

Cech (co)homology

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Contents

1	Introduction	2
2	Background	3
2.1	Direct sum and direct product	3
2.2	Category theory	4
2.2.1	Dual category	7
2.2.2	Direct and inverse systems	8
2.2.3	Morphisms between systems	10
2.2.4	Finality properties of subsystems	12
2.3	Nerve of covering	14
2.3.1	Abstract simplicial complex	14
2.3.2	Coverings	15
3	Cech homology	19
3.1	Well-definedness of the Cech homology groups	20
3.2	Induced homomorphisms	20
3.3	Dimension axiom	21

Chapter 1

Introduction

Chapter 2

Background

In this chapter we will recall a few details about algebra, category theory and abstract simplicial complexes. We assume that the reader is familiar with basic concepts of homology and cohomology theory and constructions related to them.

2.1 Direct sum and direct product

In this section we will recall a few details about direct sums and direct products.

Definition 2.1. Let $\{A_i\}_{i \in J}$ be a family of groups. The direct product of these groups is the Cartesian product $\prod_{i \in J} A_i$ where addition is defined component-wise $(a+b)_i = a_i + b_i$

Definition 2.2. Let $\{A_i\}_{i \in J}$ be a family of groups. The direct sum of these groups is the subgroup of direct product given by $\bigoplus_{i \in J} A_i = \{x \in \prod_{i \in J} A_i \mid x_i \neq 0 \text{ for only finitely many } i\}$.

For direct sums and direct products we have following universal properties:

Lemma 2.3. *Let $A = \prod_{i \in J} A_i$ be the direct sum and D an arbitrary group. Then for every set of homomorphisms $\{f_i : D \rightarrow A_i\}_{i \in J}$ there exists a unique homomorphism $f : D \rightarrow A$ for which condition $pr_i \circ f = f_i$ holds for every $i \in J$.*

Proof. Proved in [6] theorem 3.7. □

Lemma 2.4. *Let $A = \bigoplus_{i \in J} A_i$ be the direct sum and D an arbitrary group. Then for every set of homomorphisms $\{f_i : A_i \rightarrow D\}_{i \in J}$ there exists a unique homomorphism $f : A \rightarrow D$ for which the condition $f \circ j_i = f_i$ holds for every $i \in J$.*

Proof. Proved in [6] theorem 3.6. □

2.2 Category theory

In this thesis we will present theory in a categorical way, which will make the construction more general. First lets recall the definition of a category.

Definition 2.5. A category C consist of the following ingredients: A class of objects $\text{ob}(C)$, a class of morphisms $\text{hom}(C)$ for which and for every objects A, B in $\text{ob}(C)$ there exists a subclass $\text{Hom}(A, B)$ and a rule of composition $\text{Comp} : \text{Hom}(A, B) \times \text{Hom}(B, D) \rightarrow \text{Hom}(A, D)$ for which following conditions hold:

- (1) Composition is associative. Let $f : A \rightarrow B$, $g : B \rightarrow D$ and $h : D \rightarrow E$ be morphisms between objects then $(f \circ g) \circ h = f \circ (g \circ h)$
- (2) For every object $A \in \text{ob}(C)$ there exists identity morphism $1_A \in \text{hom}(A, A)$ for which the following condition holds: Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be arbitrary morphisms then $f \circ 1_A = f$ and $1_A \circ g = g$.

Definition 2.6. We say that a category D is a subcategory of category C . If the following conditions are satisfied:

- (1) Collection of objects $\text{ob}(D)$ is subcollection of $\text{ob}(C)$
- (2) Collection of morphisms $\text{hom}(D)$ is subcollection of $\text{hom}(C)$.

Definition 2.7. Let C be a category and A, B its objects and let $f : A \rightarrow B$ be some morphism between them. We say that f is isomorphism, if there exists a morphism $g : B \rightarrow A$ for which equations $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$ hold.

It is easy to see that topological spaces with continuous functions and abelian groups with homomorphisms form a category. We denote those categories by TOP and AB. Next we will define the concept of a functor between categories.

Definition 2.8. Let C and C' be categories and F map between them. We say that F is functor between categories if the following conditions hold:

- (1) For every object $A \in \text{ob}(C)$ there exists a unique object $F(A) \in C'$.
- (2) Let A and B be objects in $\text{ob}(C)$ and $f : A \rightarrow B$ a morphism between them. Then there exists a unique morphism $F(f) : F(A) \rightarrow F(B)$.
- (3) if f and g are morphisms in $\text{hom}(C)$, then the following condition holds: $F(f) \circ F(g) = F(f \circ g)$.

- (4) The identity element is mapped to the identity element. Let A be an object in $\text{ob}(C)$ and 1_A the identity morphism corresponding to it. Then $F(1_A)$ is identity morphism of $F(A)$.

Homology and cohomology groups form a functor from TOP to AB. For details see Rotman [1]. Now we will introduce a new concept called natural transformation.

Definition 2.9. Let C and D be categories and $F : C \rightarrow D$ and $G : C \rightarrow D$ functors between them. Then the following family of functors is a natural transformation:

$$\{\phi_X : F(X) \rightarrow G(X)\}_{X \in C}$$

if the following conditions hold:

- (1) For every object $X \in C$ there exists a unique morphism $\phi_X : F(X) \rightarrow G(X)$
- (2) For every morphism $f : X \rightarrow Y$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\quad \phi_X \quad} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\quad \phi_Y \quad} & G(Y) \end{array}$$

or in other words $\phi_Y \circ F(f) = G(f) \circ \phi_X$ holds.

Next we will define the concepts of limit and colimit for arbitrary categories, which will be later applied to directed and inverse systems.

Definition 2.10. Let C and J be categories, then we say that the category C is indexed by the category J if there exists a functor $F : J \rightarrow C$.

We call the functor F described above an indexed functor. Because the functor F has the information about the domain J and the range C , we can treat functor F as the subcategory $F(J)$ of C .

