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Section 2.5

Non homogeneous systems

A linear system $Ax=b$ is called non homogeneous if $b \neq 0$, that is, if $b_i \neq 0$ for some i. In other words, it is not an homogeneous linear system.

For example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \neq 0.$$

We want to check that the system is not inconsistent (i.e., that it has at least one solution). Then, if so, we want to find all solutions. (Either the unique one, or the infinitely many).

Recall, we learned how to find a particular solution x_p. (called p in the book)

We also learned how to find all solutions of the homogeneous systems

$$x_h = \alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \cdots \alpha_p x_{h_p}$$

where $p = n - r$

$r = \text{rank } A$ = # basic columns

= # of pivots

p = number of free variables.

We also have linearity

$$Ax + Ay = A(x+y)$$

so $A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$

$$x = x_p + \alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \cdots + \alpha_p x_{h_p}$$

are all solutions of $Ax = b$ with different values of $\alpha_1, \alpha_2 \cdots \alpha_p$

For this example

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 4 & 2 & 4 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 0 & -2 \end{array}\right]$$

$m_{21} = \frac{-2}{1} = -2$

$E_2 - 2E_1$

Rank $A = 2$

basic columns $1, 3$

free variables x_2, x_4

For particular solution:

set free variables $= 0$ solve

for x_1 x_3

$$2x_3 = -2 \qquad x_3 = -1$$

$$x_1 + \underbrace{x_2}_{=0} + \underbrace{x_3}_{=-1} + \underbrace{x_4}_{=0} = 3$$

$$x_1 = 3 - (-1) = 4$$

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

check!

as by product $\begin{vmatrix} 3 \\ 4 \end{vmatrix} = 4 \begin{vmatrix} 1 \\ 2 \end{vmatrix} - 1 \begin{vmatrix} 1 \\ 4 \end{vmatrix}$

b is a linear combination of the basic columns

For the homogeneous system

$$[A|0] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 4 & 2 & 0 \end{array}\right] \to [E|0] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

We write basic variables in terms of the free variables. x_2 x_4

$$2x_3 = 0 \Rightarrow x_3 = 0$$

$$x_1 = -x_2 - \underbrace{x_3}_{=0} - x_4 = -x_2 - x_4$$

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \alpha_1 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{x_{h_1}} + \alpha_2 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{x_{h_2}} = \alpha_1 x_{h_1} + \alpha_2 x_{h_2}$$

$$\alpha_1, \alpha_2 \in \mathbb{R}$$

Example

$$2x_1 + 2x_2 + 2x_3 = 4$$
$$x_1 + 2x_2 + 3x_3 = 1$$
$$4x_1 \qquad + x_3 = 7$$

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$$

matrix of coefficients — right hand side vector

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 4 \\ 1 & 2 & 3 & 1 \\ 4 & 0 & 1 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & -4 & -3 & -1 \end{array}\right]$$

$$m_{21} = -\frac{a_{21}}{a_{11}} = \frac{-1}{2} \qquad E_2' = E_2 - \frac{1}{2} E_1$$

$$m_{31} = -\frac{a_{31}}{a_{11}} = \frac{-4}{2} = -2 \qquad E_3' = E_3 - 2E_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & -5 \end{array}\right]$$

$$m_{32} = -\frac{a^*_{32}}{a^*_{22}} = -\frac{-4}{1} = 4$$

$$E_3'' = E_3' + 4E_2'$$

A is $m \times n$ $\quad 3 \times 3$

$n = 3$

rank = 3 $\quad$ # of pivots

No Free Variables

x_p particular solution by back substitution

($x_h = 0$
the only solution to the homogeneous system)

$$x_3 = \frac{-5}{5} = -1$$

$$x_2 = \left(-1 - 2 \cdot (-1)\right) / 1$$

$$= -1 + 2 = 1$$

$$x_1 = \left(4 - 2x_2 - 2x_3\right) / 2$$

$$= \left(4 - 2.1 - 2 \cdot (-1)\right) / 2 = 4/2 = 2$$

$$x_p = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \\ -1 \end{vmatrix}$$

check!

Another example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 4 & 2 & 2 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & -2 \end{array}\right]$$

$m_{21} = -\frac{1}{1} = -1 \qquad m_{31} = -\frac{2}{1} = -2$

$E_2 - E_1 \qquad E_3 - E_1$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ row of zeros. consistent

$m_{32} = -\frac{2}{1} = -2 \qquad E_3' - 2E_2'$

We do a particular solution first

Rank = 2, Free variables $x_3 \ x_4 \ x_5$

set $x_3 = x_4 = x_5 = 0$

back substitution $x_2 = -1$

$x_1 = (3 - x_2)/1 = 3 - (-1) = 4$

$$X_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

check!

