MA30287: Mathematics of Planet Earth Assessed Coursework

Climate change poses a significant challenge to our planet, and understanding the complex interactions among various climate parameters is essential in developing accurate models for prediction and mitigation. In this report, we delve into an in-depth exploration of the latitude-dependent Energy Balance Model (EBM). By conducting a series of investigations, we analyse both steady state and unsteady behaviour of key climate parameters and their impact on Earth's ice-age states and energy transport dynamics. Our investigative focus is centred on comprehending the roles played by the solar constant (Q), transport coefficient (k), critical temperature (T_C), albedo (a), and outgoing radiation (A & B) within Earth's climate system and the subsequent sensitivity of this system to variations in these parameters.

The resources utilised in this report encompass seminal works by Budyko (1969), Warren and Schneider (1979), and Cess (1976). Both Budyko's and Warren & Schneider's models signify initial efforts to simulate Earth's climate system. As time has progressed, climate models have developed greater sophistication. The parameters employed in modern climate models may diverge from those used by Budyko, Warren & Schneider, and Cess, demonstrating the growth in our knowledge of the climate system.
For investigations into steady state solutions, we used the following steady-state temperature equation

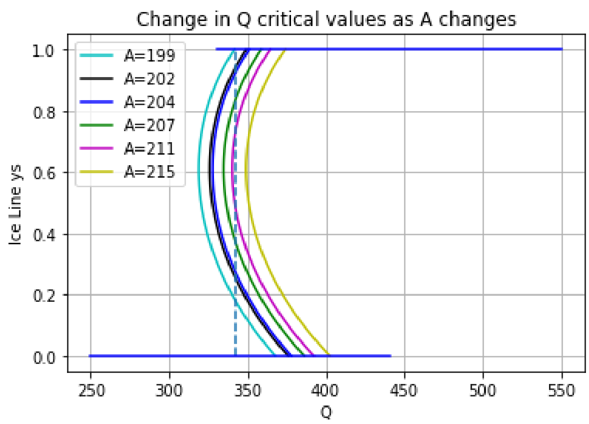
$$T^*\ (y)\ =\ \frac{k\overline{T^*} + Qs(y)(1-a(y)) - A}{B+k}$$

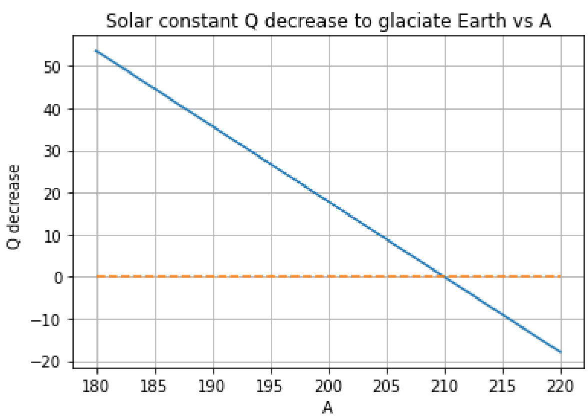
Investigation 1. Effect of parameters on the ice-age state.
We aim to explore changes in the ice line and determine critical values of the solar constant Q for specific states. Our initial goal is to identify the decrease in solar constant required for a partially-iced Earth to become fully glaciated, with the ice line reaching the Equator (y_s=0). Assuming the planet starts in a partially-iced state with an ice line near the North Pole, we use Q=342Wm^2 as it corresponds to an ice line around 0.95. Using the default values for k, A, and B in the EBM model, we calculated the decrease in solar constant needed to glaciate Earth by determining Q with y_s=0 using the equation:

$$Q\ =\ \frac{(T_c + \frac{A}{B})(B+k)}{s(y_s)[1-\frac{1}{2}(ai+aw\quad)] + \frac{k}{B}(1-\bar{a})}$$

Our result was **327.71 Wm^2**, indicating a decrease of **14.29 Wm^2** (or 4.18%) from the partially iced state. Next, we examined the increase required for the ice line to retreat from the Equator once the Earth is fully glaciated. We did this by gradually increasing the solar constant and noted how much of an increase was required before the ice retreated from the equator. We repeated this process several times to ensure the results were consistent. We found an increase of **48.02 Wm^2** (or 14.65%) is necessary for the ice line to recede.

