Hw3 Solution

Q05:

Solution:

According to the Second Law of Thermodynamics, the entropy of the (closed) system of the blocks will increase until it reaches a maximum. Thus, the entropy of the system at any time after the blocks come into contact will be greater than $S_1 + S_2$. Thus, $S > 45$ J/K.

P16:

Solution:

1100, 1010, 1001, 0110, 0101, 0011, 2000, 0200, 0020, 0002.

There are 10 arrangements of the quanta. Check this with:

$$\begin{aligned} \Omega &= \frac{(q+N-1)!}{q!(N-1)!} \\ &= \frac{5!}{2!3!} = \frac{20}{2} = 10 \end{aligned}$$

This confirms that we thought of all possible arrangements.

P22:

Solution:

(a) The number of ways Ω to arrange q quanta among N one-dimensional oscillators is

$$\begin{aligned} \Omega &= \frac{(q+N-1)!}{q!(N-1)!} \\ &= \frac{(4+6-1)!}{4!(6-1)!} \\ &= 126 \end{aligned}$$

(b) The probability of being in the state 000004 is $1/126$. If you have 48,000 objects, then the number of objects in the state 000004 is approximately $(1/26) * 48,000 = 381$.

P25:

Solution:

(a)

$$\begin{aligned} S &= kln(\Omega) \\ &= \left(1.381 \times 10^{-23}\ \frac{\text{J}}{\text{K}}\right) \ln(60) \\ &= 5.65 \times 10^{-23}\ \text{J/K} \end{aligned}$$

(b)

$$\begin{aligned} S &= kln(\Omega) \\ &= \left(1.381 \times 10^{-23}\ \frac{\text{J}}{\text{K}}\right) \ln(122) \\ &= 6.62 \times 10^{-23}\ \text{J/K} \end{aligned}$$

(c) At any time later than this, $S > S_1 + S_2 = 1.23 \times 10^{-23}$ J/K. Thus, at this instant, $S = S_A + S_B = 1.23 \times 10^{-23}$ J/K.

P27:

Solution:

Since ΔT is small, assume $T \approx T_i$.

For copper $T_i = 50°\text{C} + 273\text{K} = 323$ K.

For aluminum $T_i = 45°\text{C} + 273\text{K} = 318$ K.

(a)

$$\begin{aligned}\Delta S_{\text{Al}} &= \frac{Q}{T} \\ &\approx \frac{2500\text{ J}}{318\text{ K}} \\ &\approx 7.86\text{ J/K}\end{aligned}$$

(b)

$$\begin{aligned}\Delta S_{\text{Cu}} &= \frac{Q}{T} \\ &\approx \frac{-2500\text{ J}}{323\text{ K}} \\ &\approx -7.74\text{ J/K}\end{aligned}$$

(c) The container is insulated. Therefore, there is no interaction with the surroundings, and it is a closed system. Thus, the "Universe" in this case is the system of the aluminum and copper blocks.

$$\begin{aligned}\Delta S_{\text{sys}} &= \Delta S_{\text{Al}} + \Delta S_{\text{Cu}} \\ &= 7.86\text{ J/K} - 7.74\text{ J/K} \\ &= 0.12\text{ J/K}\end{aligned}$$

The entropy of the Universe increased as expected according to the Second Law of Thermodynamics.

(d) The change in the energy of the Universe is

$$\begin{aligned}\Delta E_{\text{sys}} &= \Delta E_{\text{Al}} + \Delta E_{\text{Cu}} \\ &= 2500\text{ J} + -2500\text{ J} \\ &= 0\end{aligned}$$

The energy of the Universe is constant, as expected from the First Law of Thermodynamics.

