Constructions of Čech and their applications to fixed point theorems

Yury Elkin

Contents

Chapter 1

Introduction

Chapter 2

Background

In this chapter we will recall a few details about algebra, category theory and abstract simplicial complexes. We assume that the reader knows the basics of singular homology and cohomology theory and constructions related to them.
We will implement structures like modules, vector spaces and groups from the algebra and we assume that the reader is familiar with this kind of structures. Modules play central role as we are interested in homology as the groups with coefficients G can be interpreted as G-modules. In the applications we will use homology groups with coefficients in $\mathbb{Q}$, which form a vector space structure. Thus some theory from linear algebra will be used.

2.1 Direct sum and direct product

In this section we will recall a few details about direct sums and direct products. In this section we will present the theory only for modules. The constructions and theorems for the abelian groups can be obtained by forgetting the multiplicative operation.

Definition 2.1. Let $\{A_i\}_{i\in J}$ be a family of R-modules. The direct product of these groups is the Cartesian product $\prod_{i\in J} A_i$ where addition and a multiplicative operation are defined component-wise:

$$(a+b)_i = a_i +_i b_i \text{ and } (r \cdot a)_i = r \cdot a_i$$

Definition 2.2. Let $\{A_i\}_{i\in J}$ be a family of R-modules. The direct sum of these modules is the submodule of direct product given by

$$\bigoplus_{i\in J} A_i = \{x \in \prod_{i\in J} A_i \mid x_i \neq 0 \text{ for only finetely many } i\}.$$

For direct sums and direct products we have following universal properties:

Lemma 2.3. *Let $A = \prod_{i\in J} A_i$ be the direct sum of R-modules and let D be an arbitrary R-module. Then for every set of module homomorphisms $\{f_i : D \to A_i\}_{i\in J}$ there exists a unique module homomorphism $f : D \to A$ for which the condition $pr_i \circ f = f_i$ holds for every $i \in J$.*

Proof. Proved in [7] theorem 3.7. □

Lemma 2.4. *Let $A = \bigoplus_{i\in J} A_i$ be the direct sum of R-modules and D an R-module. Then for every set of module homomorphisms $\{f_i : A_i \to D\}_{i\in J}$ there exists a unique module homomorphism $f : A \to D$ for which the condition $f \circ j_i = f_i$ holds for every $i \in J$.*

Proof. Proved in [7] theorem 3.6. □

2.2 Category theory

In this thesis we will present theory in a categorical way, which will make the construction more general. First lets recall the definition of a category.

Definition 2.5. A category C consist of the following ingredients: A class of objects $\mathrm{ob}(C)$, a class of morphisms $\hom(C)$ for which and for every objects A, B in $\mathrm{ob}(C)$ there exists a subclass $Hom(A, B)$ and a rule of composition $Comp : Hom(A, B) \times Hom(B, D) \to Hom(A, D)$ for which following conditions hold:

(1) Composition is associative. Let $f : A \to B$, $g : B \to D$ and $h : D \to E$ be morphisms between objects then $(f \circ g) \circ h = f \circ (g \circ h)$.

(2) For every object $A \in \mathrm{ob}(C)$ there exists identity morphism $1_A \in \hom(A, A)$ for which the following condition holds: Let $f : A \to B$ and $g : B \to A$ be arbitrary morphisms then $f \circ 1_A = f$ and $1_A \circ g = g$.

Definition 2.6. We say that a category D is a subcategory of category C, if the following conditions are satisfied:

(1) The collection of objects $\mathrm{ob}(D)$ is a subcollection of $\mathrm{ob}(C)$.

(2) The collection of morphisms $\hom(D)$ is a subcollection of $\hom(D)$.

Definition 2.7. Let C be a category and A, B its objects and let $f : A \to B$ be some morphism between them. We say that f is an isomorphism, if there exists a morphism $g : B \to A$ for which the equations $g \circ f = id_A$ and $f \circ g = id_B$ hold.

It is easy to see that topological spaces with continuous functions and abelian groups with homomorphisms form a category. We denote those categories by TOP and AB. Next we will define the concept of a functor between categories.

Definition 2.8. Let C and C' be categories and let F be a map between them. We say that F is a functor between categories if the following conditions hold:

(1) For every object $A \in \mathrm{ob}(C)$ there exists a unique object $F(A) \in C'$.

(2) Let A and B be objects in $\mathrm{ob}(C)$ and $f : A \to B$ a morphism between them. Then there exists a unique morphism $F(f) : F(A) \to F(B)$.

(3) If f and g are morphisms in $\hom(C)$, then the following condition holds: $F(f) \circ F(g) = F(f \circ g)$.

(4) The identity element is mapped to the identity element. Let A be an object in $\mathrm{ob}(C)$ and 1_A the identity morphism corresponding to it. Then $F(1_A)$ is the identity morphism of $F(A)$.

Definition 2.9. We say that a functor $F : C \to D$ is injective, if for every two distinct object a and b in category C , the functor F maps them to distinct objects in category D.

Singular homology and cohomology groups form a functor from TOP to AB. For details see Rotman [1]. Now we will introduce a new concept called natural transformation.

Definition 2.10. Let C and D be categories and let $F : C \to D$ and $G : C \to D$ be functors. Then the following family of functors is a natural transformation:

$$\{\phi_X : F(X) \to G(X)\}_{X \in C}$$

if the following conditions hold:

(1) For every object $X \in C$ there exists a unique morphism $\phi_X : F(X) \to G(X)$

(2) For every morphism $f : X \to Y$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\phi_X"'] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\phi_Y"'] & G(Y) \end{tikzcd}$$

or in other words $\phi_Y \circ F(f) = G(f) \circ \phi_X$ holds.

Next we will define the concepts of limit and colimit for arbitrary categories, which will be later applied to direct and inverse systems.

Definition 2.11. Let C and J be categories. We say that the category C is indexed by the category J if there exists a functor $F : J \to C$.

We call the functor F described above an indexing functor. It is easy to see that the functor induces a subcategory $F(J)$ of C. In the case where the indexing functor F is injective and we can equate $F(J)$ with J.

Definition 2.12. Let C be a category indexed by J and let $F : J \to C$ be the indexing functor. Let N be a fixed object of the category C. We define the cone from N to F to be an indexed family of morphisms

$$\Omega_N = \{\omega_X : N \to F(X)\}_{X \in J}$$

which satisfies the following property: If $f : X \to Y$ is a morphism in C then the following diagram commutes

$$\begin{tikzcd} & N \arrow[dl, "\omega_X"'] \arrow[dr, "\omega_Y"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$

From now on we will denote the cone structure described above by $\triangle(F, N, \Omega_N)$.

Definition 2.13. Let C be a category which is indexed by a category J and let F be the indexing functor. We say that the object $D \in \mathrm{ob}(C)$ is a limit of the functor F if the following conditions are satisfied:

(1) There exists a family of morphisms Ω_D which induce the cone structure $\triangle(F, D, \Omega_D)$.

(2) If $\triangle(F, N, \phi)$ is any other cone structure, then there exists a unique morphism $u : N \to D$ which makes the following diagram commute.

$$\begin{tikzcd} & N \arrow[d, dashed] \arrow[ddl, "\omega_X"'] \arrow[ddr, "\omega_Y"] & \\ & D \arrow[dl, "\phi_X"] \arrow[dr, "\phi_Y"'] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$

Next we will prove that if we have two limits N and D of a functor F then there exists an isomorphism between those two objects. In other words N and D can be identified and the limit of the functor F can be denoted simply as $\varinjlim C$.

Theorem 2.14. *Let $F : J \to C$ be an indexing functor. Let (N, ω) and (D, ϕ) be limits of a functor F. Then there exists an isomorphism between them.*

Proof. Because N and D are both limits there exist morphisms $u : N \to D$ and $v : D \to N$ as above. Because of the symmetry it is enough to show that $v \circ u = id_N$. This follows directly from the equation $\omega_X \circ v \circ u = \phi_X \circ u = \omega_X$. Because the diagram commutes for id_N then by the uniqueness condition we see that $v \circ u = id_N$. □

2.2.1 Dual category

To simplify definitions we will define the concept of a dual category.

Definition 2.15. The dual of any concept in category theory can be obtained in the following way:

(1) Interchange every occurrence of source with target

(2) Reorder every composition. That is, replace $a \circ b$ with $b \circ a$.

Next we will give a few important examples of duality.

Definition 2.16. Let $F : J \to C$ be the indexing functor. We define cocone to be the dual structure of the cone.

We will denote the cocone over an object N as $\bigtriangledown(F, N, \Omega_N)$

Definition 2.17. Let C be a category which is indexed by a category J and let F be the indexing functor. We say that the object $D \in \mathrm{ob}(C)$ is a colimit of the functor F if the following conditions are satisfied:

(1) There exists a family of morphisms Ω_D which induces a cocone structure $\bigtriangledown(F, D, \Omega_D)$.

(2) If $\bigtriangledown(F, N, \phi)$ is any other cocone structure, then there exists a unique morphism $u : D \to N$ which makes the following diagram commute.

$$\begin{tikzcd}
 & N & \\
 & D \arrow[u, dashed] & \\
F(X) \arrow[uur, "\phi_X"] \arrow[ur, "\omega_X"'] & & F(Y) \arrow[uul, "\phi_Y"'] \arrow[ul, "\omega_Y"] \arrow[ll, "F(f)"]
\end{tikzcd}$$

Like in the case of limits, colimits are unique.

Theorem 2.18. *Let F be an indexing functor and let (N, ϕ) and (D, ω) be colimits of F. Then there exists an isomorphism between the objects N and D.*

Proof. Because N and D are both colimits there exist morphisms $u : N \to D$ and $v : D \to N$ as above. Because of the symmetry it is enough to show that $u \circ v = id_D$ holds. This follows from the equation $u \circ v \circ \omega_X = u \circ \phi_X = \omega_X$. The diagram commutes for id_D and thus by the uniqueness condition we see that $u \circ v = id_D$. $\square$

2.3 Systems

2.3.1 Construction

In this section we will give a categorical definition of direct and inverse systems. It appears that those concepts are dual to each other. We will begin by defining a quasi-ordering relation:

Definition 2.19. Let λ be a category and let $a \leqslant b$ be a relation in the object class $\mathrm{ob}(\lambda)$. We say that this relation is a quasi-ordering if the following conditions hold:

(1) $a \leqslant a$ for all $a \in \mathrm{ob}(\lambda)$.

(2) If $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ for all $a, b, c \in \mathrm{ob}(\lambda)$.

Definition 2.20. Let λ be a category with a quasi-order relation. We say that the collection of morphisms correspond to the relation, if the classes $\hom(a, b)$ are of the following form:

$$\hom(a,b) = \begin{cases} \{\lambda_b^a(i)\}_{i \in I(a,b)} & \text{if } b \leqslant a \\ \varnothing & \text{else.} \end{cases}$$

We assumed that λ is a category, so every $\lambda_b^a(i) \circ \lambda_c^b(j)$ is well-defined. We denote the composition as $\lambda_c^a(i * j)$, where $*$ is operator of the indexation.

2.3.1.1 Inverse systems

Definition 2.21. Let λ be a quasi-ordered category with the corresponding class of morphisms. Let C be a category and let $I : \lambda \to C$ be a covariant functor for which the following conditions are satisfied:

(1) For every non-empty class of morphisms $\hom(a, b)$ there corresponds a morphism I_b^a in the category C for which $I(\hom(a, b)) = \{I_b^a\}$ holds.

(2) Every element in the class of morphisms of the form $\hom(a, a)$ is mapped to the morphism $id_{I(a)}$

Then we say that I is an inverse system indexed by the category λ.

Lemma 2.22. *Inverse system functors are well-defined.*

Proof. Let (I, λ) be arbitrary inverse system, then following conditions hold:

(1) Combining conditions (1) and (2) of the definition 2.21 and using the fact that every identity morphism can be found in collection $\hom_\lambda(a, a)$ we see that the condition $I(id_a) = id_{I(a)}$ is satisfied.

(2) The condition $I(\lambda^a_b(i)) \circ I(\lambda^b_c(j)) = I^a_b \circ I^b_c = I^a_c = I(\lambda^a_c(i * j)) = I(\lambda^a_b(i) \circ \lambda^b_c(j))$

□

Earlier we defined the concept of limit for general indexing functors. We can interpret inverse system as indexing functors, taking λ to be the indexing set. In general systems the limit may not exist. However, for our purpose it is enough to show that such limit exist in the categories of modules.

Theorem 2.23. *Let $I : \lambda \to AB$ be an inverse system of R-modules. Then the limit of this functor is the submodule $L = \{x \in \prod_{i \in \mathrm{ob}(\lambda)} I_i \mid x_a = I^b_a x_b$ for all $a \leqslant b\}$*

Proof. Let v be a family of module homomorphisms $\{v_a = pr_a : L \to I(a)\}_{a \in \lambda}$ which together with the group L induce the cone $\triangle(I, L, v)$. Let K be any other group and let ϕ be a family of module homomorphisms $\{\phi_a : K \to I(a)\}_{a \in \lambda}$ which induces the cone $\triangle(I, K, \phi)$.
We construct a module homomorphism $u : K \to L$ for which $v_a \circ u = \phi_a$ holds. Using lemma 2.3 we find a unique homomorphism $u' : K \to \prod_{i \in J} I_i$ for which $pr_a \circ u' = \phi_a$ is satisfied. Let u'' be the homomorphism, where the image is restricted to subgroup L. We will show that the homomorphism is well-defined, specifically u' maps every element inside the subgroup L. Assume that b and a are elements which satisfy the condition $a \leqslant b$. Then the following equation holds

$$I^b_a \circ pr_b \circ u' = I^b_a \circ \phi_b = \phi_a = pr_a \circ u'.$$

We can define u to be u''. The uniqueness property of the homomorphism follows directly from the fact that u' is unique. □

2.3.1.2 Directed systems

Definition 2.24. Let λ be a quasi-ordered category with the corresponding class of morphisms. Let C be a category and let $I : \lambda \to C$ be a contra-variant functor for which the following conditions are satisfied:

(1) For every non-empty class of morphisms $\hom(a, b)$ there corresponds a morphism D_a^b in the category C for which $D(\hom(a, b)) = \{D_a^b\}$ holds.

(2) Every element in the class of morphisms of the form $\hom(a, a)$ is mapped to the morphism $id_{I(a)}$

Then we say that D is a directed system indexed by the category λ.

Lemma 2.25. *Directed system functors are well-defined.*

Proof. The proof of this claim is almost identical to proof of Lemma 2.22. □

Example 2.26. Let SET be the category of sets with order relation defined by $U \leqslant V \Leftrightarrow U \subset V$. Now for every U, V we define $D_U^V : U \to V$ to be the inclusion from U to V. Clearly this forms a direct system.

