**Comparison of First and Second Order Piecewise Reconstruction Schemes Using the Advection Equation**

Reconstruction schemes for modelling fluid mechanical problems can be divided into first order, second order, and higher order accurate schemes. First and second order schemes are often referred to as constant and linear piecewise reconstruction schemes, respectively.

The following comparisons use a second order time accurate scheme known as the Characteristic Correction of Upwind States. Given a slope of the wave calculated using some function, this linear reconstruction method divides the cell size and slope by two to predict the solution in the centre of the cell. With the advection equation which, given a positive velocity v, works by shifting the wave constructed by the piecewise linear function to the right by some spatial step $\Delta x=v*\Delta t$, the solution at the cell centre becomes the solution at the cell interface after half a timestep.

$$u\_inter_{i+1/2}^{n}(t+\Delta t) = u_i^n + s_i\left[\frac{\Delta x}{2} - \frac{v\Delta t}{2}\right]$$

Here, i relates to the spatial component x and n relates to the time component t. The cell interface values are then input into the finite volume discretisation of the advection equation to give the numerical solution.

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x}\left[(v * u\_inter_{i+1/2}^{n}) - (v * u\_inter_{i-1/2}^{n})\right]$$

The slopes have been calculated using seven different methods. These are Godunov's method, Fromm's method, Beam-Warming method, Lax-Wendroff method, as well as the Minmod limiter, van Leer limiter and MC limiter. Limiters differ from non-limiting second order functions in that they are piecewise functions that provide different equations depending on factors such as whether the wave has both positive, negative, or differing gradients at both ends of a given section of cells. Also note that while Godunov's method is the only first order method, inputting a constant slope of zero into the characteristic correction of upwind states simply reduces it back to a first order upwind scheme. As such, the following compares one first-order scheme to six different second-order schemes, with and without the use of limiters.

Part One compares the performance of all seven schemes using discontinuous initial conditions. Part Two analyses the seven schemes over a smooth, continuous wavefront. Part Three compares the Minmod limiter both with and without advancing the cell interface values by a half-timestep using two different timesteps adjusted via the Courant Factor.

**PART ONE**

Figures 1 and 2 compare the seven methods using discontinuous initial conditions after time t=1 in coded units.

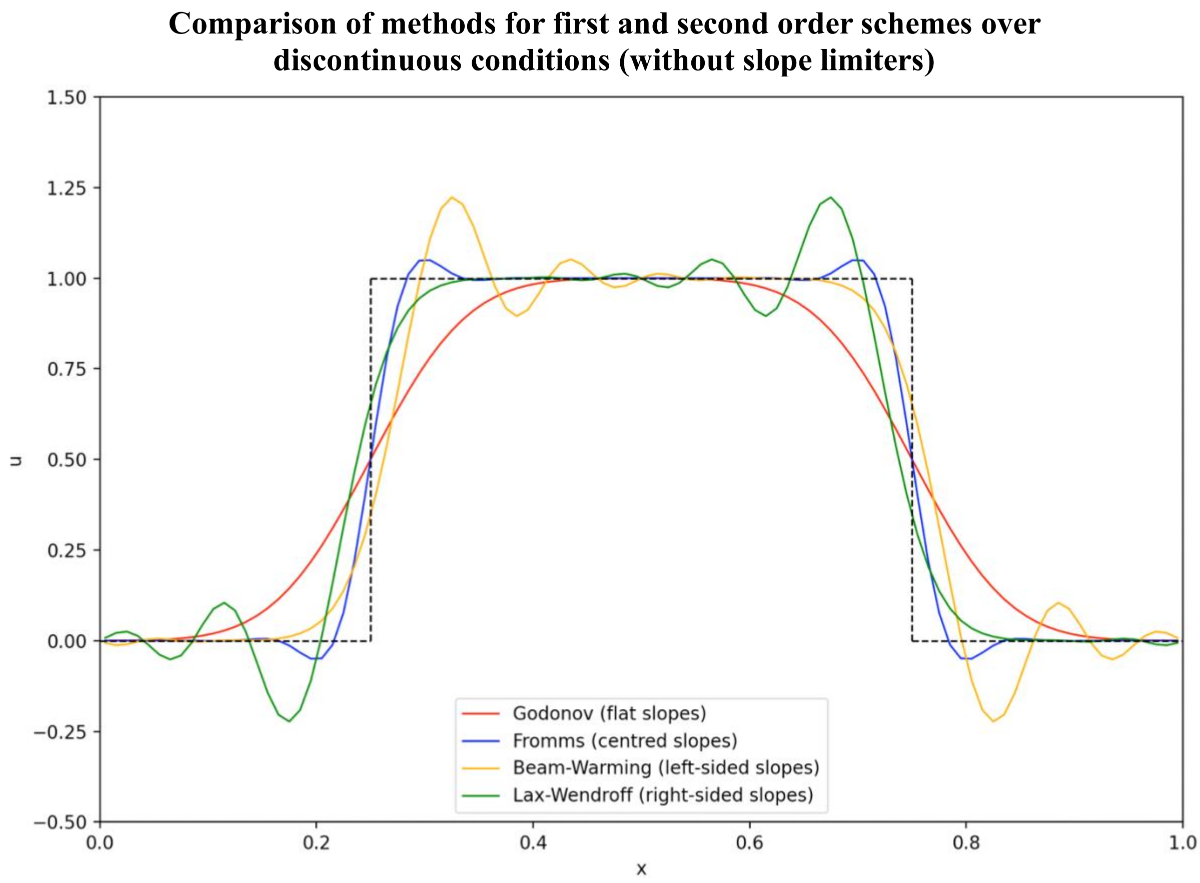


*Figure 1. The 1D advection equation modelled using discontinuous initial conditions of u=1 for 0.25<x≤0.75 and u=0 elsewhere. The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