P31:

Solution:

The specific heat capacity per atom is

$$C = \frac{1}{N_{\text{atoms}}} \frac{dE_{\text{sys}}}{dT}$$

From Problem 29, $E = \frac{1}{4}a^2T^2$. Take the derivative of E with respect to T.

$$\begin{aligned} E &= \frac{1}{4}a^2T^2 \\ \frac{dE}{dT} &= \frac{1}{2}a^2T \end{aligned}$$

Substitute this into the expression for C.

$$\begin{aligned} C &= \frac{1}{N_{\text{atoms}}} \frac{dE_{\text{sys}}}{dT} \\ &= \frac{1}{N_{\text{atoms}}} \frac{1}{2}a^2T \end{aligned}$$

This is the specific heat capacity in J/K/atom.

P33:

Solution:

(a) The energy of one quantum is also the change in energy of the system when it gains one quantum of energy, so $\Delta E = 4 \times 10^{-21}$ J.

For 5 quanta and 18 oscillators, the number of ways to distribute the energy is

$$\begin{aligned}\Omega &= \frac{(5+18-1)!}{5!17!} \\ &= 26334\end{aligned}$$

and the entropy is

$$\begin{aligned}S &= k\ln(\Omega) \\ &= (1.381 \times 10^{-23}\ \frac{\mathrm{J}}{\mathrm{K}})\ln(26334) \\ &= 1.4054 \times 10^{-22}\ \mathrm{J/K}\end{aligned}$$

For 6 quanta distributed among the same number of oscillators, the number of ways to distribute the energy is

$$\begin{aligned}\Omega &= \frac{(6+18-1)!}{6!17!} \\ &= 100947\end{aligned}$$

and the entropy is

$$\begin{aligned}S &= k\ln(\Omega) \\ &= (1.381 \times 10^{-23}\ \frac{\mathrm{J}}{\mathrm{K}})\ln(100947) \\ &= 1.591 \times 10^{-22}\ \mathrm{J/K}\end{aligned}$$

The change in entropy for a change in energy of one quanta is $\Delta S = 1.591 \times 10^{-22}$ J/K $- 1.4054 \times 10^{-22}$ J/K $=$ 1.856×10^{-23} J/K.

The temperature is approximately

$$\begin{aligned}\frac{1}{T} &\approx \frac{\Delta S}{\Delta E_{\mathrm{int}}} \\ T &\approx \frac{\Delta E_{\mathrm{int}}}{\Delta S} \\ &\approx \frac{4 \times 10^{-21}\ \mathrm{J}}{1.856 \times 10^{-23}\ \mathrm{J/K}} \\ &\approx 215.5\ \mathrm{K}\end{aligned}$$

(b) Repeat the above calculations for 8 and 9 quanta of energy. The change in energy (1 quantum) is the same, but we need to calculate the change in entropy.

For 8 quanta, $\Omega = 1,081,575$ and $S = 1.9183 \times 10^{-22}$ J/K.

For 9 quanta, $\Omega = 3,124,550$ and $S = 2.0648 \times 10^{-22}$ J/K.

This gives an approximate temperature of $T \approx \frac{4\times10^{-21}\text{ J}}{1.465\times10^{-23}\text{ J/K}} = 273.1$ K.

(c) The heat capacity (per atom) is

$$C = \frac{\Delta E_{\text{atom}}}{\Delta T}$$

where $\Delta E_{atom} = \frac{\Delta E_{\text{system}}}{N_{\text{atoms}}}$. The system in this case is the nanoparticle consisting of 6 atoms, and the change in the energy of the nanoparticle is in steps of one quantum of energy, $\Delta E_{\text{nanoparticle}} = (8.5-5.5)(4\times10^{-21}\text{ J}) = 1.2\times10^{-20}$ J. So, the change in the energy of an atom is $(1/6)(1.2 \times 10^{-20}\text{ J}) = 2 \times 10^{-21}$ J.

The heat capacity in this range of energies is approximately

$$\begin{aligned} C &= \frac{\Delta E_{\text{atom}}}{\Delta T} \\ &= \frac{2 \times 10^{-21}\text{ J}}{(273.1 - 215.5)\text{ K}} \\ &= 3.47 \times 10^{-23}\text{ J/K/atom} \end{aligned}$$

This is less than the high temperature limit of $3k = 4.2 \times 10^{-23}$ J/K as expected.

