Section 1.1 1.1 → 1.3

1.1 **Exercise 1.1.** We roll a fair die twice. Describe a sample space Ω and a probability measure P to model this experiment. Let A be the event that the second roll is larger than the first. Find the probability $P(A)$ that the event A occurs.

Sample space is $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

Total number of elements in $\Omega = 6^2 = 36$

Since this is a fair die, probability measure $P(B) = \frac{1}{36}$ for each $B \in \Omega$

The event A is $A = \left\{ \begin{array}{r} (1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,3),(2,4),(2,5),(2,6), \\ (3,4),(3,5),(3,6), \\ (4,5),(4,6), \\ (5,6) \end{array} \right\}$

and total number of elements in A is 15 so $P(A) = \frac{15}{36}$

1.2 **Exercise 1.2.** For breakfast Bob has three options: cereal, eggs or fruit. He has to choose exactly two items out of the three available.

(a) Describe the sample space of this experiment.

Hint. What are the different possible outcomes for Bob's breakfast?

(b) Let A be the event that Bob's breakfast includes cereal. Express A as a subset of the sample space.

(a) Since Bob is choosing 2 out of 3 items, the sample space is

$\Omega = \{\{\text{cereal}, \text{eggs}\}, \{\text{cereal}, \text{fruit}\}, \{\text{eggs}, \text{fruit}\}\}$

(b) According to (a), there are 2 possible outcomes that include cereal

$A = \{\{\text{cereal}, \text{eggs}\}, \{\text{cereal}, \text{fruit}\}\}$

1.3 **Exercise 1.3.**

(a) You flip a coin and roll a die. Describe the sample space of this experiment.

(b) Now each of 10 people flips a coin and rolls a die. Describe the sample space of this experiment. How many elements are in the sample space?

(c) In the experiment of part (b), how many outcomes are in the event where nobody rolled a five? How many outcomes are in the event where at least one person rolled a five?

(a) Sample space of flipping a coin $\Omega_1 = \{H, T\}$

Sample space of rolling a die $\Omega_2 = \{1,2,3,4,5,6\}$

↳ Sample space of the experiment $\Omega = \Omega_1 \times \Omega_2$

$= \{H, T\} \times \{1,2,3,4,5,6\}$

(b) Sample space when 1 person flips a coin then rolls a die is
$\Omega_1 = \{H, T\} \times \{1, 2, 3, 4, 5, 6\}$ with a total of $2(6) = 12$ elements
Therefore, when 10 people do the experiment, the sample space is
$\Omega = (\Omega_1)^{10} = (\{H, T\} \times \{1, 2, 3, 4, 5, 6\})^{10}$ with 12^{10} elements

(c) (i) If nobody rolls a 5, the new sample space is
$\Omega = (\{H, T\} \times \{1, 2, 3, 4, 6\})^{10}$ with $2^{10} \cdot 5^{10} = 10^{10}$ elements

(ii) If at least 1 person rolls a 5, the number of outcomes is the result when subtracting the number of outcomes that nobody rolls a 5 from the total outcomes: $12^{10} - 10^{10}$ elements

Appendix B

B.1 $\longrightarrow$ B.7

B.1

Exercise B.1. Let A, B and C be subsets of the set Ω.

(a) Let D be the set of elements that are in exactly two of the sets A, B and C. Using unions, intersections and complements express D in terms of A, B, and C.

(b) Let E be the set of elements that are in at least two of the sets A, B and C. Using unions, intersections and complements express E in terms of A, B, and C.

There may be more than one way to answer these questions.

(a) Let P be the set of elements in A and B but not $C \rightarrow P = A \cap B \cap C^c$

Let Q be the set of elements in A and C but not $B \rightarrow Q = A \cap C \cap B^c$

Let R be the set of elements in B and C but not $A \rightarrow R = B \cap C \cap A^c$

Therefore, the union of these sets gives the set D

$$D = P \cup Q \cup R = (A \cap B \cap C^c) \cup (A \cap C \cap B^c) \cup (B \cap C \cap A^c)$$

(b) The set D would satisfy the set E, in addition to the elements that are in all 3 sets A, B, C

$$E = D \cup (A \cap B \cap C) =$$
$$= (A \cap B \cap C^c) \cup (A \cap C \cap B^c) \cup (B \cap C \cap A^c) \cup (A \cap B \cap C)$$

B.2

Exercise B.2. Let A, B and C be subsets of the set Ω. Various other sets are described below in words. Use unions, intersections and complements to express these in terms of A, B and C. Drawing Venn diagrams may help.