As depicted in Figure 1, many of the reconstruction schemes introduce spurious oscillations containing overshoots and undershoots around the discontinuities. Out of the four schemes shown above, the flat slopes reconstruction associated with Godunov's scheme performs the best, with no spurious oscillations and neither undershooting nor overshooting the minimum and maximum values given by the initial conditions.

In comparison, Fromm's scheme tends to undershoot the minimum values and overshoot the maximum values around the discontinuities. However, compared to Gonunov's scheme, Fromm's scheme holds to the true value longer prior to the discontinuity, and returns faster after the discontinuity. This is also true for the Lax-Wendroff and Beam-Warming schemes. As such, both Fromm's method and Godunov's method have good and bad elements.

Both the Beam-Warming scheme and the Lax-Wendroff scheme perform equally poorly, with growing oscillations leading to large over and undershooting and instability. The Lax-Wendroff scheme introduces these oscillations prior to the discontinuity, while the Beam-Warming scheme produces these after the discontinuities, which is a product of how the slopes are calculated around these points.

Without limiters, the Godunov's first order scheme tends to be more accurate and stable than second order schemes, particularly Beam-Warming and Lax-Wendfroff, around discontinuities.

Fromm's second order scheme performs similarly to Godunov's in that they both appear to be stable, but with differing strengths and weaknesses.

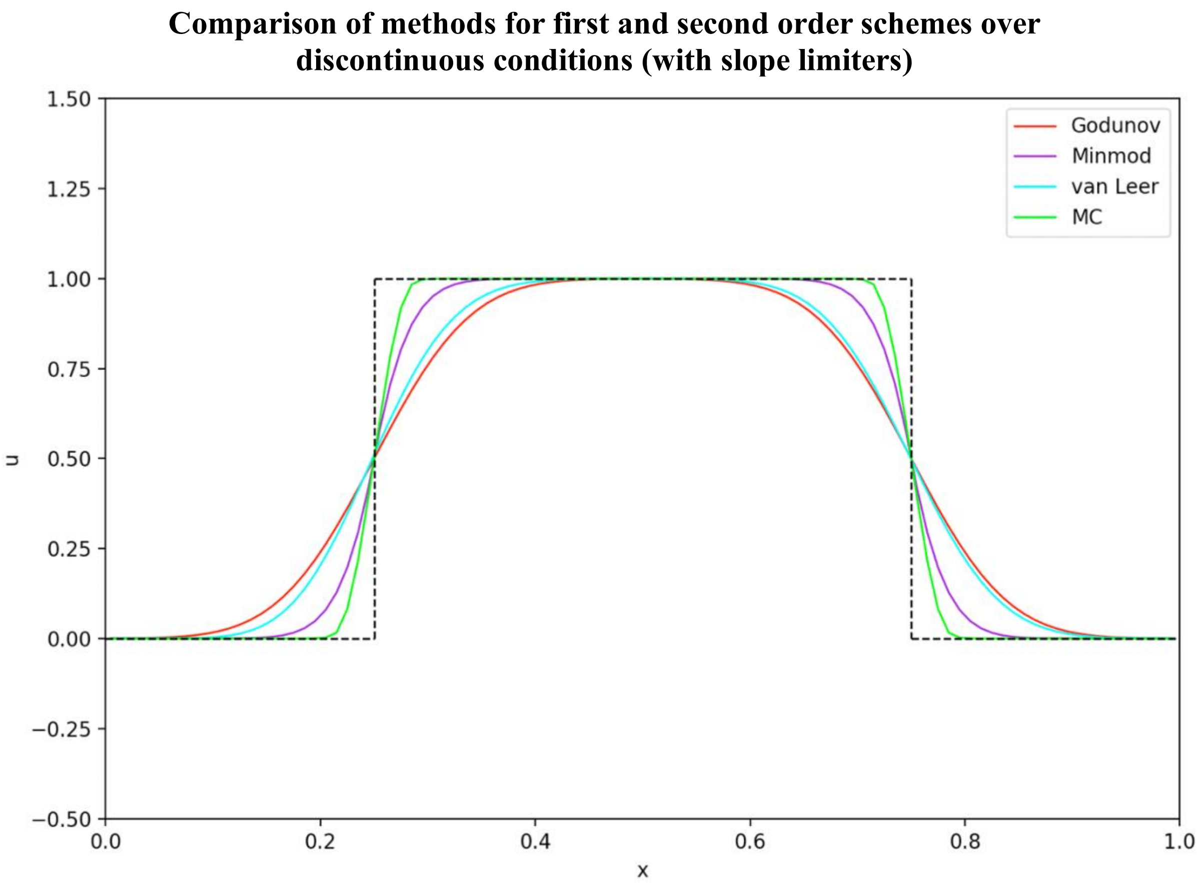


*Figure 2. The 1D advection equation modelled using discontinuous initial conditions of u=1 for 0.25<x≤0.75 and u=0 elsewhere. The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

Figure 2 depicts three limiting schemes and compares them to Godunov's method already depicted in Figure 1. Comparing the two figures, the most profound impact of the limiter function is its ability to minimise any spurious oscillations commonly produced by second-order schemes. In addition, they are also less diffusive than Godunov's method, due to the higher-order of accuracy.

