Limit Equilibria of Uni-dimensional Bargaining Games: A Core Refinement*

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Abstract

In uni-dimensional policy space, the median voter theorem provides a unique equilibrium characterization under simple majority rule. But for general decision procedures, the core is generically non-unique. We make a selection from the core by taking the limit equilibrium of a standard bargaining game as players become perfectly patient. Our refinement is characterized by asymmetric Nash bargaining with endogenous bargaining weights, between two players whose identities depend on the decision rule. The limit policy often (but not always) coincides with the ideal policy of some agent, not necessarily the median. The model provides foundations for endogenous separation of agents into cohesive voting blocs represented by non-median factional leaders.

Key Words: Bargaining, Endogenous Factions, Core, Equilibrium Refinement, Supermajority Rules.
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1 Introduction

Since the seminal works of Black (1948) and Downs (1957), a long literature has developed that applies the median voter theorem to study decision-making by committees under simple majority rule in a uni-dimensional policy environment. (See, for example, Meltzer and Richard (1981), Epple and Romer (1991), Besley and Coate (1997), amongst many others.) In many settings, however, decisions are made using procedures other than simple majority rule. For example, a Committee's decision rule may require a larger degree of consensus (i.e. super-majority rule), or may confer veto power (either absolute or qualified) upon some subset of players, or – as in the case of the U.S. legislative process – may involve some combination of the two (see Krehbiel (2010)). Notwithstanding their prevalence, there has been relatively little work on outcomes when the committee deliberates under these alternative rules (although, see Banks and Duggan (2000), McCarty (2000), and Winter (1996)). A significant reason for this disparity is that, for general decision procedures, the *core*[1] is generically non-unique, and existing theory provides no guidance about how to select from amongst the multiplicity.

In this paper, we provide bargaining foundations that select a unique focal policy from within the core. We use the canonical bargaining framework of Baron and Ferejohn (1989) to model committee behavior under a broad class of decision-making procedures, and characterize the policy that is chosen in the limit as the cost of delay is made arbitrarily small. This limit equilibrium has a simple characterization: it is the outcome of bilateral asymmetric Nash bargaining between two particular agents (whom we call the *left* and *right decisive agents*, and whose identities depend on the decision rule) with endogenous weights that depend on the ideal policies of all players and their respective recognition probabilities in the bargaining game. Under some arrangements, but not always, this core policy will coincide with the ideal policy of some agent, potentially, but not necessarily, the median.

[1]The *core* is the set of unbeatable policies given the decision rule. In a one-dimensional world with simple majority rule, the *core* is simply the ideal policy of the median agent.

Our bargaining approach is motivated by the insight that, when the cost of delay is exactly zero, every policy in the core can be sustained as an equilibrium of the bargaining game. By contrast, we show that, whenever delay has some positive cost, the game admits a unique equilibrium, and that in the limit as these costs tend to zero, equilibrium proposals converge to a focal policy. This policy is the unique policy in the core that is robust to introducing small costs to bargaining delay. Our result is similar Cho and Duggan (2009), who show that the limit equilibrium under simple majority rule selects the ideal policy of the median voter. Thus, we provide an analogue to the median voter theorem for generic decision rules.

Interestingly, we show that the focal policy is chosen as if the players endogenously separate into two factions led by the left and right decisive voters, and where policy is determined by bilateral asymmetric Nash bargaining between the factional leaders, with bargaining weights proportional to faction size. Thus, our model also provides foundations for the endogenous emergence of two factions, whose members have heterogeneous preferences, but nevertheless delegate policy bargaining to non-median factional leaders.

Examples of situations in which this framework has applications abound. In the legislative context, many jurisdictions operate under simple majority rule, but endow the executive or the judiciary, with a (qualified) veto. The United Nations Security Council is a particularly interesting case, where a simple majority of the fifteen member body is required, but each of the five permanent member nations can exercise an absolute veto. Similarly, many countries (as well as the European Union) have bicameral legislatures and require majorities in both chambers to pass legislation. In the United States, this feature is enhanced by effectively requiring a three-fifths super-majority to pass legislation in the Senate. Many countries require a super-majorities in the legislature to amend the Constitution, and in federal systems, there is often a requirement for a 'double-majority' (i.e. majorities in a majority of states/provinces). In the context of changes to monetary policy, some central banks have consensus requirements, and others endow the chairman with an absolute veto (see Riboni and Ruge-Murcia (2010)). In a quite different context, there are strong incentives for judges to achieve higher levels of consensus when adjudicating on panels. Moreover, Parameswaran, Cameron and Kornhauser (2018) argue that, in the context of the U.S. Supreme Court, whenever decisions are non-unanimous, the requirement that an opinion be joined by a simple majority of justices to establish precedent creates an effective super-majority requirement amongst justices in the dispositional majority.

Let $X \subset \mathbb{R}$ be a uni-dimensional policy space. There is a committee or legislature consisting of n people that may adopt a policy $x \in X$ according to some decision procedure d. As is standard in the literature, agents are assumed to have preferences that are concave and single-peaked over the policy space. We specialize these preferences further by assuming that agents' preferences satisfy the Spence-Mirrlees condition (i.e. the single-crossing property)[2]. This assumption embeds, as a special case, models where agents' preferences are characterized by a common convex loss function (e.g. absolute value loss or quadratic loss) centered at each agent's ideal policy. Such preferences are commonly used in the applied literature.

Given the Spence-Mirrlees condition, it can be shown that, in equilibrium, the coalitions that support and reject any proposal are both connected. This implies that there will be a connected coalition of agents that supports *every* equilibrium proposal. We refer to the agents at the boundaries of this subset as the *left* and *right* decisive agents[3], respectively. For example, under q-majority rule (with $q > \frac{1}{2}$), these are the players who are q^{th} from the right and left, respectively. We show that the Spence-Mirrlees condition guarantees the existence of a unique stationary sub-game perfect equilibrium. Predtetchinski (2011) shows that as $\delta \to 1$, equilibrium proposals in generic uni-dimensional bargaining games that embed our framework converge to a unique limit.[4] We can interpret this limit alternatively as the limit as agents become arbitrarily patient, or as delay becomes costless, or as the probability of

[2]Formally, let i and$j$ be two agents with ideal policies $x_i > x_j$ Then $u_i'(x) \geq u_j'(x)$ for each $x \in X$.

[3]Cardona and Ponsati (2011) refer to them as the *left* and *right boundary players*.

[4]In fact, Predtetchinski (2011) shows more strongly that, absent the single-crossing property when there are generically multiple equilibria, proposals in all equilibria converge to same limit.

exogenous breakdown in negotiations vanishes.

In this paper, we provide a simple and intuitive characterization of this limit equilibrium. We show that the limit equilibrium policy is characterized by asymmetric bilateral Nash bargaining between the left and right decisive agents, with endogenous bargaining weights. The bargaining weights depend on the recognition probabilities of all agents, and the spatial alignment of their ideal policies. Over a non-trivial range of recognition probabilities, the limit is shown to coincide with the ideal policy of *some* agent, provided this policy lies between the ideal policies of the left and right decisive agents. As in Cho and Duggan (2009), we show that under simple majority rule with an odd number of players, the limit equilibrium always selects the median agent's ideal policy. However, for generic decision procedures, the pivotal agent need not be the median. Indeed, under unanimity rule, there is some arrangement of recognition probabilities and ideal policies that makes every agent, other than the most extreme ones, pivotal.

Interestingly, we show that the limit equilibrium coincides with the equilibrium of a simple *faction formation* game. In this alternative conception, each agent must choose how to allocate her support between two factions L and R, led by the left and right decisive agents. The resulting policy is determined by bilateral asymmetric Nash bargaining between the factional leaders, with weights proportional to the (recognition probability weighted) sizes of each faction. Hence, the choice of which faction to support affects its bargaining strength in the policy making stage, and thus, the policy that results. Naturally, in equilibrium, most agents will give their entire support to one faction or the other — whichever would pull the policy closer to their ideal. The limit policy will coincide with the ideal policy of some agent just in case *that* agent finds it optimal to split her support between the two factions. Otherwise, the equilibrium policy will be the one that results from Nash Bargaining when every agent allocates all of their support to one faction or the other. Hence, our model provides micro-foundations for the endogenous separation of heterogeneous actors into two groups (e.g. left and right blocs on the bench, political parties or factions in the legislature, inflation 'hawks' and 'doves' on monetary policy committees etc.) that behave cohesively. Furthermore, it explains which leaders emerge within each faction.

Our focus on the limit equilibrium is motivated by a desire to make fair comparisons between the outcomes of bargaining models, and approaches that predict outcomes in the core.[5] The latter implicitly assume that non-core outcomes would be replaced, which is consistent with counter-proposals not being too expensive. Indeed, Banks and Duggan (2006) show that, when $\delta = 1$, every policy in the core can be sustained as an equilibrium of a bargaining game in no-delay. However, as we will show, the bargaining model admits a unique equilibrium whenever delay has some positive cost. The equilibrium correspondence exhibits a failure of lower-hemicontinuity at $\delta = 1$, and this presents a natural opportunity for equilibrium selection. The limit equilibrium selects the unique focal equilibrium from amongst the core that is robust to making counter-proposals slightly costly.

The intuition for the main result is as follows: we have already argued that, in equilibrium, any policy that is accepted must have the support of both the left and right decisive voters. Let $\underline{x}$ be the lowest policy that the right decisive agent will accept, and $\overline{x}$ be the highest policy that the left decisive agent will accept. Then, the equilibrium acceptance set is the interval $[\underline{x}, \overline{x}]$. Given the convergence result in Predtetchinski (2011), it follows that $\lim_{\delta \to 1} \underline{x} = \mu = \lim_{\delta \to 1} \overline{x}$. Let x_j be the ideal policy of player j and conjecture that $\mu \in (x_{i-1}, x_i)$ for some i. Then, for δ large enough, $x_{i-1} < \underline{x} < \overline{x} < x_i$, and so, in equilibrium, agents $1, ..., i-1$ will propose $\underline{x}$ and agents $i, ..., n$ will propose $\overline{x}$. It is as if the players separate into two factions, with all members of the same faction making the same equilibrium proposals. Moreover, by construction, $\underline{x}$ and $\overline{x}$ were jointly determined by the preferences of the left and right decisive agents. Hence, we can think of the left decisive agent as bargaining on behalf of the

[5] In doing so, we do not dispute the reality that decision-makers may exhibit (significant) present bias. However, we point to the ubiquity of median voter reasoning, and note that median voter results are equivalent to assuming decision-making by perfectly patient agents. Hence, to the extent that one finds the median voter logic compelling, one must also accept the validity of models with patient legislators.

left faction, and the right decisive agent as bargaining on behalf of the right faction, with the bargaining weights being proportional to the recognition probabilities of all of the agents in their respective factions. As $\delta \to 1$, this coincides with the asymmetric Nash bargaining solution between the left and right decisive agents (see Binmore, Rubinstein and Wolinsky (1986)).

Notice that the separation into factions was endogenous to the equilibrium. Hence, for the asymmetric Nash bargaining solution to be equilibrium consistent, it must be that this solution indeed lies in the interval (x_{i-1}, x_i) — so that players separate into the factions as conjectured. We show, in the paper, that there is a unique player i^* that determines the composition of factions, in equilibrium. There are two possibilities. For a range of recognition probabilities and ideal policy arrangements, the factions $L = \{1, ..., i^* - 1\}$ and $R = \{i^*, ..., n\}$ induce a recognition probability weighted Nash Bargaining solution that is equilibrium consistent — i.e. which falls in the required interval (x_{i^*-1}, x_{i^*}). If so, the limit equilibrium is precisely this Nash Bargaining solution. Outside this range of parameters, the following problem arises: If i^* is conjectured to be in the left faction, then the location of the induced Nash Bargaining solution will cause i^* to defect to the right faction, and vice versa. Player i^* is *pivotal*. The only possibility is that the limit equilibrium coincides with i^*'s ideal policy, x_{i^*}.

We provide comparative statics on the behavior of limit equilibria. Several results are particularly noteworthy. First, we examine the effect of proposal power on equilibrium outcomes. Fixing the profile of ideal policies, we show that any policy in the core can be achieved as the limit equilibrium for a suitably chosen vector of recognition probabilities. In particular, any player whose ideal policy lies within the core may be pivotal under some arrangement of recognition probabilities. However, if recognition probabilities are bounded away from zero (e.g. if players are recognized with equal probability), then only 'moderate' policies in the core — those not too close to either x_l or x_r — can be implemented.

Second, we perform comparative statics on the decision rule, by focusing on the special case of q-majority rules, and examine how equilibrium policies vary with the size of the required (super)-majority. Since policy outcomes are sensitive to the arrangement of players' ideal policies, we assume preference profiles are drawn from some distribution, and examine the statistical properties of equilibrium outcomes. As the super-majority requirement becomes more demanding, we find that the limit equilibrium becomes concentrated within a narrower band of policies. Larger super-majority requirements generate less extreme and less dispersed equilibrium outcomes.