(a) The set of elements that are in each of the three sets.
(b) The set of elements that are in A but neither in B nor in C.
(c) The set of elements that are in at least one of the sets A or B.
(d) The set of elements that are in both A and B but not in C.
(e) The set of elements that are in A, but not in B or C or both.

(a) $A \cap B \cap C$

(b) $A \cap B^c \cap C^c$

(c) $A \cap B$

(d) $A \cap B \cap C^c$

(e) $A \cap B^c \cap C^c$

B.3

Exercise B.3. Let Ω be the set $\{1, 2, \ldots, 100\}$, and let A, B and C be the following subsets of Ω:

$A = \{$positive even numbers which are at most 100$\}$,
$B = \{$two-digit numbers where the digit 5 appears$\}$,
$C = \{$positive integer multiples of 3 which are at most 100$\}$,
$D = \{$two-digit numbers such that the sum of the digits is 10$\}$.

List the elements of each of the following sets.

(a) $B \setminus A$
(b) $A \cap B \cap C^c$
(c) $((A \setminus D) \cup B) \cap (C \cap D)$

$A = \{2, 4, 6 \ldots 100\}$
$B = \{15, 25, 35, 45, 50, 51 \ldots 95\}$
$C = \{3, 6, 9 \ldots 99\}$
$D = \{19, 28, 37, 46, 55, 64, 73, 82, 91\}$

(a) $B \setminus A = \{15, 25, 35, 45, 51, 53, 55, 57, 59, 65, 75, 85, 95\}$

(b) $A \cap B = \{50, 52, 54, 56, 58\}$

$A \cap B \cap C^c = \{50, 52, 54, 56, 58\} \cap C^c = \{50, 52, 56, 58\}$

(c) $C \cap D = \emptyset$ because the sum of the digits cannot be both 10 and multiples of 3.

so $((A \setminus D) \cup B) \cap (C \cap D) = ((A \setminus D) \cup B) \cap \emptyset = \emptyset$ so no elements

$$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$$

B.4 **Exercise B.4.** Show that identity (B.2) is true.

Let $\omega \in \Omega$

$$\begin{aligned}
\omega \in \left(\bigcap_i A_i\right)^c &\Leftrightarrow \omega \notin \bigcap_i A_i && \text{by definition of complements} \\
&\Leftrightarrow \omega \notin A_i \text{ for each index } i && \text{by definition of intersection} \\
&\Leftrightarrow \omega \in A_i^c \text{ for each index } i && \text{by definition of complements} \\
&\Leftrightarrow \omega \in \bigcup A_i^c && \text{by definition of union}
\end{aligned}$$

QED

B.5 **Exercise B.5.** If A and B are sets then let $A\triangle B$ denote their symmetric difference: the set of elements that are in exactly one of the two sets. (In words this is "A or B, but not both.")

(a) Express $A\triangle B$ from A and B using unions, intersections and complements.

(b) Show that for any three sets A, B and C we have

$$A\triangle(B\triangle C) = (A\triangle B)\triangle C.$$

(a) $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$

(b)
$$\begin{aligned}
A\triangle(B\triangle C) &= [A \cap (B\triangle C)^c] \cup [A^c \cap (B \triangle C)] \\
\text{from part (a)} &= [A \cap ((B\cap C) \cup (B^c \cap C^c))] \cup [A^c \cap ((B \cap C^c) \cup (B^c \cap C))] \\
\text{distributive law} &= [(A \cap (B \cap C)) \cup (A \cap (B^c \cap C^c))] \cup [(A^c \cap (B \cap C^c)) \cup (A^c \cap (B^c \cap C))] \\
\text{associative law} &= [(A \cap B \cap C) \cup (A \cap B^c \cap C^c)] \cup [(A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)] \\
&= (A \cap B \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)
\end{aligned}$$

$$\begin{aligned}
(A \triangle B)\triangle C &= [(A\triangle B)^c \cap C] \cup [(A \triangle B) \cap C^c] \\
\text{from part (a)} &= [((A \cap B) \cup (A^c \cap B^c)) \cap C] \cup [((A \cap B^c) \cup (A^c \cap B)) \cap C^c] \\
\text{distributive law} &= [((A \cap B) \cap C) \cup ((A^c \cap B^c) \cap C)] \cup [((A \cap B^c) \cap C^c) \cup ((A^c \cap B) \cap C^c)] \\
\text{associative law} &= [(A \cap B \cap C) \cup (A^c \cap B^c \cap C)] \cup [(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \\
&= (A \cap B \cap C) \cup (A^c \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)
\end{aligned}$$

