Linear Algebra. Daniel B. Szyld

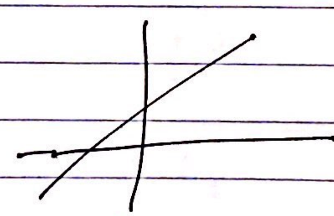
1.1. Introduction

During the course we will study:

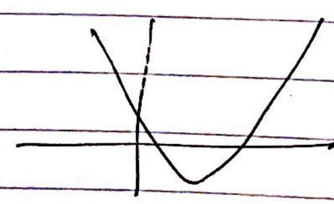
Linear Equations
Matrix Algebra
Vector Spaces
Linear transformations
Properties of all these

Extremely useful. Applied to all fields of science and engineering. Also key to solving nonlinear problems

linear $y = ax + b$



nonlinear examples $y = ax^2 + b$



$y = ax^n + b$

$y = ax^n + bx^3 + cx^2 + dx + e$

$y = \sin x$

$y = e^x$

First Example.

Section 1.2. D. B. Szyld

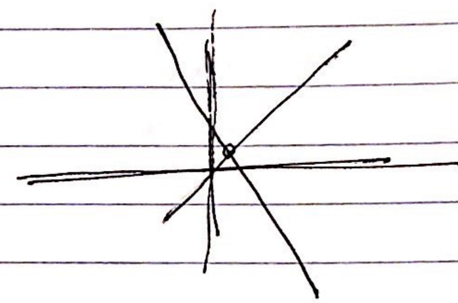
<u>two</u> equations. <u>Two</u> unknowns

$$3x + y = 4$$
$$2x - y = 1$$

e.g. x price of 1 apple
y price of 1 orange

Rewrite

$$y = -3x + 4$$
$$y = 2x - 1$$



Solution $(x, y) = (1, 1)$

In general we have $\underline{\underline{m}}$ equations, $\underline{\underline{n}}$ unknowns
(unknowns or variables)

the unknowns are $x_1\ x_2\ x_3, \dots, x_n$

equation 1 $\quad x_i$ "x sub i" $\quad i$ is the <u>index</u>

$$E_1: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$E_2: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$E_m: \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

b_i are <u>known</u>. $\{b_i\}$ is the <u>right hand side</u>

a_{ij} are called the <u>coefficients</u>

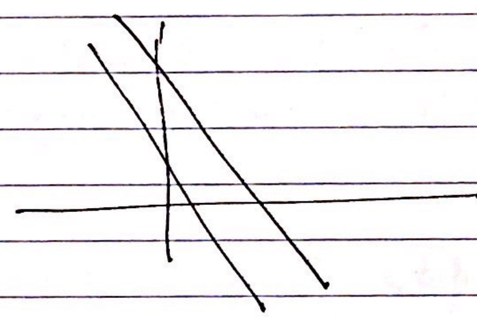
Three posibilities

1. Unique solution, as in example above (p. 2)

2. No solution

examples

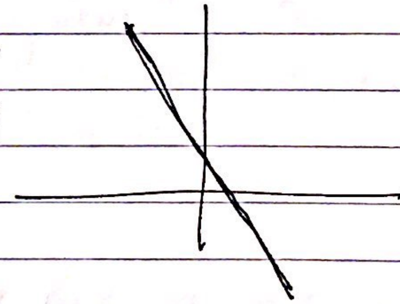
$$3x + y = 4 \qquad 3x + y = 4$$
$$3x + y = 5 \qquad 6x + 2y = 2$$



3. Infinitely many solutions

Example

$$6x + 2y = 8$$
$$3x + y = 4$$



Important observation. It is not possible to have a finite number of solutions bigger than one.
($1 < k < \infty$. k solutions not possible!)

We will spend some time learning how to solve these linear systems of equations.

$$S = \begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{Bmatrix} \quad \text{system of } m \text{ equations (and } n \text{ unknowns)}$$

The idea of the solution method (or algorithm) is due to Carl Friederich Gauss about 200 years ago.

Find another system S' with the same solution [called an equivalent system] but easier to solve.

How? By transforming S step by step by elementary operations on the rows until S' is "upper triangular" in the $n \times n$ (square) case or "upper echelon in the $m \times n$ case $(m \neq n)$.

Let me do an example and we come back to the method

$x_1 = x$
$x_2 = y$

Same example from p. 2.

$$3\,x_1 + 1\,x_2 = 4$$
$$2\,x_1 - 1\,x_2 = 1$$

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 2 & -1 & 1 \end{array}\right]$$

Replace E_2 by $E_2 - \frac{2}{3} E_1$

so as to "eliminate" the coefficient $a_{21} = 2$

obtain $3\,x_1 + 1\,x_2 = 4$

$$-\frac{5}{3} x_2 = -\frac{5}{3}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ & -\frac{5}{3} & -\frac{5}{3} \end{array}\right]$$

Since $-1 - \frac{2}{3} \cdot 1 = -\frac{5}{3}$

and $1 - \frac{2}{3} \cdot 4 = \frac{3-8}{3} = -\frac{5}{3}$

Solve S' by "back substitution"

$$x_2 = -5/3 \,/\, -5/3 = 1$$

then first equation is

$$3\,x_1 + 1 = 4$$
$$x_1 = (4-1)/3 = 1$$

$x_1 = 1$ solution
$x_2 = 1$

Each step of the method is an <u>elementary operation</u>

i.e., of the form [sets of solutions always maintained]

(1) Interchange Row i with Row j

(2) Replace equation E_i with a multiple of it, i.e. with αE_i, $\alpha \neq 0$.