Theorem 2.27. *Let $D : \lambda \to AB$ be a direct system of R-modules. For every $a \in \mathrm{ob}(\lambda)$ let $i_a : D(a) \to \bigoplus_{a \in \mathrm{ob}(\lambda)} D(a)$ be the inclusion and let G be the submodule of $D = \bigoplus_{a\in\lambda} D(a)$ which is generated by elements of the form $\{r \cdot (i_a x_a - i_b D_a^b x_a)\}$, where $x \in D$ and $r \in R$. Then the colimit of this system is the quotient R-module $L = \bigoplus_{a\in\lambda} D(a)/G$.*

Proof. We denote p to be the projection map: $p : \bigoplus_{a\in\mathrm{ob}(\lambda)} D(a) \to L$. Let v be the family of module homomorphisms

$$\{v_a = p \circ i_a : D(a) \to L\}_{a\in\mathrm{ob}(\lambda)}$$

which induces the cocone $\nabla(D, L, v)$ and let K be any other module with family of module homomorphisms $\phi = \{\phi_a : D(a) \to K\}_{a\in\mathrm{ob}(\lambda)}$ which induces the cocone $\nabla(I, K, \phi)$. We have to prove that there exists a unique module homomorphism $u : L \to K$ for which the condition $u \circ v_a = \phi_a$ holds for all $a \in \mathrm{ob}(\lambda)$. We will first show that such module homomorphism exists
Using lemma 2.4 we can find a unique module homomorphism $u' : \bigoplus_{a\in\lambda} D(a) \to K$ for which $u' \circ i_a = \phi_a$ holds. Now we see that for every generator of the subgroup G the following equations hold:

$$u'(i_a x_a - i_b D_a^b x_a) = \phi_a(x_a) - \phi_b D_a^b x_a = \phi_a(x_a) - \phi_a(x_a) = 0.$$

Thus by linearity we see that every element in the module L vanishes on map u'. Now we can define u in such way that it maps every element $x + D$ to element $\phi'(x)$. Now we see that

$$u \circ v_a = u \circ p \circ i_a = u' \circ i_a = \phi_a.$$

To prove that the homomorphism u is unique we notice that if $x \in L$, then $x = \sum_{a \in A \subset D} v_a(x_a) + G$. Then the following equations hold

$$u(x) = u(\sum_{a \in A \subset D} v_a(x_a)) = \sum_{a \in A \subset D} u \circ v_a(x_a) = \sum_{a \in A \subset D} \phi_a(x_a)$$

Thus $u(x)$ is determined by the sum of the module homomorphisms ϕ_a. It follows that the map u is unique. □

Remark 2.28. In the theorems above we showed that, there exists a limit for the inverse system functor and colimit for the corresponding functor of directed system. Because it is clear from the context, which of the limits is being used from now on we will denote both concepts for systems simply as lim .

We proved that limit can be defined for the systems of R-modules. The corresponding theorem can be obtained for the systems of abelian groups, by interpreting them as R-modules, where the ring R is trivial.

2.3.2 Morphisms between systems

To be able to define morphisms between Čech homology and cohomology groups we need the concept of a limit homomorphism. In this section we will define the limit morphism between systems and investigate properties of it. Concepts used in this chapter can be presented in a more general way. However, for our use it is enough to restrict ourselves to the direct and inverse system. Unless otherwise stated, in this section we assume that all the systems are such that the limit exists for them.

Definition 2.29. Let C and D be categories with order relation. A functor $\phi : C \to D$ is order preserving if for every pair $a \leqslant b$ in the category C, the condition $\phi(a) \leqslant \phi(b)$ is satisfied.

Definition 2.30. Let C and D be categories with order relation. A functor $\phi : C \to D$ is backward order preserving, if for every pair $a \leqslant b$ in the category D, there exists $a' \leqslant b'$ for which $\phi(a') = a$ and $\phi(b') = b$ holds.

Let I be an inverse system and $\phi : \lambda' \to \lambda$ an order preserving functor. Then the inverse system $(I\phi, \lambda')$ consist of objects $\{I(\phi(a)) \mid a \in \mathrm{ob}(\lambda')\}$ and morphisms $\{I_{\phi(a)}^{\phi(b)} \mid a \leqslant b\}$. For directed systems we define the system $(D\phi, \lambda')$ respectively. The next definition is a general statement for both directed and inverse systems.

Definition 2.31. Let (S, λ) and (S', λ') be systems and let $\phi : \lambda' \to \lambda$ be an order preserving functor. Let $\{f_a : S(\phi(a)) \to S'(a) \mid a \in \lambda\}$ be a natural transformation between $S\phi$ and S'. Then we say that $\{f_a\}_{a\in\lambda}$ is an inverse system of morphisms corresponding to the functor ϕ from the system S into the system S'.

Respectively we have following definition of directed system of the morphisms.

Definition 2.32. Let (S, λ) and (S', λ') be systems and let $\phi : \lambda \to \lambda'$ be an order preserving functor. Let $\{f_a : S(a) \to S'(\phi(a)) \mid a \in \lambda\}$ be a natural transformation between I and $I'\phi$. Then we say that $\{f_a\}_{a\in\lambda}$ is a directed system of morphisms corresponding to the functor ϕ from the system S into the system S'.

We will next give definition of limit morphism. We denote the morphisms of both limits and colimits as ρ_a for every a in the category which spans the limit.

Definition 2.33. Let f be a system of morphisms between systems (S, λ) and (S', λ'). Let $\lim f : \lim S \to S'$ be a morphism, then it is a limit morphism if the corresponding condition in the table is satisfied:

	directed system	inverse system
directed system of morphisms	$\lim f \circ \rho_a = \rho_{\phi(a)} \circ f_a$	$\rho_a \circ \lim f = f_a \circ \rho_{\phi(a)}$
inverse system of morphisms	$\lim f \circ \rho_{\phi(a)} = \rho_a \circ f_a$	$\rho_{\phi(a)} \circ \lim f = f_a \circ \rho_a$

It is not clear from the definitions that there exists an unique morphism. However, in the most practical one can find a unique limit homomorphism. The statements of theorems regarding existence of limit morphism can be compressed in the following table. Assume that the systems are such for which limit exists. Then the following assumptions guarantee existence of limit of morphisms.

	directed system	inverse system
directed system of morphisms	always exists unique	cofinality required
inverse system of morphisms	cofinality required	always exists unique

2.3.2.1 Morphisms between inverse systems

We denote all such transformations between systems (I, λ) and (I', λ') as Transf(I, I'). We will show that inverse systems together with the transformations form a category

Lemma 2.34. *Let (I, λ), (I', λ') and (I'', λ'') be inverse systems and let $\phi : \lambda' \to \lambda$ and $\phi' : \lambda'' \to \lambda'$ functors between systems. Let $\{f_a : I(\phi(a)) \to I'(a) \mid a \in \phi'(\lambda'')\}$ and $\{g_a : I'(\phi'(a)) \to I''(a) \mid a \in \lambda''\}$ be inverse systems of morphisms corresponding to ϕ and ϕ'. Then $\{g_a f_{\phi(a)} : I(\phi(\phi'(a))) \to I''(a) \mid a \in \lambda''\}$ is inverse system of morphisms corresponding to composition $\phi \circ \phi'$.*

Proof. The claim follows directly from the following diagram:

$$\begin{tikzcd}
I(\phi(\phi'(b))) \arrow[r, "f_{\phi'(b)}"'] \arrow[d, "I^{\phi''(\phi'(b))}_{\phi''(\phi'(a))}"] & I'(\phi'(b)) \arrow[r, "g_b"'] \arrow[d, "I^{\phi'(b)}_{\phi'(a)}"] & I''(b) \arrow[d, "I^b_a"] \\
I(\phi(\phi'(a))) \arrow[r, "f_{\phi'(a)}"'] & I'(\phi'(a)) \arrow[r, "g_a"'] & I''(a)
\end{tikzcd}$$

□

Theorem 2.35. *Inverse systems over category C together with inverse systems of morphisms form a category.*

Proof. The composition of two such transformation is a transformation by Lemma 2.34. For every inverse system, the identity morphism is defined as transformation $\mathrm{id}_a : I(a) \to I(a)$ corresponding to a functor ϕ which maps every elements to itself. □

In the next theorem we will prove that it is possible to define the limit of morphisms between systems in a unique way. We recall that existence of limit implies that for every objects of a category I there exists a morphism $\rho_a : \lim I \to I(a)$, for which the triangle condition is satisfied.

Lemma 2.36. *Let (I, λ) and (I', λ') be inverse systems. Let $\{f_a\}$ be inverse system of morphisms corresponding to that pair. Then there exists a unique morphism $\lim f : \lim I \to \lim I'$ for which the condition $\rho_{\phi(a)} \circ \lim f = f_a \circ \rho_a$ holds for every $a \in \lambda'$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
& & \lim I' \arrow[dl, "\rho'_a"'] \arrow[dr, "\rho'_b"] & \\
& I'(a) \arrow[rr, "I'^a_b"'] & & I'(b) \\
& \lim I \arrow[uur, dashed] \arrow[dl, "\rho_{\phi(a)}"] \arrow[dr, "\rho_{\phi(b)}"'] & & \\
I(\phi(a)) \arrow[uur, bend left] \arrow[rr, "I^{\phi(a)}_{\phi(b)}"'] & & I(\phi(b)) \arrow[uur, bend right] &
\end{tikzcd}$$

Diagram 2.3.2.1

To prove that the limit morphism exists we form the following triangle out of the diagram described above:

$$
\begin{tikzcd}
 & \lim I \arrow[d, dashed] \arrow[ddl, "f_a\rho_{\phi(a)}"'] \arrow[ddr, "f_b\rho_{\phi(b)}"] & \\
 & \lim I' \arrow[dl, "\rho'_a"] \arrow[dr, "\rho'_b"'] & \\
I'(a) \arrow[rr, "I'^{a}_{b}"'] & & I'(b)
\end{tikzcd}
$$

Then by definition of limit there exists a unique morphism $\lim f : \lim I \to \lim I'$ for which the diagram commutes. $\square$

Lemma 2.37. *Let (I, λ) and (I', λ') be inverse systems. Let $\{f_a : I(a) \to I(\phi(a))\}$ be a directed system of morphisms corresponding to a functor $\phi : \lambda \to \lambda'$ and let $R = \lim(I', \phi(\lambda))$. Then there exists a morphism $\lim f : \lim(I, \lambda) \to R$ for which the condition $\lim f \circ u_{I,a} = u_{I',\phi(a)} \circ f_a$ holds for every $a \in \lambda'$.*

Proof. Let a' and b' be an arbitrary elements for which condition $a' \leqslant b'$ holds. By assumption there exists $a \leqslant b$, for which $\phi(a) = a'$ and $\phi(b) = b'$.

$$
\begin{tikzcd}
 & & R \arrow[dl, "\rho'_{a'}"] \arrow[dr, "\rho'_{b'}"'] & \\
 & I'(a') \arrow[rr, "I'^{a'}_{b'}"'] & & I'(b') \\
 & \lim I \arrow[uur, dashed] \arrow[dl, "\rho_a"] \arrow[dr, "\rho_b"'] & & \\
I(a) \arrow[uur] \arrow[rr, "I^{a}_{b}"'] & & I(b) \arrow[uur] &
\end{tikzcd}
$$

Diagram 2.3.2.2

To prove that the limit morphism exists we form the following triangle out of the diagram described above:

$$
\begin{tikzcd}
 & \lim I \arrow[d, dashed] \arrow[ddl, "f_a\rho_a"'] \arrow[ddr, "f_b\rho_b"] & \\
 & \lim R \arrow[dl, "\rho'_{a'}"] \arrow[dr, "\rho'_{b'}"'] & \\
I'(a') \arrow[rr, "I'^{a'}_{b'}"'] & & I'(b')
\end{tikzcd}
$$

Now from properties of limit we can conclude that there exists a unique $\lim f : \lim I \to R$. $\square$

Lemma 2.38. *Let (I, λ) be an inverse system with property of having a limit and let $\phi : \lambda \to \lambda$ be a functor which maps every object to itself. Let $h = \{id_a\}$ be an inverse system of morphisms corresponding to the functor ϕ. Then $\lim h$ is an identity morphism.*

Proof. Let $a \in \lambda$ be an arbitrary element and $u_a : I(a) \to \lim I$ be the universal morphism corresponding to it. Then clearly $\mathrm{id}_{\lim I} \circ u_a = u_a \circ \mathrm{id}_a$ holds and thus by uniqueness property the limiting morphism is $\mathrm{id}_{\lim I}$. □

For composition of limits we have the following results:

Lemma 2.39. *Let (I, λ), (I', λ') and (I'', λ'') be inverse systems and let $\phi : \lambda' \to \lambda$ and $\phi' : \lambda'' \to \lambda'$ be functors between systems. Let $\{f_a : I(\phi(a)) \to I'(a) \mid a \in \phi'(\lambda'')\}$ and $\{g_a : I'(\phi'(a)) \to I''(a) \mid a \in \lambda''\}$ be systems of morphisms corresponding to ϕ and ϕ'. Let $\lim(f \circ g) : I \to I''$ be the limit of the family $\{g_a f_{\phi(a)} : I(\phi(\phi'(a))) \to I''(a) \mid a \in \lambda''\}$. Then $\lim(f \circ g) = \lim f \circ \lim g$ holds.*

Proof. To prove this we will use previous lemma and following commutative diagram:

$$\begin{tikzcd}
I(\phi(\phi'(a))) \arrow[r, "f_{\phi'(a)}"'] & I'(\phi'(a)) \arrow[r, "g_a"'] & I''(a) \\
\lim I \arrow[u, "\rho_{I,\phi\phi' a}"'] \arrow[r, dashed, "\lim f"'] & \lim I' \arrow[u, "\rho_{I',\phi' a}"'] \arrow[r, dashed, "\lim g"'] & \lim I'' \arrow[u, "\rho_{I'',a}"']
\end{tikzcd}$$

By definition of limit morphism the condition $\rho_{I'',a} \circ \lim(g \circ f) = (g_a \circ f_{\phi(a)}) \circ \rho_{I,\phi(\phi'(a))}$ holds for the limit of the composition. By uniqueness of the limit homomorphism it is enough to show that the composition of two limit functors satisfies the condition. This follows from the following equations:

$$\rho_{I'',a} \circ \lim g \circ \lim f = \lim g \circ \rho_{I',\phi'(a)} \circ f_{\phi'(a)} = g_a \circ f_{\phi'(a)} \circ \rho_{I,\phi(\phi'(a))}$$

□

Lemma 2.40. *Let (I, λ),(I', λ') and (I'', λ'') be inverse systems and let $\phi : \lambda' \to \lambda$ be an order preserving functor and $\psi : \lambda' \to \lambda''$ backward order preserving functor. Let $f = \{f_a : I(\phi(a)) \to I'(a) \mid a \in \lambda'\}$ and $g = \{g_a : I'(a) \to I''(\psi(a)) \mid a \in \lambda'\}$ be systems of morphisms corresponding to ϕ and ψ.*

Lemma 2.41. *Let (I, λ) and (I, λ') be inverse systems with property of having a limit and let $\phi : \lambda' \to \lambda$ be an order preserving injective functor. Let $f = \{f_a\}_{a \in \lambda'}$ be a family of morphisms between the systems corresponding to the functor ϕ. Assume that every f_a in the collection is an isomorphism, then $\lim\{f_a\}$ is an isomorphism.*

Proof. By assumption for every f_a there exists a morphism $g_{\phi(a)}$, for which $f_a \circ g_{\phi(a)} = \mathrm{id}_{\phi(a)}$ and $g_{\phi(a)} \circ f_a = \mathrm{id}_a$. Because ϕ is injective, we can now define $\psi : \phi(\lambda) \to \lambda'$ in such way that it maps $\phi(a)$ to a. Now collection $g = \{g_{\phi(a)}\}_{a \in \lambda'}$ forms an inverse system of morphisms, which corresponds to ψ. Applying Lemma 2.39 and Lemma 2.38 we obtain the following equations:

$$\lim f \circ \lim g = \lim(f \circ g) = \lim(\mathrm{id}) = \mathrm{id} \text{ and similarly } \lim g \circ \lim f = \mathrm{id}$$

Thus $\lim f$ is an isomorphism with inverse $\lim g$. □

Content of this section can be compressed in the following theorem

Theorem 2.42. *Let $I(C)$ be a category inverse systems over category C. Assume that limit exists for every object in $I(C)$ and we have equated all the isomorphic objects in the category C. Then the operator* lim *is a functor from category of inverse systems to the image category of the corresponding systems.*

Proof. By theorem 2.14 and assumption every inverse system is being mapped to a unique object in C and by theorem 2.36 every natural transformation between systems is being mapped to unique limit morphism.

(1) $\lim(id) = id$ holds by Lemma 2.38

(2) $\lim(f) \circ \lim(g) = \lim(f \circ g)$ holds by Lemma 2.39.

□

Now we would like to define limit for the system of homomorphisms between inverse systems of groups.

Lemma 2.43. *Let (I, λ) and (I', λ') be inverse systems of R-modules. Then*

$$f : \lim_{\rightarrow} I \to \lim_{\rightarrow} I' : f(x) = \prod_{a \in \lambda} f_a(proj_{\phi(a)}(x))$$

is a well defined module homomorphism between the limit modules.