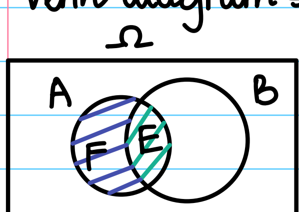
$\therefore A\triangle(B\triangle C) = (A\triangle B)\triangle C$ QED

B.6 **Exercise B.6.** Suppose that A and B are subsets of Ω. Let $E = A \cap B$ and $F = A \cap B^c$.

(a) Show that E and F are disjoint, i.e. that $E \cap F = \varnothing$.

(b) Show that $A = E \cup F$.

(a) Venn diagram shows $E \cap F = \varnothing$



(b)
$$\begin{aligned}
A &= A \cap \Omega \\
&= A \cap (B \cup B^c) && \text{definition of union \& complements} \\
&= (A \cap B) \cup (A \cap B^c) && \text{distributive law} \\
&= E \cup F
\end{aligned}$$

QED

Exercise B.7. Consider the following sets:

$$A = \{1, 2, 3, 4, 5, 6, 7\}, \quad B = \{2, 4, 6, 8\}, \quad C = \{1, 3, 4, 7\}.$$

(a) Decide whether the set $D = \{1, 3, 7\}$ can be expressed from A, B and C using unions, intersections and complements.

(b) Decide whether the set $E = \{2, 4\}$ can be expressed from A, B and C using unions, intersections and complements.

$\boxed{B.7}$

(a) Yes, $D = C \cap B^c$

(b) When the element 2 shows up in either A or B, the element 6 is there as well. Hence, it is impossible to separate these elements using set operations.

$\therefore$ The set E cannot be expressed

Section 1.2 $\quad 1.4 \to 1.8$

1.4

Exercise 1.4. Every day a kindergarten class chooses randomly one of the 50 state flags to hang on the wall, without regard to previous choices. We are interested in the flags that are chosen on Monday, Tuesday and Wednesday of next week.

(a) Describe a sample space Ω and a probability measure P to model this experiment.

(b) What is the probability that the class hangs Wisconsin's flag on Monday, Michigan's flag on Tuesday, and California's flag on Wednesday?

(c) What is the probability that Wisconsin's flag will be hung at least two of the three days?

(a) $\Omega = \{(x_1, x_2, x_3) \mid x_i \in \{50 \text{ state flags}\} \text{ for } i = 1 \dots 50\}$

This is an ordered sampling with replacement ($n = 50$, $k = 3$)

$\#\Omega = n^k = 50^3$. Each sample point has an equal probability

$P(B) = \frac{1}{50^3}$ for each $B \in \Omega$

(b) The 3-tuple (Wisconsin, Michigan, California) is a particular sample point so $P((\text{Wisconsin, Michigan, California})) = \frac{1}{50^3}$

(c) The possible outcomes that Wisconsin is on:

(i) Monday & Tuesday: $1 \times 1 \times 49 = 49$

(ii) Monday & Wednesday: $1 \times 49 \times 1 = 49$

(iii) Tuesday & Wednesday: $49 \times 1 \times 1 = 49$

(iv) all 3 days: $1 \times 1 \times 1 = 1$

total outcomes $= 49 + 49 + 49 + 1 = 148$

$\therefore P(\text{Wisconsin is on} \geq 2 \text{ days}) = \frac{148}{\#\Omega} = \frac{148}{50^3}$

1.5

Exercise 1.5. In one type of state lottery 5 distinct numbers are picked from $1, 2, 3, \dots, 40$ uniformly at random.

(a) Describe a sample space Ω and a probability measure P to model this experiment.

(b) What is the probability that out of the five picked numbers exactly three will be even?