Third, as an extension, we consider an alternative specification of the model, where policy reverts to a 'status quo' following disagreement, and where the status-quo need not be the worst possible outcome. We examine the effect of changing the reversion policy on equilibrium outcomes. We show that, when the status quo lies outside the core, a change that makes the reversion policy more extreme, will cause the equilibrium policy to become more moderate (in the sense of moving further into the middle of the core).

This paper builds on existing work that studies bargaining in a one-dimensional framework. Banks and Duggan (2000), extending Baron and Ferejohn (1989), show, in a framework that embeds ours, that no-delay equilibria exist and that equilibria must be in no-delay, when $\delta < 1$. Banks and Duggan (2006) extend the analysis to scenarios where disagreement involves reversion to a status quo policy. Cho and Duggan (2003) show that stationary equilibria in a one-dimensional framework are unique when preferences are quadratic, but that multiple pay-off variant equilibria may exist for generic concave preferences. Limiting attention to q-majority rules, Cardona and Ponsati (2011) show that equilibria will be unique if preferences are characterized by a common loss function centered at each agent's ideal policy. We generalize Cardona and Ponsati's result by both allowing for more general decision rules and requiring only that preferences satisfy the Spence-Mirrlees condition, rather than insisting that they be characterized by identical loss functions. Predtetchinski (2011) shows that for generic concave preferences, the limit equilibria will be unique, even if, for $\delta < 1$, the game admits multiple equilibria. Moreover, Predtetchinski (2011) characterizes this limit equilibrium as the generalized zero of a characteristic function. In the analysis below, we provide an explicit characterization of this generalized root, and show that it corresponds to a bilateral asymmetric Nash bargaining solution, with endogenous weights that are pinned down by the model parameters. Other papers establish the properties of uni-dimensional bargaining under q-majority rule, given alternative or particular bargaining protocols. For example, Cardona and Ponsatí (2007) study equilibria when the recognition rule follows a deterministic process. Herings and Predtetchinski (2010) study equilibria under unanimity rule when the recognition rule follows a Markov process.

This paper also contributes to a literature that demonstrate the links between outcomes of non-cooperative bargaining games and other solution concepts. Binmore, Rubinstein and Wolinsky (1986) show that the equilibria of Rubinstein's (1982) two player alternating-offers bargaining model converge to the Nash bargaining solution as players become perfectly patient. Cho and Duggan (2009) show, in a one-dimensional model under simple majority rule with multiple players and random recognition, that the limit equilibrium selects the median voter's ideal policy. Imai and Salonen (2000) show in a one-dimensional framework with players grouped into opposing factions, that equilibria are characterized by a 'Representative Nash solution'. In this paper, we show that, in the limit equilibrium, it is as if the players endogenously separate into opposing factions, and that the limit equilibrium is the result of Nash bargaining between representatives of these two factions. Hart and Mas-Colell (1996), Laruelle and Valenciano (2008) and Miyakawa (2008) show that, when the space of preferences is comprehensive[6], the limit equilibrium corresponds to the solution of a generalized Nash bargaining problem of the form: $\Pi_i \left(u_i\left(x\right) - u_i\left(d\right)\right)^{p_i}$ — i.e. the solution maximizes the 'weighted' product of the agents' surplus from the bargain, where the weights

[6]Let u be a payoff vector and let U be the utility possibility set. The payoff space is *comprehensive* if $u \in U$ and $u' \leq u$ implies that $u' \in U$.

correspond to recognition probabilities. Laruelle and Valenciano (2008) additionally demonstrate the relationship of this solution to the Shapley-Shubik value (see Shapley and Shubik (1954)). As should be clear, payoffs in a one-dimensional space with concave preferences are not comprehensive. This distinction explains the differences in the results between those papers and ours.

The remainder of this paper is organized as follows: Section 2 outlines the bargaining framework. Section 3 characterizes the limit equilibrium and explores several properties of this limit. Section 4 provides several extensions, and Section 5 concludes.

2 Framework

Let $X \subset \mathbb{R}$ be a compact interval denoting a set of outcomes. There is a set of agents, $N = \{1, ..., n\}$, who bargain over the outcome to be implemented. Each player $i \in N$ has expected utility preferences represented by a concave and strictly quasi-concave utility index $u_i(x)$ that achieves a maximum at x_i, and is differentiable almost everywhere.[7] The players are ordered by their ideal policies, so that $x_1 \leq x_2 \leq ... \leq x_n$. Preferences have a representation satisfying the Spence-Mirrlees condition, so that $x_i > x_j$ implies $u_i'(x) \geq u_j'(x)$ for every x.

Let d be a decision rule. A coalition $C \subseteq N$ is *decisive* under the decision rule if the assent of each agent $i \in C$ is sufficient to have some policy adopted. For example, in the American legislative system, a coalition is decisive if it contains either: (i) the president, (at least) a simple majority of legislators in the House of Representatives, and (at least) a three-fifths majority of Senators; or (ii) (at least) a two-thirds majority of legislators in each chamber. We denote by $\mathcal{C}(d)$ the set of decisive coalitions implied by decision rule d. Our framework admits a broad range of decision rules, subject to the requirement that $\mathcal{C}(d)$ is (i) *monotone*

[7] We allow for non-differentiability at the ideal policy. This enables the analysis to accommodate absolute value loss functions, which are commonly used in the literature.

($C \in \mathcal{C}$ and $C \subset C'$ implies $C' \in \mathcal{C}$), and (2) *proper* ($C \in \mathcal{C}$ implies $N \setminus C \notin \mathcal{C}$). Thus, our framework is consistent with many commonplace decision procedures, including:

- q-majority rule for any $q > \frac{n}{2}$ ($\mathcal{C} = \{C \subseteq N | |C| \geq q\}$), which includes simple majority rule ($q = \lceil \frac{n+1}{2} \rceil$) and unanimity rule ($q = n$) as special cases.
- *coalitional* rules ($\cap_{C \in \mathcal{C}} C \neq \emptyset$) that endow a subset of players with an absolute veto. For example, the U.N. Security Council rule requires a majority of all 15 members, as well as the support of each of the five permanent members.
- *oligarchic* rules ($\cap_{C \in \mathcal{C}} C \in \mathcal{C}$), which include *dictatorial* rules as a special case.

Naturally, the framework can also accommodate more complicated rules, including hybrid rules, such as in the American legislative system, which is the union of a coalitional rule (where the president has a credible veto threat) and a super-majority rule (where the Congress can over-ride the president's veto).

The bargaining protocol is the standard procedure in Baron and Ferejohn (1989) and Banks and Duggan (2000). There are potentially infinitely many bargaining rounds. In a given round of bargaining, each player i is recognized with probability $p_i > 0$ to propose a policy. After observing the proposal, all players simultaneously vote to either accept or reject the proposal. Acceptance requires that the proposal receive the assent of some decisive coalition $C \in \mathcal{C}$. If so, the policy is implemented, and the bargaining game ends. In the event of disagreement, the players adjourn and reconvene for another round of bargaining in the following period. This process continues until agreement is reached. Players discount the future at a common rate $\delta \in [0, 1)$.[8] We make the assumption, common in the literature (see Banks and Duggan (2000), Cardona and Ponsati (2011), Predtetchinski (2011), amongst

[8]The bargaining framework admits an alternative interpretation in which, following disagreement, the bargaining game exogenously terminates with probability $1 - \delta$. The discount factor, then, captures the likelihood of their being additional opportunities for negotiation.

many others), that $u_i(x) \geq 0$ for all $x \in X$ and normalize the disagreement payoff to 0. This implies that every policy is preferred to disagreement. In section 4, we consider an extension in which disagreement entails reversion to a status quo policy, $x_{sq} \in X$.

A strategy for player i is a pair $s_i = (y_i, A_i)$, where $y_i \in X$ is the policy proposed whenever i is recognized as the proposer, and $A_i \subset X$ is the set of policies that player i will accept ('the acceptance set'). We solve for stationary sub-game perfect equilibria. We limit attention to strategies that satisfy the weak dominance property; each player supports a proposal only if the utility from having the policy implemented is at least as large as the utility from disagreement.

3 Analysis

3.1 Equilibrium in the Bargaining Game

We begin by characterizing the equilibrium of the bargaining game for any $\delta < 1$. In a slightly more general framework than ours, Banks and Duggan (2000) establish that equilibria must be in *no-delay*, and that no-delay equilibria exist. The acceptance set for each player, then, is given by:

$$A_i = \left\{ y \in X | \, u_i(y) \geq \delta \sum_j p_j u_i(y_j) \right\}$$

Since u is strictly quasi-concave, there exist thresholds $\underline{x}_i < x_i < \overline{x}_i$ such that $A_i = \left[\underline{x_i}, \overline{x_i}\right]$. Letting $E[y] = \sum_j p_j y_j$, notice by the concavity of u_i that $E[y] \in A_i$ for each i. Hence, every player will accept the proposal coinciding with the expected offer in the continuation game.

For any decisive coalition $C \in \mathcal{C}$, let $A_C = \cap_{i \in C} A_i$ be the set of proposals that will be accepted by coalition C, and let $A = \cup_{C \in \mathcal{C}} A_C$ be the social acceptance set. Since each A_i is a convex set, then so is every A_C. Moreover, $E[y] \in A_C$ for every C, since $E[y] \in A_i$ for every i, and so A is a convex set $[\underline{x}, \overline{x}]$.

Let $l = \max_{C\in\mathcal{C}} \min_{i\in C}\{i\}$ and $r = \min_{C\in\mathcal{C}} \max_{i\in C}\{i\}$, and note that $l \leq r$.[9] We refer to l and r as the *left* and *right decisive voters*, respectively, and give them the following interpretation: Although, nothing requires that decisive coalitions be connected, given the Spence-Mirrlees condition, we can show that any coalition arising in equilibrium will be connected. Since the set of decisive coalitions is proper, this implies that there is a (connected) subset of agents whose support is required in *every* possible equilibrium. The left and right decisive voters are the left-most and right-most members of this set. For example, under q-majority rule, the left and right decisive voters are the ones who are q^{th} from the right $(l = n - q + 1)$ and right $(r = q)$, respectively.

Given the Spence-Mirrlees condition, it follows that the social acceptance set is $A = \left[\underline{x_r}, \overline{x_l}\right]$. Hence, we have the following result, which generalizes Proposition 2 in Cardona and Ponsati (2011):

Proposition 1. *There exists a unique stationary equilibrium. In equilibrium, there is no delay, and strategies are characterized by two thresholds, $\underline{x_r}$ and $\overline{x_l}$ (with $\underline{x_r} < \overline{x_l}$) satisfying:*

1. *When recognized, player i proposes:*

$$y_i = \begin{cases} \underline{x_r} & x_i < \underline{x_r} \\ x_i & x_i \in \left[\underline{x_r}, \overline{x_l}\right] \\ \overline{x_l} & x_i > \overline{x_l} \end{cases}$$

where $\underline{x_r} < x_r$ and $\overline{x_l} > x_l$ satisfy:

$$\underline{x_r} = \min\{y \in X | u_r(y) \geq \delta \sum_j p_j u_r\left(y_j\right)\}$$

$$\overline{x_l} = \max\{y \in X | u_l(y) \geq \delta \sum_j p_j u_l\left(y_j\right)\}$$

[9] To see this, note that by the definitions of l and r, and since $\mathcal{C}$ is monotone, then $\{1, .., r\}$ and $\{l, .., n\}$ are both decisive coalitions. Then, since $\mathcal{C}$ is proper, it must be that $r \geq l$.

2. Player i's acceptance set is: $A_i = \left\{ y \in X |\ u_i(y) \geq \delta \sum_j p_j u_i(y_j) \right\}$.

We provide an intuitive account of this result. We have previously shown that if the expected proposal in the continuation game, $E[y]$, is offered, it will receive unanimous support. In fact, by continuity of preferences and discounting, all policies within a neighborhood of this proposal will also receive unanimous support. Then, starting from $E[y]$, player i can pull the offer toward her ideal policy without immediately losing the support of a decisive coalition. For concreteness, suppose $x_i < E[y]$ so that the policy is being pulled to the left. As the proposal moves further from $E[y]$, it becomes less desirable from the perspective of players whose ideal policies were above $E[y]$. The single-crossing condition along with the concavity of u implies that player n will be the first player whose support is lost, then player $n-1$, and so on. By construction, $\{1, ..., r\}$ is a decisive coalition, whilst any coalition excluding all members in $\{r, .., n\}$ cannot be decisive. Hence, the proposer will continue to pull the proposal towards her ideal until either her ideal policy is reached, or she would lose the support of the right decisive player. Analogously, if $x_i > E[y]$, then the proposer can continue to pull the policy up until she would lose the support of the left decisive player.