Proof. We have to show that the image is in the subgroup

$$L = \{x \in \prod_{i \in \lambda'} I'(i) \mid x_a = I_a^b x_b \text{ for all } a \leqslant b\}.$$

Let b be some index, then: $I_a'^b x_b' = I_a'^b f_b(x_{\phi(b)}) = f_b I'^{\phi(b)}_{\phi(a)}(x_{\phi(b)}) = f_a x_{\phi(a)} = x_a'$. In this equation we used the property which says that the morphisms define a natural transformation. □

2.3.2.2 Morphisms between directed systems

The proof of all the following theorems can be obtained by using the method of dualization described in the definition 2.15 on the proof of Lemma 2.36 and Lemma 2.39 respectively.

Theorem 2.44. *Direct systems over category C together with directed system of morphism form a category.*

Lemma 2.45. *Let (D, λ) and (D', λ') be directed systems for which the limit exists. Let $\{f_a\}$ be directed system of morphisms corresponding to that pair. Then there exists a unique morphism $\lim f : \lim D \to \lim D'$ for which the condition $u_{D',\phi(a)} \circ \lim f = f_a \circ u_{D,a}$ holds for every $a \in \lambda'$.*

$$
\begin{tikzcd}
 & & & D'(\phi(n)) & \\
 & & & \lim D' \arrow[u] & \\
 & & D'(\phi(a)) \arrow[uur, "\omega'_{\phi(a)}"] \arrow[ur, "\rho'_{\phi(a)}"] \arrow[rr, "D'^{\phi(a)}_{\phi(b)}"'] & & D'(\phi(b)) \arrow[uul, "\omega'_{\phi(b)}"'] \arrow[ul, "\rho'_{\phi(b)}"'] \\
 & D(n) \arrow[uuurr] & & & \\
 & \lim D \arrow[u] \arrow[uuurr, dashed] & & & \\
D(a) \arrow[uur, "\omega_a"] \arrow[ur, "\rho_a"'] \arrow[rr, "D^a_b"'] \arrow[uuurr] & & D(b) \arrow[uul, "\omega_b"'] \arrow[ul, "\rho_b"] \arrow[uuurr] & &
\end{tikzcd}
$$

Diagram 2.3.2.3

Lemma 2.46. *Let (D, λ), (D', λ') and (D'', λ'') be directed systems with limits and let $\phi : \lambda \to \lambda'$ and $\phi' : \lambda' \to \lambda''$ functors between systems. Let $\{f_a : D(a) \to D'(\phi(a)) \mid a \in \lambda\}$ and $\{g_a : D'(a) \to D''(\phi'(a)) \mid a \in \phi(\lambda)\}$ be families of morphisms corresponding to ϕ and ϕ'. Let $\lim(f \circ g) : I \to I''$ be the limit of the family $\{g_{\phi(a)} f_a : D(a) \to D''(\phi'\phi(a)) \mid a \in \lambda\}$. Then $\lim(f \circ g) = \lim f \circ \lim g$ holds.*

$$
\begin{tikzcd}
D(a) \arrow[r, "f_a"'] \arrow[d, "u_{I,a}"] & D'(\phi(a)) \arrow[r, "g_{\phi(a)}"'] \arrow[d, "u_{I',\phi a}"] & D''(\phi'(\phi(a))) \arrow[d, "u_{I'',\phi'\phi a}"] \\
\lim D \arrow[r, dashed, "\lim f"'] & \lim D' \arrow[r, dashed, "\lim g"'] & \lim D''
\end{tikzcd}
$$

Like in the case of Inverse system, it will be shown that the limit operator is a functor between Directed systems and some category C. Nothing new is learned from this theorem, however it provides a summary of previous lemmas and can be used to shorten notation.

Theorem 2.47. *Let $D(C)$ be a category of directed systems over category C. Assume that limit exists for every object in $D(C)$ and we have equated all the isomorphic objects in the category C. Then the operator* lim *is a functor from category of directed systems to the image category of the corresponding systems.*

2.3.3 Finality properties of subsystems

Final and cofinal functors can be defined in a more general way [6]. However, because we are mainly interested in properties of systems, in this section we will investigate the properties of those specific functors only in that special case. In the case of general categories, cofinality includes a concept of connectedness defined in [6] page 217. In our definition we will use a stronger version of that concept.

Definition 2.48. Let λ be a category with a quasi-order relation on its objects. We say that λ is directed category, if for every two objects $a \in \lambda$ and $b \in \lambda$ we find an object $c \in \lambda$ for which the relations $a \leqslant c$ and $b \leqslant c$ hold.

Definition 2.49. Let λ be a category with a quasi-order relation on its objects and λ' its directed subcategory for which following condition holds:

$$\text{For every } a \in \lambda \text{ there exists } b \in \lambda' \text{ for which } a \leqslant b.$$

Then we say that λ' is dense in λ.

Definition 2.50. Let λ be a directed category with a quasi-order relation on its objects and let λ' be its subcategory. Then we say that λ' is a cofinal subcategory of λ, if λ' is dense in λ.

Lemma 2.51. *Let λ be a directed category and λ' its dense subcategory. Then λ' is a directed category.*

Proof. Let α and β be objects of λ. Then there exists $t \in \lambda$ for which $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$ hold. By density property there exists $\gamma' \in \lambda'$ for which condition $\gamma \leqslant \gamma'$ holds. Now claim follows directly from the equations $\alpha \leqslant \gamma \leqslant \gamma'$ and $\beta \leqslant \gamma \leqslant \gamma'$. □

2.3.3.1 Cofinality in inverse systems

Theorem 2.52. *Let (I, λ) be an inverse system and let (I', λ') be its subsystem generated by a cofinal subcategory λ. Assume that there exists a limit object for I'. Then the limit of the system (I, λ) is* $\lim I'$.

Proof. Let $\lim I'$ be the limit of the inverse system I'. It is enough to show that $\lim I'$ is also the limit of I. We will first show that $\lim I'$ forms a cone in I and then prove that for any other cone induced by an object N in I there exists a unique morphism $u : N \to \lim I'$.
Let a be an arbitrary object of the indexing category λ. Then by the definition of cofinal subcategory there exists element $a' \in \lambda'$ for which $a \leqslant a'$ holds. Thus there exists a morphism $I_a^{a'}$. Because $\lim I'$ is the limit of the inverse system I' it forms a cone $\triangle(I', \lim I', \rho)$, where ρ is some collection of morphisms $\{\rho_a : \lim I' \to I'(a)\}_{a \in \mathrm{ob}(\lambda')}$. Using ρ we form a new collection of morphisms ρ' defined by $\{I_a^{a'} \circ \rho_{a'}\}_{a \in \mathrm{ob}(\lambda)}$. Because $I_a^{a'} \circ \rho_{a'}$ can vary on the choice of a, the composition may be not uniquely determined. However, if a'' is another object of λ' for which the relation $a \leqslant a''$ holds, by lemma 2.51 there exists an object $t \in \lambda'$ for which the conditions $a \leqslant t$ and $a'' \leqslant t$ are satisfied. Because the following equations hold

$$I_a^{a'} \circ \rho_{a'} = I_a^{a'} \circ I_{a'}^{t} \circ \rho_t = I_a^{t} \circ \rho_t = I_a^{a''} \circ I_{a''}^{t} \circ \rho_t = I_a^{a''} \circ \rho_{a''}$$

we can conclude that $I_a^{a'} \circ \rho_{a'} = I_a^{a''} \circ \rho_{a''}$ and thus the family of homomorphisms is well-defined and $\lim(I')$ forms a cone $\triangle(I, \lim I', \rho')$
Now if there exists another cone $\triangle(I, N, \omega)$, we have to construct a morphism $u : N \to \lim I'$ and prove that it is unique. Let a and b be objects of λ, then there exists objects a' and b' in λ' for which relations $a \leqslant a'$ and $b \leqslant b'$ hold. Restricting the collection ω to the morphisms indexed by λ' we get the cone $\triangle(I', N, \omega')$ of I'. Because $\lim I'$ is the limit of the functor I', there exists a unique homomorphism $u : N \to \lim I'$. It is left to prove that the following diagram commutes:

$$\begin{tikzcd}
 & & N \arrow[d, dashed] \arrow[ddl, "\omega_{a'}"'] \arrow[ddr, "\omega_{b'}"] \arrow[dddll, "\omega_a"'] \arrow[dddrr, "\omega_b"] & & \\
 & & \lim I' \arrow[dl, "\rho'_a"] \arrow[dr, "\rho'_b"'] & & \\
 & I'(a') \arrow[rr, "I'^{a'}_{b'}"'] \arrow[dl, "I_a^{a'}"'] & & I'(b') \arrow[dr, "I_b^{b'}"] & \\
I(a) \arrow[rrrr, "I_b^a"'] & & & & I(b)
\end{tikzcd}$$

Especially we have to prove that $I_a^{a'} \circ w_{a'} = w_a$. This claim follows directly from the fact that $\triangle(I, N, \omega)$ is a cone. Thus the u defined above is the map we were searching for.

$\square$

Lemma 2.53. *Let (I, λ) and (I', λ') be inverse systems with property of having a limit and let $\phi : \lambda' \to \lambda$ be a functor. Let $f = \{f_\alpha\}_{a \in \lambda'}$ be an inverse system of morphisms corresponding to ϕ and let ς be cofinal subcategory of λ', for which $\phi(\varsigma)$ is cofinal in λ. Then limit morphism $\lim f'$ of $\{f_a\}_{a \in \varsigma}$ forms a following commutable diagram:*

$$\begin{tikzcd}
\lim(I, \lambda) \arrow[r, "\lim f"'] \arrow[d, leftrightarrow] & \lim(I', \lambda') \arrow[d, leftrightarrow] \\
\lim(I, \phi(\varsigma)) \arrow[r, "\lim f'"'] & \lim(I', \varsigma)
\end{tikzcd}$$

where the vertical lines are isomorphisms.

Proof. By assumption there exists the limit for systems (I', λ') and (I', ς). Using Lemma 2.52 we notice that the limit of system (I', ς) is also limit of the system (I', λ') and by uniqueness Lemma 2.14 there exists an isomorphism $v : \lim(I', \varsigma) \to \lim(I', \lambda')$. Using same arguments we obtain an isomorphism $w : \lim(I, \phi(\varsigma)) \to \lim(I, \lambda)$. By using the property of limit functor $\lim f$ and taking the indexation over elements $a \in \psi$ we obtain the following commutable diagram:

$$\begin{tikzcd}
I(\phi(a)) \arrow[r, "f_a"'] \arrow[d, "u_{(I(\lambda), \phi(a))}"] & I'(a) \arrow[d, "u_{(I'(\lambda'), a)}"] \\
\lim(I, \lambda) \arrow[r, "\lim f"'] & \lim(I', \lambda')
\end{tikzcd}$$

We define $g : \lim(I', \phi(\psi)) \to \lim(I', \psi)$ as the composition $v^{-1} \circ \lim f \circ w$. Now from proof of Lemma 2.14 we obtain the following two diagrams:

$$\begin{tikzcd}
 & I'(a) \arrow[dl, "u_{(I'(\varsigma), a)}"'] \arrow[dr, "u_{(I'(\lambda'), a)}"] & \\
\lim(I', \varsigma) \arrow[rr, "v"'] & & \lim(I', \lambda')
\end{tikzcd}
\quad
\begin{tikzcd}
 & I(\phi(a)) \arrow[dl, "u_{(I(\phi(\varsigma)), \phi(a))}"'] \arrow[dr, "u_{(I(\lambda), \phi(a))}"] & \\
\lim(I, \phi(\varsigma)) \arrow[rr, "w"'] & & \lim(I, \lambda)
\end{tikzcd}$$

We will now show that the morphism g satisfies the condition of limit homomorphism between systems $(I, \phi(\varsigma))$ and (I, ς). Consider the following equation :

$$\begin{aligned} g \circ u_{(I(\lambda),\phi(a))} &= v^{-1} \circ \lim f \circ w \circ u_{(I(\lambda),\phi(a))} \\ &= v^{-1} \circ \lim f \circ u_{(I(\lambda),\phi(a))} \\ &= v^{-1} \circ u_{(I'(\lambda'),a)} \circ f_a \\ &= u_{(I'(\varsigma),a)} \circ f_a \end{aligned}$$

Because limit morphisms are by Lemma 2.36 unique, we conclude $g = \lim f'$. By definition of morphism g, the equation $\lim f \circ w = v \circ \lim f'$ holds. $\square$

Corollary 2.54. *Let (I, λ) and (I', λ') be inverse systems and let $\phi : \lambda' \to \lambda$ be an injective functor. Let $\{f_\alpha\}_{a \in \lambda'}$ be an inverse system of morphisms corresponding to ϕ and let ς be some subcategory of λ. Then if the following 3 conditions are satisfied, the limit homomorphism $\lim f = \{f_\alpha\}_{a \in \lambda'}$ is isomorphism.*

(1) ς is cofinal subcategory of λ.

(2) $\phi(\varsigma)$ is cofinal subcategory of λ'.

(3) f_α is isomorphism for every $\alpha \in \varsigma$

Proof. From Lemma 2.41 we know that the limit morphism of inverse system over morphisms $\{f_a\}_{a \in \varsigma}$ is isomorphism. Using Lemma 2.53 $\lim f$ can be presented as composition of three isomorphisms. Using the fact that composition of two isomorphism is isomorphism we conclude that $\lim f$ is an isomorphism.

$\square$

2.3.3.2 Cofinality in directed systems

Theorem 2.55. *Let (D, λ) be a directed and let (D', λ') be its subsystem generated by a cofinal subcategory λ. Assume that there exists a limit object for D'. Then the limit of the system (D, λ) is $\lim D'$.*

Proof. Proof of the claim is almost identical to the proof above. Expect every source is replaced with target and cone structure with cocone. We draw a picture and leave it to the reader to fill in the details.

$$
\begin{tikzcd}
 & & N & & \\
 & & \lim D' \arrow[u, dashed] & & \\
 & D'(a') \arrow[uur, "\omega_{a'}"] \arrow[ur, "\rho'_a"] \arrow[rr, "D'^{a'}_{b'}"'] & & D'(b') \arrow[uul, "\omega_{b'}"'] \arrow[ul, "\rho'_b"'] & \\
D(a) \arrow[ur, "D_a^{a'}"] \arrow[uuurr, "\omega_a"] \arrow[rrrr, "D_b^a"'] & & & & D(b) \arrow[ul, "D_b^{b'}"'] \arrow[uuull, "\omega_b"']
\end{tikzcd}
$$

□

Theorem 2.56. *Let (D, λ) and (D', λ') be directed systems and let $\phi : \lambda \to \lambda'$ be an injective functor. Let $\{f_\alpha\}_{a\in\lambda}$ be directed system of morphisms corresponding to ϕ and let ς be some subcategory of λ. Then if the following 3 conditions are satisfied, the limit homomorphism $\lim f = \{f_\alpha\}_{a\in\lambda'}$ is isomorphism.*

(1) ς is cofinal subcategory of λ.

(2) $\phi(\varsigma)$ is cofinal subcategory of λ'.

(3) f_α is isomorphism for every $\alpha \in \varsigma$

Proof. The proof of this claim is dual of the Corollary 2.54. □

2.3.4 Special properties of Abelian systems

In this section it will be proved that the exactness of directed system of R-modules is preserved under the limit functor.

