Chapter 2 First-order DE

§2.1 Solution Curves without a Solution

Solution Curve: The graph of a solution ϕ of an ODE is called a solution curve on I.

Since ϕ is a differentiable function, it is continuous on I.

$\therefore$ no breaks and it has a tangent line at each point $(x, y(x))$.

§2.1.1 Direction fields (SKIP).

§2.12. Autonomous First-order Equations

Assume x is the independent variable and y is the dependent variable.

An ODE in which the independent variable does NOT appear is called an autonomous equation.

1st order autonomous equation is of the form

$$\frac{dy}{dx} = f(y) \qquad (2).$$

Def: Critical point (equilibrium point).

A real number c is a critical point of the autonomous DE (2) if it is a zero of f, that is $f(c) = 0 \Rightarrow \frac{dy}{dx} = 0$.

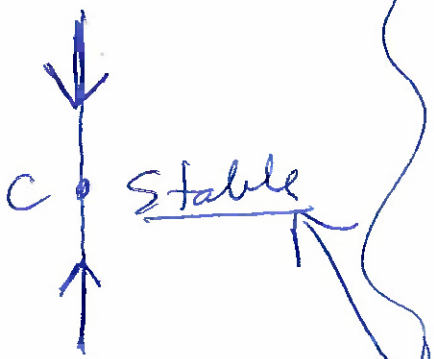
i.e $y = c$ a constant function.

If c is a critical point of (2), then $y(x) = c$ is a constant solution of the ADE; ↓ graph is a horizontal line.

We will use the sign diagram of the derivative to generate a (one-dimensional) phase portrait. ⇝ phase line (vertical line ↑)

Phase portrait is a diagram that illustrates in which direction (a) solution curve travels above and below the equilibrium points.

Attractors and Repellers



$\lim_{x \to \infty} y(x) \neq c$

unstable

an Attractor

Attractors – all solution $y(x)$ of $\frac{dy}{dx} = f(y)$ that start from an initial point (x_0, y_0) sufficient close to c (not beyond another equilibrium point) exhibit the asymptotic behavior $\lim_{x \to \infty} y(x) = c$.

semistable

Asy. behavior depends on where y_0 is in relation to c.

$\lim_{x \to \infty} y(x) \neq c$

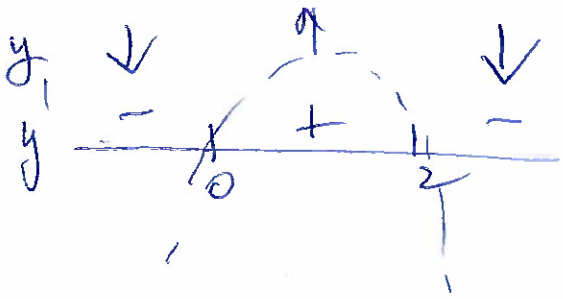
$\lim_{x \to \infty} y(c) = c$

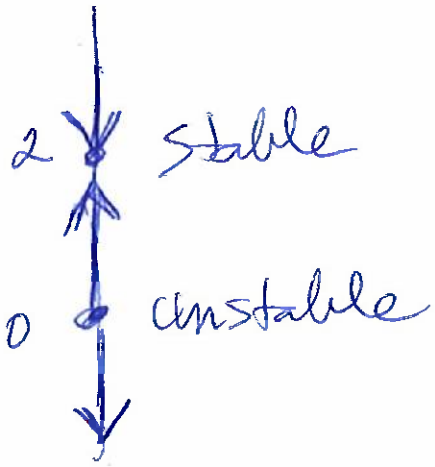
Example: Find the critical points and the phase portrait of the given Autonomous 1st-order DE. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the xy-plane determined by the graphs in the equilibrium solutions.

eg $y' = 2y - y^2 = y(2-y)$

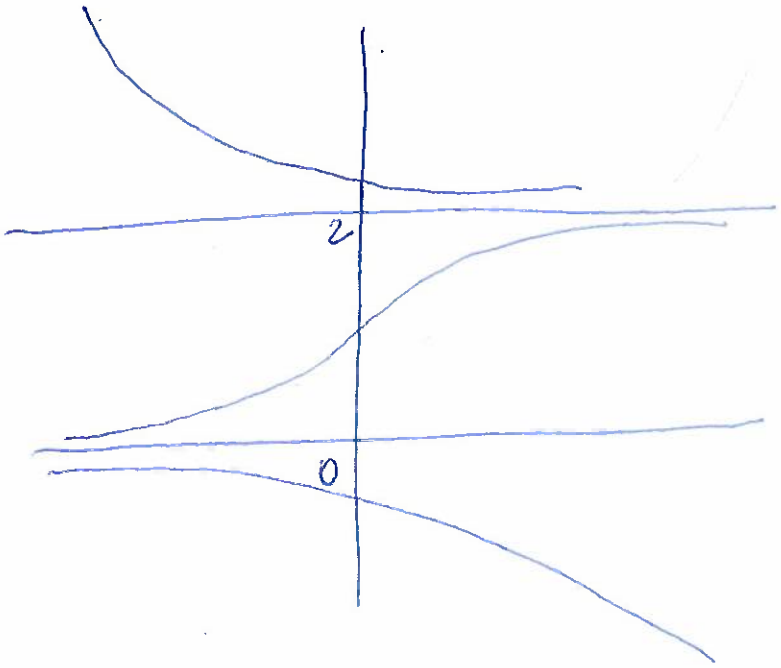
$y' = y(2-y) \overset{\text{set}}{=} 0$ ← CP

$y = 0, y = 2$





phase portrait



When you go from one equilibrium point to another, there is only one way to do this (given phase line & IP).