(a) This is an unordered list of 5 random numbers

$\Omega = \{\{s_1, s_2, s_3, s_4, s_5\} \mid s_i \in \{1, 2, \dots 40\}, s_i \neq s_j \text{ for all } i \neq j\}$

$P(\{s_1, s_2, s_3, s_4, s_5\}) = \frac{1}{\#\Omega} = \frac{1}{\binom{40}{5}} = \frac{5!\,35!}{40!}$

For each $A \subseteq \Omega$, $P(A) = \frac{\#A}{\#\Omega} = \frac{\#A}{\binom{40}{5}}$

(b) $\binom{20}{3}$ ways to choose a set of 3 even numbers $1 \to 40$

$\binom{20}{2}$ ways to choose a set of 2 odd numbers $1 \to 40$

$$P(\text{exactly 3 even numbers}) = \frac{\binom{20}{3}\binom{20}{2}}{\binom{40}{5}} = \frac{475}{1443}$$

Exercise 1.6. We have an urn with 3 green and 4 yellow balls. We choose 2 balls randomly without replacement. Let A be the event that we have two different colored balls in our sample.

1.6

(a) Describe a possible sample space with equally likely outcomes, and the event A in your sample space.
(b) Compute $P(A)$.

(a) This is an unordered sample of 2 different balls

$\Omega = \{\{s_1, s_2\} \mid s_i \in \{1, 2 \ldots 7\}, s_i \neq s_j \text{ for all } i \neq j\}$

The event A comprised of 2 possible arrangements: 1 green + 1 yellow and 1 yellow + 1 green

(b) $$P(A) = \frac{\#A}{\#\Omega} = \frac{3(4) + 4(3)}{7(6)} = \frac{4}{7}$$

Exercise 1.7. We have an urn with 3 green and 4 yellow balls. We draw 3 balls one by one without replacement.

(a) Find the probability that the colors we see in order are green, yellow, green.
(b) Find the probability that our sample of 3 balls contains 2 green balls and 1 yellow ball.

1.7

(a) This is an ordered sampling without replacement

$\Omega = \{(s_1, s_2, s_3) \mid s_i \in \{1, 2 \ldots 7\}, s_i \neq s_j, i \neq j\}$

Let A be the event that the colors are green, yellow, green in order which is a particular outcome

$$P(A) = \frac{\#A}{\#\Omega} = \frac{3 \times 4 \times 2}{7 \times 6 \times 5} = \frac{4}{35}$$

(b) This is an unordered sampling without replacement

$\Omega = \{\{s_1, s_2, s_3\} \mid s_i \in \{1, 2 \ldots 7\}, s_i \neq s_j, i \neq j\}$

Let A be the event that the sample of 3 balls contain 2 green & 1 yellow

$$P(A) = \frac{\#A}{\#\Omega} = \frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = \frac{12}{35}$$

Exercise 1.8. Suppose that a bag of scrabble tiles contains 5 Es, 4 As, 3 Ns and 2 Bs. It is my turn and I draw 4 tiles from the bag without replacement. Assume that my draw is uniformly random. Let C be the event that I got two Es, one A and one N.

1.8

(a) Compute $P(C)$ by imagining that the tiles are drawn one by one as an ordered sample.
(b) Compute $P(C)$ by imagining that the tiles are drawn all at once as an unordered sample.

(a) This is an ordered sampling without replacement

$$P(C) = \frac{\#C}{\#\Omega} = \frac{5 \times 4 \times 4 \times 3}{14 \times 13 \times 12 \times 11} \binom{4}{2}\binom{2}{1}\binom{1}{1} = \boxed{\frac{120}{1001}}$$

with $\binom{4}{2}$ ways to position 2 Es, $\binom{2}{1}$ ways to position 1 A and $\binom{1}{1}$ way for 1 N

(b) This is an unordered sampling without replacement

$$P(C) = \frac{\#C}{\#\Omega} = \frac{\binom{5}{2}\binom{4}{1}\binom{3}{1}}{\binom{14}{4}} = \boxed{\frac{120}{1001}}$$

Section 1.3 $\quad$ 1.9 $\longrightarrow$ 1.11

1.9

Exercise 1.9. We break a stick at a uniformly chosen random location. Find the probability that the shorter piece is less than 1/5th of the original.