Out of seven methods tested on the square wave, the three limiters perform best, with MC the clear winner of the three, followed by Minmod and van Leer, which performs only slightly better than the first order Godunov method. Fromm's method is less diffusive than Godunov's, but has a tendency to overshoot the maximums and undershoot the minimums. The Beam-Warming and Lax-Wendfroff methods produce growing oscillations around the discontinuities are therefore are both unstable.

**PART TWO**

Figures 3-5 depict Godunov, Fromm's, Beam-Warming and Lax-Wendfroff schemes on smooth sinusoidal wavefronts after time t=1 in coded units, with Figures 4 and 5 showing closeups of the peak and trough of the sine wave. Similarly, Figures 6-8 depict the same sinusoidal initial conditions, but are modelled using Minmod, van Leer and MC limiters , and contrasted against Godunov's scheme as a point of reference.

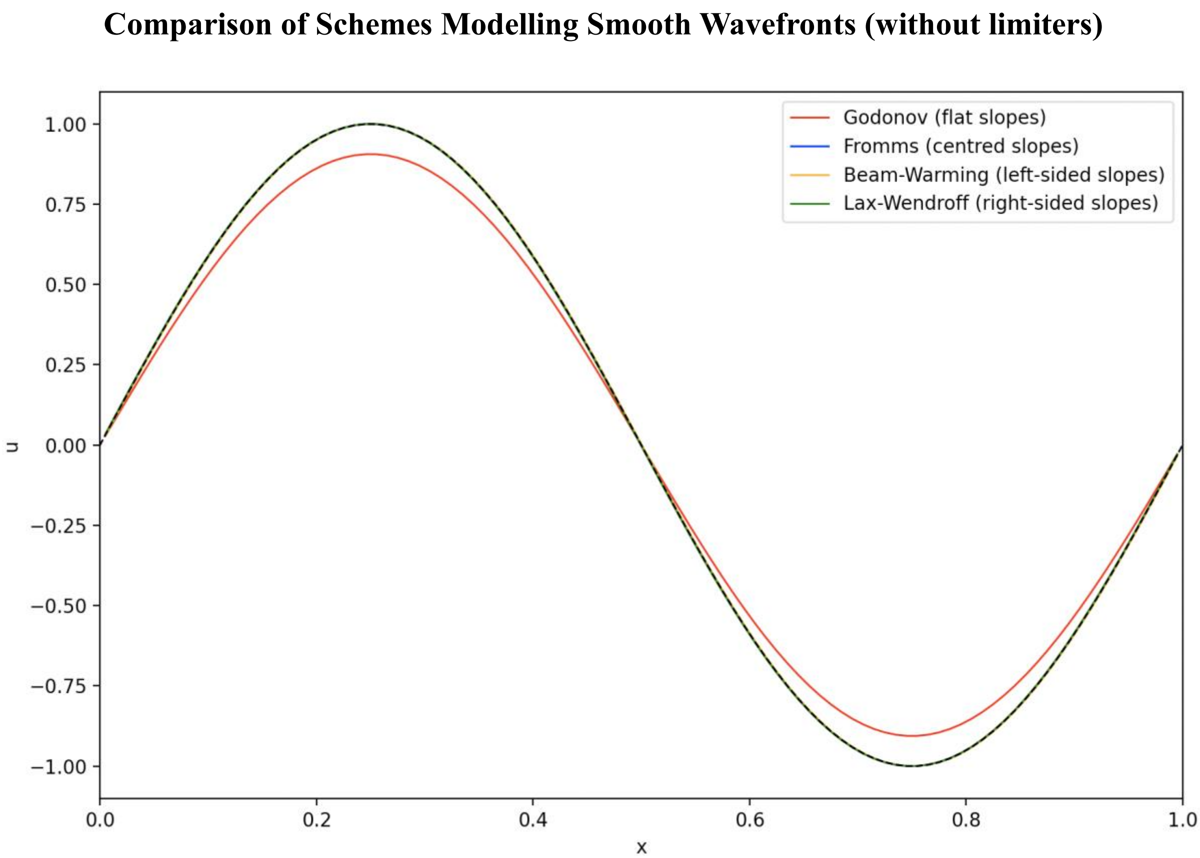


*Figure 3. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

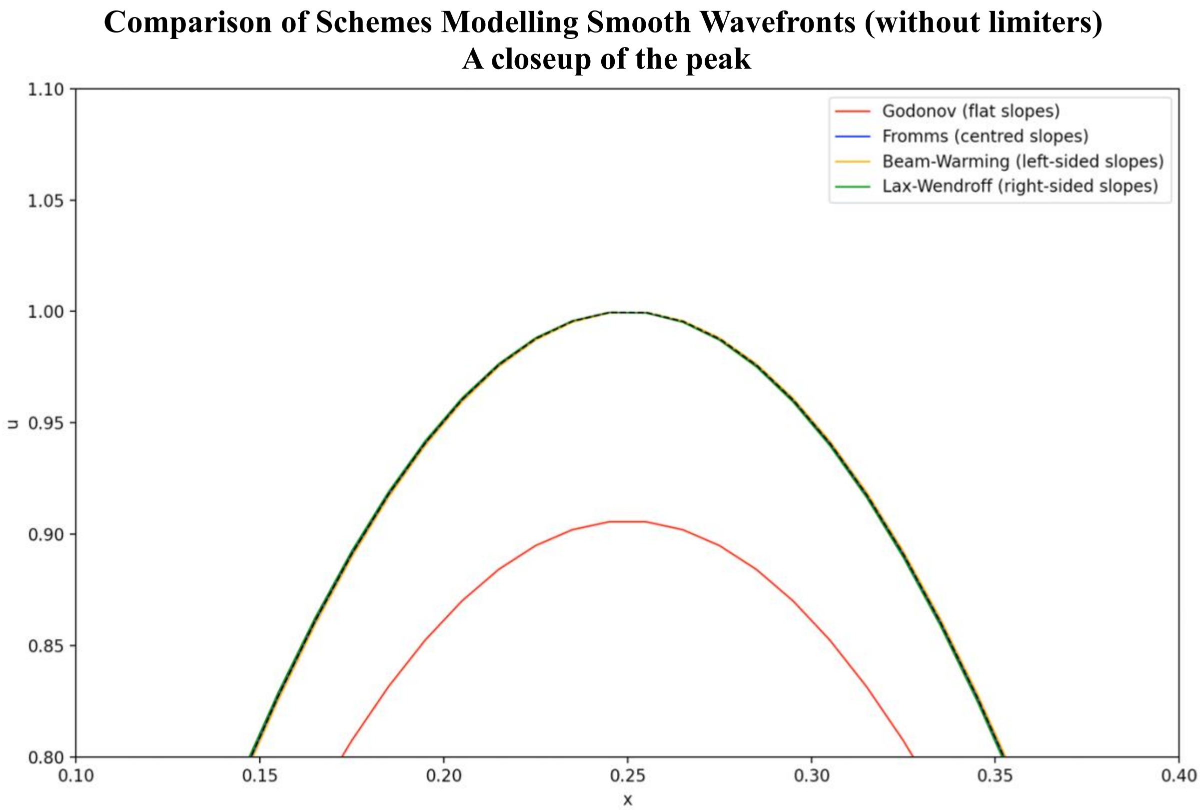


*Figure 4. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

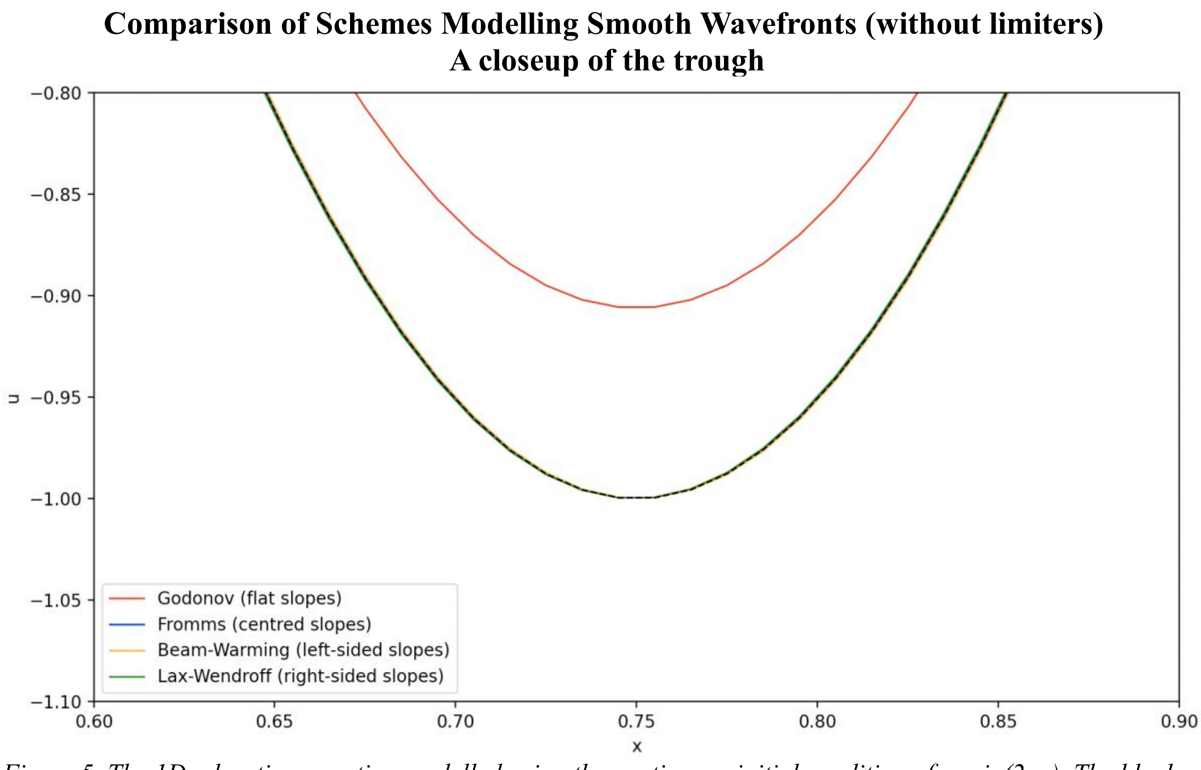


*Figure 5. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

As seen in Figures 3-5, the three second-order schemes are remarkably good for modelling continuous sinusoidal waves, with all three second-order schemes matching the black dotted line of the initial condition after time t=1. Godunov's scheme still suffers from the same diffusion problem seen in Part One, with maximum and minimum values only at approximately 90% of their original values at t=0.