In this paper, we focus on equilibrium behavior as $\delta \to 1$. We can interpret this limit in several ways. On the one hand, this is the equilibrium that obtains as the players become perfectly patient. We can also think of this as capturing bargaining under a dynamic where players can make counter-proposals arbitrarily quickly, or when the cost of making counter-proposals becomes negligible. Similarly, it might represent the case where the risk that negotiations break down following disagreement, becomes arbitrarily small. The following result reproduces Theorem 3.6 from Predtetchinski (2011).

Proposition 2. *There exists $\mu \in X$ s.t.* $\lim_{\delta \to 1} \underline{x_r} = \mu = \lim_{\delta \to 1} \overline{x_l}$.

An implication of Proposition 2 is that, as $\delta \to 1$, equilibrium proposals converge and become independent of the identity of the proposer. The intuition for this result is straight-forward.

When $\delta < 1$, it is costly for players to reject proposals, and this empowers the proposer to pull the policy in her favored direction. As $\delta \to 1$, players can costlessly wait for more favorable counter-proposals, and so the proposer's agenda-setting prerogative disappears.

The limit equilibrium of the bargaining game is related to the notion of the core. Given any decision rule d, the core $\mathcal{K}(d)$ refers to the set of policies for which there does not exist a different policy that is strictly preferred by some decisive coalition. In the one-dimensional context, the core is simply the subset of policies between the ideal policies of the left and right decisive agents; $\mathcal{K} = [x_l, x_r]$. Now, if n is odd and the decision procedure requires a simple majority ($\frac{n+1}{2}$), then the core is simply the ideal policy of the median player; $\mathcal{K} = \{x_{med}\}$. This is the median voter result, due to Black (1948). Similarly, if the decision rule is dictatorial, then $l = r = i$ where i is the dictator, and $\mathcal{K} = \{x_i\}$. For most other decision rules (and even under simple majority rule, whenever n is even), the core is generically an interval, and so contains a continuum of policies.

Banks and Duggan (2000) show that there is an equivalence between the core and bargaining equilibria when $\delta = 1$. Formally, when $\delta = 1$, the bargaining game (generically) admits a continuum of equilibria, each associated with a given policy in the core that is proposed and accepted by all players. Combining this result with the uniqueness result above and the convergence result in Predtetchinski (2011) gives the following insight: For every $\delta < 1$, the bargaining game admits a unique equilibrium, and as $\delta \to 1$, the equilibrium proposals converge to a unique outcome. However, at $\delta = 1$, the set of equilibrium policies explodes to include the entire core. The equilibrium correspondence exhibits a failure of lower-hemicontinuity at $\delta = 1$. Our refinement exploits this failure of lower-hemicontinuity. The limit equilibrium selects the unique core policy that is the robust in the sense of continuing to be an equilibrium of a bargaining game when delay becomes slightly costly.

3.2 The Faction Formation Game

We have already foreshadowed that the limit equilibrium is characterized by an asymmetric Nash bargaining solution with endogenous weights. At this juncture, we briefly digress and consider a different game, which will help build intuition for the bargaining weights that arise in equilibrium.

Consider the following two stage game, which we dub the *faction formation* game. In the first stage, each player i allocates his support between two factions L and R. Let $\rho_i \in [0,1]$ be the amount of support given to faction R, and $1-\rho_i$ be the support given to faction L. Let $\rho = \sum_i p_i \rho_i$ be the total (recognition probability weighted) support for faction R, and $1-\rho$ be the total support for faction L. In the second stage, a policy b is selected according to asymmetric Nash bargaining between the left and right decisive agents, with weights proportional to the total supports for their respective factions.

It should be clear that the second stage outcome follows mechanically from the first stage choices of each agent. We have:

$$b(\rho) = \arg\max_{b \in X} u_l(b)^{1-\rho} u_r(b)^{\rho}$$

If the ideal policies of the left and right decisive agents coincide, then the faction formation game is trivial. Any profile of first stage strategies is equilibrium consistent, and the second stage outcome is simply the ideal policy of the decisive agents. The problem becomes more interesting when the decisive agents have different ideal policies. If so, then in the second stage, the chosen policy $b(\rho)$ is monotonically increasing in ρ. (In fact, $b(\rho)$ increases from x_l to x_r, as ρ increases from 0 to 1.) Then, in the first stage, a player will not optimally split her support between the factions, unless doing so causes the resulting second stage policy to coincide with her ideal policy. (Otherwise, there is a strict improvement from reallocating support towards the faction in the direction of the player's ideal policy.) This implies the following proposition:

Proposition 3. *There exists a unique policy b^* chosen in any Nash Equilibrium of the Faction Formation game. Furthermore, if $x_l < x_r$, then in any equilibrium, $\rho_i = 0$ for any player i with $x_i < b^*$, and $\rho_i = 1$ for any player i with $x_i > b^*$.*

Proposition 3 shows that the policy chosen in any equilibrium of the faction formation game is unique. Moreover, if $x_l < x_r$, the equilibrium strategies of all players whose ideal policies do not coincide with the equilibrium policy are also unique. There are two potential sources of multiple equilibria. First, if $x_l = x_r$, then any profile of support levels is equilibrium consistent. Second, if $x_l < x_r$ and there are several players whose ideal policies coincide with the equilibrium policy, then two profiles $(\rho_1, ..., \rho_n)$ and $(\rho'_1, ..., \rho'_n)$ may be equilibrium consistent provided that $\sum_{i:x_i=b^*} \rho_i = \sum_{i:x_i=b^*} \rho'_i$. In either case, the induced policy is the same.

We seek an explicit characterization of the equilibrium policy b^*. Since this policy is the same in all equilibria, it suffices to focus on equilibrium strategies that are 'monotone' in the sense that ρ_i is weakly increasing in i *and* $\rho_i \in (0,1)$ for at most i. As Corollary 1 demonstrates, to characterize the equilibrium policy, it then suffices to simply identify the first agent for whom $\rho_i > 0$.

For each $i \in \{1, ..., n\}$, let $P_i = \sum_{j=i}^{n} p_j$ be the probability that the recognized proposer is either agent i or some agent to her right. Let $b_{i-1,i} = b(P_i) = \arg\max_{b \in X} u_l(b)^{1-P_i} u_r(b)^{P_i}$ be the policy that results when the agents separate into factions $L = \{1, .., i-1\}$ and $R = \{i, ..., n\}$.[10] Since $x_l \leq x_r$, we should expect the solution to be weakly closer to x_l as the bargaining power of agent l increases. It is easy to show that $b_{i-1,i} \leq b_{j-1,j}$ whenever $i > j$, and that this inequality is strict whenever $x_l < x_r$.

[10] Although slightly cumbersome, we choose this notation to emphasize to the reader exactly which players are in which factions.

Corollary 1. *Let* $i^* = \min\{i|\, x_i > b_{i,i+1}\}$*. Then, the equilibrium policy is* $b^* = \min\{x_{i^*}, b_{i^*-1,i^*}\}$*.*

To characterize the equilibrium policy of the faction formation game, it suffices to identify agent i^*. It must be that $x_{i^*} > b_{i^*,i^*+1}$, and so i^* would not want to join the L faction. If $x_{i^*} \leq b_{i^*-1,i^*}$ then i^* would also not want to give all her support to the R faction. i^* is pivotal and $b^* = x_{i^*}$; the equilibrium policy coincides with i^*'s ideal policy. Else, $x_{i^*} > b_{i^*-1,i^*}$, and $b^* = b_{i^*-1,i^*}$; the equilibrium policy is the result of Nash Bargaining when the players perfectly separate into connected factions $\{1, ..., i^*-1\}$ and $\{i^*, ..., n\}$. In the former case, $\rho_{i^*} \in (0,1)$, whilst in the latter case, $\rho_{i^*} = 1$.

3.3 Limit Equilibria

We now return our attention to characterizing the limit equilibria of our bargaining game. By Proposition 2, we know that as $\delta \to 1$, the equilibrium proposals of all players converge. Let $\mu = \lim_{\delta\to 1} y_i$ for each i. To build intuition for the main result, suppose $\mu \in (x_{i-1}, x_i)$. Then, for δ large enough, it must be that $x_{i-1} < \underline{x_r} < \mu < \overline{x_l} < x_i$, so that all players $j \leq i-1$ propose $y_j = \underline{x_r}$ and all players $j \geq i$ propose $y_j = \overline{x_l}$. Furthermore, we know that $\underline{x_r}$ and $\overline{x_l}$ are jointly pinned down by the preferences of players l and r. We can think of the equilibrium as resulting from a bargain between players l and r, where l negotiates on behalf of all agents $j < i$ and r negotiates on behalf of all agents $j \geq i$. The main result of this paper is that, as $\delta \to 1$, the equilibrium policy converges to the asymmetric Nash Bargaining solution that gives bargaining weights to l and r in proportion to the cumulative recognition probabilities of the players in their 'faction'.

As before, for $i = 1, ..., n$, let $P_i = \sum_{j=i}^{n} p_j$ be the probability that the proposer is weakly to the right of player i. For each i, let $b_{i-1,i}$ be defined as in the previous subsection. Recall, this is the Nash bargaining solution that obtains when the factions are $\{1, .., i-1\}$ and $\{i, ..., n\}$.

Proposition 4. *Let* $i^* = \min\{i | \, x_i > b_{i,i+1}\}$ *. Then the limit equilibrium is characterized by:*

$$\mu = \min\{b_{i^*-1,i^*}, x_{i^*}\}$$

To make sense of Proposition 4, return to the heuristic argument above. $b_{i-1,i}$ is the solution to the Nash bargaining problem between players l and r, when the bargaining strengths of l and r are $1 - P_i = \sum_{j<i} p_j$ and $P_i = \sum_{j \geq i} p_j$, respectively. These bargaining weights were motivated by the idea that, for δ sufficiently large, all players $j < i$ would choose $y_j = \underline{x_r}$ and all players $j \geq i$ would propose $y_j = \overline{x_l}$. Consistency requires $x_{i-1} < b_{i-1,i} < x_i$, otherwise the factions would not be $\{1, ..., , i-1\}$ and $\{i, ..., n\}$. Now, if $i < i^*$, then by construction, $x_{i-1} \leq x_i \leq b_{i,i+1} \leq b_{i-1,i}$, which is inconsistent with the above logic. Similarly, if $i > i^*$, then by construction $b_{i-1,i} \leq x_{i-1} \leq x_i$, which is also inconsistent with the above logic. This explains the focus on $i = i^*$.

Identifying player i^* is necessary, but not sufficient, to characterize the limit equilibrium. From here, there are two possibilities: (i) $x_{i^*-1} < b_{i^*-1,i^*} < x_{i^*}$ and (ii) $x_{i^*-1} \leq x_{i^*} \leq b_{i^*-1,i^*}$ (with at least one inequality strict). The former case is consistent with our story, and implies that $\mu = b_{i^*-1,i^*}$. In the latter case, there is a problem. If we believe player i^* is in r's faction, then the Nash bargaining solution selects a limit policy $\mu > x_{i^*}$ that would cause player i^* to want to be in l's faction. Similarly, if we believed player i^* to be in l's faction, then the Nash bargaining solution would choose limit policy $\mu < x_{i^*}$ which would cause player i^* to be in r's camp. Player i^* is *pivotal*. The only possibility is that the limit equilibrium coincides with his ideal policy, x_{i^*}. (Note, this second scenario precisely coincides with the outcome in the faction formation game where the pivotal agent splits his support between the faction.)

Proposition 4 gives a simple and tractable characterization of the limit equilibrium. The equilibria of bargaining games are typically characterized as the fixed points of some mapping. In general, these are difficult to compute, making their use in applied models cumbersome. Proposition 4 reduces the characterization of the limit equilibrium to a simple decision problem — the modeler need not solve a complicated game nor find the fixed points to some complicated system. The limit equilibria follow immediately from the first order conditions to a straight-forward decision problem.

Example 1. Let $n = 7$ and suppose acceptance requires 'almost unanimity' (i.e. a super-majority of size one less than unanimity). Then $l = 2$ and $r = 6$. Suppose $u_i(y) = 1 - |y - x_i|$, with $x_1 \leq .... \leq x_7$, and normalize $x_2 = 0$ and $x_6 = 1$. Finally, suppose $p_i = \frac{1}{7}$ for each i, so that each player is recognized to propose with equal probability. It follows that:

$$b_{i-1,i} = P_i x_l + (1 - P_i) x_r + (2P_i - 1)$$

so that $b_{i-1,i} = \frac{8-i}{7}$, for each i. Hence, we have:

$$\mu = \begin{cases} \frac{5}{7} & x_3 > \frac{5}{7} \\ x_3 & \frac{4}{7} \leq x_3 \leq \frac{5}{7} \\ \frac{4}{7} & x_3 < \frac{4}{7} < x_4 \\ x_4 & \frac{3}{7} \leq x_4 \leq \frac{4}{7} \\ \frac{3}{7} & x_4 < \frac{3}{7} < x_5 \\ x_5 & \frac{2}{7} \leq x_5 \leq \frac{3}{7} \\ \frac{2}{7} & x_5 < \frac{2}{7} \end{cases}$$

Player 4 is the median, and indeed when $x_4 \in \left[\frac{3}{7}, \frac{4}{7}\right]$, the limit equilibrium selects the median player's ideal policy. However under alternative arrangements of the players' ideal policies, the limit equilibrium policy may also coincide with the ideal policies of players 3 and 5. By contrast, there is no arrangement of ideal policies under which the limit policy coincides with the ideal policies of players 1, 2, 6 or 7. In the remaining cases, the limit policy coincides with the Nash bargaining that result from the players cleanly separating into discrete factions.

