Birthday in a class

1) we have $\mathbb{P}_n = \mathbb{P}(C_n \geq 1) = 1 - \mathbb{P}(C_n = 0)$

and $\{C_n = 0\} = \bigcup_{i=1}^{n-1} \{$at least n-tuple and no $(n+1)$ tuple$\}$

so : $\mathbb{P}(\{C_n = 0\}) = \sum_{i=1}^{n-1} p_i$

thus $\boxed{\mathbb{P}_n = 1 - \sum_{i=1}^{n-1} p_i}$

and $\mathbb{P}_n = 1 - \sum_{i=1}^{n-2} p_i - p_{n-1} = \mathbb{P}_{n-1} - p_{n-1}$.

2) For the case $n=2$, we can ask what is the expected number of pairs of students that share the same birthday in our class of 30?

→ Enumerating the students from 1 to 30,
we call E_{ij} the event that students i and j have the same birthday
and $X_{ij} = \mathbb{1}_{E_{ij}} = \begin{cases} 1 & \text{if students } i \text{ and } j \text{ share the same birthday.} \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow E(X_{ij}) = E(\mathbb{1}_{E_{ij}}) = 1 \times \mathbb{P}(E_{ij}) + 0 \times \mathbb{P}(\bar{E}_{ij}) = \mathbb{P}(E_{ij}) = \frac{1}{365}$

thus: $E(C_2) = E\left(\sum_{1 \leq i < j \leq 30} X_{ij}\right) = \sum_{1 \leq i < j \leq 30} E(X_{ij}) = \frac{30 \times 29}{2} \cdot \frac{1}{365}$

(we have used $\sum_{1 \leq i < j \leq 30} = \binom{30}{2} = \frac{30!}{(30-2)!\,2!}$ (the number of unique pairs in a set of size n, subject to the commutative property)).

→ by analogy and by enumerating students from 1 to 30.
and considering the event $E_{i_1 i_2 \ldots i_n}$: the students $i_1 \ldots i_n$ have the same birthday
we get $\boxed{E(C_n) = \binom{30}{n} \frac{1}{365^{n-1}}}$

→ $E(C_2) = \frac{30 \times 29}{2} \times \frac{1}{365} \simeq 1,192.$

→ $E(C_3) = \frac{1}{(365)^2} \times \frac{30 \times 29 \times 28}{3 \times 2} \simeq 0,030$

→ $E(C_4) = \frac{1}{(365)^3} \times \frac{30 \times 29 \times 28 \times 27}{4 \times 3 \times 2} \simeq 0,000563 \simeq 5,6 \times 10^{-4}.$

3) we have: $\{C_2=0\} = \{$ there is no pair of students with the same birthday $\}$
$= \{$ all birthdays of the 30 students are distinct $\}$

$\to$ the total number of possible combinations is $N_{total} = 365^{30}$

and the number of combinasions with distinct dates is :

$$N = 365 \times 364 \times \dots \times (365-29)$$

So $\mathbb{P}(C_2=0) = \dfrac{N}{N_{total}} = \dfrac{365 \times 364 \times \dots \times 236}{365^{30}} \simeq 0{,}294$

and $\mathbb{P}_2 = 1 - \mathbb{P}(C_2=0) = 1 - 0{,}294 \simeq 0{,}706$.

4) we have $N_{total} = 365^{30}$

$P_2 = \mathbb{P}(\{$ at least a pair and no triplet with the same birthday $\})$

and $\{$ at least a pair and no triplets $\} = \bigcup_{i=1}^{15} \{ i$ pairs and no triplets $\}$

$\to$ the number of combinasions with i pairs and not triplet is:

$$\underbrace{\frac{\prod_{j=0}^{i-1} \binom{30-2j}{2}}{i!}}_{\text{choice of the } i\text{-pairs}} \times \underbrace{(365)\times \dots \times (365-29-i)}_{\text{choice of the dates.}}$$

(30-i choice todo of distinct dates)
$\to$ because we choose for 2 students for each pair.

so $$\boxed{P_2 = \frac{1}{365^{30}} \cdot \sum_{i=1}^{15} \frac{\prod_{j=1}^{i} \binom{32-2j}{2}}{i!} \prod_{k=0}^{29-i} (365-k)}$$

5) Please see the last page.

6) $\mathbb{P}_3 = \mathbb{P}_2 - P_2 = 0{,}706 - 0{,}678 \simeq 0{,}025$.

7) $P_3 = \mathbb{P}(\{$ at least a triplet and no 4-tuple have the same birthday $\})$
$= \mathbb{P}(\bigcup_{i=1}^{10} \{ i$-triplet and no 4-tuple $\})$

the corresponding number of combinasions with i triplet and no 4-tuple (quadruplet) is:

$$\underbrace{\frac{\prod_{j=1}^{i}\binom{33-3j}{3}}{i!}}_{\text{choice of the } i\text{-triplets}}\left[\underbrace{\prod_{k=0}^{29-2i}(365-k)}_{\substack{\text{choice of distinct dates for the tirplets}\\ \text{and the rest as a singletons(no pairs)}}}+\underbrace{\sum_{l=1}^{\left[\frac{30-3i}{2}\right]}\frac{\prod_{m=1}^{l}\binom{30-3i+2-2m}{2}}{l!}}_{\text{choice of } l\text{-pairs}}\underbrace{\prod_{n=0}^{29-2i-l}(365-n)}_{29-2i-l \text{ choice of dates.}}\right]$$

- there is a possibility to include $\left[\frac{30-3i}{2}\right]$ pairs.