Theorem 2.57. *Let (D, λ), (D', λ') and (D'', λ'') be directed systems of R-modules. Let $(\{f_\alpha\}, \phi)$ be a directed system of morphisms from D to D' and let $(\{g_\alpha\}, \phi')$ be a directed system of morphism from D' to D'', in such way that $\phi(\lambda)$ is cofinal in λ' and $\phi('\phi(\lambda))$ is cofinal in λ''. Assume that the following sequence is exact:*

$$D(\alpha) \underset{f_\alpha}{\longrightarrow} D'(\phi(\alpha)) \underset{f_{\phi(\alpha)}}{\rightarrow} D''(\phi'(\phi(\alpha))$$

for every $\alpha \in \lambda$. Then the following sequence is also exact:

$$\lim D \underset{\lim f}{\longrightarrow} \lim D' \underset{\lim g}{\longrightarrow} \lim D''$$

Proof. $\ker(\lim g) \subset \operatorname{im}(\lim f)$: Recall that module G is generated by elements which are of the form $\{r \cdot (i_a x_a - i_b D_a^b x_a)\}$ Let $x \in \ker(\lim g)$. Element x has representation of

$$x = \sum_{a \in \tau} p'_a(i'_a(x_a)) + G'$$

where τ is some finite subindexation of λ' and $x_a \in D'$. The following equation holds:

$$\sum_{i \in \tau} i''_{\phi'(a)}(g_a(x_a)) = \lim g(\sum_{i \in \tau} i'_a(x_a)) = \lim g(x) = 0$$

Now $i''_{\phi(a)}(g_a(x_a)) \in G''$ for all $a \in \lambda'$. It follows that $D^b_{\phi'(a)}(g_a(x_a)) = 0$ for some $\phi(a) \leqslant b$. Because $\phi'(\phi(\lambda))$ is cofinal in this category, we can find $a' \in \phi(\lambda)$ for which $\phi'(a) \leqslant b \leqslant \phi'(a')$. Now

$$D_b^{\phi'(a')} \circ D^b_{\phi'(a)}(g_a(x_a)) = 0 \Rightarrow D^{\phi'(a')}_{\phi'(a)}(g_a(x_a)) = 0.$$

From the naturalness property we obtain $g_{a'}(D_a^{a'}(x_a)) = 0$. Then by assumption there exists $f_q(y_q) = D_a^{a'}(x_a)$, where $\phi(q) = a'$. Define element $y = \sum_q p_q i_q(y_q)$.

$$\lim f(y) = \lim f(\sum_q p_q i_q(y_q)) = \sum_{\phi(q)} p_q i_q(f_q(y_{\phi(q)})) = \sum_{\phi(q)} p_{a'} i_{a'}(D_a^{a'}(x_a)) = \sum_{\tau} p_a i_a(D_a^{a'}(x_a))$$

$\operatorname{im}(\lim f) \subset \ker(\lim g)$: Let $a \in \lambda$, then $g_{\phi(a)} \circ f_a = 0$. The claim follows directly by taking limit functor over both sides, we get

$$\lim g \circ \lim f = \lim(g \circ f) = \lim(0) = 0$$

□

2.4 Nerve of covering

In this chapter we will give a definition of an abstract simplicial complex.

2.4.1 Abstract simplicial complex

We assume that the reader knows already basic facts about simplicial complexex. If not we suggest to take a look at [1].

Definition 2.58. Let X be a set and let $\mathcal{A} = \{S_i\}_{i \in J}$ be some collection of finite subsets of the set X. We say that $\mathcal{A}$ forms an abstract simplicial complex over X, if a subset of any set S_i in the collection $\mathcal{A}$ is in $\mathcal{A}$.

The abstract simplicial complex defined above is denoted as $\Gamma(X, \mathcal{A})$.

Definition 2.59. Assume that the set X together with the collection $\mathcal{A} = \{S_i\}_{i \in I}$ is an abstract simplicial complex. Let A be some subset of X and let $\mathcal{A}' = \{S'_j\}_{j \in I'}$ be some collection of finite subsets of the space A for which the condition

$$S'_j \in \mathcal{A}' \Leftrightarrow S'_j \subset S_i \text{ for some } i \in I$$

is satisfied. Then we say that $\Gamma(A, \mathcal{A}')$ is a subcomplex of $\Gamma(X, \mathcal{A})$.

From now on we will denote the abstract simplicial pair $(\Gamma(X\mathcal{A}), \Gamma(A, \mathcal{A}'))$ simply as $\Gamma((X, A), \alpha)$, where α is the pair of collections $(\mathcal{A}, \mathcal{A}')$. In the following example we will see that a simplicial complex defined in the geometric way is an abstract simplicial complex.

Example 2.60. Let s be a simplicial structure, with edge set $\text{edge}(s)$ and $\text{simplices}(s)$ a collection of simplices. Then the abstract simplicial complex is obtained by forgetting the geometric structure.

We will now define corresponding geometric structure for an abstract simplicial complex. This will give us the interface of original simplicial complex and will allow us to use the theory behind it.

Definition 2.61. Let J be a set and let $\Gamma(J, \mathcal{A})$ be an abstract simplicial complex induced by the vertex set J and some collection of abstract simplices $\mathcal{A}$. Then the geometric realization of the abstract simplicial complex $\mathcal{S}(J, \mathcal{A})$ is a subset of the set $[0, 1]^J$ for which the following conditions are satisfied: Let $x \in \mathcal{S}(J, \mathcal{A})$ and let $\text{pr}_j : [0, 1]^J \to [0, 1]$ be the projection $\text{pr}_j(x) = x_j$ then:

(1) Relation $\text{pr}_j(x) > 0$ holds only for finitely many $j \in J$.

(2) $\sum_{j \in J} \text{pr}_j(x) = 1$.

(3) The set $\{j \in J \mid \text{pr}_j(x) > 0\}$ belongs to collection $\mathcal{A}$.

The topology of the geometric realization is defined as the subset topology of the space $[0, 1]^J$ together with its natural topology.

For the abstract simplicial pair $\Gamma((X, A), \alpha)$, we use the definition of geometric realization $\mathcal{S}((X, A), \alpha)$ separately for $\Gamma(X, \mathcal{A})$ and $\Gamma(A, \mathcal{A})$ and treat the set $[0, 1]^A$ as a subset of $[0, 1]^X$, where for every element $a \in [0, 1]^A$ the projection $\text{pr}_i(a) = 0$ for every index $i \in X \backslash A$.

Lemma 2.62. *The geometric realization $\mathcal{S}((X, A), \alpha)$ of an abstract simplicial complex is a simplicial complex.*

Proof. The following conditions are satisfied:

(1) The collection of simplexes of the simplicial complex is the geometric realizations of sets S_i of the collection $\mathcal{A}$. Respectively the subsimplex consists of the geometric realizations S_i' of collection $\mathcal{A}'$.

(2) Any face is in the collection by the definition of an abstract simplicial complex

(3) Intersection of any two simplices in the collection is clearly a face or the empty set.

Thus $\mathcal{S}((X, A), \alpha)$ is a simplicial complex. □

The simplicial map for the geometric constructions is defined in the Rotman. We will give definition of simplicial map to abstract simplicial complex.

Definition 2.63. Let (X, A) and (Y, B) be pairs of abstract simplicial complexes. We say that $f : (X, A) \to (Y, B)$ is simplicial if it is satisfies following two conditions:

(1) f is a map between edges(X) and edges(Y)

(2) f maps every element in edges(A) to set edges(B).

Lemma 2.64. *Let (X, A) and (Y, B) be pairs of abstract simplicial complexes and let $f : (X, A) \to (Y, B)$ be a simplicial map between the complexes. Let (X', A') and (Y', B') be geometric realizations of the abstract simplicial complexes (X, A) and (Y, B). Then the map $f_* : (X', A') \to (Y', B')$ induced by the images of the edges mapped by the function f and defined to be piecewise linear on all the other elements is a well-defined simplicial map.*

Proof. Let x be an arbitrary element of (X, A), by definition x has a unique representation $x = \sum_{i \in J} r_i a_i$, where r_i are coefficients and a_i belong to the set of edges. By piecewise linearity

$$f_*(x) = f_*(\sum_{i \in J} r_i a_i) = \sum_{i \in J} r_i f_*(a_i)$$

Because we assumed that the map f simplicial, we see that the sum is in the corresponding subset of $[0, 1]^J$. Thus $f_*(x)$ is uniquely determined. It is easy to see that the map is simplicial. Let $\{a_{t(1)}, ..., a_{t(n)}\}$ be some simplex in the collection, then it is mapped to simplex $\{f(a_{t(1)}), ..., f(a_{t(n)})\}$ by definition of the map f. □

Definition 2.65. Let $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ be simplicial maps. Then we say that the maps are contiguous if for every simplex $s = \{a_1, ..., a_n\}$ in (X, A) there exists a simplex k in (Y, B) for which the relations span $(f(a_1), ..., f(a_n)) \subset k$ and span $(g(a_1), ..., g(a_n)) \subset k$ hold.

Lemma 2.66. *If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are contiguous maps then they are homotopic.*

Proof. We define $H : X \times [0, 1] \to Y$ in the following way:

$$H(x, t) = tf(x) + (1 - t)g(x).$$

It is easy to see that $H(x, 1) = f(x)$ and $H(x, 0) = g(x)$. Because $f(x)$ and $g(x)$ both belong to the same simplex, it follows that $H(x, t)$ is well defined at every point. $\square$

2.4.1.1 Simplicial operators

Definition 2.67. Let S be some simplex and let X be the collection of its faces. We define operator $\triangleright : 2^X \to X$ to denote a simplex which is obtained by the intersection of all simplexes for which the union of all the faces in the domain are contained.

2.4.2 Coverings

We begin by recalling the definition of a covering

Definition 2.68. Let X be a topological space and let $\{A_i\}_{i \in J}$ be a collection of open subsets of the space X. Then we say that $\{A_i\}_{i \in J}$ is a covering of X if $\bigcup_{i \in J} A_i = X$.

We will denote the set of all open coverings of the space X by $\mathrm{COV}(X)$. Because in homology we are interested in a pair of spaces we give the definition of covering for the pair.

Definition 2.69. Let (X, A) be a topological pair. Then we say that $(\{U_i\}_{i \in J}, \{U_i\}_{i \in J'})$ is an open covering of (X, A) if the following conditions hold:

(1) $\{U_i\}_{i \in J}$ is an open covering of X

(2) $J' \subset J$.

(3) $\{U_i\}_{i \in J'}$ covers A

A covering defined above will be from now on denoted as $\mathcal{C}(U_i, J, J')$

We define the collection of all coverings of the pair (X, A) to be $\mathrm{COV}(X, A)$. Next we will see that we can define quasi-order for coverings of a topological pair (X, A) in a natural way. However, first we have to recall the definition of a refinement.

Definition 2.70. Let $\alpha = \mathcal{C}(U_i, I, I')$ and $\beta = \mathcal{C}(V_j, J, J')$ be coverings of some pair (X, A). Then the covering β is called a refinement of the covering α, if for every $V_j \in \beta$ there exists some $U_i \in \alpha$ for which $V_j \subset U_i$ holds.

Now we are ready to define a quasi-order in the collection of coverings of an arbitrary topological space.

Lemma 2.71. *Let $COV(X, A)$ be the collection of all coverings of the pair (X, A). Then the order $\leqslant$ defined by $\alpha \leqslant \beta$ if and only if β is a refinement of α, is a quasi-order.*

Proof. Any covering is a refinement of itself and if we have coverings for which $a \leqslant b$ and $b \leqslant c$ holds, then for every set U_c we can find U_b for which $U_c \subset U_b$ and $U_b \subset U_a$ for some U_a. Then we see that $U_c \subset U_a$ so actually c is refinement of a. □

2.4.2.1 Category of coverings

Definition 2.72. Let (X, A) be a topological pair and let α and β be its coverings such that $\alpha \leqslant \beta$. We define $\mathcal{P}(\alpha, \beta)$ to be the collection of all the following projection maps $p_\alpha^\beta : \beta \to \alpha$: Every $U_i \in \beta$ is mapped to some set $U_i' \in \alpha$ for which $U_i \subset U_i'$ holds.

We will now show that the family of projection maps is closed under composition.

Lemma 2.73. *Let α, β and γ be coverings of some topological pair, for which relation $\alpha \leqslant \beta \leqslant \gamma$ holds. Let p_α^β and p_β^γ be some projection maps. Then $p_\alpha^\beta \circ p_\beta^\gamma$ is also a projection map.*

Proof. Let $U \in \gamma$, then if $U' = p_\beta^\gamma(U) \in \beta$ and $U'' = p_\alpha^\beta \circ p_\beta^\gamma(U)$, by the definition condition $U \subset U' \subset U''$ holds . By properties of inclusion we can conclude that $U \subset U''$ and thus $p_\alpha^\beta \circ p_\beta^\gamma$ is a projection map. □

Lemma 2.74. *The collection of coverings $COV(X, A)$ is a category defined in the following way:*

(1) The objects of $COV(X, A)$ are all open coverings of the pair (X, A).

(2) We define $Hom(\beta, \alpha)$ to be $\mathcal{P}(\alpha, \beta)$, if $\alpha \leqslant \beta$. Else we define $Hom(\beta, \alpha)$ to be the empty set.

(3) The composition of maps is defined to be the usual composition of functions.

Proof. By Lemma 2.73 the composition of two projection maps is a projection map and thus it is well-defined. For every α there exists a well-defiend projection map p_α^α which acts like the identity map on $\mathrm{COV}(X, A)$. The associativity condition holds, because every morphism is a function. □

2.4.2.2 Directness property of coverings

For the order relation in the category of coverings we have following theorem

Lemma 2.75. *Let (X, A) be a topological pair. The category $COV(X, A)$ with the quasi-order defined in definition 2.70 is directed.*

Proof. Let $\alpha = \mathcal{C}(U_i, I, I')$ and $\beta = \mathcal{C}(V_j, J, J')$ be objects of $\mathrm{COV}(X, A)$. We will define a covering $\gamma = \mathcal{C}(W_a, I \times J, I' \times J')$ to be covering which consists of the following elements:

$$W_{(i,j)} = U_i \cap V_j$$

We will prove now that the covering constructed above is satisfies the condition of this lemma. Let $W \in \gamma$, then $W = U \cap V$ for some $U \in \alpha$ and $V \in \beta$. Thus $W \subset U$ and $W \subset V$. Because we chose W to be an arbitrary element of the collection γ, relations $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$ hold. □

2.4.2.3 Operations on coverings

We define the following fusioning operator for coverings

Definition 2.76. Let (X, A) be a topological pair and let β be some collection of coverings $\{\mathcal{C}(U_i^n, J_n, J_n') \mid n \in H\}$. Then we say that α is fusioned by the collection β, if it is spanned by the elements U_i^n and indexed by $L = \bigcup_{i \in H} J_i$.

Definition 2.77. Let (X, A) and (Y, B) be topological pairs. Let $\alpha = \mathcal{C}(\alpha_i, J, J')$ be a covering of pair (X, A) and let $\{\beta(x) \mid x \in J\}$ be a collection of coverings $\mathcal{C}(\beta_i, H_n, H_n')$ of the pair (Y, B) indexed by J. The stacked covering $\gamma = \alpha \times \{\beta(x) \mid x \in J\}$ of space $(X \times Y, A \times B)$ consists of elements of the form $a_i \times b_j$, where j belongs to collection of the covering $\beta(i)$. The covering is formally spanned by the index sets

$$H = \bigcup_{x \in J} H_x \text{ and } H' = \bigcup_{x \in J'} H_x'.$$

Let $f : H \to J$ be a function which tells in which set of the union does some specific element H_x belong to. The covering γ is of the form $\mathcal{C}(V_{f(i)} \times U_i^{f(i)}, H, H')$.

We call the collection $\{\beta(x)\}_{x \in J}$ stacks of the covering. It is also notable that the fusioned covering over the collection $\{\beta(x) \mid x \in J\}$ is same as the collection which we get from γ by composing it with projection function $\mathrm{pr}_Y : (X \times Y, A, \times B) \to (Y, B)$.

2.4.3 The construction

Now we are ready to give the definition for the nerve of the covering.

