrm{e}^{-H\alpha}$$

as $\alpha \to +\infty$. In the static coordinates, the conformal factor is

$$\Omega = \frac{H}{\cosh(Ht)} \frac{1}{\sqrt{F_t(r)}},$$

where $F_t(r) = 1 - H^2 r^2 \tanh^2(Ht)$, so that keeping r fixed, we have

$$|\tilde{\phi}| \lesssim \Omega \lesssim_r \mathrm{e}^{-Ht}$$

as $t \to +\infty$ [3].

First Coefficient

We now compute the first coefficient of the asymptotic expansion, using the conformal method as set out in [3]. Suppose that $S_3[\tilde{\phi}] := \|\partial_\alpha \tilde{\phi}\|_{H^2}^2 + \|\tilde{\phi}\|_{H^3}^2$ is small initially. Then also $S_3[\hat{\phi}] := \|\partial_\tau \hat{\phi}\|_{H^2}^2 + \|\hat{\phi}\|_{H^3}^2$ is small initially, and hence

$$|\nabla\!\!\!\!/ \hat{\phi}|^2 = (\partial_\zeta \hat{\phi})^2 + \frac{1}{\sin^2 \zeta} |\nabla^{\mathbb{S}_2} \hat{\phi}|^2$$

has a continuous limit on $\mathscr{I}^+$. This implies that $\partial_\zeta \hat{\phi}$ and $(\sin\zeta)^{-1} |\nabla^{\mathbb{S}_2} \hat{\phi}|$ also have continuous limits on $\mathscr{I}^+$.

It is now useful to compute the derivatives of the static coordinates defined by (2.5) with respect to the coordinates on the Einstein cylinder:

$$\begin{aligned} \frac{\partial t}{\partial \tau} &= H^{-1} F(r)^{-1/2} \cosh(Ht) & \frac{\partial t}{\partial \zeta} &= r F(r)^{-1/2} \sinh(Ht) \\ \frac{\partial r}{\partial \tau} &= r F(r)^{1/2} \sinh(Ht) & \frac{\partial r}{\partial \zeta} &= H^{-1} F(r)^{1/2} \cosh(Ht). \end{aligned} \tag{3.1}$$

Recalling that $\hat{\phi} = \Omega^{-1}\tilde{\phi}$, and that Ω is a function of τ but not of ζ, we see that $\partial_\zeta \tilde{\phi} = \partial_\zeta \hat{\phi}$ and $\partial_\tau \tilde{\phi} = \partial_\tau(\Omega\hat{\phi}) = \Omega \partial_\tau \hat{\phi} + (\partial_\tau \Omega)\hat{\phi}$. Then using (3.1), we can write

$$\partial_\zeta \hat{\phi} = \Omega^{-1} \left(\frac{\partial t}{\partial \zeta} \partial_t \tilde{\phi} + \frac{\partial r}{\partial \zeta} \partial_r \tilde{\phi} \right) = \Omega^{-1} \Big(r F(r)^{-1/2} \sinh(Ht) \partial_t \tilde{\phi} + H^{-1} F(r)^{1/2} \cosh(Ht) \partial_r \tilde{\phi} \Big) \tag{3.2}$$

and

$$\begin{aligned} \partial_\tau \hat{\phi} + (\partial_\tau \Omega)\Omega^{-1}\hat{\phi} = \Omega^{-1}\partial_\tau \tilde{\phi} &= \Omega^{-1} \left(\frac{\partial t}{\partial \tau} \partial_t \tilde{\phi} + \frac{\partial r}{\partial \tau} \partial_r \tilde{\phi} \right) \\ &= \Omega^{-1} \Big(H^{-1} F(r)^{-1/2} \cosh(Ht) \partial_t \tilde{\phi} + r F(r)^{1/2} \sinh(Ht) \partial_r \tilde{\phi} \Big). \end{aligned} \tag{3.3}$$

We now wish to study the e^{-Ht} component of $\tilde{\phi}$,

$$\tilde{\varphi}_1 := \mathrm{e}^{Ht} \tilde{\phi}.$$

Rewriting our two equations (3.2) and (3.3) in terms of $\tilde{\varphi}_1$, we obtain

$$\mathcal{O}\big(\mathrm{e}^{-Ht}\big) = r F(r)^{-1/2} \sinh(Ht) \mathrm{e}^{-Ht} (\partial_t \tilde{\varphi}_1 - H\tilde{\varphi}_1) + H^{-1} F(r)^{1/2} \cosh(Ht) \mathrm{e}^{-Ht} \partial_r \tilde{\varphi}_1$$

and

$$\begin{aligned} \mathcal{O}\big(\mathrm{e}^{-Ht}\big) - F(r)^{1/2} \sinh(Ht) \mathrm{e}^{-Ht} \tilde{\varphi}_1 &= H^{-1} F(r)^{-1/2} \cosh(Ht) \mathrm{e}^{-Ht} (\partial_t \tilde{\varphi}_1 - H\tilde{\varphi}_1) \\ &\quad + r F(r)^{1/2} \sinh(Ht) \mathrm{e}^{-Ht} \partial_r \tilde{\varphi}_1. \end{aligned}$$

Taking the limit $t \to +\infty$, these equations become

$$\begin{aligned} 0 &\approx Hr \partial_t \tilde{\varphi}_1 - H^2 r \tilde{\varphi}_1 + F \partial_r \tilde{\varphi}_1, \\ -HF\tilde{\varphi}_1 &\approx \partial_t \tilde{\varphi}_1 - H\tilde{\varphi}_1 + HrF\partial_r \tilde{\varphi}_1, \end{aligned}$$

where $\approx$ denotes equality at $t = +\infty$. Solving these equations algebraically, we find that $\partial_t \tilde{\varphi}_1 \approx 0$, and

$$H^2 r \tilde{\varphi}_1 \approx F(r) \partial_r \tilde{\varphi}_1. \tag{3.4}$$

As noted above, $(\sin\zeta)^{-1}|\nabla^{\mathfrak{s}_2}\hat{\phi}| = \Omega^{-1}(\sin\zeta)^{-1}|\nabla^{\mathfrak{s}_2}\tilde{\phi}|$ has a continuous limit on $\mathscr{I}^+$. Using the fact that $\sin\zeta = Hr\,\mathrm{sech}(Ht)F_t(r)^{-1/2}$, we see that

$$\left|\frac{1}{r}\nabla^{\mathfrak{s}_2}\tilde{\phi}\right| \lesssim_r \mathrm{e}^{-2Ht}$$

as $t\to+\infty$. This shows that, at $t=+\infty$, the function $\tilde{\varphi}_1$ is independent of the angular coordinates. This means that (3.4) is an ordinary differential equation in r, and hence has solution

$$\tilde{\varphi}_1(r) \approx \frac{1}{\sqrt{F(r)}}\tilde{\varphi}_1(0).$$

We thus conclude that there exists a constant c such that

$$\tilde{\phi} \sim cF(r)^{-1/2}\mathrm{e}^{-Ht} + \mathcal{O}(\mathrm{e}^{-2Ht})$$

as $t\to+\infty$.

