2020-MA

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1) Let $\mathcal{P}(\mathbb{R})$ denote the power set of $\mathbb{R}$, equipped with the metric
$$d(U,V) = \sup_{x\in\mathbb{R}} |\chi_U(x) - \chi_V(x)|,$$
where χ_U and χ_V denote the characteristic functions of the subsets U and V, respectively, of $\mathbb{R}$. The set $\{\{m\} : m \in \mathbb{Z}\}$ in the metric space $(\mathcal{P}(\mathbb{R}), d)$ is [GATE 2020]
   a) bounded but not totally bounded
   b) totally bounded but not compact
   c) compact
   d) not bounded
2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n}\chi_{(n,n+1)}(x),$$
where $\chi_{(n,n+1)}$ is the characteristic function of the interval $(n, n+1]$. For $\alpha \in \mathbb{R}$, let $S_\alpha = \{x \in \mathbb{R} : f(x) > \alpha\}$. Then [GATE 2020]
   a) $S_{\frac{1}{2}}$ is open
   b) $S_{\frac{\sqrt{3}}{2}}$ is not measurable
   c) S_0 is closed
   d) $S_{\frac{1}{\sqrt{2}}}$ is measurable
3) For $n \in \mathbb{N}$, let $f_n, g_n : (0,1) \to \mathbb{R}$ be functions defined by
$$f_n(x) = x^n \quad \text{and} \quad g_n(x) = x^n(1-x).$$
Then [GATE 2020]
   a) $\{f_n\}$ converges uniformly but $\{g_n\}$ does not converge uniformly
   b) $\{g_n\}$ converges uniformly but $\{f_n\}$ does not converge uniformly
   c) both $\{f_n\}$ and $\{g_n\}$ converge uniformly
   d) neither $\{f_n\}$ nor $\{g_n\}$ converge uniformly
4) Let u be a solution of the differential equation $y' + xy = 0$ and let $\phi = u\psi$ be a solution of the differential equation $y'' + 2xy' + \left(x^2+2\right)y = 0$ satisfying $\phi(0) = 1$ and $\phi'(0) = 0$. Then $\phi(x)$ is [GATE 2020]
   a) $\left(\cos^2 x\right)e^{-\frac{x^2}{2}}$
   b) $(\cos x)\, e^{-\frac{x^2}{2}}$
   c) $\left(1+x^2\right)e^{-\frac{x^2}{2}}$
   d) $(\cos x)\, e^{-x^2}$
5) For $n \in \mathbb{N} \cup \{0\}$, let y_n be a solution of the differential equation
$$xy'' + (1-x)y' + ny = 0$$
satisfying $y_n(0) = 1$. For which of the following functions $w(x)$, the integral
$$\int_0^{\infty} y_p(x)\, y_q(x)\, w(x)\, dx, \quad (p \neq q)$$
is equal to zero? [GATE 2020]

a) e^{-x^2}
b) e^{-x}
c) xe^{-x^2}
d) xe^{-x}

6) Suppose that

$$X = \{(0,0)\} \cup \left\{\left(x, \sin\frac{1}{x}\right) : x \in \mathbb{R} \setminus \{0\}\right\}$$

and

$$Y = \{(0,0)\} \cup \left\{\left(x, x\sin\frac{1}{x}\right) : x \in \mathbb{R} \setminus \{0\}\right\}$$

are metric spaces with metrics induced by the Euclidean metric of $\mathbb{R}^2$. Let B_X and B_Y be the open unit balls around $(0,0)$ in X and Y, respectively. Consider the following statements: [GATE 2020]

I. The closure of B_X in X is compact.
II. The closure of B_Y in Y is compact.

a) both I and II are true
b) I is true II is false
c) I is false II is true
d) both I and II are false

7) If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a function such that $f(z) = f\left(\frac{z}{|z|}\right)$ and its restriction to the unit circle is continuous, then [GATE 2020]

a) f is continuous but not necessarily analytic
b) f is analytic but not necessarily a constant function
c) f is a constant function
d) $\lim\limits_{z\to 0} f(z)$ exists

8) For a subset S of a topological space, let $\operatorname{Int}(S)$ and $\overline{S}$ denote the interior and closure of S, respectively. Then which of the following statements is TRUE? [GATE 2020]

a) If S is open, then $S = \operatorname{Int}(S)$
b) If the boundary of S is empty, then S is open
c) If the boundary of S is empty, then S is not closed
d) If $\overline{S} \setminus S$ is a proper subset of the boundary of S, then S is open

9) Suppose $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ are the smallest topologies on $\mathbb{R}$ containing S_1, S_2 and S_3, respectively, where

$$S_1 = \left\{\left(a, a + \frac{\pi}{n}\right) : a \in \mathbb{Q}, n \in \mathbb{N}\right\},$$

$$S_2 = \{(a,b) : a < b,\ a,b \in \mathbb{Q}\},$$

$$S_3 = \{(a,b) : a < b,\ a,b \in \mathbb{R}\}.$$

Then [GATE 2020]

a) $\mathcal{T}_3 \supseteq \mathcal{T}_1$
b) $\mathcal{T}_3 \supseteq \mathcal{T}_2$
c) $\mathcal{T}_1 = \mathcal{T}_2$
d) $\mathcal{T}_1 \subseteq \mathcal{T}_2$

10) Let $M = \begin{pmatrix} \alpha & 3 & 0 \\ \beta & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Consider the following statements:

I: There exists a lower triangular matrix L such that $M = LL^t$, where L^t denotes the transpose of L.
II: Gauss-Seidel method for $Mx = b$ $\left(b \in \mathbb{R}^3\right)$ converges for any initial choice $x_0 \in \mathbb{R}^3$.

Then:

[GATE 2020]

a) I is not true when $\alpha > \frac{9}{2}, \beta = 3$
b) I is true when $\alpha = 5, \beta = 3$
c) II is not true when $\alpha > \frac{9}{2}, \beta = -1$
d) II is not true when $\alpha = 4, \beta = \frac{3}{2}$

11) Let I and J be the ideals generated by $\left\{5, \sqrt{10}\right\}$ and $\left\{4, \sqrt{10}\right\}$ in the ring $\mathbb{Z}[\sqrt{10}] = \left\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\right\}$, respectively. Then: [GATE 2020]

a) both I and J are maximal ideals
b) I is a maximal ideal but J is not a prime ideal
c) I is not a maximal ideal but J is a prime ideal
d) neither I nor J is a maximal ideal

12) Suppose V is a finite dimensional vector space over $\mathbb{R}$. If W_1, W_2 and W_3 are subspaces of V, then which of the following statements is **TRUE**?

[GATE 2020]

a) If $W_1 + W_2 + W_3 = V$ then $\text{span}(W_1 \cup W_2) \cup \text{span}(W_2 \cup W_3) \cup \text{span}(W_3 \cup W_1) = V$
b) If $W_1 \cap W_2 = \{0\}$ and $W_1 \cap W_3 = \{0\}$, then $W_1 \cap (W_2 + W_3) = \{0\}$
c) If $W_1 + W_2 = W_1 + W_3$, then $W_2 = W_3$
d) If $W_1 \neq V$, then $\text{span}(V \setminus W_1) = V$

13) Let $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$. The system

$$\begin{aligned} x_1 - 2x_2 + \alpha x_3 &= 8 \\ x_1 - x_2 + x_4 &= \beta \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

has NO basic feasible solution if:

[GATE 2020]

a) $\alpha < 0, \beta > 8$
b) $\alpha > 0, 0 < \beta < 8$
c) $\alpha > 0, \beta < 0$
d) $\alpha < 0, \beta < 8$