Definition 2.78. Let X be some set, then the operator $\mathrm{CNI}(X)$ is the following subset of the set 2^X:

$$V \in \mathrm{CNI}(X) \Leftrightarrow V \text{ is finite and intersection over all its elements is non-empty}$$

Definition 2.79. Let X be a topological space with an open covering $\mathcal{U}(X) = \{U_i\}_{i \in J}$. Then the nerve of this covering is the following abstract simplicial complex:

(1) The vertex set is defined to be the collection $\mathcal{U}(X)$.

(2) The simplex set of the abstract simplicial complex is defined to be $\mathrm{CNI}(\mathcal{U}(X))$.

The structure described above can be denoted as $\Gamma(\mathcal{U}(X), \mathrm{CNI}(\mathcal{U}(X)))$. However, to simplify notation we will from now on denote the nerve of the covering as $\Gamma(X)_{\mathcal{U}(X)}$.

Lemma 2.80. *The abstract simplicial complex $\Gamma(X)_{\mathcal{U}(X)}$ described above is well-defined.*

Proof. We have to show that any subset of an element in the collection $\mathrm{CNI}(\mathcal{U}(X))$ is in the collection. This follows from the fact that if $\bigcap_{i \in J} U_i \neq \varnothing$, then also $\bigcap_{i \in J'} U_i \neq \varnothing$ for any subcollection $J' \subset J$. $\square$

We will now give the definition of the nerve of the covering for topological pairs.

Definition 2.81. Let $\mathcal{L} = \mathcal{C}(U_i, J, J')$ be a covering of some topological pair (X, A). We denote $\mathcal{L}_X$ to be the covering taking elements from $\mathcal{L}$ indexed by J and $\mathcal{L}_A$ to be the covering, where we use elements from $\mathcal{L}$ indexed by J' . Then the nerve of this covering $\mathcal{S}(X, A)_{\mathcal{L}}$ is the simplicial pair $(\mathcal{S}(\mathcal{L}_X, \mathrm{CNI}(\mathcal{L}_X)), \mathcal{S}(\mathcal{L}_A, \mathrm{CNI}(\mathcal{L}_A))$.

Lemma 2.82. *The simplicial pair described above is well-defined.*

Proof. By Lemma 2.80, the simplicial complexes are well-defined. We only have to show that $\mathcal{S}(\mathcal{L}_A, \mathrm{CNI}(\mathcal{L}_A))$ is a subsimplex of $\mathcal{S}(\mathcal{L}_X, \mathrm{CNI}(\mathcal{L}_X))$. This follows directly from the definition of the pair of coverings, as we defined the covering $\mathcal{L}_A$ to be a sub covering of the covering $\mathcal{L}_X$. $\square$

Lemma 2.83. *Let $\beta = \mathcal{C}(V_i, I, I')$ and $\alpha = \mathcal{C}(U_i, I, I')$ be coverings of (X, A) with $\alpha \leqslant \beta$ and let p be a projection map belonging to $\mathcal{P}(\alpha, \beta)$. Let the map $p'^{\beta}_{\alpha} : \Gamma(X, A)_\beta \to \Gamma(X, A)_\alpha$ be a map between abstract simplicial complexes induced by the edges $V_i \in \beta$ and the map p. The map p'^{β}_{α} is a well-defined simplicial map.*

Proof. Well-definiteness follows directly from the fact that the inducing map p is well-defined. Let s be a simplex in $\Gamma(X, A)_\beta$ spanned by some sets $\{U_i\}_{i\in J}$ such that $\bigcap_{i\in J} U_i \neq \varnothing$. Every U_i is mapped to some vertex V_i in $\Gamma(X, A)_\alpha$ for which $U_i \subset V_i$. Intersection $\bigcap_{i\in J} V_i$ is clearly non-empty, so edges of every simplex span some simplex in the image. □

All the lemmas and proofs have been presented only for the abstract simplicial complexes. The geometric realization of the nerve of the covering will be denoted simply as $\mathcal{S}(X, A)_\alpha$. The concept of projection map is being used in many different contexts. However, because it is clear from the context which definition is used in all the contexts we will denote the projection map to be simply p_α^β. It is also worth noticing that from lemma 2.64 it follows that the induced map p_α^β between the geometric realizations $\mathcal{S}(X, A)_\beta$ and $\mathcal{S}(X, A)_\alpha$ is also well-defined and simplicial.
Projection maps are not uniquely determined by coverings α and β, but any of those maps are contiguous thus they define the same homomorphisms between corresponding homology groups of spaces $\mathcal{S}(X, A)_\alpha$ and $\mathcal{S}(X, A)_\beta$.

Lemma 2.84. *Let α and β be any coverings of (X, A) for which the condition $\alpha \leqslant \beta$ is satisfied. Then any two projection maps p_α^β and $p_\alpha'^\beta$ are contiguous.*

Proof. To prove that the maps are contiguous we form a simplex in $\mathcal{S}(Y, B)$ and prove that the images of both maps belong to it. Every vertex x_i in $\mathcal{S}(X, A)$ is mapped to $x'_{s(i)}$ and $x''_{d(i)}$ by p_α^β and $p_\alpha'^\beta$ respectively. Let U_i be the open sets corresponding to vertexes x_i. Now for every simplex $\{x_{r(0)}, ...x_{r(n)}\}$ the corresponding intersection $\bigcap_{i\in J} U_i$ is non-empty. Let U'_i correspond to $x'_{s(i)}$ and U''_i to $x''_{d(i)}$. We see that $\bigcap_{i\in J} U'_i \cap \bigcap_{i\in J} U''_i \neq \varnothing$ and thus the corresponding simplex $\{x'_{s\circ r(1)}, ..., x'_{s\circ r(n)}, x''_{d\circ r(1)}, ...x''_{d\circ r(n)}\}$ where the duplicates are removed is in the collection. In the case where the simplex belongs to $\mathcal{S}(A)$ it is easy to see that any projection map maps them to some simplex which lies inside $\mathcal{S}(B)$. □

Corollary 2.85. *Let (X, A) be a topological pair. The projection maps in collection $\mathcal{P}_{(X,A)}(a, b)$ define same homomorphisms between singular homology groups.*

Proof. The claim follows directly by using lemma 2.84 and the fact that contiguous simplicial maps induce the same homomorphisms between homology groups □

Thus we can define the unique homomorphisms $I_\alpha^\beta : H_q(\mathcal{S}(X, A)_\beta) \to H_q(\mathcal{S}(X, A)_\alpha)$ for every $q \in \mathbb{N}$.

Definition 2.86. Let $f : (X, A) \to (Y, B)$ and let $\beta = \mathcal{C}(U_i, J_\beta, J'_\beta)$ be a covering of (Y, A). Let K_β be indexation of the elements U_i for which the preimage under map f is non-empty. The covering $f^{-1}\beta$ is a covering of (X, A) which is formed by the collection $\mathcal{C}(V_i, K_\beta, K'_\beta)$, where $V_i = f^{-1}U_i$ for all $i \in K_\beta$.

Note that the collection $\alpha = f^{-1}\beta$ above is defined in a such way, that for every element $U_i \in \beta$ we make copy, take preimage of it and add it to the collection α.

Lemma 2.87. *Let $f : (X, A) \to (Y, B)$ be a function between pairs and let $\beta = \mathcal{C}(U_i, I, I')$ be a covering of (Y, B). Let α be the covering $f^{-1}\beta$ of (X, A). Then the induced map $f_* : \mathcal{S}(X, A)_\alpha \to \mathcal{S}(Y, B)_\beta$, defined by $f^{-1}U_i \to U_i$ is well-defined inclusion.*

Proof. Let $\{V_{j_1}, .., V_{j_n}\}$ be arbitrary simplex in the nerve $\mathcal{S}(X, A)_\alpha$, for which $V_i = f^{-1}U_i$ holds. Then for the edges of the corresponding abstract simplex the intersection $\bigcap_{i \in J_\beta} f^{-1}U_i$, where J_β is finite index set over elements of covering β, is non-empty. It implies that $\bigcap_{i \in J_\beta} U_i \neq \varnothing$, thus because edges were defined in one to one correspondence the function defined this way is well-defined inclusion. □

For projection maps and induced homomorphisms we have the following useful lemma.

Lemma 2.88. *Let α be a covering of the pair (Y, B) and let β be a covering of the pair (Y, B) for which $\alpha \leqslant \beta$ holds and let $f : (X, A) \to (Y, B)$ be a function between pairs. Then the pre-image $f^{-1}\beta$ of the covering β is a refinement of $f^{-1}\alpha$.*

Proof. This follows directly from the fact that if $U \subset V$, then also $f^{-1}U \subset f^{-1}V$. □

Lemma 2.89. *Let $f : (X, A) \to (Y, B)$ and let α and β be coverings of (Y, B) such that $\alpha \leqslant \beta$. Let $\alpha' = f^{-1}\alpha$ and let $\beta' = f^{-1}\beta$ and let p_α^β be a projection map for the coverings. We denote the inclusion maps induced by function f by $f_\alpha : \mathcal{S}(X, A)_{\alpha'} \to \mathcal{S}(Y, B)_\alpha$. Then there exists a projection map $p'^{\beta'}_{\alpha'}$ for which the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{S}(X, A)_{\beta'} \arrow[r, "p'^{\beta'}_{\alpha'}"'] \arrow[d, "f_\beta"] & \mathcal{S}(X, A)_{\alpha'} \arrow[d, "f_\alpha"] \\
\mathcal{S}(Y, B)_\beta \arrow[r, "p_\alpha^\beta"'] & \mathcal{S}(Y, B)_\alpha
\end{tikzcd}$$

Proof. Lemma 2.88 states that $f^{-1}\beta$ is refinement of $f^{-1}\alpha$. We define map $p'^{\beta'}_{\alpha'}$ to be the map which takes every U_i to element $f_a^{-1}(p_\alpha^\beta \circ f_\beta(U_i))$. We have to show that the map is well-defined. We recall that p_α^β maps every set to some larger set. Thus from the equations

$$U_i \subset f^{-1}f(U_i) \subset f^{-1}p_\alpha^\beta f(U_i)$$

we can conclude that every element in the collection β' is mapped to non-empty element of α' and the map is well-defined. The second part of the claim follows directly from Lemma 2.87, which states that the maps f_α and f_β are inclusions between the corresponding simplices. □

2.5 Čech homology

In this section we will give construction of Čech homology. It will be shown that Čech homology satisfies almost all the Eilenberg-Steenrod axioms. We will show that the Čech homology is a functor in details and provide a reference for the proofs of all the other axioms.

2.5.1 Functorial properties

2.5.1.1 The construction

Definition 2.90. Let α be an element of $\mathrm{COV}(X, A)$ and let G be an abelian group. Then we can define the group $H_{q,\alpha}(X, A; G)$ using the singular homology groups in the following way:

$$H_{q,\alpha}(X, A; G) = H_q(\mathcal{S}(X, A)_\alpha; G).$$

Let α, β be coverings of (X, A) for which relation $\alpha \leqslant \beta$ holds. Let p^β_α be some projection map between the coverings. Let

$$I^\beta_\alpha : H_{q,\beta} \to H_{q,\alpha}$$

be the homomorphisms induced by p^β_α. Since all projection maps p^β_α are contingous, it follows that I^β_α is independent of the choice of p^β_α.

Lemma 2.91. *Let $q \in \mathbb{N}$ be a fixed number. Let (X, A) be a topological pair and let G be an abelian group. Then the groups $H_{q,\alpha}(X, A; G)$ together with maps I^α_β form an inverse system of groups.*

Proof. Lemma 2.74 says that $\mathrm{COV}(X, A)$ with the morphisms induced by the quasi-order relation forms a category. Let $I : \mathrm{COV}(X, A) \to \mathrm{Ab}$ be a functor which maps every covering to a corresponding abelian group $H_{q,\alpha}(X, A; G)$ and every morphism between the coverings to the induced homomorphism. Using lemmas 2.85 and 2.22 we see that the functor defined in this way is a well-defined functor and forms an inverse system. □

Now we are ready to define the Čech homology groups $\check{\mathrm{H}}_n$. We denote the spanning functor of the inverse system defined above as $I_q(X, A; G)$.

Definition 2.92. Let (X, A) be a topological pair and let G be an abelian group. Then the Čech homology groups with coefficients G are defined to be inverse limits of the corresponding functors I. Simply denoted

$$\check{\mathrm{H}}_q(X, A; G, q) = \lim_{\rightarrow} I_q(X, A; G)$$

2.5.1.2 Induced homomorphisms

In this section we will investigate properties of induced homomorphisms. Earlier we defined Čech homology groups to be the limits of the groups induced by coverings of pairs. In section 2.3.2 we defined the concept of a limit homomorphism. In the next theorem we will show that such homomorphism exists.

Definition 2.93. Let $f : (X, A) \to (Y, B)$ be a continuous function between pairs of spaces and let $\phi : \mathrm{COV}(Y, B) \to \mathrm{COV}(X, A)$ be a functor, which maps every α to $f^{-1}\alpha$. Lemma states that 2.88 ϕ is order preserving. We denote

$$\mathcal{A}(f) = \{f_\alpha \mid \alpha \text{ is covering of the pair } (Y, B)\}$$

to be the family of inclusions defined in section 2.4.3 corresponding to functor ϕ.

Theorem 2.94. *Let $f : (X, A) \to (Y, B)$ be a continuous function between pairs of spaces. Then the family $\mathcal{A}(f)$ induces homomorphisms between corresponding Čech homology groups:*

$$f'_{\alpha,q} : H_q(\mathcal{S}(X, A)_{f^{-1}\alpha}; G) \to H_q(\mathcal{S}(Y, B)_\alpha; G)$$

We denote a family of functions indexed by coverings of (Y, B) described above with $\mathcal{A}_q$. Then every such collection is a natural transformation of the Čech homology groups and the limit function is well defined.

Proof. Let β and α be some coverings of the pair (Y, B). We denote by α' and β' the preimage categories of α and β under the map f respectively. The diagram of lemma 2.89 shows that the corresponding maps between coverings commute. Thus we have the following commutable diagram:

$$\begin{tikzcd}
H_q(\mathcal{S}(X, A)_{\beta'}) \arrow[r, "I^{\beta'}_{\alpha'}"'] \arrow[d, "f'_{\beta,q}"] & H_q(\mathcal{S}(X, A)_{\alpha'}) \arrow[d, "f'_{\alpha,q}"] \\
H_q(\mathcal{S}(Y, B)_{\beta}) \arrow[r, "I^{\beta}_{\alpha}"'] & H_q(\mathcal{S}(Y, B)_{\alpha})
\end{tikzcd}$$

The fact that the collection $\mathcal{A}_q$ is a natural transformation of the groups follows directly from the above diagram. Thus by lemma 2.36 there exists a limit homomorphism

$$f_q : \check{\mathrm{H}}_q(X, A; G) \to \check{\mathrm{H}}_q(Y, B; G)$$

□

We will now show that Čech homology groups are functors.

Theorem 2.95. *Čech homology groups are functors from the category of topological pairs to the category of abelian groups*

Proof. (1) Every pair of spaces is mapped uniquely to some abelian group.

(2) The limit of an identity map is an identity homomorphism.

(3) Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ be continuous maps between pairs. We know that f and g induce homomorphisms

$$f_* : H_*(X, A) \to H_*(Y, B) \text{ and } g_* : H_*(X, A) \to H_*(Z, C).$$

Thus using axioms of singular homology theory we see that $(g \circ f)_* = g_* \circ f_*$ holds for the singular homology groups. Then by using theorem 2.39 we conclude that the limit homomorphism $\lim((g \circ f)_*)$ equals $\lim g_* \circ \lim f_*$.

□

2.5.1.3 Boundary map

In this section we give construction of the boundary map $\partial_n : \check{H}_n(X, A) \to \check{H}_{n-1}(A)$ and prove that it induces a natural transformation between the corresponding functors.

Definition 2.96. Let (X, A) be a topological pair. Then we denote the inverse system formed by $(\{H_q(\mathcal{S}(A)_\alpha)\}, \mathrm{COV}(X, A))$ simply as $sI_q(X, A)$

Because I_q is inverse system of abelian groups, the limit $\lim sI_q(X, A)$ is well-defined for all q.