P41:

Solution:

(a) Apply the Momentum Principle, treating the system as a single particle located at its center of mass.

$$\vec{F}_{net} = \frac{\Delta\vec{p}}{\Delta t}$$

This a vector equation, so write it in component form. Only the x-direction is needed. $\vec{F}_{net} =< F, 0, 0 >$, $\vec{v}_i =< v_i, 0, 0 >$, and $\vec{v}_f =< v_f, 0, 0 >$. Thus, in the x-direction.

$$\begin{aligned} F &= \frac{(M+m)v_f - (M+m)v_i}{\Delta t} \\ v_f &= v_i + \frac{F}{(M+m)}\Delta t \end{aligned}$$

(b) Apply the Angular Momentum Principle to the wheel. Since the wheel is a cylinder rotating about its center of mass, its moment of inertia about this rotation axis is $I = \frac{1}{2}MR^2$. Define the +z direction to be out of the page. The torque on the wheel due to F is in the –z direction, the torque on the wheel due to f is in the +z direction, and the initial and final angular velocities of the wheel are in the –z direction. Apply the Angular Momentum Principle to the wheel about the z axis.

$$\begin{aligned} \tau_{net,z} &= \frac{\Delta L_z}{\Delta t} \\ -FR + fR &= \frac{I(-\omega_f) - I(-\omega_i)}{\Delta t} \\ -(F+f)R\Delta t &= -I\omega_f + I\omega_i \\ \omega_f &= \frac{I\omega_i + (F+f)R\Delta t}{I} \\ &= \omega_i + \frac{(F+f)R\Delta t}{\frac{1}{2}MR^2} \\ &= \omega_i + \left(\frac{2(F+f)}{MR}\right)\Delta t \end{aligned}$$

(c) Apply the Energy Principle.

First, consider the point-particle system. Treat the system of box, wheel, and block as a single particle located at its center of mass. Then the Energy Principle applied to this particle is

$$\begin{aligned}\Delta K_{trans} &= W \\ \Delta K_{trans} &= F|\Delta\vec{r}_{cm}| \\ \Delta K_{trans} &= Fx\end{aligned}$$

Next, consider the actual system of the box, wheel, and block. The system gains translational kinetic energy, rotational kinetic energy, and thermal energy. The work done is the dot product of the applied external force and the displacement through which the applied force acts.

$$\begin{aligned}W &= \Delta K_{trans} + \Delta K_{rot} + \Delta E_{therm} \\ Fd &= Fx + \left(\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2\right) + \Delta E_{therm} \\ \Delta E_{therm} &= F(d-x) - \frac{1}{2}I(\omega_f^2 - \omega_i^2) \\ &= F(d-x) - \frac{1}{4}MR^2(\omega_f^2 - \omega_i^2)\end{aligned}$$

(d) Use the definition of specific heat capacity to calculate the change in temperature of the block and wheel.

$$\begin{aligned}C &= \frac{\Delta E_{atom}}{\Delta T} \\ &= \frac{\Delta E_{sys}/N_{atoms}}{\Delta T}\end{aligned}$$

The number of atoms in the block and wheel is

$$\begin{aligned}N_{atoms} &= (1.2\text{ kg})\left(\frac{1\text{ mol}}{0.0558\text{ kg}}\right)\left(6.022\times10^{23}\text{ mol}^{-1}\right) \\ &= 1.30\times10^{25}\text{ atoms}\end{aligned}$$

Estimate C using $C = 3(k) = 4.14\times10^{-23}$ J/K/atom.

The change in temperature of the system for the given change in thermal energy is

$$\begin{aligned}C &= \frac{\Delta E_{sys}/N_{atoms}}{\Delta T} \\ \Delta T &= \frac{8\times10^4\text{ J}/1.3\times10^{25}\text{ atoms}}{4.14\times10^{-23}\text{ J/K/atom}} \\ &= 149\text{ K}\end{aligned}$$

Thus, the final temperature is $T_f = 350\text{ K} + 149\text{ K} = 499\text{ K}$.