Definition 2.11. Let C be a category indexed by J and let $F : J \rightarrow C$ be the indexing functor. Let N be fixed object of the category C . We define the cone from N to F to be an indexed family of morphisms

$$\Omega_N = \{\omega_X : N \rightarrow F(X)\}_{X \in J}$$

which satisfies following property: If $f : X \rightarrow Y$ is a morphism in C then the following diagram commutes

$$\begin{array}{ccc}
 & N & \\
 \omega_X \swarrow & & \searrow \omega_Y \\
 F(X) & \xrightarrow{F(f)} & F(Y)
 \end{array}$$

From now on we will denote the cone structure described above by $\Delta(F, N, \Omega_N)$.

Definition 2.12. Let C be a category which is indexed by a category J and let F be the indexing functor. We say the object $D \in \text{ob}(C)$ is a limit of the functor F if following conditions are satisfied:

- (1) There exists family of morphisms Ω_D which induce cone structure $\Delta(F, D, \Omega_D)$.
- (2) If $\Delta(F, N, \phi)$ is any other cone structure, then there exists unique morphism $u : N \rightarrow D$ which makes the following diagram commute.

$$\begin{array}{ccc}
 & N & \\
 \omega_X \swarrow & \downarrow & \searrow \omega_Y \\
 & D & \\
 \phi_X \swarrow & & \searrow \phi_Y \\
 F(X) & \xrightarrow{F(f)} & F(Y)
 \end{array}$$

Next we will prove that if we have two limits N and D of a functor F then there exists an isomorphism between those two objects. In other words N and D can be identified and the limit of functor F can be denoted simply as $\varinjlim C$.

Theorem 2.13. Let $F : J \rightarrow C$ be an indexing functor. Let N and D be limits of a functor F . Then there exists an isomorphism between them.

Proof. Because N and D are both limits there exists morphisms $u : N \rightarrow D$ and $v : D \rightarrow N$ as above. Because of the symmetry it is enough to show that $u \circ v = id_N$. This follows from the equations $\omega_X \circ u \circ v = \phi_X \circ u = \omega_X$ and $u \circ v \circ \omega_Y = u \circ \phi_Y = \omega_Y$. Because the diagram commutes for id_N then by the uniqueness condition we see that $u \circ v = id_N$. \square

2.2.1 Dual category

To simplify definitions we will define the concept of dual category.

Definition 2.14. Let ω be a statement in this model. We get a dual statement ω^{op} by following procedure:

- (1) Interchange every occurrence of source with target
- (2) Reorder every composition. That is replace $a \circ b$ with $b \circ a$.

Next we will give a few important example of a dual statement.

Definition 2.15. Let $F : J \rightarrow C$ be the indexing functor. We define cocone to be the dual structure of the cone.

We will denote the cocone over object N as $\nabla(F, N, \Omega_N)$

Definition 2.16. Let C be a category which is indexed by a category J and let F be the indexing functor. We say the object $D \in \text{ob}(C)$ is a colimit of the functor F if the following conditions are satisfied:

- (1) There exists family of morphisms Ω_D which induce cocone structure $\nabla(F, D, \Omega_D)$.
- (2) If $\nabla(F, N, \phi)$ is any other cone structure, then there exists unique morphism $u : N \rightarrow D$ which makes the following diagram commute.

$$\begin{array}{ccccc}
 & & N & & \\
 & \nearrow \phi_X & \uparrow \text{---} & \nwarrow \phi_Y & \\
 & & D & & \\
 & \nwarrow \omega_X & \text{---} & \nearrow \omega_Y & \\
 F(X) & \xleftarrow{F(f)} & F(Y) & &
 \end{array}$$

Like in the case of limits, colimits are unique.

Theorem 2.17. Let F be indexing functor and let N and D be colimits of a functor F . Then there exists isomorphism between objects N and D .

Proof. Because N and D are both colimits there exist morphisms $u : N \rightarrow D$ and $v : D \rightarrow N$ as above. Because of the symmetry it is enough to show that $u \circ v = id_N$ holds. This follows from the equations $u \circ v \circ \omega_X = u \circ \phi_X = \omega_X$ and $u \circ v \circ \omega_Y = u \circ \phi_Y = \omega_Y$. The diagram commutes for id_N and thus by the uniqueness condition we see that $u \circ v = id_N$. \square

2.2.2 Direct and inverse systems

In this section we will give a categorical definition of direct and inverse systems. It appears that those concepts are dual of each other. We will begin by defining quasi-ordering relation:

Definition 2.18. Let λ be a category and let $a \leq b$ be a relation in an object class $\text{ob}(\lambda)$. We say that this relation is a quasi-ordering if the following conditions hold:

- (1) $a \leq a$ for all $a \in \text{ob}(\lambda)$
- (2) if $a \leq b$ and $b \leq c$ then $a \leq c$ for all $a, b, c \in \text{ob}(\lambda)$

Definition 2.19. Let λ and C be categories and $D : \lambda \rightarrow C$ an indexing functor. We say that functor D is a direct system indexed by λ if the following condition hold:

- (1) For every pair $a \leq b$ in $\text{ob}(\lambda)$ there exists unique morphism $D_a^b : D(a) \rightarrow D(b)$.
- (2) For every triple $a \leq b \leq c$, the morphisms commute in a following way: $D_b^c \circ D_a^b = D_a^c$.
- (3) For every $a \in \lambda$: D_a^a is identity morphism id_a .

The dual of direct system is inverse system.

Definition 2.20. Let λ and C be categories and $I : \lambda \rightarrow C$ an indexing functor. We say that functor I is an inverse system indexed by λ if the following condition hold:

- (1) For every pair $a \leq b$ in $\text{ob}(\lambda)$ there exists unique morphism $I_a^b : I(b) \rightarrow I(a)$.
- (2) For every triple $a \leq b \leq c$, the morphisms commute in a following way: $I_a^b \circ I_b^c = I_a^c$.
- (3) For every $a \in \lambda$: I_a^a is identity morphism id_a .