With these critical values established, our objective is to investigate how variations in *k, A* and *B,* and the albedo formulation influence these critical values and their impact on the climate model.



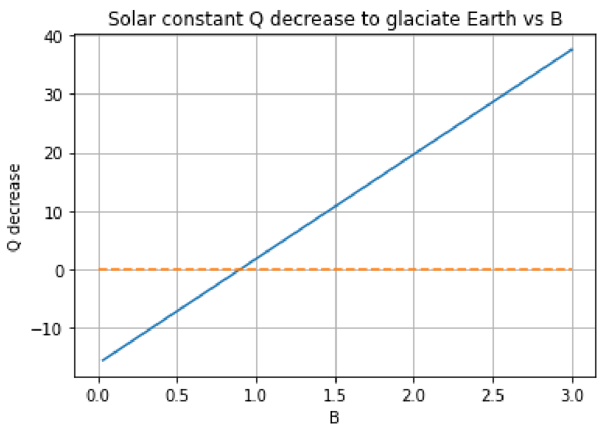


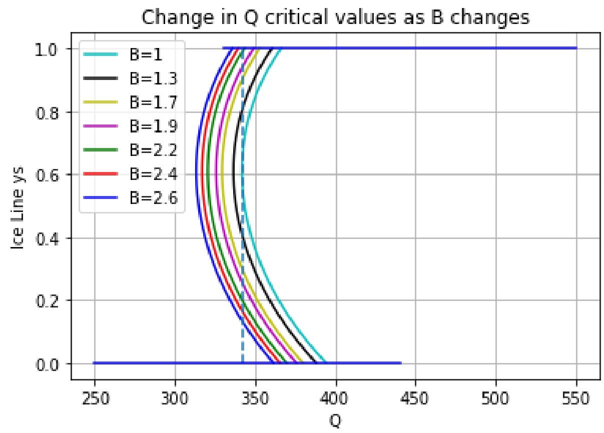
Solar constant Q's critical values sensitivity to the value of A

The first graph illustrates that changing A affects the position of the curve, but not the curvature of the bifurcation. As A increases, the minimum Q approaches our default Q of 342Wm^2, which means it is more sensitive to a decrease in the solar constant, minimising the decrease needed to glaciate Earth. For instance, at a specific value of A, around A=210Wm^2 glaciation occurs at our initial solar constant Q=342 Wm^2 without any further adjustments, indicated by the dotted line in the *Decrease vs A* graph. In this graph, a negative on the y-axis indicates an increase. Analysing various values of A from the first graph, we found that at A=198 Wm^2, a 46.97 Wm^2 increase is required for the ice line to begin retreating. At A=210 Wm^2, a 50.11 Wm^2 increase is needed. These values correspond to a 14.65% increase, consistent with the original value of A. This indicates that once Earth is fully glaciated, the value of A does not impact the increase required for the ice line to retreat.

Solar constant Q's critical values sensitivity to the value of B

Modifying B does not affect the curvature; however, decreasing B leads to an increase in the solar constant values. As B increases, a greater decrease is needed to glaciate Earth. This is illustrated in the two graphs. Similar to A, B also yields a critical value for Q, where Q=342 Wm^2, requires no increase or decrease for glaciation. This is depicted in the graph and occurs near B=1. Q is quite sensitive to changes in B values; an approximate 45% decrease in B results in glaciation at 342 Wm^2, while an approximate 4.95% increase in A leads to the same outcome. This suggests that the solar constant is more sensitive to A than B.