Let L denote the total length of the stick $\rightarrow \Omega = [0, L]$

Let A be the event that the shorter piece is less than $1/5$th of the original

$\therefore A = \{x \in \Omega \mid x \leq L/5 \text{ or } x \geq 4L/5\}$

Since $\{x \leq \frac{L}{5}\} \cap \{x \geq \frac{4L}{5}\} = \emptyset$, they are disjoint probabilities

$$\therefore P(A) = P\left(x \leq \frac{L}{5}\right) + P\left(x \geq \frac{4L}{5}\right) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Exercise 1.10. We roll a fair die repeatedly until we see the number four appear and then we stop. The outcome of the experiment is the number of rolls.

(a) Following Example 1.16 describe a sample space Ω and a probability measure P to model this situation.

(b) Calculate the probability that the number four never appears.

1.10

(a) $\Omega = \{1, 2, 3 \ldots\} \cup \{\infty\}$

Let k denote the number of rolls to see the first 4

∞ denote the event that 4 never appears

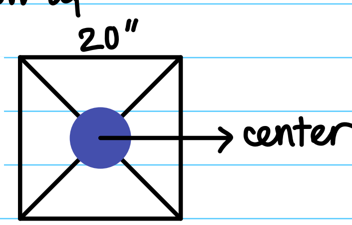
$$P(k) = \underbrace{\frac{5}{6} \times \frac{5}{6} \times \ldots}_{k-1 \text{ terms}} \times \frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}, \quad k = 1, 2, 3 \ldots$$

(b)
$$1 = P(\infty) + \sum_{i=1}^{k} P(k) = P(\infty) + \sum_{i=1}^{k} \overbrace{\frac{1}{6}}^{a}\Big(\overbrace{\frac{5}{6}}^{r}\Big)^{k-1} = P(\infty) + \overbrace{\frac{1/6}{1 - 5/6}}^{\frac{a}{1-r}} = P(\infty) + \frac{\cancel{1/6}}{\cancel{1/6}}^{\,1}$$

$\therefore P(\infty) = 0 \qquad \therefore$ 4 will eventually show up

1.11

Exercise 1.11. We throw a dart at a square shaped board of side length 20 inches. Assume that the dart hits the board at a uniformly chosen random point. Find the probability that the dart is within 2 inches of the center of the board.



Let A be the event that the dart is within 2 inches of the board's center

i.e: the dart lies in the circle with radius of 2 inches

$$P(A) = \frac{\text{area of circle}}{\text{area of square}} = \frac{\pi(2)^2}{20^2} = \frac{4\pi}{400} = \frac{\pi}{100} \approx 0.03142$$

Section 1.4   1.12 → 1.15

1.12

Exercise 1.12. We roll a fair die repeatedly until we see the number four appear and then we stop.

(a) What is the probability that we need at most 3 rolls?

(b) What is the probability that we needed an even number of die rolls?

(a) Let k denote the number of rolls to see the first 4

∞ denote the event that 4 never appears

$$P(k) = \underbrace{\frac{5}{6} \times \frac{5}{6} \times \dots}_{k-1 \text{ terms}} \times \frac{1}{6} = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1}, \quad k = 1, 2, 3 \dots$$

$$P(\leq 3 \text{ rolls}) = P(1) + P(2) + P(3) = \frac{1}{6}\left(\frac{5}{6}\right)^0 + \frac{1}{6}\left(\frac{5}{6}\right)^1 + \frac{1}{6}\left(\frac{5}{6}\right)^2$$

$$= \frac{1}{6}\left(1 + \frac{5}{6} + \frac{5^2}{6^2}\right) = \frac{91}{216}$$

(b)
$$P(\text{even number of rolls}) = \sum_{k=1}^{\infty} P(2k) = \sum_{k=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k-1} = \sum_{k=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{-1}\left(\frac{5}{6}\right)^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{6} \times \frac{6}{5} \times \left(\frac{25}{36}\right)^k = \sum_{k=1}^{\infty} \frac{1}{5}\left(\frac{25}{36}\right)^k = \frac{1/5\,(25/36)}{1 - 25/36} = \frac{5/36}{11/36} = \frac{5}{11}$$

1.13

Exercise 1.13. At a certain school, 25% of the students wear a watch and 30% wear a bracelet. 60% of the students wear neither a watch nor a bracelet.

(a) One of the students is chosen at random. What is the probability that this student is wearing a watch or a bracelet?

(b) What is the probability that this student is wearing both a watch and a bracelet?

