Monday 8/28

Office hours MWF 1:30 - 2:30

Phone # 215 983 3816

- 2 midterms 50%
- final 35%
- quizzes 12% every 2 weeks on Friday
- HW 3% due every Friday

Q1	Q2	T1	Q3	Q4	T2	Final
9/8/23						

Chapter 1: Experiments w/ random outcomes

1.1 Sample spaces & probabilities

Def The set of all possible outcomes of a random experiment is called a sample space Ω

e.g.: Toss a coin $\Omega = \{H, T\}$

$A = \{H\}$

$B = \{T\}$

$C = \emptyset \longrightarrow P(\emptyset) = 0$

$D = \{H, T\} \longrightarrow P(D) = 1$ head or tail

Def A subset of Ω is called an event

Let $\mathcal{F}$ the class of all events in Ω $\quad \mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$

- sigma field
- set of sets

Def 1.1 A fn $P: \mathcal{F} \to \mathbb{R}$ is called a prob. measure if:

(i) $0 \leq P(A) \leq 1$ for $A \in \mathcal{F}$

(ii) $P(\Omega) = 1$, $P(\emptyset) = 0$

(iii) If $A_1, A_2 \dots A_n$ is a sequence of $\overbrace{\text{pairwise disjoint events}}^{\text{no intersection}}$ then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$P\Big(\overset{A_1}{\bigcirc} \cup \overset{A_2}{\bigcirc} \cup \ldots \overset{A_n}{\bigcirc} \ldots\Big) = \sum_{i=1}^{\infty} P(A_i)$$

$A_1, A_2 \ldots A_n$ are said to be pairwise disjoint if $A_i \cap A_j = \emptyset$ for all $i \neq j$

Def → $(\Omega, \mathcal{F}, P)$ is called a prob. space

e.g.: $P(\{H\}) = 1/3$, $P(\{T\}) = 2/3$ > disjoint & union gives $P(\{H,T\}) = 1$, $P(\emptyset) = 0$ satisfy prob. measure

Notation → $P(\{H\}) = P(H)$ but $P(\emptyset) \neq P(0)$

$\{\ \} = \emptyset \neq \{\emptyset\}$

e.g.: Roll a die $\Omega = \{1, 2, 3, 4, 5, 6\}$. Suppose this is a fair dice

$P(\{1\}) = 1/6 \ldots P(\{6\}) = 1/6 \longrightarrow P(\{2,4,6\}) = 1/2 = P(\{\text{even \#s}\})$

$\{2\} \cap \{4\} = \emptyset$
$\{4\} \cap \{6\} = \emptyset$
$\{2\} \cap \{6\} = \emptyset$
> pairwise disjoint $\xrightarrow{\text{by def}} P(\{2\} \cup \{4\} \cup \{6\})$
$= P(\{2\}) + P(\{4\}) + P(\{6\})$
$= 1/6 + 1/6 + 1/6 = 1/2$

Appendix B Set Notations & Operations

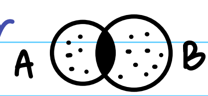
Let A, B be sets

(1) $A \subseteq B$ if every element of A is an element of B

(2) $A = B$ if $A \subseteq B$ and $B \subseteq A$

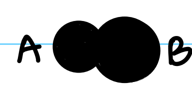
(3) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ $x \in A$ if x is an element of A

↳ event that both A & B occur

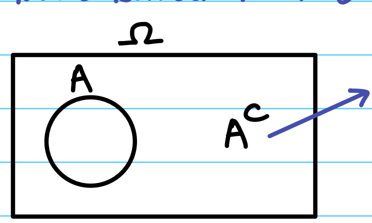


(4) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

↳ event that either A or B occurs

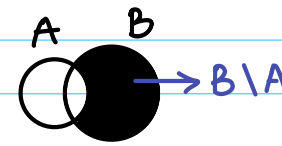


$A \subseteq \Omega$



$A^c = \{x \mid x \in \Omega,\ x \notin A\}$

$B \setminus A = \{x \mid x \in B,\ x \notin A\}$



(5) $A \times B = \{\underbrace{(a,b)}_{\text{ordered pair}} \mid a \in A,\ b \in B\}$

! Equations:

(1) De Morgan's Law $\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c$ $\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$

(2) Distribution's Law

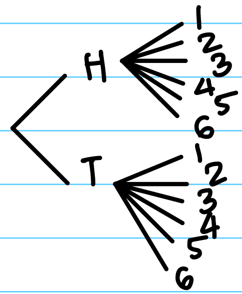
$$A \cap \left(\bigcup_{i=1}^{\infty} B_i \right) = \bigcup_{i=1}^{\infty} (A \cap B_i) \qquad A \cup \left(\bigcap_{i=1}^{\infty} B_i \right) = \bigcap_{i=1}^{\infty} (A \cup B_i)$$

e.g: Toss a coin then roll a die. Find the sample space

$$A \times B = \{(H,1), \dots (H,6), (T,1), \dots (T,6)\} = \Omega$$

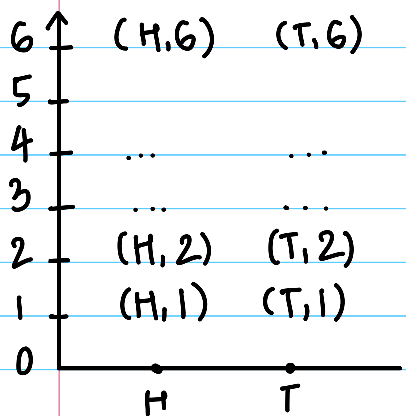
A: $\{H, T\}$ B: $\{1,2,3,4,5,6\}$

Tree diagram



$$\Omega = \{H,T\} \times \{1,2,3,4,5,6\}$$
$$= \{(H,1) \dots (H,6), (T,1) \dots (T,6)\}$$

→ each is a sample point



Wednesday 8/30

1.2 Random sampling

Population size $= n$, sample size $= k$

	Sampling w/o replacement	Sampling w/ replacement
Permutations (ordered)	$\underbrace{n \;\; n-1 \;\; n-2 \dots n-(k-1)}_{k}$ ($n-k+1$) $= n(n-1)(n-2)\dots(n-k+1)$ $= \boxed{\dfrac{n!}{(n-k)!} = {}_nP_k}$	$\underbrace{n \;\; n \;\; n \;\dots\; n}_{k}$ $= \boxed{n^k}$
Combinations (unordered)	$\boxed{{}_nC_k = \dfrac{n!}{(n-k)!\,k!}}$ or $\binom{n}{k}$	$\boxed{\binom{n+k-1}{k} = \binom{n+k-1}{n-1}}$

$$\boxed{{}_{K}P_{K} = \frac{K!}{(K-K)!} = \frac{K!}{0!} = \frac{K!}{1} = K!}$$

e.g: $n = 3$

Population set $= \{a, b, c\}$

Choose 2 $\longrightarrow K = 2$

Permutations	Combinations
(a,b), (b,a)	$\{a,b\}$
(a,c), (c,a)	$\{a,c\}$
(b,c), (c,b)	$\{b,c\}$

$$({}_{n}C_{K})\,K! = {}_{n}P_{K}$$

$$\longrightarrow {}_{n}C_{K} = \frac{{}_{n}P_{K}}{K!} = \frac{n!}{(n-K)!\,K!}$$

e.g: n chidren. How many ways can you arrange them in order?

③ ② ① $= 3! = 6 = {}_{3}P_{3} \longrightarrow$ 3 choose 3 permutations

e.g: $n = 3$ $\{ch, s, v\}$. Choose 12 w/ replacement. How many combinations? ↷ no order

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

$\hookrightarrow \binom{14}{2} = \binom{14}{12}$ how?

1.10 Suppose our urn contains 5 balls labeled 1, 2, 3, 4, 5. Sample 3 balls w/ replacement & produce an ordered list of the # drawn

(a) Find the sample space

$\Omega = S \times S \times S = \{(s_1, s_2, s_3) \mid s_i \in S \text{ for } i = 1, 2, 3\}$

(b) Find prob. that the sample is $(2, 1, 5)$

$$P(\{2,1,5\}) = \frac{1}{5.5.5} = \frac{1}{125}$$

1.11 Choose 3 w/o replacement in order

(a) $\Omega = \{(s_1, s_2, s_3) \mid s_i \in S,\ i = 1, 2, 3 \ \&\ s_i \neq s_j \text{ for all } i \neq j\}$

(b) $P((2,1,5)) = \dfrac{1}{\#\Omega} = \dfrac{1}{\underbrace{5(4)(3)}_{{}_{5}P_{3}}} = \dfrac{1}{60}$

notation for unorder

1.12 Choose 3 w/o replacement unordered samples

(a) $\Omega = \{\{s_1, s_2, s_3\} \mid s_i \in S,\ s_i \neq s_j \text{ for all } i \neq j\}$
$= \{$3-element subsets of $S\}$

(b) $$P(\{\{2,1,5\}\}) = \frac{1}{{}_5C_3} = \frac{1}{\frac{5!}{(5-3)!\,3!}} = \frac{2!\,3!}{5!} = \frac{1}{10}$$

1.13 Suppose we have a class of 24 children.

(a) Every day a random student is chosen to lead the class to lunch, w/o regards to previous choices. Find prob. that Cassidy was chosen on M & W and Aaron on T

$$P = \frac{(1)(1)(1)}{(24)(24)(24)} = \frac{1}{24^3}$$

(b) 3 students are chosen randomly to be class president, vice president & treasurer. No student can hold more than 1 office

(i) P(Mary is president, Cory vice president, Matt is treasurer) → in order

$$= \frac{1}{24(23)(22)}$$

(ii) $$P(\text{Ben is president or vice-president}) = \frac{1\cancel{(23)}\cancel{(22)}}{24\cancel{(23)}\cancel{(22)}} + \frac{\cancel{23}(1)\cancel{(22)}}{24\cancel{(23)}\cancel{(22)}} = \frac{1}{12}$$

(c) $$P(\text{committee of } 3) = \frac{1}{{}_{24}C_3}$$

(d) $$P(\text{Mary in committee}) = \frac{{}_{23}C_2}{{}_{24}C_3} = 1/8$$ → need 2 more positions besides Mary

e.g.: 3 g, 2 r

Choose 2 at random w/o replacement

$$P(1g, 1r) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}$$ → unorder

the same

combination for arrangement

$$P(1g, 1r) = \underset{g\ r}{\frac{3.2}{5.4}} + \underset{r\ g}{\frac{2.3}{5.4}} = \frac{3.2}{5.4} \cdot \underbrace{\binom{2}{1}\binom{1}{1}}$$ → order

↳ P(1g, 1r in order)

1.3 Infinitely many outcomes

$\boxed{1.16}$ Flip a fair coin until the 1^{st} tail comes up. Record # of flips required

(a) Find $\Omega = \{1, 2, \ldots\} \cup \{\infty\}$

(b) $P(K) = \dfrac{\overset{H}{1}\ \overset{H}{1}\ \overset{H}{1}\ \overset{T}{1}}{\underbrace{2\ 2\ 2\ 2 \ldots}_{K}} = \dfrac{1}{2^K}$, $K = 1, 2, 3 \ldots$

$$\sum_{K=0}^{\infty} ar^K = \frac{a}{1-r} \quad (a = \tfrac{1}{2})$$

$$1 = P(\infty) + \sum_{K=1}^{\infty} P(K) = P(\infty) + \sum_{K=1}^{\infty} \frac{1}{2^K} = P(\infty) + \frac{1/2}{1-1/2} = P(\infty) + 1$$

$P(\infty) \to 0$

$\therefore$ Tail will always come up

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Ω is a discrete set

Ω has countably many elements

↳ means either finite or countably infinite

if $A = \{x_1, x_2 \ldots\}$

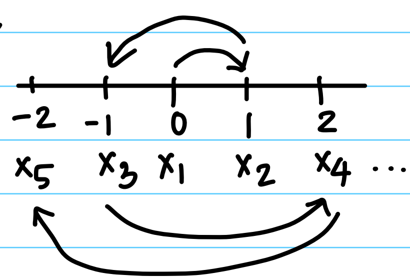
e.g. $\mathbb{N} = \{1, 2, 3 \ldots\}$ is countable

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2 \ldots\}$

$\mathbb{Q} = \{\text{rational numbers}\}$

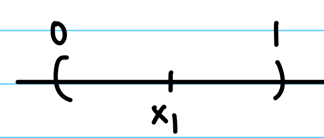
$= \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$

↳ countable



		p: 1	2	3	4	5 ...
q	1	1 1/1	2/1 2	3/1 4	4/1	5/1
	2	3 1/2	2/2	3/2	4/2	5/2
	3	5 1/3	2/3	3/3	4/3	5/3
	4	1/4	2/4	3/4	4/4	5/4
	5	1/5	2/5	3/5	4/5	5/5
	...					

BUT $(0,1) = \{x \in \mathbb{R}.\ 0 < x < 1\}$



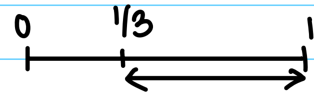
uncountable

Example for uncountable experiment:

$\boxed{1.17}$ We pick a real # uniformly at random from the closed unit interval $[0,1]$. Let X denote the # chosen.