The characterization of the limit equilibrium as the solution to an asymmetric Nash bargaining problem between the left and right decisive players admits the following nice interpretation. Although the bargaining game is played by n agents, each with potentially distinct preferences, as players become patient, they separate into two factions and delegate the bargaining to a single representative agent. The composition of the equilibrium factions is endogenous to the model, and depends on the players' recognition probabilities and the spatial alignment of their ideal policies. The bargaining framework, thus provides micro-foundations for the separation of diverse agents into cohesive coalitions or voting blocs, which are represented by agents with non-median preferences. Such behavior is a commonly observed feature in legislature as well as on the U.S. Supreme Court.

Our characterization of the limit equilibrium coincides with that of Predtetchinski (2011). He shows that the limit equilibrium is the generalized root of the characteristic function defined by[11]:

$$\xi\left(x\right)=\frac{u_l'\left(x\right)}{u_l\left(x\right)}\sum_{j:x_j\leq x}p_j+\frac{u_r'\left(x\right)}{u_r\left(x\right)}\sum_{j:x_j>x}p_j$$

Predtetchinski shows that this function is strictly decreasing and has discontinuities. Suppose the generalized root of this function is x^*. There are two possibilities: either (i) $\xi\left(x^*\right)=0$ or (ii) $\lim_{x\uparrow x^*}\xi\left(x\right)>0>\lim_{x\downarrow x^*}\xi\left(x\right)$. In the first case, x^* is an actual root of the characteristic function. In the second case, ξ has a discontinuity at x^* such that it is positive below x^* and negative above. By inspection, the first case corresponds to the bilateral asymmetric Nash bargaining solution between agents l and r, when the bargaining weights on l and r are $\sum_{j:x_j\leq x}p_j$ and $\sum_{j:x_j>x}p_j$, which is precisely $b_{i-1,i}$ if $x^*\in\left(x_{i-1},x_i\right)$. Furthermore, since each u_i is continuously differentiable, the only points of discontinuity of $\xi\left(x\right)$ occur at the ideal policies x_j of agents who are recognized with positive probability. (At these points, there is a discontinuous jump in the cumulative probabilities $\sum_{j:x_j>x}p_j$ and $\sum_{j:x_j\leq x}p_j$.) Hence, the

[11] In fact, his paper provides a more general definition of the characteristic function, for a world where preferences need not be differentiable and where the Spence-Mirrlees condition need not be satisfied. The expression below gives the special case of the characteristic function when those properties hold.

second case, where x^* is a generalized root of ξ must correspond to the ideal policies of some agent who is recognized with positive probability.

Predtetchinski (2011) (on p. 536), limiting attention to preferences that are differentiable and characterized by a common loss function centered at each agent's ideal policy (à la Cardona and Ponsati (2011)), nevertheless claims that the limit policy is generically characterized by neither the asymmetric Nash bargaining solution, nor the median agent's ideal policy. The second part of the claim is partially true — the limit policy may coincide with the ideal policy of any agent whose ideal policy lies within the core, which includes, but is not necessarily restricted to be, the median player. The reason why Predtetchinski and we differ in the first part is that he focuses on Nash bargaining problems of the form: $\max \Pi_j u_j (x)^{p_j}$ — i.e. of the form considered by Laruelle and Valenciano (2008) and Miyakawa (2008). As we showed, the relevant bargaining problem is not between all n players, but simply the bilateral problem between the left and right decisive players alone. We additionally provide a direct characterization of the limit policy in terms of i^*.

3.4 Properties of Limit Equilibria

We now consider some salient properties of limit equilibria.

Lemma 1. *Consider a decision rule under which $l = r$. Then $\mu = x_l = x_r$. In particular:*

- *If n is odd, and decisions are made by simple majority rule, then $l = \frac{n+1}{2} = r$, and the limit policy coincides with the median agent's ideal ($\mu = x_{\frac{n+1}{2}}$).*
- *If i is a dictator, then $l = i = r$, and the limit policy coincides with the dictator's ideal.*

Whenever $l = r$, the core is a singleton, and so selection from the core becomes trivial. This is consistent with Cho and Duggan (2009) and Predtetchinski (2011), who show that when

n is odd and agreement requires a simple majority, then the core is simply the ideal policy of the median voter.

Next, consider a decision rule under which $l < r$ (e.g. a super-majority rule, or simple majority rule with a non-median veto player such as the U.N. Security Council rule). If $x_l = x_r$, then $x_l = x_i = x_r$ for every $i \in \{l+1, ..., r-1\}$. The core remains a singleton, and Lemma 1 obtains. The situation becomes more interesting if $x_l < x_r$, so that the core is an interval. From herein, we assume that $x_l < x_r$.

Lemma 2. *Consider any decision rule d for which $l < r$, and suppose $x_l < x_r$. There exist recognition probabilities $p_1, ..., p_n$ and an arrangement of ideal policies $x_1 \leq ... \leq x_n$ with $x_l < x_r$ s.t. player i is pivotal for each $i \in \{l+1, ..., r-1\}$. Moreover, if player i is pivotal, then $x_i \in (x_l, x_r)$.*

Lemma 2 states that, whenever the core is a continuum, every agent whose ideal policy is in the interior of the core (i.e. strictly between the ideal policies of the left and right decisive agents) is pivotal under some assignment of recognition probabilities. This may include the median player (for example, if n is odd, and the decision rule is a super-majority). But it will also include other agents. We see this in Example 1, above. In that example, there are 7 agents and the decision rule requirement requires a super-majority of one less than unanimity. The left and right decisive agents are players 2 and 6, respectively, and the median agent is player 4. The lemma states that, under some arrangement of ideal policies and recognition probabilities, each of players 3, 4 and 5 will be pivotal. The example describes various arrangements of ideal policies under which each of these outcomes will obtain, assuming uniform recognition probabilities.

The intuition for Lemma 2 is evident in Example 1. Since the limit policy is the result of Nash bargaining between l and r, it must be that $\mu \in [x_l, x_r]$ (and it is generically in the interior of this set). But this means that l and all the players to his left (i.e. agents 1 and

2 in the example) will necessarily join the left faction, and r and all the players to her right (agents 6 and 7) will necessarily join the right faction. The only players who might be pivotal are those whose ideal policies are in the interior of the core (i.e players 3,4 and 5).

Lemma 2 speaks to the pivotality of an agent whose ideal policy lies in the core. The next two lemmas demonstrate more generally which core policies may be selected in the limit equilibrium.

Lemma 3. *Fix a vector of recognition probabilities. The limit equilibrium generically selects a policy in the interior of the core. (Formally, $\mu \in [b_{r-1,r}, b_{l,l+1}] \subset (x_l, x_r]$.) Furthermore, if preferences are everywhere differentiable and if recognition probabilities are bounded away from zero, then the limit equilibrium is bounded away from the boundary of the core.*[12]

Lemma 3 shows that the limit policy will generically be in the interior of the core (although, given the recognition rule, it may be arbitrarily close to either endpoint). In fact, the Lemma makes the stronger claim that the range in which the equilibrium policy will be found is bounded by the policies that result from Nash bargaining when the coalitions are $\{1, .., r-1\}$ and $\{r, ..., n\}$, on the one hand, and $\{1, .., l\}$ and $\{l+1, ..., r\}$, on the other. The intuition is very similar to that in Lemma 2.

Additionally, Lemma 3 shows that, if agents' recognition probabilities are bounded away from zero, then the selected policy will be bounded away from the boundary of the core. This result is particularly salient in much of the applied literature, where uniform recognition probabilities are commonly assumed. Under these conditions, the Lemma implies that the limit policy will likely be 'moderate', in the sense of being towards the middle of the core, rather than at the extremes.

Lemma 4. *Let $x_1, .., x_n$ be an arbitrary arrangement of ideal policies. For each policy $y \in (x_l, x_r)$ in the interior of the core, there exists an assignment of recognition probabilities $p_1, ..., p_n$ s.t. the limit policy is y.*

[12]Formally, there exists $\hat{x}_l > x_l$ and $\hat{x}_r < x_r$ s.t. $\hat{x}_l < b_{r-1,r} \leq< b_{l,l+1} < \hat{x}_r$.

Lemma 4 is a counterpoint to Lemma 3. Whereas Lemma 3 shows that, given some recognition rule, the limit policy will generically be contained in the interior of the core, Lemma 4 shows that any interior policy can be sustained as a limit equilibrium given appropriately chosen recognition probabilities. Moreover, this is true regardless of the arrangement of players' ideal policies. Hence, Lemma 4 demonstrates that the salient feature in generating limit policies is recognition probabilities rather than the ideal policies of agents. Indeed, the 'symmetric' claim is not true – fixing arbitrary recognition probabilities, it is not the case that any policy in the interior of the core can be sustained as an equilibrium given some appropriate alignment of ideal policies.

3.5 Super-majority Requirements

In this section, we limit attention to q-majority rules. Doing so enables us to perform comparative statics on the decision rule, by studying the implications of varying the the super-majority requirement. The basic insight is that, as the super-majority rule becomes more demanding, the limit policy selected in the bargaining game becomes more 'moderate'. This is made clear in the following example:

Example 2. Let $n = 3$, and suppose players are recognized with equal probability, $p_i = \frac{1}{3}$. Preferences are given by $u_i(y) = 1 - |y - x_i|$, with $0 = x_1 \leq x_2 \leq x_3 = 1$. The limit equilibria under different q-majority requirements, with $q \in \{2, 3\}$, are given by:

- If $q = 2$ (i.e. simple majority rule), then $\mu = x_2$.
- If $q = 3$ (i.e. unanimity rule), then $l = 1$ and $r = 3$. Then $b_{i-1,i} = \frac{4-i}{3}$, and:

$$\mu = \begin{cases} \frac{1}{3} & x_2 < \frac{1}{3} \\ x_2 & \frac{1}{3} \leq x_2 \leq \frac{2}{3} \\ \frac{2}{3} & x_2 > \frac{2}{3} \end{cases}$$

Example 2 characterizes the limit equilibria under both simple majority and unanimity, as the alignment of ideal policies varies. Under simple majority rule, the limit equilibrium is uniquely the median player's ideal policy, and this will be true even if the median agent's ideal policy is relatively 'extreme' (i.e. $x_2 \notin \left[\frac{1}{3}, \frac{2}{3}\right]$). Under unanimity ($q = 3$), the median player continues to be pivotal when her ideal policy is 'moderate'. However, if the median player's ideal policy becomes sufficiently extreme, she ceases to be pivotal. The limit policy becomes the Nash Bargaining solution that results from player 2 giving all of her support to the faction closest to her ideal. For example, if $x_2 < \frac{1}{3}$, then player 2 will join faction L, and the equilibrium policy will be $b_{2,3} = \frac{1}{3}$.

To analyze this issue more formally, take any continuous distribution F whose support is a closed interval, and admits a density f. Let $X_1 \leq, ..., \leq X_n$ be the (reverse) order statistics associated with n independent draws from F. The vector $X = (X_1, ..., X_n)$ represents the (ordered) profile of ideal policies in a random sample of agents. Fix a vector of recognition probabilities $p = (p_1, ..., p_n)$ such that $p_i > 0$ for each i. For any ordered vector of ideal policies $x = (x_1, ..., x_n)$, let $\mu_q^n(x)$ be the limit policy, as defined in Proposition 4, under q -majority rule. Then, given a random ordered vector X, let $M_q^n = \mu_q^n(X)$ be a random variable corresponding to the equilibrium policy under q-majority rule. Let $G_q^n(y)$ be the distribution function for M_q^n. Since $\mu_q^n(x)$ is a continuous function, then the support of $\mu_q^n(x)$ will also be a closed interval.

Let $q' > q \geq \frac{1}{2}$, so that q' is a stronger super-majority requirement than q. As before, let $l = n - q + 1$ and $r = q$ denote the left and right decisive agents when the majority requirement is q, and let $l' = n - q' + 1$ and $r' = q'$ be the analogues when the majority requirement is q'. We know that μ_q^n depends crucially on x_l and x_r, and similarly μ_q^n depends crucially on $x_{l'}$ and $x_{r'}$. Moreover, for any realization of ideal polices $x = (x_1, .., x_n)$, it must be that $x_{l'} \leq x_l \leq x_r \leq x_{r'}$ (and these inequalities will be strict, almost surely).