Second Coefficient

We now go beyond the results of [3] by computing the second coefficient of the asymptotic expansion. To find this next coefficient, we evaluate the second derivatives,

$$\begin{aligned}
\frac{\partial^2 t}{\partial\tau^2} &= \frac{1}{2}H^{-1}(1+H^2r^2)F(r)^{-1}\sinh(2Ht) & \frac{\partial^2 r}{\partial\tau^2} &= r(H^2r^2+F(r)\cosh(2Ht))\\
\frac{\partial^2 t}{\partial\zeta^2} &= \frac{1}{2}H^{-1}(1+H^2r^2)F(r)^{-1}\sinh(2Ht) & \frac{\partial^2 r}{\partial\zeta^2} &= -r\\
\frac{\partial^2 t}{\partial\tau\partial\zeta} &= \frac{1}{2}H^{-1}F(r)^{-1}\sinh(2Ht) & \frac{\partial^2 r}{\partial\tau\partial\zeta} &= rF(r)^{-1}\cosh(2Ht).
\end{aligned} \tag{3.5}$$

We can use these results to compute

$$\begin{aligned}
\Omega\partial_\zeta^2\hat{\phi} &= \frac{\partial^2 t}{\partial\zeta^2}\partial_t\tilde{\phi} + \left(\frac{\partial t}{\partial\zeta}\right)^2\partial_t^2\tilde{\phi} + 2\frac{\partial t}{\partial\zeta}\frac{\partial r}{\partial\zeta}\partial_t\partial_r\tilde{\phi} + \frac{\partial^2 r}{\partial\zeta^2}\partial_r\tilde{\phi} + \left(\frac{\partial r}{\partial\zeta}\right)^2\partial_r^2\tilde{\phi}\\
&= \frac{1}{2}H^{-1}(1+H^2r^2)F(r)^{-1}\sinh(2Ht)\partial_t\tilde{\phi} + r^2F(r)^{-1}\sinh^2(Ht)\partial_t^2\tilde{\phi} + H^{-1}r\sinh(2Ht)\partial_t\partial_r\tilde{\phi}\\
&\quad - r\partial_r\tilde{\phi} + H^{-2}F(r)\cosh^2(Ht)\partial_r^2\tilde{\phi},
\end{aligned} \tag{3.6}$$

and, using the result $\Omega^{-1}(\partial_\tau\Omega) = -F(r)^{1/2}\sinh(Ht)$,

$$\begin{aligned}
\Omega\partial_\tau^2\hat{\phi} &= \Omega\partial_\tau^2\tilde{\phi} - 2\Omega^{-1}(\partial_\tau\Omega)\partial_\tau\tilde{\phi} + 2\Omega^{-2}(\partial_\tau\Omega)^2\tilde{\phi} - \tilde{\phi}\\
&= \frac{\partial^2 t}{\partial\tau^2}\partial_t\tilde{\phi} + \left(\frac{\partial t}{\partial\tau}\right)^2\partial_t^2\tilde{\phi} + 2\frac{\partial t}{\partial\tau}\frac{\partial r}{\partial\tau}\partial_t\partial_r\tilde{\phi} + \frac{\partial^2 r}{\partial\tau^2}\partial_r\tilde{\phi} + \left(\frac{\partial r}{\partial\tau}\right)^2\partial_r^2\tilde{\phi}\\
&\quad - 2\Omega^{-1}(\partial_\tau\Omega)\left(\frac{\partial t}{\partial\tau}\partial_t\tilde{\phi} + \frac{\partial r}{\partial\tau}\partial_r\tilde{\phi}\right) + 2\Omega^{-2}(\partial_\tau\Omega)^2\tilde{\phi} - \tilde{\phi}\\
&= \frac{1}{2}H^{-1}(3-H^2r^2)F(r)^{-1}\sinh(2Ht)\partial_t\tilde{\phi} + H^{-2}F(r)^{-1}\cosh^2(Ht)\partial_t^2\tilde{\phi} + H^{-1}r\sinh(2Ht)\partial_t\partial_r\tilde{\phi}\\
&\quad - r(1-2H^2r^2-2F(r)\cosh(2Ht))\partial_r\tilde{\phi} + r^2F(r)\sinh(Ht)\partial_r^2\tilde{\phi} + (2F(r)\sinh^2(Ht)-1)\tilde{\phi},
\end{aligned} \tag{3.7}$$

and finally,

$$\begin{aligned}
\Omega\partial_\tau\partial_\zeta\hat{\phi} &= \Omega\partial_\tau\partial_\zeta\tilde{\phi} - \Omega^{-1}(\partial_\tau\Omega)\partial_\zeta\tilde{\phi}\\
&= \frac{\partial^2 t}{\partial\tau\partial\zeta}\partial_t\tilde{\phi} + \frac{\partial t}{\partial\zeta}\frac{\partial t}{\partial\tau}\partial_t^2\tilde{\phi} + \left(\frac{\partial t}{\partial\tau}\frac{\partial r}{\partial\tau} + \frac{\partial t}{\partial\zeta}\frac{\partial r}{\partial\zeta}\right)\partial_t\partial_r\tilde{\phi} + \frac{\partial^2 r}{\partial\tau\partial\zeta}\partial_r\tilde{\phi} + \frac{\partial r}{\partial\zeta}\frac{\partial r}{\partial\tau}\partial_r^2\tilde{\phi}\\
&\quad - \Omega^{-1}(\partial_\tau\Omega)\left(\frac{\partial t}{\partial\zeta}\partial_t\tilde{\phi} + \frac{\partial r}{\partial\zeta}\partial_r\tilde{\phi}\right)\\
&= \left(r^2\sinh(2Ht) + \frac{1}{2}H^{-1}F(r)^{-1}\sinh(2Ht)\right)\partial_t\tilde{\phi} + \left(rF(r)^{-1}\cosh(2Ht) + \frac{1}{2}H^{-1}F(r)\sinh(2Ht)\right)\partial_r\tilde{\phi}\\
&\quad + \frac{1}{2}H^{-1}rF(r)^{-1}\sinh(2Ht)\partial_r^2\tilde{\phi} + \frac{1}{2}H^{-1}rF(r)^{-1}\sinh(2Ht)\partial_t^2\tilde{\phi} + H^{-1}r\sinh(2Ht)\partial_t\partial_r\tilde{\phi}.
\end{aligned} \tag{3.8}$$