**Comparison of Schemes Modelling Smooth Wavefronts (with limiters)**

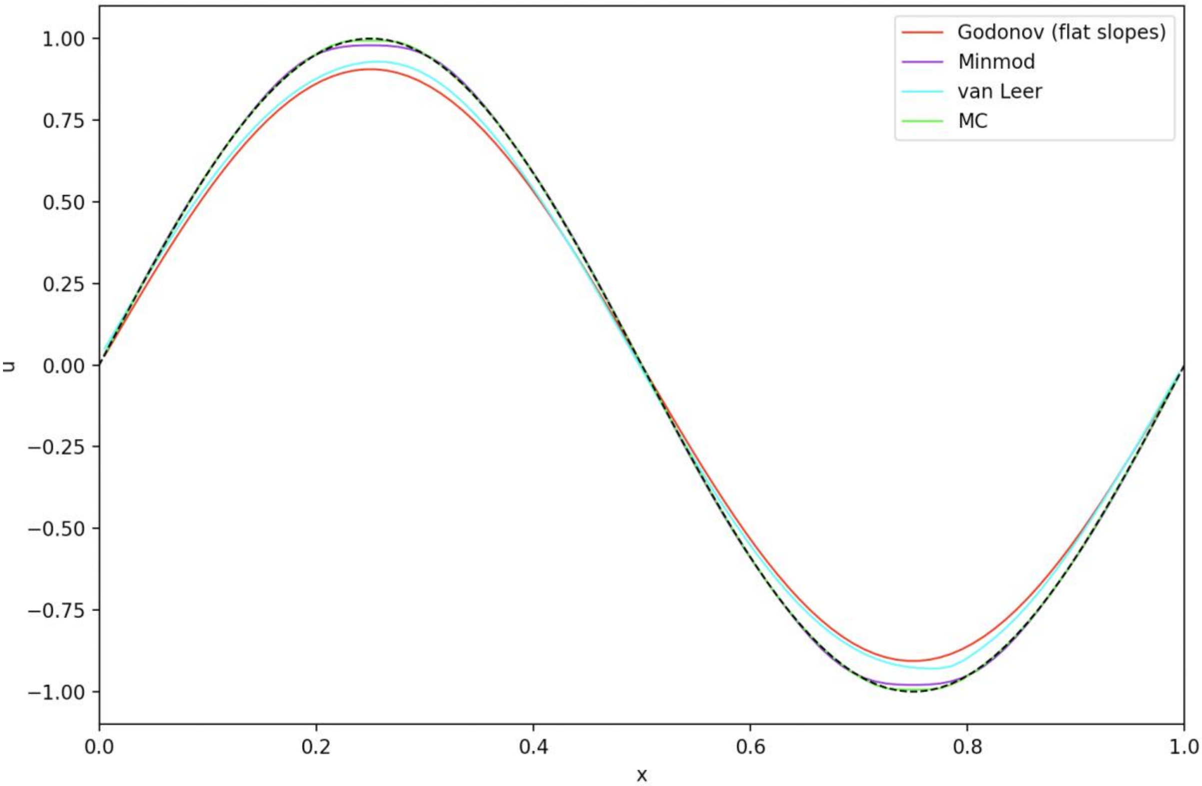


*Figure 6. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

**Comparison of Schemes Modelling Smooth Wavefronts (with limiters)**
**A closeup of the peak**

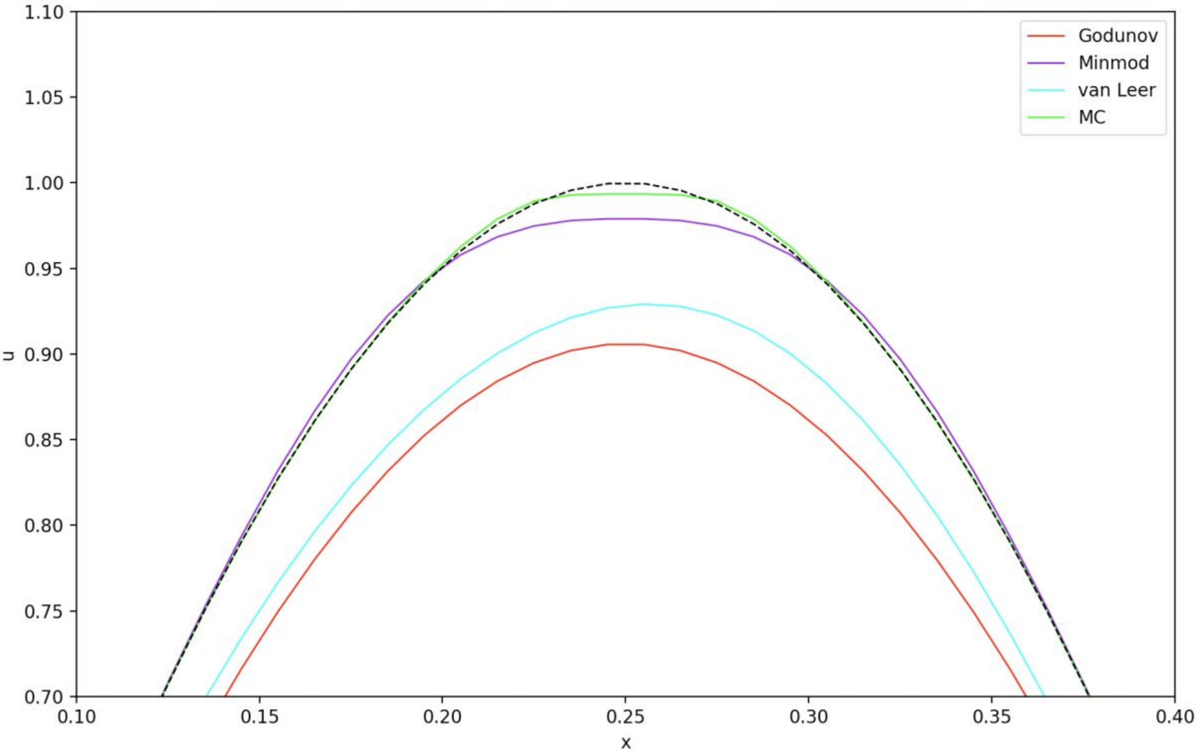


*Figure 7. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

**Comparison of Schemes Modelling Smooth Wavefronts (with limiters)**
**A closeup of the trough**

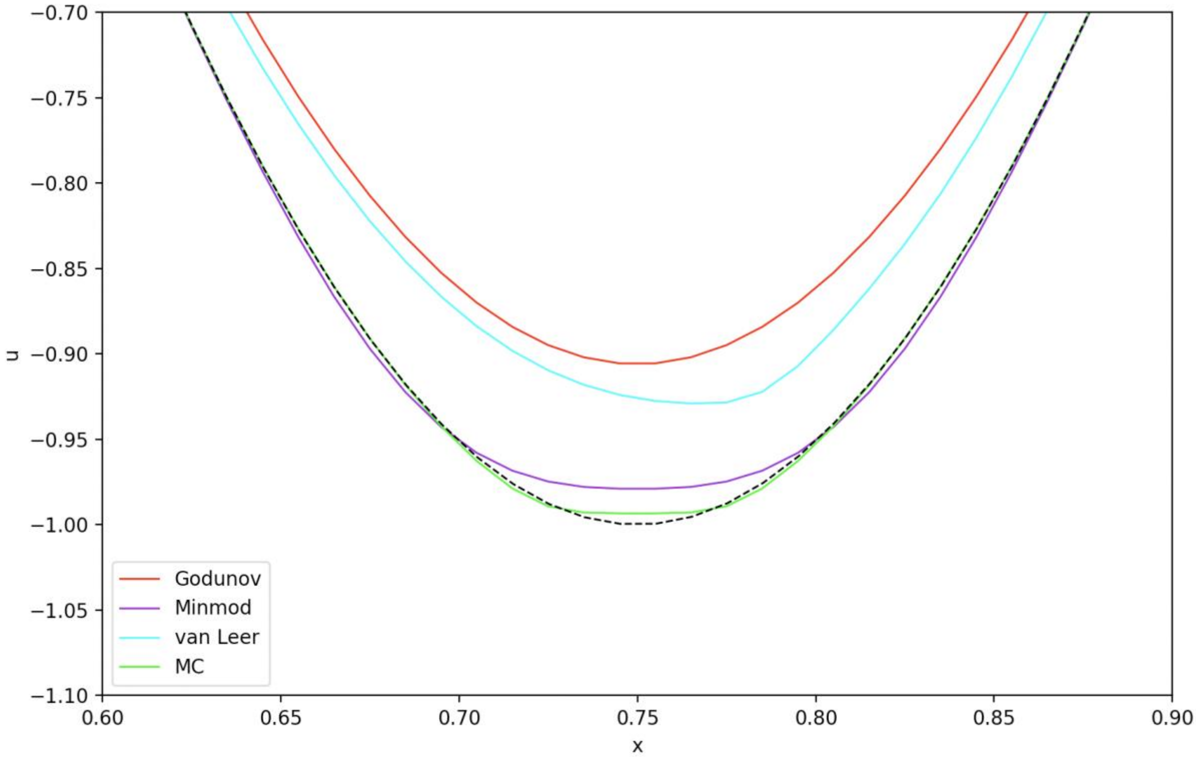

*Figure 8. The 1D advection equation modelled using the continuous initial condition of u=sin(2πx). The black dotted line represents these initial conditions and provides a comparison of accuracy for the four linear reconstruction schemes. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5.*

Similar to Part One, Figures 6-8 show the limiters suffering from much less diffusion compared to the first order Godunov scheme, although some diffusion is still present. However, unlike in Part One, the limiter methods are performing worse than the second-order Fromm's method, Beam-Warming method and Lax-Wendroff method.

