Mini HW #10

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(1)

if $\frac{size}{cap} = 1$ or $\frac{size}{cap} = \frac{1}{2}$, push() = pop() = O(size)

else push() = pop() = O(1)

(2)

let $\alpha = \frac{size}{cap}$, and define

$$\phi(D_i) = \begin{cases} 2 \times size - cap & \text{if } \alpha \geq \frac{1}{2} \\ \frac{cap}{2} - size & \text{if } 0 < \alpha < \frac{1}{2} \end{cases}$$

if cap = size = 0, $\alpha = 1$.

$\because$ we always have

$$size = \alpha \times cap$$

$\therefore \phi(D_i) \geq 0$

push():

if $(1 \geq \alpha_i \geq \frac{1}{2})$ but $(0 < \alpha_{i-1} < \frac{1}{2})$, then

$$\begin{aligned} \widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\ &= 1 + (2 \times size_i - cap_i) - (\frac{cap_{i-1}}{2} - size_{i-1}) \\ &= 1 + (2(size_{i-1} + 1) - cap_{i-1}) - (\frac{cap_{i-1}}{2} - size_{i-1}) \\ &= 3 \times size_{i-1} - \frac{3}{2}cap_{i-1} + 3 \\ &= 3 \times \alpha_{i-1} \times cap_{i-1} - \frac{3}{2}cap_{i-1} + 3 \\ &< 3 \end{aligned}$$

else if ($\alpha_{i-1} = 1$ and $\alpha_i = \frac{1}{2}$), then

$$\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= size_i + (2 \times size_i - cap_i) - (2 \times size_{i-1} - cap_{i-1}) \\
&= (size_{i-1} + 1) + (2(size_{i-1} + 1) - 2 \times cap_{i-1}) - (2 \times size_{i-1} - cap_{i-1}) \\
&= size_{i-1} - cap_{i-1} + 3 \\
&= 3
\end{aligned}$$

else

$$\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= 1 + (\frac{cap_i}{2} - size_i) - (\frac{cap_{i-1}}{2} - size_{i-1}) \\
&= 1 + (\frac{cap_i}{2} - size_i) - (\frac{cap_i}{2} - (size_i - 1)) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= 1 + (2 \times size_i - cap_i) - (2 \times size_{i-1} - cap_{i-1}) \\
&= 1 + (2 \times size_i - cap_i) - (2(size_i - 1) - cap_i) \\
&= 3
\end{aligned}$$

The amortized cost of a **push()** is at most 3.

pop():

if ($\alpha_{i-1} \geq \frac{1}{2}$) but ($\frac{1}{4} < \alpha_i < \frac{1}{2}$), then

$$\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= 1 + (\frac{cap_i}{2} - size_i) - (2 \times size_{i-1} - cap_{i-1}) \\
&= 1 + (\frac{cap_{i-1}}{2} - (size_{i-1} - 1)) - (2size_{i-1} - cap_{i-1}) \\
&= (-3) \times size_{i-1} + \frac{3}{2}cap_{i-1} + 2 \\
&= (-3) \times \alpha_{i-1} \times cap_{i-1} + \frac{3}{2}cap_{i-1} + 2 \\
&< 2
\end{aligned}$$

else if ($\alpha_{i-1} = \frac{1}{4}$ and $\alpha_i = \frac{1}{2}$), then

$$
\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= (size_i + 1) + (2 \times size_i - cap_i) - (\frac{cap_{i-1}}{2} - size_{i-1}) \\
&= (size_i + 1) + (2 \times size_i - cap_i) - (cap_i - (size_i + 1)) \\
&= 4 \times size_i - 2 \times cap_i + 2 \\
&= 2
\end{aligned}
$$

else

$$
\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= 1 + (\frac{cap_i}{2} - size_i) - (\frac{cap_{i-1}}{2} - size_{i-1}) \\
&= 1 + (\frac{cap_i}{2} - size_i) - (\frac{cap_i}{2} - (size_i + 1)) \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\widehat{c_i} &= c_i + \phi(D_i) - \phi(D_{i-1}) \\
&= 1 + (2 \times size_i - cap_i) - (2 \times size_{i-1} - cap_{i-1}) \\
&= 1 + (2(size_{i-1} - 1) - cap_{i-1}) - (2 \times size_{i-1} - cap_{i-1}) \\
&= -1
\end{aligned}
$$

The amortized cost of a **pop()** is at most 2.

Thus, the time complexity of **push()**, **pop()** are amortized O(1).