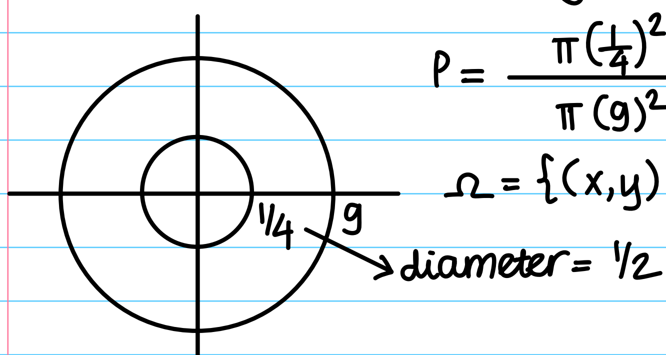
(a) Sample space $\Omega = [0,1]$

(b) $P(\frac{1}{3} < x < 1) = \frac{2}{3}$



(c) Let $0 < a < b < 1$. $P(a < X < b) = \frac{b-a}{1} = b-a$

$\boxed{1.18}$ Consider a dartboard in the shape of a disk w/ radius 9 inches. The bullseye is a disk of diameter 1/2 inch in the middle of the board. What is prob. that a dart randomly thrown on the board hit bullseye?



$$P = \frac{\pi(\frac{1}{4})^2}{\pi(9)^2} = \frac{1/16}{81} = \frac{1}{16(81)}$$

$$\Omega = \{(x,y) \mid x^2+y^2 \leq 81\}$$

1.4 Consequences of the rules of prob.

(1) Decomposing an event

Suppose $A = A_1 \cup A_2 \cup \ldots$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$

$$\boxed{P(A) = P(A_1) + P(A_2) + \ldots}$$

e.g.:

30	r
20	g
10	y

Choose 2 w/o replacement

disjoint prob. to sum

$P(\text{exactly } 1r \text{ or exactly } 1y) = P(1r, 1g) + P(1r, 1y) + P(1g, 1y)$

$$= \frac{\binom{30}{1}\binom{20}{1}}{\binom{60}{2}} + \frac{\binom{30}{1}\binom{10}{1}}{\binom{60}{2}} + \frac{\binom{20}{1}\binom{10}{1}}{\binom{60}{2}} = \frac{30(20) + 30(10) + 20(10)}{60 \cdot 59 / 2 \cdot 1}$$

(2) Inclusion-Exclusion principle

$$\boxed{P(A \cup B) = P(A) + P(B) + P(A \cap B)}$$ if A, B disjoint: $P(A \cap B) = 0$

P(exactly 1r $\cup$ exactly 1y)

= P(exactly 1r) + P(exactly 1y) − P(exactly 1r $\cap$ exactly 1y)

$$= \frac{\binom{30}{1}\binom{30}{1}}{\binom{60}{2}} + \frac{\binom{10}{1}\binom{50}{1}}{\binom{60}{2}} - \frac{\binom{30}{1}\binom{10}{1}}{\binom{60}{2}}$$

$$\boxed{\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C) \end{aligned}}$$

$$\begin{aligned} P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) \\ - P(A_1 \cap A_2) - P(A_2 \cap A_3) - \ldots - P(A_1 \cap A_n) \\ + P(A_1 \cap A_2 \cap A_3) + P(A_2 \cap A_3 \cap A_4) + \ldots \\ - P(A_1 \cap A_2 \cap A_3 \cap A_4) \ldots \end{aligned}$$

$$\boxed{P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < i_k \le n} (-1)^{k-1} P(A_1 \cap A_2 \ldots \cap A_k)}$$

General Inclusion-Exclusion principle

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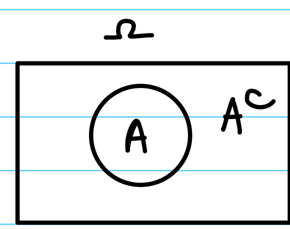
(3) Rule of complements

$$\boxed{P(A) + P(A^c) = 1}$$



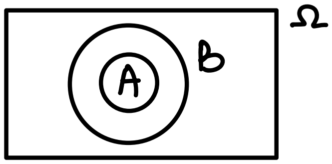
Proof $\rightarrow$ $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

$\therefore P(A^c) = 1 - P(A)$

1.21 Roll a fair die 4 times. What is the prob. that some numbers appear more than once?

2 3 2 5 + ...

$$P(A) = 1 - P(\text{all 4 numbers are diff}) = 1 - \frac{6 \times 5 \times 4 \times 3}{6 \times 6 \times 6 \times 6} = \frac{13}{18}$$



(4) Monotonicity of Prob. $\boxed{\text{If } A \subseteq B, \text{ then } P(A) \leq P(B)}$

Proof $\rightarrow$ $P(B) = \underbrace{P(A \cup (B \setminus A))}_{\text{disjoint union}} = P(A) + P(B \setminus A) \geq P(A) + 0 = P(A)$

$\boxed{1.22}$ P(repeat coin flips eventually yields tails)

$= 1 - P(\text{never get a tail})$

$0 \leq P(\text{never get a tail}) \leq P(1^{st}\ n \text{ tosses must be all heads})$

$$= \frac{1 \times 1 \times 1 \ldots}{\underbrace{2 \times 2 \times 2 \times \ldots}_{n}} = \frac{1}{2^n} \text{ for all } n = 1, 2 \ldots$$

$\frac{1}{2^n} \searrow 0$ as $n \to \infty$

$\therefore P(\text{never get a tail}) = 0$ (squeeze theorem)

$\therefore P(\text{eventually tail}) = 1$

$\boxed{1.27}$ Suppose n people randomly exchange (random permutations) their hats. What is prob. that no one gets his own hat?

$P(A) = 1 - P(\geq 1 \text{ get his own hat}) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right)$

Let $A_i = \{i\text{th person gets his own hat}\}$

fixed, random for other $(n-1)$ people

$$P(A_1) = \frac{1 \times (n-1)!}{n \times n \times n \ldots} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_1 \cap A_2) = \frac{1 \times 1 \times (n-2)!}{n \times n \ldots \times n} = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = \frac{1}{n} + \frac{1}{n} - \frac{1}{n(n-1)}$$

$$\rightarrow P(A) = 1 - \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} P(A_{i_1} \cap \ldots \cap A_{i_k})\right)$$

$$= 1 - \left(\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} \frac{(n-k)!}{n!}\right)$$

$$= 1 - \left(\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!}\right)$$

$\hookrightarrow$ counting how many terms between 1 & n

$$= 1 + \sum_{k=1}^{n} (-1)^k \frac{\cancel{n!}}{\cancel{(n-k)!}\,k!} \quad \frac{\cancel{(n-k)!}}{\cancel{n!}} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

General: $\sum_{k=0}^{k} \frac{x^k}{k!} = e^k$ $\therefore P(A) = e^{-1} = \frac{1}{e} \approx \frac{1}{2.7}$

What about:

$P(1^{st}$ person not get his$) = 1 - \frac{1}{n}$

$P(2^{nd}$ ______________ $) = 1 - \frac{1}{n}$

$\vdots$

$P(n^{th}$ ______________ $) = 1 - \frac{1}{n}$

$\Bigg\}$ P(no one gets his) $= \cancel{\left(1-\frac{1}{n}\right)^n} \to e^{-1}$

General: $\left(1+\frac{x}{n}\right)^n = e^{\ln\left(1+\frac{x}{n}\right)^n} = e^{n\ln\left(1+\frac{x}{n}\right)} = e^x$

$\boxed{1.15}$ An urn contains 4 balls: 1 white, 1 green, 2 red. We draw 3 balls w/ replacement. Find the prob. that we did not see all 3 colors.

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(a) Define the event $W = \{$white ball did not appear$\}$ & use in-exclusion

1 w
1 g
2 r

$P(W) = \frac{3\times3\times3}{4\times4\times4} = \frac{3^3}{4^3}$

$R = \{$red ball did not appear$\} \to P(R) = \frac{2^3}{4^3}$

$G = \{$green ball did not appear$\} \to P(G) = \frac{3^3}{4^3}$

$P(W\cap R) = \frac{1^3}{4^3} = P(G\cap R);\ P(W\cap G) = \frac{2^3}{4^3};\ P(W\cap R\cap G) = \frac{0}{4^3} = 0$

$\therefore P(W\cup R\cup G) = P(W) + P(R) + P(G) - P(W\cap R) - P(W\cap G) - P(G\cap R) + P(W\cap R\cap G)$

$$= \frac{3^3 + 2^3 + 3^3 - 1 - 1 - 2^3 + 0}{4^3}$$

(b) Compute prob. by considering the complement

total combination wgr, rwg...

$P(\text{not see all } 3) = 1 - P(\text{see all}) = 1 - \frac{\overset{w}{1} \times \overset{r}{2} \times \overset{g}{1} \times 3!}{4^3}$

1.5 Random variables

Def Let $(\Omega, \mathcal{F}, P)$ be a prob. space. A random variable X is a real-valued fn defined on Ω

$$X: \Omega \to \mathbb{R}$$

e.g.: Toss a coin, let $X = \#$ heads.

$\Omega = \{H, T\}$ $\quad X(H) = 1, \quad X(T) = 0$

$$\Omega = \{H, T\} \xrightarrow{X} \mathbb{R}$$

X is called the **Bernoulli random variable**.

Def Let X be a random variable.

Let $\mu_X(B) = P(\omega \in \Omega \mid X(\omega) \in B)$, $B \subseteq \mathbb{R}$

then μ_X is called the **prob. distribution of X**

$(\mathbb{R}, \mathcal{B}, \mu_X)$

- $\mathbb{R}$ = set of all real #
- $\mathcal{B}$ = set of all subsets of $\mathbb{R}$
- μ_X = prob. measure on $\mathcal{B}$

e.g.: $\mu_X(B) = P(\omega \in \{H, T\} \mid X(\omega) \in B)$

$B = (2, 6)$

can only be 1 or 0

$\mu_X((2,6)) = P(\omega \in \{H, T\} \mid 2 < X(\omega) < 6) = 0$

$\mu_X(\{1\}) = P(\omega \in \{H, T\} \mid X(\omega) = 1) = P(\text{head}) = 1/2$

$\mu_X(\{0\}) = P(\text{tail}) = 1/2$

$\mu_X((-2, 1)) = P(X = 0) = P(\text{tail}) = 1/2$

$\hookrightarrow -2 < X(\omega) < 1$

$\mu_X((-2, 1]) = P(X = 0 \text{ or } X = 1) = P(\text{head or tail}) = 1$

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, \mu_X)$$

Def A random variable X is called discrete if X has only countable possible values.

If X is discrete w/ possible values in $\{x_1, x_2 \ldots\}$, we let $P_X(x_i) = P(X = x_i)$

then $P_X(x_i)$ is called the **prob. mass fn of X**

e.g.: $P_X(0) = P(X = 0) = 1/2$ $\qquad$ $P_X(\cdot)$ is called the prob. mass fn of X

$P_X(1) = P(X = 1) = 1/2$

e.g: Roll a die 2 times. Let S = sum of outcomes of 2 dies. Find the prob. mass fn of X ?

$S = 2, 3, 4 \ldots 12$

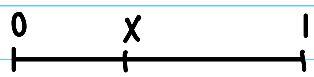
S	2	3	4	5	6	7	8	9	10	11	12
$P_S(K)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$P(S=3 \text{ or } S=5) = P(S=3) + P(S=5) = \frac{2}{36} + \frac{4}{36} = \frac{1}{6}$$

$$P(S = \text{even \#s}) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{1}{2}$$

Monday 9/18

$\boxed{1.9}$



$$P(\text{shorter piece is less than } 1/5 \text{ of the original})$$
$$= P(X < 1/5) + P(X > 4/5) = P(0 < X < 1/5) + P(4/5 < X < 1)$$
$$= 1/5 + 1/5 = 2/5$$

$\boxed{1.10}$ (a) $\Omega = \{1, 2, 3 \ldots\} \cup \{\infty\}$

$$P(K) = \frac{5 \times 5 \times 5 \times \ldots \times 1}{6^K} = \frac{5^{K-1}}{6^K}, \quad K = 1, 2, 3 \ldots$$

$$\text{(b) } P(\infty) = 1 - \sum_{K=1}^{\infty} \frac{5^{K-1}}{6^K} = 1 - \frac{1/6^{\,(K=1)\ a}}{1 - \underset{r}{5/6}} = 0$$

OR $P(\text{not } 4, \ldots, \text{not } 4) = \frac{5^K}{6^K}$ $\therefore 0 \le P(\infty) \le \frac{5^K}{6^K} \xrightarrow{} 0$ for all K $\therefore P(\infty) = 0$ by squeeze

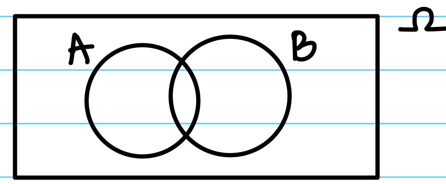
$P(K)$ = part of X

$$A \subseteq \Omega \rightarrow P(A) = \sum_{K \in A} P(K)$$

2.1 Conditional Prob.

Def Let $(\Omega, \mathcal{F}, P)$ be a prob. space. Let $A, B \subseteq \Omega$. Suppose $P(B) \neq 0$. Then the conditional prob. of A given that B has occurred is defined

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$



$$\boxed{P(B \mid B) = 1}$$

e.g: Toss a fair coin 3 times.

$$P(\underbrace{\text{exactly 2 heads}}_{A} \mid \underbrace{1^{st} \text{ coin is H}}_{B}) = \frac{P(\text{exactly 2 heads} \cap 1^{st} \text{ head})}{P(1^{st} \text{ head})}$$

$$= \frac{P(HHT) + P(HTH)}{1/2} = \frac{1/8 + 1/8}{1/2} = \boxed{\frac{1}{2}}$$

(1) Multiplication Rule $\boxed{P(A \cap B) = P(A).P(B \mid A)}$

Proof $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} \longrightarrow P(A \cap B) = P(A).\ P(B \mid A)$

2.7 $\boxed{8 \text{ (r) } 4 \text{ (w)}}$ Draw 2 w/o replacement

$$P(r, r) = P(\underbrace{1^{st} R}_{A} \cap \underbrace{2^{nd} R}_{B}) = P(1^{st} R).\ P(2^{nd} R \mid 1^{st} R) = \frac{8}{12} \times \frac{7}{11}$$

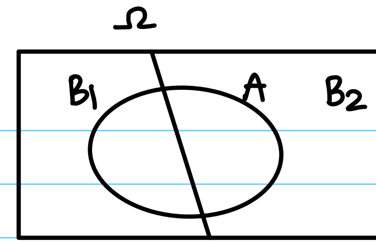
e.g: Draw 4 cards from a deck of 52 cards w/o replacement

$$P(1^{st} \text{ diamond ace}, 2^{nd} \text{ queen}, 3^{rd} \text{ king}, 4^{th} \text{ 9}) = \frac{1}{52} \times \frac{4}{51} \times \frac{4}{50} \times \frac{4}{49}$$

$$\boxed{P(A \cap B \cap C) = P(A).\ P(B \mid A).\ P(C \mid A \cap B)}$$

$$\boxed{P(A_1 \cap A_2 \cap \ldots A_n) = P(A_1).\ P(A_2 \mid A_1).\ P(A_3 \mid A_1 \cap A_2) \ldots P(A_n \mid A_1 \cap \ldots A_{n-1})}$$

Suppose $\{B_1, B_2\}$ is a partition of Ω then



$$\boxed{P(A) = P(B_1).\, P(A \mid B_1) + P(B_2).\, P(A \mid B_2)}$$

$B_1 \cup B_2 = \Omega$
$B_1 \cap B_2 = \emptyset$

Proof $\rightarrow$

$$P(A) = P((A \cap B_1) \cup (A \cap B_2)) \quad \text{disjoint}$$
$$= P(A \cap B_1) + P(A \cap B_2)$$
$$= P(B_1).\, P(A \mid B_1) + P(B_2).\, P(A \mid B_2)$$

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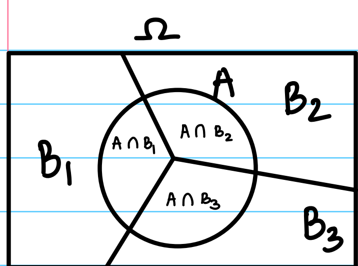
Theorem $\rightarrow$ Law of total prob.