(a) Let A be the event that the students wear neither a watch nor bracelet

$\therefore P(A) = 0.6$

$\therefore A^c$ be event that the students wear a watch or bracelet

$\therefore P(A^c) = 1 - P(A) = 1 - 0.6 = 0.4$

(b) Let W be the event that the students wear a watch $\therefore P(W) = 0.25$

B be the event that the students wear a bracelet $\therefore P(B) = 0.3$

$A^c = W \cap B \rightarrow P(A^c) = P(W \cap B) = 0.4$

$P(W \cup B) = P(W) + P(B) - P(W \cap B)$ by Inclusion - Exclusion principle

$= 0.25 + 0.3 - 0.4 = 0.15$

$\boxed{1.14}$

Exercise 1.14. Assume that $P(A) = 0.4$ and $P(B) = 0.7$. Making no further assumptions on A and B, show that $P(AB)$ satisfies $0.1 \le P(AB) \le 0.4$.

By Inclusion - Exclusion principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.4 + 0.7 - P(A \cap B) = 1.1 - P(A \cap B)$$

$$\therefore P(A \cap B) = 1.1 - P(A \cup B)$$

$$A \cup B \subseteq \Omega \longrightarrow P(A \cup B) \le P(\Omega) = 1$$

$$B \subseteq A \cup B \rightarrow P(B) = 0.7 \le P(A \cup B)$$

$$\therefore 0.7 \le P(A \cup B) \le 1$$

$$-0.7 \geqslant -P(A \cup B) \geqslant -1$$

$$1.1 - 0.7 \geqslant 1.1 - P(A \cup B) \geqslant 1.1 - 1$$

$$0.4 \geqslant P(A \cap B) \geqslant 0.1 \quad \text{or} \quad 0.1 \le P(A \cap B) \le 0.4$$

QED!

Exercise 1.15. An urn contains 4 balls: 1 white, 1 green and 2 red. We draw 3 balls with replacement. Find the probability that we did not see all three colors. Use two different calculations, as specified by (a) and (b) below.

$\boxed{1.15}$

(a) Define the event $W = \{\text{white ball did not appear}\}$ and similarly for G and R. Use inclusion-exclusion.

(b) Compute the probability by considering the complement of the event that we did not see all three colors.

1w
2r
1g

(a)

$$P(W) = \frac{3 \times 3 \times 3}{4 \times 4 \times 4} = \frac{3^3}{4^3} \qquad P(W \cap R) = \frac{1 \times 1 \times 1}{4 \times 4 \times 4} = \frac{1}{4^3}$$

$$P(G) = \frac{3 \times 3 \times 3}{4 \times 4 \times 4} = \frac{3^3}{4^3} \qquad P(W \cap G) = \frac{2 \times 2 \times 2}{4 \times 4 \times 4} = \frac{2^3}{4^3} \qquad P(W \cap R \cap G) = \frac{0}{4^3} = 0$$

$$P(R) = \frac{2 \times 2 \times 2}{4 \times 4 \times 4} = \frac{2^3}{4^3} \qquad P(R \cap G) = \frac{1 \times 1 \times 1}{4 \times 4 \times 4} = \frac{1}{4^3}$$

$$P(W \cup R \cup G) = P(W) + P(R) + P(G) - P(W \cap R) - P(W \cap G) - P(R \cap G) + P(W \cap R \cap G)$$

$$= \frac{3^3 + \cancel{2^3} + 3^3 - 1 - \cancel{2^3} - 1 + 0}{4^3} = \frac{27 + 27 - 2}{64} = \frac{52}{64} = \boxed{\frac{13}{16}}$$

(b)

$$P(W \cup R \cup G) = 1 - P(W^c \cup R^c \cup G^c) = 1 - \frac{1 \times 1 \times 2 \times 3!}{4 \times 4 \times 4} = 1 - \frac{3}{16} = \boxed{\frac{13}{16}}$$

Section 1.5 $\quad 1.16 \rightarrow 1.19$

1.16

Exercise 1.16. We flip a fair coin five times. For every heads you pay me \$1 and for every tails I pay you \$1. Let X denote my net winnings at the end of five flips. Find the possible values and the probability mass function of X.