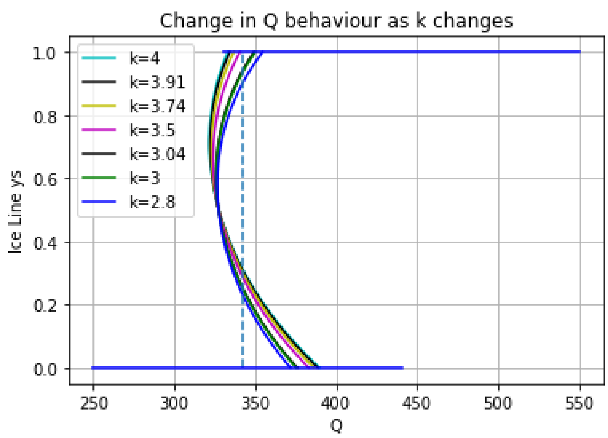


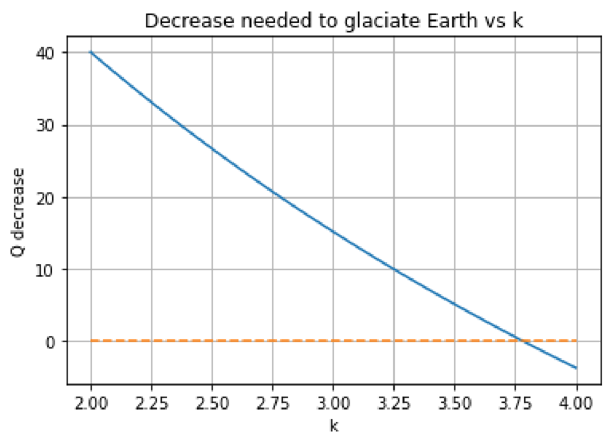


The range of increases needed for the ice line to retreat from B=1 to B=2.6 varies from 49.59Wm^2 to 46.18Wm^2 . As with A, these increases all correspond to a 14.65% increase, indicating that the increase required is not affected by varying B values.

Solar constant Q's critical values sensitivity to the value of k

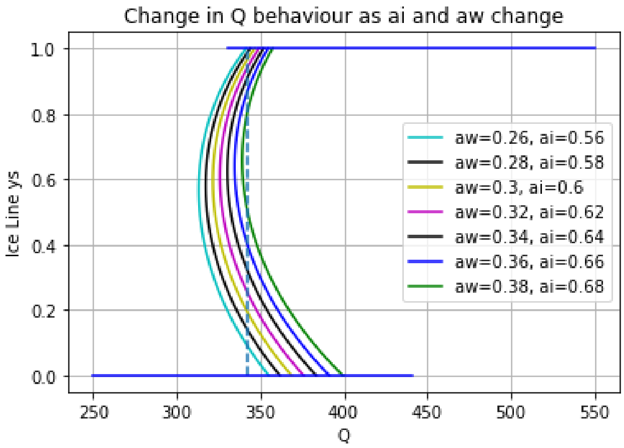
As illustrated in the graph, the behaviour of Q with respect to k is distinct from other variables since k also affects the curvature. The *Decrease vs k* graph shows that as *k* increases, the decrease required to glaciate Earth diminishes. Around k=3.75*,* which is a 23% increase from k=3.04, a solar constant of 342 Wm^2 would glaciate Earth, and any k above 3.75 would necessitate an increase for glaciation. The increase needed for the ice line to retreat ranges from 49.13Wm^2 to 44.30Wm^2 between k=2.8 and k=4. These increases correspond to a 15.24% and 12.81% change. Therefore, unlike other constants, *k* influences the increase required for the ice line to retreat, with the necessary increase decreasing as *k* increases.





<u>Solar constant Q's critical values sensitivity to the values of the albedo formulation</u>

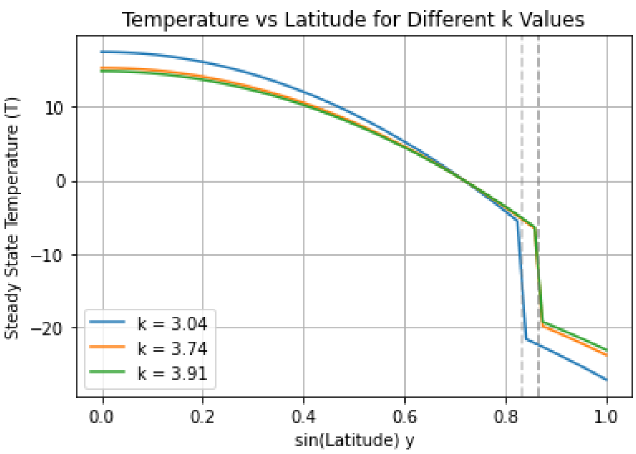
Albedo is a measure of how much energy a surface reflects. The albedo formulation involves two constants, a_i and a_w, allowing us to examine its changes in two different ways: changing a_i and a_w in the same direction (both increased or both decreased), or altering them differently. When one constant is increased and the other is decreased, they balance each other out when applying the $\frac{1}{2}(a_w+a_i)$ formulation. However, when both constants are changed in the same direction, the graph shows a modification in the curvature of the solar constant and the critical value. As a_w and a_i decrease, the critical value needed for glaciation drops, leading to an increased requirement for glaciation.



