1 Model

Symmetric Coalition-Competition Game

[Now I just gave an arbitrary name for such game. We should search it has been given a name before, or see if we can propose a better name.]

We consider a symmetric coalition-competition game of n players, labeled from 1 to n. The game proceeds in two stages. In the first stage, the players form a number of coalitions. A coalition structure (CS) is a partition of the players into one or more coalitions. For instance, when $n = 8$, some examples of CSs are $\{\{1,2,3\},\{4,5,6\},\{7,8\}\}$, $\{\{1,7\},\{2\},\{3,5,8\},\{4,6\}\}$, and $\{\{1,2,3,4,5,6,7,8\}\}$. The last CS is where all players form one single coalition, which is usually called a *grand coalition*.

Once a CS is formed, the game proceeds to the second stage, where each coalition effectively acts as a meta-player making decisions in a *reward system*.[1] In this note, we focus on reward systems modeled as non-cooperative games, such as Cournot competitions and Tullock contests, that admit a unique Nash equilibrium determining the rewards to the meta-players. More generally, a reward system might not be a game — it could be a market with rewards determined at competitive equilibrium, or it could be governed by externally imposed or even artificially constructed rules. Each meta-player/coalition's reward is then divided equally among its member players.

In the symmetric game, all players are treated equally. As a result, two CSs that are equivalent up to a relabeling of players yield the same reward distribution. Also, within a given CS, all players belonging to coalitions of the same size receive the same rewards. For example, consider the two CSs $\{1,2,3\},\{4,5,6\},\{7,8\}$ and $\{1,3,7\},\{2,5,8\},\{4,6\}$ which are equivalent up to a relabeling of players. The rewards received by players $1,2,3,4,5,6$ in the first CS are the same as those received by players $1,3,7,2,5,8$ in the second CS. [Haiqing raised a good question about how this model differs from the standard cooperative game model underpinning Shapley and core values. We should clarify this in a later version.]

Example 1. Suppose there are 8 players forming a CS $\{\{1,7\},\{2\},\{3,5,8\},\{4,6\}\}$. The four meta-players then participate in a Cournot competition. Let x_i denote the quantity of goods produced by the i-th meta-player, and let $x = \sum_{i=1}^{4} x_i$. The Cournot competition has a linear price curve: each unit of good is sold for $p = 14 - 8x$ dollars. Producing one unit of the good costs 2 dollars.

The Cournot competition admits a unique Nash equilibrium, at which each meta-player produces 0.3 unit of good, and receives a reward of 0.72 dollar. As the reward of each coalition is divided equally among its member players, player 2 receives 0.72 dollar, players $1,7,4,6$ each receives 0.36 dollar, and players $3,5,8$ each receives 0.24 dollar.

Next, suppose the coalition $\{3,5,8\}$ is split into two coalitions $\{3\}$ and $\{5,8\}$. At the new Cournot equilibrium, each of the five meta-players produces 0.25 unit of good, and receives a reward of 0.5 dollar. Players $2,3$ each receives 0.5 dollar, while each of the other players receive 0.25 dollar. Observe that player 3 receives more reward, but every other player receives less.

Players aim to maximize their individual rewards and therefore act strategically when forming coalitions in the first stage. We consider a dynamic model of coalition formation and characterize the conditions under which a CS is *stable*.

Dynamic Betrayal Model

In general, coalition formation dynamics can involve both splitting and merging of coalitions, making the analysis complex. We focus on a simpler model in which only splitting is allowed. Initially, all

[1] Throughout this note, we use the terms "coalition" and "meta-player" interchangeably.

players belong to a single grand coalition, which may undergo a sequence of splits. This process continues until no player has an *incentive* to leave her current coalition and form a smaller one; we will define incentive formally below. In each coalition of size $i \geq 2$, there is a player who may initiate a *betrayal* by pulling at most $\lfloor i/2 \rfloor$ players (including herself) away from the rest of the coalition. [We need to justify why setting $\lfloor i/2 \rfloor$ as a threshold, or saying this is arbitrary and can be adjusted with the context.] Before moving on, we present a simple example from Ray's book to demonstrate the concept of *farsightedness* and how it determines stability of a CS.

