Homework 1

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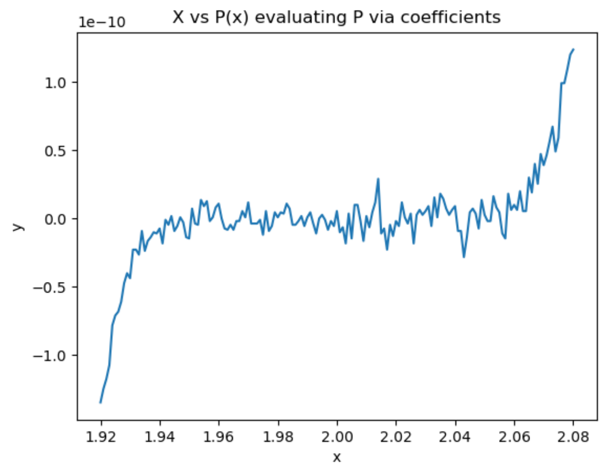
September 5, 2024

1 Question 1

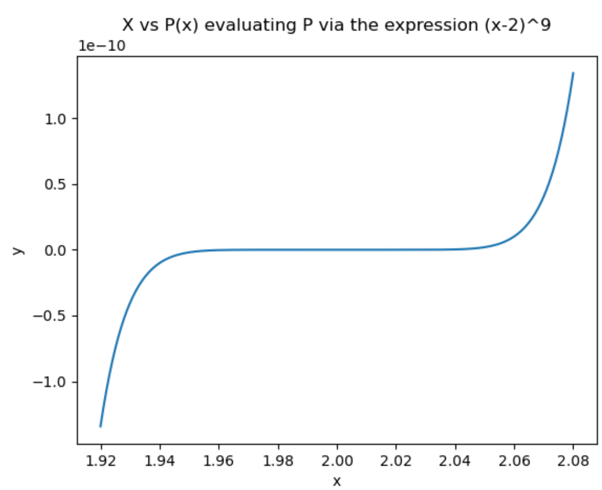
Considering the polynomial:

$$p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512 \quad (1)$$

I. Plot $p(x)$ for x ranging from 1.920 to 2.080 with steps 0.001 via the coefficients (See Appendix for code)



II. Plot $p(x)$ via the expression (See Appendix for code)



III. The difference between the two plots is that the first plot is not smooth due to the loss of decimals during floating point arithmetic. The correct plot between the two is the second using the expression $p(x) = (x-2)^9$ because the expression with coefficients will loose a lot of floating points during the multiple additions and subtractions of the numbers with many decimal points. Using the shorter expression works better because there is only one subtraction.

2 Question 2

How I would calculate the following to avoid cancellation and why

I. $\sqrt{x+1} - 1$ for $x \simeq 0$

$$(\sqrt{x+1}-1)\frac{-\sqrt{x+1}-1}{-\sqrt{x+1}-1} = \frac{-(x+1)+1}{-\sqrt{x+1}-1} = \frac{x}{\sqrt{x+1}+1} \tag{2}$$

This modification takes away the concern that the very small number would be lost due to operations. Rather than the small x being ignored in the addition leading to an immediate 0, the small x will be on its own on the numerator.

II. $\sin x - \sin y$ for $x \simeq y$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \tag{3}$$

This modifications avoids cancellation because rather than risking the evaluation to 0 from x and y being so similar and taking the sin of them, subtracting y from x before taking the sin eliminates cancellation.

III. $\frac{1-\cos x}{\sin x}$ for $x \simeq 0$

$$\frac{1-\cos x}{\sin x} = \frac{1-\cos x}{\sin x} * \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{\sin x(1+\cos x)} = \frac{\sin^2 x}{\sin x(1+\cos x)} = \frac{\sin x}{1+\cos x} \tag{4}$$

This calculation eliminates the concern that the calculation will evaluate to zero because the numerator is no longer at concern for cancelling out.