Once Earth is fully glaciated, the increase needed for the ice line to start retreating ranges from 37.85 Wm^2 (a_w=0.26, a_i=0.56) to 62.87 Wm^2 (a_w=0.38, a_i=0.68), corresponding to a 12.47% to 16.93% increase. Thus, modifying a_w and a_i affects the required increase for the ice line to retreat, with the necessary increase rising as a_i and a_w are increased.

Investigation 2. On the energy transport.

As mentioned earlier, Budyko's k=3.91$Wm^{-2}°C^{-1}$ value and Warren & Schneider's k=3.74$Wm^{-2}°C^{-1}$ value represent early attempts to understand and simulate the Earth's climate system, whilst default value k=3.04$Wm^{-2}°C^{-1}$ reflects the current climate system better. To explore the sensitivity of the model's climate prediction to the transport coefficient k, we varied k and observed how the climate predictions changed. We used the default values of albedo, A, B, and solar constant Q in the model for this investigation.

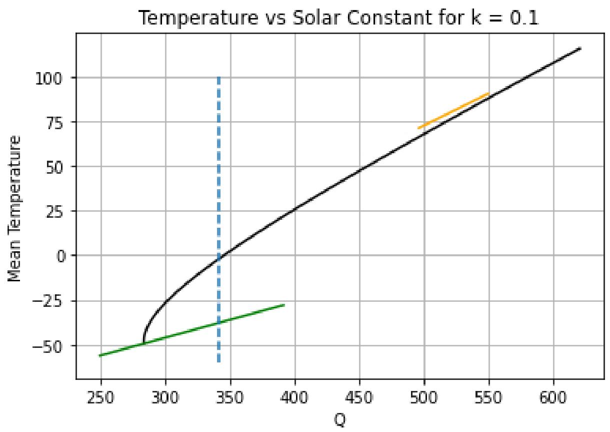


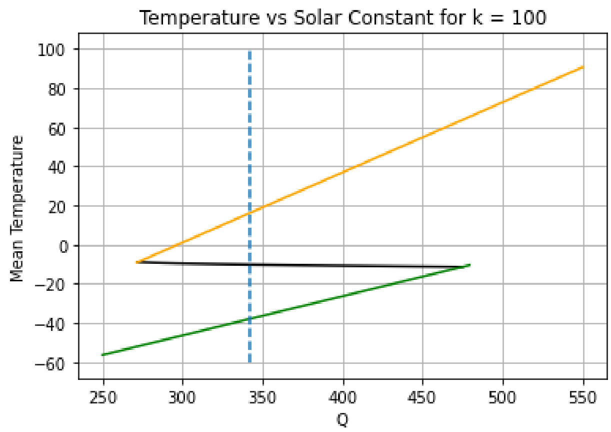
For all values of k, the temperature decreases from the equator (y = 0) towards the poles (y = 1). This is expected since the equator receives more solar radiation than higher latitudes. As the value of k increases, the temperature gradient between the equator and the poles becomes less pronounced. This indicates that a higher k value corresponds to more efficient heat transport from low to high latitudes, leading to a more uniform temperature distribution. The ice line moves towards the equator as k decreases. This means that when the heat transport is less efficient, the ice cover extends further towards the equator.

The transport coefficient, k, plays a vital role in balancing Earth's climate system energy. It pertains to physical properties governing energy and matter transport within the climate system, such as thermal conductivity, diffusivity, and viscosity.

To examine the climate variations in response to changes in the solar constant for very small k and very large k, we create separate graphs plotting the steady-state temperature against the solar constant (Q). This allows us to compare the sensitivities of the climate system for small k and large k values.

For our analysis, we've chosen k=0.1 to represent a small k and k=100 to represent a large k.





The graphs clearly display a notable difference in how the steady-state temperature changes as the solar constant increases between small k and large k. In each graph, a blue line represents the default solar constant of $342Wm^{-2}$.

When k is small, the temperature exhibits a marked increase as the solar constant rises, indicating high climate sensitivity to changes in the solar constant. The gradient becomes less steep near a solar constant of $342Wm^{-2}$, and continues to increase consistently beyond that point.
When k is large, temperature changes less dramatically, remaining relatively stable around the critical temperature of -10°C as the solar constant increases.