In addition, both Minmod and van Leer have a flattening effect at the peaks and troughs, although MC is affected by this less than Minmod. Interestingly, the van Leer method also has the effect of offsetting the minimum value to the right, such that the smooth wavefronts become more pointed and lopsided.

It appears that, at least for smooth continuous waves, the limiting schemes perform worse the second-order non-limiting schemes, suffering from issues of diffusion as well as flattened or offset peaks and troughs. Fromm's, Beam-Warming and Lax-Wenddroff methods all performed equally best with no visible issues with error.

**PART THREE**

Figure 9 plots the characteristic correction of upwind states on the square wave, but without advancing the cell interfaces by a half timestep. Comparisons are made between a Courant Factor 0.5 and 0.8, where the equation of the Courant Factor is given by the following:

$$cfac = v\frac{\Delta t}{\Delta x}$$

Because the velocity and spatial step is held constant in coded units of v=1 and Δx=0.01, the timestep increases with increasing Courant Factor.

Similarly for comparison, Figure 10 plots the characteristic correction of upwind states on the square wave while using the half-timestep correction and Courant Factors of 0.5 and 0.8.

**Minmod scheme without advancing cell interfaces by a half-timestep**

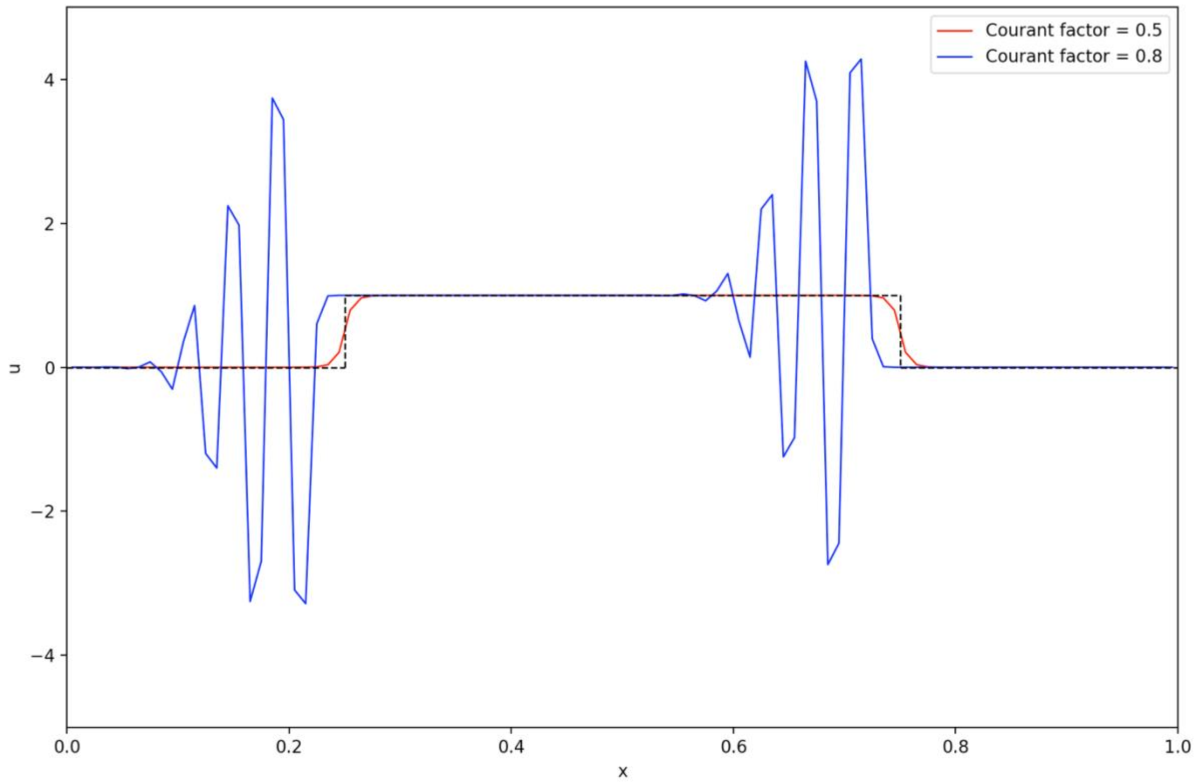


*Figure 9. The 1D advection equation modelled using discontinuous initial conditions of u=1 for 0.25<x≤0.75 and u=0 elsewhere. The black dotted line represents these initial conditions and provides a comparison of accuracy for the Minmod scheme. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and a Courant factor of 0.5 and 0.8.*