Example 2.21. Let SET be the category of sets with order relation defined by $U \leq V \Leftrightarrow U \subset V$. Now for every U, V we define $D_U^V : U \rightarrow V$ to be just inclusion from U to V . Clearly this forms a direct system.

Earlier we defined the concept of limit for general indexing functors. We can interpret direct and inverse system as indexing functors, taking the λ to be indexing set. In general systems the limit may not exist. However, for our purpose it is enough to show that such limit can be found in category of abelian groups.

Theorem 2.22. Let $D : \lambda \rightarrow AB$ be a direct system of groups. Let $i_a : D(a) \rightarrow \bigoplus_{a \in \text{ob}(\lambda)} D(a)$ let G be the subgroup of $D = \bigoplus_{a \in \lambda} D(a)$ with is generated by $\{i_a x_a - i_b D_a^b x_a\}$. Then the colimit of this system is an object $L = \bigoplus_{a \in \lambda} D(a)/G$.

Proof. We denote p to be the projection map: $p : \bigoplus_{a \in \text{ob}(\lambda)} D(a) \rightarrow L$. Let v be the family of morphisms $\{v_a = p \circ i_a : D(a) \rightarrow L\}_{a \in \text{ob}(\lambda)}$ which induces cocone $\nabla(I, L, v)$ and let K be any other object with family of morphisms $\phi = \{\phi_a : D(a) \rightarrow K\}_{a \in \text{ob}(\lambda)}$ which induces cocone $\nabla(I, K, \phi)$. We have to prove that there exists unique homomorphism $u : L \rightarrow K$ for which condition $u \circ v_a = \phi_a$ holds for all $a \in \text{ob}(\lambda)$. We will first show that such homomorphism exists

Using lemma 2.4 we can find unique homomorphism $u' : \bigoplus_{a \in \lambda} D(a) \rightarrow K$ for which $u' \circ i_a = \phi_a$ holds. Now we see that for every generator of subgroup G the following equations hold:

$$u'(i_a x_a - i_b D_a^b x_a) = \phi_a(x_a) - \phi_b D_a^b x_a = \phi_a(x_a) - \phi_a(x_a) = 0.$$

Now we can define u in such way that it maps every element $x + D$ to element $\phi'(x)$. We will denote the projection map from $\bigoplus_{a \in \lambda} D(a)$ to L by p . Now we see that

$$u \circ v_a = u \circ p \circ i_a = u' \circ i_a = \phi_a.$$

To prove that the homomorphism u we notice that if $x \in L$, then $x = \sum_{a \in A \subset D} v_a(x_a) + G$. Let x be arbitrary element then, the following equations hold

$$u(x) = u\left(\sum_{a \in A \subset D} v_a(x_a)\right) = \sum_{a \in A \subset D} u \circ v_a(x_a) = \sum_{a \in A \subset D} \phi_a(x_a)$$

Thus $u(x)$ is determined by the sum of homomorphisms ϕ_a . It follows that the map u is unique. \square

Theorem 2.23. *Let $I : \lambda \rightarrow AB$ be a inverse system of groups. Then the limit of this functor is group $L = \{x \in \prod_{i \in J} I_i \mid x_a = I_a^b x_b \text{ for all } a \leq b\}$*

Proof. Let v be a family of homomorphisms $\{v_a = pr_a : L \rightarrow I(a)\}_{a \in \lambda}$ which together with group L induce cone $\triangle(I, L, v)$. Let K be any other group and let ϕ be family of homomorphisms $\phi = \{\phi_a : K \rightarrow I(a)\}_{a \in \lambda}$ which induces cone $\triangle(I, K, \phi)$.

We construct function $u : K \rightarrow L$ for which $v_a \circ u = \phi_a$ holds. Using lemma 2.3 we find unique homomorphism $u' : K \rightarrow \prod_{i \in J} I_i$ for which $pr_a \circ u' = \phi_a$ is satisfied. Let u'' be the homomorphism, where the image is restricted to subgroup L . We will show that the homomorphism is well-defined, specifically u' maps every element inside the subgroup L . Assume that b and a are elements which satisfy $a \leq b$ condition. Then the following equation hold

$$I_a^b \circ pr_b \circ u' = I_a^b \circ \phi_b = \phi_a = pr_a \circ u'.$$

We can define u to be u'' . The uniqueness property of the homomorphism follows directly from the fact that u' is unique. \square

2.2.3 Morphisms between systems

2.2.3.1 Morphisms between inverse systems

To be able to define morphisms between Čech homology groups we need the concept of a limit homomorphism. In this section we will define limit morphism between systems and investigate properties of it. Concepts used in this chapter can be presented in a more general way. However, for our use case its enough to restrict ourselves to the direct and inverse system.

Definition 2.24. Functor $\phi : X \rightarrow Y$ is order preserving if for every elements a and b in set X for which $a \leq b$, the condition $\phi(a) \leq \phi(b)$ is satisfied.

Let I be inverse system and $\phi : \lambda' \rightarrow \lambda$ order preserving functor. Then inverse system $(I\phi, \lambda')$ consist of objects $\{I'(\phi(a)) \mid a \in \text{ob}(\lambda')\}$ and morphisms $\{I_{\phi(a)}^{\phi(b)} \mid a \leq b\}$.

Definition 2.25. Let (I, λ) and (I', λ') be inverse systems and $\phi : \lambda' \rightarrow \lambda$ an order and limit preserving functor. Let $\{f_a : I(\phi(a)) \rightarrow I'(a) \mid a \in \lambda\}$ be natural transformation between $I\phi$ and I' . Then we say that $\{f_a\}_{a \in \lambda}$ is an inverse system of morphisms corresponding to functor ϕ of the system I into system I' .