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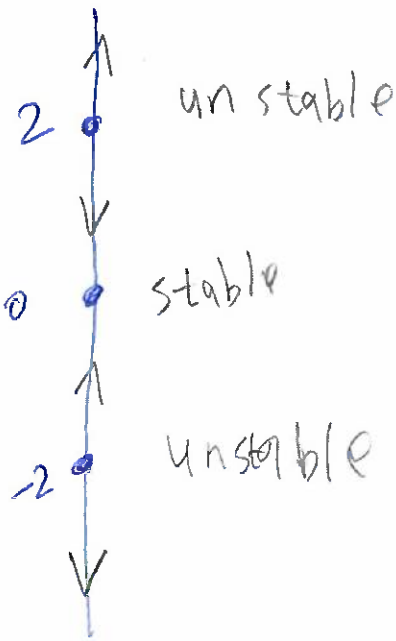
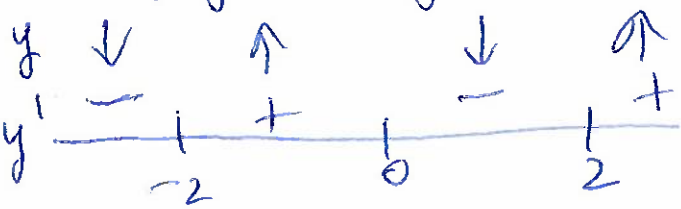
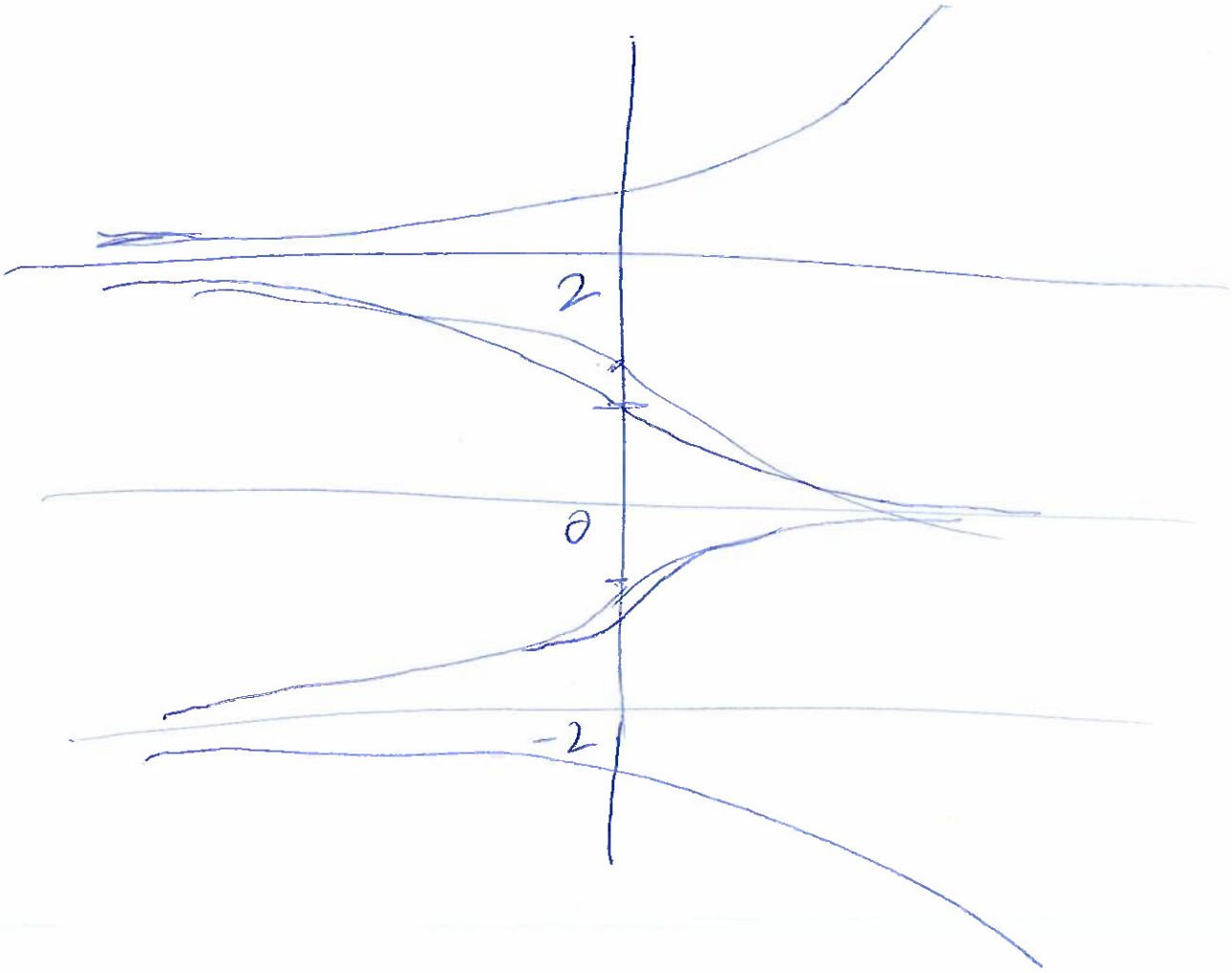
e.g. $y' = y^3 - 4y = y(y^2-4) = y(y+2)(y-2)$

$y' \overset{\text{set}}{=} 0 \Rightarrow y = -2, y = 0, y = 2$ CP





$y+2$	−	+	+	+
y	−	−	+	+
$y-2$	−	−	−	+



$y'' = 3y^2 y' - 4y'$

$= y'(3y^2 - 4)$ set $= 0$

$y = -2$

$y = 0$

$y = 2$ Not required.

$\rightarrow y = \pm \frac{2}{\sqrt{3}} \approx \pm 1.154$