Definition 2.97. Let (X, A) be a topological pair and let $\partial_\alpha : H_q(\mathcal{S}(X, A)_\alpha) \to H_{q-1}(\mathcal{S}(A)_\alpha)$ be a boundary map as in the definition of singular homology for every $\alpha \in \mathrm{COV}(X, A)$. By defining $\phi : \mathrm{COV}(X, A) \to \mathrm{COV}(X, A)$ to be just the identity map and noting that every ∂_α is a natural transformation, we obtain an inverse system of morphisms between systems $I_q(X, A)$ and $sI_{q-1}(X, A)$. We denote the limit morphisms of the system as ∂'_q.

Definition 2.98. Let ϕ : COV(X,A) $\to$ COV(A) be the following map: Let α be a covering of the pair (X, A) and let $\mathcal{C}(\alpha_i, J, J')$ be its decomposition. Then $\phi(\alpha)$ is defined to be the covering $\mathcal{C}(\alpha'_i, P', \varnothing)$, where the corresponding open sets satisfy $\alpha'_i = \alpha_i \cap A$ and $P' \subset J'$ is indexation of non-empty elements α'_i. Then we say that the map ϕ is natural projection from $\mathrm{COV}(X, A)$ to $\mathrm{COV}(A)$.

Definition 2.99. Let $\phi : \mathrm{COV(X,A)} \to \mathrm{COV}(A)$ be a natural homomorphism. Let

$$\Phi_\alpha : I_q(A)(\phi(\alpha)) \to sI_q(X,A)(\alpha)$$

be an identity map spanned by ϕ for every α and $q \in \mathbb{N}$. Because every map Φ_α is an inclusion, by lemma C.2 the morphisms form an inverse system of morphisms. Thus we can define $\Phi = \lim\{\Phi_\alpha\}$.

Lemma 2.100. *Let* (X,A) *and* (Y,B) *be topological pairs and let* $f : (X,A) \to (Y,B)$ *be a continuous function. Then the following diagram commutes:*

$$\begin{tikzcd}
COV(X,A) \arrow[r, "\phi"'] & COV(A) \\
COV(Y,B) \arrow[u, "f^{-1}"'] \arrow[r, "\phi"'] & COV(B) \arrow[u, "f^{-1}|A"']
\end{tikzcd}$$

Proof. Let $\alpha \in \mathrm{COV}(X,A)$, with decomposition $\mathcal{C}(\alpha_i, J, J')$. Then $\phi(\alpha) = \mathcal{C}(A \cap \alpha_i, P', \varnothing)$ and $f^{-1}(\phi(\alpha)) = \mathcal{C}(f^{-1}(A \cap \alpha_i), K', \varnothing)$, where $K' \subset P'$, for which $f^{-1}(A \cap \alpha_i)$ is non-empty. On the other hand $f^{-1}\alpha = \mathcal{C}(f^{-1}\alpha_i, T, T')$, where T is indexation of non-empty elements $f^{-1}\alpha_i$ and $\phi(f^{-1}(\alpha)) = \mathcal{C}(f^{-1}(A \cap \alpha_i), R', \varnothing)$. It is left to proof that $R' = K'$, this follows directly from the set-theoretical relation $f^{-1}U_i \cap A = f^{-1}(U_i \cap B)$. Thus the claim holds. ∎

Lemma 2.101. *Let* (X,A) *be a topological pair and* $f : (X,A) \to (Y,B)$ *a continuous function. Then the following diagram commutes:*

$$\begin{tikzcd}
sI_q(X,A)(f^{-1})\alpha \arrow[d, "f_\alpha"] & I_q(A)(f^{-1}\phi(\alpha)) \arrow[l, "\Phi_{f^{-1}\alpha}"] \arrow[d, "f_{\phi(\alpha)}"] \\
sI_q(Y,B)(\alpha) & I_q(B)(\phi(\alpha)) \arrow[l, "\Phi_\alpha"]
\end{tikzcd}$$

Proof. By Lemma 2.100 the diagram is well-defined. The claim follows directly from the fact that morphisms Φ_α are inclusions. ∎

Lemma 2.102. *The homomorphism* Φ *defined above is isomorphism between* $\lim I_q(X,A)$ *and* $\lim sI_q(X,A)$.

Proof. By definition ϕ is injective and surjective. Every identity map is an isomorphism, thus to apply Corollary 2.54 it is enough to show that $\phi(\mathrm{COV}(X, A))$ is cofinal in $\mathrm{COV}(A)$. Let $\alpha \in \mathrm{COV}(A)$ be arbitrary covering with decomposition $\mathcal{C}(\alpha_i, J, \varnothing)$. We construct covering $\beta \in \mathrm{COV}(X, A)$ in the following way. For every α_i we find an open set β_i of X, for which $\beta_i \cap A = \alpha_i$. Let K be indexation of elements α_i, which is formed by adding one element 0 to index set J for which $\alpha_0 = X$ holds. Then we define β to be covering $\mathcal{C}(\beta_i, K, K')$, where K' is the corresponding subindexation of K. Now clearly $\alpha \leqslant \phi(\beta)$ holds. Thus by the Corollary 2.54 homomorphism Φ is isomorphism. $\square$

Definition 2.103. Boundary homomorphism $\partial_q : \check{H}_q(X, A) \to \check{H}_{q-1}(A)$ is defined as $\partial_q = \Phi^{-1} \circ \partial'_q$.

Theorem 2.104. *The boundary homomorphism ∂_q is natural transformation between $\check{H}_q(X, A)$ and $\check{H}_{q-1}(A)$*

Proof. From singular homology theory we know that the following diagram commutes:

$$\begin{tikzcd} H_q(\mathcal{S}(X,A)_{f^{-1}\alpha}) \arrow[r, "\partial'_{f^{-1}\alpha}"'] \arrow[d, "f_\alpha"] & H_q(\mathcal{S}(A)_{f^{-1}\alpha}) \arrow[d, "f_\alpha|H_q(\mathcal{S}(A)_{f^{-1}\alpha})"] \\ H_q(\mathcal{S}(Y,B)_\alpha) \arrow[r, "\partial'_\alpha"'] & H_q(\mathcal{S}(B)_\alpha) \end{tikzcd}$$

Because limit if functor, we can take limit over the diagram. Combining the diagram above with diagram of Lemma **??** composed with limit functor and Lemma 2.102 we obtain the following diagram:

$$\begin{tikzcd} \check{H}_q(X,A) \arrow[r, "\partial'_q"'] \arrow[d, "\lim f"] & \lim sI_{q-1}(X,A) \arrow[d, "\lim f'"] & \check{H}_{q-1}(A) \arrow[l, "\Phi"'] \arrow[d, "\lim f"] \\ \check{H}_q(Y,B) \arrow[r, "\partial'_q"'] & \lim sI_{q-1}(Y,B) & \check{H}_{q-1}(B) \arrow[l, "\Phi"'] \end{tikzcd}$$

Where $\lim f' = \lim f | \lim sI_{q-1}(X, A)$. The claim follows directly from this. $\square$

2.5.2 Other properties

In this section we will introduce most important properties of the Čech homology functor and show that in some special circumstances it satisfies all the Eilenberg-Steenrod axioms.

However, because the theorems are technical we will omit the proofs and give the references to them instead.

Theorem 2.105. ***Dimension Axiom of Čech homology***. *Let $X = \{x\}$ be a space consisting of one point. Then the Čech homology groups $\check{\mathrm{H}}_n(X;G)$ are trivial groups for $n \geqslant 1$ and isomorphic to G when $n = 0$.*

Proof. For the space X there exists only one open covering $\alpha = \{x\}$ and thus the inverse system of the corresponding singular homology groups is trivial. It is easy to see that the limit of this system is the group $H_q(\mathcal{S}(X)_a;G)$ together with the identity homomorphisms. The simplex $\mathcal{S}(X)_\alpha$ consists only of one edge and thus is homeomorphic to a one point space. From the dimension axiom of singular homology it is easy to see that

$$\check{\mathrm{H}}_n(X;G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n \geqslant 1 \end{cases}$$

□

Theorem 2.106. ***Homotopy Axiom of Čech homology***

Proof. Proof is presented in the section B.3. □

Theorem 2.107. ***Excision Axiom of Čech homology***

Proof. Proof of this claim can be found in Eilenberg [2] theorem IX.6.1. □

Theorem 2.108. ***Additivity axiom of Čech homology***

Proof. Idea of the proof: Let $X = \bigsqcup_{i\in J}\{A_i\}$. Then it can be shown that $\mathrm{COV}'(X) = \bigsqcup_{i\in J}\{\mathrm{COV}(A_i)\}$ is cofinal in $\mathrm{COV}(X)$. It is easy to see that an abstract simplex induced by $\mathrm{COV}'(X)$ is disjoint union of simplices spanned by covering $\mathrm{COV}(A_i)$, thus applying the additivity axiom of singular homology theory to the simplex we obtain the claim. □

The exactness axiom is not satisfied in general for the Čech homology. However for our purpose the following special case is essential:

Theorem 2.109. ***Exactness axiom of Čech homology***. *Let (X, A) be a compact pair and let G be a vector space, then the Čech homology functor satisfies the exactness axiom.*

Proof. Proof of this claim can be found in Eilenberg [2] theorem IX.7.6. □

2.6 Čech cohomology

Like in the case of Čech homology, we will provide detailed proofs only for the functonrial properties of the Čech cohomology.

2.6.1 Functorial properties

2.6.1.1 The construction

Definition 2.110. Let α be an element of $\mathrm{COV}(X, A)$ and let G be an abelian group. Then we can define the group $H^q_\alpha(X, A; G)$ using the singular cohomology groups in the following way:

$$H^q_\alpha(X, A; G) = H^q(\mathcal{S}(X, A)_\alpha; G).$$

Let α, β be coverings of (X, A) for which relation $\alpha \leqslant \beta$ holds. Let p^β_α be some projection map between the coverings. Let

$$D^\alpha_\beta : H_{q,\beta} \to H_{q,\alpha}$$

be the homomorphisms induced by p^β_α. Since all projection maps p^β_α are contingous, it follows that D^α_β is independent of the choice of p^β_α.

Lemma 2.111. *Let $q \in \mathbb{N}$ be a fixed number. Let (X, A) be a topological pair and let G be an abelian group. Then the groups $H^q_\alpha(X, A; G)$ together with maps D^α_β form an inverse system of groups.*

Proof. Lemma 2.74 says that $\mathrm{COV}(X, A)$ with the morphisms induced by the quasi-order relation forms a category. Let $I : \mathrm{COV}(X, A) \to \mathrm{Ab}$ be a functor which maps every covering to a corresponding abelian group $H^q_\alpha(X, A; G)$ and every morphism between the coverings to the induced homomorphism. Using lemmas 2.85 and 2.22 we see that the functor defined in this way is a well-defined functor and forms an inverse system. □

The directed system functor defined above will be denoted as $D^q(X, A; G)$.

Definition 2.112. Let (X, A) be a topological pair and let G be an abelian group. Then the Čech cohomology groups with coefficients G are defined to be inverse limits of the corresponding functors D. Simply denoted

$$\check{\mathrm{H}}^q(X, A; G, q) = \lim_{\to} I_q(X, A; G)$$

2.6.1.2 Induced homomorphisms

Theorem 2.113. *Let $f : (X, A) \to (Y, B)$ be a continuous function between pairs of spaces. The family $\mathcal{A}(f)$ induces homomorphisms between corresponding Čech homology groups:*

$$f^q_\alpha : H^q(\mathcal{S}(Y, B)_\alpha; G) \to H^q(\mathcal{S}(X, A)_{f^{-1}\alpha}; G)$$

We denote a family of functions indexed by coverings of (Y, B) described above with $\mathcal{A}^q$. Then every such collection is a natural transformation of the Čech homology groups and the limit function is well defined.

Proof. Let β and α be some coverings of the pair (Y, B). We denote by α' and β' the preimage categories of α and β under the map f respectively. The diagram of lemma 2.89 shows that the corresponding maps between coverings commute. Thus we have the following commutable diagram:

$$\begin{tikzcd}
H^q(\mathcal{S}(X,A)_{\beta'}) & H^q(\mathcal{S}(X,A)_{\alpha'}) \arrow[l, "D^{\beta'}_{\alpha'}"'] \\
H^q(\mathcal{S}(Y,B)_{\beta}) \arrow[u, "f^q_\beta"'] & H^q(\mathcal{S}(Y,B)_{\alpha}) \arrow[l, "D^{\beta}_{\alpha}"] \arrow[u, "f^q_\alpha"']
\end{tikzcd}$$

The fact that the collection $\mathcal{A}^q$ is a natural transformation of the groups follows directly from the above diagram. Thus by lemma 2.45 there exists a limit homomorphism

$$f^q : \check{\mathrm{H}}^q(Y, B; G) \to \check{\mathrm{H}}^q(X, A; G)$$

□

Theorem 2.114. *Čech cohomology groups are contra-variant functors from the category of topological pairs to the category of abelian groups*

Proof. (1) Every pair of spaces is mapped uniquely to some abelian group.

(2) The limit of an identity map is an identity homomorphism.

(3) Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ be continuous maps between pairs. We know that f and g induce homomorphisms

$$f^* : H^q(Y, B) \to H^q(X, A) \text{ and } g^* : H^q(Z, C) \to H^q(Y, B) \text{ for every } q \in \mathbb{N}.$$

Thus using axioms of singular homology theory we see that $(g \circ f)^* = f^* \circ g^*$ holds for the singular homology groups. Then by using theorem 2.39 we conclude that

$$\lim((g \circ f)^*) = \lim(f^* \circ g^*) = \lim f^* \circ \lim g^*.$$

□

2.6.1.3 Boundary maps

Definition 2.115. Let (X, A) be a topological pair. Then we denote the directed system formed by $(\{H^q(\mathcal{S}(A)_\alpha)\}, \mathrm{COV}(X, A))$ simply as $sD^q(X, A)$

Definition 2.116. Let (X, A) be a topological pair and let $\partial_\alpha : H^{q-1}(\mathcal{S}(A)_\alpha) \to H^q(\mathcal{S}(X, A)_\alpha)$ be a boundary map as in the definition of singular cohomology for every $\alpha \in \mathrm{COV}(X, A)$. By defining $\phi : \mathrm{COV}(X, A) \to \mathrm{COV}(X, A)$ to be just the identity map and noting that every ∂^α is a natural transformation, we obtain a directed system of morphisms between systems $D^q(X, A)$ and $sD^{q-1}(X, A)$. We denote the limit morphisms of the system as ∂_*^q

Definition 2.117. Let $\phi : \mathrm{COV(X,A)} \to \mathrm{COV}(A)$ be a natural homomorphism. Let

$$\Phi^\alpha : D^q(X, A)(\alpha) \to D^q(A)(\phi(\alpha))$$

be an identity map spanned by ϕ for every α and $q \in \mathbb{N}$. Because every map Φ^α is an inclusion, by Lemma C.2 the morphisms form an inverse system of morphisms. Thus we can define $\Phi^{\mathrm{op}} = \lim\{\Phi^\alpha\}$.

Lemma 2.118. *Let (X, A) be a topological pair and $f : (X, A) \to (Y, B)$ a continuous function. Then the following diagram commutes:*

$$\begin{tikzcd}
sD^q(X, A)(f^{-1}\alpha) \arrow[r, "\Phi^{f^{-1}\alpha}"'] & D^q(A)(f^{-1}\phi(\alpha)) \\
sD^q(Y, B)(\alpha) \arrow[u, "f^\alpha"'] \arrow[r, "\Phi^\alpha"'] & D^q(B)(\phi(\alpha)) \arrow[u, "f^{\phi(\alpha)}"']
\end{tikzcd}$$

Proof. By Lemma 2.100 the diagram is well-defined. The claim follows directly from the fact that morphisms Φ^α are inclusions. $\square$

Lemma 2.119. *The homomorphism Φ defined above is isomorphism between* $\lim sD^q(X, A)$ *and* $\lim D^q(X, A)$.

