9.2.9

EE24BTECH11020 - Ellanti Rohith

Question: Find the solution of the differential equation $y^2y' + y^2 + 1 = 0$ and verify if $x + y = \tan^{-1}(y)$ is a solution.

Theoretical Solution:

Given the differential equation:

$$y^2\frac{dy}{dx} + y^2 + 1 = 0 \tag{0.1}$$

Rearranging:

$$y^2\frac{dy}{dx} = -y^2 - 1 \tag{0.2}$$

Dividing both sides by y^2:

$$\frac{dy}{dx} = -1 - \frac{1}{y^2} \tag{0.3}$$

This simplifies to:

$$\frac{dy}{-1 - \frac{1}{y^2}} = dx \tag{0.4}$$

$$\frac{-y^2 dy}{y^2 + 1} = dx \tag{0.5}$$

Adding and subtracting 1 in the numerator of LHS:

$$\frac{(-y^2 + 1 - 1)dy}{y^2 + 1} = dx \tag{0.6}$$

$$-dy + \frac{dy}{y^2 + 1} = dx \tag{0.7}$$

Integrating both sides:

$$\int -dy + \frac{dy}{y^2 + 1} = \int dx \tag{0.8}$$

$$-y + \tan^{-1}(y) = x + C \tag{0.9}$$

Let the initial condition be $x_0 = 0$, $y_0 = 0$, then $C = 0$, and we have:

$$x + y = \tan^{-1}(y) \tag{0.10}$$

Thus, $x + y = \tan^{-1}(y)$ is indeed a solution.
Simulated solution:
The difference equation is given by:

$$y_{n+1} = y_n + h\frac{dy}{dx} \tag{0.11}$$

$$y_{n+1} - y_n = h\frac{dy}{dx} \tag{0.12}$$

Given that:

$$\frac{dy}{dx} = -1 - \frac{1}{y^2} \tag{0.13}$$

The difference equation becomes:

$$y_{n+1} = y_n + (-1 - \frac{1}{y^2})h \tag{0.14}$$

However, this method provides inaccurate results since $\frac{dy}{dx} \to \infty$ as $y \to 0$. To fix this, we use the finite differences algorithm as follows:
The difference equation is:

$$y_{n+1} = y_n + h \tag{0.15}$$

$$x_{n+1} = x_n + h\frac{dx}{dy} \tag{0.16}$$

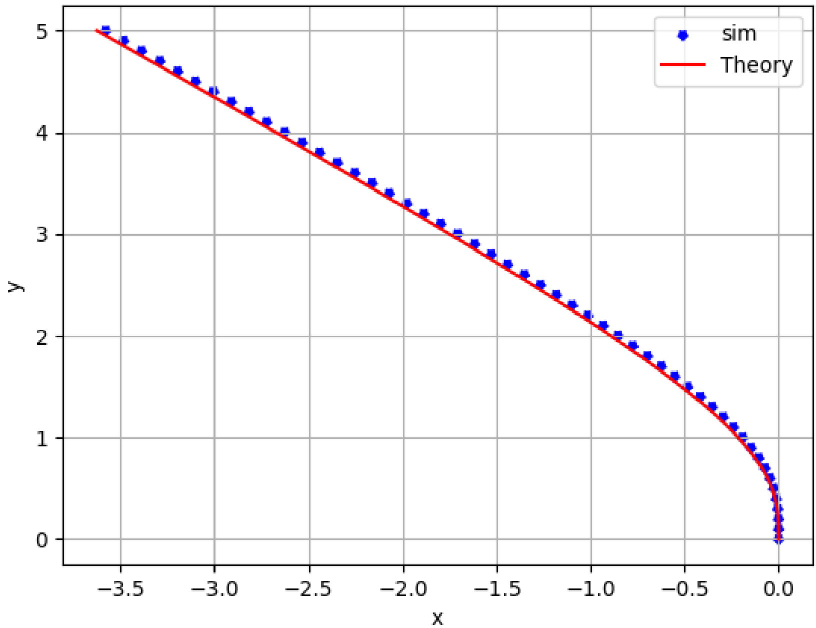
Given that:

$$\frac{dx}{dy} = \frac{-y^2}{1 + y^2} \tag{0.17}$$

The difference equation becomes:

$$x_{n+1} - x_n = \frac{-hy^2}{1 + y^2} \tag{0.18}$$

Taking initial point as $x_0 = 0, y_0 = 0$ and iterating difference equation.
Verifying Algoritham,



Thus, We have numerically verified $x + y = \tan^{-1}(y)$ is a solution to given differential equation.