We now consider the e^{-2Ht} component of $\tilde{\phi}$,

$$\tilde{\varphi}_2 := \mathrm{e}^{2Ht}(\tilde{\phi} - \tilde{\varphi}_1 \mathrm{e}^{-Ht}).$$

Rewriting (3.6), (3.7) and (3.8) in terms of $\tilde{\varphi}_2$, we obtain

$$\begin{aligned}\Omega\partial_\zeta^2\hat{\phi} = \mathrm{e}^{-2Ht}\bigg(&\tilde{\varphi}_1(0)F^{-3/2}\big\{(1+H^2r^2)\cosh(Ht) - 2H^2r^2\sinh(Ht)\big\}\\ &+ F^{-1}r^2\sinh^2(Ht)\partial_t^2\tilde{\varphi}_2 + H^{-2}F\cosh^2(Ht)\partial_r^2\tilde{\varphi}_2 + H^{-1}r\sinh(2Ht)\partial_r\partial_t\tilde{\varphi}_2\\ &+ H^{-1}F^{-1}\sinh(Ht)\big\{(1+H^2r^2)\cosh(Ht) - 4H^2r^2\sinh(Ht)\big\}\partial_t\tilde{\varphi}_2 - r(1+2\sinh(2Ht))\partial_r\tilde{\varphi}_2\\ &- F^{-1}\big\{(1+H^2r^2)\sinh(2Ht) - 4H^2r^2\sinh^2(Ht)\big\}\tilde{\varphi}_2\bigg),\end{aligned}$$

$$\begin{aligned}\Omega\partial_\tau^2\hat{\phi} = \mathrm{e}^{-2Ht}\bigg(&\tilde{\varphi}_1(0)F^{-3/2}\big\{(1+H^2r^2)\sinh(Ht) - 2\cosh(Ht)\big\}\\ &+ H^{-2}F^{-1}\cosh^2(Ht)\partial_t^2\tilde{\varphi}_2 + r^2F\sinh^2(Ht)\partial_r^2\tilde{\varphi}_2 + H^{-1}r\sinh(2Ht)\partial_r\partial_t\tilde{\varphi}_2\\ &+ \frac{1}{2}H^{-1}F^{-1}\big\{(3-H^2r^2)\sinh(2Ht) - 8\cosh^2(Ht)\big\}\partial_t\tilde{\varphi}_2\\ &- r\big\{1 - 2H^2r^2 - 2F\cosh(2Ht) + 2\sinh(2Ht)\big\}\partial_r\tilde{\varphi}_2\\ &+ F^{-1}\big\{2 + H^2r^2F + (3 - 2H^2r^2 + H^4r^4)\cosh(2Ht) - (3-H^2r^2)\sinh(2Ht)\big\}\tilde{\varphi}_2\bigg),\end{aligned}$$

$$\begin{aligned}\Omega\partial_\tau\partial_\zeta\hat{\phi} = \mathrm{e}^{-2Ht}\bigg(&-2\tilde{\varphi}_1(0)HrF^{-3/2}\mathrm{e}^{-Ht}\\ &+ \frac{1}{2}H^{-1}F^{-1}\sinh(2Ht)\partial_t^2\tilde{\varphi}_2 + \frac{1}{2}H^{-1}r\sinh(2Ht)\partial_r^2\tilde{\varphi}_2 + \big\{F + (1 + H^2r^2\cosh(2Ht))\big\}\partial_r\partial_t\tilde{\varphi}_2\\ &+ \frac{1}{2}rF^{-1}\big\{(3-H^2r^2)\cosh(2Ht) - 4\sinh^2(Ht) - F\big\}\partial_t\tilde{\varphi}_2\\ &+ H^{-1}\big\{(1+H^2r^2)\cosh(2Ht) + F(1-\sinh(2Ht))\big\}\partial_r\tilde{\varphi}_2\\ &+ HrF^{-1}\big\{F + 2\sinh(2Ht) - (3-H^2r^2)\cosh(2Ht)\big\}\tilde{\varphi}_2\bigg).\end{aligned}$$

Taking the limit $t \to +\infty$ and multiplying each equation by $4H^2F$ yields

$$0 \approx -H^2r^2\partial_t^2\tilde{\varphi}_2 - F^2\partial_r^2\tilde{\varphi}_2 - 2HrF\partial_r\partial_t\tilde{\varphi}_2 - 3H^3r^2\partial_t\tilde{\varphi}_2 + 4H^2rF\partial_r\tilde{\varphi}_2 + 2H^2F\tilde{\varphi}_2 \tag{3.9a}$$
$$0 \approx \partial_t^2\tilde{\varphi}_2 + H^2r^2F^2\partial_r^2\tilde{\varphi}_2 + 2HrF\partial_r\partial_t\tilde{\varphi}_2 - H(1+H^2r^2)\partial_t\tilde{\varphi}_2 - 4H^4r^3F\partial_r\tilde{\varphi}_2 - 2H^4r^2F\tilde{\varphi}_2 \tag{3.9b}$$
$$0 \approx Hr\partial_t^2\tilde{\varphi}_2 + HrF^2\partial_r^2\tilde{\varphi}_2 + (1+H^2r^2)F\partial_r\partial_t\tilde{\varphi}_2 - H^2r(1+H^2r^2)\partial_t\tilde{\varphi}_2 - 4H^3r^2F\partial_r\tilde{\varphi}_2 - 2H^3rF\tilde{\varphi}_2. \tag{3.9c}$$

Assuming that $\partial_t\tilde{\varphi}_2 \approx 0$, all time derivatives in (3.9) vanish, and from any of the three equations we obtain

$$F\partial_r^2\tilde{\varphi}_2 - 4H^2r\partial_r\tilde{\varphi}_2 - 2H^2\tilde{\varphi}_2 \approx 0.$$

This equation has solution

$$\tilde{\varphi}_2(r) \approx \frac{\tilde{\varphi}_2(0) + r\tilde{\varphi}_2'(0)}{F(r)},$$

where we have suppressed any potential angular dependence.