Let $B_1, B_2, \ldots B_n$ be a partition of Ω. Suppose $P(B_i) > 0 \ \forall i$ then

$$\boxed{P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i)\, P(A \mid B_i)}$$



$B_i \cap B_j = \emptyset \ \forall i \neq j$

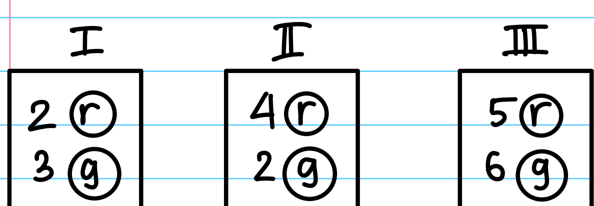
$\bigcup_{i=1}^{n} B_i = \Omega$

Proof $\rightarrow$

$$A = A \cap \Omega = A \cap (B_1 \cup B_2 \cup \ldots B_n)$$
$$= (A \cap B_1) \cup (A \cap B_2) \cup \ldots (A \cap B_n) \quad \text{disjoint union}$$

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i)\, P(A \mid B_i) \quad \text{by multiplication rule}$$

e.g:



We first choose 1 box at random then we choose a ball from the selected box at random

$$P(\text{red ball}) = P(I)\,P(\text{red} \mid I) + P(II)\,P(\text{red} \mid II) + P(III)\,P(\text{red} \mid III)$$

$$= \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{4}{6} + \frac{1}{3} \times \frac{5}{11}$$

w/o replacement $\leftarrow$ If $P(1r, 1g) = \frac{1}{3} \times \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} + \frac{1}{3} \times \frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} + \frac{1}{3} \times \frac{\binom{5}{1}\binom{6}{1}}{\binom{11}{2}}$

Monday 9/25

2.2 Bayes' formula

I	II
2g, 1r	2r, 3y

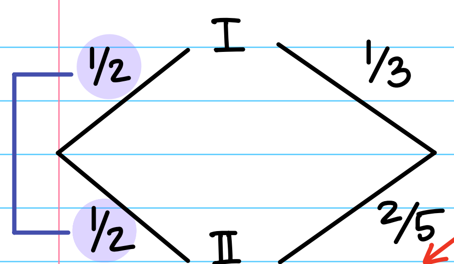
2.12 We have 2 urns: urn I (2g, 1r), urn II (2r, 3y). An urn is picked at random & a ball is drawn from it.

(a) $P(\text{red ball}) = P(I)\,P(\text{red} \mid I) + P(II)\,P(\text{red} \mid II)$ by law of total prob

$$= \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{5} = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$$

(b) Given that the ball is red, P(ball from urn II)

prior prob.



backwards

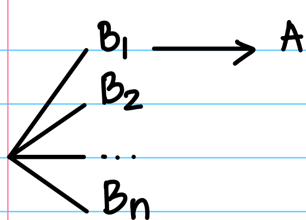
$$P(II \mid \text{red}) = \frac{P(II \cap \text{red})}{P(\text{red})} = \frac{\overset{1/2}{P(II)}\,\overset{2/5}{P(\text{red} \mid II)}}{P(I)P(\text{red} \mid I) + P(II)P(\text{red} \mid II)} = \frac{1/5}{11/30} = \frac{6}{11}$$

multiplication rule

posterior prob.

$P(I \mid \text{red}) = \frac{5}{11}$ → also makes sense since urn I has fewer red balls

General formula:

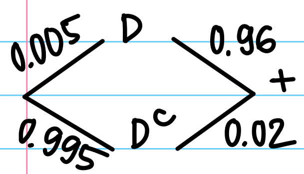


$$\boxed{P(B_i \mid A) = \frac{P(B_i)P(A \mid B_i)}{\sum_{j=1}^{n} P(B_j)P(A \mid B_j)}} \quad \forall\, i = 1, 2 \ldots n$$

Bayes' formula: Let $\{B_1, B_2 \ldots B_n\}$ be a partition of Ω then:

$$\boxed{P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i)P(A \mid B_i)}{P(A)}}$$

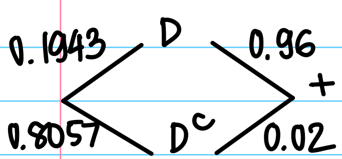
2.14 Suppose we have a medical test that detects a particular disease 96% of the time, but gives a false positive 2% of the time. Assume that 0.5% of the population carries the disease. If a random person tests positive for the disease, P(actually carry the disease)?



$$P(D|+) = \frac{P(D)P(+|D)}{P(D)P(+|D) + P(D^c)P(+|D^c)} = \frac{0.005 \times 0.96}{0.005 \times 0.96 + 0.995 \times 0.02} = 0.1943$$

small # bc D^c weigh heavier

What if:



$$P(D|+) = \frac{0.1943 \times 0.96}{0.1943 \times 0.96 + 0.8057 \times 0.02} = 0.9205$$

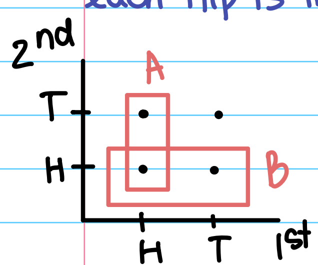
Wednesday 9/27

2.3 Independence

Def: Let A, B be events. $\boxed{\text{A, B are independent if } P(A \cap B) = P(A)P(B)}$

e.g.: $P(H,H) = P(1^{st}\ H) \times P(2^{nd}\ H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

each flip is independent of each other



$\Omega = \{H,T\} \times \{H,T\}$

$A = \{1^{st} \text{ coin} = H\}$ $\longrightarrow$ $P(A \cap B) = \frac{1}{4} = P(A)P(B)$

$B = \{2^{nd} \text{ coin} = H\}$

$\boxed{\text{If A, B independent, } P(A|B) = P(A)}$

Proof:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$\boxed{\text{If A, B independent \& disjoint, } P(A \cap B) = P(\emptyset) = 0 = P(A)P(B)}$

$\hookrightarrow$ $P(A) = 0$ or $P(B) = 0$

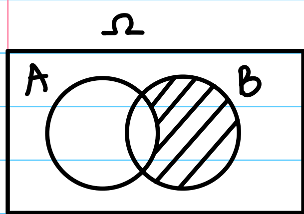
Theorem $\rightarrow$ Suppose A, B independent then:

(1) A^c & B are independent

(2) A & B^c are independent

(3) A^c & B^c are independent

Proof (1) $\rightarrow$



$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c) \end{aligned}$$

QED!

$\therefore A^c, B$ are independent

$\boxed{2.19}$ Suppose we have an urn w/ 4r and 7g balls. Choose 2 balls w/ replacement. Let $A = \{1^{st}$ ball is red$\}$, $B = \{2^{nd}$ ball is green$\}$

(a) Show that A, B are independent

$$P(A \cap B) = P(1^{st} \text{ red} \cap 2^{nd} \text{ green}) = \frac{4 \times 7}{11 \times 11} = \frac{28}{121}$$

$$P(A) = \frac{4}{11}, \quad P(B) = \frac{\cancel{11} \times 7}{\cancel{11} \times 11} = \frac{7}{11} \longrightarrow P(A \cap B) = P(A)P(B)$$

$\therefore A, B$ are independent

not care abt 1^{st} ball

(b) Are A, B independent if the selection of balls are w/o replacement?

$$P(A \cap B) = \frac{4 \times 7}{11 \times 10}$$

$$P(A) = \frac{4}{11}, \quad P(B) = \frac{\overset{R}{4} \times \overset{G}{7}}{11 \times 10} + \frac{\overset{G}{7} \times \overset{G}{6}}{11 \times 10} = \frac{7}{11}$$

$P(A \cap B) \neq P(A)P(B)$

$\therefore A, B$ are dependent

Monday 10/2

Def $\rightarrow$: Events A, B, C are independent (mutually independent) if

(1) $P(A \cap B \cap C) = P(A)P(B)P(C)$

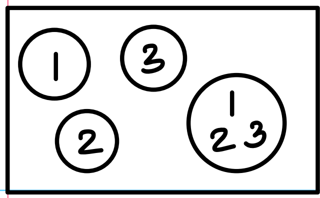
(2) $P(A \cap B) = P(A)P(B)$

(3) $P(B \cap C) = P(B)P(C)$

(4) $P(A \cap C) = P(A)P(C)$

(2)–(4): pairwise independent

Note: $(1) \xrightarrow[\text{imply}]{\text{not}} (2)(3)(4)$ $\qquad$ $(2)(3)(4) \xrightarrow[\text{imply}]{\text{not}} (1)$

e.g: 

Draw 1 ball at random. $A_i = \{\#i \text{ written on the ball}\}$

$$P(A_1) = \frac{2}{4} = \frac{1}{2} = P(A_2) = P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{1}{4} = P(A_1 \cap A_3) = P(A_2 \cap A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq P(A_1)P(A_2)P(A_3)$$

$\therefore$ Pairwise independence does not imply all 3 independence

Def $\rightarrow$

Events $A_1, A_2 \dots A_n$ are independent (mutually independent) if
$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \dots P(A_{i_k}) \quad \text{for all } k = 2, 3 \dots n$$
$$1 \leq i_1 < i_2 < \dots < i_k \leq n$$

Theorem $\rightarrow$

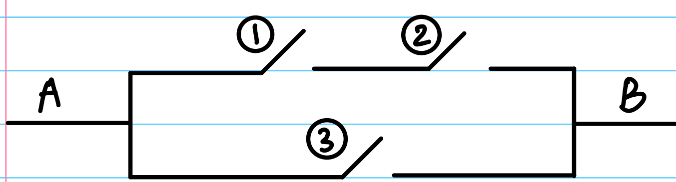
Suppose $A_1, A_2 \dots A_n$ are independent then:
$$P(A_{i_1}^* \cap \dots \cap A_{i_k}^*) = P(A_{i_1}^*) \dots P(A_{i_k}^*) \quad \text{for all } k = 2, 3 \dots n$$
& $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Have $A^* = A$ or A^c

e.g: Suppose A, B, C are independent then

$P(A \cap B^c) = P(A)\,P(B^c)$ $\quad\quad$ $P(A \cap B^c \cap C) = P(A)P(B^c)P(C)$

$P(A^c \cap C^c) = P(A^c)\,P(C^c)$ $\quad\quad$ ...

2.26



S_i = event that switch i is closed

$P(S_i) = P_i$

Suppose switches open/close independently

$$\begin{aligned}
P(A \rightarrow B) &= P(S_3 \cup (S_1 \cap S_2)) \\
&= P(S_3) + P(S_1 \cap S_2) - P(S_3 \cap S_1 \cap S_2) && \text{inclusion – exclusion} \\
&= P(S_3) + P(S_1)P(S_2) - P(S_3)P(S_1)P(S_2) && \text{independent events} \\
&= P_3 + P_1 P_2 - P_1 P_2 P_3
\end{aligned}$$

Def $\rightarrow$

Let $X_1, X_2 \dots X_n$ be random variables defined on the same prob. space. Then $X_1, X_2 \dots X_n$ are independent if

$$\boxed{P(X_1 \in B_1, X_2 \in B_2 \dots X_n \in B_n) = P(X_1 \in B_1) \dots P(X_2 \in B_n) \text{ for all } B_i \subseteq \mathbb{R}}$$

Theorem → Discrete random variables $X_1, X_2 \dots X_n$ are independent iff'

$$P(X_1 = x_1, X_2 = x_2, \dots X_n = x_n) = P(X_1 = x_1) \dots P(X_n = x_n)$$

for all possible values $x_1, x_2 \dots x_n$ of the random variables

e.g: $X = 0$ or 1 X and Y are independent if
$Y = 0$ or 1

$$P(X=0, Y=0) = P(X=0)P(Y=0)$$
$$P(X=0, Y=1) = P(X=0)P(Y=1)$$
$$P(X=1, Y=0) = P(X=1)P(Y=0)$$
$$P(X=1, Y=1) = P(X=1)P(Y=1)$$

$\boxed{2.13}$ Suppose $P(A) = 1/3$, $P(B) = 1/3$, $P(A \cap B^c) = 2/9$. Decide whether A and B are independent.

$P(B^c) = 2/3$

$P(A \cap B^c) = P(A)P(B^c) = 1/3 \times 2/3 = 2/9$

$\therefore A, B^c$ are independent $\therefore A, B$ are independent by theorem

2.4 Independent trials

Def → An experiment is called a Bernoulli trial if it has exactly 2 possible outcomes (success / failure)

$P(\text{success}) = p$ $p + q = 1$

$P(\text{failure}) = 1 - p = q$ $0 \leq p \leq 1$, $0 \leq q \leq 1$

Let X = # of successes $\longrightarrow X = 0, 1$

$P(X=1) = P(\text{success}) = p$

$P(X=0) = P(\text{failure}) = q$

pmf: $\left.\begin{array}{l} P_X(0) = q \\ P_X(1) = p \end{array}\right\}$ Bernoulli distribution $0 \leq p \leq 1$ is a parameter

Wednesday 10/4

$\boxed{2.16}$ $\Omega = \{(s_1, s_2, s_3) \mid s_i = H, T\}$

$A_1 = \{(H, *, *)\}$ $\longrightarrow P(A_1) = 1/2$

$A_2 = \{(H, T, *), (T, H, *)\}$ $\longrightarrow P(A_2) = 2/4 = 1/2$

$A_3 = \{(H,T,T),(T,H,T),(T,T,H),(H,H,H)\} \longrightarrow P(A_3) = 4/8 = 1/2$

$P(A_1 \cap A_2) = \frac{1}{4} = P(A_1)P(A_2)$

$P(A_1 \cap A_3) = \frac{2}{8} = \frac{1}{4} = P(A_1)P(A_3)$

$P(A_2 \cap A_3) = \frac{2}{8} = \frac{1}{4} = P(A_2)P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = \frac{1}{8} = P(A_1)P(A_2)P(A_3)$ $\therefore A_1, A_2, A_3$ are independent

$\boxed{2.18}$ (a) $P(X=i) = \frac{10}{99-9} = \frac{1}{9}$; $P(Y=j) = \frac{9}{90} = \frac{1}{10}$