In next theorem we prove that it is possible to define limit of the morphism between systems in an unique way. We recall that exists of limit implies that for every objects of category I there exists unique functor $u_{I,a} : \lim I \rightarrow I(a)$.

Lemma 2.26. Let (I, λ) and (I', λ') be inverse systems for which limit exists. Let $\{f_a\}$ be inverse system of morphisms corresponding to that pair. Then there exists unique morphism $\lim f : \lim I \rightarrow \lim I'$ for which and for every $a \in \lambda'$ condition $\lim f \circ u_{I,\phi(a)} = u_{I',a} \circ f_a$ holds.

Proof. Consider following diagram

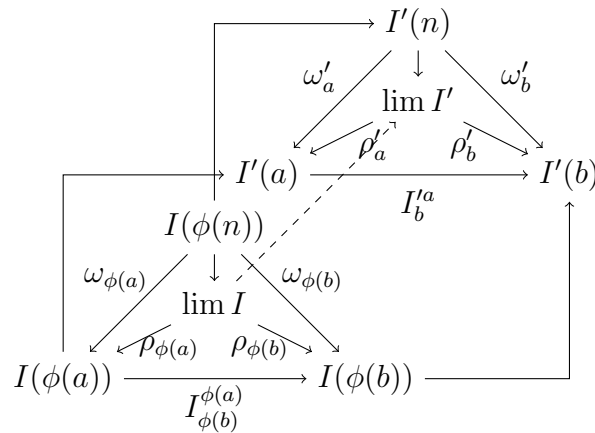


Diagram 2.2.3.1

To prove that the limit morphism exists we form following triangle out of the diagram described above:

$$\begin{array}{ccc}
& \lim I & \\
f_a \rho_{\phi(a)} \swarrow & \downarrow & \searrow f_b \rho_{\phi(b)} \\
& \lim I' & \\
\rho'_a \swarrow & & \searrow \rho'_b \\
I'(a) & \xrightarrow{I'_a} & I'(b)
\end{array}$$

Then by definition of limit there exists unique morphism $\lim f : \lim I \rightarrow \lim I'$ for which the diagram commutes. We will prove now that the universal property $\lim f \circ u_{I, \phi(a)} = u_{I', a} \circ f_a$ holds for limit morphism. By diagram 2.2.3.1 and fact that the family of function $\{f_a\}$ is natural transformation or in other words condition $\omega'_a \circ f_n = f_a \circ \omega_{\phi(a)}$ holds following diagram commutes:

$$\begin{array}{ccc}
& \lim I(\phi(n)) & \\
f_a \omega_{\phi(a)} \swarrow & \downarrow & \searrow f_b \omega_{\phi(b)} \\
& \lim I' & \\
\rho'_a \swarrow & & \searrow \rho'_b \\
I'(a) & \xrightarrow{I'_a} & I'(b)
\end{array}$$

Both maps $\lim f \circ u_{I, \phi(a)}$ and $u_{I', a} \circ f_a$ complete the diagram. Thus by uniqueness of the map corresponding to limit, we can conclude that the functions are same. \square

For composition of two limits we have following result:

Lemma 2.27. *Let (I, λ) , (I', λ') and (I'', λ'') be inverse systems with limits and let $\phi : \lambda' \rightarrow \lambda$ and $\phi' : \lambda'' \rightarrow \lambda'$ functors between systems. Let $\{f_a : I(\phi(a)) \rightarrow I'(a) \mid a \in \phi'(\lambda'')\}$ and $\{g_a : I'(\phi(a)) \rightarrow I''(a) \mid a \in \lambda''\}$ be family of morphisms corresponding to ϕ and ϕ' . Let $\lim(f \circ g) : I \rightarrow I''$ be limit of the family $\{g_a f_{\phi(a)} : I(\phi\phi'(a)) \rightarrow I''(a) \mid a \in \lambda''\}$. Then $\lim(f \circ g) = \lim f \circ \lim g$ holds.*

Proof. To prove this we will use previous lemma and following commutable diagram:

$$\begin{array}{ccccc}
I(\phi(\phi'(a))) & \xrightarrow{f_{\phi'(a)}} & I'(\phi'(a)) & \xrightarrow{g_a} & I''(a) \\
\downarrow u_{I, \phi\phi'a} & & \downarrow u_{I', \phi'a} & & \downarrow u_{I'', a} \\
\lim I & \xrightarrow{\lim f} & \lim I' & \xrightarrow{\lim g} & \lim I''
\end{array}$$

By definition of limit morphism condition $\lim(f \circ g) \circ u_{I, \phi(\phi'(a))} = u_{I'', a} \circ (g_a \circ f_{\phi(a)})$ holds for limit of the composition. By uniqueness of limit homomorphism it is enough to show that the composition of two limit functors satisfies the condition. This follows from following equations:

$$\lim g \circ \lim f \circ u_{I, \phi(\phi'(a))} = \lim g \circ u_{I, \phi'(a)} \circ f_{\phi'(a)} = u_{I'', a} \circ g_a \circ f_{\phi(a)}$$

□

Now we would like to define limit for the system of homomorphisms between inverse systems of groups.

Lemma 2.28. *Let (I, λ) and (I', λ') be inverse systems of groups. Then*

$$f : \lim_{\rightarrow} I \rightarrow \lim_{\rightarrow} I' : f(x) = \prod_{a \in \lambda} f_a(\text{proj}_{\phi(a)}(x))$$

is well defined function between limit groups.

Proof. We have to show that the image is in the subgroup

$$L = \{x \in \prod_{i \in \lambda'} I'(i) \mid x_a = I_a^b x_b \text{ for all } a \leq b\}.$$

Let b be some index, then: $I_a^b x'_b = I_a^b f_b(x_{\phi(b)}) = f_b I_{\phi(a)}^{t\phi(b)}(x_{\phi(b)}) = f_a x_{\phi(a)} = x'_a$. In this equation we used property which says that the morphisms define natural transformation.