Third Coefficient

We can use the same method to compute the third coefficient, by evaluating the third derivatives $\Omega\partial_\tau^3\hat{\phi}$, $\Omega\partial_\tau^2\partial_\zeta\hat{\phi}$, $\Omega\partial_\tau\partial_\zeta^2\hat{\phi}$ and $\Omega\partial_\zeta^3\hat{\phi}$. We omit the details as the calculation is very similar to those shown above, and merely state that upon taking the limit $t \to +\infty$, we obtain the four equations

$$0 \approx H^3r^3\partial_t^3\tilde{\varphi}_3 - F^3\partial_r^3\tilde{\varphi}_3 + 3H^2r^2F\partial_t^2\partial_r\tilde{\varphi}_3 + 3HrF^2\partial_t\partial_r^2\tilde{\varphi}_3 + 3H^2r(1-2H^2r^2)\partial_t^2\tilde{\varphi}_3$$

$$
\begin{aligned}
&- 9H^2rF^2\partial_r^2\tilde{\varphi}_3 + 3H(1-5H^2r^2)F\partial_t\partial_r\tilde{\varphi}_3 - 9H^2(1-2H^2r^2)F\partial_r\tilde{\varphi}_3 \\
&- H^3r(12-11H^2r^2)\partial_t\tilde{\varphi}_3 + 3H^4r(3-2H^2r^2)\tilde{\varphi}_3,
\end{aligned} \tag{3.10a}
$$

$$
\begin{aligned}
0 \approx\ & H^2r^2\partial_t^3\tilde{\varphi}_3 - HrF^3\partial_r^3\tilde{\varphi}_3 + Hr(2+H^2r^2)F\partial_t^2\partial_r\tilde{\varphi}_3 + (1+2H^2r^2)F^2\partial_t\partial_r^2\tilde{\varphi}_3 \\
&+ H(1-3H^2r^2+H^4r^4)\partial_t^2\tilde{\varphi}_3 - 9H^3r^2F^2\partial_r^2\tilde{\varphi}_3 - 3H^2r(1+3H^2r^2)F\partial_t\partial_r\tilde{\varphi}_3 \\
&- H^2(3+3H^2r^2-5H^4r^4)\partial_t\tilde{\varphi}_3 - 9H^3r(1-2H^2r^2)F\partial_r\tilde{\varphi}_3 + 3H^5r^2(3-2H^2r^2)\tilde{\varphi}_3,
\end{aligned} \tag{3.10b}
$$

$$
\begin{aligned}
0 \approx\ & Hr\partial_t^3\tilde{\varphi}_3 - H^2r^2F^3\partial_r^3\tilde{\varphi}_3 + (1+2H^2r^2)F\partial_t^2\partial_r\tilde{\varphi}_3 + Hr(2+H^2r^2)F^2\partial_t\partial_r^2\tilde{\varphi}_3 \\
&- 9H^4r^3F^2\partial_r^2\tilde{\varphi}_3 - H(1+7H^2r^2+4H^4r^4)F\partial_t\partial_r\tilde{\varphi}_3 - H^2r(2+H^2r^2)\partial_t^2\tilde{\varphi}_3 \\
&- H^3r(3-2H^4r^4)\partial_t\tilde{\varphi}_3 - 9H^4r^2(1-2H^2r^2)F\partial_r\tilde{\varphi}_3 + 3H^6r^3(3-2H^2r^2)\tilde{\varphi}_3,
\end{aligned} \tag{3.10c}
$$

$$
\begin{aligned}
0 \approx\ & \partial_t^3\tilde{\varphi}_3 - H^3r^3F^3\partial_r^3\tilde{\varphi}_3 + 3HrF\partial_t^2\partial_r\tilde{\varphi}_3 + 3H^2r^2F^2\partial_t\partial_r^2\tilde{\varphi}_3 - 9H^5r^4F^2\partial_r^2\tilde{\varphi}_3 \\
&- 3H\partial_t^2\tilde{\varphi}_3 - 3H^2r(1+3H^2r^2)F\partial_t\partial_r\tilde{\varphi}_3 + H^2(2-6H^2r^2+3H^4r^4)\partial_t\tilde{\varphi}_3 \\
&- 9H^5r^3(1-2H^2r^2)F\partial_r\tilde{\varphi}_3 + 3H^7r^4(3-2H^2r^2)\tilde{\varphi}_3.
\end{aligned} \tag{3.10d}
$$

As before, assuming that $\partial_t\tilde{\varphi}_3 \approx 0$, all time derivatives vanish and from any of the four equations above, we obtain

$$F^3\partial_r^3\tilde{\varphi}_3 + 9H^2r\partial_r^2\tilde{\varphi}_3 + 9H^2(1-2H^2r^2)F\partial_r\tilde{\varphi}_3 - 3H^4r(3-2H^2r^2)\tilde{\varphi}_3 \approx 0$$

This equation has solution

$$\tilde{\varphi}_3(r) \approx \frac{\tilde{\varphi}_3(0) + r\tilde{\varphi}_3'(0) + r^2\tilde{\varphi}_3''(0)}{F(r)^{3/2}}.$$

From the pattern observed up to the third order, we conjecture that

$$\tilde{\varphi}_n(r) \approx \frac{\tilde{\varphi}_n(0) + r\tilde{\varphi}_n'(0) + r^2\tilde{\varphi}_n''(0) + \cdots + r^{n-1}\tilde{\varphi}_n^{(n-1)}(0)}{F(r)^{n/2}}. \tag{3.11}$$

3.2 Asymptotic Decomposition via Direct Substitution

In this section we attempt to compute the coefficients of the asymptotic expansion of $\tilde{\phi}$ via direct consideration of the conformal wave equation on dS_4 in various coordinate systems.