Lemma 5. *Let $q' > q$. Then supp$\{M_{q'}^n | X_{l'} = x_{l'}, X_{r'} = x_{r'}\} \subset$ supp$\{M_q^n | X_{l'} = x_{l'}, X_{r'} =$*

$x_{r'}\}$ *for every realization* $(x_{l'}, x_{r'})$ *of* $(X_{l'}, X_{r'})$*, where the set inclusion is strict.*

Lemma 5 states that, conditional upon *any* realization of $(x_{l'}, x_{r'})$, the range of possible equilibrium policies that obtain under the more demanding majority requirement q' is a strict subset of those that obtain under the less demanding requirement q. Since the supports of both distributions are intervals, this implies that less demanding majority requirements admit more extreme outcomes. As the super-majority requirement becomes larger, the set of potential outcomes becomes concentrated within a narrower range. Moreover, this is not just true 'on average', but for every realization of the ideal policies of the left and right decisive agents in the more demanding problem.

We interpret Lemma 5 as saying that larger super-majority requirements produce less dispersed outcomes. In fact, when $n = 3$ or $n = 4$, we have an even stronger claim. We say distribution H_1 is a *simple-spread*[13] of H_2 (or that the distributions are *simply-intertwined*[14]) if there exists some $\hat{x}$ such that $H_2(x) \leq H_1(x)$ whenever $x < \hat{x}$ and $H_2(x) \geq H_1(x)$ whenever $x > \hat{x}$. In words, H_1 is a simple spread of H_2 if H_1 puts more weight in the tails than H_2 does. There is a clear sense in which a distribution that is a simple spread of another is more dispersed.[15]

Lemma 6. *Suppose* $n = 3$ *or* $n = 4$*. Then conditional upon any realization* $(x_{l'}, x_{r'})$ *of* $(X_{l'}, X_{r'})$*,* G^n_q *is a simple spread of* $G^n_{q'}$*.*

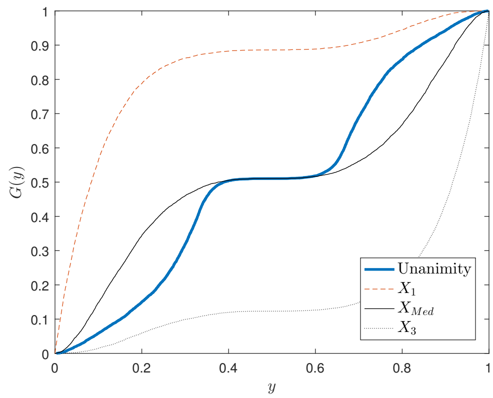
Lemma 5 showed that equilibrium outcomes become concentrated within a narrower range of policies as the super-majority requirement is made stronger. Lemma 6 strengthens Lemma 5 by showing that, even within this narrower band, the stronger super-majority requirement

[13] see Eeckhoudt, Gollier and Schlesinger (1995) and Johnson and Myatt (2006)

[14] see Hammond (1974)

[15] A simple-spread is a distinct notion from that of a mean-preserving spread. A mean-preserving simple-spread is a mean-preserving spread, although the converse need not be true. (Formally, if H_1 is a simple spread of H_2, and $E[H_1] = E[H_2]$, then H_1 is a mean-preserving spread of H_2.) Moreover, in general, simple-spreads need not be mean-preserving.

Figure 1: Distribution of Limit Policies under Different Super-majority Requirements



concentrates more weight in the 'middle' of the band, whereas the weaker majority requirement puts more in the tails. Again, the result holds for every realization of $(x_{l'}, x_{r'})$. For $n \geq 5$, a proof of the result remains elusive, although simulations suggest that the result continues to be true. Similarly, for all n, simulations suggest that G_q^n is a simple spread of $G_{q'}^n$, *unconditionally*. We can see this in the following example, which is a stochastic variant of Example 2:

Example 3. Let $n = 3$, and suppose players are recognized with equal probability, $p_i = \frac{1}{3}$. Preferences are given by $u_i(y) = 1 - |y - x_i|$. Ideal policies be drawn from a distribution F on $[0, 1]$, where $f(z) = 12\left(z - \frac{1}{2}\right)^2$. The thin lines represent the distributions of the order statistics. In particular, the solid thin line represents the distribution of the median voter's ideal policy, which corresponds to the (unconditional) distribution of equilibrium policies under simple majority rule (i.e. it is G_2^3). The thick line represents the distribution of equilibrium policies under unanimity rule (i.e. it is G_3^3). It is easily verified that $G_2^3(y) < G_3^3(y)$ for all $y \in (0, \frac{1}{2})$ and $G_2^3(y) > G_3^3(y)$ for all $y \in (\frac{1}{2}, 1)$. Clearly, the distribution of limit policies under simple majority rule is a simple spread of the distribution under unanimity. (In fact, given the strong symmetry of the setup, it is a mean-preserving spread.)

4 Extensions

4.1 Reversion to Status Quo

In the baseline model, we made the common assumption that for every player, disagreement is worse than accepting any given policy. We now consider an alternative framework in which, after disagreement, an exogenous 'reversion policy' is implemented (see Banks and Duggan (2006)). It is common in the literature to refer to this reversion policy as a 'status quo' policy , but we note that policy need not revert to an existing policy.

For the most part, the results from the baseline model carry over to the status quo model. Since preferences are unique up to an affine transformation, we can always normalize the status quo utility to zero – which gives the two models a very similar appearance. An important difference is that, even after doing so, payoffs may be negative in the status quo setting, and this may affect the possibility of agreement. Banks and Duggan (2006) show that, if the reversion policy lies outside the core, then equilibria must be in no delay. If so, then all of the results from the previous section carry over exactly. Importantly, the limit equilibrium remains the policy that results from Nash bargaining between the left and right decisive agents, where the procedure now maximizes the product of the gains for each agent over the status quo.

If the status quo lies within the core, then the behavior of the equilibrium changes. Banks and Duggan (2006) show that, whilst a no-delay equilibrium will continue to exist, there will also be equilibria with delay. However, Theorem 7 of Banks and Duggan (2006) shows that in every such equilibrium, the only policy that is socially acceptable is the status quo policy itself. An agent may either propose the status quo policy and it will be accepted, or can propose some other policy which will be rejected, resulting in reversion to the status quo anyway. Hence, whether there is delay or not, the policy that prevails in every round of bargaining is simply the status quo policy. Banks and Duggan refer to this as a 'static' equilibrium. If we limit attention to no-delay equilibria, Proposition 1 continues to hold – although now the acceptance set is a singleton for every δ. Our main result (Proposition 4) continues to be true, although the result is now trivial, in that the status quo policy, which is the limit policy, is the only policy that is weakly Pareto improving for l and r.

We now turn our attention to comparative statics in the status quo. Let $\Delta u_i(y, x_{sq}) = u_i(y) - u_i(x_{sq})$ denote the utility improvement for player i from policy y over the status quo x_{sq}. We say preference improvements are log-super-modular if $x_i > x_j$ implies $\frac{\Delta u_i(y', x_{sq})}{\Delta u_i(y, x_{sq})} > \frac{\Delta u_j(y', x_{sq})}{\Delta u_j(y, x_{sq})}$ whenever $y' > y$. Roughly speaking, log-supermodularity of utility improvements implies that the coefficient of absolute risk aversion decreases for each agent as policy moves towards the agent's ideal.[16] [17]

Lemma 7. *For status quo policies within the core,* $\frac{\partial \mu}{\partial x_{sq}} = 1$*. For status quo policy outside the core, if utility improvements are log-supermodular, then* $\frac{\partial \mu}{\partial x_{sq}} < 0$*.*

Lemma 7 states that, for x_{sq} outside the core, as the status quo becomes more extreme, the equilibrium policy becomes more moderate. To make sense of this proposition, first suppose x_{sq} is slightly above x_r. Then the potential utility gains for player r are much smaller than the potential gains for player l. Since the Nash bargaining solution maximizes the (weight) product of these utility gains, it must be that player r achieves the lion's share of her potential utility gains. The Nash bargaining solution will be very close to player r's ideal policy. The fact that the gains from the bargain are small relative to her outside option endow player r with significant bargaining power. Now, as the status quo moves further to

[16]When utility is increasing, the coefficient of absolute risk aversion is defined by $A_i(y) = -\frac{u_i''(y)}{u_i'(y)}$ which takes a positive value if the agent is risk averse. The appropriate analogue when utility is decreasing is: $A_i(y) = \frac{u_i''(y)}{u_i'(y)}$. Log-supermodularity implies that when policy moves in a direction that improves utility, the coefficient of absolute risk aversion decreases. If utility is increasing, then this exactly coincides with the notion of decreasing absolute risk aversion.

[17]To see this, suppose without loss of generality, that policy $y > x_{sq}$ is an improvement over the status quo. By the mean value theorem, there exists $\gamma \in (x_{sq}, y)$ such that $\Delta u_i(y, x_{sq}) = u_i'(\gamma)(y - x_{sq})$. Then $\frac{\partial}{\partial y}(\log \Delta u_i(y, x_{sq})) = \frac{u_i''(\gamma)}{u_i'(\gamma)} \cdot \frac{\partial \gamma}{\partial y} + \frac{1}{y - x_{sq}}$. It is easily verified that $\frac{\partial \gamma}{\partial y} > 0$. Hence, the expression will be increasing in the ideal policy of the agent if $-\frac{u_i''(\gamma)}{u_i'(\gamma)}$ is decreasing.

the right, becoming more extreme, the potential utility gains for both players increase. Log super-modularity implies that the gains for player r are larger than the gains for player l. But, following the same logic as above, this *reduces* the bargaining of player r. The Nash bargaining solution will move further from the ideal policy of player r and towards the ideal policy of player l. Hence, as the status quo policy becomes more extreme, the equilibrium policy becomes more moderate.

4.2 Continuum of Players

Proposition 4 showed that, for any decision rule, the limit equilibrium was characterized either by the ideal policy of some agent, or the asymmetric Nash Bargaining solution after the agents optimally separated into two distinct factions. We showed additionally that a player would only be be pivotal, if in the faction formation game, they had a positive mass of support which they would ideally split between the factions. Intuitively, as the number of players increases, each agent's sway in the game (as measured by their recognition probability) will typically become smaller, and so accordingly will the preference profiles that make them pivotal. We formalize this intuition in the limit as the number of players becomes arbitrarily large.

Suppose there are a continuum of players, indexed by their ideal policies $x \in X$. Let $P(x)$ denote the probability that a player with ideal policy no larger than x is recognized as the proposer. Let $u(y, x)$ denote the utility index of a type-x agent, given policy y. As before, u is differentiable and concave in y for each x, and satisfies the Spence-Mirrlees condition. (Formally, $x' > x$ implies $u_y(y, x') \geq u_y(y, x)$ for all $y \in X$. It suffices that $u_{yx} \geq 0$.) Bargaining follows the protocol in Baron and Ferejohn (1989). Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and let $\mathcal{C} \subset \mathcal{B}(X)$ be the set of decisive coalitions. Let $x_l = \sup_{C\in\mathcal{C}} \inf x \in C\{x\}$ and $x_r = \inf_{C\in\mathcal{C}} \sup x \in C\{x\}$ denote the ideal policies of the left and right decisive agents, respectively.

The results in Section 3.1 on equilibrium in the finite player bargaining game (e.g. Proposition 1) continue to hold in the continuum setting. The equilibrium is pinned down by the preferences of two decisive agents, with ideal policies x_l and x_r. The analogue of the main result on the limit equilibrium (Proposition 4) now becomes:

Proposition 5. *The limit equilibrium is characterized by a pair* (μ, ϕ) *which satisfy:* $\phi = P(\mu)$ *and*

$$\mu = \arg\max_{y} u\left(y, x_l\right)^{\phi} u\left(y, x_r\right)^{1-\phi}$$

As Proposition 5 shows, with a continuum of players, the limit policy is both the ideal policy of the agent at the ϕ^{th} quantile of the recognition distribution P, and the outcome of asymmetric Nash Bargaining between the left and right decisive agents, when the weights are ϕ and $1 - \phi$. We can think of all agents with ideal policies to the left of μ as joining the left faction, and all agents with ideal policies to the right as joining the right faction. Indeed, ϕ in the continuum player model is analogous to player i^* in the finite agent model. However, unlike the baseline model, the 'pivotal' agent cannot sway the policy outcome by switching her support from one faction to the other, or by dividing her support between the factions in different proportions. This follows because in the continuation game, each agent is an atom. Her pivotality is non-generic, and (unlike the baseline model) if the model were slightly perturbed, she would no longer be pivotal.

5 Conclusion

In a uni-dimensional policy space, the median voter theorem provides a simple and unique characterization of equilibrium outcomes, when decisions are made by simple majority rule. For general decision rules, however, the core generically contains a continuum of policies, and existing theory provides little guidance about how to select from amongst these. In this paper, we used a bargaining approach to select a robust policy from amongst the multiplicity.