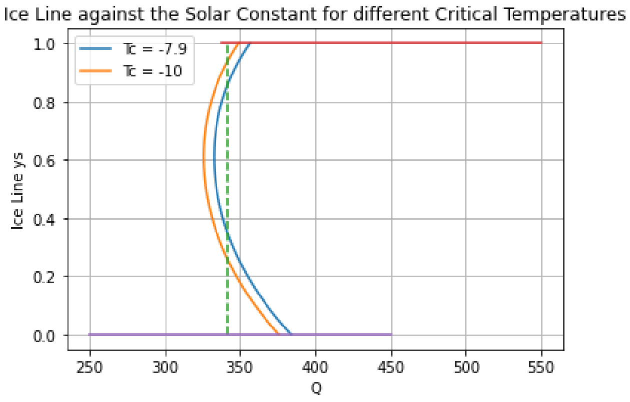
In summary, as k increases, heat transport becomes more efficient, stabilising temperature and resulting in a climate less sensitive to solar constant variations. A very small k creates a sensitive, unstable climate system, while a large k yields a stable, less sensitive system.

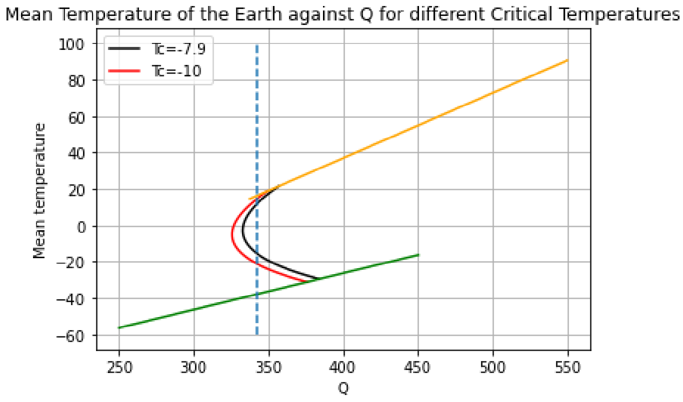
Physical principles can help explain the climate system's behaviour by highlighting the transport coefficient's impact on the atmosphere and ocean. Thus, real-world k values lie between these extremes and the actual climate system is influenced by various factors beyond heat transport efficiency alone.

Investigation 3. On the solar constant.

Observations show that the critical temperature for land is 0°C and for oceans -13°C. We will vary the critical temperature and investigate how the climate is affected by changes in the solar constant, while keeping the other parameter values to their defaults.

According to research (Webb, 2021) 71% of the Earth's surface is covered by water, so if we were to take an 'average' critical temperature for the Earth using the suggested critical values, we would find T_C=-9.23°C. This is very close to the default value of -10°C, so this is trivial to investigate. Instead, we can take the Northern hemisphere as an example, where 60.9% is covered by water. This gives our critical temperature of: -7.9°C. We can now find how solar radiation affects the ice line and average temperature of the Earth.





For our model of the Northern hemisphere, the two states for the ice line at Q=342Wm^2 converge towards each other from our original ice lines which have a critical temperature of -10°C.
The stable ice line moves from 0.940 to 0.8591 as the critical temperature changes from -10°C to -7.9°C, and the unstable ice line from 0.256 to 0.357.

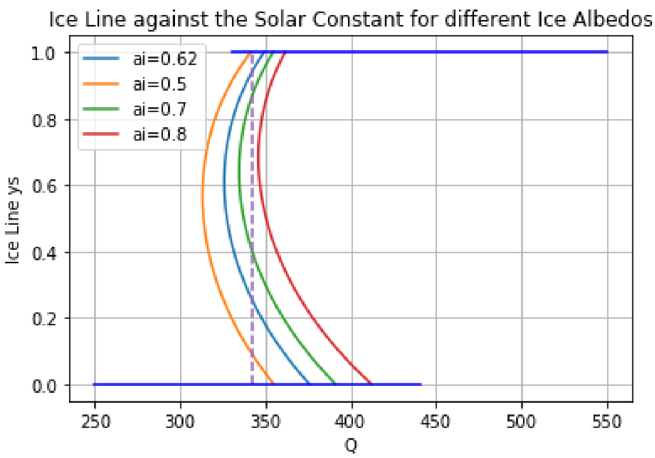
The distance between degrees of latitude is 69 miles (Dempsey, 2023), not allowing for variations due to the Earth's elliptical shape. The angle between the two steady state ice lines is 10.84 degrees, which means that the altered critical temperature for the Northern Hemisphere gives an ice line which is 748 miles further advanced than the default value.

