Cech (co)homology

Yury Elkin

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Background

In this chapter we will a few details about algebra, category theory and abstract simplicial complexes. We assume that reader is familiar with basic concepts of homology and cohomology theory and constructions related to them.

2.1 Direct sum and direct product

In this section we will recall a few details about Direct sums and direct products

Definition 2.1. Let $\{A_i\}_{i \in J}$ be family of groups. Direct product of groups is the cartesian product $\prod_{i \in J} A_i$ where addition is defined componentwise $(a + b)_i = a_i +_i b_i$

Definition 2.2. Let $\{A_i\}_{i \in J}$ be family of groups. Direct sum of groups is subgroup of direct product $\bigoplus_{i \in J} A_i = \{x \in \prod_{i \in J} A_i \mid x_i \neq 0 \text{ for only finetely many } i\}$

For direct sums and direct products we have following universal properties:

Lemma 2.3. *Let $A = \prod_{i \in J} A_i$ be direct sum and D arbitary group. Then for every set homomorphisms $\{f_i : D \to A_i\}_{i \in J}$ there exists unique homomorphism $f : D \to A$ for which and for every $i \in J$ condition $pr_i \circ f = f_i$ holds.*

Proof. trivial □

Lemma 2.4. *Let $A = \bigoplus_{i \in J} A_i$ be direct sum and D arbitary group. Then for every set homomorphisms $\{f_i : A_i \to D\}_{i \in J}$ there exists unique homomorphism $f : A \to D$ for which and for every $i \in J$ condition $f \circ i_i = f_i$ holds.*

Proof. trivial □

2.2 Category theory

In this thesis we will present theory in a categorical way, which will make the construction more general. First lets recall definition of a category.

Definition 2.5. A category C consist of a following ingredients: A class of objects ob(C), class of morphisms hom(C) for which and for every object A, B in ob(C) there exists subclass $Hom(A, B)$ and a rule of composition $Comp : Hom(A, B) \times Hom(A, D) \rightarrow Hom(A, D)$. For which following conditions hold:

(1) Composition is associative. Let $f : A \rightarrow B$, $g : B \rightarrow D$ and $h : D \rightarrow E$ be functors between categories then $(f \circ g) \circ h = f \circ (g \circ h)$

(2) For every category $A \in$ ob(C) there exists identity functor $1_A \in$ hom(A, A) for which following condition hold: Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be arbitrary functors then $f \circ 1_A = f$ and $1_A \circ g = g$.

It is easy to see that topological spaces with continuous functions and abelian groups with homomorphisms form a category. We denote those categories by TOP and AB. Next we will define concept of functor between categories.

Definition 2.6. Let C and C' be categories and F rule between them. We say that F is functor between categories if following conditions hold:

(1) For every object $A \in$ ob(C) there exists unique element $F(A) \in C'$.

(2) Let A and B be objects in ob(C) and $f : A \rightarrow B$ functor between them. Then there exists unique functor $F(f) : F(A) \rightarrow F(B)$.

(3) if f and g are morphisms in hom(C), then following condition holds: $F(f) \circ F(g) = F(f \circ g)$.

(4) Identity element is mapped to identity element. Let A be object in ob(C) and 1_A identity functor corresponding to it. Then $T(1_A)$ is identity functor of $T(A)$.

All homotopy, homology and cohomology groups form functor from TOP to AB. For details see Rotman [1]. Now we will introduce new concept called natural transformation

Definition 2.7. Let C and D be categories and $F : C \rightarrow D$ and $G : C \rightarrow D$ functors between them. Then the following family of functors is natural transformation:

$$\{\phi_X : F(X) \rightarrow G(X)\}_{X \in C}$$

if following conditions hold:

(1) For every object $X \in C$ there exists unique morphism $\phi_X : F(X) \to G(X)$

(2) For every functor $f : X \to Y$: $\phi_Y \circ F(f) = G(f) \circ \phi_X$

Next we will define limit and colimit concepts for arbitary categories, which will be later applied to directed systems.