□

2.2.4 Finality properties of subsystems

Final and cofinal functors can be defined in a more general way [5]. However, because we are mainly interested in properties of systems, in this section we will investigate the properties of those specific functors only in that special case. In the case of general categories, cofinality included a concept of connectedness defined in [5] page 217. In our definition we will use a stronger version of that concept.

Definition 2.29. Let λ be a category with quasi-order relation on its objects and let λ' be its subcategory. Then we say that ω is cofinal subcategory of λ , if for every two objects $a, b \in \text{ob}(\lambda)$ there exist element in $t \in \text{ob}(\lambda')$ in such a way that conditions $a \leq t$ and $b \leq t$ hold.

Now for the inverse system we have following theorem:

Theorem 2.30. *Let (I, λ) be inverse system and let (I', λ') be its subsystem generated by connected cofinal subcategory λ . Assume that there exists limit object for I' . Then the limit of the system (I, λ) is $\lim I'$.*

Proof. Let $\lim I'$ be limit of inverse system I' . It is enough to show that $\lim I'$ is also the limit of I . We will first show that $\lim I'$ forms a cone in I and then prove that for any other cone induced by object N in I there exists unique morphism $u : N \rightarrow \lim I'$

Let $I(b)$ be arbitrary object of inverse system I . Then by definition of cofinal subcategory there exists element $a' \in I'$ for which $a \leq a'$ holds. Thus there exists a morphisms $I_a^{a'}$. Because $\lim I'$ is limit of inverse system I' it forms cone $\Delta(I', \lim I', \rho)$, where ρ is some collection of morphisms $\{\rho_a : \lim I' \rightarrow I'(a)\}_{a \in \text{ob}(\lambda')}$. Using ρ we form a new collection of morphisms ρ' defined by $\{I_a^{a'} \circ \rho_{a'} : \lim I' \rightarrow I(a)\}_{a \in \text{ob}(\lambda)}$. Because $I_a^{a'} \circ \rho_{a'}$ can vary on the choice of a , the composition may be not uniquely determined. However, if a'' is another element for which relation $a \leq a''$ holds, then there exists element t for which conditions $a \leq t$ and $a'' \leq t$ are satisfied. Because the following equations hold

$$I_a^{a'} \circ \rho_{a'} = I_a^{a'} \circ I_{a'}^t \circ \rho_t = I_a^t \circ \rho_t = I_a^{a''} \circ I_{a''}^t \circ \rho_t = I_a^{a''} \circ \rho_{a''}$$

we can conclude that $I_a^{a'} \circ \rho_{a'} = I_a^{a''} \circ \rho_{a''}$ and thus the family of the homomorphisms is well-defined and $\lim(I')$ forms a cone $\Delta(I, \lim I', \rho')$

Now if there exists another cone $\Delta(I, N, \omega)$, we have construct map $u : N \rightarrow \lim I'$ and prove that it is unique. Let a and b be objects of λ , then there exists objects a' and b' for which relations $a \leq a'$ and $b \leq b'$ hold. Restricting collection ω to the morphisms indexed by λ' we get cone $\Delta(I', N, \omega')$ of I' . Because $\lim I'$ is the limit of the functor I' , there exists a unique homomorphism $u : N \rightarrow \lim I'$. It is left to prove that following diagram commutes:

$$\begin{array}{ccccc}
 & & N & & \\
 \omega_a \swarrow & & \downarrow & \searrow \omega_{b'} & \omega_{a'} \\
 & \lim I' & & & \\
 \swarrow \omega_{a'} & & \rho'_a & \searrow \rho'_b & \\
 I'(a') & \xrightarrow{I_{b'}^{a'}} & I'(b') & & \\
 \swarrow I_a^{a'} & & & \searrow I_b^{b'} & \\
 I(a) & \xrightarrow{I_b^a} & I(b) & &
 \end{array}$$

Especially we have to prove that $I_a^{a'} \circ \omega_{a'} = \omega_a$. This claim follows directly from the fact that $\Delta(I, N, \omega)$ is a cone. Thus the u defined above is the map we were searching for. \square

2.3 Nerve of covering

In this chapter we will give abstract definition of simplicial complex.

2.3.1 Abstract simplicial complex

We assume that reader knows already basic facts about simplicial complexes if not we suggest to take a look at [1]. We start by defining infinite dimensional simplicial complex.

Definition 2.31. Let X be a set and $S = \{S_i\}_{i \in J}$ be some finite collection of subsets of the set X . We say that S forms abstract simplicial complex over X , if subset of any set S_i in the collection S is in the collection.

For abstract simplicial complex we define following geometric structure which will give us the interface of original simplicial complex.

Definition 2.32. The geometric $\mathcal{S}(X)_S$ realization of the abstract simplicial complex is a subset of the set $[0, 1]^X$ for which the following conditions are satisfied: Let $x \in \mathcal{S}(X)$ then

- (1) Relation $pr_i(x) > 0$ holds only for finitely many indexes $i \in J$.
- (2) $\sum_{i \in J} pr_i(x) = 1$.
- (3) Set $\{i \in J \mid pr_i(x) > 0\}$ belongs to collection S .

Lemma 2.33. Let X be a set and $S = \{S_i\}_{i \in J}$ be some collection of subsets of the set X . Let $\mathcal{S}(X)_S$ be its geometric realization. Then $\mathcal{S}(X)_S$ is simplicial complex.

Proof. We define collection simplexes of the simplicial complex to be the geometric realizations of sets S_i in the collection. Any face is in the collection by definition of abstract simplicial complex. Also any intersection of two simplexes is in the collection. Thus $\mathcal{S}(X)_S$ is simplicial complex. \square

Definition 2.34. Let set X together with collection $S = \{S_i\}_{i \in I}$ be abstract simplicial. Let A be some subset of X and $S' = \{S'_i\}_{i \in I}$ some collection of its subsets. Then we say that $\mathcal{S}(A)_{S'}$ is subsimplex of $\mathcal{S}(X)_S$.