8)b:

$$P_3=\frac{1}{365^{30}}\sum_{i=1}^{10}\frac{\prod_{j=1}^{i}\binom{33-3j}{3}}{i!}\left[\prod_{k=0}^{29-2i}(365-k)+\sum_{l=1}^{\left[\frac{30-3i}{2}\right]}\frac{\prod_{m=1}^{l}\binom{30-3i+2-2m}{2}}{l!}\prod_{n=0}^{29-2i-l}(365-n)\right]$$

9) we can express P_n as:

$$P_n=\frac{1}{365^{30}}\times\sum_{i=1}^{\left[\frac{30}{n}\right]}\frac{\prod_{j=1}^{i}\binom{(30+n)-n\times j}{n}}{i!}\left[\prod_{k=0}^{29-(n-1)-i}(365-k)+\sum_{l=1}^{\left[\frac{30-n\times i}{2}\right]}\frac{\prod_{m=1}^{l}\binom{30-n\times i+2-2m}{2}}{l!}\prod_{n'=0}^{29-2i-l}(365-n')\right.$$
$$\left.+\dots+\sum_{l=1}^{\left[\frac{30-n\times i}{n-1}\right]}\frac{\prod_{m=1}^{l}\binom{30-n\times i+(n-1)(m-1)}{n-1}}{l!}\cdot\prod_{n'=0}^{29-(n-1)i-(n'-2)\times l}(365-n')\right]$$

10) We have: P_{30} is the probability that all students share the same date.

$P_{30}=EC_{30}=\mathbb{P}(C_{30}=1)$.

→ we have 365 possible combinaisons; so:

$$EC_{30}=P_{30}=\frac{365}{365^{30}}=\frac{1}{365^{29}}.$$

From question ②, we have: $E\,C_{29}=\frac{1}{(365)^{23}}\times\binom{30}{23}=\frac{30}{(365)^{23}}$

→ $P_{29}=\mathbb{P}(C_{29}\geq 1)$:

the number of combinaisons with 29 dates that are similar is:

$$365\times\left(1+\binom{30}{1}\times 364\right)$$

so $$\boxed{P_{29}=\frac{30\times 364+1}{365^{29}}}$$

11) the given table summerizes the results from EC_n and P_n for all values of n.
it shows mainly that the relative difference of EC_n and P_n are more and more smaller as the rank n increases. More general, they give the same orders of

magnitudes. So we can get the same information as the computation of $\mathbb{P}_n$ by computing EC_n that is more simple and intuitive.

12) For $n \geq 3$: $\mathbb{P}(C_n \geq 2)$ becomes negligible with respect to $\mathbb{P}(C_n = 1)$

and
$$E(C_n) = \sum_{k=1} \mathbb{P}(C_n \geq k)$$
$$\simeq \mathbb{P}(C_n \geq 1) + o(\mathbb{P}(C_n \geq 1))$$
$$\simeq \mathbb{P}_n.$$

So EC_n and $\mathbb{P}_n$ are close for $n \geq 3$.

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Question 5 :

```
from scipy.special import comb
import math

def calculate_p2():
    total_sum = 0
    for i in range(1, 16):
        inner_product = 1
        for j in range(1, i+1):
            inner_product *= comb(32-2*j, 2)
        total_sum += (inner_product / math.factorial(i)) * math.prod(range(365, 365-30+i, -1))
    p_2 = 1 / (365**30) * total_sum
    return p_2

result = calculate_p2()
print("p_2 =", result)
```

```
p_2 = 0.677785738974939
```

Question 8 :

```
def binomial_coefficient(n, k):
    return math.factorial(n) // (math.factorial(k) * math.factorial(n - k))

def p_3():
    p_sum = 0
    for i in range(1, 11):
        prod1 = 1
        for j in range(1, i + 1):
            prod1 *= binomial_coefficient(33 - 3 * j, 3)
        prod2 = 1
        for k in range(30 - 2 * i):
            prod2 *= (365 - k)
        sum_term = prod2
        for l in range(1, (30 - 3 * i) // 2 + 1):
            prod3 = 1
            for m in range(1, l + 1):
                prod3 *= binomial_coefficient(30 - 3 * i + 2 - 2 * m, 2)
            prod4 = 1
            for n in range(30 - 2 * i - l):
                prod4 *= (365 - n)
            sum_term += prod3 * prod4 / math.factorial(l)
        p_sum += prod1 * sum_term / math.factorial(i)
    p_3 = 1 / (365 ** 30) * p_sum
    return p_3
print(p_3())
```

```
0.027998224290220405
```