The response of glaciers and ice-sheets to seasonal and climatic changes

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A glacier is treated as an essentially one-dimensional flow system, which is, continuously along its length, either gaining new material by snowfall or losing it by melting and evaporation. The flow at each point, i.e. the volume passing a given point per unit time, is assumed to be a function of the thickness and surface slope. It is then shown that a region of uniform longitudinal compressive strain-rate is temporarily unstable. Lighthill & Whitham's theory of kinematic waves is applied to study the response of a simple model to a sudden change in rate of accumulation (snowfall). All parts of the glacier thicken, but the lower parts thicken unstably until a kinematic wave arrives to restore stability. The thickening of the lower parts, and the advance of the glacier, can be very great for only a small change in accumulation; the analysis thus explains why glaciers are such sensitive indicators of climate.

The velocity and diffusion of kinematic waves is discussed; they travel at 2 to 5 times the surface speed of the ice. Any part of a glacier or ice-sheet has a characteristic response time to changes in accumulation—about 5000 yr for the Antarctic ice-sheet as a whole, and 3 to 30 yr for a typical glacier. The amplitude and phase of the response of a simple glacier to high frequency (seasonal) and low-frequency (climatic) periodic changes in accumulation is calculated. (The periodic changes in accumulation are the Fourier components of the complicated variation that actually occurs.) The upper part behaves in a simple way; the lower part, on the other hand, shows not only a direct response, but also a delayed response due to the arrival of kinematic waves from the upper part. The two components interfere and give wide scope for variation in behaviour from one glacier to another.

A further instability, discussed by Bodvarsson, which arises from an increase of accumulation with altitude, is included in the theory. It is also shown how the theory is to be applied in a valley of changing width to calculate the development of any disturbance produced by climatic change. Thus the theory enables one to calculate, in principle, the response of any glacier to any climatic change.

1. Introduction

Glaciers are extremely sensitive indicators of climate, for a very slight climatic variation is sufficient to cause a considerable advance or retreat of the ice. By the successive advances and retreats of the glaciers the changing climate of the past has left its record on the landscape, in the forms of moraines, ancient and more recent, areas of changed vegetation, and in other features familiar to the geologist. To read this record we need to know, in some detail, how a glacier responds to a change in climate. The fact of the response is known, but the mechanism by which it occurs is obscure. The theory given in the following pages is based on the concept of a glacier as an essentially one-dimensional flow system, which is continuously throughout its length either receiving new material by snowfall or losing material by melting and evaporation. Much of the work is based on the theory of kinematic waves developed by Lighthill & Whitham (1955), which they have fruitfully applied to flood waves on rivers, a subject already extensively studied, to road traffic flow, and to the group velocity phenomenon. Kinematic wave phenomena also play a

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part in crystal growth (F. C. Frank, private communication) and in chromatography.

The wider problem of the relation between glaciers and climate has two aspects: the influence of the climate on the glaciers, which is a question of the kinematics and mechanics of ice flow; and the reverse effect, namely, the influence of the glaciers on the climate, which is a question of meteorology. We deal here only with the first, the non-meteorological, aspect.

2. Kinematic wave theory

Consider the following model of a glacier or ice-sheet. A sheet of ice rests on a slope of inclination β and flows down the slope, in the x direction, the sheet being of unlimited extent in the direction perpendicular to x. β is supposed to be a slowly varying function of x, and the thickness $h(x,t)$ is supposed to be such that $(\partial h/\partial x)_t \ll 1$. Then the surface slope $\alpha(x,t)$, downhill in the x direction, is given by

$$\alpha = \beta - \partial h/\partial x. \tag{1}$$

The volume of ice, for unit breadth, passing a given point in unit time is denoted by q, and is called the flow. The condition that there is no change in volume of the ice is then

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = a, \tag{2}$$

where $a(x,t)$ is the rate of accumulation at the surface. Specifically, a is the rate of addition of ice to the upper surface by snowfall and avalanching, measured as thickness of ice per unit time. Negative a implies melting or evaporation of ice (ablation) from the upper (or lower) surface of the glacier.

Let us first assume that q is a given function of x and h. Then, following Lighthill & Whitham (1955), we multiply both sides of equation (2) by the quantity

$$c = (\partial q/\partial h)_x, \tag{3}$$

and obtain

$$c\frac{\partial q}{\partial x} + \frac{\partial q}{\partial t} = ac. \tag{4}$$

This is the kinematic wave equation. If we draw in the (x,t) plane the paths $\mathrm{d}x/\mathrm{d}t = c$, namely the characteristics of the equation, we have simply

$$\mathrm{D}q/\mathrm{D}x = a, \tag{5}$$

where $\mathrm{D}/\mathrm{D}x$ means differentiation following a characteristic. Along these paths q changes at the rate a per unit distance. If a were zero, q would be constant on the paths—that is, there would be waves of constant q moving with velocity c. c is in general different from the mean velocity of the ice itself, which is given by $u = q/h$. In figure 1, which shows a typical $q:h$ relation, c is the slope of the tangent at the representative point P; u, on the other hand, is the slope of OP. It is essential to understand that by a 'wave' in this context is meant, not a moving wave-form, but simply a point, carrying with it some specified property, in this case constancy of q, which moves through the medium at a speed different from that of the medium itself. There may, or may not, be a recognizable disturbance at the point concerned.

Waves of q are not so readily visualized as waves of h, but the mathematical description is simpler in terms of q. When necessary we can always change from q to h by using the $q:h$ relation for the point x in question.

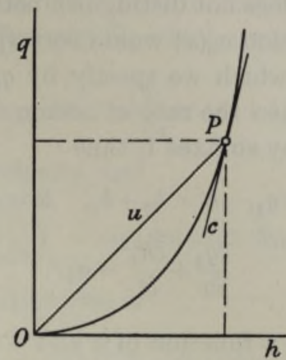

FIGURE 1. Typical relation between flow q and thickness h. The wave velocity is the slope of the tangent at P, while the ice velocity is the slope of the chord OP.

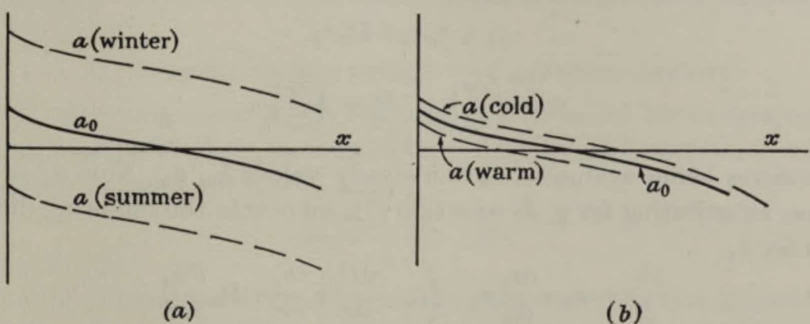

FIGURE 2. The rate of accumulation $a(x, t)$ is divided into a part $a_0(x)$ which is independent of t, and a part $a_1(x, t)$ which is approximately independent of x. The figure shows the x dependence of a and a_0 for (a) seasonal changes, and (b) long-term climatic changes.

It is convenient to refer to perturbations of a from a certain steady rate $a_0(x)$, so that

$$a(x,t) = a_0(x) + a_1(x,t), \quad \text{say.} \tag{6}$$

$a_0(x)$ will always refer to the rate of accumulation averaged over a complete year, or over a longer period—that is, it represents 'net accumulation'. $a_0(x)$ changes from positive to negative as we go down the glacier, being zero at the firn line (snow-line) (figure 2). $a_1(x,t)$ on the other hand, represents the departure from $a_0(x)$ produced either by seasonal variations during the year or by longer period variations of climate. If seasonal variations are under consideration (figure $2a$), $a_1(x,t)$ will be positive over the whole glacier during the winter, denoting snowfall over and above the average annual rate $a_0(x)$, and negative over the whole glacier during the summer. Thus a high value of a_1 can be due either to a high rate of precipitation in the winter or to high temperatures in the summer. Whereas $a_0(x)$ changes sign down the glacier, a_1 may be expected to be comparatively uniform with x.

If, on the other hand, long-term changes in climate are being studied, $a_1(x,t)$ will represent long-term changes in net accumulation, and, once again, a_1 may be expected to be comparatively uniform down the glacier. As figure $2b$ shows, a change in a will produce a change in the position of the firn line, defined now by $a = 0$. An increase in net accumulation may be due either to an increase in snow precipitation or to a decrease in mean annual temperature; the present theory covers both possibilities and does not distinguish between them.