Example 2. There are 3 players. The second stage of the coalition-competition game is a Cournot competition with linear price curve $p = 14 - x$ and production cost of 2 dollars per unit. It is a standard exercise to show that if there are m meta-players, the reward of each meta-player at Nash equilibrium is $\frac{144}{(m+1)^2}$.

When the 3 players form a grand coalition, the reward of each player is $\frac{144}{(1+1)^2} \times \frac{1}{3} = 12$ dollars. If a player betrays and form a new coalition by herself, the reward of this player becomes $\frac{144}{(2+1)^2} = 16$ dollars. So, in *short sight*, it would seem the player is motivated to betray. However, if she considers the further response of the other two players – a *far-sight* consideration, she would not betray. The reason is after her betrayal, each of the other two players receives reward of $\frac{144}{(2+1)^2} \times \frac{1}{2} = 8$ dollars. If the two players split, then each of them receives $\frac{144}{(3+1)^2} = 9$ dollars, which are more than 8 dollars, so they are motivated to split. After their split, the player who betrays the grand coalition receives 9 dollars of reward, which is less than 12 dollars that she would receive if she stayed in the grand coalition. To conclude, the far-sight consideration ensures no player would betray the grand coalition in first place, making the grand coalition stable.

Due to the symmetry discussed above, when specifying the rewards of meta-players, we only need to distinguish CSs which are not equivalent upon relabeling. A cleaner way to represent these non-equivalent CSs is using integer partition (IP) of the positive integer n. An IP of n is a multiset of positive integers whose sum is n. For example, when $n = 8$, some IPs are $[4, 3, 1]$, $[3, 3, 2]$, and $[2, 2, 2, 1, 1]$. Each integer i in an IP corresponds to one coalition of size i. Our algorithm below will proceed by enumerating all IP of n, and computing certain *value functions* for them. A huge advantage of doing so is to avoid enumerating the set of CSs of n players, whose size is larger than $(\frac{n}{\log n})^n$, while the set of IPs of n has a much smaller size of at most $e^{\Theta(\sqrt{n})}$.

We will define two value functions $V^{\mathsf{D}}, V^{\mathsf{P}}$, which we call the *default value function* and the *pessimistic value function* respectively. They will be used to define stable CSs. $V^{\mathsf{D}}(S, i)$ is the reward received by any player in a size-i coalition when the CS is fixed. For instance, in Example 2, $V^{\mathsf{D}}([3], 3) = 12$, $V^{\mathsf{D}}([2, 1], 1) = 16$, $V^{\mathsf{D}}([2, 1], 2) = 8$ and $V^{\mathsf{D}}([1, 1, 1], 1) = 9$.

$V^{\mathsf{P}}(S, i)$ is the pessimistically anticipated reward (PAR) a player in a size-i coalition might eventually get, after *any* sequence of *incentivized betrayals* by any players. By incentivized betrayal, we refer to a betrayal initiated by a player such that her PAR with the new CS is strictly better than her default value with the current CS. For instance, in Example 2, $V^{\mathsf{P}}([2, 1], 1) = 9$, because the player in the singleton coalition anticipates that the other coalition is incentivized to split, so the eventual reward she gets is her default reward with the IP $[1, 1, 1]$.

Since coalitions can only be split but not merged in the dynamic betrayal model, V^{P} can be defined recursively. Let U be the IP $[1, 1, \ldots, 1]$, where 1 occurs n times. This corresponds to the CS where each player forms a coalition by herself. Since no further betrayal can occur from U, $V^{\mathsf{P}}(U, 1) = V^{\mathsf{D}}(U, 1)$. Then $V^{\mathsf{P}}(S, i)$ for any IP $S \neq U$ and $i \in S$ will be defined recursively, going from IPs with the most number of coalitions to the least (the grand coalition).

Before moving on, we define a few notations which will help with simplifying our presentation:

- Given a multiset S and $i \in S$, let $S - i$ denote the multiset formed by removing from S one occurrence of i. For example, when $S = [2, 2, 2, 2, 1]$, $S - 2 = [2, 2, 2, 1]$ and $S - 1 = [2, 2, 2, 2]$.

- Given a multiset S, $i \in S$ and $1 \le j < i$, let $\mathcal{B}(S,i,j)$ denote the multiset formed by splitting one occurrence of i in S into j and $i-j$. For example, when $S = [5,5,2]$, $\mathcal{B}(S,5,2) = [5,3,2,2]$.