P47:

Solution:

The mass of a helium atom is $\left(\frac{0.004\ \text{kg}}{\text{mol}}\right)\left(\frac{1\ \text{mol}}{6\times10^{23}\ \text{atoms}}\right) = 6.7\times10^{-27}$ kg.

v_{rms} at room temperature is

$$\begin{aligned}\bar{v^2} &= \frac{3\ kT}{m} = \frac{3\left(1.381\times10^{-23}\ \frac{\text{J}}{\text{K}}\right)(293\ \text{K})}{6.7\times10^{-27}\ \text{kg}} = 1.81\times10^{6}\ \text{m}^2/\text{s}^2\\ v_{rms} &= \sqrt{\bar{v^2}} = 1350\ \text{m/s}\end{aligned}$$

The mass of a nitrogen molecule is

$$m = \left(\frac{0.028\ \text{kg}}{\text{mol}}\right)\left(\frac{1\ \text{mol}}{6\times10^{23}\ \text{atoms}}\right) = 4.67\times10^{-26}\ \text{kg}$$

v_{rms} at room temperature is

$$\begin{aligned}\bar{v^2} &= \frac{3\ kT}{m} = \frac{3\left(1.381\times10^{-23}\ \frac{\text{J}}{\text{K}}\right)(293\ \text{K})}{4.67\times10^{-26}\ \text{kg}} = 2.6\times10^{5}\ \text{m}^2/\text{s}^2\\ v_{rms} &= \sqrt{\bar{v^2}} = 510\ \text{m/s}\end{aligned}$$

P54:

Solution:

(a) An atom in a monatomic gas only has translational kinetic energy. Thus, the average translational kinetic energy per atom is

$$\overline{K}_{trans} = \frac{3}{2}kT$$

If $Q = 580$ J of energy is added to 1 mole of gas, then the average increase in energy per atom is 580 J$/6.022\times10^{23}$ mol^{-1} 9.63×10^{-22} J. The change in temperature corresponding to this increase in energy per atom is

$$\begin{aligned}\Delta T &= \frac{9.63\times10^{-22}\ \text{J}}{\frac{3}{2}k}\\ &= \frac{9.63\times10^{-22}\ \text{J}}{\frac{3}{2}(1.381\times10^{-23}\ \frac{\text{J}}{\text{K}})}\\ &= 46.5\ \text{K}\end{aligned}$$

(b) The rotational energy of the gas contributes $\frac{2}{2}k$ to the specific heat of the gas. Thus, the specific heat at high temperatures will be approximately $C = \frac{3}{2}k + \frac{2}{2}k = \frac{5}{2}k$.

$$\begin{aligned}C &= \frac{\Delta E_{atom}}{\Delta T}\\ \Delta T &= \frac{\Delta E_{atom}}{C}\\ &= \frac{9.63\times10^{-22}\ \text{J}}{\frac{5}{2}(1.381\times10^{-23}\ \frac{\text{J}}{\text{K}})}\\ &= 27.9\ \text{K}\end{aligned}$$

Note that this is 3/5 (i.e. $\frac{3/2}{5/2}$) of the change in temperature for the monatomic gas.

(c) The rotational energy terms contribute $\frac{2}{2}k$ to the specific heat of the gas, and the vibrational energy terms contribute $\frac{2}{2}k$. Thus, the specific heat at high temperatures will be approximately $C = \frac{3}{2}k + \frac{2}{2}k + \frac{2}{2}k = \frac{7}{2}k$.

This results in a change in temperature that is $3/7$ (i.e. $\frac{3/2}{7/2}$)of the change in temperature for the monatomic gas. Therefore, $\Delta T = 3/7(46.5 \text{ K}) = 19.9 \text{ K}$.

Examine the results. Since the specific heat for a diatomic gas is larger than a monatomic gas, its temperature increases less for a given increase in thermal energy, which is consistent with the calculations.