A steady rate of accumulation $a_0(x)$ would correspond to a certain steady-state configuration of the glacier, which we specify by $q_0(x)$, $h_0(x)$. $\alpha_0(x)$. Then, from equation (2), $\mathrm{d}q_0/\mathrm{d}x = a_0$. When the rate of accumulation is not $a_0(x)$ but $a(x,t)$, we denote the perturbations by suffixes 1, thus

$$q = q_0+q_1, \quad h = h_0+h_1, \quad \alpha = \alpha_0+\alpha_1.$$

Then, from (2),

$$\frac{\partial q_1}{\partial x}+\frac{\partial h_1}{\partial t} = a_1. \tag{7}$$

Instead of assuming that q is a function of x and h only, we now make the more general assumption that q is a given function of x, h and α. Then we have, for small perturbations h_1 and α_1 at the point x,

$$q_1 = c_0 h_1 + D_0 \alpha_1, \tag{8}$$

where

$$c_0 = \left(\frac{\partial q}{\partial h}\right)_0, \quad D_0 = \left(\frac{\partial q}{\partial \alpha}\right)_0, \tag{9}$$

the derivatives being evaluated at the steady values h_0, α_0. Now $\alpha_1 = -\partial h_1/\partial x$, and hence, substituting for q_1 in equation (7), we obtain the following differential equation for h_1

$$\frac{\partial h_1}{\partial t} = a_1 - \frac{\mathrm{d}c_0}{\mathrm{d}x}h_1 - \left(c_0 - \frac{\mathrm{d}D_0}{\mathrm{d}x}\right)\frac{\partial h_1}{\partial x} + D_0\frac{\partial^2 h_1}{\partial x^2}. \tag{10}$$

The four terms on the right-hand side of equation (10) may be interpreted as follows. The term a_1 tells us that h_1 increases at the rate given by the perturbation in accumulation. The second term represents an exponential change of h_1, which we shall discuss in detail in a moment. The third term represents a kinematic wave of constant h_1 which moves with velocity $(c_0 - \mathrm{d}D_0/\mathrm{d}x)$ in the positive x direction. The fourth term represents an outward spreading of a disturbance h_1 in accordance with the diffusion equation, the diffusivity being D_0.

As a preliminary example of a function $q(x, h, \alpha)$ we may take the case, also considered by Weertman (1958), of a glacier whose forward motion consists entirely of slipping on its bed. The velocity is thus uniform with depth, but at the same time the glacier may extend or compress in the longitudinal direction. We suppose that, at a given x, the forward velocity, $u = q/h$, is given by Weertman's formula (1957)

$$u = (\tau/A)^m, \tag{11}$$

where τ is the shear stress on the bed, A is a measure of the roughness of the bed, and m is a constant equal to $\frac{1}{2}(n+1)$. n is the power in the creep law of ice,

$$\text{rate of strain} \propto (\text{stress})^n, \tag{12}$$

and is found experimentally by Glen (1955) to be in the neighbourhood of 3 or 4. Thus $m \simeq 2$.

Since (Nye 1952*a*)

$$\tau = \rho g h \sin\alpha,$$

we have for the function q,

$$q = uh = (\rho g/A)^m h^{m+1} \sin^m \alpha \quad \text{at given } x. \tag{13}$$

The wave velocity c_0 and diffusivity D_0 which occur in equation (10) are thus in this case

$$c_0 = \left(\frac{\partial q}{\partial h}\right)_0 = (m+1)\frac{q_0}{h_0} = (m+1)\,u_0 \simeq 3u_0, \tag{14}$$

where u_0 is the steady-state velocity, and

$$D_0 = \left(\frac{\partial q}{\partial \alpha}\right)_0 = mq_0 \cot\alpha_0 \simeq 2q_0 \cot\alpha_0. \tag{15}$$

The coefficient of $-h_1$ in equation (10) is

$$\frac{\mathrm{d}c_0}{\mathrm{d}x} = (m+1)\frac{\mathrm{d}u_0}{\mathrm{d}x} = (m+1)\,r_0 \simeq 3r_0, \tag{16}$$

where $r_0 = \mathrm{d}u_0/\mathrm{d}x$ is the rate of longitudinal strain in the glacier in the steady state.

3. Example of the effect of a climatic change

Without restricting ourselves to the pure sliding model let us now consider a region of a glacier in which $\mathrm{d}c_0/\mathrm{d}x$ is uniform, and let us look for solutions of equation (10) for which $\partial h_1/\partial x = 0$. Then we have

$$\mathrm{d}h_1/\mathrm{d}t = a_1 - \gamma_0 h_1, \tag{17}$$

where $\gamma_0 = \mathrm{d}c_0/\mathrm{d}x$.

Suppose the glacier is initially in the steady state under the rate of accumulation $a_0(x)$, and that the rate of accumulation is suddenly increased by a constant amount a_1. Then the solution of equation (17) gives the following exponential relation between h_1 and t

$$h_1 = (a_1/\gamma_0)\,(1 - \mathrm{e}^{-\gamma_0 t}). \tag{18}$$

The case $\gamma_0 > 0$ is straightforward. For the pure sliding model, by equation (16), $\gamma_0 = (m+1)\,r_0$, and thus $\gamma_0 > 0$ implies uniform longitudinal extension of the glacier. We see that the thickness initially increases at the rate a_1, and then approaches a new steady state, with thickness $h_0 + a_1/\gamma_0$, exponentially with time constant γ_0^{-1}. For a region of a glacier in which $r_0 = 0{\cdot}1\,\mathrm{yr}^{-1}$, a typical value, the time constant would be about 3 years. The case $\gamma_0 < 0$, on the other hand, gives an interesting and unexpected result. In the pure sliding model $\gamma_0 < 0$ implies uniform longitudinal compression. The thickness initially increases at the rate a_1, but no new steady state is approached and the thickness continues to increase exponentially without limit.

This unstable behaviour when $\gamma_0 < 0$ is also manifest when $a_1 = 0$. For if the accumulation is held steady at a_0 we have

$$\mathrm{d}h_1/\mathrm{d}t = -\gamma_0 h_1. \tag{19}$$

This means that if, for any reason, the glacier is disturbed from its steady-state thickness, the disturbance h_1 spontaneously increases in an unstable way. A simple argument to show this inherent instability in a region of uniform longitudinal compression is given in appendix A.

Now we do not expect to find in nature a system in an unstable steady state. In order to pursue the matter we must consider, not just an isolated region of a glacier, but the whole glacier. An idealized case is shown in figure 3. The upper part of the glacier down to the point F is supposed to be uniformly extending, while the lower part is uniformly compressing. (Since $\mathrm{d}q_0/\mathrm{d}x = a_0$, q_0 is a maximum at the firn line, where $a_0 = 0$; a glacier thus tends to flow fastest near the firn line, so that extension in the accumulation area and compression in the ablation area is typical behaviour.) Initially the glacier is in a steady state under a rate of accumulation $a_0(x)$; the thickness is $h_0(x)$, and hence $h_1 = 0$. Suppose the rate of accumulation now changes suddenly to $a_0(x) + a_1$. The glacier will begin to thicken everywhere at the rate a_1, but after a short time the lower part, which is responding unstably, will have thickened more than the upper part. There must accordingly be a region around F where $\partial h_1/\partial x \neq 0$, and where our solutions break down. This will be the beginning of a kinematic wave which will move down the glacier—and which will, as we shall see, restore a steady state.

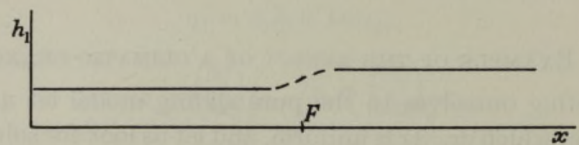

Figure 3. An idealized glacier which is uniformly extending along its length down to the point F, and uniformly compressing below F. When the rate of accumulation suddenly increases, the resulting increase in thickness occurs stably above F and unstably below F.

If the accumulation suddenly decreases (say, by increased melting) so that a_1 is negative, the surface falls everywhere, but the lower part begins to fall unstably. The kinematic wave is then reversed in sense, being thicker on the upstream side.

It is clear from these general considerations that we must study the glacier as a whole if we are to understand its response to climatic changes, and, in particular, we must study the passage of kinematic waves down its length. We are fortified in this conclusion by the fact that waves travelling with speeds greater than that of the ice have been observed, or their existence has been inferred, in a number of glaciers (Sharp 1954): notably in the Hintereis Glacier (Blümcke & S. Finsterwalder 1905; Deeley & Parr 1914) and in the glaciers of the Yakutat Bay area, Alaska (Tarr & Martin 1914; Miller 1958) in the early 1900's; in the Mer de Glace in 1891–95 (Vallot 1900; Lliboutry 1958); and at the present time both in the Mer de Glace and in the neighbouring Glacier des Bossons (R. Finsterwalder 1959), and in Nisqually Glacier, Washington, U.S.A. (Johnson 1953, 1957; Harrison 1956, 1957).† Particularly large waves have been thought to be responsible for such spectacular advances as that of the Black Rapids Glacier, Alaska, in 1936–7, which moved forward 3 miles within 5 months (Hance 1937; Geist & Péwé 1954), and of the Muldrow Glacier, Alaska, in 1956–7, which advanced nearly 4 miles in less than a year (Péwé 1957).

† Dr Mark F. Meier tells me that there are indications that a new wave was forming in 1959 just below the firn limit of Nisqually Glacier.

The existence of travelling waves on a glacier was first deduced theoretically by S. Finsterwalder (1907)†. Fifty years later the present writer pointed out (Nye 1958) that such waves were a consequence of the general kinematic wave theory of Lighthill & Whitham, while at the same time Weertman (1958), unaware of Finsterwalder's work and of the general theory, independently inferred the existence of travelling waves on theoretical grounds and investigated a number of specific models.

From the standpoint of kinematic wave theory there is nothing new in the present analysis; it is merely the application to glaciers of certain aspects of Lighthill & Whitham's general theory.

4. Further kinematic wave theory

For deriving the apparent instability of the compression region of a glacier it is most direct to work in terms of h_1, as we have done; but for further developments it is decidedly more advantageous to work in terms of q_1.