$i = 1, 2 \ldots 9$ $\qquad\qquad\qquad$ $j = 0, 1, 2 \ldots 9$

$P(X=i \cap Y=j) = \frac{1}{90} = P(X=i)P(Y=j) \rightarrow X, Y$ are independent

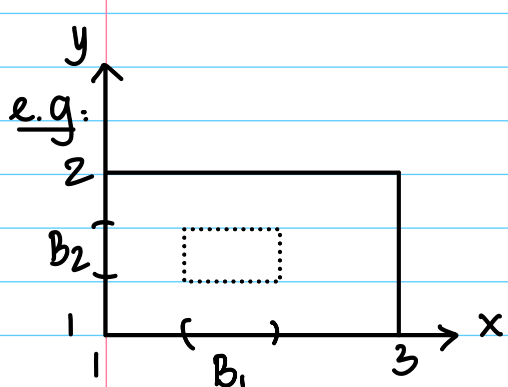
(b) $P(X=2, Z=1) = 0$

$P(X=2) = \frac{1}{9}$, $P(Z=1) = \frac{1}{90}$

$\Big\}$ $P(X=2, Z=1) \neq P(X=2, Z=1)$
$\rightarrow X, Z$ are not independent

$\boxed{2.19}$ (b) $P(X_2=5) = P(X_1 \neq 5)P(X_2=5 \mid X_1 \neq 5) + P(X_1=5)P(X_2=5 \mid X_2=5)$
$= \frac{6}{7} \times \frac{1}{6} + \frac{1}{7} \times 0 = \frac{1}{7}$

e.g.:



$P(X \in B_1 \cap Y \in B_2) = \frac{|B_1||B_2|}{(3-1)(2-1)}$

$P(X \in B_1) = \frac{|B_1|(2-1)}{(3-1)(2-1)} = \frac{|B_1|}{3-1}$

$P(Y \in B_2) = \frac{|B_2|(3-1)}{(3-1)(2-1)} = \frac{|B_2|}{2-1}$

$\therefore P(X \in B_1 \cap Y \in B_2) = P(X \in B_1)P(Y \in B_2)$

$\therefore X, Y$ are independent

Monday
10/9

X has a Bernolli distribution $\longrightarrow X \sim Ber(p)$

Consider n independent Bernolli trials. Let S_n be # of successes in n trials
$S_n = 0, 1, 2 \dots n$
Find $P_{S_n}(k)$, $k = 0, 1, 2 \dots n$

s s s f f f f
p p p q q q q

$$\boxed{P_{S_n}(k) = \binom{n}{k} p^k q^{n-k} \;;\; k = 0, 1, 2 \dots n}$$

$k = 3 \longrightarrow p^3 q^{n-3}$

$S_n \sim Bin(n, p)$ (binomials distribution)

e.g: S_n, $n = 1 \longrightarrow Bin(1, p) = Ber(p)$
$S_n = X_1 + X_2 + \dots + X_n$ where $X_1, X_2 \dots X_n$ are independent
$X_i \sim Ber(p)$

2.33 What is the prob. that 5 rolls of a fair die yield 2 or 3 sixes?
$n = 5$ independent trials

$p = P(\text{six}) = \frac{1}{6}$, $q = P(\text{not six}) = \frac{5}{6}$

$P(S_5 = 2 \text{ or } S_5 = 3) = P(S_5 = 2) + P(S_5 = 3)$

$$= \binom{5}{2} p^2 q^3 + \binom{5}{3} p^3 q^2 = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

Geometric distributions
Let N = # trials needed to get a first success in a sequence of independent Bernolli trials.
$N = \{1, 2, \dots n\} \cup \{\infty\}$

f f f f f s
q q q q q p

$$\boxed{P_N(k) = P(N = k) = q^{k-1} p, \; k = 1, 2, 3 \dots}$$

$$P_N(N = \infty) = P_N(\infty) = 1 - \sum_{k=1}^{\infty} P_N(k) = 1 - \sum_{k=1}^{\infty} pq^{k-1} = 1 - p \cdot \frac{1}{1-q}$$

$$= 1 - \frac{p}{p} = 1 - 1 = 0 \text{ if } q \neq 1$$

If $q = 1$: $P_N(\infty) = 1$ $\quad \therefore N \sim Geom(p)$

$$\overbrace{q\ \ q\ \ q\ \ q\ \ q\ \ q}^{n}$$

Suppose $N \sim Geom(p)$. Find

(1) $P(N > k) = q^k$; $k = 0, 1, 2 \ldots$

(2) $P(N \leq k) = 1 - P(N > k) = 1 - q^k$

$\boxed{2.36}$ Roll a pair of dice until you get either a sum of 5 or a sum of 7. What is the prob. you get 5 first ?

S	2	3	4	5	6	7	8	9	10	11	12
$P_S(k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$p = P(S = 5 \text{ or } S = 7) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36}$$

$$q = 1 - p = 1 - \frac{10}{36} = \frac{26}{36}$$

$$\underset{q}{\bigcirc}\ \underset{q}{\bigcirc}\ \underset{q}{\bigcirc}\ \underset{q}{\bigcirc}\ \underset{q}{\bigcirc}\ \overset{4/36}{\underset{p}{\bigcirc}} \rightarrow q^{k-1} \times \frac{4}{36}$$

$$P(\text{stop at sum} = 5) = \sum_{k=1}^{\infty} q^{k-1} \frac{4}{36} = \sum_{k=1}^{\infty} \left(\frac{26}{36}\right)^{k-1} \frac{4}{36} = \frac{4}{36} \times \frac{1}{1 - \frac{26}{36}} = \boxed{\frac{2}{5}}$$

OR $$P(\text{sum} = 5 \mid \text{sum} = 5 \text{ or sum} = 7) = \frac{\frac{4}{36}}{\frac{4}{36} + \frac{6}{36}} = \frac{2}{5}$$

Grouping

Suppose $A_1, A_2 \ldots A_n$ are mutually independent.
Suppose we construct events $B_1, B_2 \ldots B_k$ by applying set operations to $A_1 \ldots A_n$ but in such a manner that 2 diff B's never use the same A_i
Then $B_1, B_2 \ldots B_k$ are independent

A_1, A_2, A_3
$B_1 = A_1 \cap A_2$
$B_2 = A_3^c$

A_1, A_2, A_3, A_4, A_5
$$\left.\begin{array}{l} B_1 = (A_1 \cap A_4^c) \cup A_5 \\ B_2 = A_2 \cup A_3^c \end{array}\right\} B_1, B_2 \text{ independent}$$

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CHAPTER 3

3.1 Prob. distributions of random variables

Def: Discrete random variables $X \longrightarrow$ distribution is described by its pmf $P_X(k)$

e.g: Toss a fair coin 3 times. Let X = # of heads. $X \sim Bin(\underset{n}{3}, \underset{p}{\tfrac{1}{2}})$

$X = 0, 1, 2, 3$

pmf $P_X(0) = \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ $\quad$ $P_X(2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$

$P_X(1) = \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = \frac{3}{8}$ $\quad$ $P_X(3) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$

(a) $P(X = 0, 1) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

(b) $P(\text{at most 1 head}) = P(X = 0 \text{ or } 1) = \frac{1}{2}$

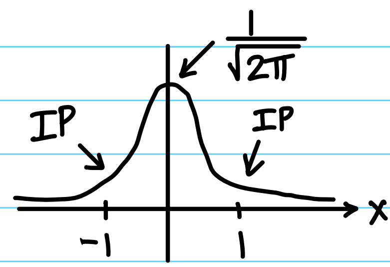
(c) $P(\text{at least 1 head}) = P(X = 1, 2, 3) = 1 - P(X=0) = 1 - \frac{1}{8} = \frac{7}{8}$

For a continuous random variable X, we use the $\underbrace{\text{prob. density fn}}_{\text{pdf}}$ $f_X(x)$

$\boxed{P(X \in B) = \int_B f_X(x)\,dx}$ for all $B \subseteq \mathbb{R}$

e.g: $X \sim$ standard normal distribution.

pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $-\infty < x < \infty$



Properties: pdf $f_X(x)$

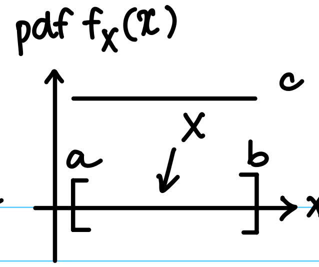
(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = P(X \in \mathbb{R}) = 1$

(iii) $\int_a^b f_X(x)\,dx = P(a < X < b)$ or $P(a \leq X \leq b)$

(iv) $P(X = a) = \int_a^a f_X(x)\,dx = 0$

uniform distribution



e.g: $X \sim \text{unif}[a,b] \longrightarrow X$ is a continuous random variable

(a) Find the pdf of X

$$f_X(x) = \begin{cases} c, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

(b) Find c

$$\int_a^b f_X(x) = 1 = \int_a^b c\,dx = cx\Big|_a^b = c(b-a) \longrightarrow c = \frac{1}{b-a}$$

$$\therefore f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leqslant x \leqslant b \\ 0, & \text{otherwise} \end{cases}$$

$f_X(x) =$ pdf of $X \sim \text{unif}(a,b)$

OR $f_X(x) \sim \text{unif}(a,b)$

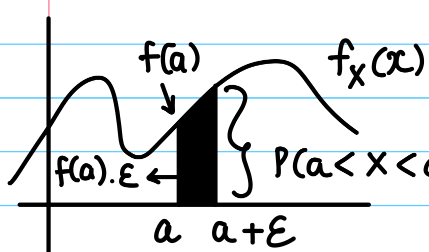
e.g: Suppose $X \sim$ standard normal

$$P(X>0) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{2}$$

← because total distribution is 1

∴ right half distribution is ½

Infinitesimal method $\boxed{P(a < X < a+\varepsilon) \approx f_X(a).\varepsilon}$ for a small $\varepsilon > 0$



$$P(a<X<a+\varepsilon) = \int_a^{a+\varepsilon} f_X(x)dx$$

e.g: Suppose a random variable X has pdf $f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Compare $P(0.5 < X < 0.51)$ with $f(0.5)(0.01)$

$$P(0.5 < X < 0.51) = \int_{0.5}^{0.51} f_X(x)dx = \int_{0.5}^{0.51} 3x^2 dx = x^3\Big|_{0.5}^{0.51} = 0.51^3 - 0.5^3 = 0.007651$$

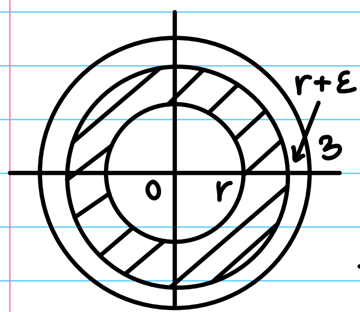
$$f(0.5)(0.01) = 3(0.5)^2(0.01) = 0.0075$$

Error = 0.000151

$$\lim_{\varepsilon \to 0} \frac{f_X(a).\varepsilon - P(a < X < a+\varepsilon)}{\varepsilon} = 0$$

$$\therefore \boxed{f_X(a) = \lim_{\varepsilon \to 0} \frac{P(a < X < a+\varepsilon)}{\varepsilon}}$$ infinitesimal method

e.g:



$X \sim$ unif on the disk w/ radius 3

$R = \sqrt{x_1^2 + x_2^2}$, $X = (x_1, x_2)$

Find $f_R(r)$ pdf of R

$$f_R(r) = \lim_{\varepsilon \to 0} \frac{P(r < R < r+\varepsilon)}{\varepsilon} = \frac{1}{\varepsilon} \cdot \frac{\pi(r+\varepsilon)^2 - \pi r^2}{\pi . 3^2} = \frac{2r+\varepsilon}{9}$$

$$\uparrow 2r/9$$

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$f_X(x)$ is pdf if $P(X \in B) = \int_B f_X(x)\,dx$

e.g: Suppose the pdf of X is $f(x) = \begin{cases} 3e^{-3x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

(a) Find $P(1 < X < 2)$

$$P(1 < X < 2) = \int_1^2 3e^{-3x}\,dx = 3\int_1^2 \frac{e^{-3x}}{-3}\,dx = e^{-3x}\Big|_2^1 = e^{-3} - e^{-6}$$

(b) $P(X > 3) = \int_3^\infty 3e^{-3x}\,dx = e^{-3x}\Big|_\infty^3 = e^{-9} - 0 = e^{-9}$

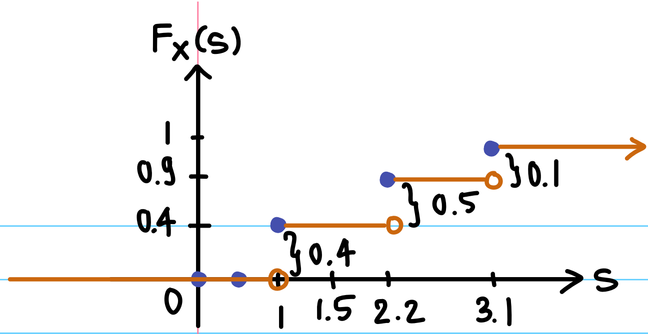
3.2 Cumulative distribution function cdf

Def $\rightarrow$ Let X be a random variable. $F_X(s) = P(X \le s)$, $s \in \mathbb{R}$ is called the cdf of X

e.g: Suppose X is discrete w/ pmf

X	1	2.2	3.1
$P_X(k)$	0.4	0.5	0.1

Find cdf of X. $F_X(s) = P(X \le s)$



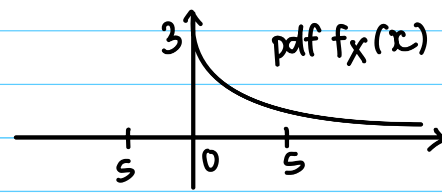
$F_X(0) = P(X \le 0) = 0$

$F_X(0.5) = P(X \le 0.5) = 0$

$F_X(1) = P(X \le 1) = P(X = 1) = 0.4$

$F_X(1.5) = P(X \le 1.5) = P(X = 0) + P(X = 1) = 0.4$

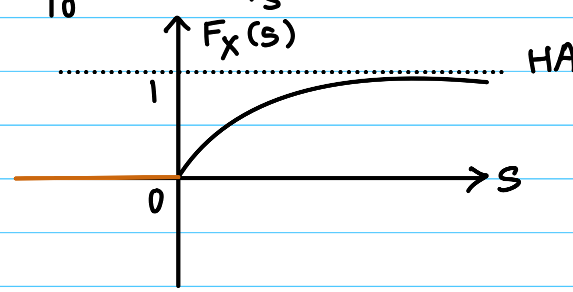
e.g.: Suppose the pdf of X is $f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$. Find cdf $F_X(s)$



$$F_X(s) = P(X \le s) = \int_{-\infty}^{s} f_X(x)\,dx = \begin{cases} 0 & \text{if } s \le 0 \\ \int_0^s f_X(x)\,dx & \text{if } s > 0 \end{cases}$$

$$\int_0^s f_X(x)\,dx = \int_0^s 3e^{-3x}\,dx = 3\frac{e^{-3x}}{-3}\Big|_0^s = e^{-3x}\Big|_s^0 = 1 - e^{-3s},\ s > 0$$

$$F_X(s) = \begin{cases} 0, & s \le 0 \\ 1 - e^{-3s}, & s > 0 \end{cases}$$



Properties

(1) $F_X(s)$ is nondecreasing

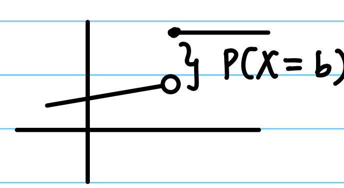
(2) $\lim_{s \to \infty} F_X(s) = 1$ $\qquad$ $\lim_{s \to -\infty} F_X(s) = 0$