Using the canonical bargaining framework of Baron and Ferejohn (1989), we characterized the unique equilibrium strategies when agents are impatient. We showed that, as agents were made arbitrarily patient (δ to 1), equilibrium proposals converged to a unique policy in the core. By contrast, when $\delta = 1$, there were a continuum of equilibria, each corresponding to a particular policy in the core. We exploited this failure of lower-hemicontinuity of the equilibrium correspondence to select a robust policy from within the core. Moreover, we showed that our refinement has a simple and elegant characterization: it is the policy that results from asymmetric Nash bargaining between the left and right decisive players.

This focal policy coincides with the equilibrium of a *faction formation* game. It is chosen as if the players endogenously separated into two factions led by the left and right decisive agents, with bargaining strengths that depend upon factional size. Our model thus provides foundations for the emergence of two cohesive groups, whose members have heterogeneous preferences, but nevertheless delegate decision making to (non-median) leaders. This result has clear applications in explaining the emergence of coalitions in a variety of settings.

Our framework provides a unique, simple and tractable approach for applied theoretical modelling in a variety of settings, and allows for the analysis and exploration of myriad decision procedures including many that are richer than simple majority rule. For example, a companion paper (see Parameswaran and Rendleman (2018)) explores the effect of adopting various decision rules on the level of redistribution and public goods provision, and compares these to the benchmark analysis in Meltzer and Richard (1981). Applications to a variety of other contexts suggest themselves.

6 Appendix

Proof of Proposition 1. The proof follows the strategy in Cardona and Ponsati (2011). Suppose the equilibrium social acceptance set is $[\underline{x}, \overline{x}]$. It is immediate that the equilibrium proposals are given by:

$$y_i = \begin{cases} \underline{x} & x_i < \underline{x} \\ x_i & x_i \in [\underline{x}, \overline{x}] \\ \overline{x} & x_i > \overline{x} \end{cases}$$

Let $U_i(\underline{x}, \overline{x}) = u_i(\underline{x}) \sum_{k:x_k<\underline{x}} p_k + \sum_{k:x_k\in[\underline{x},\overline{x}]} p_k u_i(x_k) + u_i(\overline{x}) \sum_{j:x_k>\overline{x}} p_k$ be the expected utility of agent i in the continuation game. Let the individual acceptance sets $A_i = [\underline{x}_i, \overline{x}_i]$ be defined as in the statement of the proposition.

Step 1. We first show that in any equilibrium, $\underline{x} = \underline{x}_r$ and $\overline{x} = \overline{x}_l$. Take any i, j with $x_i < x_j$. The following claims are true:

1. Suppose $u_i(\underline{x}) \leq \delta U_i(\underline{x}, \overline{x})$. Then $u_j(\underline{x}) < \delta U_j(\underline{x}, \overline{x})$.
2. Suppose $u_j(\overline{x}) \leq \delta U_j(\underline{x}, \overline{x})$. Then $u_i(\overline{x}) < \delta U_i(\underline{x}, \overline{x})$.

We prove (1), and note that (2) is proved analogously. Suppose (1) is not true. Then there exists some $x_i < x_j$ s.t. $u_i(\underline{x}) \leq \delta U_i(\underline{x}, \overline{x})$ and $u_j(\underline{x}) \geq \delta U_j(\underline{x}, \overline{x})$. This implies:

$$u_i(\underline{x}) - u_j(\underline{x}) \leq \delta [U_i(\underline{x}, \overline{x}) - U_j(\underline{x}, \overline{x})]$$

Now, by the Spence-Mirlees condition, $\frac{\partial}{\partial x}[u_i(x) - u_j(x)] \leq 0$. Hence:

$$\begin{aligned} U_i(\underline{x}, \overline{x}) - U_j(\underline{x}, \overline{x}) &= \sum_{k:x_k<\underline{x}} p_k [u_i(\underline{x}) - u_j(\underline{x})] \\ &\quad + \sum_{k:x_k\in[\underline{x},\overline{x}]} p_k [u_i(x_k) - u_j(x_k)] + \sum_{k:x_k>\overline{x}} p_k [u_i(\overline{x}) - u_j(\overline{x})] \\ &= u_i(\underline{x}) - u_j(\underline{x}) \end{aligned}$$

Then $u_i(\underline{x}) - u_j(\underline{x}) \leq \delta [U_i(\underline{x}, \overline{x}) - U_j(\underline{x}, \overline{x})] \leq \delta [u_i(\underline{x}) - u_j(\underline{x})]$, which cannot be, since $\delta < 1$. This verifies (1).

Now, suppose $\underline{x} < \underline{x}_r$. Then, $u_r(\underline{x}) < \delta U_r(\underline{x}, \overline{x})$, and so, by (1), $u_i(\underline{x}) < \delta U_i(\underline{x}, \overline{x})$ for all $i > r$. But, by the construction of r, we know that every decisive coalition must contain at least one agent in $\{r, ..., n\}$. Hence, the coalition that supports $\underline{x}$ will not be decisive, which cannot be. Hence $\underline{x} \geq \underline{x}_r$. Next, since $u_r(\underline{x}_r) \geq \delta U_r(\underline{x}, \overline{x})$, taking the contrapositive of (1), $u_i(\underline{x}_r) > \delta U_i(\underline{x}, \overline{x})$ for all $i < r$. Again, by construction and monotonicity, we know that $1, .., r \in \mathcal{C}$. Since this coalition that would accept $\underline{x}_r$, and so $\underline{x} \leq \underline{x}_r$. Hence $\underline{x} = \underline{x}_r$. By a similar argument, we can show that $\overline{x} = \overline{x}_l$.

Step 2. Next, we show that the bargaining game admits a unique equilibrium. Let $\overline{\zeta}_l(\theta) = \max\{x \in X | \, u_l(x) \geq \delta U_l(\theta, x)\}$ and $\underline{\zeta}_r(\theta) = \min\{x \in X | \, u_r(x) \geq \delta U_r(x, \theta)\}$. Naturally, if $(\underline{x}, \overline{x})$ are a pair of equilibrium thresholds, we must have: $\underline{x} = \underline{\zeta}_r(\overline{x})$ and $\overline{x} = \overline{\zeta}_l(\underline{x})$. Let $H(\underline{x}) = \underline{\zeta}_r\left(\overline{\zeta}_l(\underline{x})\right)$. Then $(\underline{x}, \overline{x})$ is an equilibrium if $\underline{x}$ is a fixed point of H and $\overline{x} = \overline{\zeta}_l(\underline{x})$. Since u is continuous, so are $\overline{\zeta}_r$ and $\underline{\zeta}_l$. Hence, by Brouwer's fixed point theorem, H admits a fixed point.

We need to show that this fixed point is unique. For concreteness, denote $X = [\underline{z}, \overline{z}]$. Let $P(x) = \sum_{k: x_k \leq x} p_k$. Implicitly differentiating the function that defines $\overline{\zeta}_l(\theta)$, we have:

$$\overline{\zeta}_l'(\theta) = \begin{cases} \frac{\delta P(\theta)}{1-\delta+\delta P\left(\overline{\zeta}_l(\theta)\right)} \cdot \frac{u_l'(\theta)}{u_l'\left(\overline{\zeta}_l(\theta)\right)} & \overline{\zeta}_i(y) < \overline{z} \\ 0 & \overline{\zeta}_i(y) = \overline{z} \end{cases}$$

Similarly, we have:

$$\underline{\zeta}_r'(\theta) = \begin{cases} \frac{\delta(1-P(\theta))}{1-\delta P\left(\underline{\zeta}_r(\theta)\right)} \cdot \frac{u_r'(\theta)}{u_r'\left(\underline{\zeta}_r(\theta)\right)} & \underline{\zeta}_r(\theta) > \underline{z} \\ 0 & \underline{\zeta}_r(\theta) = \underline{z} \end{cases}$$

Let $(\underline{x}, \overline{x})$ be equilibrium thresholds (which implies $\overline{\zeta}(\underline{x}) = \overline{x}$ and $\underline{\zeta}(\overline{x}) = \underline{x}$). Then:

$$H'(\underline{x}) = \begin{cases} \frac{\delta(1-P(\overline{x}))}{1-\delta P(\underline{x})} \cdot \frac{\delta P(\underline{x})}{1-\delta+\delta P(\overline{x})} \cdot \frac{u_r'(\overline{x})}{u_r'(\underline{x})} \cdot \frac{u_l'(\underline{x})}{u_l'(\overline{x})} & \underline{z} < \underline{x} < \overline{x} < \overline{z} \\ 0 & \underline{z} = \underline{x} \text{ or } \overline{x} = \overline{z} \end{cases}$$

We seek to show that $H'(\underline{x}) < 1$ at any fixed point $\underline{x}$. Notice that this is immediate if $\underline{z} = \underline{x}$ or $\overline{x} = \overline{z}$. Suppose $\underline{z} < \underline{x} < \overline{x} < \overline{z}$. Then $H'(\underline{x})$ is the product of 4 terms, the first two of which are positive and less than 1. It suffices then to show that the product of the third and fourth terms is also less than 1.

Suppose $H(\underline{x}) \geq 1$. Then at least one of $\left|\frac{u_r'(\overline{x})}{u_r'(\underline{x})}\right| > 1$ or $\left|\frac{u_l'(\underline{x})}{u_l'(\overline{x})}\right| > 1$. There are several cases to consider. First, suppose $\left|\frac{u_r'(\overline{x})}{u_r'(\underline{x})}\right| > 1$. Since $\underline{x} < x_r$ then $u_r'(\underline{x}) > 0$ by the concavity of u. If $\underline{x} < \overline{x} \leq x_r$, then concavity implies $0 \leq u_r'(\overline{x}) \leq u_r'(\underline{x})$, which contradicts $\left|\frac{u_r'(\overline{x})}{u_r'(\underline{x})}\right| > 1$. Hence $\underline{x} < x_r < \overline{x}$, and so $u_r'(\overline{x}) < 0$. Suppose additionally $x_l \leq \underline{x} < \overline{x}$. Then $u_l'(\underline{x}) < 0$ and $u_l'(\overline{x}) < 0$. Hence $\frac{u_r'(\overline{x})}{u_r'(\underline{x})} < 0$, and $\frac{u_l'(\underline{x})}{u_l'(\overline{x})} > 0$, and so $H < 0$, which cannot be. Hence $\underline{x} < x_l \leq x_r < \overline{x}$. Then, by the Spence-Mirrlees condition, $0 < u_l'(\underline{x}) < u_r'(\underline{x})$ and $u_l'(\overline{x}) < u_r'(\overline{x}) < 0$, and so:

$$\frac{u_l'(\underline{x})}{u_l'(\overline{x})} \cdot \frac{u_r'(\overline{x})}{u_r'(\underline{x})} = \frac{u_l'(\underline{x})}{u_r'(\underline{x})} \cdot \frac{u_r'(\overline{x})}{u_l'(\overline{x})} < 1$$

Hence $H < 1$, which cannot be, and so $\left|\frac{u_r'(\overline{x})}{u_r'(\underline{x})}\right| \leq 1$. By a similar logic, we show that $\left|\frac{u_l'(\underline{x})}{u_l'(\overline{x})}\right| \leq 1$. Hence our initial supposition was wrong; $H'(\underline{x}) \ngeq 1$. Hence, $H' < 1$ and so H admits a unique fixed point. $\square$

Proof of Proposition 2. We offer a different proof to Predtetchinski (2011). The acceptance set is $A = [\underline{x_r}, \overline{x_l}]$, where $\underline{x_r} = \min\{x \in X | u_r(x) \geq \delta U_r(x, \overline{x_l})\}$, and $\overline{x_l} = \max\{x \in X | u_l(x) \geq \delta U_l(\underline{x_r}, x)\}$. Now, by construction $u_l(\underline{x_r}) \geq u_l(\overline{x_l}) \geq U_l(\underline{x_r}, \overline{x_l})$, since l will accept $\underline{x_r}$. Then, since u is strictly quasi-concave, $u_l(y) > u_l(\overline{x_l})$ for all $y \in (\underline{x_r}, \overline{x_l})$. Similarly, $u_r(y) > u_r(\underline{x_r})$ for all $y \in (\underline{x_r}, \overline{x_l})$. Hence $U_l(\underline{x_r}, \overline{x_l}) > u_l(\overline{x_l})$ and $U_r(\underline{x_r}, \overline{x_l}) > u_r(\underline{x_r})$ whenever $\underline{x_r} < \overline{x_l}$.

w, for every $\delta < 1$, $\frac{u_l(\overline{x_l})}{U_l(\underline{x_r}, \overline{x_l})} = \delta = \frac{u_r(\underline{x_l}}{U_r(\underline{x_r}, \overline{x_l})}$, and so as $\delta \to 1$, we need $U_l(\underline{x_r}, \overline{x_l}) - u_l(\overline{x_l}) \to 0$ and $U_r(\underline{x_r}, \overline{x_l}) - u_r(\underline{x_r}) \to 0$. But this requires $\overline{x_l} - \underline{x_r} \to 0$. Hence $A = [\underline{x_r}, \overline{x_l}] \to [\mu, \mu]$ as $\delta \to 1$. $\square$

Proof of Proposition 3. First, take the case when $x_l = x_r$. Then, for every ρ, the unique maximizer of the Nash bargaining problem is $b = x_l = x_r$. (This follows, since the Nash bargaining problem maximizes $u_l(b)^{1-\rho} u_r(b)^{\rho} = u_l(b)$, and this is uniquely maximized when $b = x_l$.