**Minmod limiter with half-timestep**

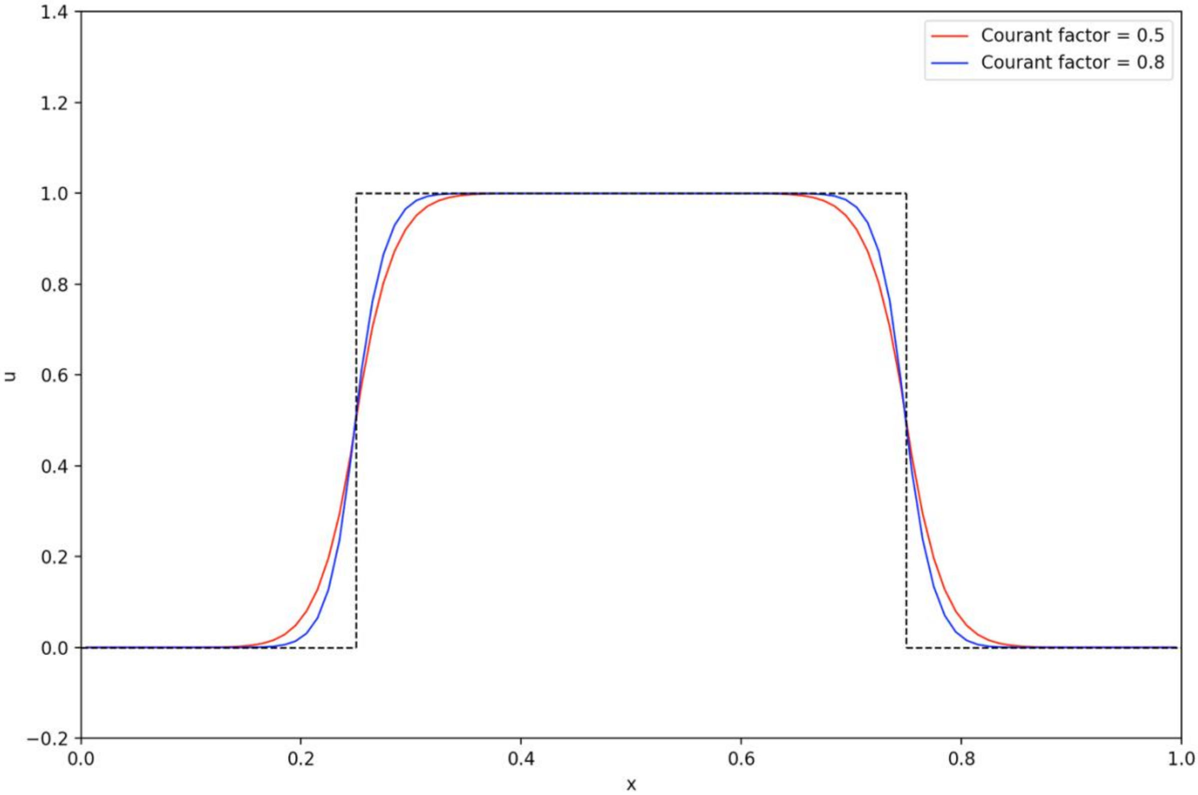

*Figure 10. The 1D advection equation modelled using discontinuous initial conditions of u=1 for $0.25<x\leq 0.75$ and u=0 elsewhere. The black dotted line represents these initial conditions and provides a comparison of accuracy for the Minmod scheme. The plot depicts the schemes at t=1 in coded units with 100 equally spaced points in the x direction, a velocity of 1.0 and Courant factors of both 0.5 and 0.8.*

Figure 9 shows that, without the half-timestep correction, we see stability only when using smaller timesteps. This is because without the half-timestep correction, the value of u at the cell interfaces is one whole timestep behind the solution we aim to calculate in the middle of the cell. However, with the half-timestep correction shown in Figure 10, the value of u at the cell interfaces is only half a timestep behind. Essentially, without the half-timestep correction the second-order time accurate scheme reverts to a first order scheme. Therefore, using *whole* timesteps must be counteracted by using *smaller* timesteps (i.e. smaller $\Delta t$) in order to achieve similar levels of accuracy and stability. However, one can get away with using larger timesteps while maintaining stability and accuracy by using the half-timestep correction, thereby making the characteristic correction of upwind states more computationally efficient with the half-timestep correction.

## CONCLUSION

The characteristic correction of upwind states was implemented with seven different slope calculation methods and tested on square waves, smooth waves, and without advancing the cell interface values by a half timestep. It was found that limiting schemes performed best on the square wavefronts due to their ability to resist spurious oscillations and instability, with the MC limiter performing best overall. The first-order Godunov method was stable but suffered from diffusion.

On smooth wavefronts, second-order non-limiting schemes performed best, with no observable error. All other schemes suffered from diffusion or spurious flatting or skewing of peaks and troughs.

Last, compared to not advancing cell interface values, advancing the cell interface values by a half timestep meant that larger timesteps can be used while maintaining a similar level or accuracy and stability. Without the half-timestep, the timestep must remain small to prevent instability in square waves, thereby resulting in a less computationally efficient system.