The second graph shows us that the state for the average temperature of the Earth is slightly cooler for the new critical temperature of -7.9°C.

We'll now assess how different albedos for the ice covered areas of the Earth affect the climate.

A surface with a larger albedo will reflect more of the incoming radiation, absorbing less. This is an important factor in determining the Earth's climate, as a lower surface albedo will lead to the Earth retaining more energy and heating up. To see the effect of this on the ice line position, we will vary the ice albedo in the given range of 0.5-0.8 and use a similar code to above to plot the ice line for varying incoming radiation, Q.

Ice Albedo	**Stable Ice Line**	**Unstable Ice Line**
0.5	1.0039	0.0919
0.62	0.9395	0.2562
0.7	0.8657	0.4037
0.8	No Solution	No Solution



At an albedo of 0.5, with our default 342Wm^2 value of Q, the Earth is in an ice-free state, as the ice line is greater than 1. This is due to the surface of any ice absorbing enough energy that it doesn't reach the critical temperature of -10°C to form an ice line at any latitude.

At an albedo of 0.8 the Earth is in an total-ice state, as we can see the curve is to the right of the Q=342Wm^2 line. In between, we can see that the ice line advances as albedo increases from 0.5 to 0.8. Using the theory that the distance between degrees of latitude is 69 miles (2), the advancement in the ice line caused by a change in albedo of 0.62 to 0.7 would be 690 miles.

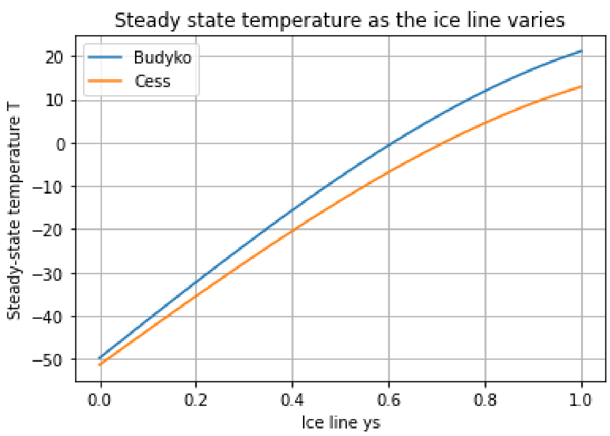
Investigation 4. On the outgoing radiation

Parameters A and B influence the Earth's temperature via outgoing radiation. We would expect that as outgoing radiation is varied from the model's steady state, increased energy emission cools the Earth and conversely, decreased emission raises temperature.

The climate's reaction to varying outgoing radiation can cause a positive feedback mechanism to occur. Cooling temperatures due to increased outgoing radiation encourages ice formation, thus increasing the amount of reflected incoming radiation and leading to further emitted radiation and cooling. This loop occurs similarly with a decrease in outgoing radiation: warming temperatures, diminishing ice cover, increased absorption of solar radiation and further warming.

These positive feedback mechanisms affect the stability of the climate when it is in an unstable steady state as these small perturbations are amplified, causing the climate to rapidly change towards a different equilibrium point. However a change in outgoing radiation can also be considered as a negative feedback mechanism. In the example of constants A, B that are dependent on the amount of clouds present, a rise in temperature could also increase the amount of cloud cover and solar radiation reflection, offsetting the initial perturbation and stabilising temperature.

