Demostraciones

(a) $\nabla(\phi\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$

Sean $\phi = \phi(x, y, z)$ y $\psi = \psi(x, y, z)$. Por definición del gradiente:

$$\nabla(\phi\psi) = \big(\partial_x(\phi\psi), \partial_y(\phi\psi), \partial_z(\phi\psi)\big).$$

Aplicamos la regla del producto en cada componente:

$$\begin{aligned}
\partial_x(\phi\psi) &= (\partial_x\phi)\psi + \phi(\partial_x\psi),\\
\partial_y(\phi\psi) &= (\partial_y\phi)\psi + \phi(\partial_y\psi),\\
\partial_z(\phi\psi) &= (\partial_z\phi)\psi + \phi(\partial_z\psi).
\end{aligned}$$

Entonces:

$$\nabla(\phi\psi) = \big((\partial_x\phi)\psi + \phi(\partial_x\psi),\ (\partial_y\phi)\psi + \phi(\partial_y\psi),\ (\partial_z\phi)\psi + \phi(\partial_z\psi)\big).$$

Agrupando términos:

$$\nabla(\phi\psi) = \psi\,(\partial_x\phi, \partial_y\phi, \partial_z\phi) + \phi\,(\partial_x\psi, \partial_y\psi, \partial_z\psi).$$

Reconociendo gradientes:

$$\boxed{\nabla(\phi\psi) = \psi\,\nabla\phi + \phi\,\nabla\psi}$$

(d) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$

Sea $\mathbf{a} = (a_1, a_2, a_3)$. El rotacional es:

$$\nabla \times \mathbf{a} = (\partial_y a_3 - \partial_z a_2,\ \partial_z a_1 - \partial_x a_3,\ \partial_x a_2 - \partial_y a_1).$$

Ahora tomamos la divergencia:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{a}) &= \partial_x(\partial_y a_3 - \partial_z a_2) + \partial_y(\partial_z a_1 - \partial_x a_3) + \partial_z(\partial_x a_2 - \partial_y a_1)\\
&= \partial_x\partial_y a_3 - \partial_x\partial_z a_2 + \partial_y\partial_z a_1 - \partial_y\partial_x a_3\\
&\quad + \partial_z\partial_x a_2 - \partial_z\partial_y a_1.
\end{aligned}$$

Agrupamos:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{a}) = {} & (\partial_x\partial_y a_3 - \partial_y\partial_x a_3)\\
& + (\partial_y\partial_z a_1 - \partial_z\partial_y a_1)\\
& + (\partial_z\partial_x a_2 - \partial_x\partial_z a_2).
\end{aligned}$$

Las derivadas mixtas se pueden intercambiar de orden, por lo que cada par se anula:

$$\boxed{\nabla \cdot (\nabla \times \mathbf{a}) = 0}$$

(f) $\nabla \times (\nabla \times \mathbf{a})$

Sea nuevamente $\mathbf{a} = (a_1, a_2, a_3)$.
Primero:

$$\nabla \times \mathbf{a} = (\partial_y a_3 - \partial_z a_2,\ \partial_z a_1 - \partial_x a_3,\ \partial_x a_2 - \partial_y a_1).$$

Calculamos componente a componente.

Componente 1

$$(\nabla \times (\nabla \times \mathbf{a}))_1 = \partial_y((\nabla \times \mathbf{a})_3) - \partial_z((\nabla \times \mathbf{a})_2).$$

Sustituimos:

$$(\nabla \times \mathbf{a})_3 = \partial_x a_2 - \partial_y a_1, \qquad (\nabla \times \mathbf{a})_2 = \partial_z a_1 - \partial_x a_3.$$

Entonces:

$$\begin{aligned}(\nabla \times (\nabla \times \mathbf{a}))_1 &= \partial_y(\partial_x a_2 - \partial_y a_1) - \partial_z(\partial_z a_1 - \partial_x a_3) \\ &= \partial_y \partial_x a_2 - \partial_y^2 a_1 - \partial_z^2 a_1 + \partial_z \partial_x a_3.\end{aligned}$$

Reordenando:

$$(\nabla \times (\nabla \times \mathbf{a}))_1 = \partial_x(\partial_x a_1 + \partial_y a_2 + \partial_z a_3) - (\partial_x^2 a_1 + \partial_y^2 a_1 + \partial_z^2 a_1).$$

Componente 2

$$\begin{aligned}(\nabla \times (\nabla \times \mathbf{a}))_2 &= \partial_z(\partial_y a_3 - \partial_z a_2) - \partial_x(\partial_x a_2 - \partial_y a_1) \\ &= \partial_z \partial_y a_3 - \partial_z^2 a_2 - \partial_x^2 a_2 + \partial_x \partial_y a_1.\end{aligned}$$

Reordenando:

$$(\nabla \times (\nabla \times \mathbf{a}))_2 = \partial_y(\partial_x a_1 + \partial_y a_2 + \partial_z a_3) - (\partial_x^2 a_2 + \partial_y^2 a_2 + \partial_z^2 a_2).$$

Componente 3

$$\begin{aligned}(\nabla \times (\nabla \times \mathbf{a}))_3 &= \partial_x(\partial_z a_1 - \partial_x a_3) - \partial_y(\partial_y a_3 - \partial_z a_2) \\ &= \partial_x \partial_z a_1 - \partial_x^2 a_3 - \partial_y^2 a_3 + \partial_y \partial_z a_2.\end{aligned}$$

Reordenando:

$$(\nabla \times (\nabla \times \mathbf{a}))_3 = \partial_z(\partial_x a_1 + \partial_y a_2 + \partial_z a_3) - (\partial_x^2 a_3 + \partial_y^2 a_3 + \partial_z^2 a_3).$$

Juntando las tres componentes:

$$\nabla \times (\nabla \times \mathbf{a}) = \big(\partial_x(\nabla \cdot \mathbf{a}), \partial_y(\nabla \cdot \mathbf{a}), \partial_z(\nabla \cdot \mathbf{a})\big) - \begin{pmatrix} \partial_x^2 a_1 + \partial_y^2 a_1 + \partial_z^2 a_1 \\ \partial_x^2 a_2 + \partial_y^2 a_2 + \partial_z^2 a_2 \\ \partial_x^2 a_3 + \partial_y^2 a_3 + \partial_z^2 a_3 \end{pmatrix}.$$