Proof. By proof of the corresponding Lemma for Čech homology, the conditions for Theorem 2.56 are satisfied. The claim follows directly from this. $\square$

Definition 2.120. Boundary homomorphism $\partial^q : \check{H}^{q-1}(A) \to \check{H}^q(X, A)$ is defined as $\partial^q = \partial_*^q \circ \Phi^{-1}$.

Theorem 2.121. *The boundary homomorphism ∂^q is natural transformation between $\check{H}^{q-1}(A)$ and $\check{H}^q(X, A)$.*

Proof. We obtain the following diagram from singular cohomology theory:

$$\begin{tikzcd}
H^q(\mathcal{S}(X, A)_{f^{-1}\alpha}) & H^q(\mathcal{S}(A)_{f^{-1}\alpha}) \arrow[l, "\partial'^{f^{-1}\alpha}"] \\
H^q(\mathcal{S}(Y, B)_\alpha) \arrow[u, "f^\alpha"'] & H^q(\mathcal{S}(B)_\alpha) \arrow[l, "\partial'^\alpha"] \arrow[u, "f^\alpha|H_q(\mathcal{S}(A)^{f^{-1}\alpha})"']
\end{tikzcd}$$

Because limit is functor in category of directed systems, we can take limit over the diagram. Combining the diagram above with diagram of Lemma 2.118 composed with limit functor and Lemma 2.119 we obtain the following diagram:

$$\begin{array}{ccccc}
\check{H}^q(X,A) & \xleftarrow[\partial^q_*]{} & \lim sD^{q-1}(X,A) & \xrightarrow[\Phi^{\mathrm{op}}]{} & \check{H}^{q-1}(A) \\
\uparrow{\scriptstyle \lim f} & & \uparrow{\scriptstyle \lim f'} & & \uparrow{\scriptstyle \lim f} \\
\check{H}^q(Y,B) & \xleftarrow[\partial^q_*]{} & \lim sD^{q-1}(Y,B) & \xrightarrow[\Phi^{\mathrm{op}}]{} & \check{H}^{q-1}(B)
\end{array}$$

Where $\lim f' = \lim f | \lim sD^{q-1}(X,A)$. The claim follows directly from this. ☐

2.6.2 Other properties

Theorem 2.122. ***Dimension Axiom of Čech cohomology****. Let $X = \{x\}$ be a space consisting of one point. Then the Čech clhomology groups $\check{\mathrm{H}}^n(X;G)$ are trivial groups for $n \geqslant 1$ and isomorphic to G when $n = 0$.*

Proof. For the space X there exists only one open covering $\alpha = \{x\}$ and thus the directed system of the corresponding singular homology groups is trivial. It is easy to see that the limit of this system is the group $H^n(\mathcal{S}(X)_a;G)$ together with the identity homomorphisms. The simplex $\mathcal{S}(X)_\alpha$ consists only of one edge and thus is homeomorphic to a one point space. From the dimension axiom of singular homology it is easy to see that

$$\check{\mathrm{H}}^n(X;G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{if } n \geqslant 1 \end{cases}$$

☐

Theorem 2.123. ***Homotopy Axiom for Čech cohomology***

Proof. Proof is presented in the section B.3. ☐

Theorem 2.124. ***Excision Axiom for Čech cohomology***

Proof. Proof of this claim can be found in Eilenberg [2] theorem IX.6.1. ☐

Theorem 2.125. ***Additivity axiom for Čech cohomology***

Proof. Follows directly from proof of the Additivity axiom for Čech cohomology, expect we use the additivity axiom of singular homology in the end of the proof. ☐

Unlike Čech homology, the cohomology satisfies the Exactness axiom.

Theorem 2.126. ***Exactness axiom for Čech cohomology***

Proof. Proof of this claim can be found in Eilenberg [2] theorem IX.7.6. ☐

2.7 Čech homology with compact carriers

Because the Čech homology does not satisfy the exactness axiom for all spaces, we will present a modication of the theory using compact carriers. In this section we assume that the coefficent group G has vector space structure. The idea of the construction is to use a directed system of compact pairs, where every pair is mapped to the corresponding Čech homology group and use limit functor to define the groups. Recall that the Čech homology functor satisfies all of the Eilenberg-Steenrod axioms for compact pairs with coefficents in vector spaces. We will show that the limit functor preserves all the properties and thus it will be shown the functor with compact carriers is a homology theory.

2.7.1 Functorial properties

Lemma 2.127. *Let $\mathscr{C}$ be collection of all compact spaces. Then relation induced by*

$$(X, A) \leqslant (Y, B) \Leftrightarrow X \subset Y \text{ and } A \subset B$$

is quasi-order. We denote the relation simply as $R(\mathscr{C})$

Proof. Follows directly from the fact order-relation is transitive. $\square$

Theorem 2.128. *Compact spaces together with inclusions as morphisms form a category $CAT(\mathscr{C})$ corresponding to relation $R(\mathscr{C})$.*

Proof. The composition of two inclusions is inclusions. The identity morphism defined as $id : (X, A) \to (X, A)$ is clearly an inclusion. Also, the transitivity axiom is satisfied, because inclusions are functions. It is easy to see that there exists an inclusion between pairs (X, A) and (Y, B) only and only if $X \subset Y$ and $A \subset B$. Thus the category corresponds to relation. $\square$

Every such inclusion $i : (X, A) \to (Y, B)$ induces homomorphism

$$i_* : \check{H}(X, A) \to \check{H}(Y, B)$$

between the corresponding groups.

Definition 2.129. Let (X, A) be a topological pair, then a directed system $cD_q(X, A)$ is defined in the following way:

(1) We define the spanning category $\lambda(X, A)$ to be subcategory of CAT($\mathscr{C}$) consisting of all compact pairs (U, V), for which $U \subset X$ and $V \subset A$ holds.

(2) The functor $cD_q(X, A)$ maps every object and morphism to image of functor $\check{H}_q$

Because $cD_q(X, A)$ is a directed system of abelian groups, the limit of it is well defined. Recall that the Lemma 2.47 states that the limit is functor in category of directed systems.

Definition 2.130. The Čech groups with compact carriers is defined as

$$\bar{H}_q(X, A) = \lim cD_q(X, A)$$

We aim to proof that the Čech groups with compact carriers can be extended to a functor. Thus we have to define induced homomorphisms. We do it in the following way:

Definition 2.131. Let $f : (X, A) \to (Y, B)$ be a continuous function and let

$$\phi : cD_q(X, A) \to cD_q(Y, B)$$

be functor, which maps a compact pair (U, V) to pair $(F(U), F(V))$. This functor is well-defined, because f is continuous and following from properties of the functions it is order-preserving. Define system

$$\{f_\alpha : cD_q(X, A)(\alpha) \to cD_q(Y, B)(\phi(\alpha)) \mid \alpha \in \lambda(X, A)\}$$

where every $f_{(U,V)} = f|(U, V)$. We say that the system described above is natural respect to function f.

Remark 2.132. From the lemma C.2 and fact that $\check{H}_q(X, A)$ is functor, it follows that every family of morphism $\{f_a : cD_q(X, A) \to cD_q(Y, B)\}$ is natural transformation.

Definition 2.133. We define $\bar{H}$ to be functor from category of topological pairs TOP^2 to category of abelian groups Ab, where every object (X, A) is mapped to $\bar{H}(X, A)$ and every function f to limit $\lim\{f_\alpha\}$ of natural directed system corresponding to f.

The well-definedness of the functor defined above follows directly from the theorem 2.47.

2.7.2 Boundary homomorphism and exactness

To define an exact sequence we need the following two systems

Definition 2.134. Let (X, A) be a topological pair and let $\lambda(X, A)$ be as in the definition 2.129. Let $sT : \lambda(X, A) \to \lambda(A, \varnothing)$ be a functor which maps every compact pair (K, K') to pair $(K', \varnothing)$. Then we define the system $sD_q(X, A)$ as a system for which we use $cD_q(A)$ as the functor and which is spanned by elements $sT \circ \lambda(X, A)$.

Similarly for system over first parameter we define

Definition 2.135. Let (X, A) be a topological pair. Let $fT : \lambda(X, A) \to \lambda(X, \varnothing)$ be a functor which maps every compact pair (K, K') to pair $(K, \varnothing)$. Then we define the system $fD(X, A)$ as a system for which we use $cD_q(X)$ as the functor and which is spanned by elements $fT \circ \lambda(X, A)$.

Lemma 2.136. *Let*

Definition 2.137. Let (X, A) be a topological pair and let

$$\phi : cD_q(X, A) \to cD_{q-1}(A)$$

be a map which maps every $(U, V) \in cD_q(X, A)$ to $(V, \varnothing) \in cD_{q-1}(A, \varnothing)$. The functor is clearly order preserving. Define a directed system of morphisms as

$$\{\partial'_{(U,V)} : cD_q(X, A)(U, V) \to cD_q(A)(V)\}$$

where every $\partial'_{(U,V)}$ is the corresponding natural transformation of Čech homology.

Theorem 2.138. $\partial_q = \lim(\partial'_q)$ *forms a natural transformation between* $\bar{H}_q(X, A)$ *and* $\bar{H}_{q-1}(A)$

Proof. Follows directly from the fact ∂_q is natural transformation, applying functor lim to the diagram of transformations we obtain the claim. □

Theorem 2.139. ***Exactness*** *for Cech homology with compact carriers*

Proof. Let (X, A) be a topological pair. For every compact pair $(K, K') \subset (X, A)$, the following sequence is exact:

$$\cdots \longrightarrow \check{H}_n(K') \xrightarrow[i_*]{} \check{H}_n(K) \xrightarrow[j_*]{} \check{H}_n(K, K') \xrightarrow[\partial_n(K)]{} \check{H}_{n-1}(K') \longrightarrow \cdots$$

The idea is to apply theorem 2.57 to every connected triple in the sequence. () □

2.7.3 Other properties

Theorem 2.140. ***Homotopy axiom*** *for Cech homology with compact carriers*

Proof. Let $f, g : (X, A) \to (Y, B)$ be homotopic functions. Then $(f|(U, V))_* = (g|(U, V))_*$ for all the induced homomorphisms of Čech groups for all compact pairs $(U, V) \subset (X, A)$. Taking limit of the induced maps we obtain the claim. □

Chapter 3

Applications

3.1 Tools

3.1.1 Lefschetz number

3.2 Lefschetz fixed-point theorem

[Theory of this will be presented later]

3.3 Kakutani fixed point theorem

The fixed point theorem of Kakutani can be derived for Lefchetz fixed-point theorem.

Definition 3.1. Let X be vector space with topological structure and K topological field. If vector addition $+ : X \times X \to X$ and scalar multiplication $\cdot : K \times X \to X$ are continuous we say that X is topological vector space over field K or shortly $L(X, K)$

Topological vector space has following property:

Theorem 3.2. *Kakutani fixed point theorem. Let S be a non empty, compact and convex subset of a locally convex Hausdorff space. Let $f : S \to 2^s$ be a multivalued function on S which has closed graph and the property that $f(x)$ is convex and non-empty for all $x \in S$. Then the set of fixed points of f is non-empty and compact.*

3.4 Continuous game theory

In this section we will apply theory developed in previous chapters to game theory. We define n player continuous game structure $Cgs(J, \mathcal{A})$ in a following way:

Definition 3.3. Let J be finite indexing set and $\mathcal{A}$ collection of continuous utility functions $\{u_i : X_i \to \mathbb{R}\}_{i \in J}$ with a compact domain. Then we say that $Cgs(J, \mathcal{A})$ is a continuous game.

Now we give definition of Nash equilibrium for Cgs.

Definition 3.4. Let $Cgs(J, \mathcal{A})$ be continuous game we say that point $x^* \in \prod_{i \in J} X_i$ is Nash equilibrium if for every $k \in J$ and every $x'_k \in X_k$ following condition holds:

$$u_k(x*) \geqslant u_k(x_1, ..., x'_k, ..., x_n)$$

Next we will prove important lemma which says that game has equilibrium only and only if specific function has fixed point.

Lemma 3.5. *$Cgs(n, \mathcal{A})$ has Nash equilibrium only and only if following set value function*

$$g : \prod_{i \in J} X_i \to \prod_{i \in J} 2^{X_i} : g_k(x) = \operatorname*{argmax}_{t \in X_k} u_k(x_1, ...x_k, t, x_{k+1}, ..., x_n)$$

has a fixed point $x \in g(x)$

Proof. Trivial (Just check definitions) □

3.4.1 General version of resource optimization problem

Consider a game where a set of players compete in a multiple competitions at the same. Every player has limited energy and they can allocate it freely. Every competition is won by a player who allocated most energy in it and the goal of the players is to maximize the amount of the competitions they win.

Definition 3.6. We define the continuous game structure of the game described above as follows:

1. The choice space for every player consists of continuous functions $f : [0, 1] \to \mathbb{R}$ for which $\int_0^1 f(x)dx = 1$
2. The evaluation function $u_n(x) : \prod_{i \in J} X_i \to [0, 1]$ is defined to be:

$$u_n(f) = m^*(\{x \in [0, 1] \mid f_n(x) \geqslant f_i(x)\})$$

We will show that in some cases there exists an equalibrium.

Appendices

Appendix A

Eilenberg–Steenrod axioms

For homology theories we have the following interface:

Definition A.1. A homology theory consists of a family of the functors from category of topological pairs to category of abelian groups $\{H_n\}_{n\in\mathbb{N}}$ and natural transformations

$$\{\delta_n : H_n(X, A) \to H_{n-1}(A, \varnothing)\}_{n\in\mathbb{N}}.$$

We denote $H_n(X)$ to be $H_n(X, \varnothing)$. For the homological theory the following axioms are satisfied:

(1) **Homotopy axiom:** If maps $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are homotopic, then $H_n(f) = H_n(g)$ for all n.

(2) **Excision axiom:** If (X, A) is a topological pair and U subset of the space X for which $\bar{U} \subset \mathring{A}$ holds, then the inclusion $(X\backslash U, A\backslash U) \subset (X, A)$ induces the following isomorphism for all n:

$$H_n(i) : H_n(X\backslash U, A\backslash U) \to H_n(X, A)$$

(3) **Dimension axiom:** If X is one point space, then $H_n(X)$ is trivial group for all $n \neq 0$.

(4) **Additivity axiom:** Let X be a topological space, which is homeomorphic to a space formed by disjoint union of spaces in the collection $\{A_i\}_{i\in J}$, then group $H_n(X)$ is isomorphic to $\bigoplus_{i\in J} A_i$

(5) **Exactness axiom:** Let $i : A \to X$ and $j : (X, \varnothing) \to (X, A)$ be inclusion, then the following sequence is exact:

$$\cdots \longrightarrow H_n(A) \xrightarrow[i_*]{} H_n(X) \xrightarrow[j_*]{} H_n(X, A) \xrightarrow[\partial_n]{} H_{n-1}(A) \longrightarrow \cdots$$

In the case functor being contra-variant we reverse the morphisms in the following way:

$$\cdots \longleftarrow H_n(A) \underset{i_*}{\longleftarrow} H_n(X) \underset{j_*}{\leftarrow} H_n(X,A) \underset{\partial_n}{\leftarrow} H_{n-1}(A) \longleftarrow \cdots$$

Remark A.2. In the definition above we didn't specify whether the functor is contra-variant or covariant. In the case where we call a theory the homology theory, we imply that the functor is covariant. In the case of cohomology, the functor is contra-variant.

Remark A.3. The functors defined above map every pair (X, A) into some abelian group. However, in many cases the algebraic object has extra properties.

Appendix B

Technical details of Čech constructions

In this section we will present the theory of the Čech in a more detail and prove some of the technical theorems related to the homology and cohomology of Čech.

B.1 Chain homotopy

In this section we recall definition of chain homotopy and an important theorem related to it.