Let q be a function of x, h and α, as before, and multiply the continuity equation (2) by

$$c = \left(\frac{\partial q}{\partial h}\right)_{\alpha,x}. \tag{20}$$

Then

$$c\frac{\partial q}{\partial x} + \left(\frac{\partial q}{\partial h}\right)_{\alpha,x}\frac{\partial h}{\partial t} = ac,$$

or

$$c\frac{\partial q}{\partial x} + \frac{\partial q}{\partial t} - \left(\frac{\partial q}{\partial \alpha}\right)_{h,x}\frac{\partial \alpha}{\partial t} = ac.$$

Now from equation (1),

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial^2 h}{\partial x\,\partial t}.$$

Hence, substituting this value, we have

$$c\frac{\partial q}{\partial x} + \frac{\partial q}{\partial t} + D\frac{\partial^2 h}{\partial x\,\partial t} = ac, \tag{21}$$

where

$$D = \left(\frac{\partial q}{\partial \alpha}\right)_{h,x}. \tag{22}$$

Note that the expressions $\partial q/\partial h$ for c and $\partial q/\partial \alpha$ for D agree with the definitions of c_0 and D_0 given in equation (9).

By substituting for $\partial h/\partial t$ in (21) from the equation of continuity (2) we obtain the following differential equation for q

$$\frac{\partial q}{\partial x} + \frac{1}{c}\frac{\partial q}{\partial t} - \frac{D}{c}\left(\frac{\partial^2 q}{\partial x^2} - \frac{\partial a}{\partial x}\right) = a. \tag{23}$$

† *Erratum.* In (Nye 1958) I said that Finsterwalder did not envisage the formation of 'shock waves' from comparatively gradual increases in feed of the glacier, which is one of the consequences of the Lighthill–Whitham theory. This is incorrect; further study of his paper shows that he was, in fact, well aware of the possibility.

We may now write, as before, $q = q_0+q_1$ and $a = a_0+a_1$, where q_0 and a_0 are the steady-state values, but it is important to notice that the perturbation is not now restricted to be small. Since $\mathrm{d}q_0/\mathrm{d}x = a_0$, we have

$$\frac{\partial q_1}{\partial x}+\frac{1}{c}\frac{\partial q_1}{\partial t}-\frac{D}{c}\left(\frac{\partial^2 q_1}{\partial x^2}-\frac{\partial a_1}{\partial x}\right) = a_1. \tag{24}$$

This is the equation for the finite perturbation $q_1(x,t)$ produced by a change $a_1(x,t)$ in rate of accumulation. For small perturbations it leads to equation (10).

D, as we have seen, is a coefficient of diffusion. We shall begin by neglecting diffusion and then later, having studied the case $D = 0$, we can discuss how diffusion will modify the results. Without diffusion equation (24) becomes

$$\frac{\mathrm{D}q_1}{\mathrm{D}x} = \frac{\partial q_1}{\partial x}+\frac{1}{c}\frac{\partial q_1}{\partial t} = a_1(x,t). \tag{25}$$

so that, along a wave path, q_1 changes at the rate a_1 per unit distance.

5. Effect of a sudden uniform climatic change on an ideal glacier

We are now in a position to solve explicitly the problem posed by figure 3. We neglect diffusion for the moment, and consider the case where a_1 is a constant. For simplicity we suppose a_1 to be such that q_1 is small. This means that the kinematic wave velocity at each point is essentially fixed by the steady-state configuration, and we may write it as $c_0(x)$; it is only infinitesimally changed by the perturbation.

Let the glacier start at $x = 0$ and finish at $x = 1$, and let the steady state be such that

$$c_0(x) = \begin{cases} \epsilon x & (0 \leqslant x \leqslant \tfrac{1}{2}), \\ \epsilon(1-x) & (\tfrac{1}{2} \leqslant x \leqslant 1), \end{cases} \tag{26}$$

where ϵ is a positive constant. When the forward motion of the ice is entirely by slipping on the bed, this assumption implies, by equation (16), that the glacier above $x = \frac{1}{2}$ is extending uniformly at the rate $\epsilon/(m+1)$, and below $x = \frac{1}{2}$ it is compressing at an equal rate.

We begin by constructing the paths of the waves, the characteristics, in the (x,t) plane. On a given path $\mathrm{d}x/\mathrm{d}t = c_0(x)$; hence the equation of the path is

$$t = \int_{x_0}^{x} \frac{\mathrm{d}x}{c_0(x)},$$

where x_0 is the value of x when $t = 0$. x_0 labels the path. For $0 \leqslant x \leqslant \frac{1}{2}$, region I in figure 4, we put $c_0 = \epsilon x$ and integrate to obtain

$$x = x_0\,\mathrm{e}^{\epsilon t}. \tag{27}$$

These paths are plotted in figure 4. For $x > \frac{1}{2}$ we have to distinguish between paths which come in across $x = \frac{1}{2}$ (region II) and those which originate on the $t = 0$ axis (region III). For region II we have

$$t = \int_{x_0}^{\frac{1}{2}} \frac{\mathrm{d}x}{\epsilon x}+\int_{\frac{1}{2}}^{x} \frac{\mathrm{d}x}{\epsilon(1-x)},$$

or

$$4x_0(1-x) = \mathrm{e}^{-\epsilon t}. \tag{28}$$

For region III we have
$$t = \int_{x_0}^{x} \frac{dx}{\epsilon(1-x)},$$
or
$$1-x = (1-x_0)\,e^{-\epsilon t}. \tag{29}$$

We now have to integrate equation (25) along these paths. Initially the glacier is unperturbed, so that $q_1 = 0$ at $t = 0$. This means that $q_1 = 0$ at the beginning of each path, that is, at $x = x_0$. Therefore, since a_1 is constant, equation (25) integrates to
$$q_1 = a_1(x - x_0). \tag{30}$$
We wish to express this result in terms of h_1, and, since q_1 is small and $D_0 = 0$, we have, from equation (8)
$$h_1 = q_1/c_0.$$
Hence
$$h_1 = a_1(x-x_0)/c_0. \tag{31}$$

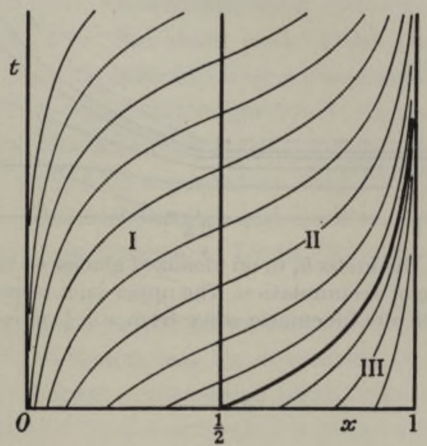

FIGURE 4. Wave paths in the (x, t) plane for an idealized glacier.

It remains to substitute the expressions for x_0 and c_0 in terms of x and t. Thus, for region I, from equations (26) and (27),
$$h_1 = (a_1/\epsilon)(1 - e^{-\epsilon t}); \tag{32}$$
for region II, from equations (26) and (28),
$$h_1 = \frac{a_1}{\epsilon}\left\{\frac{x}{1-x} - \frac{1}{4(1-x)^2}\,e^{-\epsilon t}\right\}; \tag{33}$$
for region III, from equations (26) and (29),
$$h_1 = (a_1/\epsilon)(e^{\epsilon t} - 1). \tag{34}$$

It will be observed that the solutions for regions I and III are simply the stable and unstable solutions already obtained in § 3; they correspond to equation (18) with $\gamma_0 = \pm\epsilon$, respectively. The complete solution is shown in figure 5*a*.† Initially

† The solutions in figures 5*a* and *b* were first obtained by Dr W. B. Kamb, not using the kinematic wave approach but working from equation (10), with $D_0 = 0$, and using the Laplace transform method.

h_1 increases at the rate a_1 everywhere. Above $x = \frac{1}{2}$ the glacier is seen to behave uniformly at all times, and h_1 approaches the value a_1/ϵ exponentially. Any point below $x = \frac{1}{2}$, on the other hand, responds unstably, by equation (34), until the

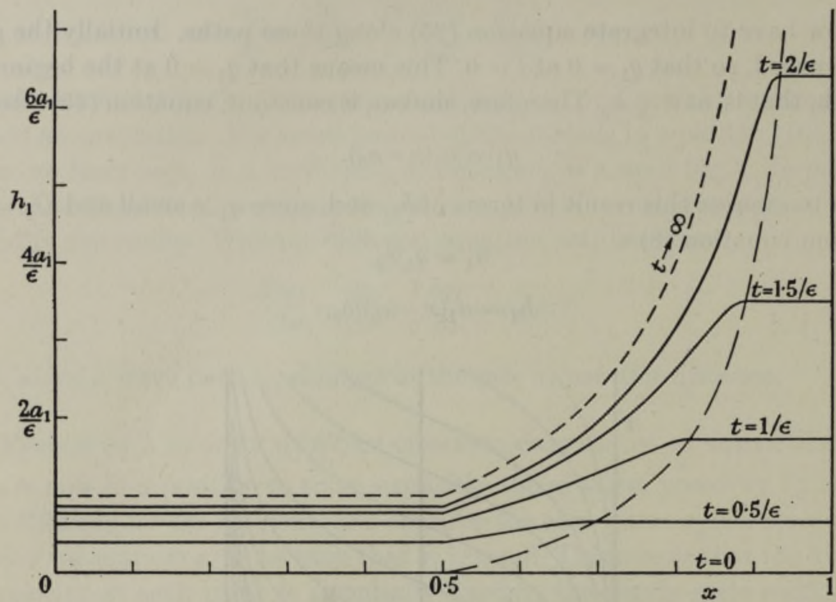

FIGURE 5*a*. The increase in thickness h_1 in an idealized glacier at various times after a sudden uniform increase in rate of accumulation. The upper part responds stably; the lower part responds unstably, until the kinematic wave from $x = \frac{1}{2}$ arrives.