3 Question 3

Find the second degree Taylor polynomial $P_2(x)$ for $f(x)$ about $x_0 = 0$

$$f(x) = (1+x+x^3)\cos x \tag{5}$$

Using the formula for Taylor polynomial we can find $P_2(x)$. This requires $f'(x_0)$ and $f''(x_0)$, and $f^3(x)$ and $f^4(x)$ for error finding later.

$$f'(x) = (1+3x^2)\cos x - (1+x+x^3)\sin x \tag{6}$$
$$f''(x) = (-x^3+5x-1)\cos x - (2+6x^2)\sin x \tag{7}$$
$$f^3(x) = (x^3-17x+1)\sin x + (3-9x^2)\cos x \tag{8}$$
$$f^4(x) = (x^3-35x+1)\cos x + (12x^2-20)\sin x \tag{9}$$

Therefore $f'(0) = 1$, $f''(0) = -1$. (6) and (7) are used in order to find the error, where we need (7) to find where (6) is at its max, which is at $x = 0.018183$. Using the formula for Taylor polynomials:

$$P_2(x) = 1 + x - \frac{x^2}{2} \tag{10}$$

A. Using $P_2(.5)$ to approximate $f(.5)$ we get $P_2(.5) = 1.375$ Using the Taylor error formula, the upper bound for the error $|f(0.5) - P_2(0.5)|$ is:

$$|f(0.5) - P_2(0.5)| \leq \frac{f^3(z)|x|^3}{3!} = \frac{f^3(0.018183)(0.5)^3}{3!} = 0.06268939232668799 \tag{11}$$

while the actual error $|f(0.5) - P_2(0.5)| = 0.05107166307185573$. The actual error is a 100th of a decimal place away from the upper bound for error found through Taylor's error formula.

B. A bound on the error when $P_2(x)$ is used to approximate $f(x)$ can be found by filling in the formula $|f(x) - P_2(x)|$ and using the triangle inequality:

$$|f(x) - P_2(x)| = |(1+x+x^3)\cos x - (1+x-\frac{x^2}{2})| \leq |(1+x+x^3)\cos x| + |1+x-\frac{x^2}{2}|$$
$$\leq |(1+x+x^3)| + |1+x-\frac{x^2}{2}| \leq 2+2x+\frac{x^2}{2}+x^3$$

Therefore,

$$|f(x) - P_2(x)| \leq 2+2x+\frac{x^2}{2}+x^3 \tag{12}$$

C. Approximating $\int_0^1 f(x)dx$ by the Taylor polynomial:

$$\int_0^1 1+x-\frac{x^2}{2}dx = x+\frac{x^2}{2}-\frac{x^3}{6}\Big|_0^1 = \frac{4}{3} \tag{13}$$

D. To estimate the error in the integral in C, we can integrate the error bound function in B from 0 to 1.

$$\int_0^1 2+2x+\frac{x^2}{2}+x^3 = 2x+x^2+\frac{x^3}{6}+\frac{x^4}{4}\Big|_0^1 = \frac{41}{12} \tag{14}$$

4 Question 4

Consider the quadratic equation $ax^2+bx+c=0$ with $a=1, b=-56, c=1$.

A. Using the equation above and the assumption we can calculate the square root with 3 correct decimals, we can compute the relative errors for the two roots through the standard formula.

$$r_1 = \frac{56+\sqrt{56^2-4}}{2} = \frac{56+55.964}{2} = 55.982 \tag{15}$$

$$r_2 = \frac{56-\sqrt{56^2-4}}{2} = \frac{56-55.964}{2} = 0.018 \tag{16}$$

The more accurate roots are $r_1 = 55.98213716$ and $r_2 = 0.01786284$ Hence, the relative errors can be determined:

$$r_1 : \frac{|55.98213716 - 55.982|}{55.98213716} = 2.45 * 10^-6 \tag{17}$$

$$r_2 : \frac{|0.01786284 - 0.018|}{0.01786284} = .00768 \tag{18}$$

The relative error for r_1 is small meaning that it is a good approximation. The relative error for r_2 is large so it is a bad approximation, making r_2 the "bad" root.