Next, suppose $x_l < x_r$. Then since u is concave, the Nash bargaining problem

$$\max_b u_l(b)^{1-\rho} u_r(b-x)^{\rho}$$

has a unique maximizer $b(\rho)$. Moreover, $b(\rho)$ is strictly increasing in ρ.

Let $(\rho_1, ..., \rho_n)$ be an equilibrium vector of support levels, and let $b(\rho)$ be the corresponding Nash bargaining solution. Then, if $x_i < b(\rho)$, it must be that $\rho_i = 0$, or else agent i could choose some $\rho_i' < \rho_i$ and move the policy closer to her ideal. Similarly, if $x_i > b(\rho)$ then $\rho_i = 1$.

Now, let $(\rho_1, ..., \rho_n)$ and $(\rho_1', ..., \rho_n')$ be equilibria and suppose the implied policies $b(\rho)$ and $b(\rho')$ are distinct. WLOG, suppose $b(\rho) < b(\rho')$. Let $L = \{i | \, \rho_i = 0\}$ and $L' = \{i | \, \rho_i' = 0\}$ be the sets of agents allocating all of their support to the left faction, in each equilibrium, and define R and R' similarly. Clearly, if $\rho_i < 1$, then $\rho_i' = 0$. (To see this, note that, if $\rho_i < 1$, then $x_i \leq b(\rho) < b(\rho')$, which by the previous discussion requires $\rho_i' = 0$.) Hence, $\rho = \sum_i \rho_i \geq \sum_i \rho_i' = \rho'$. Then, by the monotonicity property, $b(\rho) \geq b(\rho')$. But this contradicts the assumption that $b(\rho) < b(\rho')$. Hence, there cannot be multiple equilibria with distinct equilibrium policies. □

Proof of Corollary 1. Let $\{\rho_1, ..., \rho_n\}$ be equilibrium strategies that are monotone and satisfy $\rho_i \in (0, 1)$ for at most one agent, and let $\rho = \sum_i p_i \rho_i$. Let $i^* = \min\{i | \rho_i > 0\}$. Then, since $\rho_j = 0$ for all $j < i^*$ and $\rho_j = 1$ for all $j > i^*$, $P_{i^*+1} < \rho \leq P_{i^*}$. Next, since $b(\rho)$ is increasing in ρ, this implies $b_{i^*, i^*+1} < b^* \leq b_{i^*-1, i^*}$. Now, since $\rho_{i^*} > 0$, $x_{i^*} \geq b^*$. Hence

$x_{i^*} > b_{i^*,i^*+1}$. Moreover, since $\rho_j = 0$ for any $j < i^*$, it must be that $b_{j,j+1} \geq b_{i^*,i^*} \geq b^* \geq x_j$. Hence $i^* = \min\{i | x_i > b_{i+1}\}$.

There are two possibilities. If $\rho_{i^*} < 1$, then $b^* = x_{i^*} \leq b_{i^*-1,i^*}$ (since $b(\rho)$ is increasing in ρ), whereas if $\rho_{i^*} = 1$, then $b^* = b_{i^*-1,i^*} \leq x_{i^*}$. It follows that $b^* = \min\{x_{i^*}, b_{i^*-1,i^*}\}$. □

Proof of Proposition 4 . Take any $i \in \{1, ..., n\}$, and suppose $\mu \in (x_{i-1}, x_i)$. Then, by Proposition 2, there exists $\bar{\delta} < 1$ s.t. for $\delta > \bar{\delta}$, $x_{i-1} < \underline{x}_r(\delta) < \overline{x}_l(\delta) < x_i$. (For clarity, we make explicit the dependence of $\underline{x}_r$ and $\overline{x}_l$ on δ.) Then, by Proposition 1, all players $j \in \{1, .., i-1\}$ will propose $\underline{x}_r$ and all players $j \in \{i, .., n\}$ will propose $\overline{x}_l$. Again by Proposition 1, this implies that:

$$u_r(\underline{x}_r) = \delta\left[(1-P_i)\,u_r(\underline{x}_r) + P_i u_r(\overline{x}_l)\right] \tag{1}$$

$$u_l(\overline{x}_l) = \delta\left[(1-P_i)\,u_l(\underline{x}_r) + P_i u_l(\overline{x}_l)\right] \tag{2}$$

where $P_i = \sum_{j \geq i} p_j$. By the implicit function theorem, this system of equations pins down $\underline{x}_r$ and $\overline{x}_l$ in terms of the model parameters.

Now, let $E[y] = (1-P_i)\,\underline{x}_r + P_i \overline{x}_l$. Note, by construction, that $\underline{x}_r < E[y] < \overline{x}_l$. Then $\overline{x}_l - E[y] = \frac{1-P_i}{P_i}(E[y] - \underline{x}_r)$. We effect the following change of variables: Let $\varepsilon = E[y] - \underline{x}_r$. Then, we have: $\underline{x}_r = E[y] - \varepsilon$ and $\overline{x}_l = E[y] + \frac{1-P_i}{P_i}\varepsilon$. Equations (1) and (2) become:

$$(1-\delta(1-P_i))\,u_r(E[y]-\varepsilon) = \delta P_i u_r\left(E[y] + \frac{1-P_i}{P_i}\varepsilon\right) \tag{3}$$

$$(1-\delta P_i)\,u_l\left(E[y] + \frac{1-P_i}{P_i}\varepsilon\right) = \delta(1-P_i)\,u_l(E[y]-\varepsilon) \tag{4}$$

By the implicit function theorem, and since u is continuously differentiable, we have:

$$\begin{bmatrix} (1-\delta(1-P_i))\,u_r'(\underline{x}_r) - \delta P_i u_r'(\overline{x}_l) & -(1-\delta(1-P_i))\,u_r'(\underline{x}_r) - \delta(1-P_i)\,u_r'(\overline{x}_l) \\ (1-\delta P_i)\,u_l'(\overline{x}_l) - \delta(1-P_i)\,u_l'(\underline{x}_r) & \left(\frac{1-P_i}{P_i} - \delta(1-P_i)\right)u_l'(\overline{x}_l) + \delta(1-P_i)\,u_l'(\underline{x}_r) \end{bmatrix} \begin{pmatrix} \frac{\partial E[y]}{\partial \delta} \\ \frac{\partial \varepsilon}{\partial \delta} \end{pmatrix} =$$

$$\left(\begin{array}{c}(1-P_i)\,u_r\left(\underline{x}_r\right)+P_i u_r\left(\overline{x}_l\right)\\ P_i u_l\left(\overline{x}_l\right)+(1-P_i)\,u_l\left(\underline{x}_r\right)\end{array}\right)$$

Taking limits as $\delta \to 1$, we have:

$$\left[\begin{array}{cc}0 & -u_r'(\mu)\\ 0 & \frac{1-P_i}{P_i}u_l'(\mu)\end{array}\right]\left(\begin{array}{c}\lim_{\delta\to 1}\frac{\partial E[y]}{\partial\delta}\\ \lim_{\delta\to 1}\frac{\partial\varepsilon}{\partial\delta}\end{array}\right)=\left(\begin{array}{c}u_r(\mu)\\ u_l(\mu)\end{array}\right)$$

These imply that:

$$\lim_{\delta\to 1}\frac{\partial\varepsilon}{\partial\delta}=-\frac{u_r(\mu)}{u_r'(\mu)}=\frac{P_i}{1-P_i}\frac{u_l(\mu)}{u_l'(\mu)}$$

The second equality provides an equation that uniquely defines the limit equilibrium.

Next, we note that equation defining μ_i coincides with the first order condition of the i^{th} Nash Bargaining problem. Recall, that problem was: $\max_{y\in X} u_l(y)^{1-P_i}\,u_r(y)^{P_i}$. Since utilities are concave, the maximizer must be the solution to the first order condition: $(1-P_i)\frac{u_l'(b_{i-1,i})}{u_l(b_{i-1,i})}+P_i\frac{u_r'(b_{i-1,i})}{u_r(b_{i-1,i})}=0$. Re-arranging gives the desired result.

Notice that $b_{i-1,i}$ is increasing in P_i. (To see this, re-arrange the first order condition to give: $\frac{u_l'(b_{i-1,i})}{u_r'(b_{i-1,i})}\cdot\frac{u_r(b_{i-1,i})}{u_l(b_{i-1,i})}=-\frac{P_i}{1-P_i}$. We know that $b\in[x_l,x_r]$. By single-peakedness, over this region we know that $u_l(b)$ is strictly decreasing in b and $u_r(b)$ is strictly increasing in b, and so $\frac{u_r(b)}{u_l(b)}$ is strictly decreasing in b. Similarly, by concavity of u, $u_l'(b)$ is decreasing in b and $u_r'(b)$ is increasing in b, and so $\frac{u_r'(b)}{u_l'(b)}$ is weakly decreasing in b. Hence, the left hand term is strictly decreasing in b. The right hand term is also strictly decreasing in P. Hence, as P increases, so must b.) Then, since P_i is decreasing in i, it follows that $b_{i-1,i}$ is decreasing is as well.

Since we conjectured $\mu\in(x_{i-1},x_i)$, then the limit equilibrium policy coincides with i^{th} Nash Bargaining solution provided that $x_{i-1}<b_{i-1,i}<x_i$. Now, since x_i is increasing and $b_{i-1,i}$ is decreasing in i, then by the definition of i^*, $x_i<b_{i,i+1}$ for all $i<i^*$ and $x_i\geq b_{i,i+1}$ for all $i\geq i^*$. Moreover, for $i<i^*$, $x_{i-1}\leq x_i<b_{i,i+1}\leq b_{i-1,i}$, which is inconsistent. Similarly, for $i>i^*$, $b_{i-1,i}\leq x_{i-1}\leq x_i$, which is inconsistent. Hence, if $b_{i-1,i}\in(x_{i-1},x_i)$, then $i=i^*$.

Note however, that the converse need not be true. Setting $i = i^*$ gives two possibilities: (i) $x_{i^*-1} < b_{i^*-1,i^*} < x_{i^*}$, or (ii) $x_{i^*-1} \leq x_{i^*} \leq b_{i^*-1,i^*}$ (with at least one inequality strict). The former case is equilibrium consistent, and since the equilibrium is unique, we have $\mu = b_{i^*-1,i^*}$.

Suppose the latter case prevails. It follows that the limit equilibrium is not contained in any of the open intervals (x_{i-1}, x_i), and so $\mu \in \{x_1, ..., x_n\}$. (In fact, since $\underline{x}_r < x_r$ and $\overline{x}_l > x_l$ for all δ, and since $\lim_{\delta\to1} \underline{x}_r = \mu = \lim_{\delta\to1} \overline{x}_l$, then $x_l \leq \mu \leq x_r$, and so $\mu \in \{x_l, ..., x_r\}$.) Suppose $\mu = x_i$ for some $i \in \{l, ..., r\}$. Let $I = \{j | x_j = x_i\}$ and denote $I = \{i^-, ..., i^+\}$, where $i^- \leq j \leq i^+$ for all $j \in I$. (Obviously, I may be a singleton, in which case $i^- = i = i^+$.) Let $\Pi_i^- = \sum_{j<i^-} p_j$ and $\Pi_i^+ = \sum_{j>i^+} p_j$ and $\Pi_i = \sum_{j\in I} p_j$. Then, for δ sufficiently large, (1) becomes:

$$u_r(\underline{x}_r) = \delta \left[\Pi_i^- u_r(\underline{x}_r) + \Pi_i u_r(x_i) + \Pi_i^+ u_r(\overline{x}_l)\right]$$

Since $\underline{x}_r < x_i < \overline{x}_l$, there exists $\gamma \in (0, 1)$ s.t. $x_i = \gamma \underline{x}_r + (1 - \gamma)\overline{x}_l$. We can write (1) as:

$$\begin{aligned} u_r(\underline{x}_r) = {} & \delta \left[(\Pi_i^- + \Pi_i\gamma)u_r(\underline{x}_r) + (\Pi_i^+ + \Pi_i(1-\gamma))u_r(\overline{x}_l)\right] \\ & + \delta \left[\Pi_i\gamma(u_r(\underline{x}_r) - u_r(x_i)) + \Pi_i(1-\gamma)(u_r(\overline{x}_l) - u_r(x_i))\right] \end{aligned} \tag{5}$$

Notice (5) is the sum of two terms, with the first term being analogous to the expression in (1), and the second term being a 'correction' term.