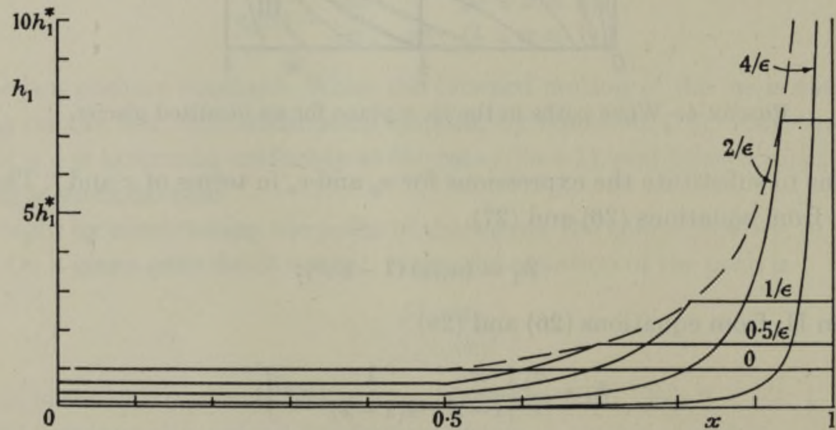

FIGURE 5*b*. Showing the history, in an idealized glacier, of an initially uniform perturbation in thickness h_1. The temporary instability of the lower half is relieved by the arrival of the kinematic wave generated at $x = \frac{1}{2}$.

kinematic wave generated initially at $x = \frac{1}{2}$ reaches it. The perturbation then obeys equation (33). After a long time the glacier again reaches a steady state, namely

$$\left.\begin{aligned} h_1 &= a_1/\epsilon \qquad (x < \tfrac{1}{2}), \\ h_1 &= \frac{a_1}{\epsilon}\frac{x}{1-x} \quad (x > \tfrac{1}{2}). \end{aligned}\right\} \tag{35}$$

It is interesting to interpret the steady state in terms of the kinematic waves. It is clear from figure 4 that, after a long time, only those waves with vanishingly small x_0 remain upon the glacier. The rest are accumulated at $x = 1$. Thus the steady state may be found by setting $x_0 = 0$, and if we do this in equation (31) we immediately obtain equations (35).

It will be seen that, in the solution, $h_1 \to \infty$ at $x = 1$; this is inadmissible, if only because we have restricted h_1 to be small. It is clear from equation (31) that the divergence occurs because we arbitrarily put $c_0 = 0$ at the end of the glacier. However, as we shall show in § 6, c_0 is not in practice zero at the snout of a glacier, and it would therefore be more realistic if we cut off the solution before reaching $x = 1$. We may do this without violating any conditions, for the steady-state length of the glacier is fixed by the distribution of $a_0(x)$, and not specifically by the assumed distribution of $c_0(x)$. We shall return to this point in § 6.

When a_1 is negative, so that the accumulation suddenly decreases, the sign of h_1 is everywhere reversed. Thus the lower part of the glacier falls more rapidly than the upper part, and stability is restored by a wave originating at $x = \frac{1}{2}$.†

If a_1 is such that the resulting q_1 is no longer small, we proceed as follows. Equation (25) holds for finite q_1. The difference is that since c is a function of q the paths of the waves can now no longer be regarded as fixed independently of the values of q carried by the waves. It is necessary to construct the wave paths step by step, starting at $t = 0$, and adjusting their slope $1/c$ to the value of q that they are currently carrying.

With the solution of figure 5*a* before us we can now make some generalizations applicable to cases where diffusion can be neglected. First, we learn that the instability of a region of uniform compressive strain-rate is only temporary, and that stability may be restored by the arrival of a kinematic wave from higher up the glacier. The general relation which applies is equation (25), which in integrated form is

$$q_1 = \int_{x_0}^{x} a_1(x,t)\,\mathrm{d}x \quad \text{along a wave path,} \tag{36}$$

where x_0 denotes the point at which the wave originates. For a small perturbation

$$h_1 = \frac{1}{c_0}\int_{x_0}^{x} a_1(x,t)\,\mathrm{d}x \quad \text{along a wave path,} \tag{37}$$

† It is interesting to calculate the corresponding solution for the case where $a_1 = 0$, and where h_1 is initially a constant, say h_1^*. This corresponds to keeping the climate constant and artificially adding or subtracting a uniform thin layer of ice on the top surface of the glacier. By taking h_1^* negative we may approximate the effect of a single abnormally hot summer, which is assumed to leave the glacier surface lower than its normal level by a uniform amount. h_1^* positive would correspond to a single abnormally snowy winter. The question is, how does the surface return to its normal level after being subjected to a uniform perturbation such as this? The solution is given in appendix B and figure 5*b*. It will be seen from the figure that the upper part of the glacier returns stably to its original thickness, but the lower part grows unstably, by equation (19). A kinematic wave is then generated at $x = \frac{1}{2}$, which moves down the glacier and restores stability. Eventually the whole glacier returns to the state $h_1 = 0$. However, the lowest parts can grow very considerably before the wave reaches them—it is rather like the cracking of a whip. In view of this inherent instability it is hardly surprising that glacier snouts often seem to behave very erratically.

and if a_1 is a constant, this is simply equation (31). We wish to emphasize that this equation applies regardless of the exact shape of the wave paths. It follows that, for a given sudden climatic change specified by $a_1(x)$, the two factors which will produce a large response h_1 at the point x are (i) long distance of travel of the particular wave which is at x at the moment considered, and (ii) small wave velocity at the point x. A particularly favourable place for a large change of thickness would therefore be the stagnant snout of a long glacier, for this would provide the small wave velocity and long travel distance that we need. A thickening at the snout end would, of course, be accompanied by an advance of the glacier down its valley, and this aspect is considered in detail in the next section.

Trim lines on the sides of glacier valleys, representing profiles of maximum advance, are commonly observed. They are usually high above the present glacier surface near the snout, but they run closer to the surface in the higher regions of the glacier. This is in general agreement with what would be deduced from the steady-state solution shown in figure $5a$.† As the solution clearly shows, a small change in rate of accumulation can produce a very large thickening of the lowest parts of the glacier—and this seems to be the fundamental reason why glaciers are such sensitive indicators of climatic change.

6. Advance and retreat of the snout

It is worth examining the extreme end of the glacier a little more closely, because our considerations have not yet enabled us to say how far the glacier advances down its valley in consequence of a sudden increase $a_1(x)$ in rate of accumulation.

Provided we are only interested in the final steady state, the answer to this question can be given very simply, and independently of kinematic wave theory, as follows. The steady-state length of the glacier L_0, is given by

$$\int_0^{L_0} a_0 \, \mathrm{d}x = 0, \tag{38}$$

since all the material added in the upper part is lost in the lower part. q_0 is zero at the end of the glacier, but, when the increased rate of accumulation $a_0(x)+a_1(x)$ has prevailed for a long time, there will be an additional flow $\int_0^{L_0} a_1 \mathrm{d}x$ at $x = L_0$ (we are assuming the upper end of the glacier remains stationary). In consequence, the snout will have advanced a distance L_1, measured parallel to its surface (figure 6). If the rate of ablation at the snout is $-A_0$, it follows that, for small a_1,

$$\int_0^{L_0} a_1 \mathrm{d}x = -A_0 L_1, \tag{39}$$

and, if a_1 is uniform,

$$\frac{L_1}{L_0} = -\frac{a_1}{A_0}. \tag{40}$$

Thus, if the rate of net ablation at the snout is 5 m/yr and the rate of net accumulation increases uniformly by 0·5 m/yr, the glacier eventually increases its total length by 10%.

† I am indebted to Professor R. P. Sharp for this remark.

To investigate the history of such an advance we must return to the kinematic wave theory. The first point to note here is that, at the snout of a glacier, $\partial h/\partial x$ is not small, as assumed up to now, and our basic equations must therefore be re-examined. We proceed as follows. x is to be measured now specifically along the *upper* surface of the glacier; h is defined as the thickness measured perpendicularly to the upper surface; q is the volume per unit time passing through a section drawn perpendicular to the upper surface; and a is the rate of accumulation at the upper surface. With these definitions equation (2) continues to hold. u is defined as q/h, and is the mean velocity component parallel to the surface. We shall assume that, to a sufficient approximation, q is a function of x and h (thereby ignoring diffusion). The kinematic wave equation (25) for a finite perturbation q_1 then follows.

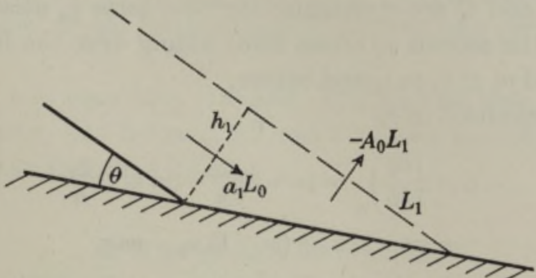

FIGURE 6. The advance of the end of a glacier.

At the end of the glacier $q_0 = 0$; h_0 is also zero at the end, but u_0 is not zero. (This is a fact of observation, and it implies that the relation between q and h at the snout is not, for example, a simple power dependence.) Now, since at the snout $q_0 = h_0 = 0$, it follows that, in the limit, $(\mathrm{d}q/\mathrm{d}h)_0 = q_0/h_0$; that is $c_0 = u_0$. Thus c_0 is not zero at the snout, and there is therefore no divergence of h_1.

