Contraction Algorithm

The Analysis

Design and Analysis
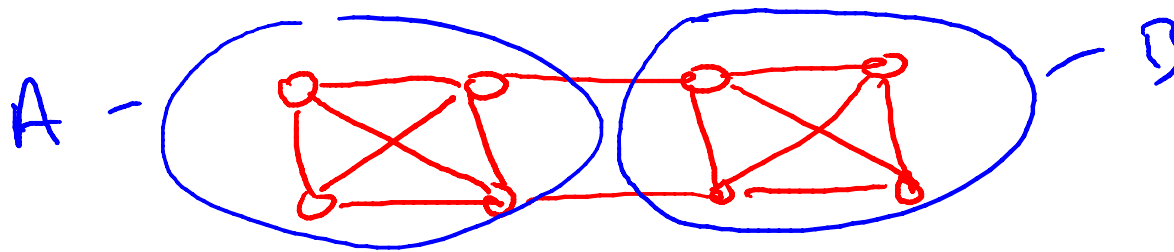of Algorithms I

The Minimum Cut Problem

Input: An undirected graph G = (V, E).
[parallel edges ⊂⊃ allowed]
[See other video for representation of input]

Goal: Compute a cut with fewest number of crossing edges.
(a min cut)

Random Contraction Algorithm

[due to Karger, early 90s]

While there are more than 2 vertices:
- pick a remaining edge (u,v) uniformly at random
- merge (or "contract") u and v into a single vertex
- remove self-loops

return cut represented by final 2 vertices.

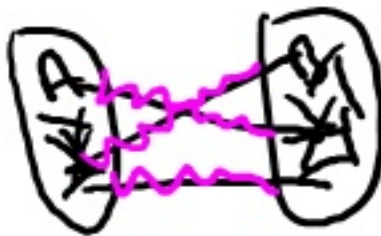Tim Roughgarden

The Setup

Question: what is the probability of success?
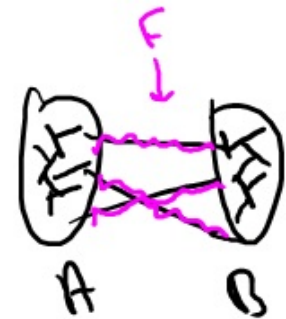
Fix a graph G = (V, E) with n vertices, m edges.

Fix a minimum cut (A, B).

Let k = # of edges crossing (A, B). (Call these edges F)

What Could Go Wrong?

1. Suppose an edge of F is contracted at some point
   $\Rightarrow$ algorithm <u>will not</u> output (A,B).

2. Suppose only edges inside A or inside B get
   contracted $\Rightarrow$ algorithm <u>will</u> output (A, B).

<u>Thus</u>:   Pr [output is (A, B)] = Pr [never contracts an edge of F]

Let S_i = event that an edge of F contracted in iteration i.

Goal: Compute $\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge \ldots\ldots \wedge \neg S_{n-2}]$

What is the probability that an edge crossing the minimum cut (A, B) is chosen in the first iteration (as a function of the number of vertices n, the number of edges m, and the number k of crossing edges)?

○ k/n

○ k/m

$$\Pr[S_1] = \frac{\#\text{ of crossing edges}}{\#\text{ of edges}} = \frac{k}{m}$$
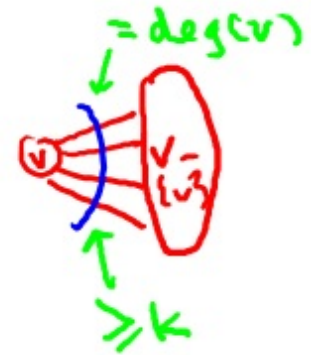
○ k/n^2

○ n/m

The First Iteration

<u>Key Observation</u>: degree of each vertex is at least k

# of incident edges

<u>Reason</u>: each vertex v defines a cut ({v}. V-{v}).

Since $\sum_v degree(v) = 2m,$ we have $m \geq \frac{kn}{2}$

$\geq kn$

Since $\Pr[S_1] = \frac{k}{m}, \Pr[S_1] \leq \frac{2}{n}$

$= deg(v)$

$v - \{v\}$

$\geq k$

The Second Iteration

<u>Recall:</u> $\Pr[\neg S_1 \wedge \neg S_2] = \Pr[\neg S_2 | \neg S_1] \cdot \Pr[\neg S_1]$

$$= 1 - \frac{k}{\text{\# of remaining edges}} \qquad \geq \left(1 - \frac{2}{n}\right)$$

what is this?

Note: all nodes in contracted graph define cuts in G
(with at least k crossing edges).
➢ all degrees in contracted graph are at least k
So: # of remaining edges $\geq \frac{1}{2}k(n-1)$

So $\Pr[\neg S_2 | \neg S_1] \geq 1 - \dfrac{2}{(n-1)}$

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All Iterations

In <u>general</u>:

$$\Pr[\neg S_1 \wedge \neg S_2 \wedge \neg S_3 \wedge \dots \wedge \neg S_{n-2}]$$

$$= \Pr[\neg S_1] \Pr[\neg S_2 | \neg S_1] \Pr[\neg S_3 | \neg S_2 \wedge \neg S_1] \dots \Pr[\neg S_{n-2} | \neg S_1 \wedge \dots \wedge \neg S_{n-3}]$$

$$\geq (1 - \tfrac{2}{n})(1 - \tfrac{2}{n-1})(1 - \tfrac{2}{n-2}) \dots (1 - \tfrac{2}{n-(n-4)})(1 - \tfrac{2}{n-(n-3)})$$

$$= \tfrac{n-2}{n} \cdot \tfrac{n-3}{n-1} \cdot \tfrac{n-4}{n-2} \dots \tfrac{2}{4} \cdot \tfrac{1}{3} = \tfrac{2}{n(n-1)} \geq \tfrac{1}{n^2}$$

<u>Problem</u>: <u>low</u> success probability! (<u>But</u>: non trivial)

recall $\simeq 2^n$ cuts !

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Repeated Trials

<u>Solution</u>: run the basic algorithm a large number N times, remember the smallest cut found.

<u>Question</u>: how many trials needed?

Let T_i = event that the cut (A, B) is found on the ith try.
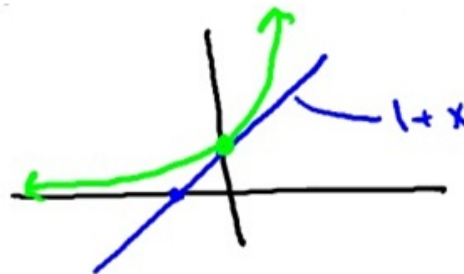
➢ by definition, different T_i's are independent

<u>So</u>:

$$\Pr[\text{all N trails fail}] = \Pr[\neg T_1 \wedge \neg T_2 \wedge ... \wedge \neg T_N]$$

$$= \prod_{i=1}^{N} \Pr[\neg T_i] \leq (1 - \frac{1}{n^2})^N$$

By independence !

Repeated Trials (con'd)

Calculus fact: $\forall$ real numbers x, $1+x \le e^x$

$\Pr[\text{all trials fail}] \le (1 - \frac{1}{n^2})^N$

$-1+x$

So: if we take $N = n^2, \Pr[\text{all fail}] \le \left(e^{-\frac{1}{n^2}}\right)^{n^2} = \frac{1}{e}$

If we take $N = n^2 \ln n, \Pr[\text{all fail}] \le \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}$

Running time: polynomial in n and m but slow $(\Omega(n^2 m))$

But: can get big speed ups (to roughly $O(n^2)$) with more ideas.

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