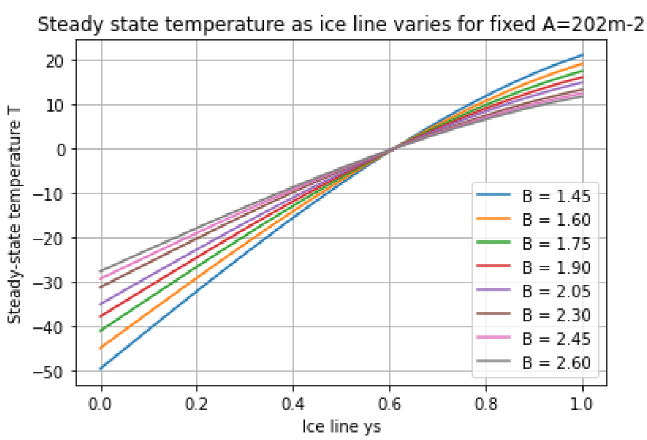
We investigated the effect of outgoing radiation on the climate by plotting the steady state temperature as ice line location varies from the equator to the North Pole for different A, B constants. We used Budyko's constants A = 202W m−2 and B = 1.45W m−2 ◦C −1 and Cess' constants A = 212W m−2 and B = 1.6W m−2 ◦C −1 to make this comparison.



The plot illustrates that the temperature of Earth rises as we transition from an ice covered state to an ice free one, with the ice line retreating from the equator to the North Pole for both Budyko and Cess' constants. As Cess' constants cause more outgoing radiation than Budyko's in the EBM, we generally observe a lower average temperature in comparison to Budkyo's as expected. These significantly differing profiles indicate the climate's sensitivity to varying outgoing radiation.

Effect on climate by only varying B

We also took note of the growing difference in temperatures (between the models across latitude) which implied the effect of a feedback mechanism. We investigated this by fixing A, as it is not dependent on temperature nor latitude, and varying B - which does.



Since A is fixed and not dependent on these variables, all models share the same ice line location when the steady state temperature is 0 Celsius. We observed that as B increased, the steady state temperature at the equator ice line decreased and increased at the North Pole ice line. Furthermore the temperature difference between these models were more prevalent near the equator and the poles, plotting a polarised temperature distribution. This implied that B physically interprets the strength of the ice-albedo feedback mechanism.

In comparison to Cess' larger constant A=212Wm-2 in the previous plot, the sensitivity of the climate appears to be less significant to changes in B. This suggests that the strength of the ice-albedo feedback mechanism impacts climate sensitivity more when outgoing radiation is lower.

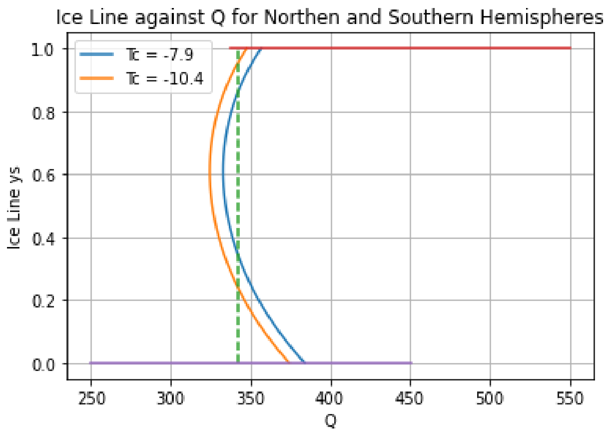
Investigation 5. Steady and unsteady states

We want to investigate the possibility of non-symmetric steady states for the ice line about the equator:

The proportion of land and ocean differs from the northern and southern hemisphere. The northern hemisphere is approximately 40% land, the southern hemisphere is approximately 20% land (1). As a result the hemispheres will have different average critical temperatures, leading to non-symmetric ice lines about the equator. Using the values from investigation 3 for the critical temperature of land and ocean of 0°C and -13°C respectively, we can see that the average critical temperature for the hemispheres are:

$$T_C\,\text{north} = (0 \times 0.4) - (13 \times 0.6) = -7.8°C \qquad T_C\,\text{south} = (0 \times 0.2) - (13 \times 0.8) = -10.4°C$$

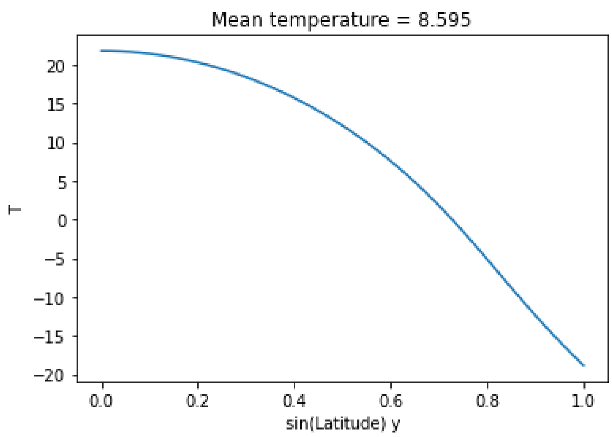
We can now use these values to find the ice lines for each hemisphere using the same method as in investigation 3, given in the graph.