Now the theory we have developed is valid for a finite q_1, and hence it applies at the extreme end of the glacier, even though $q_0 = 0$ there. The value of $q_1(t)$ at $x = L_0$ is then properly calculated as $q_1 = \int_{x_0}^{L_0} a_1(x)\,\mathrm{d}x$. But q_1 at $x = L_0 + L_1$ is zero, and hence, integrating equation (25) for the small increment of path L_1, we have

$$L_1 = -\frac{1}{A_0}\int_{x_0}^{L_0} a_1\,\mathrm{d}x. \tag{41}$$

This gives the increase in length L_1 at the time when the wave which started at $x = x_0$ reaches the end of the glacier. For the steady state, all waves come from $x_0 = 0$ and we regain equation (39).

L_1 may also be expressed as $h_1 \cot\theta$, where h_1 is evaluated at $x = L_0$, and θ is the angle at the snout (figure 6). Thus the changes in length of the glacier are simply proportional to the changes of h_1 at the original position of the snout. (This gives an alternative way of deriving equation (41), for we have

$$L_1 = h_1 \cot\theta = \frac{q_1}{c_0}\cot\theta = \frac{q_1}{u_0}\cot\theta = \frac{\cot\theta}{u_0}\int_{x_0}^{L_0} a_1\,\mathrm{d}x = -\frac{1}{A_0}\int_{x_0}^{L_0} a_1\,\mathrm{d}x.)$$

7. Values of c and D for various flow models

We return now to the main part of the glacier, and shall specifically exclude the extreme end from our considerations. In § 2 we looked at a particular example of a functional relation $q(x, h, \alpha)$ which corresponded to a glacier whose forward motion consisted entirely of sliding on its bed. We now wish to calculate the values of c and D for the more general situation where differential shear motion within the ice of one layer over another makes a significant contribution to the forward velocity. If we assume that the shear motion is given by the equation for laminar flow (Nye 1952*b*), it follows that, at a given x,

$$q = q_d + q_b = Fh^{n+2} \sin^n \alpha + Gh^{m+1} \sin^m \alpha. \tag{42}$$

In this equation F and G are constants; the first term q_d arises from differential shear motion, and the second q_b arises from sliding over the bed (equation (13)). n is about 3 or 4 and $m \simeq 2$, as noted before.

Then, from the definition of c,

$$c = \left(\frac{\partial q}{\partial h}\right)_\alpha = (n+2)\frac{q_d}{h} + (m+1)\frac{q_b}{h}$$

$$= (n+2)\, u_d + (m+1)\, u_b, \quad \text{say,} \tag{43}$$

where u_d is the mean velocity due to shear motion, and u_b is the velocity of sliding. The mean forward velocity of the ice is $u = u_d + u_b$. Thus c always lies between $(m+1)\,u$ and $(n+2)\,u$, that is between about $3u$ and $6u$, depending upon the relative proportions of the two forms of motion. This is the result derived in an earlier paper (Nye 1958).

Since the surface velocity u_s, say, is more readily observable than u, it is useful to express c in terms of u_s and u_b. It is readily shown (Nye 1952*b*) that

$$(n+2)\, u_d = (n+1)\,(u_s - u_b).$$

Hence, substituting for u_d in (43), we find

$$c = (n+1)\,(u_s - u_b) + (m+1)\, u_b, \tag{44}$$

in agreement with Weertman (1958). Thus c always lies between $(m+1)\, u_s$ and $(n+1)\, u_s$, that is between about $3u_s$ and $5u_s$. This theoretical result is in general agreement with the observed velocities of waves of increased thickness. (However, a wave of decreased *velocity* apparently passed down the Hintereis Glacier between 1900 and 1903 at a speed of 20 to 150 times u_s (Blumcke & S. Finsterwalder 1905; Deeley & Parr 1914).)

We turn now to the diffusivity D, defined by $D = (\partial q/\partial \alpha)_h$. With the model defined by equation (42) we find

$$D = (nu_d + mu_b)\, h \cot \alpha. \tag{45}$$

Thus D lies between m and n, that is between about 2 and 4, times $uh \cot \alpha$.

Alternatively, in terms of u_s and u_b,

$$D = \left\{\frac{n(n+1)}{n+2}\,(u_s - u_b) + mu_b\right\} h \cot \alpha, \tag{46}$$

so that D lies between $n(n+1)/(n+2)$ and m, that is between about 2 and 3·3, times $u_s h \cot\alpha$.

When a longitudinal strain-rate r is present in the glacier, the flow will no longer be given by equation (42), because the non-linearity of the creep law (12) causes a strong interaction between the longitudinal strain-rate and the differential shear motion (Nye 1957). The effect of the longitudinal strain-rate is to reduce the effective viscosity at all depths, and thus to increase the shear motion. The direction of the effect on the wave velocity is most easily seen by considering the extreme case when r is so large that, so far as slow shear motions are concerned, the ice behaves as if it had constant viscosity. Then $n = 1$ in equations (43) and (44), and m, being equal to $\frac{1}{2}(n+1)$, may fall from 2 to 1. The limits for c in terms of u are then $2u$ to $6u$; in terms of u_s they are $2u_s$ to $5u_s$. The lower limit, corresponding to a very high rate of longitudinal strain, would be exceptional, and we may perhaps take $c = 4u_s$ as a mean figure.

By putting $n = 1$ in equations (45) and (46), we find that D now lies between 1 and 4 times $uh\cot\alpha$, and between 0·7 and 3·3 times $u_s h \cot\alpha$. As a mean figure we may perhaps take $D = 3u_s h \cot\alpha$.

8. The effect of diffusion

To study the effect of diffusion we must return to equation (21) or (24). From equation (21) it is seen that diffusion is negligible if the product $(D/c)\,\partial^2 h/\partial x\,\partial t$ is sufficiently small. The results of the last section show that D/c is roughly $\frac{3}{4}h\cot\alpha$, and thus diffusion will be more important for small slope angles.

Equation (24) for a finite perturbation may be written as

$$\frac{\mathrm{D}q_1}{\mathrm{D}t} = ca_1 - D\frac{\partial a_1}{\partial x} + D\frac{\partial^2 q_1}{\partial x^2}. \tag{47}$$

We shall treat the terms in D as correction terms, since this seems to be a reasonable procedure in many practical cases. Without diffusion q_1 increases at the rate ca_1 per unit time along a characteristic. Equation (47) shows that the rate of increase of q_1 is changed slightly by diffusion, the change depending on the values of q_1 on neighbouring characteristics ($\partial^2 q_1/\partial x^2$), and on the variation, if any, of a_1 with x. If q_1 is a local maximum, so that $\partial^2 q_1/\partial x^2 < 0$, diffusion will cause q_1 to increase more slowly. Conversely, if q_1 is a local minimum, q_1 will increase more rapidly. In this way diffusion smooths out the values of q_1. In the example of § 5 diffusion will round off the sharp corners seen in the curves of h_1 in figures 5*a* and *b*.

If a_1 is a given function of x and t, equation (47) may be integrated numerically along the characteristics. Suppose that $q_1(x)$ is given at $t = 0$ for the range of x covered by the glacier, say 0 to L. Then, q being known, the slope of the characteristics at $t = 0$ is known. The right-hand side of (47) is evaluated for $t = 0$, and hence the increment Δq_1 along the characteristics after a time Δt is calculated. With the new values of q_1 so calculated the slopes of the characteristics at time Δt are found, and so the solution is extended step by step. This gives $q_1(x, t)$ in the region of the (x, t) plane which lies to the right of the characteristic passing through the origin.

If q_1 is small, h_1 may be found as follows. From equation (8), with $\alpha_1 = -\partial h_1/\partial x$, we have

$$h_1 = \frac{q_1}{c_0}+\frac{D_0}{c_0}\frac{\partial h_1}{\partial x}. \tag{48}$$

Therefore, as a first approximation h_1 is calculated as q_1/c_0. We then calculate $\partial h_1/\partial x$, and substitute into equation (48) to find a second approximation for h_1, and so on.

To calculate $q_1(x,t)$ for the remainder of the (x,t) plane we shall assume that $q_1(t)$ is known at $x = 0$. A suitable numerical procedure must then be devised for use with the equation

$$\frac{\mathrm{D}q_1}{\mathrm{D}x} = a_1-\frac{D}{c}\frac{\partial^2 h_1}{\partial x\,\partial t}. \tag{49}$$

If there is a sudden, sustained, change in rate of accumulation, a_1 is a function of x only. Equation (47) then simplifies to

$$\frac{\mathrm{D}q^*}{\mathrm{D}t} = D\frac{\partial^2 q^*}{\partial x^2},$$

where $q^* = q_1-\int a_1\,\mathrm{d}x$. In this case, therefore, the numerical procedure would be a little easier. The steady state which is eventually reached is readily calculated in terms of q_1. From equation (7) we have simply

$$\frac{\mathrm{d}q_1}{\mathrm{d}x} = a_1(x); \quad q_1 = \int_0^x a_1(x)\,\mathrm{d}x+q_1^0, \tag{50}$$

where q_1^0 is the final value of q_1 at $x = 0$, a quantity which must be specified before the solution is completely determined. To obtain the result in terms of h_1 is much more complicated. q^* is seen to be the difference between the flow at time t and the final steady flow.

9. Response to periodic changes

In this section we consider the effect of a simple harmonic variation of a_1, putting

$$a_1 = a'\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{51}$$

where a' is a constant. This equation can be interpreted in two ways. (1) The changes in a_1 can be those due to the alternations of winter and summer, in which case the angular frequency ω corresponds to a period of 1 yr. a is then the rate at which ice is actually being added or subtracted at a given instant, and a_0 is the average annual rate (net accumulation). (2) The changes in a_1 can be those due to a long-term climatic cycle, so that ω corresponds to a period measured in tens or in thousands of years. In this case a refers to net accumulation and a_1 represents the change in this rate from the long-term mean.

