MATH 3326 – Spring 2024

Lab Assignment 3 - KEY

Names: ____

____

1. A 5×3 matrix $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ has all non-zero columns and $\vec{a}_3 = 5\vec{a}_1 + 7\vec{a}_2$. Identify a non-trivial solution to $A\vec{x} = \vec{0}$.

   one version of A could be $A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Solving $A\vec{x} = \vec{0}$ we have $x_3 = x_3, x_2 = -7x_3$ and $x_1 = -5x_3$ so our non-trivial solution is $\vec{x} = x_3 \begin{pmatrix} -5 \\ -7 \\ 1 \end{pmatrix}$. Note, this is the solution regardless of the A you make according to the description of A.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

   $$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

   Construct a matrix A so that $T(\vec{x}) = A\vec{x}$ for all vectors $\vec{x}$.

   $A = \begin{pmatrix} -1 & 4 \\ 3 & -1 \end{pmatrix}$

3. Construct the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$

   Where $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix}$ and $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 6 \\ 1 \\ 8 \end{pmatrix}$

   $A = \begin{pmatrix} 3 & 1 \\ 1 & 6 \\ 4 & 1 \\ 1 & 8 \end{pmatrix}$

4. For square matrices A, B, is it always true that $(A + B)^2 = A^2 + 2AB + B^2$? Explain why/why not.
   No. $(A + B)^2 = (A + B)(B + A) = AB + AA + BB + BA = A^2 + AB + BA + B^2$ and $AB \neq BA$

5. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & h \\ k & 1 \end{pmatrix}$$

For what values (if any) of $k, h \in \mathbb{R}$:

Note: $AB = \begin{pmatrix} 1+k & h+1 \\ k & 1 \end{pmatrix}$

(a) do matrices A and B commute?

$BA = \begin{pmatrix} 1 & 1+h \\ k & k+1 \end{pmatrix}$ so we need $k = 0$ and h can be any number.

(b) is the product AB equal to I_2?

$k = 0, h = -1$

(c) is the product AB equal to the 2×2 zero matrix $0_{2\times 2}$?

Not possible because $ab_{2,2} = 1$ and it must be 0.

6. Compute the inverse of the matrix, where $c \in \mathbb{R}$. For what values of c does the matrix have an inverse?

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & -1 & c \end{pmatrix}$$

First note that as we rref $(A|I)$ we get to the step $\begin{pmatrix} 1 & 0 & 2 & | & 0 & 1/2 & 0 \\ 0 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 0 & c+1 & | & 1 & 0 & 1 \end{pmatrix}$ so $c \neq -1$.

Continuing the augmentation we find $A^{-1} = \begin{pmatrix} -2/(c+1) & 1/2 & -2/(c+1) \\ -1/(c+1)+1 & 0 & -1/(c+1) \\ 1/(c+1) & 0 & 1/(c+1) \end{pmatrix}$

7. Indicate whether the statements are true or false. A is an $n \times n$ matrix.

(a) If $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y} \in \mathbb{R}^n$, then A cannot be invertible.
True - if $A\vec{x} = A\vec{y}$ then $A^{-1}A\vec{x} = A^{-1}A\vec{y} \implies \vec{x} = \vec{y}$ but if $\vec{x} \neq \vec{y}$ then there is no inverse of A.

(b) If for some $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has more than one solution, then A is invertible.
False - if there is more than one solution that means we had a free variable $\implies$ a row with all zeros for a square matrix.

(c) Every elementary matrix is invertible.
True by nature of elementary matrix definition.