If $\mathcal{S}(X)$ is simplicial complex and $\mathcal{S}(A)$ its subcomplex they form topological pair which we will denote by $(\mathcal{S}(X)_S, \mathcal{S}(A)_{S'})$. Next we will recall the definition of simplicial map.

Definition 2.35. Let (X, A) and (Y, B) be simplicial complexes and f map between pairs. Then we say that map f is simplicial if for any simplex in X the images of vertexes of the simplex span some simplex in Y and images of vertexes of simplex in A span some simplex in B .

Definition 2.36. Let $f : (X, A) \rightarrow Y$ and $g : (X, A) \rightarrow (Y, B)$ be simplicial maps then we say that maps are contiguous if for every simplex $s = \{a_1, \dots, a_n\}$ in $X(A)$ there exists simplex k in $Y(B)$ for which $\text{span}(f(a_1), \dots, f(a_n)) \subset k$ and $\text{span}(g(a_1), \dots, g(a_n)) \subset k$

Lemma 2.37. If $f : (X, A) \rightarrow (Y, B)$ and $g : (X, A) \rightarrow (Y, B)$ are contiguous maps then they are homotopic.

Proof. We define $H : X \times [0, 1] \rightarrow Y$ in a following way:

$$H(x, t) = tf(x) + (1 - t)g(x)$$

It is easy to see that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, also because $f(x)$ and $g(x)$ both belong to same simplex $H(x, t)$ is well defined at every point. \square

2.3.2 Coverings

We begin by recalling definition of covering

Definition 2.38. Let X be a topological space and $\{A_i\}_{i \in J}$ collection of its open subsets. Then we say that $\{A_i\}_{i \in J}$ is covering of X if $\bigcup_{i \in J} A_i = X$.

We will denote set of all open coverings of space X by $\text{COV}(X)$. Because in homology we are interested in pair of spaces we give definition of covering for the pair.

Definition 2.39. Let (X, A) be topological pair. Then we say that $\{(U_i, V_i)\}_{i \in J}$ is covering of (X, A) if $V_i \subset U_i$, $\{U_i\}_{i \in J}$ covers X and $\{V_i\}_{i \in J}$ covers A .

Respectively we will define set of all coverings of pair (X, A) to be $\text{COV}(X, A)$. Now we are ready to define nerve of the covering.

Definition 2.40. Let $(U_i, V_i)_{i \in J}$ be covering of some topological pair (X, A) . Then nerve of this covering is following abstract simplicial complex $\mathcal{S}(X, A)$:

- (1) We define vertex set of the complex $\mathcal{S}(X)$ to be $\{U_i\}_{i \in J}$ and vertex set of subcomplex $\mathcal{S}(A)$ to be $\{V_i\}_{i \in J}$.
- (2) Let $I \subset J$, then for every such I for which $\bigcap_{i \in I} U_i \neq \emptyset$ define simplex in simplicial complex $\mathcal{S}(X)$. In other words $\mathcal{S}(X)$ consist of simplexes defined by all non-empty intersection of spanning sets. In subcomplex \mathcal{A} we define simplex for every non-empty intersection $\bigcap_{i \in I} V_i \cap A$ where I is some subspace of J .

It is easy to see that the simplicial complex is well defined. We can interpret simplicial complex $\mathcal{S}(X)$ as collection of faces of one big complex, then it is enough to show that every face of any simplex in the collection belongs to the collection. We see that if $\bigcap_{i \in N} U_i \neq \emptyset$, then for every $M \subset N$ intersection of $\bigcap_{i \in M} U_i$ is non empty. In coverings of topological pair (X, A) we can define quasi order in a natural way. First we will recall definition of refinement.

Definition 2.41. Let $\alpha = \{(U_i, V_i)\}_{i \in I}$ and $\beta = \{(U'_i, V'_i)\}_{i \in J}$ be coverings of some pair (X, A) . Then covering β is called refinement of α if for every $(U'_i, V'_i) \in \beta$ there exists some $(U_i, V_i) \in \alpha$ for which $U'_i \subset U_i$ and $V'_i \subset V_i$.

Now we are ready to define the quasi order in the collection of coverings of an arbitrary topological space.

Lemma 2.42. Let $COV(X, A)$ be collection of all coverings of the pair (X, A) , then order defined by $\alpha \leq \beta$ if and only if β is refinement of α is quasi-order.

Proof. Any covering is refinement of itself and if we $a \leq b$ and $b \leq c$ then for every set U_c we can find U_b for which $U_c \subset U_b$ and $U_b \subset U_a$ for some U_a . Then we see that $U_c \subset U_a$ so actually c is refinement of a . \square

Definition 2.43. Let β and α be coverings for which $\alpha \leq \beta$. Then we define $\mathcal{P}(\alpha, \beta)$ to be collection of the following projection maps $p_\alpha^\beta : \beta \rightarrow \alpha$: Every $U_i \in \beta$ is mapped to some set $U'_i \in \alpha$ for which $U_i \subset U'_i$ holds.

We will now show that the family of projection maps is closed under composition.

Lemma 2.44. Let p_α^β and p_β^γ be some projection maps. Then $p_\alpha^\beta \circ p_\beta^\gamma$ is also projection map.

Proof. Let $U \in \gamma$, then there exists $U' \in \beta$ and $U'' \in \alpha$ for which condition $U \subset U' \subset U''$ is satisfied. By properties of inclusion $U \subset U''$ and thus $p_\alpha^\beta \circ p_\beta^\gamma$ is a projection map. \square

Lemma 2.45. The collection of coverings $COV(X, A)$ is a category defined in the following way:

- (1) The objects of $COV(X, A)$ are all open coverings of the pair (X, A) .
- (2) We define $Hom(\beta, \alpha)$ to be $\mathcal{P}(\alpha, \beta)$, if $\alpha \leq \beta$. Else we define $Hom(\beta, \alpha)$ to be empty set.
- (3) The composition of maps is defined to be the usual composition of functions.