For X_h

$$[E|0] = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$x_2 = 0$

$x_1 = -x_3 - x_4 - x_5$

$$X_h = \begin{bmatrix} -x_3 - x_4 - x_5 \\ 0 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\alpha_1}_{x_3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\alpha_2}_{x_4} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \underbrace{\alpha_3}_{x_5} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

check. $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

general solution

$$\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For example $\alpha_1 = 1$ $\alpha_2 = 0$ $\alpha_3 = 0$

$$\begin{vmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$ check

another solution

One more $\alpha_1 = 2$ $\alpha_2 = 1$ $\alpha_3 = -1$

$$\begin{vmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -2 \\ 0 \\ 2 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} + \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{vmatrix} = \begin{vmatrix} 2 \\ -1 \\ 2 \\ 1 \\ -1 \end{vmatrix}$$ check!

can do like this infinitely many

Back to HW1 .(4).

2 x 2 system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_p = \begin{vmatrix} 2 \\ 1 \end{vmatrix} \quad \text{or} \quad x_p = \begin{vmatrix} 3 \\ 2 \end{vmatrix}$$

We can write 4 equations for the 6 unknowns

$a_{11} \quad a_{12} \quad a_{21} \quad a_{22} \quad b_1 \quad b_2$

$$a_{11} 2 + a_{12} 1 = b_1$$
$$a_{21} 2 + a_{22} 1 = b_2.$$
$$a_{11} 3 + a_{12} 2 = b_1$$
$$a_{21} 3 + a_{12} 2 = b_2$$

$$2 a_{11} + 1 a_{12} + 0 a_{21} + 0 a_{22} - 1 b_1 + 0 b_2 = 0$$
$$0 a_{11} + 0 a_{12} + 2 a_{21} + 1 a_{22} + 0 b_1 - 1 b_2 = 0$$
$$3 a_{11} + 2 a_{12} + 0 a_{21} + 0 a_{22} - 1 b_1 + 0 b_2 = 0$$
$$0 a_{11} + 0 a_{12} + 3 a_{21} + 2 a_{22} + 0 b_1 - 1 b_2 = 0$$

$4 \times 6 \quad A = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 \\ 3 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 0 & -1 \end{bmatrix} \qquad Ax = 0$

homogeneous system

$$[A|0] = \left[\begin{array}{cccccc|c} 2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 3 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & -1 & 0 \end{array}\right]$$

$$m_{31} = -\frac{a_{31}}{a_{11}} = -\frac{3}{2} \qquad E_3 - \frac{3}{2}E_1$$

$$\rightarrow \left[\begin{array}{cccccc|c} 2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{\text{row interchange}} \left[\begin{array}{cccccc|c} 2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 0 & -1 & 0 \end{array}\right]$$

$3 - \frac{3}{2} \cdot 2 = 0$

$2 - \frac{3}{2} \cdot 1 = \frac{1}{2}$ $\qquad -1 - \frac{3}{2}(-1) = -1 + \frac{3}{2} = \frac{1}{2}$

same numbers $\quad m_{43} = -\frac{a^*_{43}}{a^*_{33}} = -\frac{3}{2} \qquad E_4'' - \frac{3}{2}E_3''$

$$\rightarrow \left[\begin{array}{cccccc|c} 2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right]$$

4 pivots, rank 4. 2 free variables

x_5, x_6 (that is b_1, b_2)

We note first that since it is a homogeneous system 0 is a solution

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the trivial solution

To find all possible solutions, write basic variables in terms of the free variables x_5, x_6

last equation $\frac{1}{2} x_4 + \frac{1}{2} x_6 = 0$

$$x_4 = -x_6 \quad [a_{22} = -b_2]$$

3rd equation $2x_3 + x_4 - x_6 = 0$

$$2x_3 = x_6 - x_4$$

$$2x_3 = x_6 - (-x_6) = 2x_6$$

$$x_3 = x_6 \quad [a_{21} = b_2]$$

Similarly, 2nd eq. $\frac{1}{2} x_2 + \frac{1}{2} x_5 = 0$

$$x_2 = -x_5 \quad [a_{12} = -b_1]$$

1st eq. $2x_1 + x_2 - x_5 = 0$

$$2x_1 = x_5 - x_2 = 2x_5$$

$$x_1 = x_5 \quad [a_{11} = b_1]$$

b_1, b_2 free variables (like α_1, α_2) take any value

i) $b_1 = \alpha \quad a_{11} = \alpha \quad a_{12} = -\alpha$

$b_2 = \beta \quad a_{21} = \beta \quad a_{22} = -\beta$

$$\begin{bmatrix} \alpha & -\alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

all possible linear systems with solutions

$$\begin{vmatrix} 2 \\ 1 \end{vmatrix} \quad \begin{vmatrix} 3 \\ 2 \end{vmatrix}$$

Indeed $\quad 2\alpha - \alpha = \alpha \qquad 3\alpha - 2\alpha = \alpha$

$2\beta - \beta = \beta \qquad 3\beta - 2\beta = \beta$

For example $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} x = \begin{vmatrix} 1 \\ 2 \end{vmatrix}$

$$\begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix} x = \begin{vmatrix} 2 \\ 5 \end{vmatrix} \qquad \begin{bmatrix} 3 & -3 \\ 7 & -7 \end{bmatrix} x = \begin{vmatrix} 3 \\ 7 \end{vmatrix}$$

$$\begin{vmatrix} 1/5 & -1/5 \\ 0 & 0 \end{vmatrix} x = \begin{vmatrix} 1/5 \\ 0 \end{vmatrix}$$