Definition 2.8. Let C be indexed category in a such way that there exists category J with functor $F : J \to C$. Let N be fixed object of $\mathrm{ob}(C)$. We define cone from N to F to be indexed family of morphisms

$$\{\omega_X : N \to F(X)\}_{X \in J}$$

which satisfies following property: Let $f : X \to Y$ be morphism in C then the following diagram commutes

$$\begin{tikzcd}
& N \arrow[dl, "\omega_X"'] \arrow[dr, "\omega_Y"] & \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}$$

We denote this structure by (N, ω)

Definition 2.9. Let C be indexed category which is indexed by category J and functor F. Let D be object of category C which forms cone together with function ϕ. We say that (D, ϕ) is limit of C if following property holds: Let (N, ω) be any other cone of category C, then there exists unique morphism $u : N \to D$ in such way that following diagram commutes:

$$\begin{tikzcd}
& N \arrow[ddl, "\omega_X"'] \arrow[d, dashed] \arrow[ddr, "\omega_Y"] & \\
& D \arrow[dl, "\phi_X"] \arrow[dr, "\phi_Y"'] & \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}$$

Next we will proof that we have two limits N and D of category C then there exists isomorphism between those two objects. In other words N and D can be identified and the limit of category C can be denoted simply as $\varinjlim C$

Theorem 2.10. *Let N and D be limits of category C. Then there exists isomorphism between them.*

Proof. Because N and D are both limits there exist morphisms $u : N \to D$ and $v : D \to N$. Because of the symmetry it is enough $u \circ v = id_N$. This follows from equations: $\omega_X \circ u \circ v = \phi_X \circ u = \omega_X$ and $u \circ v \circ \omega_Y = u \circ \phi_Y = \omega_Y$. Because the diagram commutes for id_N by uniqueness condition we see that $u \circ v = id_N$. $\square$

2.2.1 Dual of category

To simplify definitions we will define concept of dual of category. In this section we will interpret category as model of first order logic

Definition 2.11. Let ω be statement in this model. We get dual statement ω^{op} by following procedure:

(1) Interchange every occurrence of source with target

(2) Reorder every composition. That is replace $a \circ b$ with $b \circ a$.

Next we will give important example of dual statement.

Definition 2.12. Colimit of category C is dual statement of limit of C. We define co-cone to be dual of cone. Let J be index category of C then we say that co-cone D is colimit of category C if for every other co-cone N there exists unique morphism $u : D \to N$ in such way that the diagram below commutes

$$\begin{tikzcd}
 & N & \\
 & D \arrow[u, dashed] & \\
F(X) \arrow[uur, "\omega_X"] \arrow[ur, "\phi_X"'] & & F(Y) \arrow[uul, "\omega_Y"'] \arrow[ul, "\phi_Y"] \arrow[ll, "F(f)"]
\end{tikzcd}$$

Like limits colimits are unique.

Theorem 2.13. *Let N and D be colimits of category C. Then there exists isomorphism between them.*

Proof. Because N and D are both colimits there exist morphisms $u : N \to D$ and $v : D \to N$. Because of the symmetry it is enough $u \circ v = id_N$. This follows from equations: $u \circ v \circ \omega_X = u \circ \phi_X = \omega_X$ and $u \circ v \circ \omega_Y = u \circ \phi_Y = \omega_Y$. Because the diagram commutes for id_N by uniqueness condition we see that $u \circ v = id_N$. $\square$

2.2.2 Direct and inverse systems

In this section we will categorical definition of direct and inverse systems. It appears that those concepts are dual of each other. We will begin by defining Quasi ordering relation:

Definition 2.14. Let $a \leqslant b$ be relation in set λ. We say that this relation is quasi-ordering if following conditions hold:

(1) $a \leqslant a$ for all $a \in \lambda$

(2) if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ for all $a, b, c \in \lambda$

Definition 2.15. Let λ be quasi-ordered set and $\mathcal{L}$ a category then a direct λ-system consist of functor D which assigns unique object $D(\lambda) \in \mathcal{L}$ and a family of morphism in such way that for every pair $a \leqslant b$ there exists unique morphism $D_a^b : D(a) \to D(b)$. The family of morphisms satisfies following conditions:

(1) For every triple $a \leqslant b \leqslant c$ in λ functions commute in a following way: $D_b^c \circ D_a^b = D_a^c$

(2) For every $a \in \lambda$: $D_a^a = id_a$

The structure in above definition can be also viewed as category C in such way that $\mathrm{ob}(C) = D(\lambda)$ and for every $D(a), D(b) \in \mathrm{ob}(C)$ there exists exactly one morphism in $\mathrm{hom}(D(a), D(b)$. In other words λ is indexation of subcategory of $\mathcal{L}$. We can define dual concept of above definition in a following way:

Definition 2.16. Let λ be quasi-ordered set and $\mathcal{L}$ category. Then inverse system is dual of a direct system. Let I be functor from λ to $\mathcal{L}$ in such way that for every pair $a \leqslant b$ there exists unique morphism $I_a^b : I(b) \to I(a)$ and following properties hold: $I_b^c \circ I_a^b = I_a^c$ and $I_a^a = id_a$.

Example 2.17. Let SET be category of sets with order relation in such way that $U \leqslant V \Leftrightarrow U \subset V$ now for every U, V we define $D_U^V U \to V$ to be just inclusion from U to V. Clearly this forms direct system.

Next we define a bit more complex category

Definition 2.18. Direct system together with natural transformations between them form. We will denote elements of this category shortly $(\lambda, \mathcal{L})$

Taking dual and using functor I instead of D we see that natural transformation with inverse system form category. We will denote this category by $(\lambda, \mathcal{L}^{op})$. Now we can define limit of the category $(\lambda, \mathcal{L})$. We see that for every quasi-order λ and object k in $\mathcal{L}^{op}$ we can define inverse system by mapping every element λ to the object k and every functor between objects in λ to identity functor. This forms inverse system which we will denote by K. Now natural transformations $\phi_a : D(a) \to K$ between K and arbitrary element in $(\lambda, \mathcal{L})$ induces following: co(cone)

$$\begin{tikzcd} & K & \\ D(a) \arrow[ur, "\phi_a"] \arrow[rr, "D_a^b"'] & & D(b) \arrow[ul, "\phi_b"'] \end{tikzcd}$$

So we can define (co)limit of this structure in unique way if it exists. We will now give important example of this construction and show that in case category is AB there exists such limit. To prove this result we first recall following lemma:

Theorem 2.19. *Let $D : \lambda \to AB$ be a direct system of groups. Let G be the subgroup of $\bigoplus_{a\in\lambda} D(a)$ with is generated by $\{i_a x_a - i_b D_a^b x_a\}$. Then limit of this system is $L = \bigoplus_{a\in\lambda} D(a)/G$*

Proof. Let $\{v_a : D(a) \to L\}$ induce cone L and K be any other cocone induced by functors $\{\phi_a : D(a) \to K\}$. We have to prove that there exists unique homomorphism $u : L \to K$ for which condition $u \circ v_a = \phi_a$ holds for all $a \in \mathrm{ob}(\mathcal{L})$. Lets first assume that such homomorphism exists and prove that it is unique. Because images of functors v_a generate space L it follow that u must be unique
We will now construct morphism u. Using lemma 2.4 we can find unique homomorphism $u' : \bigoplus_{a\in\lambda} D(a) \to K$ for which $u' \circ i_a = \phi_a$ holds. Now we see that for every generator of subgroup G $u'(i_a x_a - i_b D_a^b x_a) = \phi_a(x_a) - \phi_b D_a^b x_a = \phi_a(x_a) - \phi_a(x_a) = 0$. Now we can define u in such way that every element $x + D$ maps to element $\phi'(x)$. We will denote the projection map from $\bigoplus_{a\in\lambda} D(a)$ to L by p. Now we see that $u \circ v_a = u \circ p \circ i_a = u' \circ i_a = \phi_a$. □

Theorem 2.20. *Let $I : \lambda \to AB$ be a inverse system of groups. Then limit of this system is group $L = \{x \in \prod_{i\in J} I_i \mid x_a = I_b^a x_b \text{ for all } a \leqslant b\}$*