We can see that the ice line for the southern hemisphere is closer to the pole than the northern hemisphere. In the same way as was calculated in investigation 3, the ice line is modelled to be 370 miles further from the pole in the Northern hemisphere than in the Southern hemisphere.

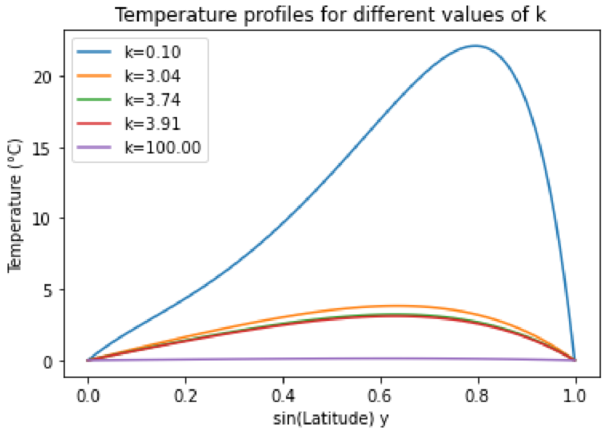
However, this is not the case. In fact, 90% of the Earth's ice is at the South Pole (Gallo, 2021), due to Antarctica being land and the arctic circle is sea ice. This shows the shortcomings of this modelling approach, it is not good enough to just take an average land proportion for the Earth or the hemispheres, we must take into account specifically where the land mass is.

The temperature-dependent albedo function in the original model assumes only three possible albedo states, occurring when the temperature is above, below, or exactly at the critical temperature of -10°C. However, in real life, there is a gradual transition between water and ice albedos as temperature changes. The discontinuous model exhibits a sharp shift between ice and water albedos, resulting in a steep jump in the temperature profile at the critical temperature. This is exhibited in the graph of investigation 2. We aim to explore the implications of utilising a smooth, continuous albedo function on the typical steady-state solutions and their overall behaviour within the climate system.



We are interested in seeing how this steady state solution changes when we implement a continuous albedo function instead. As demonstrated in the graph below, the continuous albedo function leads to a much smoother transition between the albedos. There is no drastic shift at the critical temperature, and the temperature profile appears more consistent with real-world physical processes. This smoother temperature profile reflects the fact that the shift in albedo is not instantaneous or perfect in the real world. The continuous albedo function can potentially enhance the reliability and accuracy of climate model predictions.

To further investigate the stability properties as value k changes, we explored the temperature profiles for different values of k.



The graph shows that as k increases, the change in temperature across the latitude becomes smaller and more consistent. A uniform temperature distribution suggests that the climate system is less sensitive to perturbations, indicating a more stable climate system.

Conversely, as for the very small value of k, the temperature becomes much less steady as it increases constantly from 0°C to about 22°C and drops suddenly as it approaches the north pole.

In conclusion, the climate system becomes more stable as k increases, and becomes less stable as k decreases.

In conclusion, our investigations indicate that the Earth’s climate is highly sensitive to changes in the parameter. It is important to take into account the current physical properties to address climate change effectively. In summary, our key findings include:

- ➔ Critical ice-age values are highly sensitive to parameters, with solar constant Q being more responsive to A than B.
- ➔ Heat transport coefficient k influences the solar constant, affecting the ice line and climate stability.
- ➔ Varying critical temperature and ice albedo significantly impact the Earth's climate, influencing average temperatures and ice line advancement by hundreds of miles.
- ➔ Parameters A and B affect climate stability and sensitivity through outgoing radiation, with higher B values leading to stronger feedback mechanisms.
- ➔ A more realistic model should account for landmass distribution and employ a continuous albedo function for enhanced accuracy.

Our research highlights the need to examine climate sensitivity to refine models, inform policy making, and develop targeted interventions. This understanding also helps identify critical thresholds and potential tipping points, which, if crossed, could have abrupt and irreversible consequences. In essence, grasping the sensitivity of Earth's climate to various parameters is a cornerstone of global efforts to address climate change and protect our planet's future.

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