Global Coordinates

We begin by considering the conformal wave equation in the global coordinates $(\alpha, \omega^{(3)})$. In these coordinates, the metric is given by

$$\tilde{g} = \mathrm{d}\alpha^2 - \frac{1}{H^2}\cosh^2(H\alpha)\mathfrak{s}_3.$$

This has determinant $\det\tilde{g} = -(\cosh^6(H\alpha)/H^6)\det\mathfrak{s}_3$, so that the box operator corresponding to this metric is

$$
\begin{aligned}
\tilde{\Box} &= \frac{1}{\sqrt{|\det\tilde{g}|}}\partial_\mu\Big(\sqrt{|\det\tilde{g}|}\tilde{g}^{\mu\nu}\partial_\nu\Big) \\
&= \frac{1}{\frac{1}{H^3}\cosh^3(H\alpha)\sqrt{\det\mathfrak{s}_3}}\partial_\alpha\left(\frac{1}{H^3}\cosh^3(H\alpha)\sqrt{\det\mathfrak{s}_3}\partial_\alpha\right) - \frac{1}{\sqrt{\det\mathfrak{s}_3}}\partial_\omega\Big(\sqrt{\det\mathfrak{s}_3}g^{\omega\omega'}\partial_{\omega'}\Big) \\
&= \partial_\alpha^2 + 3\tanh(H\alpha)\partial_\alpha - \frac{H^2}{\cosh^2(H\alpha)}\slashed{\nabla}^2.
\end{aligned}
$$

Thus the wave equation in these coordinates is

$$0 = \tilde{\Box}\tilde{\phi} + 2H^2\tilde{\phi} = \partial_\alpha^2\tilde{\phi} + 3\tanh(H\alpha)\partial_\alpha\tilde{\phi} - \frac{H^2}{\cosh^2(H\alpha)}\slashed{\nabla}^2\tilde{\phi} + 2H^2\tilde{\phi}.$$

Assuming that $\tilde{\phi}$ has an asymptotic expansion of the form

$$\tilde{\phi} \sim \sum_{n=1}^{\infty} c_n(\omega^{(3)})\mathrm{e}^{-nH\alpha} \qquad \text{as } \alpha \to +\infty,$$

we obtain
$$\tilde{\Box}\tilde{\phi} + 2H^2\tilde{\phi} = \sum_{n=1}^{\infty} \mathrm{e}^{-nH\alpha}\left[\left(n^2H^2 + 3nH\tanh(H\alpha) + 2H^2\right)c_n - \frac{H^2}{\cosh^2(H\alpha)}\not{\nabla}^2 c_n\right],$$
which indicates that every coefficient c_n must satisfy the equation
$$\left(n^2H^2 + 3nH\tanh(H\alpha) + 2H^2\right)c_n - \frac{H^2}{\cosh^2(H\alpha)}\not{\nabla}^2 c_n = 0.$$
We note that the timelike coordinate α appears along with derivatives with respect to the spatial coordinates in this equation, which means that we cannot simply take $\alpha \to +\infty$ in this case. To avoid having to deal with this additional difficulty, we will instead consider the equation in the static coordinates, where the equation is in fact separable in the timelike and spatial coordinates.

Static Coordinates

We now perform the same calculation using the static coordinates $(t, r, \omega^{(2)})$. In these coordinates, the metric takes the form
$$\tilde{g} = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2\mathfrak{s}_2,$$
with $F(r) = 1 - H^2r^2$. This is defined only in region I of the Penrose diagram, where $r \in (0, 1/H)$. Clearly $\det\tilde{g} = r^2\det\mathfrak{s}_2$, so that in these coordinates,
$$\tilde{\Box} = F(r)^{-1}\partial_t^2 - \frac{1}{r^2}\partial_r\left(r^2F(r)\partial_r\right) - \frac{1}{r^2}\nabla^2_{\mathfrak{s}_2}.$$
If we assume that $\tilde{\phi}$ has an asymptotic expansion of the form
$$\tilde{\phi} \sim \sum_{n=1}^{\infty} c_n(r, \omega^{(2)})\mathrm{e}^{-nHt} \qquad \text{as } t \to +\infty,$$
then we obtain
$$0 = \tilde{\Box}\tilde{\phi} + 2H^2\tilde{\phi} \sim \sum_{n=1}^{\infty} \mathrm{e}^{-nHt}\left[\left(n^2F(r)^{-1} + 2\right)H^2c_n - \frac{1}{r^2}\partial_r\left(r^2F(r)\partial_r c_n\right) - \frac{1}{r^2}\nabla^2_{\mathfrak{s}_2}c_n\right] \quad \text{as } t \to +\infty. \tag{3.12}$$
We note that in this case, the wave equation is separable in t and the other coordinates, so we can take $t \to +\infty$ and simply get a PDE for c_n on $\mathscr{I}^+$. This is in contrast to the case above using global coordinates, where there is coupling between the timelike coordinate α and the angular coordinates inside the equation. In fact, this decoupling is a direct consequence of the staticity of the metric.[1] The PDE obtained from (3.12) may be written explicitly as
$$\frac{1}{r}\partial_r\left(r^2(1 - H^2r^2)\partial_r c_n\right) - r\left(2 + \frac{n^2}{1 - H^2r^2}\right)H^2c_n + \frac{1}{r}\nabla^2_{\mathfrak{s}_2}c_n = 0.$$
We seek a separable solution, writing $c_n = F^{-n/2}R_n(r)\Theta_n(\omega^{(2)})$. Separating variables, we obtain
$$\frac{r^2}{R_n}\left[F\frac{\mathrm{d}^2R_n}{\mathrm{d}r^2} + \frac{\mathrm{d}R_n}{\mathrm{d}r}\frac{2}{r}\left(1 + (n-2)H^2r^2\right) + R_nH^2(3n - (n^2+2))\right] = \lambda = -\frac{1}{\Theta_n}\nabla^2_{\mathfrak{s}_2}\Theta_n.$$
For the angular component, we have
$$\nabla^2_{\mathfrak{s}_2}\Theta_n + \lambda\Theta_n = 0.$$
The solutions to these are the spherical harmonics $Y_{l,m}$ where $l \in \mathbb{N}_0, m \in \mathbb{Z}, -l \leq m \leq l$ and $\lambda = l(l+1)$. For the radial component, we substitute $z := Hr$ and obtain an ODE for $R_n(r)$
$$\frac{\mathrm{d}^2R_n}{\mathrm{d}z^2} + \frac{\mathrm{d}R_n}{\mathrm{d}z}\left(\frac{2}{z} + \frac{1-n}{z-1} + \frac{1-n}{z+1}\right) + R_n\left(-\frac{\lambda}{z^2} + \frac{\frac{1}{2}(3n - 2 - n^2 - \lambda)}{z+1} + \frac{\frac{1}{2}\left(n^2 + 2 - 3n + \lambda\right)}{z-1}\right) = 0. \tag{3.13}$$

[1] This is documented in Appendix 4.