Proof. Let $\{v_a : L \to I(a)\}_{a\in\lambda}$ induce cone L and K be any other cone induced by functors $\{\phi_a : K \to I(a)\}_{a\in\lambda}$. We construct function $u : K \to L$ for which $v_a \circ u = \phi_a$. By lemma 2.3 we know that there exists unique homomorphism $u' : K \to \prod_{i\in J} I_i$ for which $pr_a \circ u' = \phi_a$. It is easy to see that u' maps actually every element to L. Assume that β and α are elements which satisfy $a \leqslant b$ condition. Then $I_b^a \circ pr_b \circ u' = I_b^a \circ \phi_b = \phi_a = pr_a \circ u'$. We can define u to be u'. □

2.2.3 Cofinal functors

We will first define cofinality in abstract way.

Definition 2.21. A functor $L : J' \to J$ is cofinal if for every element $k \in J$ there exists $j' \in J'$ and morphism $Lj' \to k$.

Definition 2.22. Let C be category, the objects of which are induced by quasi-ordered set λ and let A be its subcategory. The subcategory A is cofinal subset of C if for every element $\beta \in X$ we can find element $\alpha \in A$ for which condition $\beta \leqslant \alpha$ holds.

Now for inverse and direct systems we have following theorem:

Theorem 2.23. *Let D be direct system and T its subsystem, in such way that $ob(T)$ is cofinal in $ob(D)$. Assume that D has limit. Then T has limit too which is isomorphic to limit of D.*

Proof. We have following situation: □

2.3 Nerve of covering

In this chapter we will give abstract definition of simplicial complex.

2.3.1 Abstract simplicial complex

We assume that reader knows already basic facts about simplicial complexes if not we suggest to take a look at [1]. We start by defining infinite dimensional simplicial complex.

Definition 2.24. Let X be set of vertices. Then abstract simplicial complex consist of collection $\{S_i\}_{i\in I}$ of finite dimensional simplexes which have edges in set X. The collection has property which is for every simplex face of it belongs to the collection. For every point in abstract simplicial complex $\mathcal{S}(X)$ we define coordinates in a following way: Let $x \in \bigcup_{i\in I} S_i$ then $x \in S_i$ for some $i \in I$. Then for point x we define coordinates $x = \sum_{x_i \in X} a_i x_i$ for which $a_i > 0$ only for finitely many $i \in I$ and $\sum_{i\in I} a_i = 1$ holds.

It is easy to see that coordinates are well defined. If x belongs to two different simplex S and S' then x belongs to finite simplicial complex spanned by generators of $S \cup S'$. Thus the statement reduces to finite dimensional case. Because face of any simplex in the collection belongs to collection we see that simplicial complex spanned by generators $S \cup S'$ is union of some faces of simplex which we get by taking vertices to be $S \cup S'$ and thus x has well defined coordinates.

Definition 2.25. Let set X together with collection $\mathcal{A} = \{S_i\}_{i\in I}$ be simplicial complex. Then we say that A is subcomplex if vertices of it are spanned by X and it consist of subfamily of $\mathcal{A}$.

If $\mathcal{S}(X)$ is simplicial complex and $\mathcal{S}(A)$ its subcomplex they form topological pair which we will denote by $(\mathcal{S}(X), \mathcal{S}(A))$ Next we will recall the definition of simplicial map.

Definition 2.26. Let (X, A) and (Y, B) be simplicial complexes and f map between pairs. Then we say that map f is simplicial if for any simplex in X the images of vertexes of the simplex span some simplex in Y and images of vertexes of simplex in A span some simplex in B.