In general, equation (51) represents one Fourier component of the complicated variation of a with t that actually occurs. Since the equations are linear in h_1 and q_1, we can study the response of the glacier or ice-sheet to each Fourier component separately, and the final result is simply the sum of these responses.

(a) *The upper regions*

Consider first a region where $\mathrm{d}c_0/\mathrm{d}x$ is a positive constant ϵ, as, for example, in the upper part of the model glacier of § 5. Now if the rates of extension of the upper and lower surfaces of the glacier are equal (a reasonable approximation), so that $\mathrm{d}u_s/\mathrm{d}x = \mathrm{d}u_b/\mathrm{d}x$, equation (44) shows that $\epsilon \simeq (m+1)\,r_0$, where r_0 is the rate of extension in the steady state. Thus the region of constant $\mathrm{d}c_0/\mathrm{d}x$ that we are considering is approximately a region of uniform rate of longitudinal extension.

As a first approximation we neglect diffusion. Then the example of § 5 shows that, if our region constitutes the upper part of the glacier, no disturbance is propagated back into it from downstream, and it may therefore be treated in isolation. We need only consider solutions in which $\partial h_1/\partial x$ and $\partial^2 h_1/\partial x^2$ are zero, and equation (10) then becomes

$$\mathrm{d}h_1/\mathrm{d}t = a'\,\mathrm{e}^{\mathrm{i}\omega t} - \epsilon h_1 \quad (h_1 \text{ small}).$$

The solution is

$$h_1 = h_1^*\,\mathrm{e}^{-\epsilon t} + \frac{a'}{\epsilon + \mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{52}$$

where h_1^* is a constant. After a time, long compared with ϵ^{-1}, the transient disappears, and we are left with the solution

$$h_1 = \frac{a'}{\epsilon + \mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{53}$$

or

$$h_1 = \frac{a'}{\sqrt{(\epsilon^2+\omega^2)}}\,\mathrm{e}^{\mathrm{i}(\omega t - \phi)}, \quad \tan\phi = \frac{\omega}{\epsilon}, \quad 0 < \phi < \tfrac{1}{2}\pi. \tag{54}$$

Thus the changes in thickness lag behind the changes in rate of accumulation by the phase angle ϕ. The system is exactly analogous to an electrical circuit containing a condenser and a resistance in series; a_1 is proportional to the applied voltage, and h_1 corresponds to the resulting charge on the condenser.

(i) *Antarctica and Greenland*

The response of a large ice-sheet, such as that covering Antarctica or Greenland, to seasonal changes in precipitation is readily seen from equation (54).

The value of ϵ may be estimated as follows. We have $q_0 = u_0 h_0$, where u_0 is the average velocity in the steady state. Hence, differentiating with respect to x and putting $\mathrm{d}q_0/\mathrm{d}x = a_0$, we obtain

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = \frac{a_0}{h_0} - \frac{u_0}{h_0}\frac{\mathrm{d}h_0}{\mathrm{d}x}.$$

But $\epsilon \simeq (m+1)\,\mathrm{d}u_0/\mathrm{d}x$. Hence

$$\epsilon \simeq (m+1)\left(\frac{a_0}{h_0} - \frac{u_0}{h_0}\frac{\mathrm{d}h_0}{\mathrm{d}x}\right). \tag{55}$$

For central Antarctica we put

$$m = 2, \quad a_0 = 0{\cdot}10\,\mathrm{m/yr}, \quad h_0 = 3000\,\mathrm{m}, \quad u_0 = 20\,\mathrm{m/yr}, \quad \text{and} \quad -\mathrm{d}h_0/\mathrm{d}x = 3\times10^{-3},$$

and find $\epsilon \simeq 1{\cdot}6\times10^{-4}\,\mathrm{yr}^{-1}$. For annual changes $\omega = 2\pi\,\mathrm{yr}^{-1}$. Hence $\omega \gg \epsilon$, and equation (54) becomes essentially

$$h_1 = \frac{a_1}{\omega}\,\mathrm{e}^{\mathrm{i}(\omega t - \frac{1}{2}\pi)}. \tag{56}$$

h_1 lags behind a_1 by 90° in phase. This is exactly what one would expect, for h_1 will increase fastest when a_1 is a maximum (during the winter). The response time of the ice-sheet is $\epsilon^{-1} \simeq 5000$ yr, and this is so long compared with 1 yr that the seasonal changes of a_1 make no sensible impression upon the flow of the sheet.

The response will be the same for a climatic cycle whose period $\tau = 2\pi/\omega$ is short compared with $2\pi/\epsilon$, that is, much less than 30 000 yr: h_1 will lag behind a_1 by nearly 90° in phase. It is important to notice that the time of maximum rate of accumulation will *not* be the time of maximum thickness of the ice sheet; if the cycle had a period of 1000 yr the greatest thickness would be reached about 250 yr after the time of greatest rate of accumulation.

On the other hand, for a climatic cycle whose period is about 30 000 yr the phase lag is 45°, and for periods long compared with 30 000 yr the phase lag approaches zero. In this last case it is more useful to consider the time lag Δt, rather than the phase lag ϕ. We have

$$\Delta t = \frac{\phi}{2\pi}\tau = \frac{1}{\omega}\tan^{-1}\frac{\omega}{\epsilon} \simeq \frac{1}{\epsilon} \quad (\omega \ll \epsilon).$$

Thus, for these very long-period changes, the time lag between maximum rate of accumulation and maximum thickness approaches the response time ϵ^{-1}, which is about 5000 yr.

If the rate of accumulation changes permanently by a_1, equation (18) shows that the resultant final change in thickness is $h_1 = a_1/\epsilon$. If, for example, the rate of accumulation in Antarctica increased by 1 cm/yr (an increase of about 10 %) the resultant change of thickness would eventually be about 60 m.†

(ii) *The upper parts of a glacier*

The difference between the upper part of a glacier and a large ice-sheet, as regards its response to changes in accumulation, is that the response time of the glacier is much smaller. It is much smaller simply because the rate of longitudinal extension is much greater—for it is this that allows the glacier to adjust itself more quickly to new conditions.

Let us first apply equation (54) to a typical glacier for which r_0 in the upper parts is 0·1 yr^{-1}. This gives $\epsilon \simeq (m+1)r_0 \simeq 0{\cdot}3$ yr^{-1}. For seasonal changes $\omega = 2\pi$ yr^{-1}. Hence, for seasonal changes $\phi = \tan^{-1}\omega/\epsilon = 87°$, and the amplitude of h_1 is approximately a'/ω. a' may be interpreted as the maximum rate of accumulation (or ablation) at the firn line during the year, and this might be 2 cm/day or 7 m/yr in a typical case. The amplitude of h_1 is then $7/2\pi \simeq 1$ m. It is to be noted that this amplitude is measured as thickness of ice rather than thickness of snow. The phase angle is practically 90° (as for Antarctica), and it will be near 90° in most practical cases. Thus even for $\epsilon = 0{\cdot}5$ yr^{-1}, which is a high value, ϕ is 74°.

† A different way of estimating the change is to assume that the ice-sheet is on a horizontal bed of uniform roughness and fixed width. The steady-state height is then found to be proportional to $a_0^{1/(2m+1)}$(Nye 1959). Thus, with these assumptions, a 10 % increase in accumulation gives a $10/(2m+1)$ % increase in h. With $h_0 = 3000$m, this gives $h_1 = 60$m. There is thus good agreement between the two different methods of estimation.

In relation to seasonal changes it is necessary to remember that the present theory ignores the possibly important effect of changes in the amount of melt water on the glacier bed. More melt water may help to lubricate the bed and make the glacier move faster. Such an effect could perhaps be included as a periodic change in A in equation (11).

As an example of a long-period climatic change suppose that τ, the period, is 100 yr and r_0 is $0{\cdot}1\,\mathrm{yr}^{-1}$. Then $\phi = 12°$ and h_1 lags behind a_1 by only this amount (3 yr). On the other hand, if r_0 were $0{\cdot}01\,\mathrm{yr}^{-1}$, ϕ would be 65° (18 yr); thus the glacier would not be keeping pace with the changing climate. For a very long-term change, say with $\tau = 1000\,\mathrm{yr}$, and with $r_0 = 0{\cdot}1\,\mathrm{yr}^{-1}$, $\phi = 1°$ (that is, 3 yr, which is the response time) and the amplitude of h_1 in this case is a'/ϵ. In an extreme case a' might be of the order of the net accumulation, say 5 m/yr, and the amplitude of h_1 would be $5/0{\cdot}1 = 50\,\mathrm{m}$. This shows that, even if, in a 1000 yr cycle, the net accumulation reached zero, the glacier would not necessarily disappear completely, but could live on its reserves, as it were, until the climate became more favourable again. Whether or not it managed to survive would depend upon the rate of extension r_0. If this were very small the amplitude of h_1 would be large and the glacier would temporarily disappear.