(3) $F_X(s)$ is right continuous $\left(\lim_{s \to a^+} F_X(s) = F_X(a)\right)$

Notations $\quad F_X(a+) = \lim_{s \to a^+} F_X(s)$ $\qquad$ $F_X(a-) = \lim_{s \to a^-} F_X(s)$

(4) $P(a \le X \le b) = F_X(b) - F_X(a)$

(5) $P(X < b) = F_X(b-)$



(6) $P(X = b) = F_X(b) - F_X(b-) =$ jump size at b

N_A	Type A
N_B	Type B

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Hypergeometric(N, N_A, n)

$N = N_A + N_B$

Choose n balls w/o replacement

Let X = # of the type A balls in the sample $\longrightarrow X \sim \text{Hypergeom}(N, N_A, n)$

$X = 0, 1, 2 \dots n$

pmf of X: $P_X(k) = P(X=k) = \dfrac{\binom{N_A}{k}\binom{N_B}{n-k}}{\binom{N}{n}} = \dfrac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$, $k = 0, 1, 2 \dots n$

e.g: $\binom{2}{3} = 0$

Monday 10/23

3.3 Expectation

(1) Expectation of a discrete random variable

Def $\rightarrow$ Let X be discreet random variable. The expectation of X is defined by

$$\boxed{E[X] = \underbrace{\sum_k k P(X=k)}_{\text{weighted sum}} = \sum_k k P_X(k)}$$

(1) $E[c] = c$

(2) $E[aX] = aE[X]$

e.g: Suppose $P_X(k)$ is given by

X	0	1.2	2.1
$P_X(k)$	0.3	0.4	0.3

Find $E[X] = 0 \times 0.3 + 1.2 \times 0.4 + 2.1 \times 0.3 = 1.11$

$\downarrow$

expected value of X / mean of X

$E[\underset{\overset{||}{x^2}}{g(x)}] = \sum_k g(k) P_X(k) = \sum_k k^2 P_X(k) = 0^2 \times 0.3 + 1.2^2 \times 0.4 + 2.1^2 \times 0.3$

e.g: Let $X \sim \text{Bin}(n,p)$. Find $E[X] = \sum_{k=0}^{n} k P_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$

$= \sum_{k=0}^{n} k \cdot \dfrac{n!}{k!(n-k)!} p^k q^{n-k} \xrightarrow[\text{is } 0]{\text{1st term}} \sum_{k=1} \dfrac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$

k starts from 1

$s = k-1$

$s = 0 \to n-1$

$$= \sum_{s=0}^{n-1} \frac{n\boxed{(n-1)!}}{\boxed{(n-1-s)!\,s!}} p \cdot p^{k-1} q^{(n-1)-(k-1)} = \sum_{s=0}^{n-1} n\binom{n-1}{s} p \cdot p^s \cdot q^{n-1-s}$$

$\binom{n-1}{s}$

$$= np \underbrace{\sum_{s=0}^{n-1}\binom{n-1}{s} p^s q^{n-1-s}}_{Bin(n-1,p)} = np \times 1 = np$$

total distribution = 1

e.g.: $n=1 \to X \sim Ber(p)$

X	0	1
$P_X(k)$	q	p

$E[X] = 0.q + 1.p = p \overset{n=1}{=\!=\!=} n.p$

e.g.: $X \sim Geom(p)$. $E[X] = \sum_{k=1}^{\infty} k . P_X(k) = \sum_{k=1}^{\infty} k . q^{k-1} p = p \sum_{k=1}^{\infty} (q^k)' = p\left(\sum_{k=1}^{\infty} q^k\right)'$

$$= p\left(\sum_{k=0}^{\infty} q^k\right)' = p.\left(\frac{1}{1-q}\right)' = p\left[(1-q)^{-1}\right]' = p(-1)(-1)(1-q)^{-2} = \frac{p}{p^2} = \frac{1}{p}$$

$\left(1 + \sum_{k=1}^{\infty} q^k\right)'$ ← derivative stays the same whether starting at 0 or 1

$k=0$

(2) Expectation of continuous random variable

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx} \quad f_X(x) = \text{pdf of } X \quad \boxed{E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx}$$

e.g.: $X \sim Unif[a,b]$. $E[X] = \frac{a+b}{2}$

a — X — b ← mean

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2} \checkmark$$

Wednesday 10/25

3.3 Median and quantiles

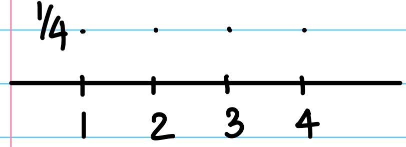
Def → The median of a random variable is any real value m that satisfies

$$\boxed{P(X \le m) \ge 1/2 \;\&\; P(X \ge m) \ge 1/2}$$

e.g: Suppose pmf of X is

X	1	2	3	4
$P_X(K)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

. Find median of X



$P(X \le m) \geq \frac{1}{2}$ & $P(X \geq m) \geq \frac{1}{2}$ — $m = 2$ ✓, $m = 2.5$ ✓, $m = 3$ ✓

m is any # in $[2, 3]$

e.g: Suppose $X \sim Exp(\lambda)$, $\lambda = 2$. Find median of X

$$X \sim Exp(\lambda) \to f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \xrightarrow{\lambda = 2} \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$P(X \geq m) \geq \frac{1}{2} \to \int_m^\infty 2e^{-2x}dx = e^{-2x}\Big|_\infty^m = e^{-2m} - \cancel{e^{-2\infty}}^{\,0} = e^{-2m} \underset{=}{\text{set}} \frac{1}{2}$$

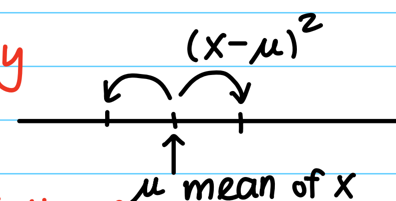
$$\therefore -2m = \ln\frac{1}{2} = -\ln 2 \quad \therefore m = \frac{\ln 2}{2}$$

3.4 Variance

Def → The variance of a random variable X is defined by

$$\boxed{\sigma_X^2 = Var(X) = E[(X-\mu)^2]} \text{ where } \mu = E[X]$$

σ_X^2 = std deviation of X



(1) $Var(X) = E[X^2] - (E[X])^2$

Proof → By def: $Var(X) = E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2]$

$$= E[X^2] - 2\mu E[X] + E[\mu^2]$$

$$= E[X^2] - 2\mu\mu + \mu^2 = E[X^2] - \mu^2 = E[X^2] - E[X]^2 \text{ QED!}$$

e.g: $X \sim Bin(n, p) \to \mu = E[X] = np$. Find $Var(X)$

By (1): $Var(X) = E[X^2] - \mu^2$

$$E[X(X-1)] = \sum_{K=0}^{n} K(K-1)\, P_X(K) = \sum_{K=0}^{n} K(K-1)\binom{n}{K} p^K q^{n-K}$$

$$= \sum_{K=0}^{n} K(K-1)\frac{n!}{K!(n-K)!} p^K q^{n-K} = \sum_{K=2}^{n} \frac{n!}{(K-2)!(n-K)!} p^K q^{n-K}$$

↖ start at 2 after cancelling terms

$$s = k-2 \qquad = \sum_{s=0}^{n-2} \frac{n(n-1)\boxed{(n-2)!}}{\boxed{(n-s-2)!\, s!}} p^{s+2} q^{(n-2)-s} = n(n-1)p^2 \overbrace{\underbrace{\sum_{s=0}^{n-2} \binom{n-2}{s} p^s q^{(n-2)-s}}_{Bin(n-2,\,p)}}^{\text{Total distribution} = 1}$$

$\binom{n-2}{s}$

$$= n(n-1)p^2$$

$$E[X^2] = E[X(X-1)] + E[X] = n(n-1)p^2 + np$$

$$Var(X) = E[X^2] - \mu^2 = n(n-1)p^2 + np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np(1-p) = npq$$

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$$X \sim exp(\lambda) \longrightarrow f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \begin{aligned} u &= x & dv &= e^{-\lambda x} \\ du &= dx & v &= \frac{e^{-\lambda x}}{-\lambda} \end{aligned}$$

(a) Find $E[X] = \mu = \int_0^\infty x f(x)dx = \int_0^\infty x\lambda e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$

$$= \lambda\left(-x\frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda}dx\right) = \cancel{\infty e^{-\infty}}^{\,0} + \lambda \cdot \frac{1}{\lambda} \cdot \frac{1}{-\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= -\frac{1}{\lambda}(\cancel{e^{-\infty}}_{\,0} - 1) = \frac{1}{\lambda}$$

(b) Find $Var(E) = E[X^2] - \mu^2$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = x^2(-e^{-\lambda x})\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx$$

$$= \frac{2}{\lambda}\int_0^\infty x\lambda e^{-\lambda x} dx = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\therefore Var(E) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

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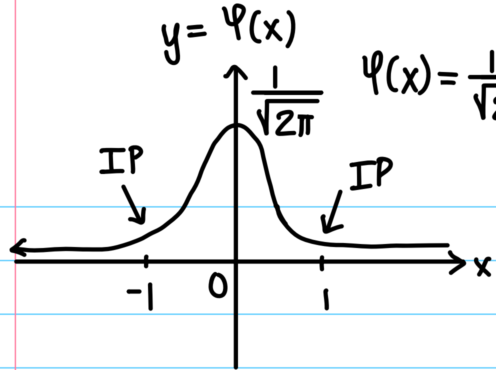
3.5 Gaussian Distribution

Def $\rightarrow$ A random variable Z has the std normal distribution if Z has density fn

$$\text{pdf of } Z = \boxed{\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty} \rightarrow Z \sim N(0,1)$$



$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\sigma_Z^2 = \text{variance of } Z = 1$

$\downarrow$

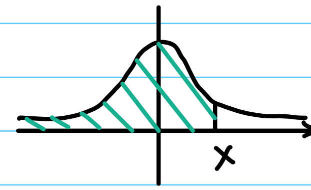
$Z \sim N(0,1)$

$\downarrow$

$\mu_Z = \text{mean of } Z = 0$

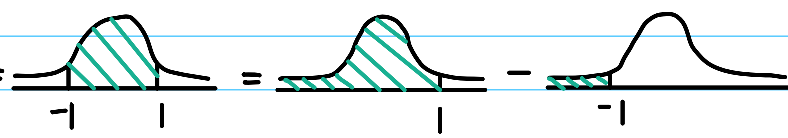
$P(a \le Z \le b) = \int_a^b \varphi(x)\,dx = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ = Area under the curve in $[a,b]$

Let $\Phi(x) = P(-\infty < Z \le x)$ = cdf of $Z = \int_{-\infty}^{x} \varphi(u)\,du$



e.g: (a) $P(-\infty < Z \le 1) = \Phi(1) = 0.8413$ (Appendix E)

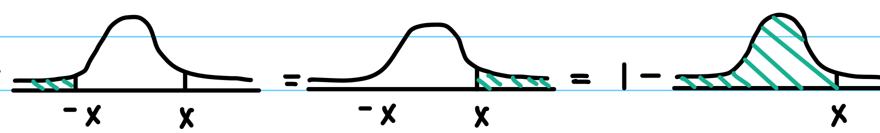
(b) $P(-1 \le Z \le 1) =$



$= \Phi(1) - \Phi(-1)$

$= \Phi(1) - (1 - \Phi(1)) = 2 \times 0.8413 - 1 = 0.6826$

$\Phi(-x) =$



$= 1 - \Phi(x)$

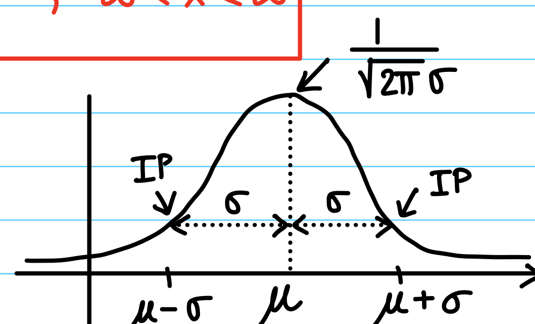
$$\boxed{\Phi(-x) = 1 - \Phi(x)}$$

Def $\rightarrow$ A random variable X has the normal distribution w/ mean μ & variance σ^2

pdf of X = $\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \ -\infty < x < \infty}$

$E[X] = \mu$

$Var(X) = \sigma^2$ $\quad \rightarrow X \sim N(\mu, \sigma^2)$



Theorem $\rightarrow$ Suppose $X \sim N(\mu_X, \sigma_X^2)$

Let $Y = aX + b$

Then $\boxed{Y \sim N(a\mu_X + b,\ a^2\sigma_X^2)}$ $\begin{cases} \mu_Y = a\mu_X + b \\ \sigma_Y^2 = a^2\sigma_X^2 \end{cases}$

Application → $\boxed{\text{If } X \sim N(\mu_X, \sigma_X^2) \text{ then } Z = \dfrac{X-\mu_X}{\sigma_X} \sim N(0,1)}$

Proof →

$$Z = \frac{1}{\sigma_X} X - \frac{\mu_X}{\sigma_X} \rightarrow a = \frac{1}{\sigma_X},\ b = \frac{-\mu_X}{\sigma_X}$$

$$\left.\begin{aligned} \mu_Z &= a\mu_X + b = \frac{1}{\sigma_X}\mu_X - \frac{\mu_X}{\sigma_X} = 0 \\ \sigma_Z^2 &= a^2\sigma_X^2 = \left(\frac{1}{\sigma_X}\right)^2 \sigma_X^2 = 1 \end{aligned}\right\} \therefore Z \sim N(0,1) \quad \text{QED!}$$

e.g: Let $X \sim N(\underset{\mu_X}{2}, \underset{\sigma_X^2}{9})$. $P(1 \le X \le 4) = P\left(\frac{1-2}{3} \le \frac{X-\mu_X}{\sigma_X} \le \frac{4-2}{3}\right)$

$= P(-0.3333 \le Z \le 0.6666)$

$= \Phi(0.6666) - \Phi(-0.3333)$

$= \Phi(0.6666) - (1 - \Phi(0.3333)) = 0.7454 + 0.6293 - 1 = 0.3747$

Proof theorem →

(1) Find cdf of Y

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right),\ a > 0 = F_X\left(\frac{y-b}{a}\right)$$

(2) Find pdf of Y

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = F_X'\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{\frac{1}{-2\sigma_X^2}\left(\frac{y-b}{a} - \mu_X\right)^2} \cdot \frac{1}{a} = \frac{1}{\sqrt{2\pi}\,\sigma_X a} e^{\frac{-1}{2\sigma_X^2 a^2}(y - b - a\mu_X)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\underbrace{\sigma_X a}_{\sigma_Y}} e^{\frac{-1}{2\sigma_X^2 a^2}[y - \underbrace{(b + a\mu_X)}_{\mu_Y}]^2} \rightarrow Y \sim N(a\mu_X + b, a^2\sigma_X^2) \quad \text{QED!}$$

Wednesday 11/2

(2) $Var(aX+b) = a^2 Var(X)$

Proof →

$$Var(aX+b) = E[(aX + b - \mu_{aX+b})^2] \text{ where } \mu_{aX+b} = E[aX+b]$$
$$= aE[X] + b = a\mu_X + b$$

$\therefore Var(aX+b) = E[(aX+b-a\mu_X-b)^2] = E[a^2(X-\mu_X)^2] = a^2 Var(X)$ QED!