Definition 2.27. Let $f : (X, A) \to Y$ and $g : (X, A) \to (Y, B)$ be simplicial maps then we say that maps are contiguous if for every simplex $s = \{a_1, ..., a_n\}$ in $X(A)$ there exists simplex k in $Y(B)$ for which $\text{span}(f(a_1), ..., f(a_n)) \subset k$ and $\text{span}(g(a_1), ..., g(a_n)) \subset k$

Lemma 2.28. *If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are contiguous maps then they are homotopic.*

Proof. We define $H : X \times [0, 1] \to Y$ in a following way:

$$H(x, t) = tf(x) + (1 - t)g(x)$$

It is easy to see that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, also because $f(x)$ and $g(x)$ both belong to same simplex $H(x, t)$ is well defined at every point. □

2.3.2 Coverings

We begin by recalling definition of covering

Definition 2.29. Let X be a topological space and $\{A_i\}_{i\in J}$ collection of its open subsets. Then we say that $\{A_i\}_{i\in J}$ is covering of X if $\bigcup_{i\in J} A_i = X$.

We will denote set of all open coverings of space X by $\text{COV}(X)$. Because in homology we are interested in pair of spaces we give definition of covering for the pair.

Definition 2.30. Let (X, A) be topological pair. Then we say that $\{(U_i, V_i)\}_{i\in J}$ is covering of (X, A) if $V_i \subset U_i$, $\{U_i\}_{i\in J}$ covers X and $\{V_i\}_{i\in J}$ covers A.

Respectively we will define set of all coverings of pair (X, A) to be $\text{COV}(X, A)$. Now we are ready to define nerve of the covering.

Definition 2.31. Let $(U_i, V_i)_{i\in J}$ be covering of some topological pair (X, A). Then nerve of this covering is following abstract simplicial complex $\mathcal{S}(X, A)$:

(1) We define vertex set of the complex $\mathcal{S}(X)$ to be $\{U_i\}_{i\in J}$ and vertex set of subcomplex $\mathcal{S}(A)$ to be $\{V_i\}_{i\in J}$.

(2) Let $I \subset J$, then for every such I for which $\bigcap_{i\in I} U_i \neq \varnothing$ define simplex in simplicial complex $\mathcal{S}(X)$. In other words $\mathcal{S}(X)$ consist of simplexes defined by all non-empty intersection of spanning sets. In subcomplex $\mathcal{A}$ we define simplex for every non-empty intersection $\bigcap_{i\in I} V_i \cap A$ where I is some subspace of J.

It is easy to see that the simplicial complex is well defined. We can interpret simplicial complex $\mathcal{S}(X)$ as collection of faces of one big complex, then it is enough to show that every face of any simplexes in the collection belongs to the collection. We see that if $\bigcap_{i\in N} U_i \neq \varnothing$, then for every $M \subset N$ intersection of $\bigcap_{i\in M} U_i$ is non empty. In coverings of topological pair (X, A) we can define quasi order in a natural way. First we will recall definition of refinement.

Definition 2.32. Let $\alpha = \{(U_i, V_i)\}_{i\in I}$ and $\beta = \{(U_i', V_i')\}_{i\in J}$ be coverings of some pair (X, A). Then covering β is called refinement of α if for every $(U_i', V_i') \in \beta$ there exists some $(U_i, V_i) \in \alpha$ for which $U_i' \subset U_i$ and $V_i' \subset V_i$.

Now we define quasi order in a following way: $\alpha \leqslant \beta$ if β is refinement of α. Clearly the quasi order is well defined. Any covering is refinement of itself and if we $a \leqslant b$ and $b \leqslant c$ then for every set U_c we can find U_b for which $U_c \subset U_b$ and $U_b \subset U_a$ for some U_a. Then we see that $U_c \subset U_a$ so actually c is refinement of a.

Definition 2.33. Let $F : (X, A) \to (Y, B)$ and let β be covering of (Y, A), then define $f^{-1}\beta$ to be covering of (X, A) which is induced by elements $(f^{-1}U_i, f^{-1}V_i)_{i\in J_\beta}$.