5 heads $\rightarrow X = \$5$ $\quad$ 2 heads $\rightarrow X = \$-1$

4 heads $\rightarrow X = \$3$ $\quad$ 1 head $\rightarrow X = \$-3$ $\quad \therefore X = \{-5, -3, -1, 1, 3, 5\}$

3 heads $\rightarrow X = \$1$ $\quad$ 0 head $\rightarrow X = \$-5$ $\quad$ ↳ possible values

Let k denote the number of heads out of 5 flips

$$P_X(k) = \frac{P(k \text{ heads from 5 flips})}{2\times 2\times 2\times 2\times 2} = \binom{5}{k} 2^{-5}$$

$\therefore$ Probability mass function of X:

$$P(X=-5) = P(0 \text{ head}) = \binom{5}{0} 2^{-5} = 2^{-5} = \frac{1}{32}$$

$$P(X=-3) = P(1 \text{ head}) = \binom{5}{1} 2^{-5} = \frac{5}{32}$$

$$P(X=-1) = P(2 \text{ heads}) = \binom{5}{2} 2^{-5} = \frac{10}{32} = \frac{5}{16}$$

$$P(X=1) = P(3 \text{ heads}) = \binom{5}{3} 2^{-5} = \frac{5}{16}$$

$$P(X=3) = P(4 \text{ heads}) = \binom{5}{4} 2^{-5} = \frac{5}{32}$$

$$P(X=5) = P(5 \text{ heads}) = \binom{5}{5} 2^{-5} = \frac{1}{32}$$

1.17

Exercise 1.17. An urn contains 4 red balls and 3 green balls. Two balls are drawn randomly.

(a) Let Z denote the number of green balls in the sample when the draws are done without replacement. Give the possible values and the probability mass function of Z.

(b) Let W denote the number of green balls in the sample when the draws are done with replacement. Give the possible values and the probability mass function of W.

$\boxed{4 \text{ r} \quad 3 \text{ g}}$

(a) $Z = \{0, 1, 2\}$

$$P_Z(0) = P(\text{no green}) = \frac{\binom{4}{2}}{\binom{7}{2}} = \frac{2}{7}$$

$$P_Z(1) = P(1 \text{ green}, 1 \text{ red}) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{4}{7}$$

$$P_Z(2) = P(2\text{ greens}) = \frac{\binom{3}{2}}{\binom{7}{2}} = \frac{1}{7}$$

(b) $W = \{0, 1, 2\}$

$$P_W(0) = P(\text{no green}) = \frac{4\times 4}{7\times 7} = \frac{16}{49}$$

$$P_W(1) = P(1\text{ green, }1\text{ red}) = \frac{4\times 3\times 2!}{7\times 7} = \frac{24}{49}$$

$$P_W(2) = P(2\text{ greens}) = \frac{3\times 3}{7\times 7} = \frac{9}{49}$$

Exercise 1.18. The statement

SOME DOGS ARE BROWN

1.18 has 16 letters. Choose one of the 16 letters uniformly at random. Let X denote the length of the *word* containing the chosen letter. Determine the possible values and probability mass function of X.

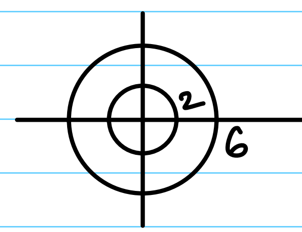
$X = \{3, 4, 5\}$

$$P_X(3) = P(X=3) = \frac{3}{16} \qquad P_X(5) = P(X=5) = \frac{5}{16}$$

$$P_X(4) = P(X=4) = \frac{8}{16} = \frac{1}{2}$$

Exercise 1.19. You throw a dart and it lands uniformly at random on a circular dartboard of radius 6 inches. If your dart gets to within 2 inches of the center I will reward you with 5 dollars. But if your dart lands farther than 2 inches away from the center I will give you only 1 dollar. Let X denote the amount of your reward in dollars. Find the possible values and the probability mass function of X.

1.19



$X = \{1, 5\}$

$$P_X(1) = P(X=1) = \frac{\pi 6^2 - \pi 2^2}{\pi 6^2} = \frac{8}{9}; \quad P_X(5) = P(X=5) = \frac{\pi 2^2}{\pi 6^2} = \frac{1}{9}$$

Section 2.1 $\quad 2.1 \to 2.8$

2.1 **Exercise 2.1.** We roll two dice. Find the conditional probability that at least one of the numbers is even, given that the sum is 8.