Proof. By lemma 2.44 the composition of two projection maps is projection map and thus is well-defined. For every α there exists well-defined projection map p_α^α which acts like identity map on the $\text{COV}(X, A)$. The associativity condition holds, because every morphism is a function. \square

Definition 2.46. Let $F : (X, A) \rightarrow (Y, B)$ and let β be covering of (Y, A) , then define $f^{-1}\beta$ to be covering of (X, A) which is induced by elements $(f^{-1}U_i, f^{-1}V_i)_{i \in J_\beta}$.

Lemma 2.47. Let $F : (X, A) \rightarrow (Y, B)$ be function between pairs and let α and β be covers for (X, A) and (Y, B) for which $\alpha = f^{-1}\beta$ holds. Then a map $p : \mathcal{S}(X, A)_\alpha \rightarrow \mathcal{S}(Y, B)_\beta$ defined by $(f^{-1}U_i, f^{-1}V_i) \rightarrow (U_i, V_i)$ is well defined.

Proof. Let $\bigcap_{i \in I} f^{-1}U_i \neq \emptyset$ for some index set $I \subset J$. We see that $\bigcap_{i \in I} U_i \neq \emptyset$, so the lemma holds. \square

Lemma 2.48. Let β and α be coverings of (X, A) in such way that $\alpha \leq \beta$ and p be a projection map belonging to $\mathcal{P}(\alpha, \beta)$. Let map $p_\alpha^\beta : \mathcal{S}(X, A)_\beta \rightarrow \mathcal{S}(X, A)_\alpha$ be a map between simplices defined by edges $U \in \mathcal{S}(X, A)_\beta$ with requirement of being piecewise linear on the other elements. Then the map p' is well-defined.

Proof. Let x be arbitrary element of $\mathcal{S}(X, A)_\beta$, then it belongs to some simplex. Let S be simplex which is intersection of all simplices in $\mathcal{S}(X, A)_\beta$ which have property that intersection with $\{x\}$ is non-empty. In simplex S point x has unique representation $x = \sum_{i \in J} r_i a_i$ where elements a_i are edges and r_i such non-negative real numbers for which $\sum_{i \in J} r_i = 1$. By definition of map p' this point is mapped to $p'(x) = \sum_{i \in J} r_i p_\alpha^\beta(a_i)$ which is uniquely determined so map p' is well-defined. \square

We denote map p_α^β described above just as p_α^β . It is clear from context which of the maps is being used. We will prove now that the map p is simplicial

Lemma 2.49. Projection maps p_α^β are simplicial.

Proof. Let s be simplex in $\mathcal{S}(X, A)_\alpha$ spanned by some sets $\{U_i\}_{i \in J}$ for which $\bigcap_{i \in J} U_i \neq \emptyset$. Every U_i is mapped to some vertex U'_i in $\mathcal{S}(X, A)_\beta$ for which $U_i \subset U'_i$. Intersection $\bigcap_{i \in J} U'_i$ is clearly non empty, so every simplex is mapped inside some simplex. \square

Projection maps are not uniquely determined by coverings α and β , but any of those maps are contiguous thus they define same homology groups.

Lemma 2.50. Let α and β be any coverings of (X, A) for which condition $\alpha \leq \beta$ is satisfied. Then projection maps p_α^β and $p_\alpha^{\beta'}$ are contiguous.

Proof. To prove that the maps are contiguous we form simplex in $\mathcal{S}(Y, B)$ and prove that images of both maps belong to it. Every vertex U_i in $\mathcal{S}(X, A)$ is mapped to U'_i and U''_i by p^β_α and p'^β_α respectively. Now if $\bigcap_{i \in J} U_i \neq \emptyset$ then also $\bigcap_{i \in J} U'_i \cap \bigcap_{i \in J} U''_i \neq \emptyset$. In case if the vertexes belong to $\mathcal{S}(A)$ it is easy to see that projection map maps them to some simplex in $\mathcal{S}(B)$. \square

By using this lemma and the fact that contiguous simplicial maps induce same homology groups we can define unique homomorphisms $I^\beta_\alpha : H_q(\mathcal{S}(X, A)_\beta) \rightarrow H_q(\mathcal{S}(X, A)_\alpha)$ for every $q \in \mathbb{N}$.

For projection maps and induced homomorphisms we have following useful lemma.

Lemma 2.51. *Let α be covering of pair (X, A) and β covering of pair (Y, B) for which $\alpha \leq \beta$ holds and $f : (X, A) \rightarrow (Y, B)$ a function between pairs. Then pre-image $f^{-1}\beta$ of covering β is refinement of $f^{-1}\alpha$.*

Proof. This follows directly from the fact that if $U \subset V$, then also $f^{-1}U \subset f^{-1}V$. \square

Lemma 2.52. *Let $f : (X, A) \rightarrow (Y, B)$ and α and β coverings of (Y, B) for which $\alpha \leq \beta$ holds. Let p^β_α be projection map for the coverings, then there exists projection map $p^{\beta'}_{\alpha'}$ for which the diagram commutes.*

$$\begin{array}{ccc} \mathcal{S}(X, A)_{\beta'} & \xrightarrow{p^{\beta'}_{\alpha'}} & \mathcal{S}(X, A)_{\alpha'} \\ \downarrow f_\beta & & \downarrow f_\alpha \\ \mathcal{S}(Y, B)_\beta & \xrightarrow{p^\beta_\alpha} & \mathcal{S}(Y, B)_\alpha \end{array}$$

Proof. Let $\{U_i\}_{i \in J}$ be some simplex in $\mathcal{S}(X, A)_{\beta'}$. For every edge U the set $f^{-1} \circ p \circ f(U)$ is non-empty because p maps $f(U)$ to larger set. Thus for every edge U_i there is corresponding edge in $\mathcal{S}(X, A)_{\alpha'}$. We can now define map $p^{\beta'}_{\alpha'} : \mathcal{S}(X, A)_{\beta'} \rightarrow \mathcal{S}(X, A)_\alpha$ in such way that edge U is mapped to the edge $f^{-1}pfU$ and we extend this map by linearity. Now by defining map $p^{\beta'}_{\alpha'}$ in such way for every simplex we get well-defined map for which the diagram commutes. \square

Chapter 3

Cech homology

In this chapter we will construct Cech homology and prove main results related to it. We will first define inverse system of homology groups of the nerves.

