§4.4 Method of Undetermined Coefficient (Guessing game)

eg $y'' - y' - 2y = e^{3x}$

Step 1 : Solution to associated homogeneous equation.

$m^2 - m - 2 = 0$ $\quad\quad m_1 = 2$

$(m-2)(m+1) = 0$ $\quad\quad m_2 = -1$

$y_h = c_1 e^{2x} + c_2 e^{-x}$

Step 2 : M of UDC : $y_p = Ae^{3x}$ guess

$y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$

plug this into the DE :

$y'' - y' - 2y = e^{3x}$

$9Ae^{3x} - 3Ae^{3x} - 2(Ae^{3x}) \stackrel{set}{=} e^{3x}$

$e^{3x}(9A - 3A - 2A) = e^{3x}$

$4A = 1 \Rightarrow A = \frac{1}{4}$

$y_p = \frac{1}{4}e^{3x}$

Step 3 : L.I ? yes . So no modification is needed.

Step 4 : $y = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4}e^{3x}$

eg $y'' - y' - 2y = 4x^2$

$m^2 - m - 2 = 0$

$(m+1)(m-2) = 0$

$m_1 = -1, \; m_2 = 2$

$y_h = C_1 e^{-x} + C_2 e^{2x}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$\therefore \; 2A - (2Ax + B) - 2(Ax^2 + Bx + C) \overset{\text{set}}{=} 4x^2$

$2A - 2Ax - B - 2Ax^2 - 2Bx - 2C = 4x^2$

$-2Ax^2 + (-2A - 2B)x + (2A - B - 2C) = 4x^2$

$-2A = 4$

$A = -2$

$-2A - 2B = 0$

$-2B = 2(-2)$

$-2B = -4$

$B = 2$

$2A - B - 2C = 0$

$2(-2) - 2 - 2C = 0$

$2C = -6$

$C = -3$

$y_p = -2x^2 + 2x - 3$

$y = C_1 e^{-x} + C_2 e^{2x} - 2x^2 + 2x - 3$

LS no modif.

eg $y'' - y' - 2y = \sin 2x$

$y_h = c_1 e^{-x} + c_2 e^{2x}$

$y_p = A \sin 2x + B \cos 2x$

$y_p' = 2A \cos 2x - 2B \sin 2x$

$y_p'' = -4A \sin 2x - 4B \cos 2x$

y''	$-4A \sin 2x$	$-4B \cos 2x$
$-y'$	$2B \sin 2x$	$-2A \cos 2x$
$-2y$	$-2A \sin 2x$	$-2B \cos 2x$

$(-6A + 2B) \sin 2x + (-2A - 6B) \cos 2x \overset{set}{=} \sin 2x$

$-6A + 2B = 1 \qquad (-2A - 6B = 0)^{-3}$

$$\begin{aligned} -6A + 2B &= 1 \\ 6A + 18B &= 0 \\ \hline 20B &= 1 \\ B &= \frac{1}{20} \end{aligned}$$

$$\begin{aligned} -6A + 2\left(\tfrac{1}{20}\right) &= 1 \\ -6A &= 1 - \tfrac{1}{10} \\ -6A &= \tfrac{9}{10} \\ A &= \frac{\cancel{9}^{3}}{10} \cdot \frac{-1}{\cancel{6}_{2}} = \frac{-3}{20} \end{aligned}$$

$y_p = -\frac{3}{20} \sin 2x + \frac{1}{20} \cos 2x$

LI - no modification

$y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{20} \sin 2x + \frac{1}{20} \cos 2x$

$L(y) = g(x) \quad , \; g(x) \neq 0$

eg Determine the form of y_p if the general solution to the homo. equation $L(y) = 0$ is

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$ (Alway check for LI)

y_h

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a) $g(x) = 2x + 3e^{8x}$

$g_1(x) = 2x \qquad g_2(x) = 3e^{8x}$

$y_{p_1} = Ax + B \qquad y_{p_2} = Ce^{8x}$

$$y = c_1 e^{2x} + c_2 e^{3x} + Ax + B + Ce^{8x}$$

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b) $g(x) = x \cos 3x$

$$y_p = (Ax + B) \sin 3x + (Cx + D) \cos 3x$$

PAIN - use columns.

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c) $g(x) = 2x^3 e^{5x}$

$$y_p = (A_3 x^3 + A_2 x^2 + A_1 x + A_0) e^{5x}$$

Can Differential Operator

Modification multiply by x^r for some r ($r \le m$) to get linear independence with the terms of y_h.

eg $y'' = 9x^2 + 2x - 1$

$m^2 = 0$

$m = 0$

mult. 2

$y_h = C_1 + C_2 x$

$\cancel{y_p = Ax^2 + Bx + C}$

$\cancel{y_p = Ax^3 + Bx^2 + Cx}$ OK

$y_p = Ax^4 + Bx^3 + Cx^2$ x^2

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eg $y'' - 5y' + 4y = 8e^x$

$y_h = C_1 e^x + C_2 e^{4x}$

$\cancel{y_p = Ae^x}$

$y_p = Axe^x$

use Diff-Operator

eg $y'' - 5y' + 6y = 2xe^{3x}$

$y_h = c_1 e^{2x} + c_2 e^{3x}$; $\cancel{y_p = e^{3x}(Ax + B)}$

$y_p = e^{3x}(Ax^2 + Bx)$ OK

$L(y_p) = 2xe^{3x}$

$D(e^{3x} y) = e^{3x} y' + 3e^{3x} y$

$= e^{3x}(y' + 3y)$

$= e^{3x}(D+3)[y]$

$(D^2 - 5D + 6)[e^{3x}(Ax^2 + Bx)]$

$= e^{3x}((D+3)^2 - 5(D+3) + 6)[Ax^2 + Bx]$

$= e^{3x}(D^2 + D)[Ax^2 + Bx]$

$(D+3)^2 = D^2 + 6D + 9$
$-5D - 15$
6

$D^2 + D + 0$

$= \cancel{e^{3x}}(4A + 2Ax + B) \stackrel{\text{set}}{=} 2x\cancel{e^{3x}}$

$2A = 2$; $2A + B = 0$

$A = 1$; $B = -2(1) = -2$

$(Ax^2 + Bx)' = 2Ax + B$

$(Ax^2 + Bx)''$

$y_p = e^{3x}(x^2 - 2x)$

$y = c_1 e^{2x} + c_2 e^{3x} + e^{3x}(x^2 - 2x)$