Definition B.1. Let (C_*, ∂) and (D_*, ζ) be some chain complexes of abelian groups. Then we say that collection f of homomorphisms $\{f_n : C_n \to D_n\}$ is chain map, if the following diagram induced by the maps commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n-1}"'] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"'] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\zeta_{n-1}"'] & D_n \arrow[r, "\zeta_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition B.2. Let (C_*, ∂) and (D_*, ζ) be some chain complexes of abelian groups and let f and g be chain maps between them. We say that f is chain homotopic to g if there exists a collection of homomorphisms $\{P_n : C_n \to D_{n+1}\}$ for which the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n-1}"'] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"'] \arrow[d, "f_n"] \arrow[dl, "P_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "P_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\zeta_{n-1}"'] & D_n \arrow[r, "\zeta_n"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Let (C_*, ∂) arbitrary chain complex of abelian groups. We denote the groups $H_n(C_*)$ to be $\ker\partial_n/\mathrm{Im}\partial_{n+1}$. If $f : C_* \to D_*$ is a chain map between chain complexes. Then we denote the induced map by f between $H_n(C_*)$ and $H_n(D_*)$ as $H_n(f)$. This map have been proved to be well-defined in [1]. For the chain homotopic maps we have the following important result.

Theorem B.3. *Let (C_*, ∂) and (D_*, ζ) be some chain complexes of abelian groups and let f and g be chain homotopic maps between them. Then $H_n(g) = H_n(f)$.*

Proof. proved in Rotman [1] theorem 5.3. □

B.2 Algebraic mappings

In this section we introduce concepts and main result of algebraic mappings. The proofs of the theorems in this section are technical and thus we skip them.

Definition B.4. Let (C_*, ∂) and (D_*, ζ) be chain complexes for which the group of order -1 is free G-module and let $f : C_* \to D_*$ be a chain map . We say that f is algebraic, if the equation $\mathrm{In}_{C(-1)}(x) = \mathrm{In}_{D(-1)}(f(x))$ holds for every $x \in C(-1)$.

Lemma B.5. *Let X and Y be simplicial complexes and let $f : X \to Y$ be a continuous maps between them. Let C_* and D_* be induced augmented chain complexes of space X and Y respectively. Then the induced map $f_* : C_* \to D_*$ is algebraic.*

Proof. Let $c = \sum_{i\in J} g_i z_i$ be an element of C_{-1}. Then by definition of the induced map $f(c) = \sum_{i\in J} g_i f(z_i)$. The image of both elements under map In is the element $\sum_{i\in J} g_i$, thus the map is algebraic. □

Definition B.6. Let X and Y be simplicial complexes. Let simplices(X) be the collection of all simplices in X and complexes(Y) be collection of all sub simplicial complexes of Y. Then function

$$C : \text{simplices}(X) \to \text{complexes}(Y)$$

is a carrier function, if every face s' of simplex s is mapped to subcomplex $C(s')$ of $C(s)$.

Definition B.7. We say that a carrier function C is acyclic, if image of every element $C(s)$ is acyclic.

Theorem B.8. *Let (X, A) and (Y, B) be simplicial pairs. Let $f : X \to Y$ and $g : X \to Y$ be algebraic maps and C an acyclic carrier, for which the restriction C to simplices which lie in A maps every element inside subsimplex B. Then the induced maps $f', g' : C_*(X)/C_*(A) \to C_*(Y)/C_*(B)$ are chain homotopic.*

Proof. Proved in [2] theorem VI.5.8. □

Corollary B.9. *Let $f, g : X \to Y$ be maps as in the theorem above, then $H_n(g) = H_n(f)$.*

Proof. Follows directly from theorem B.3 and B.8. □

B.3 Homotopy axiom

The main result of this section is the homotopy axiom of Čech homology which states that if two spaces are homotopic then they have isomorphic Čech homology groups for all $q \in \mathbb{N}$. In the beginning we will prove an important result regarding the singular homology groups of the nerve of the unit interval $[0, 1]$.

Definition B.10. Let X be a topological space and let α be a finite open covering of X. Then CLEAN(α) is a covering of the space X, which is formed by removing every such element $A \in \alpha$ for which $A \subset B$ holds for some $B \in \alpha$.

Lemma B.11. *Let X be a topological space and let α be its finite open covering. Let β be covering CLEAN(α) . Then the singular homology groups $H_q(\mathcal{S}(X)_\alpha)$ and $H_q(\mathcal{S}(X)_\beta)$ are isomorphic for every $q \in \mathbb{N}$.*

Proof. Because β is a subcollection of α, the relation $\alpha \leqslant \beta$ holds. Relation $\beta \leqslant \alpha$ holds, because if we take any element from $a \in \alpha$ which is not in β by definition we will find an element $b \in \beta$ for which the relation $a \subset b$ holds.
Thus there exist morphisms I_α^β and I_β^α in the inverse system of abelian groups induced by coverings. By definition the equations

$$I_\alpha^\beta \circ I_\beta^\alpha = I_\beta^\beta = id_\beta \text{ and } I_\beta^\alpha \circ I_\alpha^\beta = I_\beta^\alpha = id_\alpha$$

hold. Thus I_α^β is an isomorphism between $H_q(\mathcal{S}(X)_\alpha)$ and $H_q(\mathcal{S}(X)_\beta)$. □

Definition B.12. Let α be a cleaned finite open covering of $[0, 1]$, for which every element in the collection is connected. The trivial indexation of α is the following indexation: We order open intervals in the collection $(a, b) \in \alpha$ by the coordinate a and number coverings.

Lemma B.13. *Let α be an open covering of $[0,1]$ which is cleaned, finite and connected. Then the trivial indexation $[m]$ preserves the order of the coordinate b of open intervals.*

Proof. If $b_n > b_{n+1}$ for some $n \in [m-1]$, then $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ which is a contradiction by the definition of the covering α. □

Lemma B.14. *Let α be an open covering which is finite. Assume that every element in α is connected. Then the nerve $\mathcal{S}([0,1])_\alpha$ is acyclic in the singular homology.*

Proof. Using Lemma B.11 we can reduce our problem to coverings which have property that no inclusion holds for any two elements of α. We define indexation to be trivial indexation for covering α. Let $\{f_n(x) : \mathcal{S}([0,1])_\alpha \to \mathcal{S}([0,1])_\alpha\}_{n\in[m]}$ be a collection of simplicial maps which are induced by the edges in the following way:

$$f_n(v_i) = \begin{cases} v_i & \text{if } i \leqslant n \\ v_n & \text{if } i \geqslant n \end{cases}$$

We will now show that the maps defined above induce same homomorphisms between the corresponding groups. Because the indexation set is finite, it is enough to show that f_n and f_{n+1} are contiguous for every n in the collection. Let s be some simplex in the nerve $\mathcal{S}([0,1])_\alpha$. If simplex s is spanned only by elements whose index is less than n, maps f_n and f_{n+1} are both identity on it. Now assume that s contains atleast one edge the index of which is larger than $n-1$. Then the image of the element s under the map f_n can be presented as $\rhd(s', v_n)$, where s' is simplex for which index of every edge is smaller than n. There are two possible representations for the image $f_{n+1}(s)$ namely $\rhd(s', v_n, v_{n+1})$ and $\rhd(s\, v_{n+1})$. If the representation of $f_{n+1}(s)$ is $\rhd(s', v_n, v_{n+1})$, we can choose the simplex to be $\rhd(s', v_n, v_{n+1})$, as it clear from definition that $\rhd(s', v_{n+1})$ is its face. In the second case we notice that because intersection $\bigcap_{a\in s'} a \cap v_{n+1}$ is non-empty and connected. Thus it is an open interval (a_{n+1}, b_1). Because condition $(a_n, b_n) \cap (a_{n+1}, b_1) \neq \varnothing$ holds, we can choose $\rhd(s', v_n, v_{n+1})$ to be the simplex where image of s of the both functions are contained. □

Next we define a class of nice coverings for the unit interval $[0,1]$ and prove that the subcategory which consists of the coverings defined is a subcategory of the Cech inverse system.

Definition B.15. Let $\alpha = \{U_i\}_{i\in[n]}$ be a connected open covering of unit interval $[0,1]$ for which the following conditions are satisfied:

(1) $0 \in U_0$ and $1 \in U_n$

(2) $U_i \cap U_{i+1} \neq \varnothing$ for every $i \in [n-1]$

(3) $U_i \cap U_j = \varnothing$ for all $|i-j| \geqslant 1$

Then we say that covering α is regular.

Lemma B.16. *The category of the regular coverings of* $[0,1]$ *is cofinal in the* $COV([0,1])$.

Proof. Let α be arbitrary covering of $[0,1]$. We modify the covering in the following ways:

(1) We form covering β by selecting some finite subcovering of α.

(2) We split every non-connected element in the collection β into connected elements.

(3) We search for the removable elements i.e the the collection after removing the element covers the space $[0,1]$. We remove every such element and stop this process after no removable elements are found.

(4) We define the indexation to be the trivial indexation.

The condition (1) and (2) are satisfied , because γ is trivially indexed covering. Condition (3) is satisfied, because if two open intervals with are not neighbors have non-empty intersection then all the coverings between can be removed and that collection will still cover the set. Because covering γ is obtained by removing and cutting elements, the relation $\beta \leqslant \gamma$ holds. □

Earlier we defined the concept of stacked covering in more general way. For covering of the pair, where the second element is unit interval there is a natural way to define a stacked covering using regular coverings.

Definition B.17. Let (X, A) be topological pair and $\alpha = \mathcal{C}(U_x, J, J')$ be its covering. Suppose that for every element U_x there exists a regular covering $\beta(x) = \mathcal{C}(V_i^x, K, K')$ of $[0,1]$. If γ is the stacked covering over α of the pair $(X \times [0,1], A \times [0,1])$, we say that it is naturally stacked covering.

Lemma B.18. *The category of naturally stacked coverings is cofinal in* $COV((X \times [0,1], A \times [0,1]))$

Proof. We denote I to be unit interval $[0,1]$. Let γ be some covering $\mathcal{C}(Y_i, J, J')$ of pair $(X \times I, A \times I)$. We construct a stacked covering in the following way. Let (x,t) be a point of the space $X \times I$. Then by definition of the topology on Cartesian product there exists a neighborhood $W = U(x,t) \times V(x,t)$ for which $(x,t) \in W \subset Y_i$ holds for some Y_i in the collection γ. We fix x now, we denote covering induced by sets $V(x,t)$ over all $t \in I$ by $\mathcal{K}^x$. Using Lemma B.16 we find a regular covering of the unit interval

$$\mathcal{R}^x = \mathcal{C}(\mathcal{R}_i^x, P, P')$$

for which $\mathcal{K}^x \leqslant \mathcal{R}^x$ holds. For every $j \in J$ we can find subindexation $P_j \subset P$ for which the following condition holds: For every $i \in P_j$ there exists $t(i,j) \in I$ for which $U(x, t(i,j)) \times \mathcal{R}_i^x \subset Y_j$ holds. We define H_j to be the following open set $H_j^x = \bigcap_{i \in P_j} U(x, t(i,j))$ and define covering $\mathcal{A}^x$ to be

$$\mathcal{A}^x = \mathcal{C}(H_j^x, J, J').$$

The covering γ' spanned by elements $H_j^x \times \mathcal{R}_i^x$ is stacked covering over $\mathcal{A}^x$ which refines covering γ.

□

Lemma B.19. *Let γ be a naturally stacked covering of pair $(X \times [0,1])$ over α. Assume that the nerve $\mathcal{S}(X)_\alpha$ is simplex. Then the nerve $\mathcal{S}(X \times I)_\gamma)$ is acyclic in singular homology.*

Proof. Assume that the decomposition of γ is $\mathcal{C}(\gamma_i, J)$ and the decomposition of α is $\mathcal{C}(W_i, R)$. We denote by $\beta(x)$ the stacks of covering γ. Let δ be the covering $\mathcal{C}(\delta_i, K)$ obtained by fusioning all the coverings $\beta(x)$.
Let $s = \{\gamma_{j_0}, ..., \gamma_{j_n}\}$ be some simplex in the nerve $\mathcal{S}(X)_\gamma$. We denote J' to be indexation over elements $\{j_0, ...j_n\}$. By the definition we know that each $\gamma_j = \alpha_{f(j)} \times \beta_{g(j)}$ for some $\alpha_{f(j)}$ and $\beta_{g(j)}$. Let $t : g(J) \to K$ be the indexation function which tells the index of element in β_j in covering γ. The following equations hold:

$$\bigcap_{i \in J'} \gamma_i = \bigcap_{i \in J'} \alpha_i \times \bigcap_{i \in J'} \beta_{g(i)} = \bigcap_{i \in J'} \alpha_{f(i)} \times \bigcap_{i \in J'} \beta_{t(g(i))} = \bigcap_{i \in J'} \delta_{t(g(i))}$$

Note that the last equation holds because we assumed that α is simplex and thus intersection of all its elements is non-empty. By definition of stacked coverings wee see that the covering obtained by projecting elements of γ to the unit interval has exactly the same structure as the fusioned covering δ. Using the equation above we see that the simplices $\mathcal{S}([0,1])_\gamma$ and $\mathcal{S}([0,1])_\beta$ are isomorphic. By B.14 the homology groups of $[0,1]_\delta$ are trivial, thus $H_q(\mathcal{S}([0,1])_\delta)$ is trivial for all $q \in \mathbb{N}$. □

Lemma B.20. *Let (X, A) be a topological pair. Let γ be naturally stacked covering $\mathcal{C}(\gamma_i, J, J')$ of the pair over covering $\alpha = \mathcal{C}(\alpha_i, I, I')$. Then maps*

$$\{u_n : \mathcal{S}(X, A)_\alpha \to \mathcal{S}(X \times [0,1], A \times [0,1])_\gamma \mid n \in [0,1]\}$$

defined as $u_n(x) = (x, n)$ induce the same map $H_0(u_0)$ between the corresponding singular homology groups for all $n \in [0,1]$

Proof. We will show that the maps satisfy requirements of corollary **??** and use the theorem to prove that the induced maps are same for the singular homology groups. We will define the Carrier function C to map arbitrary simplex s of complex $\mathcal{S}(X, A)_\alpha$ into the following simplicial complex. Assume that s is of the form $\{\gamma_i\}_{i\in T}$, then simplicial complex $C(s)$ consists of all elements in the simplicial complex $\mathcal{S}(X \times [0,1], A \times [0,1])_\alpha$, which are of the form (U_i, Z) where $i \in T$.

$\square$

Appendix C

Set theory

Lemma C.1. *Let X and Y be some sets and $f : X \to Y$ function between them. Let $\{V_i \mid i \in J\}$ be some collections of subsets of the set Y. Assume that $\bigcap_{i \in J} f^{-1}V_i \neq \varnothing$, then $\bigcap_{i \in J} V_i \neq \varnothing$.*

Proof. There exists some $x \in \bigcap_{i \in J} f^{-1}V_i$. The conditions $x \in f^{-1}V_i$ and $f(x) \in V_i$ hold for all $i \in J$. Thus $f(x) \in \bigcap_{i \in J} V_i$ and $\bigcap_{i \in J} V_i \neq \varnothing$. □

Lemma C.2. *Let X and Y be some sets and $f : X \to Y$ be a function between the sets Let $A \subset X$ and $B \subset Y$. Then the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "f|A"'] \arrow[d, "i_A"] & B \arrow[d, "i_B"] \\ X \arrow[r, "f"'] & Y) \end{tikzcd}$$

Bibliography

[1] J. Rotman An Introduction to Algebraic Topology (Graduate Texts in Mathematics), Springer

[2] S. Eilenberg and N. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press 1952

[3] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Publishers, 1999.

[4] I. L. Glicksberg, A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points, Proc. Amer. Math. Soc. 3 (1952), 170–174.

[5] M. BAYE, G. TIAN, J. ZHOU, "The Existence of Pure Nash Equilibrium in Games with Nonquasiconcave Payoffs," Texas A M University, mimeo, 1990.

[6] Saunders Mac Lane: Categories for the Working Mathematician (Graduate Texts in Mathematics)

[7] Jokke Häsä: Algebra II, 2016.