Definition 3.1. Let α be element of $\text{COV}(X, A)$ and G abelian group. Then we can define group $H_{q,\alpha}(X, A; G)$ using the singular homology groups in a following way:

$$H_{q,\alpha}(X, A; G) = H_q(\mathcal{S}(X, A)_\alpha; G)$$

. For every coverings of α, β in (X, A) for which $\alpha \leq \beta$ we define the maps to be

$$I_\alpha^\beta : H_{q,\beta} \rightarrow H_{q,\alpha}$$

maps I_β^α are induced by some maps p_α^β which are unique up to homotopy.

Lemma 3.2. Let $q \in \mathbb{N}$ be fixed number, let (X, A) be topological pair and G abelian group. Then groups $H_{q,\alpha}(X, A; G)$ together with maps I_β^α form inverse system of groups.

Proof. First we have to show that I_α^α is identity function for every $\alpha \in \text{COV}(X, A)$. We already know that identity map between simplicial complexes is projection map and because identity map induces identity map between the complexes by uniqueness of I_α^α we see that the map has to be identity map between corresponding homology groups.

Now for every coverings α, β, γ of pair (X, A) for which relation $\alpha \leq \beta \leq \gamma$ holds we have to show that $I_\beta^\gamma \circ I_\alpha^\beta = I_\alpha^\gamma$. By lemma 2.44 the composition of corresponding projection maps $p_\beta^\gamma \circ p_\alpha^\beta$ is projection map. Thus it defines map I_α^γ . \square

Now we are ready to define the cech homology groups \check{H}_n .

Definition 3.3. Let (X, A) be topological pair and G abelian group. Then cech homology groups with coefficients G are defined to be inverse limits of $H_{q,\alpha}(X, A; G)$ where α belongs to $\text{COV}(X, A)$. Simply denoted $\check{H}_q(X, A; G) = \lim_{\rightarrow} \{H_{q,\alpha}(X, A; G)\}$

Because coverings can be infinite the limits are not necessary defined. However there is a way to evade this problem.

3.1 Well-definedness of the Čech homology groups

In this section we will provide condition for the spaces which will guarantee that the Čech groups are well-defined. ... From now on in this chapter we assume that the topological spaces are such for which the Čech groups are well-defined.

3.2 Induced homomorphisms

In this section we will investigate properties of induced homomorphism. Earlier we defined Čech homology groups to be limit of the groups induced by coverings of the pair. In section 2.2.3 we defined concept of limit homomorphism. In next theorem we will show that such homomorphism exists.

Theorem 3.4. *Let $f : (X, A) \rightarrow (Y, B)$ be continuous function between pairs of spaces. Let $\mathcal{A} = \{f_\alpha \mid \alpha \text{ is covering of pair } (Y, B)\}$ be family of functions for which every element of the open covering α is mapped to its preimage. This family induces maps between corresponding Čech homology groups.*

$$f'_{\alpha,q} : H_q(\mathcal{S}(X, A)_{f^{-1}\alpha}; G) \rightarrow H_q(\mathcal{S}(Y, B)_\alpha; G)$$

We denote a family of functions indexed by coverings of (Y, B) described above with \mathcal{A}_q . Then every such collection is natural transformation of the Čech homology groups and the limit function is well defined.

Proof. We can define a functor $\phi : \text{COV}(Y, B) \rightarrow \text{COV}(X, A)$ to map every object α to $f^{-1}\alpha$. By lemma 2.51 this functor preserves order. Let β and α be some coverings of pair (Y, B) . We denote α' and β' to be the preimage categories of α and β under map f . The diagram of lemma 2.52 shows that the corresponding maps between coverings commute. Thus we have a following commutable diagram:

$$\begin{array}{ccc} H_q(\mathcal{S}(X, A)_{\beta'}) & \xrightarrow{I_{\alpha'}^{\beta'}} & H_q(\mathcal{S}(X, A)_{\alpha'}) \\ \downarrow f'_{\beta,q} & & \downarrow f'_{\alpha,q} \\ H_q(\mathcal{S}(Y, B)_\beta) & \xrightarrow{I_\alpha^\beta} & H_q(\mathcal{S}(Y, B)_\alpha) \end{array}$$

The fact that collection \mathcal{A}_q is natural transformation of the groups follows directly from the above diagram. Thus by lemma 2.28 there exists limit homomorphism

$$f_q : \check{H}_q(X, A; G) \rightarrow \check{H}_q(Y, B; G)$$

□

We will show now that the Čech homology group is a functor.

Theorem 3.5. *Čech homology groups are functors from the category of the topological pairs to the category of abelian groups*

Proof. (1) Every pair of spaces is mapped uniquely to some abelian group.

(2) The limit of identity map is an identity homomorphism.

(3) Let $f : (X, A) \rightarrow (Y, B)$ and $g : (Y, B) \rightarrow (Z, C)$ be continuous maps between pairs. We know that $(g \circ f)_* = g_* \circ f_*$ holds for the singular homology groups. Then by using theorem 2.27 we see that the limit homomorphism $\lim((g \circ f)_*)$ equals $\lim g_* \circ \lim f_*$.

□

3.3 Dimension axiom

In this section we will prove that the Čech homology theory satisfies the dimension axiom.

Theorem 3.6.

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