For any IP $S \neq U$ and $i \in S$, let

$$\mathcal{M}(S,i) \;:=\; \max_{1 \le j \le \lfloor i/2 \rfloor} V^{\mathsf{P}}(\mathcal{B}(S,i,j),j)\ ,$$

which is the PAR of a player in a size-i coalition after she betrays by pulling j players (including herself) away from her current coalition, while she chooses the optimal j. The player is incentivized to betray only when $\mathcal{M}(S,i)$ is strictly larger than her current default value, and using "maximin" approach, she will choose one of the optimal j's. Accordingly, we define the i-betrayal-descendants of S to be

$$\mathcal{D}(S,i) \;:=\; \begin{cases} \left\{\mathcal{B}(S,i,j) \;\middle|\; V^{\mathsf{P}}(\mathcal{B}(S,i,j),j) = \mathcal{M}(S,i)\right\}, & \text{if } \mathcal{M}(S,i) > V^{\mathsf{D}}(S,i); \\ \emptyset, & \text{if } \mathcal{M}(S,i) \le V^{\mathsf{D}}(S,i). \end{cases}$$

which is the set of IPs due to a player in a size-i coalition betraying optimally. When $\mathcal{M}(S,i) \le V^{\mathsf{D}}(S,i)$, the player is not motivated to betray, so the set of the i-betrayal descendants is empty. By viewing the IPs as vertices in a directed graph, and there are directed edges from each S to IPs in $\cup_{i \in S}\mathcal{D}(S,i)$, any sink vertex in this graph (i.e., IPs S where $\cup_{i \in S}\mathcal{D}(S,i)$ empty) corresponds to a stable coalition structure. The PAR value $V^{\mathsf{P}}(S,i)$ is the least possible reward of a player in size-i coalition at any sink which is descendant from S in the above graph, and can be defined as

$$V^{\mathsf{P}}(S,i) \;:=\; \begin{cases} \min\left\{\min_{S' \in \cup_{k \in S-i}\mathcal{D}(S,k)} V^{\mathsf{P}}(S',i)\ ,\ \max\{V^{\mathsf{D}}(S,i)\ ,\ \mathcal{M}(S,i)\}\right\}, & \text{if } \cup_{k \in S-i}\,\mathcal{D}(S,k) \neq \emptyset \\ \max\{V^{\mathsf{D}}(S,i)\ ,\ \mathcal{M}(S,i)\}, & \text{otherwise.} \end{cases}$$

In the next page, we list the sink descendants from the grand coalition, when $3 \le n \le 80$ and the second-stage game is Cournot competition.

n	Cournot Linear Price	Tullock Contest
3	3	3
4	4	4
5	2,2,1	4,1
6	3,2,1	6
7	3,3,1	7
8	8	8
9	9	8,1
10	10	9,1
11	11	10,1
12	5,5,2	11,1
13	13	13
14	6,6,2	13,1
15	7,7,1	15
16	9,6,1	16
17	9,7,1	9,6,1,1
18	10,7,1	9,7,1,1
19	5,5,5,3,1	10,7,1,1
20	11,8,1	20
21	6,6,6,2,1	11,8,1,1
22	7,7,6,1,1	22
23	7,7,7,1,1	23
24	8,8,6,1,1	24
25	5,5,5,5,3,1,1	25
26	7,7,7,3,1,1	26
27	7,7,7,4,1,1	27

28	6,6,6,6,2,1,1	28
29	5,5,5,5,5,2,1,1	29
30	5,5,5,5,5,3,1,1 8,8,7,5,1,1	30
31	7,7,7,7,1,1,1	31
32	8,8,8,5,1,1,1	32
33	7,7,7,7,2,1,1,1	33
34	7,7,7,7,3,1,1,1	34
35	6,6,6,6,6,2,1,1,1	35
36	8,8,8,8,1,1,1,1	30,6
37	8,8,8,8,2,1,1,1	25,9,3
38	6,6,6,6,6,4,1,1,1,1	38
39	6,6,6,6,6,5,1,1,1,1	36,3
40	6,6,6,6,6,6,1,1,1,1	36,4
41	8,8,8,8,5,1,1,1,1	36,5
42	6,6,6,6,6,6,2,1,1,1,1	38,4
43	6,6,6,6,6,6,3,1,1,1,1	38,5
44	6,6,6,6,6,6,4,1,1,1,1	25,15,4
45	6,6,6,6,6,6,4,1,1,1,1,1	45
46	6,6,6,6,6,6,5,1,1,1,1,1	42,4
47	9,9,8,8,8,1,1,1,1,1	42,5
48	8,8,8,8,7,4,1,1,1,1,1	42,6
49	49	47,2
50	28,22	49,1