We want to solve (3.13) in the region $z \in (0,1)$. $z=0$ is a regular singular point, so we seek a Frobenius series solution. The indicial equation near $z=0$ is given by $\sigma^2+\sigma+l(l+1)=0$ so the exponents for the singular point $z=0$ are $l, -(l+1)$. These differ by an integer, so in general, the solutions are given by

$$R_{1,n,m,l} = z^l \sum_{k=0}^{\infty} a_k z^k, \qquad R_{2,n,m,l} = R_{1,n,m,l} \log z + \sum_{k=0}^{\infty} b_k z^{k-l-1}.$$

From some energy estimates, we note that R_n must be smooth near $z=0$ and $z=1$. Hence, we only consider the first solution. Substituting this into the ODE, we obtain the recurrence relations

$$a_1 = 0, \qquad a_k = \frac{(k-(n-l))(k-(n-l+1))}{k(1+k+l)} a_{k-2}. \tag{3.14}$$

The solution must be smooth near $z=0$, so we necessarily have $n \geq l$, so that the series terminates and hence we get a polynomial solution, given by the product of z^l and an even polynomial. If $n-l$ is even, then $R_{1,n,m,l}$ is a polynomial of degree $n-l-2+l=n-2$. If $n-l$ is odd, then $R_{1,n,m,l}$ is a polynomial of degree $n-l+1-2+l=n-1$. We always get the second case for any n, obtained by $l=0$ with n odd and $l=1$ with n even. The general solution is given by

$$\phi \sim \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \alpha_{n,m,l} R_{1,n,m,l} Y_{l,m} F(r)^{-n/2} \mathrm{e}^{-nHt} = \sum_{n=1}^{\infty} P_n(r) F(r)^{-n/2} \mathrm{e}^{-nHt} \quad \text{as } t \to +\infty, \tag{3.15}$$

where P_n is a polynomial in Hr of degree $n-1$ with coefficients being linear combinations of the spherical harmonics multiplied by the radial polynomial solutions of each with coefficients being related by (3.14).

3.3 Asymptotic Decomposition via Quasinormal Modes

For a linear scalar field ϕ, quasinormal modes (QNMs) capture the behaviour at late times in the sense that

$$\phi(t,x) \sim \sum \mathrm{e}^{-i\omega_j t} c_j u_j(x), \quad t \to +\infty \tag{3.16}$$

where $u_j(x)$ are the mode solutions and c_j are constants depending on initial conditions of ϕ. For our purposes, we use the definition from [4], which defines a quasinormal mode as $\omega \in \mathbb{C}$ such that there is a purely outgoing solution

$$\mathrm{e}^{-i\omega t} u(x), \quad u(r \to 1/H) \sim \mathrm{e}^{i\omega r_*}$$

to the wave equation $(\Box + 2H^2)(\mathrm{e}^{-i\omega t} u) = 0$, where r_* is the tortoise coordinate $\mathrm{d}r_* = \mathrm{d}r/F$.

The following method involves comparing a series solution of the wave equation to the QNM expansion (3.16). In doing so we obtain an exact asymptotic expansion for the scalar field ϕ, where we can simply read off the QNMs and the corresponding mode solutions.

We begin with the static metric on dS_4

$$g = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2 \mathfrak{s}_2,$$

where $F(r) = 1 - H^2 r^2$ as above. We make the substitution $t_* = t + (\log F)/2H$, giving the metric

$$g = F\, \mathrm{d}{t_*}^2 + 2Hr\, \mathrm{d}t_* \mathrm{d}r - \mathrm{d}r^2 - r^2 \mathfrak{s}_2.$$

Setting $x = r\omega^{(2)}$, $\tau = \mathrm{e}^{-Ht_*}$ and $X = Hx\tau$ we get the new metric

$$g = \frac{\mathrm{d}\tau^2 - \mathrm{d}X^2}{(\tau H)^2}.$$

Using the well-known formula

$$\Box = \frac{1}{\sqrt{|\det g|}} \partial_\mu \Big(\sqrt{|\det g|} g^{\mu\nu} \partial_\nu \Big),$$

the wave equation $(\Box + 2H^2)\phi = 0$ in the (τ, X) coordinates becomes

$$((\tau\partial_\tau)^2 - 3(\tau\partial_\tau) - \tau^2 \Delta_X)\phi + 2\phi = 0, \tag{3.17}$$

where $\Delta_X = \sum \partial^2_{X^i}$ is the spatial Laplace operator. Without the $\tau^2 \Delta_X$ term, equation (3.17) becomes

$$(\tau\partial_\tau)^2 - 3(\tau\partial_\tau)\phi + 2\phi = 0,$$

which has characteristic equation $p(\lambda) = \lambda^2 - 3\lambda + 2$, with roots $\lambda_\pm = 2, 1$. It is shown in [7] that the solution of this equation can be written as

$$\phi(\tau, X) = \sum_\pm \tau^{\lambda_\pm} u_\pm(\tau, X). \tag{3.18}$$

where the $u_\pm(\tau, X)$ are smooth functions in $[0, 1] \times \mathbb{R}^4$. We now Taylor expand $u_\pm(\tau, X)$, assuming even powers of τ as

$$u_\pm(\tau, X) = \sum_{j=0}^{\infty} \tau^{2j} u_\pm^{(j)}(X),$$

where we note that this assumption is allowed as $\lambda_\pm = 2, 1$. Substituting into (3.17) gives the expression

$$\sum_\pm \sum_{j=0}^{\infty} \big[(\lambda_\pm + 2j)^2 - 3(\lambda_\pm + 2j) + 2\big] \tau^{\lambda_\pm + 2j} u_\pm^{(j)}(X) - \sum_\pm \sum_{j=0}^{\infty} \tau^{\lambda_\pm + 2(j+1)} \Delta_X u_\pm^{(j)}(X) = 0,$$

which in turn gives the following recurrence relation for $u^{(j)}$ with $j \geq 1$:

$$p(\lambda_\pm + 2j) u_\pm^{(j)} - \Delta_X u_\pm^{(j-1)} = 0.$$

Solving this recurrence relation gives $u_\pm^{(j)}$ as

$$u_\pm^{(j)} = q_\pm^{(j)} \Delta_X^j u_\pm^{(0)}, \quad q_\pm^{(j)} = \prod_{k=1}^{j} \frac{1}{p(\lambda_\pm + 2k)}.$$