(3) Replace E_j with $E_j + \alpha E_i$

(as we did in the example)

<u>Gaussian Elimination</u> is then a (finite) sequence of elementary operations.

Thus we have the same set of solutions (unique, infinitely many, or none).

A 3x3 example (from p.5 of book)

$$
\begin{array}{ll}
E_1 & 2x_1 + 1x_2 + 1x_3 = 1 \\
E_2 & 6x_1 + 2x_2 + 1x_3 = -1 \\
E_3 & -2x_1 + 2x_2 + 1x_3 = 7
\end{array}
$$

At each step concentrate on a <u>pivot</u> position

Here, we start we the (1,1) position

$a_{11} = 2$ is the <u>pivotal element</u> (which needs to be nonzero)

① Replace E_2 with $E_2' = E_2 - 3E_1$

obtain

$$
\begin{aligned}
2x_1 + 1x_2 + 1x_3 &= 1 \\
-1x_2 - 2x_3 &= -4
\end{aligned}
$$

E_3 the same.

② Replace E_3 with $E_3' = E_3 + E_1$

If pivot zero -row interchange if not possible not unique solution

obtain

$$
\begin{aligned}
2x_1 + 1x_2 + 1x_3 &= 1 \\
-1x_2 - 2x_3 &= -4 \\
3x_2 + 2x_3 &= 8
\end{aligned}
\qquad (*)
$$

eliminated

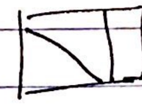
Note: in ① E_3 unchanged
in ② E_2' unchanged

Next we use a new pivot (2,2) of <u>new</u> system (*) with value $-1 \neq 0$

3 Replace E_3' with $E_3' + 3E_2'$

obtain

$$\begin{aligned} 2x_1 + 1x_2 + 1x_3 &= 1 \\ -1x_2 - 2x_3 &= -4 \\ -4x_3 &= -4 \end{aligned}$$



We have trianguladized the system

Now system S'' is easy to solve by back substitution

$\boxed{x_3 = 1}$ then $-x_2 - 2 = -4$

$\Rightarrow \boxed{x_2 = 2}$

and lastly $2x_1 + 2 + 1 = 1$

$x_1 = (1-3)/2 = -1$

$$\boxed{\begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}}$$

solution of S'', S' and S!

check! (always check!)

$$\begin{aligned} 2(-1) + 1\cdot 2 + 1\cdot 1 &= -2+2+1 = 1 \\ 6(-1) + 2\cdot 2 + 1\cdot 1 &= -6+4+1 = -1 \\ -2(-1) + 2\cdot 2 + 1\cdot 1 &= 2+4+1 = 7 \end{aligned}$$

✓

All this can be done without writing $x_1\ x_2\ x_3$ or the $=$ sign. every time.

The Matrix of coefficients

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

and the augmented matrix:

$$\left[A \mid b \right] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{array}\right]$$

Then work with the elementary row operations on these (A or $[A|b]$)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$Ax = b$ $\qquad\qquad$ $Tx = c$

$[A|b]$ $\qquad\qquad$ $[T|c]$ $\qquad\qquad$ $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ & t_{22} & \cdots & t_{2n} \\ & & \ddots & \\ & & & t_{nn} \end{bmatrix}$

If $t_{ii} \neq 0 \ \forall i$ we have a unique solution these t_{ii} are the pivots along the main diagonal.

Back substitution Algorithm

Given $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ 0 & 0 & t_{33} \cdots & t_{3n} \\ & & \ddots & t_{nn} \end{bmatrix}$ and $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$x_n = c_n / t_{nn}$$

For $i = n-1, n-2, \ldots, 1$

$$x_i = \frac{1}{t_{ii}} \left(c_i - t_{i,i+1} x_{i+1} - \cdots - t_{n} x_n \right)$$

or $$x_i = \frac{1}{t_{ii}} \left(c_i - \sum_{k=i+1}^{n} t_{ik} x_k \right)$$

Another example

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & 2 & 1 & 8 \\ -2 & 1 & 0 & -5 \end{array}\right] \xrightarrow[E_3 + 1 E_1]{E_2 - 2 E_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$[A \mid b]$

(2,2) pivot is now zero

Exchange row 3 with row 2

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right] \qquad [T \mid c]$$

<u>back substitution</u> $\quad -x_3 = -2 \qquad x_3 = -2/-1 = 2$

$$x_2 = \frac{1}{2}\left(0 - 1.2\right) = -\frac{2}{2} = -1$$

$$x_3 = \frac{1}{2}\left(5 - 1.(-1) - 1((2)\right) = \frac{4}{2} = 2$$

Can do all exercises in section 1.2.

1.2.*