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 3: [(3,)]
 4: [(4,)]
 5: [(2, 2, 1)]
 6: [(3, 2, 1)]
 7: [(3, 3, 1)]
 8: [(8,)]
 9: [(9,)]
10: [(10,)]
11: [(11,)]
12: [(5, 5, 2)]
13: [(13,)]
14: [(6, 6, 2)]
15: [(7, 7, 1)]
16: [(9, 6, 1)]
17: [(9, 7, 1)]
18: [(10, 7, 1)]
19: [(5, 5, 5, 3, 1)]
20: [(11, 8, 1)]
21: [(6, 6, 6, 2, 1)]
22: [(7, 7, 6, 1, 1)]
23: [(7, 7, 7, 1, 1)]
24: [(8, 8, 6, 1, 1)]
25: [(5, 5, 5, 5, 3, 1, 1)]
26: [(7, 7, 7, 3, 1, 1)]
27: [(7, 7, 7, 4, 1, 1)]
28: [(6, 6, 6, 6, 2, 1, 1)]
29: [(5, 5, 5, 5, 5, 2, 1, 1)]
30: [(5, 5, 5, 5, 5, 3, 1, 1), (8, 8, 7, 5, 1, 1)]
31: [(7, 7, 7, 7, 1, 1, 1)]
32: [(8, 8, 8, 5, 1, 1, 1)]
33: [(7, 7, 7, 7, 2, 1, 1, 1)]
34: [(7, 7, 7, 7, 3, 1, 1, 1)]
35: [(6, 6, 6, 6, 6, 2, 1, 1, 1)]
36: [(8, 8, 8, 8, 1, 1, 1, 1)]
37: [(8, 8, 8, 8, 2, 1, 1, 1)]
38: [(6, 6, 6, 6, 6, 4, 1, 1, 1, 1)]
39: [(6, 6, 6, 6, 6, 5, 1, 1, 1, 1)]
40: [(6, 6, 6, 6, 6, 6, 1, 1, 1, 1)]
41: [(8, 8, 8, 8, 5, 1, 1, 1, 1)]
42: [(6, 6, 6, 6, 6, 6, 2, 1, 1, 1, 1)]
43: [(6, 6, 6, 6, 6, 6, 3, 1, 1, 1, 1)]
44: [(6, 6, 6, 6, 6, 6, 4, 1, 1, 1, 1)]
45: [(6, 6, 6, 6, 6, 6, 4, 1, 1, 1, 1, 1)]
46: [(6, 6, 6, 6, 6, 6, 5, 1, 1, 1, 1, 1)]
47: [(9, 9, 8, 8, 8, 1, 1, 1, 1, 1)]
48: [(8, 8, 8, 8, 7, 4, 1, 1, 1, 1, 1)]
49: [(49,)]
50: [(28, 22)]
51: [(51,)]
52: [(52,)]
53: [(53,)]
54: [(7, 7, 7, 7, 7, 7, 6, 1, 1, 1, 1, 1, 1)]
55: [(32, 23)]
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56: [(56,)]
57: [(25, 20, 12)]
58: [(36, 22)]
59: [(36, 23)]
60: [(60,)]
61: [(34, 27)]
62: [(39, 23)]
63: [(37, 26)]
64: [(38, 26)]
65: [(39, 26)]
66: [(33, 25, 8), (21, 21, 16, 8)]
67: [(27, 27, 13)]
68: [(27, 27, 14)]
69: [(36, 24, 9)]
70: [(30, 30, 10)]
71: [(30, 30, 11)]
72: [(25, 25, 18, 4)]
73: [(33, 33, 7)]
74: [(40, 30, 4)]
75: [(27, 27, 17, 4)]
76: [(36, 31, 9)]
77: [(20, 20, 20, 13, 4)]
78: [(29, 27, 19, 3)]
79: [(39, 39, 1)]
80: [(21, 21, 21, 13, 4)]
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