(*b*) *The lower regions*

We have already seen in the example of § 5 that, to study the effect of a change in accumulation on the lower part of a glacier, we cannot consider the lower part in isolation. This is because the accompanying changes which occur in the upper part will be propagated down the glacier as kinematic waves. To gain insight into what happens we shall consider again the model of § 5 in which

$$c_0 = \begin{cases} \epsilon x & (0 \leqslant x \leqslant \frac{1}{2}), \\ \epsilon(1-x) & (\frac{1}{2} \leqslant x \leqslant 1), \end{cases} \tag{26}$$

where ϵ is a positive constant. To begin with we neglect diffusion, and we set $a_1 = a'\,\mathrm{e}^{\mathrm{i}\omega t}$. Then, since no disturbance is propagated up-stream, the long-time solution for region I (figure 4) is given by equation (53). h_1 being small, we have

$$q_1 = c_0 h_1 = \frac{c_0 a'}{\epsilon + \mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega t} \quad \text{(region I)}. \tag{57}$$

To extend the solution to region II we need to find the boundary condition on $x = \frac{1}{2}$. On $x = \frac{1}{2}$, $c_0 = \frac{1}{2}\epsilon$; hence

$$q_1 = \frac{a'\epsilon}{2(\epsilon + \mathrm{i}\omega)}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{58}$$

(Since we are only interested in the long-time solution, it is unnecessary to consider region III.) It is easier in this example to label the characteristics in region II by a time t_0 instead of by x_0. On a characteristic

$$t - \int_{\frac{1}{2}}^{x} \frac{\mathrm{d}x}{c_0(x)} = \text{constant} = t_0, \quad \text{say}. \tag{59}$$

t_0 is the value of t at which the characteristic cuts $x = \frac{1}{2}$. Putting $c_0 = \epsilon(1-x)$, and integrating, we have

$$2(1-x) = \mathrm{e}^{-\epsilon(t-t_0)} \quad \text{(region II)} \tag{60}$$

as the equations of the characteristics.

In region II, we know that along the characteristics

$$\mathrm{D}q_1/\mathrm{D}t = c_0 a_1 = \epsilon(1-x)\, a'\, \mathrm{e}^{\mathrm{i}\omega t} \quad (t_0 \text{ constant})$$

and therefore, substituting for $(1-x)$ from equation (60),

$$\mathrm{D}q_1/\mathrm{D}t = \tfrac{1}{2}\epsilon a'\, \mathrm{e}^{\epsilon t_0}\, \mathrm{e}^{(\mathrm{i}\omega-\epsilon)t} \quad (t_0 \text{ constant}).$$

Integrating, we have $$q_1 = \frac{\epsilon a'}{2(\mathrm{i}\omega-\epsilon)}\, \mathrm{e}^{\epsilon t_0}\, \mathrm{e}^{(\mathrm{i}\omega-\epsilon)t} + f(t_0),$$

or, using equation (60) again,

$$q_1 = \frac{\epsilon a'}{\mathrm{i}\omega-\epsilon}(1-x)\, \mathrm{e}^{\mathrm{i}\omega t} + f(t_0).$$

Now continuity at $x = \frac{1}{2}$ demands that when $t = t_0$ and $x = \frac{1}{2}$, q_1 is given by equation (58). This fixes $f(t_0)$, and we obtain

$$q_1 = \frac{\epsilon a'}{\mathrm{i}\omega-\epsilon}(1-x)\, \mathrm{e}^{\mathrm{i}\omega t} + \frac{\epsilon^2 a'}{\epsilon^2+\omega^2}\, \mathrm{e}^{\mathrm{i}\omega t_0}. \tag{61}$$

Now $$h_1 = \frac{q_1}{c_0} = \frac{q_1}{\epsilon(1-x)}.$$

Hence, substituting the value of t_0 from equation (59) and introducing the phase angle ϕ, we finally obtain for h_1 the equation

$$h_1 = \frac{a'}{\sqrt{(\epsilon^2+\omega^2)}}\left\{-\exp \mathrm{i}(\omega t+\phi) + \frac{\cos\phi}{1-x}\exp \mathrm{i}\omega\left(t - \int_{\frac{1}{2}}^{x}\frac{\mathrm{d}x}{c_0}\right)\right\}, \tag{62}$$

where $\tan\phi = \omega/\epsilon$, $0 < \phi < \frac{1}{2}\pi$.

The variation of h_1 is thus made up of two parts, represented by the two terms within the bracket. The first term represents the local, or direct, response. It is independent of x and is caused by the direct action of the varying rate of ablation on the glacier. In this term h_1 lags behind a_1 by a phase angle $\pi-\phi$. The second term represents a travelling wave form which moves down the glacier at the kinematic wave velocity c_0. Its amplitude varies with x, and its phase is such that it is in phase with a_1 on $x = \frac{1}{2}$. The second term of equation (61) shows that this part of the response is a wave of *constant* q_1 propagated along the characteristics. Regarded as a wave of h_1 its amplitude increases as $1/(1-x)$, because the wave velocity decreases. Just as in § 5 we need not be concerned by the infinite amplitude at $x = 1$, because the solution can be terminated before this point is reached.

(i) *Seasonal changes*

To apply equation (62) to seasonal changes in a_1 we put $\omega = 2\pi\,\mathrm{yr}^{-1}$ and, as an example, put $r_0 = -0{\cdot}1\,\mathrm{yr}^{-1}$. Then $\epsilon \simeq -(m+1)\,r_0 \simeq 0{\cdot}3\,\mathrm{yr}^{-1}$ and $\phi = 87°$. $\cos\phi = 0{\cdot}05$, and so the second term in equation (62) is very small except at the extreme end—but, as we shall see, it is damped out by diffusion long before the end is reached. Thus the response is almost entirely given by the first term: the amplitude of h_1 is approximately a'/ω, and h_1 lags behind a_1 by 93° in phase. This may be compared with the angle of 87° calculated for the upper half. Continuity between the two halves is preserved by the second term in (62).

It was mentioned that diffusion would damp out the travelling wave. This may be seen as follows. The characteristic time for diffusion of a wave of wavelength λ is $T \sim \lambda^2/4\pi^2 D_0$. λ in this case is the distance travelled by a wave in 1 yr. If the glacier velocity u_s is 100 m/yr, this distance will be about 400 m (§7). The value of D_0 may be estimated (§7) as $3u_s h_0 \cot\alpha_0$, and if we take $h_0 = 200$ m and $\alpha_0 = 4\times10^{-2}$ (2·3°), this gives $T \sim 0{\cdot}13 u_s\alpha_0/h_0 = 3\times10^{-3}$ yr. Thus, such a wave would be immediately damped out. In this connexion, however, we must bear in mind that our considerations will only strictly apply to wavelengths longer than, say, $2\pi h_0$—because only then are the changes in surface slope, which cause the diffusion, able to affect the shear stress throughout the whole thickness of the glacier.

(ii) *Climatic changes*

In a long-term climatic change both terms in equation (62) can be important. Consider, for instance, a cyclic change of period $\tau = 100$ yr, with $r_0 = +0{\cdot}1\ \text{yr}^{-1}$ in the upper part of the glacier, and $r_0 = -0{\cdot}1\ \text{yr}^{-1}$ in the lower part. Under these conditions we have already seen that h_1 in the upper part lags behind a_1 by 12° in phase. In the lower part $\phi = 12°$, and hence, so far as the first term in equation (62) is concerned, h_1 lags behind a_1 by 168°. However, $\cos\phi = 0{\cdot}98$, and so the second term is also significant. The wave it represents will of course be subject to diffusion, and, with the same values of u_s, h_0 and α_0 as before, we find $T \sim 30$ yr. In this time the wave will have travelled ~ 12 km. These figures show that, in such a case, the travelling wave constitutes an important part of the response.

In general, there will be a competition between the exponential decrease due to diffusion and the growth due to the reduction in wave velocity, which goes as $1/c_0$. So far as diffusion is concerned, we may define a 'penetration distance', $X = c_0 T$, where c_0 is here an average wave velocity in the region considered. With $T \sim 0{\cdot}13 u_s \alpha_0 \tau^2/h_0$ and $c_0 \simeq 4u_s$, we have $X \sim 0{\cdot}5 u_s^2 \alpha_0 \tau^2/h_0$. If X is measured in wavelengths, so that $X = n\lambda$, we have $n = T/\tau$. With the same values of u_s, α_0 and h_0 as before, the values of X and n for various periods τ are

$\tau = 10$ yr	$n = 0{\cdot}03$	$X = 0{\cdot}1$ km
50	0·1	3
100	0·3	10
200	0·5	40
500	1·3	300

From the above table it appears that 'wave trains' initiated by climatic cycles will normally contain less than one wavelength. At the same time it seems that cycles with periods of 50 yr or more will generate waves which can reach the end of a moderate-sized glacier. A large effect will be produced if a wave can survive diffusion for long enough for it to reach a place of low wave velocity.

The phase relation between h_1 and a_1 in the lower part of the glacier will depend on the interference between the local response and the delayed response due to the wave. For a period of around 100 yr, as we have seen, the local response can be nearly in antiphase with a_1. The phase of the travelling wave, on the other hand, varies according to the distance it has travelled. At the firn line h_1 due to the wave is in phase with a_1, at a distance $\frac{1}{4}\lambda$ down the glacier h_1 due to the wave is 90° behind a_1, and at $\frac{1}{2}\lambda$ h_1 due to the wave is 180° behind a_1. The net effect at the firn line we already know to be a small phase lag of h_1 behind a_1 (say 12°). As we go down the glacier the net effect of the two contributions is to increase this lag, until at $\frac{1}{2}\lambda$ below the firn line it is 180°.