We repeat the procedure for equation (2), and then apply the change of basis method above, and take limits as $\delta \to 1$. Since $\underline{x}_r \to x_i$ and $\overline{x}_l \to x_i$, the 'correction' term in (5) goes to zero. It follows that $\mu = b(\rho^*)$, where $\rho^* = \Pi_i^+ + \Pi_i(1 - \lim_{\delta\to1} \gamma(\delta))$. Now, there must be some k s.t. $b_{k,k+1} < b(\rho^*) = x_i < b_{k-1,k}$. Moreover, it must be that $k \in I$, since $b_{i^+,i^++1} < b(\rho^*) < b_{i^--2,i^--1}$, by construction. But then, we can choose i appropriately s.t. $b_{i,i+1} < x_i < b_{i-1,i}$. But this requires $i = i^*$. $\square$

Proof of Lemma 1. Let $\mathcal{C}$ be such that $l = r$. Notice that every limit equilibrium must be contained in the set $[b_{r-1,r}, b_{l,l+1}]$. (We prove this formally in the proof of Lemma 3.) Since $l = r$, then $x_l = x_r$, and so it follows that $b_{r-1,r} = b_{l,l+1} = x_{med}$. Hence $\mu = x_l = x_r$. $\square$

Proof of Lemma 2. Suppose $x_l < x_i < x_r$. If $\mu = x_i$, then we need $x_{i-1} \leq x_i \leq b_{i-1,i}$ with at least one inequality strict, and $b_{i,i+1} \leq x_i$. Generically, $x_i < b_{i-1,i}$, and so $b_{i,i+1} \leq x_i < b_{i-1,i}$. Since $b_{i-1,i}$ is a strictly decreasing in P_i, it suffices to assign $p_1, ..., p_n$ s.t. $b_{i-1,i} \leq x_i < b_{i,i-1}$. $\square$

Proof of Lemma 3. Let μ be the limit equilibrium. Let $i^* = \min_i \{x_i > b_{i,i+1}\}$. Then, by Proposition 4, $\mu = \min\{b_{i^*-1,i^*}, x_{i^*}\}$. Now, by construction, $b_{i-1,i} \in [x_l, x_r]$ for every i, since every Nash bargaining solution is Pareto optimal amongst the parties to the bargain. Then for every $i \leq l$ s.t. $x_i \leq x_l$, it must be that $x_i \leq x_l \leq b_{i,i+1}$, and so $i \neq i^*$. Similarly, by construction, $x_r \geq b_{r,r+1}$ and so $i^* \leq r$. Hence $i^* \in \{l+1, ..., r\}$.

Now, since $i^* \in \{l+1, ..., r\}$ and $b_{i-1,i}$ is decreasing in i, $\mu = \min\{x_{i^*}, b_{i^*-1,i^*}\} \leq b_{i^*-1,i^*} \leq b_{l-1,l}$. Suppose $\mu < b_{r-1,r}$. Then $\mu < b_{i-1,i}$ for all $i \leq r$ since $b_{i-1,i}$ is decreasing. Then, there is some $i^* \in \{l+1, .., r\}$ s.t. $b_{i^*,i^*+1} \leq x_{i^*} \leq b_{i^*-1,i^*}$ and $\mu = x_{i^*}$. But since $\mu < b_{i-1,i}$ for all $i \in \{l+1, ..., r\}$, the only possibility is $i^* = r$. Hence $\mu = x_r \geq b_{r-1,r}$, which is a contradiction. Hence $\mu \geq b_{r-1,r}$. Hence $\mu \in [b_{r-1,r}, b_{l,l+1}]$. $\square$

Proof of Lemma 4. Fix any alignment of ideal policies $x_1 \leq ... \leq x_n$ s.t. $x_l < x_r$. Take any $y \in [x_l, x_r]$. For each $P \in (0, 1)$, let $b(P) = \arg\max_y u_l(y)^{1-P} u_r(y)^P$. Since $x_l < x_r$, then $b(P)$ is strictly increasing in P. Then $b(P)$ is invertible. For any $y \in (x_l, x_r)$, let $b^{-1}(y)$ define the P s.t. $b(P) = y$. Notice that $\lim_{P \to 0} b(P) = x_l$ and $\lim_{P \to 1} b(P) = x_r$.

Now, let i^* be s.t. $x_{i^*} < y \leq x_{i^*+1}$. Then, by Proposition 4 $\mu = y$ provided that $b_{i^*-1,i^*} = y$ and $x_{i^*} < b_{i^*-1,i^*} \leq x_{i^*+1}$. But, by construction $b_{i^*-1,i^*} = b(P_{i^*})$. It suffices to assign probabilities $p_1, ..., p_n$ s.t. $\sum_{j=1}^{i^*} p_j = 1 - b^{-1}(y)$ and $\sum_{j=i^*}^{n} p_j = b^{-1}(y)$.

Next, suppose $p_i > \underline{p}$ for each i, with $\underline{p} > 0$. We know that if $\mu = b_{i-1,i}$, then $i \in \{l+1, ..., r\}$. Then $b_{l,l+1} \leq b\left(r\underline{p}\right) < x_r$ and $b_{r-1,r} \geq b\left(l\underline{p}\right) > x_l$. Hence, the set of implementable policies is contained in $\left(b\left(l\underline{p}\right), b\left(r\underline{p}\right)\right) \subset (x_l, x_r)$. $\square$

Proof of Lemma 5. Let $q' > q \geq \frac{1}{2}$ and define: $l' = n - q' + 1$, $l = n - q + 1$, $r = q$ and $r' = q'$. Notice that $l' < l \leq r < r'$.Then, $X_{l'} \leq X_l \leq X_r \leq X_{r'}$ and almost surely $X_{l'} < X_{r'}$. Take any realization $(x_{l'}, x_{r'})$ s.t. $x_{l'} < x_{r'}$. Let b_j be the Nash Bargaining solution when the coalitions are $1, .., j-1$ and $j, ..., n$ and the decisive agents are l' and r'. Since $p_j > 0$ for each j, we know by Lemma 3 that $x_{l'} < b_{r'-1,r'} < b_{l',l'+1} < x_{r'}$, and that $\mu^n_{q'} \in [b_{r'-1,r'}, b_{l',l'+1}]$. Then, since $\mu^n_{q'}$ is continuous in $\{x_1, ..., x_n\}$, $supp(M^n_{q'}) \subset [b_{r'-1,r'}, b_{l',l'+1}]$. Next, notice that conditioning on $(x_{l'}, x_{r'})$, does not restrict the values that X_l and X_r may take within the interval $[x_{l'}, x_{r'}]$. Then, given that the underlying distribution is continuous, with positive probability, $x_{l'} < x_l < x_r < b_{r'-1,r'}$. But we know that $\mu^n_q \in [x_l, x_r]$. Hence, with positive probability $\mu^n_q < \min \mu^n_{q'}$. A similar argument shows that $\mu^n_q > \max \mu^n_{q'}$ with positive probability. Hence, $supp\{M^n_{q'}|X_{l'} = x_{l'}, X_{r'} = x_{r'}\} \subset supp\{M^n_q|X_{l'} = x_{l'}, X_{r'} = x_{r'}\}$ for every realization $(x_{l'}, x_{r'})$ of $(X_{l'}, X_{r'})$. $\square$

Proof of Lemma 6. We will focus our attention on the case where $n = 4$; the proof for the $n = 3$ case follows trivially. Let $b^{n,q}_{i-1,i}$ represent the Nash Bargaining solution for n players with super-majority rule q, where the players $\{1, ..., i-1\}$ are in the left faction and $\{i, ..., n\}$ are in the right faction. For $n = 4$, we consider the equilibrium policies for $q = 3$ and $q = 4$. We wish to show that M^4_3 is a simple spread of M^4_4.

First, consider $q = 3$. By Proposition 4, the limit equilibrium policy is $b^{4,4}_{2,3}$, which simply depends on the realizations of X_2 and X_3. For $q = 4$, the characterization is less straight forward. We have:

$$\mu(x) = \begin{cases} b_{1,2}^{4,4} & x_2 > b_{1,2}^{4,4} \\ x_2 & b_{2,3}^{4,4} \leq x_2 \leq b_{1,2}^{4,4} \\ b_{2,3}^{4,4} & x_2 < b_{2,3}^{4,4} < x_3 \\ x_3 & b_{3,4}^{4,4} \leq x_3 \leq b_{2,3}^{4,4} \\ b_{3,4}^{4,4} & x_3 < b_{3,4}^{4,4} \end{cases}$$

To show that M_3^4 is a simple spread of M_4^4, we take each of the five cases that might prevail when $q = 4$ and check the corresponding outcome when $q = 3$.

- If $\mu_4^4 = b_{1,2}^{4,4}$, then $b_{1,2}^{4,4} < x_2 < b_{1,2}^{4,3}$, so $\mu_4^4 < \mu_4^3$.
- If $\mu_4^4 = x_2$, then $\mu_4^4 = x_2 < \mu_4^3$.
- If $\mu_4^4 = b_{2,3}^{4,4}$, then $x_2 < b_{2,3}^{4,4} < x_3$, so $\mu_4^4 < \mu_4^3$ iff $b_{2,3}^{4,4} < b_{2,3}^{4,3}$.
- If $\mu_4^4 = x_3$, then $\mu_4^4 = x_3 > \mu_4^3$.
- If $\mu_4^4 = b_{3,4}^{4,4}$, then $b_{3,4}^{4,4} > x_3 > b_{2,3}^{4,3}$, so $\mu_4^4 > \mu_4^3$.

It is immediate that $M_3^4 < M_4^4$ iff $b_{2,3}^{4,3} < b_{2,3}^{4,4}$. But conditional on (x_1, x_4), the former is an increasing random variable and the latter is constant, and so there is a single crossing. $\square$

Proof of Lemma 7. We wish to show that $\frac{\partial \mu}{\partial x_{sq}} < 0$. First, let $\Delta u_i(y, x_{sq}) = u_i(y) - u_i(x_{sq})$, and let ϕ represent the probability weights of Nash Bargaining problem. Now, let ψ be the log transform of the Nash Bargainng equation, i.e.

$$\psi(y, x_{sq}, \phi) = (1 - \phi) \ln [\Delta u_l(y, x_{sq})] + \phi \ln [\Delta u_r(y, x_{sq})]$$

Note that

$$\psi_y = (1-\phi)\frac{u_l'(y)}{\Delta u_l(y, x_{sq})} + \phi\frac{u_r'(y)}{\Delta u_r(y, x_{sq})}$$

and

$$\psi_{y,y} = (1-\phi)\frac{u_l'(y)}{\Delta u_l(y, x_{sq})}\frac{u_l'(x_{sq})}{\Delta u_l(y, x_{sq})} + \phi\frac{u_r'(y)}{\Delta u_r(y, x_{sq})}\frac{u_r'(x_{sq})}{\Delta u_r(y, x_{sq})}$$

We know that ψ is strictly concave in y, as u is concave in y and log is strictly concave and monotone. Hence, $\psi_{yy}(y; x_{sq}) < 0$ for all $(y; x_{sq})$.

Let $b(x_{sq}, \phi) = \arg\max_{y \in Y} \psi(y; x_{sq}, \phi)$. We know that since ψ is strictly concave, b satisfies the first order condition, i.e. $\psi_y\left(b(x_{sq}, \phi); x_{sq}, \phi\right) = 0$ for all (x_{sq}, ϕ). Let this equation be denoted by (*).

We are interested in observing how b changes in response to changes of x_{sq}. Totally differentiate (*) with respect to the status quo.

$$\psi_{y,x_{sq}}\left(b(x_{sq}, \phi), x_{sq}, \phi\right) + \psi_{y,y}\left(b(x_{sq}, \phi), x_{sq}, \phi\right)\frac{\partial b}{\partial x_{sq}} = 0$$

It follows that

$$\frac{\partial b}{\partial x_{sq}} = -\frac{\psi_{y,x_{sq}}\left(b(x_{sq}, \phi), x_{sq}, \phi\right)}{\psi_{y,y}\left(b(x_{sq}, \phi), x_{sq}, \phi\right)}$$

Since $\psi_{y,y} < 0$, we see that $\frac{\partial b}{\partial x_{sq}}$ has the same sign as $\phi_{y,x_{sq}}$. Furthermore, since

$$(1-\phi)\frac{u_l'(b)}{\Delta u_l(y, x_{sq})} = -\phi\frac{u_r'(b)}{\Delta u_r(y, x_{sq})}$$

we have:

$$\psi_{y,x_{sq}}\left(b(x_{sq}, \phi); x_{sq}, \phi\right) = \phi\frac{u_r'(b)}{\Delta u_r(y, x_{sq})}\left[\frac{u_r'(x_{sq})}{\Delta u_r(y, x_{sq})} - \frac{u_l'(x_{sq})}{\Delta u_l(y, x_{sq})}\right] > 0$$

, where the inequality follows from the fact that Δu is log-supermodular and $x_r > x_l$. $\square$

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