! Memorize p. 118 formulas + distribution forms

Monday 11/6

CHAPTER 4

Approximations of Binomial Distribution

4.1 Normal Approximation

Central limit theorem for binomials random variables

Theorem → Let $0 < p < 1$ be fixed. Suppose that $S_n \sim Bin(n,p)$. Then any fixed $-\infty \le a \le b \le \infty$ we have

$$\boxed{\lim_{n\to\infty} P\left(a \le \frac{S_n - np}{\sqrt{npq}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = P(a \le Z \le b)}$$

must

$S_n \sim Bin(n,p)$ ← distribution

$S_n \approx N(np, npq)$, $q = 1-p$ ← approximation

$Z = \frac{S_n - np}{\sqrt{npq}} \approx N(0,1)$ if $npq > 10$

4.2

A fair coin is flipped 10,000 times. Estimate the prob. that the # heads is between 4850 & 5100

S_n = # heads in n flips → $S_n \sim Bin(n,p)$; $n = 10{,}000$, $p = \frac{1}{2} = q$

→ $E[S_n] = np$, $Var(S_n) = npq$

$$P(4850 < S_n < 5100) = P\left(\frac{4850-np}{\sqrt{npq}} < \frac{S_n - np}{\sqrt{npq}} < \frac{5100-np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{4850-5000}{\sqrt{2500}} < \frac{S_n-np}{\sqrt{npq}} < \frac{5100-5000}{\sqrt{2500}}\right) \approx P(-3 < Z < 2)$$

$$= \Phi(2) - \Phi(-3) = \Phi(2) - [1-\Phi(3)] = 0.9772 + 0.9987 - 1 = 0.9759$$

Def → Let X_n, X be random variables. We say that $X_n \to X$ in distribution if

$$\boxed{P(a \le X_n \le b) \to P(a \le X \le b), \text{ as } n \to \infty}$$

Notation: $X_n \to X$ in distribution

for all a, b for which $P(X=a) = 0$ & $P(X=b) = 0$

Application If $S_n \sim Bin(n,p)$ then $\boxed{\frac{S_n - np}{\sqrt{npq}} \to Z \text{ in distribution}}$, $Z \sim N(0,1)$

Continuity Correction

4.5 Roll a fair die 720 times. Estimate the prob. that we have exactly 113 sixes

$S_n \sim Bin(n,p)$; $n = 720$, $p = \frac{1}{6}$ ← prob. getting a 6 in a trial, $q = \frac{5}{6}$

$$P(S_n = 113) = P(112.5 \le S_n \le 113.5) = P\left(\frac{112.5 - np}{\sqrt{npq}} \le \frac{S_n - np}{\sqrt{npq}} \le \frac{113.5 - np}{\sqrt{npq}}\right)$$

apply approximation not work
need to create an interval to get int value } use when a,b are too close

$$\approx P(-0.75 \le Z \le -0.65) = \Phi(-0.65) - \Phi(-0.75)$$
$$= 1 - \Phi(0.65) - 1 + \Phi(0.75)$$
$$= \Phi(0.75) - \Phi(0.65) = 0.7734 - 0.7422 = 0.0312$$

OR $P(S_n = 113) = \binom{720}{113} p^{113} q^{607} = \binom{720}{113}\left(\frac{1}{6}\right)^{113}\left(\frac{5}{6}\right)^{607}$ → but ERROR

4.2 Law of large numbers

Theorem Law of large numbers for binomial random variables $S_n \sim Bin(n,p)$

For any fixed $\varepsilon > 0$, we have:

$$\boxed{\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1}$$ ← weak law large #s

Wednesday 11/8

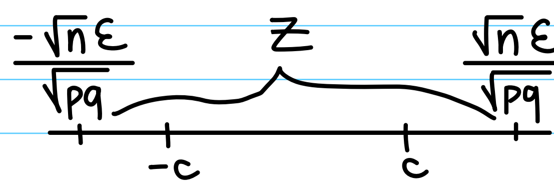
$$P\left(\frac{S_n}{n} \to p\right) = 1$$ ← strong law large #s

Proof

$$1 \ge \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = \lim_{n\to\infty} P\left(\left|\frac{S_n - np}{\sqrt{npq}}\right| < \frac{\sqrt{n}\varepsilon}{\sqrt{pq}}\right)$$

$$= \lim_{n\to\infty} P\left(\frac{-\sqrt{n}\varepsilon}{\sqrt{pq}} < \frac{S_n - np}{\sqrt{npq}} < \frac{\sqrt{n}\varepsilon}{\sqrt{pq}}\right)$$

$$\ge \lim_{n\to\infty} P(-c < Z < c), \text{ for any } c$$



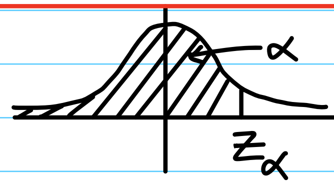
QED!

$= \Phi(c) - \Phi(-c)$, for any $c \xrightarrow{\text{Let } c \to \infty} 1 - 0 = 1 \therefore \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1$

4.3 Applications of the normal approximation

Theorem → Let $\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}}$ then $\boxed{\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) \geq \alpha}$

$z_\alpha = \alpha^{th}$ quantile of $N(0,1)$



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* Consequences of the law of large numbers:

$$\lim_{n\to\infty} P\left(\frac{S_n}{n} \in I\right) = \begin{cases} 0 & \text{if } I \text{ does not contain an open interval containing } p \\ 1 & \text{if } I \text{ contains an open interval containing } p \end{cases}$$

e.g: $\frac{S_n}{n} \to p$, $p = \frac{1}{3}$

$\lim_{n\to\infty} P\left(0.3 < \frac{S_n}{n} < 0.4\right) = 1 \to (0.3, 0.4)$

$\lim_{n\to\infty} P\left(\frac{S_n}{n} > 0.5\right) = 0 \to (0.5, \infty)$

4.11 How many times should we flip a coin w/ unknown success prob. p so that the estimate $\hat{p} = \frac{S_n}{n}$ is within 0.05 of the true p, w/ prob. ≥ 0.99?

sample mean → need to converge to p as $n \to \infty$

$P(|\hat{p} - p| < 0.05) \geq 0.99$ → need to solve for n

$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) \geq \alpha$

$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}} \to 0.05 = \frac{1}{2\sqrt{n}} z_{\frac{1+0.99}{2}} = \frac{1}{2\sqrt{n}} \underbrace{z_{0.995}}_{=2.47} \to \sqrt{n} = \frac{2.47}{2(0.05)} = 24.7$

$\to n = 24.7^2 = 610.09 \xrightarrow[\text{up to get int}]{\text{need to round}} \boxed{611}$

$$\boxed{100\alpha\% \text{ confidence interval for } p \text{ is } (\underset{\overset{\uparrow}{\frac{S_n}{n}}}{\hat{p}} - \underset{\overset{\uparrow}{\frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}}}}{\varepsilon}, \hat{p} + \varepsilon)}$$

$$P(p \in (\hat{p}-\varepsilon, \hat{p}+\varepsilon)) \geq \alpha \rightarrow P(\hat{p}-\varepsilon < p < \hat{p}+\varepsilon) \geq \alpha$$

4.12 We repeat a trial 1000 times & observe 450 successes. Find the 95% confidence interval for the true success prob. p.

Confidence interval for $p = (\hat{p}-\varepsilon, \hat{p}+\varepsilon)$, $\hat{p} = \frac{S_n}{n} = \frac{450}{1000} = 0.45$

$$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}} = \frac{1}{2\sqrt{1000}} z_{\frac{1+0.95}{2}} = \frac{1}{2\sqrt{1000}} z_{0.975} = \frac{1.96}{2\sqrt{1000}} = 0.0309$$

$$CI = (0.45 - 0.0309, 0.45 + 0.0309) = (0.4191, 0.4809)$$

Monday 11/13

4.4 Poisson Approximation

Def → Let $\lambda > 0$. A random variable X has the Poisson distribution w/ parameter λ if X is nonnegative int valued and has the pmf:

$$\boxed{P_X(k) = P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2 \ldots} \rightarrow X \sim \text{Poisson}(\lambda)$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Theorem → Let $X \sim \text{Poisson}(\lambda)$ $\boxed{E[X] = \lambda \qquad Var(X) = \lambda}$

Proof →

$$E[X] = \sum_{k=0}^{\infty} k P_X(k) = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda}$$

$s = k-1$
$k = s+1$

$$= \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{s!} e^{-\lambda} = e^{-\lambda} \cdot \lambda \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = 1 \cdot \lambda = \lambda \quad \text{QED!}$$

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1) P_X(k) = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} e^{-\lambda}$$

$s = k-2$
$k = s+2$

$$= \sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{s!} e^{-\lambda} = e^{-\lambda} \cdot \lambda^2 \overbrace{\sum_{s=0}^{\infty} \frac{\lambda^s}{s!}}^{e^{\lambda}} = e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} = \lambda^2$$

$$\therefore Var(X) = E[X^2] - E[X]^2 = E[X(X-1)] + E[X] - E[X]^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda \quad \text{QED!}$$

Theorem → Let $\lambda > 0$. Suppose $S_n \sim Bin(n, \frac{\lambda}{n})$ then $\boxed{\lim_{n\to\infty} P(S_n = k) = \frac{\lambda^k}{k!} e^{-\lambda},\ k = 0, 1, 2 \ldots}$

$$P(S_n = k) \approx \frac{\lambda^k}{k!} e^{-\lambda} \text{ if } np^2 \text{ small} < 0.05 \quad (\lambda = np)$$

Proof →

$$\lim_{n\to\infty} P(S_n = k) = \lim_{n\to\infty} \binom{n}{k} p^k q^{n-k} = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \cdot \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{\lambda^k}{k!} \cdot \frac{n(n-1)\ldots(n-k+1)}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \frac{1}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$= \lim_{n\to\infty} \frac{\lambda^k}{k!} \cdot \underbrace{\frac{n}{n}}_{1} \cdot \underbrace{\frac{n-1}{n}}_{1} \cdots \underbrace{\frac{n-k+1}{n}}_{1} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \cdot \underbrace{\frac{1}{\left(1 - \frac{\lambda}{n}\right)^k}}_{1} = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{QED!}$$

$$\hookrightarrow \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \rightarrow \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln(1+\frac{x}{n})^n} = \lim_{n\to\infty} e^{n \ln(1+\frac{x}{n})} = \lim_{n\to\infty} e^{\frac{\ln(1+\frac{x}{n})}{1/n}}$$

$y = \frac{1}{n}$

$$= \lim_{y\to 0} e^{\frac{\ln(1+xy)}{y}} \overset{LH}{=} \lim_{y\to 0} e^{\frac{x}{1+xy}} = e^x \checkmark$$

$S_n \sim Bin(n,p) \rightarrow S_n \approx N(np, npq)$ if $npq > 10$
or $S_n \approx$ Poisson (np) if np^2 small < 0.05 ↗ p very small, ↘ rare events

$\boxed{4.22}$ The proofreader of a textbook noticed that a random chosen page of the book has no typos w/ prob. 0.9. Estimate the prob. that a random chosen page contains exactly 2 typos.