For our values of $\lambda_\pm = 2, 1$ we easily find $q_\pm^{(j)}$ to be

$$q_+^{(j)} = \frac{1}{2^j (j+1)!}, \quad q_-^{(j)} = \frac{2^j j!}{(2j+1)!}.$$

This gives a full expansion for ϕ depending on the asymptotic data $(u_+^{(0)}, u_-^{(0)})$.

As $t_* \to +\infty$, if x is bounded then $\tau, X \to 0$ so we can Taylor expand $u_\pm^{(j)}(X)$. We have

$$\begin{aligned} u_\pm^{(j)}(X) &= \sum_{|\alpha|=0}^{\infty} \frac{\partial_X^\alpha u_\pm^{(j)}(0)}{\alpha!} X^\alpha \\ &= \sum_{|\alpha|=0}^{\infty} \frac{\partial_X^\alpha q_\pm^{(j)} \Delta_X^j u_\pm^{(0)}(0)}{\alpha!} X^\alpha \\ &= \sum_{|\alpha|=0}^{\infty} c_\pm^{(j,a)} X^\alpha, \end{aligned}$$

where we use multi index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $c_\pm^{(j,\alpha)} = \frac{1}{\alpha!} q_\pm^{(j)} \partial_X^\alpha \Delta_X^j u_\pm^{(0)}(0)$.

Thus the Taylor expanded ϕ is

$$\phi(\tau, X) = \sum_\pm \tau^{\lambda_\pm} \sum_{j=0}^{\infty} \tau^{2j} \sum_{|\alpha|=0}^{\infty} c_\pm^{(j,\alpha)} X^\alpha.$$

In the (t_*, x) coordinates this is

$$\phi(t_*, x) = \sum_\pm \mathrm{e}^{-\lambda_\pm H t_*} \sum_{j=0}^{\infty} \sum_{|\alpha|=0}^{\infty} \mathrm{e}^{-(2j+\alpha)H t_*} c_\pm^{(j,\alpha)} (Hx)^\alpha.$$

Letting $2j+|\alpha|=n$ we have that $\sum_{j=0}^{\infty}\sum_{|\alpha|=0}^{\infty}=\sum_{n=0}^{\infty}\sum_{2j+|\alpha|=n}$ which gives

$$\phi(t_*,x)=\sum_{\pm}\sum_{n=0}^{\infty}\mathrm{e}^{-H(\lambda_\pm+n)t_*}\sum_{2j+|\alpha|=n}c_{\pm}^{(j,\alpha)}(Hx)^{\alpha}. \tag{3.19}$$

This is a full expansion for ϕ in depending on the data $(u_+^{(0)}(0),u_-^{(0)}(0))$.

We now turn to the QNM expansion. In (t_*,x) coordinates this is given by

$$\phi(t_*,x)\sim\sum c_j\mathrm{e}^{-i\omega_j t_*}u_j, \tag{3.20}$$

where ω_j are the QNMs, $u_j=u_j(x)$ are the mode solutions and c_j are coefficients depending on initial conditions. Comparing (3.19) and (3.20) we can read of the QNMs as the ω such that

$$\mathrm{e}^{-i\omega t_*}=\mathrm{e}^{-H(\lambda_\pm+n)t_*},\quad n=0,1,2,\dots.$$

Hence the QNMs are

$$\omega=-iH(1+n),\ \omega=-iH(2+n)\text{ for }n=0,1,2,\dots$$

which we see are purely imaginary. Given a choice of $\pm$ and n we can also easily read off the mode solution $u=u(x)$ so that $\mathrm{e}^{-H(\lambda\pm+n)t_*}u$ solves equation (3.17).

We now move to static coordinates (t,x), where we have

$$\begin{aligned}\mathrm{e}^{-(\lambda_\pm+n)Ht_*}&=\mathrm{e}^{-(\lambda_\pm+n)H(t+\frac{1}{2H}\log F(r))}\\&=F(r)^{-(\lambda_\pm+n)/2}\mathrm{e}^{-(\lambda_\pm+n)Ht}.\end{aligned}$$

So our mode solutions in static coordinates are $F(r)^{(\lambda_\pm+n)/2}u(x)$ for $n=0,1,2,\dots$, where $u(x)$ is a mode solution in (t_*,x) coordinates. Our expansion of the scalar field in static coordinates is then

$$\phi(t,x)\sim\sum_{\pm}\sum_{n=0}^{\infty}\mathrm{e}^{-(\lambda_\pm+n)Ht}\sum_{2j+|\alpha|=n}c_{\pm}^{(j,\alpha)}F(r)^{(\lambda_\pm+n)/2}(Hx)^{\alpha},$$

where $r=|x|$. Moreover, since $\lambda_\pm=2,1$ we can combine $\sum_\pm\sum_n$ into a single sum over a new index n via the calculation