(c) *Summary*

We may summarize the response of a glacier to changes in rate of accumulation as follows. A glacier has a characteristic response time of approximately $1/(3r_0)$, where r_0 is the longitudinal strain rate. This will vary from point to point down the glacier, but for r_0 between 10^{-1} and $10^{-2}\,\text{yr}^{-1}$, which is a typical range, it will be between about 3 and 30 yr. The seasonal changes in accumulation have a period ($\tau = 1$ yr) which is normally considerably less than 2π times the response time. The thickness of the glacier then responds so that the maximum rate of increase occurs close to the time when the rate of accumulation is a maximum, exactly as one would expect; that is, h_1 lags behind a_1 by nearly 90° in all parts of the glacier. Any travelling wave generated is attenuated very rapidly indeed.

For longer period changes, h_1 in the upper, extending, part of the glacier tends to come into phase with a_1. The behaviour in the lower, compressing, part, on the other hand, is compounded of two separate effects. There is first a direct response, arising from the instability of a compression region, which tends to make h_1 in antiphase with a_1; but this may be more than counterbalanced by the delayed response due to the travelling wave which is propagated down from the extending regions. For very long-term changes this wave dominates (witness the example of § 5). In general, its amplitude h_1 (*a*) grows as it travels owing to the decreasing wave velocity, and (*b*) diminishes owing to diffusion. The effect observed at any point on the glacier will depend on the interference between the direct effect and the travelling wave. This interaction between the direct and the delayed response clearly gives considerable scope for variation in behaviour from one glacier to another. Wide variations are certainly observed, and indeed they have been for many years a prime subject of discussion in the glaciological literature.

10. Effect of a change in accumulation with altitude

We have assumed up to now that the rate of accumulation at a given x is purely governed by the prevailing climate. In fact there is another effect which may enter —namely that, when the thickness of the ice increases, the altitude of the surface also increases. The rate of accumulation may then change purely from this altitude effect, and without there being any change in climate. To allow for this, one may write, at a given x,

$$a_1 = \bar{a}_1 + \eta h_1, \tag{63}$$

where $\bar{a}_1$ represents the climatic change and ηh_1 arises from the altitude effect. Then the perturbation theory of § 2 is unchanged, except that, in the final equation (10), $\bar{a}_1$ replaces a_1 and the coefficient of h_1 is $(\eta - \mathrm{d}c_0/\mathrm{d}x)$ instead of $-\mathrm{d}c_0/\mathrm{d}x$. If $\eta > \mathrm{d}c_0/\mathrm{d}x$, the glacier will be unstable. Thus, a positive η tends to give an instability even in a region of longitudinal extension, if the latter is small enough. This is the instability studied by Bodvarsson (1955). It comes from the fact that, if a glacier thickens, it may thereby reach an altitude of greater accumulation, which may in turn cause an additional thickening. This type of instability, arising from positive η, is quite different from the one arising from negative $\mathrm{d}c_0/\mathrm{d}x$ which has been our main concern. (Bodvarsson (1955) studied the special case where $q(x, h, \alpha)$ is, for a given x, proportional to $h\alpha$. His perturbation equation (26) omits a term $\dfrac{\partial}{\partial x}\left(\dfrac{u}{k}\dfrac{\mathrm{d}h_s}{\mathrm{d}x}\right)$, which ought to be present, and it is this missing term which leads to the instability treated in the present paper.)

At low altitudes η may be about $10^{-2}\,\mathrm{yr}^{-1}$. Thus, for regions of a glacier where $r_0 \sim 10^{-2}\,\mathrm{yr}^{-1}$ the altitude effect may be significant. In Greenland, if η is estimated simply from the observed change of accumulation as one moves up the existing surface, η decreases to about $3 \times 10^{-4}\,\mathrm{yr}^{-1}$ near 1500 m, and then changes sign to about $-3 \times 10^{-4}\,\mathrm{yr}^{-1}$ above this level (Benson, private communication). However, the change of accumulation as the ice cap climbs to greater height is not the same thing as the change of accumulation as the explorer climbs to greater heights up the surface of the ice cap. Difficult questions are involved here, for ηh_1 will not, in general, depend only on h_1 at the point concerned, but on the changes of height which are taking place at the other points of the ice cap.

11. A glacier in a valley of variable width

A modification of the foregoing theory to take account of the finite and variable width of a glacier valley may be made in a way suggested by corresponding work on rivers, as reviewed, for example, by Lighthill & Whitham (1955).

Let $S(x, t)$ be the area of cross-section of the glacier and let $Q(x, t)$ be the volume flowing through a section in unit time. Then, since S is the volume of material per unit distance, the equation for the conservation of volume is

$$\frac{\partial Q}{\partial x} + \frac{\partial S}{\partial t} = aB, \tag{64}$$

where $B(x, t)$ is the breadth at the surface. Let us postulate that Q is a function of x, S and the local surface slope α. Then, multiplying equation (64) by

$$c = \left(\frac{\partial Q}{\partial S}\right)_{\alpha, x},$$

we have

$$c\frac{\partial Q}{\partial x} + \left(\frac{\partial Q}{\partial S}\right)_{\alpha, x}\frac{\partial S}{\partial t} = caB,$$

or

$$c\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} - \left(\frac{\partial Q}{\partial \alpha}\right)_{S, x}\frac{\partial \alpha}{\partial t} = caB,$$

or

$$c\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} + D\frac{\partial^2 h}{\partial x\,\partial t} = caB, \tag{65}$$

where h is the height of the free surface above a certain reference level (the 'stage'), and where $D = (\partial Q/\partial \alpha)_{S,x}$.

Since S is a function of h and x, we may express Q as a function of x, h and α. At a given x, $dS = B dh$, and so c and D may be written as

$$c = \frac{1}{B}\left(\frac{\partial Q}{\partial h}\right)_{\alpha,x}, \quad D = \left(\frac{\partial Q}{\partial \alpha}\right)_{h,x}. \tag{66}$$

Thus, if Q is expressed as a function of x, h and α, (65) is the equation governing the propagation of kinematic waves, the velocity c and diffusion coefficient D being given by (66).

Our conclusions for the effect of changes in accumulation will thus continue to hold for a valley of changing width, provided that the appropriate function Q and the appropriate values of c and D are used, and that aB is used in place of a. If the rate of change of Q with respect to changes in h and α is known, either by theory or by observation, at each point on a particular glacier, then c and D are known for this glacier once for all. It is then possible, in principle, by means of equation (65) to calculate the development of any disturbance produced by a change in the rate of accumulation.

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Appendix A

The following simple argument shows that a region of uniform longitudinal compressive strain-rate is unstable.

AB in figure 7 is a small length δx of the glacier in the ablation area. Let there be a steady state. Then the flow (volume per unit time) through the section AA' is $u_0 h_0$. The flow through the upper surface must be just sufficient to balance the rate of ablation $-a_0$, and so is $-a_0 \delta x$. The flow through the section BB' is then $u_0 h_0 + a_0 \delta x$.

Suppose now that an extra layer of thickness h_1 is artificially laid on top of the glacier, the rate of ablation remaining the same. Then the velocity will increase to $u_0 + u_1$, and, to the first order, the flow through AA' will increase by $u_0 h_1$, on account of the increased thickness, and by $h_0 u_1$ on account of the greater velocity. Now if $u \propto h^m$, $u_1/u_0 = m h_1/h_0$; hence the increase in flow on account of the increased velocity is $m u_0 h_1$. The total increase of flow through AA' is therefore $(m+1) u_0 h_1$. The important point is that the increase of flow though a section is proportional to the steady-state velocity at the section. If the increase of flow through BB' were equal to that through AA' the new surface would remain as it is. In fact the increase of flow through BB' is less, because the steady-state velocity is less there (compressive strain-rate). There is therefore an accumulation of material between the two sections, and the level of the surface rises. Thus the small addition to the thickness that we made is unstable.

A similar argument shows that a small reduction in thickness is also unstable.

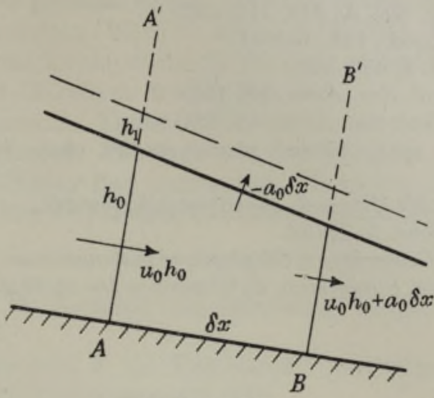

FIGURE 7

APPENDIX B

The wave velocity of the glacier is assumed to be given by equation (26), and there is initially a uniform perturbation of the thickness by an amount h_1^*. We wish to calculate the subsequent history of the perturbation. When $t = 0$, $q_1 = c_0 h_1^*$ and $x = x_0$. Hence at $t = 0$, q_1 is given by $\epsilon x_0 h_1^*$ for $0 \leqslant x_0 \leqslant \frac{1}{2}$ and $\epsilon(1-x_0)h_1^*$ for $\frac{1}{2} \leqslant x_0 \leqslant 1$. Since a_1 is zero, q_1 is constant along the wave paths, that is, it retains its initial value. Therefore q_1 is $\epsilon x_0 h_1^*$ in regions I and II (figure 4) and $\epsilon(1-x_0)h_1^*$ in region III. Dividing by c_0 to obtain h_1, and substituting in the appropriate values for c_0 and x_0 in terms of x and t from equations (26), (27), (28) and (29) gives the required solution, namely:

$$h_1 = h_1^* \,\mathrm{e}^{-\epsilon t} \qquad \text{(region I)},$$

$$h_1 = \frac{h_1^* \,\mathrm{e}^{-\epsilon t}}{4(1-x)^2} \qquad \text{(region II)},$$

$$h_1 = h_1^* \,\mathrm{e}^{\epsilon t} \qquad \text{(region III)}.$$

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