Let S_n = # typos in a page $\rightarrow S_n \approx$ Poisson(λ)
$P(S_n = 0) = 0.9 \rightarrow$ Find $P(S_n = 2)$

$$0.9 = P(S_n=0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-\lambda} \to \ln(0.9) = -\lambda \to \lambda = -\ln(0.9) = 0.1054$$

$$P(S_n=2) = \frac{\lambda^2}{2!}e^{-\lambda} = \frac{(0.1054)^2}{2}e^{-0.1054} = 0.005$$

What about $E(S_n) \approx \lambda = 0.1054$

Wednesday 11/15

Hypergeometric approximation to Binomial distribution

Let $X \sim \text{Hypergeom}(N, N_A, n)$. Suppose $\frac{N_A}{N} \to p$ as $N \to \infty$

then

$$\underbrace{\boxed{P(X=k) \to \binom{n}{k} p^k q^{n-k} \text{ as } N \to \infty}}_{\text{Bin}(n,p) \sim N(np,\, npq)}$$

Proof →

$\frac{N_B}{N} = \frac{N - N_A}{N}$

$= 1 - \cancelto{p}{\frac{N_A}{N}}$

$\to 1 - p = q$

$$P(X=k) = \frac{\binom{N_A}{k}\binom{N_B}{n-k}}{\binom{N}{n}} = \frac{N_A!}{k!(N_A-k)!} \cdot \frac{N_B!}{(n-k)!(N_B-n+k)!} \cdot \frac{n!(N-n)!}{N!}$$

$$= \frac{n!}{k!(n-k)!} \cdot \underbrace{\cancelto{p}{\frac{N_A}{N}} \cdot \frac{N_A-1}{N-1} \cdots \frac{N_A-k+1}{N-k+1}}_{k} \cdot \underbrace{\cancelto{q}{\frac{N_B}{N-k}} \cdot \frac{N_B-1}{N-k-1} \cdots \frac{N_B-n+k+1}{N-n+1}}_{N-k} \to \text{total } N \text{ terms}$$

$$\to \binom{n}{k} p^k q^{n-k} = \text{Bin}(n,p) \quad \text{QED!}$$

e.g.: $n=5$, $P(X=3) = \frac{\binom{N_A}{3}\binom{N_B}{2}}{\binom{N}{5}} = \frac{N_A(N_A-1)(N_A-2)}{3!} \cdot \frac{N_B(N_B-1)}{2!} \cdot \frac{5!}{N(N-1)(N-2)(N-3)(N-4)}$

$$= \frac{5!}{3!\,2!} \cdot \cancelto{p}{\frac{N_A}{N}} \cdot \cancelto{p}{\frac{N_A-1}{N-1}} \cdot \cancelto{p}{\frac{N_A-2}{N-2}} \cdot \cancelto{q}{\frac{N_B}{N-3}} \cdot \cancelto{q}{\frac{N_B-1}{N-4}} \to \binom{5}{3} p^3 q^2$$

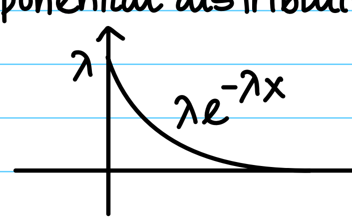
Monday 11/27

Def →

4.5 Exponential Distribution

Let $0 < \lambda < \infty$. A random variable X has the exponential distribution w/ parameter λ if X has the pdf

$$\boxed{f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad X \sim \text{Exp}(\lambda)}$$



$\boxed{E[X] = \frac{1}{\lambda} \quad Var(\lambda) = \frac{1}{\lambda^2}}$ could be used for waiting time problems

4.28 Let $X \sim Exp(\lambda)$. Let $Y = \frac{X}{2}$. Find $P(Y > t)$ for $t > 0$

$$P(Y > t) = P\left(\frac{X}{2} > t\right) = P(X > 2t) = \int_{2t}^{\infty} \lambda e^{-\lambda x}\,dx = e^{-\lambda x}\Big|_{\infty}^{2t} = e^{-2tx},\ t > 0$$

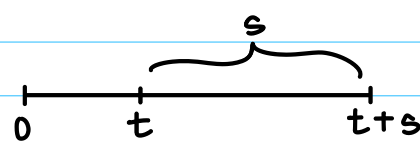
(1) $F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ (2) $P(X > t) = \begin{cases} e^{-\lambda t}, & t \geq 0 \\ 1, & t < 0 \end{cases}$

$$F_X(t) = P(X \leq t) = \int_0^t \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^t = 1 - e^{-\lambda t}$$

Theorem → **Memoryless property of exponential distribution**

Suppose $X \sim Exp(\lambda)$ then for any $s, t > 0$ we have

$$\boxed{P(X > t+s \mid X > t) = P(X > s)}$$

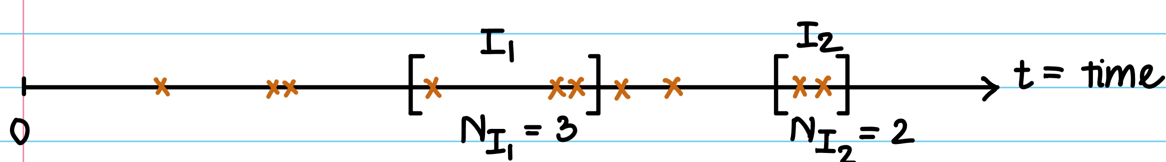


Proof →

$$P(X > t+s \mid X > t) = \frac{P(X > t+s \cap X > t)}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = \frac{e^{-\lambda t} e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s) \quad \text{QED!}$$

4.6 **Poisson Process** → application: arrival time for phone calls



Def → The Poisson process w/ intensity $\lambda > 0$ is a collection of random variables $\{N_I \mid I \subseteq [0, \infty)\}$ such that

$\boxed{\text{(1) } N_I \sim \text{Poisson}(\lambda \underbrace{|I|}_{\text{length of } I}) \qquad \text{(2) If } I_1, I_2 \ldots I_n \text{ are nonoverlapping then } N_{I_1}, \ldots N_{I_n} \text{ are mutually independent}}$

I_1, I_2 are nonoverlapping if interval $I_1 \cap$ interval $I_2 = \emptyset$

4.35 Suppose that customers arrive in a certain store according to a Poisson process w/ intensity 5/hr. Suppose that the store is open 9am – 6pm

(a) Find the prob. that no customer comes to the store within 1 hr of opening

$N_{[9,10]} \sim \text{Poisson}(\lambda \cdot 1)$

↳ length of 1 hr $\rightarrow \frac{5}{hr} \cdot 1hr = 5$

$$P(N_{[9,10]} = 0) = \frac{5^0}{0!} e^{-5} = e^{-5}$$

(b) Find the prob. that we have 2 customers between 9–10am, 3 customers between 10 – 10:30am & 5 customers between 2 – 3:30pm

$P(N_{[9,10]} = 2 \cap N_{[10,10:30]} = 3 \cap N_{[2,3:30]} = 5)$

$\overset{\text{independent}}{=\!=}$ $P(N_{[9,10]} = 2) \cdot P(N_{[10,10:30]} = 3) \cdot P(N_{[2,3:30]} = 5)$

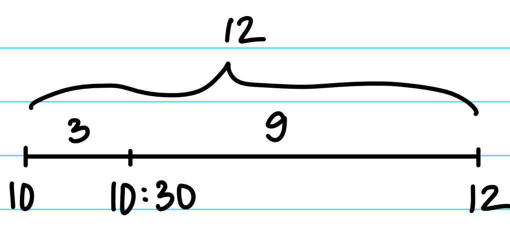
$$= \frac{5^2}{2!} e^{-5} \cdot \frac{(5/2)^3}{3!} e^{-5/2} \cdot \frac{(15/2)^5}{5!} e^{-15/2}$$

(c) Find the conditional prob. that 3 customers arrive between 10 – 10:30am given that 12 customers arrive between 10am – 12pm

$$P(N_{[10,10:30]} = 3 \mid N_{[10,12]} = 12) = \frac{P(N_{[10,10:30]} = 3 \cap N_{[10,12]} = 12)}{P(N_{[10,12]} = 12)}$$

$$= \frac{P(N_{[10,10:30]} = 3 \cap N_{[10:30,12]} = 9)}{P(N_{[10,12]} = 12)}$$

$$= \frac{\frac{(\frac{5}{2})^3}{3!} e^{-5/2} \cdot \frac{(\frac{15}{2})^9}{9!} e^{-15/2}}{\frac{10^{12}}{12!} e^{-10}}$$



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4.15 (a) $P(N_{[11,12]} > 2) = 1 - P(N_{[11,12]} \leq 2)$

$$= 1 - \frac{4^0}{0!} e^{-4} + \frac{4^1}{1!} e^{-4} + \frac{4^2}{2!} e^{-4} = 0.762$$

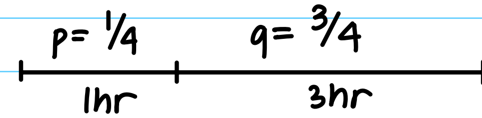
(b) $P(N_{[11,12]} = 0 \cap N_{[12,3]} \geq 10) \overset{\text{independent}}{=} P(N_{[11,12]} = 0)\, P(N_{[12,3]} \geq 10)$

$$= \frac{4^0}{0!} e^{-4} \left\{ 1 - \left(\frac{12^0}{0!} + \frac{12^1}{1!} + \dots + \frac{12^9}{9!} \right) e^{-12} \right\}$$

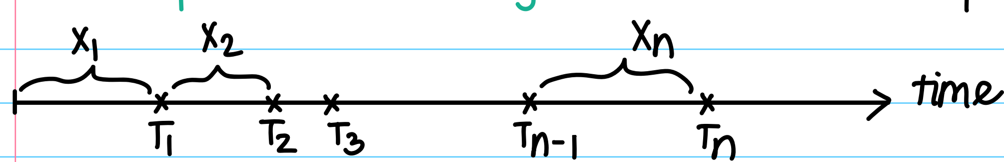
(c) $P(N_{[11,12]} = 0 \mid N_{[11,3]} = 13) = \dfrac{P(N_{[11,12]} = 0 \cap N_{[11,3]} = 13)}{P(N_{[11,3]} = 13)}$

$$= \frac{P(N_{[11,12]} = 0 \cap N_{[12,3]} = 13)}{P(N_{[11,3]} = 13)} = \frac{P(N_{[11,12]} = 0)\, P(N_{[12,3]} = 13)}{P(N_{[11,3]} = 13)}$$

$$= \frac{\frac{4^0}{0!} e^{-4} \cdot \frac{12^{13}}{13!} e^{-12}}{\frac{16^{13}}{13!} e^{-16}} = \frac{13!}{0!\,13!} \left(\frac{4}{16}\right)^0 \left(\frac{12}{16}\right)^{13} = \binom{13}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{13} \sim Bin\left(13, \frac{1}{4}\right)$$

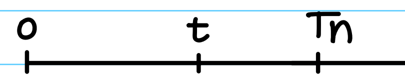


* Poisson process w/ intensity λ → arrival time of phone calls



$X_n = T_n - T_{n-1}$ $\quad$ $X_1, X_2 \dots X_n \dots$ interarrival time

Find the distribution of T



Step 1 $\quad$ Find the cdf of T_n

$$F_{T_n}(t) = P(T_n \leq t) = 1 - P(T_n > t) = 1 - P(\underbrace{N_{[0,t]}}_{\sim \lambda t} \leq n-1)$$

$$= 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Step 2 $\quad$ Find pdf of T_n

$$f_{T_n}(t) = \frac{d}{dt} F_{T_n}(t) = -\frac{d}{dt} \left(e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} \right)$$

$$= -(-\lambda) e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} - e^{-\lambda t} \sum_{k=1}^{n-1} \frac{k (\lambda t)^{k-1}}{k!} \lambda$$

$$= e^{-\lambda t}\left(\sum_{k=0}^{n-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{k=1}^{n-1} \frac{\lambda^k t^{k-1}}{(k-1)!}\right)$$

$T_n \sim \text{Gamma}(n, \lambda)$

$$= e^{-\lambda t}\left(\sum_{k=0}^{n-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{s=0}^{n-2} \frac{\lambda^{s+1} t^s}{s!}\right) = \boxed{\begin{cases} e^{-\lambda t} \cdot \dfrac{\lambda^n t^{n-1}}{(n-1)!}, & t > 0 \\ 0, & t \leq 0 \end{cases}}$$

Def → Let $r, \lambda > 0$. A random variable X has the Gamma distribution w/ parameters (r, λ) if $X \geq 0$ and has the pdf

$$\boxed{f_X(x) = \begin{cases} \dfrac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & x \geq 0 \\ 0, & x \leq 0 \end{cases}} \quad \text{where} \quad \boxed{\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx, \ r > 0}$$

is the Gamma fn

$\Gamma(n) = (n-1)!$ for $n = 1, 2, 3 \ldots$ $\qquad \Gamma(1) = 1 \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(r) = (r-1)\Gamma(r-1), \ r > 1$

Friday 12/1

5.1 Moment Generating Function

Def → The moment generating function of X is defined by $\boxed{M(t) = E[e^{tX}]}$ for real t if the expectation exists

5.2 X is a discrete random variable

X	-1	4	9
$P_X(k)$	$1/3$	$1/6$	$1/2$

$$M(t) = E[e^{tX}] = \sum_k e^{tk} P_X(k) = e^{-t} \cdot \frac{1}{3} + e^{4t} \cdot \frac{1}{6} + e^{9t} \cdot \frac{1}{2}, \ t \in \mathbb{R}$$

e.g: Suppose the moment generating fn of X is $M(t) = 0.25 + 0.3e^t + 0.2e^{5t} + 0.25e^{-2t}$. Find pmf of X

X	0	1	5	-2	← exponent
$P_X(k)$	0.25	0.3	0.2	0.25	← coeffs

e.g: $X \sim \text{Exp}(\lambda)$. Find $M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$

$$= \lambda \cdot \frac{e^{(t-\lambda)x}}{t-\lambda}\Bigg|_0^{\infty} = \frac{\lambda}{t-\lambda}\left(\cancelto{0}{e^{(t-\lambda)\infty}} - 1\right) = \boxed{\begin{cases} \dfrac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda \end{cases}}$$

Theorem → $\boxed{E[X^n] = M_X^{(n)}(0)}$ ← n^{th} derivative

Proof → (1) $M_X(t) = E[e^{tX}]$

$\frac{d}{dt} M_X(t) = E[Xe^{tX}]$

$M_X''(t) = E[X^2 e^{tX}]$

$M_X'''(t) = E[X^3 e^{tX}]$

$\cdots$

$M_X^{(n)}(t) = E[X^n e^{tX}]$

$M_X^{(n)}(0) = E[X^n]$ QED!