Lemma 2.34. *Let $F : (X, A) \to (Y, B)$ be function between pairs and let α and β be covers for (X, A) and (Y, B) for which $\alpha = f^{-1}\beta$ holds. The Then map $p : \mathcal{S}(X, A)_\alpha \to \mathcal{S}(Y, B)_\beta$ defined by $(f^{-1}U_i, f^{-1}V_i) \to (U_i, V_i)$ is well defined.*

Proof. Let $\bigcap_{i\in I} f^{-1}U_i \neq \varnothing$ for some index set $I \subset J$. We see that $\bigcap_{i\in I} U_i \neq \varnothing$, so the lemma holds. $\square$

Let β and α be coverings for which $\alpha \leqslant \beta$. Now we can define projection map from $p_\alpha^\beta : \beta \to \alpha$. For every $U_i \in \beta$ we map it to some set $U_i' \in \alpha$ for which $U_i \subset U_i'$. We can extend those maps to the corresponding maps between abstract simplicial complex.

Lemma 2.35. *Let β and α be coverings of (X, A) in such way that $\alpha \leqslant \beta$ and p be projection map defined above. Then map $p_\alpha'^\beta : \mathcal{S}(X, A)_\beta \to \mathcal{S}(X, A)_\alpha$ defined to be piecewise linear map induced by edges $U \in \mathcal{S}(X, A)_\beta$ and map p is well-defined.*

Proof. Let x be arbitary element of $\mathcal{S}(X,A)_\beta$, then it belongs to some simplex. Let S be simplex which is intersection of all simplices in $\mathcal{S}(X,A)_\beta$ which have property that intersection with $\{x\}$ is non-empty. In simplex S point x has unique representation $x = \sum_{i\in J} r_i a_i$ where elements a_i are edges and r_i such non-negative real numbers for which $\sum_{i\in J} r_i = 1$. By definition of map p' this point is mapped to $p'(x) = \sum_{i\in J} r_i p_\alpha^\beta(a_i)$ which is uniquely determined so map p' is well-defined. □

We denote map p'^β_α described above just as p^β_α. It is clear from context which of the maps is used. We will prove now that the map p is simplicial

Lemma 2.36. *Projection maps p^β_α are simplicial.*

Proof. Let s be simplex in $\mathcal{S}(X,A)_\alpha$ spanned by some sets $\{U_i\}_{i\in J}$ for which $\bigcap_{i\in J} U_i \neq \varnothing$. Every U_i is mapped to some vertex U_i' in $\mathcal{S}(X,A)_\beta$ for which $U_i \subset U_i'$. Intersection $\bigcap_{i\in J} U_i'$ is clearly non empty, so every simplex is mapped inside some simplex.

□

We will now show that the family of projection maps is closed under composition.

Lemma 2.37. *Let p^β_α and p^γ_β be some projection maps . Then $p^\beta_\alpha \circ p^\gamma_\beta$ is also projection map.*

Proof. Let $U \in \gamma$, then there exists $U' \in \beta$ and $U'' \in \alpha$ for which condition $U \subset U' \subset U''$ is satisfied. By properties of inclusion $U \subset U''$ and thus $p^\beta_\alpha \circ p^\gamma_\beta$ is projection map. □

Projection maps are not uniquely determined by coverings α and β, but any of those maps are contiguous thus they define same homology groups.

Lemma 2.38. *Let α and β be any coverings of (X,A) for which condition $\alpha \leqslant \beta$ is satisfied. Then projection maps p^β_α and p'^β_α are contiguous.*

Proof. To prove that the maps are contiguous we form simplex in $\mathcal{S}(Y,B)$ and prove that images of both maps belong to it. Every vertex U_i in $\mathcal{S}(X,A)$ is mapped to U_i' and U_i'' by p^β_α and p'^β_α respectively. Now if $\bigcap_{i\in J} U_i \neq \varnothing$ then also $\bigcap_{i\in J} U_i' \cap \bigcap_{i\in J} U_i'' \neq \varnothing$. In case if the vertexes belong to $\mathcal{S}(A)$ it is easy to see that projection map maps them to some simplex in $\mathcal{S}(B)$. □

By using this lemma and the fact that contiguous simplicial maps induce same homology groups we can define unique homomorphisms $I^\beta_\alpha : H_q(\mathcal{S}(X,A)_\beta) \to H_q(\mathcal{S}(X,A)_\alpha)$ for every $q \in \mathbb{N}$.

For projection maps and induced homomorphisms we have following useful lemma.