$$\begin{aligned}\phi(t,x)\sim&\sum_{n=0}^{\infty}\mathrm{e}^{-(1+n)Ht}\sum_{2j+|\alpha|=n}c_{-}^{(j,\alpha)}F(r)^{-(1+n)/2}(Hx)^{\alpha}\\&+\sum_{n=0}^{\infty}\mathrm{e}^{-(2+n)Ht}\sum_{2j+|\alpha|=n}c_{+}^{(j,\alpha)}F(r)^{-(2+n)/2}(Hx)^{\alpha}\\=&\,\mathrm{e}^{-Ht}\sum_{2j+|\alpha|=0}c_{-}^{(j,\alpha)}F(r)^{-1/2}(Hx)^{\alpha}+\sum_{n=1}^{\infty}\mathrm{e}^{-(1+n)Ht}\sum_{2j+|\alpha|=n}c_{-}^{(j,\alpha)}F(r)^{-(1+n)/2}(Hx)^{\alpha}\\&+\sum_{n=1}^{\infty}\mathrm{e}^{-(1+n)Ht}\sum_{2j+|\alpha|=n-1}c_{+}^{(j,\alpha)}F(r)^{-(1+n)/2}(Hx)^{\alpha}\\=&\,\mathrm{e}^{-Ht}F(r)^{-1/2}u_-^{(0)}(0)+\sum_{n=2}^{\infty}\mathrm{e}^{-nHt}\sum_{2j+|\alpha|=n-1}c_{-}^{(j,\alpha)}F(r)^{-n/2}(Hx)^{\alpha}\\&+\sum_{n=2}^{\infty}\mathrm{e}^{-nHt}\sum_{2j+|\alpha|=n-2}c_{+}^{(j,\alpha)}F(r)^{-n/2}(Hx)^{\alpha}\\=&\,\mathrm{e}^{-Ht}F(r)^{-1/2}u_-^{(0)}(0)+\sum_{n=2}^{\infty}\mathrm{e}^{-nHt}\left[\sum_{2j+|\alpha|=n-1}c_{-}^{(j,\alpha)}+\sum_{2j+|\alpha|=n-2}c_{+}^{(j,\alpha)}\right]F(r)^{-n/2}(Hx)^{\alpha}.\end{aligned}$$

Considering the expression

$$\sum_{2j+|\alpha|=n-1}x^{\alpha}+\sum_{2j+|\alpha|=n-2}x^{\alpha}$$

which appears as a sub-expression in the above result (omitting constants), we see that when $j = 0$ we sum over $|\alpha| = n-1, n-2$ and so obtain powers of r^{n-1} and r^{n-2}. Summing over $j = 1$ gives powers of r^{n-3} and r^{n-4}, and so on until $j = (n-1)/2$ (n odd) and $j = (n-2)/2$ (n even) gives $|\alpha| = 0$ and so a constant. Hence the coefficient of the term $\mathrm{e}^{-nHt}F(r)^{-n/2}$ is a polynomial in r of degree $n-1$ which we will denote as $P_n(r) = c_{n0} + c_{n1}r + \cdots c_{nn-1}r^{n-1}$, where the c_{ij} are constants in r depending on $(u_+^{(0)}(0), u_-^{(0)}(0))$.

We thus obtain the asymptotic expansion

$$\phi \sim \sum_{n=1}^{\infty} \mathrm{e}^{-nHt} F(r)^{-n/2} P_n(r) \quad \text{as } t \to +\infty.$$

4 Conclusion

This report has focused on the asymptotic decomposition of a scalar field in de Sitter space from several different angles. Firstly, the conformal method of [3] was used to derive energy estimates on de Sitter space, which were then used to derive asymptotic decay estimates for the scalar field. In Section 3 the method was used to derive the coefficients in the asymptotic expansion of the field up to $\mathcal{O}(\mathrm{e}^{-3Ht})$. Based on the pattern observed in these coefficients, a conjecture was made for the general n-th coefficient in the asymptotic expansion. The same computation was also performed by solving for the coefficients directly from the conformal wave equation, and the results were found to agree with those obtained using the conformal method.

The same problem of obtaining an asymptotic decomposition of a scalar field on de Sitter space was also studied using the quasinormal mode approach of [4]. We observed that the coefficients obtained using the conformal method were in agreement with the mode expansion obtained using quasinormal modes, which provided an important validation of our results for the case of the linear wave equation.

We conclude this report by noting that the asymptotic decomposition obtained for the scalar field via the conformal method does not only hold in the linear case, but actually also applies in the case of a charged scalar field in the nonlinear Maxwell-scalar field system. This demonstrates an advantage of developed here using the conformal method, when compared to existing methods involving quasinormal modes.

Appendix A Wave Equation for General Static Metric

We show here that the decoupling of the time derivatives and spatial derivatives in the PDE is a direct consequence of the staticity of the metric. A metric is said to be *static* if it has a timelike Killing vector which is orthogonal to a family of hypersurfaces. Such a metric may be written in the form

$$ds^2 = g_{00}(x)\,\mathrm{d}t^2 + h_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j,$$

where ∂_t is a timelike Killing vector which is orthogonal to the family of hypersurfaces described by $t = \text{constant}$.

For such a metric, we have

$$\begin{aligned}
\Box &= \frac{1}{\sqrt{|\det g|}}\partial_\mu\left(\sqrt{|\det g|}g^{\mu\nu}\partial_\nu\right)\\
&= \frac{1}{\sqrt{g_{00}}\sqrt{\det h}}\partial_t\left(\sqrt{g_{00}}\sqrt{\det h}g_{00}^{-1}\partial_t\right) + \frac{1}{\sqrt{g_{00}}\sqrt{\det h}}\partial_k\left(\sqrt{g_{00}}\sqrt{\det h}h^{kl}\partial_l\right)\\
&= g_{00}^{-1}\partial_t^2 + \frac{1}{2}g_{00}^{-1}(\partial_k g_{00})h^{kl}\partial_l + \Delta_h\\
&= g_{00}^{-1}\partial_t^2 + \frac{1}{2}(\partial_k \log g_{00})h^{kl}\partial_l + \Delta_h.
\end{aligned}$$

so that the conformal wave equation is

$$0 = \Box\phi + \frac{1}{6}R\phi = g_{00}^{-1}\partial_t^2\phi + \frac{1}{2}(\partial_k \log g_{00})h^{kl}\partial_l\phi + \Delta_h\phi + \frac{1}{6}R\phi.$$

Assuming that

$$\phi = \sum_{k=1}^{\infty} c_k \mathrm{e}^{-kHt},$$

we obtain the equation

$$\Delta_h c_k + \frac{1}{2}(\partial_m \log g_{00})h^{ml}\partial_l c_k + \left(k^2 g_{00}^{-1}H^2 + \frac{1}{6}R\right)c_k = 0.$$

We thus see that in the case of a general static metric, the dependence on the time coordinate decouples from the dependence on the spatial coordinates.

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