(2) $M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n$ → Taylor series

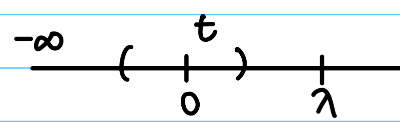
$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$

$\therefore E[X^n] = M_X^{(n)}(0)$ by Taylor series expansion

QED!

e.g: $X \sim$ Exp(λ). Find $E[X^n]$

$M_X(t) = \frac{\lambda}{\lambda - t} = \lambda(\lambda - t)^{-1}, \ t < \lambda$



$M_X'(t) = \lambda(-1)(-1)(\lambda - t)^{-2} = \frac{\lambda}{(\lambda - t)^2}$

$M_X''(t) = \lambda(-1)(-2)(\lambda - t)^{-3} = \frac{2\lambda}{(\lambda - t)^3}$

$M_X'''(t) = 2\lambda(-1)(-3)(\lambda - t)^{-4} = \frac{6\lambda}{(\lambda - t)^4}$

$\cdots$

General: $M_X^{(n)}(t) = n!\,\lambda(\lambda - t)^{-(n+1)}$

$\therefore E[X^n] = M_X^{(n)}(0) = \frac{n!\,\lambda}{\lambda^{n+1}} = \frac{n!}{\lambda^n}$

Monday 12/4 e.g: Let $Z \sim N(0,1)$

(a) Find the moment generating fn of Z

$$M_Z(t)=E[e^{tZ}]=\int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}(x^2-2tx)}dx$$

$$=\int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}(x-t)^2}}_{\text{pdf of }N(t,1)}e^{\frac{1}{2}t^2}dx = e^{t^2/2},\ -\infty<t<\infty$$

(b) Find $E[Z^n]=M_Z^{(n)}(0)$

$\therefore M_Z'(t)=te^{t^2/2}$ $\quad\therefore M_Z''(t)=e^{t^2/2}+t^2e^{t^2/2}\rightarrow$ uncontrollable

Start w/ mean

$$M_Z(t)=E[e^{tZ}]=E\left(\sum_{k=0}^{\infty}\frac{(tZ)^k}{k!}\right)=\sum_{k=0}^{\infty}\frac{E[Z^n]}{k!}t^k$$

$$e^{-t^2/2}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}t^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{t^{2n}}{2^n n!}$$

make =

only even powers for t^{2n}

$$\therefore \frac{(2n)!}{2^n n!}=\frac{(2n)(2n-1)(2n-2)\ldots 3.2.1}{2^n n!}=\frac{(2n-1)(2n-3)\ldots 2^n\cancel{(n)}\cancel{(n-1)}\ldots}{\cancel{2^n}\cancel{n!}}$$

$$=(2n-1)(2n-3)\ldots 5.3.1=(2n-1)!!$$

$$\therefore E[Z^k]=0 \text{ if } k \text{ odd } \& \ E[Z^{2n}]=\frac{(2n)!}{2^n n!} \text{ if } k \text{ even}=2n$$

Distribution	Moment generating fn	
Poisson (λ)	$e^{\lambda(e^t-1)}$	$-\infty<t<\infty$
Geom (p)	$\frac{pe^t}{1-qe^t}$	$qe^t<1\rightarrow e^t<\frac{1}{q}\rightarrow t<-\ln q$
Unif(a,b)	$\begin{cases}\frac{e^{bt}-e^{at}}{(b-a)t}, & t\neq 0\\ 1, & t=0\end{cases}$	
Ber(p)	$q+pe^t$	$-\infty<t<\infty$
Bin(n,p)	$(q+pe^t)^n$	$-\infty<t<\infty$

$\text{Exp}(\lambda)$ $\qquad \begin{cases} \dfrac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda \end{cases}$

$N(\mu, \sigma^2)$ $\qquad e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

e.g: Suppose $M_X(t) = e^{2t + 6t^2}$. Find distribution of X

$\mu = 2$

$\frac{1}{2}\sigma^2 = 6 \rightarrow \sigma^2 = 12 \qquad \therefore X \sim N(2, 12)$

Wednesday 12/6

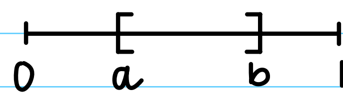
Def $\rightarrow$ Random variables X, Y are equal in distribution if $P(X \in B) = P(Y \in B)$ for all subsets $B \subseteq \mathbb{R}$

We write $X \stackrel{d}{=} Y$

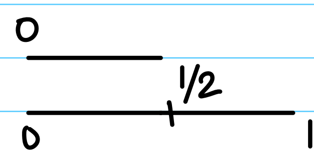
e.g: $\Omega = [0, 1]$

Let P be the uniform distribution on $[0, 1]$

$P([a, b]) = b - a$ for all $0 \leq a \leq b \leq 1$



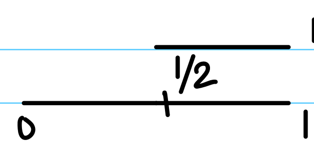
e.g: Let $X(w) = \begin{cases} 1 & \text{if } 0 \leq w \leq 1/2 \\ 0 & \text{if } 1/2 \leq w \leq 1 \end{cases}$

X 

X	0	1
$P_X(k)$	$\frac{1}{2}$	$\frac{1}{2}$

$X \sim \text{Ber}(\frac{1}{2})$

Let $Y = \begin{cases} 0 & \text{if } 0 \leq w \leq 1/2 \\ 1 & \text{if } 1/2 \leq w \leq 1 \end{cases}$

Y 

Y	0	1
$P_Y(k)$	$\frac{1}{2}$	$\frac{1}{2}$

$Y \sim \text{Ber}(\frac{1}{2})$

$\therefore X \stackrel{d}{=} Y$

e.g: $\left. \begin{array}{l} X \sim N(0, 1) \\ Y = -X \rightarrow Y \sim N(0, 1) \end{array} \right\} X \stackrel{d}{=} Y$

Theorem → Let X, Y be random variables. Suppose $M_X(t) = M_Y(t)$ for all $t \in (-\delta, \delta)$ with some $\delta > 0$ then $X \stackrel{d}{=} Y$

If $M_X(t) = M_Y(t)$ for all t in an interval that contains an open interval containing 0 then $X \stackrel{d}{=} Y$

e.g: $M_Y(t) = e^{17(e^t - 1)}$, $t \in \mathbb{R}$. Find the distribution of Y

$Y \sim$ Poisson (17)

↳ $e^{\lambda(e^t - 1)}$

e.g: Suppose a random variable Y has a moment generating fn that equals $e^{17(e^t-1)}$ for $t = 0$ only. Find the distribution of Y

→ insufficient info

↘ only for 1 value of t

5.2 Distribution of a function of a random variable

* Discrete random variables:

e.g: Suppose the pmf of X is given by

X	-1	0	1	2
$P_X(K)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Let $Y = X^2$. Find the distribution of Y

Y	0	1	4
$P_Y(K)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(Y=4) = P(X^2 = 4) = P(X=2) = 1/4$

$P(Y=1) = P(X^2 = 1) = P(X=1) + P(X=-1) = 1/2$

$P(Y=0) = P(X^2 = 0) = P(X=0) = 1/4$

* Continuous random variables:

$\boxed{5.19}$ Let $U \sim$ Unif $(0,1)$. Let $\lambda > 0$. Let $g(x) = -\frac{1}{\lambda}\ln(1-x)$. Find the dist. of $Y = g(x)$

Use cdf method

Step 1 – Find cdf

$F_Y(y) = P(Y \le y) = P(g(x) \le y) = P\left(-\frac{1}{\lambda}\ln(1-X) \le y\right) = P(\ln(1-X) \le -\lambda y)$

$= P(1 - X \geq e^{-\lambda y}) = P(0 \le X \le 1 - e^{-\lambda y})$

$$= \begin{cases} 1-e^{-\lambda y}, & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

Step 2 - Take derivative to find pdf

$$f_Y(y) = \frac{d}{dy} F_Y(y) = -e^{-\lambda y} \cdot (-\lambda) = \begin{cases} \lambda e^{-\lambda y}, & y \geqslant 0 \\ 0, & y < 0 \end{cases} \quad \therefore Y \sim \text{Exp}(\lambda)$$

$U_1, U_2 \ldots U_n \sim \text{Unif}(0,1)$

$g(U_1), g(U_2), \ldots g(U_n) \sim \text{Exp}(\lambda)$

Fact $\rightarrow$ Let $U \sim \text{Unif}(0,1)$, $Y = F_Y^{-1}(U)$ $>$ Y has the cdf of F_Y

Proof $\rightarrow$ $Y \sim \text{Exp}(\lambda) \rightarrow F_Y(y) = 1 - e^{-\lambda y}, \ y > 0$

Let $z = F_Y(y)$ solve Y

$$z = 1 - e^{-\lambda y} \rightarrow e^{-\lambda y} = 1 - z \rightarrow -\lambda y = \ln(1-z)$$

$$\therefore y = \frac{-1}{\lambda} \ln(1-z) \rightarrow F_Y^{-1}(z) = \frac{-1}{\lambda} \ln(1-z) \quad \therefore F_Y^{-1}(U) \sim \text{Exp}(\lambda)$$

Theorem $\rightarrow$ Transformation method

Suppose random variable X has pdf f_X and the fn g is differentiable, 1-to-1, and its derivative is 0 only at finitely many points. Then the pdf of $Y = g(X)$ is given by

Let $Y = g(x)$ then $f_Y(y) = f_X(g^{-1}(y)) \dfrac{1}{|g'(\underbrace{g^{-1}(y)}_{x})|}$

e.g. $X \sim N(\overset{\mu}{1}, \overset{\sigma^2}{2})$. $Y = X^3$. Find pdf of Y

$g(x) = x^3 = y$

$\rightarrow g'(x) = 3x^2$

$\rightarrow g^{-1}(y) = x = \sqrt[3]{y}$

$\rightarrow g'(g^{-1}(y)) = 3 \cdot (\sqrt[3]{y})^2 = 3y^{2/3}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{\frac{-1}{2 \cdot 2}(x-1)^2} \cdot \frac{1}{|3y^{2/3}|} = \frac{1}{6\sqrt{\pi}\,|y^{2/3}|} e^{-\frac{1}{4}(y^{1/3}-1)^2} \quad -\infty < y < \infty$$

Friday 12/8

Theorem: General case

Let X be a random variable with pdf f_X. Let $Y = g(X)$ then

$$f_Y(y) = \sum_{\substack{x:\, x=g^{-1}(y) \\ g(x)=y}} f_X(x) \frac{1}{|g'(x)|} \Bigg|_{x=g^{-1}(y)}$$

e.g.: $X \sim N(0,1)$. $Y = X^2$. Find pdf of Y

(1) General $y = x^2 \rightarrow y = g(x) = x^2 \rightarrow x = g^{-1}(y) = \pm\sqrt{y}$, $g'(x) = 2x$

$$f_Y(y) = f_X(x)\frac{1}{|g'(x)|}\Bigg|_{x=\sqrt{y}} + f_X(x)\frac{1}{|g'(x)|}\Bigg|_{x=-\sqrt{y}}$$

$$= f_X(x)\frac{1}{|2x|}\Bigg|_{x=\sqrt{y}} + f_X(x)\frac{1}{|2x|}\Bigg|_{x=-\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{|2x|}\Bigg|_{x=\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\frac{1}{|2x|}\Bigg|_{x=-\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} \cdot 2 = \begin{cases} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{2}\sqrt{y}} e^{-y/2}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

$\therefore Y \sim \text{Gamma}\left(\frac{1}{\alpha}, \frac{1}{\alpha}\right) = \chi^2(1)$

(2) Cdf

Step 1 – Find cdf of Y

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

Step 2 – Find pdf of $Y \rightarrow$ take derivative

is the pdf of normal dist.

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right) = \Phi'(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + \Phi'(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$$

$$= f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{-y/2}\frac{1}{2\sqrt{y}}\cdot 2 = \begin{cases} \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2}\sqrt{y}} e^{-y/2} & y > 0 \\ 0 & y \le 0 \end{cases}$$

$\therefore$ same result $\therefore$ use either method

$\boxed{5.28}$ Let $X \sim Exp(1)$. Let $Y = (X-1)^2$. Find pdf of Y

cdf method → $F_Y(y) = P(Y \le y) = P((X-1)^2 \le y) = P(-\sqrt{y} \le X - 1 \le \sqrt{y})$, $0 < y < \infty$

$= P(1-\sqrt{y} \le X \le 1+\sqrt{y}) = F_X(1+\sqrt{y}) - F_X(1-\sqrt{y})$

pdf: $f_Y(y) = F'_X(1+\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + F'_X(1-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = f_X(1+\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(1-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

2 conditions ←

$$= \begin{cases} e^{-1-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} + e^{-1+\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} & \text{if } 1-\sqrt{y} > 0 \\ e^{-1-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}} & \text{if } 1-\sqrt{y} \le 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{y}}\left(e^{-1-\sqrt{y}} + e^{-1+\sqrt{y}}\right) & \text{if } 0<y<1 \\ \frac{1}{2\sqrt{y}} e^{-1-\sqrt{y}} & \text{if } y \ge 1 \\ 0 & \text{if } y \le 0 \end{cases}$$

Transformation → $y = (x-1)^2 \rightarrow x = 1 \pm \sqrt{y} \rightarrow \frac{dy}{dx} = 2(x-1)$

$$f_Y(y) = f_X(x) \frac{1}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=1+\sqrt{y}} + f_X(x) \frac{1}{\left|\frac{dy}{dx}\right|}\Bigg|_{x=1-\sqrt{y}}$$

$$= \begin{cases} e^{-1-\sqrt{y}} \frac{1}{|2(x-1)|}\Bigg|_{x=1+\sqrt{y}} + e^{-1+\sqrt{y}} \frac{1}{|2(x-1)|}\Bigg|_{x=1-\sqrt{y}} & \text{if } 1-\sqrt{y} > 0 \\ e^{-1-\sqrt{y}} \frac{1}{|2(x-1)|}\Bigg|_{x=1+\sqrt{y}} + 0 & \text{if } 1-\sqrt{y} < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} e^{-1-\sqrt{y}} + \frac{1}{2\sqrt{y}} e^{-1+\sqrt{y}}, & 0 \le y \le 1 \\ \frac{1}{2\sqrt{y}} e^{-1-\sqrt{y}}, & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

Monday
12/11

REVIEW

$X \sim Gamma(r, \lambda)$ if pdf $f_X(x) = \begin{cases} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ w/ $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$, $r > 0$

Properties

(1) $\Gamma(r) = (r-1)\Gamma(r-1)$, $r > 1$

(2) $\Gamma(n) = (n-1)!$, $n = 1, 2 \ldots$

(3) $\Gamma(1) = 1$

(4) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$E[X] = \int_0^\infty \frac{x \cdot \lambda^r \cdot x^{r-1} e^{-\lambda x}}{\Gamma(r)} dx = \int_0^\infty \frac{\lambda^r \cdot x^r}{\Gamma(r)} e^{-\lambda x} dx$$

$$= \frac{\Gamma(r+1)}{\lambda \Gamma(r)} \underbrace{\int_0^\infty \lambda^{r+1} \frac{x^{r+1}}{r(r+1)} e^{-\lambda x} dx}_{\text{pdf of } Gamma(r+1, \lambda)} = \frac{\Gamma(r+1)}{\lambda \Gamma(r)} = \frac{r\Gamma(r)}{\lambda \Gamma(r)} = \frac{r}{\lambda}$$

pdf of $Gamma(r+1, \lambda)$ $\rightarrow$ $T_n \sim Gamma(n, \lambda)$ for the Poisson process w/ intensity λ

$$E[X^2] = \frac{\Gamma(r+2)}{\lambda^2 \Gamma(r)} = \frac{(r+1)\Gamma(r+1)}{\lambda^2 \Gamma(r)} = \frac{(r+1) r \Gamma(r)}{\lambda^2 \Gamma(r)} = \frac{r(r+1)}{\lambda^2}$$

$$\therefore Var(X) = E[X^2] - E[X] = \frac{r(r+1)}{\lambda^2} - \frac{r}{\lambda} = \frac{r}{\lambda^2}$$

Continuous random variables:

cdf

$$F_X(x) = \int_{-\infty}^{\infty} \underset{\substack{\uparrow \\ \text{pdf}}}{f_X(u)} du = P(X \leq u) \longrightarrow f_X(x) = F'_X(x)$$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

Discrete random variables:

cdf

$$F_X(x) = \sum_{K \leq x} \underset{\substack{\uparrow \\ \text{pmf}}}{P_X(K)} = P(X \leq x)$$

$$P_X(K) = P(X \leq K) - P(X < K) = F_X(K) - F_X(K-)$$