Lemma 2.39. *Let $f : (X, A) \to (Y, B)$ and α and β coverings of (Y, B) for which $\alpha \leqslant \beta$ holds. Then for coverings $\alpha' = f^{-1}\alpha$ and $\beta' = f^{-1}\beta$ condition $\alpha' \leqslant \beta'$ holds. Let p_α^β be projection map for the coverings, then there exists projection map $p_{\alpha'}^{\beta'}$ for which the the diagram commutes.*

$$\begin{tikzcd}
\mathcal{S}(X, A)_{\beta'} \arrow[r, "p_{\alpha'}^{\beta'}"'] \arrow[d, "f_\beta"] & \mathcal{S}(X, A)_{\alpha'} \arrow[d, "f_\alpha"] \\
\mathcal{S}(Y, B)_{\beta} \arrow[r, "p_{\alpha}^{\beta}"'] & \mathcal{S}(Y, B)_{\alpha}
\end{tikzcd}$$

Proof. Let $\{U_i\}_{i \in J}$ be some simplex in $\mathcal{S}(X, A)_{\beta'}$. For every edge U the set $f^{-1} \circ p \circ f(U)$ is non-empty because p maps $f(U)$ to larger set. Thus for every edge U_i there is corresponding edge in $\mathcal{S}(X, A)_{\alpha'}$. We can now define map $p_{\alpha'}^{\beta'} : \mathcal{S}(X, A)_{\beta'} \to \mathcal{S}(X, A)_\alpha$ in such way that edge U is mapped to the edge $f^{-1}pfU$ and we extend this map by linearity. Now by defining map $p_{\alpha'}^{\beta'}$ in such way for every simplex we get well-defined map for which the diagram commutes. $\square$

Chapter 3

Cech homology

In this chapter we will construct Cech homology and prove main results related to it. We will first define inverse system of homology groups of the nerves.

Definition 3.1. Let α be element of $\mathrm{COV}(X, A)$ and G abelian group. Then we can define group $H_{q,a}(X, A; G)$ using the singular homology groups in a following way:

$$H_{q,a}(X, A; G) = H_q(\mathcal{S}(X, A)_\alpha; G)$$

. For every coverings of α, β in (X, A) for which $\alpha \leqslant \beta$ we define the maps to be

$$I_\alpha^\beta : H_{q,\beta} \to H_{q,\alpha}$$

maps I_β^α are induced by some maps p_α^β which are unique up to homotopy.

Lemma 3.2. *Let $q \in \mathbb{N}$ be fixed number, let (X, A) be topological pair and G abelian group. Then groups $H_{q,a}(X, A; G)$ together with maps I_β^α form inverse system of groups.*

Proof. First we have to show that I_α^α is identity function for every $\alpha \in \mathrm{COV}(X, A)$. We already know that identity map between simplicial complexes is projection map and because identity map induces identity map between the complexes by uniqueness of I_α^α we see that the map has to be identity map between corresponding homology groups.
Now for every coverings α, β, γ of pair (X, A) for which relation $\alpha \leqslant \beta \leqslant \gamma$ holds we have to show that $I_\beta^\gamma \circ I_\alpha^\beta = I_\alpha^\gamma$. By lemma 2.37 the composition of corresponding projection maps $p_\beta^\gamma \circ p_\alpha^\beta$ is projection map. Thus it defines map I_α^γ. □

Now we are ready to define the cech homology groups $\check{\mathrm{H}}_n$.

Definition 3.3. Let (X, A) be topological pair and G abelian group. Then cech homology groups with coefficients G are defined to be inverse limits of $H_{q,\alpha}(X, A; G)$ where α belongs to $\mathrm{COV}(X, A)$. Simply denoted $\check{\mathrm{H}}_q(X, A; G) = \lim_{\to}\{H_{q,\alpha}(X, A; G)\}$

Because coverings can be infinite the limits are not necessary defined. However there is a way to evade this problem.

3.1 Induced homomorphisms

In this section we will investigate

3.2 Dimension axiom

In this section we will prove that the cech homology theory satisfies the dimension axiom.

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