B. To get better approximations on the roots, we can find relations for r_1 and r_2 to a, b, c

$$r_1 = \Big(\frac{-b+\sqrt{b^2-4ac}}{2a}\Big)\Big(\frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}\Big) = \frac{-2c}{b+\sqrt{b^2-4ac}} \tag{19}$$

$$r_2 = \Big(\frac{-b-\sqrt{b^2-4ac}}{2a}\Big)\Big(\frac{-b+\sqrt{b^2-4ac}}{-b+\sqrt{b^2-4ac}}\Big) = \frac{-2c}{b-\sqrt{b^2-4ac}} \tag{20}$$

Using these equations we can approximate r_2:

$$r_2 = \frac{-2}{-56-\sqrt{56^2-4}} = 0.01786 \tag{21}$$

Therefore we can see that using a different approximation that does not have subtraction in the numerator computes r_2 more accurately

5 Question 5

Consider computing $y = x_1 - x_2$ with $\tilde{x_1} = x_1 + \Delta x_1$ and $\tilde{x_2} = x_2 + \Delta x_2$ being approximations to exact values. Therefore $\tilde{y} = y + \Delta y$ where $\Delta y = \Delta x_1 - \Delta x_2$

A. To find the upper bounds on the absolute error $|\Delta y|$ and the relative error $\frac{|\Delta y|}{|y|}$ we can use the triangle inequality.

$$|\Delta y| = |\Delta x_1 - \Delta x_2| \leq |\Delta x_1| + |\Delta x_2| \tag{22}$$

Hence,

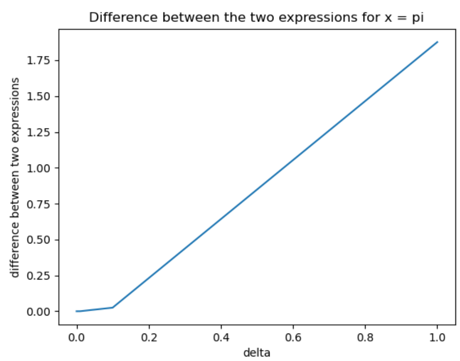
$$\frac{|\Delta y|}{|y|} \leq \frac{|\Delta x_1|}{|x_1 - x_2|} + \frac{|\Delta x_2|}{|x_1 - x_2|} \tag{23}$$

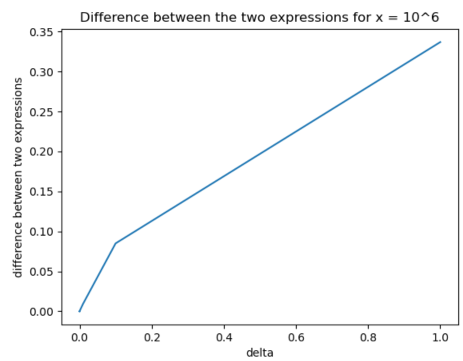
The relative error is large when there is minimal difference between x_1 and x_2 and the change in them respectively is large.

B. The expression $\cos x + \delta - \cos x$ can be manipulated to not have subtraction:

$$\cos x + \delta - \cos x = -2 \sin(2x + \delta) \sin(\delta) \tag{24}$$

Using the values $x = \pi$ and $x = 10^6$ we can experiment with the differences between the two representations of the expression. For each x we can see the differences between the expressions for $\delta = 10^-16, 10^-15, ..., 10^-1, 10^0$. (See Appendix for code)





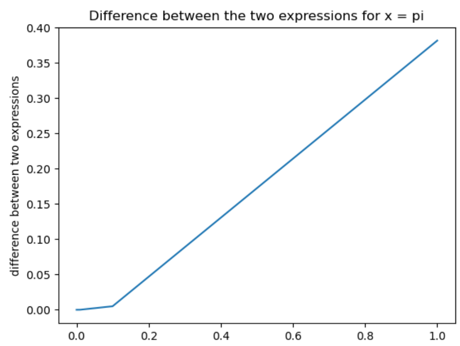
C. The Taylor expansion yields a method to approximate the expressions above stating that:

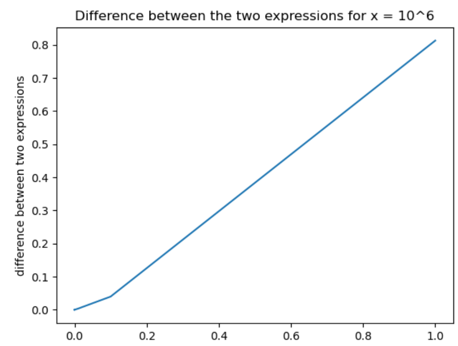
$$\cos x + \delta - \cos x = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi) \tag{25}$$

where $\xi \epsilon [x, x + \delta]$. Because we do not know the exact value of ξ, we choose to adjust the equation to an approximation instead so that we are able to create a procedure for this expression that is approximate:

$$\cos x + \delta - \cos x \approx \delta f'(x) \tag{26}$$

Using the same values for x and delta as above and using the same technique to see the accuracy of the approximation (See Appendix for code),





For large x, the expression in (15) is more accurate than the approximation (17) but for $x = \pi$ the approximation is clearly more accurate than the manipulated expression. All four graphs agree with the statement in A about when the relative error is large.

6 Appendix

Code for Question 1

```
q1.py > ...
import numpy as np
import matplotlib.pyplot as plt
import numpy.linalg as la
import math

x = np.arange(1.920,2.080,0.001)
p1 = lambda x: x**9-18*x**8+144*x**7-672*x**6+2016*x**5-4032*x**4+5376*x**3-4608*x**2+2304*x-512
p2 = lambda x: (x-2)**9
y1 = p1(x)
y2 = p2(x)
plt.plot(x,y1)
plt.title('X vs P(x) evaluating P via coefficients')
plt.xlabel('x')
plt.ylabel('y')
plt.show()

plt.plot(x,y2)
plt.title('X vs P(x) evaluating P via the expression (x-2)^9')
plt.xlabel('x')
plt.ylabel('y')
plt.show()

```

Code for Question 5b

```
x1 = math.pi
x2 = 10**6
delta = np.array([10**i for i in range(-16,1)])

def func1(x, delta):
    y1 = np.cos(x+delta)-np.cos(x)
    return y1

def func2(x,delta):
    y2 =-2*np.sin(2*x+delta)*np.sin(delta)
    return y2

y11 = func1(x1,delta)
y12 = func2(x1,delta)
y13 = abs(y11-y12)

y21 = func1(x2,delta)
y22 = func2(x2,delta)
y23 = abs(y21-y22)

plt.plot(delta,y13)
plt.xlabel("delta")
plt.ylabel("difference between two expressions")
plt.title("Difference between the two expressions for x = pi")
plt.show()

plt.plot(delta,y23)
plt.xlabel("delta")
plt.ylabel("difference between two expressions")
plt.title("Difference between the two expressions for x = 10^6")
plt.show()
```

Code for Question 5c

```
x1 = math.pi
x2 = 10**6
delta = np.array([10**i for i in range(-16,1)])

def func1(x, delta):
    y1 = np.cos(x+delta)-np.cos(x)
    return y1

def func2(x,delta):
    y2 = delta*(np.sin(x)-np.sin(x+delta))
    return y2

y11 = func1(x1,delta)
y12 = func2(x1,delta)
y13 = abs(y11-y12)

y21 = func1(x2,delta)
y22 = func2(x2,delta)
y23 = abs(y21-y22)

plt.plot(delta,y13)
plt.xlabel("delta")
plt.ylabel("difference between two expressions")
plt.title("Difference between the two expressions for x = pi")
plt.show()

plt.plot(delta,y23)
plt.xlabel("delta")
plt.ylabel("difference between two expressions")
plt.title("Difference between the two expressions for x = 10^6")
plt.show()
```