1st ex $y'' - y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n \cdot n\, x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$n \Rightarrow n+2$

$$\sum_{n=0}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^n + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2} \cdot (n+2) \cdot (n+1) - a_n \right] x^n = 0$$

$$\Rightarrow \quad a_{n+3} \cdot (n+2)(n+1) = a_n$$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}, \quad n \geq 0$$

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6}$$

$a_4, a_5, \ldots$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 \ldots$$

$$y(x) = a_0 \left(1 + \frac{1}{2} x^2 \ldots\right)$$

$$y(x) = a_0\left(1+\tfrac{1}{2}x^2 \ldots\right) + a_1\left[x + \frac{x^3}{6} + \cdots\right]$$

$$y'' - y = 0$$

(1) $y'' + xy = 0$, $y = \sum_{n=0}^{\infty} a_n x^n$

$x = a_0$ $\quad y = \sum_{n=0}^{\infty} a_n (x-a_0)^n$

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1)\, x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$n \to n+2$ $\quad$ $n \to n-1$

$$\sum_{n=0}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow a_2 \cdot 2 \cdot 1 + \sum_{n=1}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$\Rightarrow$ $2a_2 = 0$
$a_2 = 0$

$$\sum_{n=1}^{\infty} \left[a_{n+2} \cdot n+2 \cdot n+1 + a_{n-1} \right] x^n = 0$$

$\Rightarrow$ $a_{n+2} \cdot n+2 \cdot n+1 + a_{n-1} = 0$

$\Rightarrow$ $\boxed{a_{n+2} = \dfrac{-a_{n-1}}{(n+2)(n+1)}}$, $n = 1$

a

(c)

$$(x^2+1)y'' + xy' - xy = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n \, n \cdot n-1 \, x^{n-2}$$

$\Rightarrow$ $x^2 y'' + y'' + xy' - xy = 0$

$$\Rightarrow \quad x^2 \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \, x^{n-2} + \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \, x^{n-2}$$

$$+ x \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} - x \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow \quad \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \, x^n + \sum_{n=2}^{\infty} \underline{a_n \cdot n \cdot n-1 \cdot x^{n-2}} \quad n \to n+2$$

$$+ \sum_{n=1} a_n \cdot n \, \underline{x^n} - \sum_{n=0}^{\infty} a_n \cdot \underline{x^{n+1}} = 0 \quad n \to n-1$$

$$\sum_{n=1} \cdots \qquad \sum_{n=0} \cdots$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \cdot x^n + \sum_{n=0}^{\infty} a_{n+2} \cdot n+2 \cdot n+1 \cdot x^n$$

$$+ \sum_{n=1}^{\infty} a_n \cdot n \cdot x^n - \sum_{n=1}^{\infty} a_{n-1} \cdot x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n \cdot n \cdot n-1 \cdot x^n + a_2 \cdot 2 \cdot 1 + a_3 \cdot 3 \cdot 2 \cdot x + \sum_{n=2}^{\infty} a_{n+2} \cdot n+2 \cdot n+1 \cdot x^n$$

$$+ a_1 \cdot 1 \cdot x + \sum_{n=2}^{\infty} a_n \cdot n \cdot x^n - a_0 x - \sum_{n=2}^{\infty} a_{n-1} \cdot x^n = 0$$

$$\Rightarrow 2a_2 + x(6a_3 + a_1 - a_0) + \sum_{n=2}^{\infty} \left\{ a_n \cdot n \cdot n-1 \cdot + a_{n+2} \cdot n+2 \cdot n+1 + a_n \cdot n - a_{n-1} \right\} x^n = 0$$

$a_2 = 0$

$$\boxed{6a_3 = a_1 - a_0}$$

$$a_n \cdot n \cdot n-1 + a_{n+2} \cdot n+2 \cdot n+1 + a_n \cdot n - a_{n-1} = 0$$

$$\Rightarrow a_{n+2} \cdot (n+2)(n+1) = a_{n-1} - n \cdot a_n - a_n \cdot n \cdot n-1$$

$$\Rightarrow a_{n+2} = \frac{a_{n-1} - n \cdot a_n - a_n \cdot n \cdot n-1}{(n+2)(n+1)}$$

$, n \geq 1$

1 → $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$

2 → $x y'' - xy' + 2y = 0, \quad x = 0$

$f(x)$ — continuous

$y'' - xy' + 2y = 0, \quad x = 0$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

3 → $y'' + \frac{y'}{x} + \left(\frac{x^2 - 1/4}{x^2}\right) y = 0$

$x P(x), \quad x^2 (Q(x))$

$y = \sum_{n=0}^{\infty} a_n \cdot x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n \cdot (n+r) x^{n+r-1}$

$y'' = \sum_{n=2}^{\infty} a_n \cdot (n+r)(n+r-1) \cdot x^{n+r-2}$

$\Rightarrow x^2 \sum_{n=2}^{\infty} a_n (n+r)(n+r-1) \cdot x^{n+r-2} + x \sum_{n=1}^{\infty} a_n (n+r) x^{n+r-1}$

$+ \left(x^2 - \frac{1}{4}\right) \sum_{n=0}^{\infty} a_n \cdot x^{n+r} = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n+r)(n+r-1) \cdot x^{n+r} + \sum_{n=1}^{\infty} a_n \cdot (n+r) \cdot x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n \cdot x^{n+r+2} - \sum_{n=0}^{\infty} \frac{1}{4} a_n \cdot x^{n+r} = 0$$

$n \to n-2$

$$\Rightarrow (\quad) + (\quad) + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - (\quad) = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} \left[a_n (n+r)(n+r-1) + a_n (n+r) + a_{n-2} - \frac{1}{4} a_n \right] x^{n+r}$$

$a_1 (r-1) x^{r+1}$ $\quad$ $\cancel{a_1 (r+1) x}$ $\quad - \frac{1}{4} a_0 x^r - \frac{1}{4} a_1 x^{1+r} = 0$

$a_1 (1+r) - \frac{1}{4} a_1 = 0$ $\quad$ $a_0 \neq 0$, $a_1 \left(r + 1 - \frac{1}{4} \right) = 0$, $r = -3/4$

$$a_n (n+r)(n+r-1) + a_n (n+r) + a_{n-2} - \frac{1}{4} a_n = 0$$

$$a_n \left[(n+r)(n+r-1) + (n+r) - \frac{1}{4} \right] = -a_{n-2}$$

$$a_n \left[n^2 + r^2 + 2rn - n - r + n + r - \frac{1}{4} \right] = -a_{n-2}$$

$$\Rightarrow a_n = \frac{-a_{n-2}}{(n+r)^2 - 1/4}, \quad n \geq 2$$

$$a_2 = \frac{-a_0}{(2+r)^2 - \frac{1}{4}} \qquad \frac{-a_0}{\left(2 - \frac{3}{4}\right)^2 - \frac{1}{4}}$$

$a_3 =$

a_4

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad r = -\frac{3}{4}$$

$x \sin x = \ldots \quad x > 0$

$n=0$

(4) $xy'' + y\sin x = 0 \quad x=0$

$\Rightarrow y'' + \frac{y}{x}\sin x = 0$

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots$

$\sin x = x$