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Answers to HW3.

We are asked to consider $Ax = 0$ and $Ax = b$. We shall keep this in mind and perform Gaussian elimination on the augmented system $[A|b]$

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 2 & 5 & 7 & 7 & 7 \\ 3 & 6 & 6 & 6 & 9 \end{array}\right] \xrightarrow[m_{31}=3]{m_{21}=2}$$

$$\rightarrow [E|c] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We begin by answering (d):
Since we see the pivots in first and second columns, the basic columns (first and second) are: $\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}$ and $\begin{vmatrix} 2 \\ 5 \\ 6 \end{vmatrix}$.

x_3 and x_4 are the free variables (total 2).

Rank $A = 2$ (2 pivots).

# of free variables $n - r = 4 - 2 = 2$.

We proceed with (a).

We want to write all variables in terms of the free variables. From $EX = 0$ we have

first: $x_2 + 3x_3 + 3x_4 = 0$

thus $x_2 = -3x_3 - 3x_4$

second $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$$x_1 + 2(-3x_3 - 3x_4) + 2x_3 + 2x_4 = 0$$

$$x_1 - 4x_3 - 4x_4 = 0$$

thus $x_1 = 4x_3 + 4x_4$

$$x_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 + 4x_4 \\ -3x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

The general solution of $Ax = 0$ is then of the form

$$x_h = \alpha \underbrace{\begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{h_1} + \beta \underbrace{\begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{h_2} \quad \text{for } \alpha, \beta \in \mathbb{R}$$

check $4 + 2(-3) + 2 \cdot 1 + 0 = 0$

$2 \cdot 4 + 5(-3) + 7 \cdot 1 + 0 = 0$, etc.

Continuing with (b), we use $[E|c]$ to find a particular solution.

For example, we set $x_3 = x_4 = 0$ (free variables), and obtain:

$$x_2 = 1$$

$$x_1 + 2 \cdot 1 = 3 \Rightarrow x_1 = 3 - 2 = 1$$

So, $x_p = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Check: $1 \cdot 1 + 2 \cdot 1 = 3$

$2 \cdot 1 + 5 \cdot 1 = 7$

$3 \cdot 1 + 6 \cdot 1 = 9$ ✓

All possible solutions of $Ax = b$ are then given by

$$x = x_p + \alpha h_1 + \beta h_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

for all possible values of α, β.

(c) use different values of α, β.

$\alpha = 0, \beta = 0$ $x_p = \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$, $\alpha = 1, \beta = 0$ $x_p = \begin{vmatrix} 5 \\ -2 \\ 1 \\ 0 \end{vmatrix}$ $\alpha = 0, \beta = 1$ $x_p = \begin{vmatrix} 5 \\ -2 \\ 0 \\ 1 \end{vmatrix}$

$\alpha = 1, \beta = 1$ $x_p = \begin{bmatrix} 9 \\ -5 \\ 1 \\ 1 \end{bmatrix}$. Check each of them!