$\Omega = \{(s_1, s_2) \mid 1 \le s_i \le 6 \ \forall i = 1,2,3,4,5,6\} \longrightarrow \#\Omega = 36$

Let A be the event that at least one of the numbers is even

B be the event that the sum of the dices is 8

$B = \{(2,6),(6,2),(3,5),(5,3),(4,4)\} \longrightarrow P(B) = \frac{\#B}{\#\Omega} = \frac{5}{36}$

$A \cap B = \{(2,6),(6,2),(4,4)\} \longrightarrow P(A \cap B) = \frac{\#(A \cap B)}{\#\Omega} = \frac{3}{36} = \frac{1}{12}$

$\therefore P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{5/36} = \boxed{\frac{3}{5}}$

2.2 **Exercise 2.2.** A fair coin is flipped three times. What is the probability that the second flip is tails, given that there is at most one tails among the three flips?

$\Omega = \{H,T\} \times \{H,T\} \times \{H,T\} \longrightarrow \#\Omega = 2^3 = 8$

Let A be the event that 2^{nd} flip is tail

B be the event that there is at most 1 tail among 3 flips

$\therefore A = \{(H,T,H),(H,T,T),(T,T,H),(T,T,T)\}$

$B = \{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\} \to P(B) = \frac{\#B}{\#\Omega} = \frac{4}{8} = \frac{1}{2}$

$\therefore A \cap B = \{(H,T,H)\} \to P(A \cap B) = \frac{\#(A \cap B)}{\#\Omega} = \frac{1}{8}$

$\therefore P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/2} = \boxed{\frac{1}{4}}$

2.3 **Exercise 2.3.** What is the probability that a randomly chosen number between 1 and 100 is divisible by 3, given that the number has at least one digit equal to 5?

$\Omega = \{1,2,3 \ldots 100\} \to \#\Omega = 100$

Let A be the event that the chosen number is divisible by 3

B be the event that the number has ≥ 1 digit of 5

$\therefore A = \{3,6,9 \ldots 99\}$

$B = \{5,15,25 \ldots 95\} \cup \{50,51, \ldots 59\} \to \#B = 19 \to P(B) = \frac{\#B}{\#\Omega} = \frac{19}{100}$

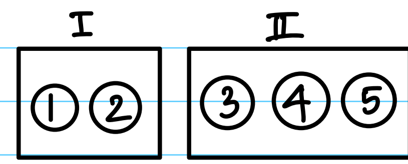
$\therefore A \cap B = \{15,45,51,54,57,75\} \to \#(A \cap B) = 6$

$$P(A \cap B) = \frac{\#(A \cap B)}{\#\Omega} = \frac{6}{100}$$

$$\therefore P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{6/100}{19/100} = \frac{6}{19}$$

2.4

Exercise 2.4. We have two urns. The first urn contains two balls labeled 1 and 2. The second urn contains three balls labeled 3, 4 and 5. We choose one of the urns at random (with equal probability) and then sample one ball (uniformly at random) from the chosen urn. What is the probability that we picked the ball labeled 5?



Let A be the event that we picked the ball labeled 5

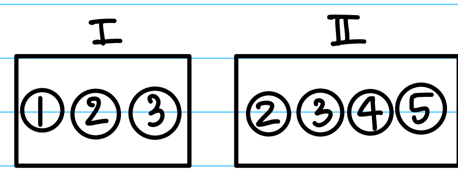
B be the event that we picked the 2^{nd} urn

$\therefore B^c$ be the event that we picked the 1^{st} urn

$$P(A) = P(A \mid B^c) \cdot P(B^c) + P(A \mid B) \cdot P(B) = 0 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

2.5

Exercise 2.5. We have two urns. The first urn contains three balls labeled 1, 2 and 3. The second urn contains four balls labeled 2, 3, 4 and 5. We choose one of the urns randomly, so that the probability of choosing the first one is 1/5 and the probability of choosing the second is 4/5. Then we sample one ball (uniformly at random) from the chosen urn. What is the probability that we picked a ball labeled 2?



Let A be the event that we picked the ball labelled 2

B be the event that we picked the 1^{st} urn

$\therefore B^c$ be the event that we picked the 2^{nd} urn

$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid B^c) \cdot P(B^c) = \frac{1}{3} \times \frac{1}{5} + \frac{1}{4} \times \frac{4}{5} = \frac{4}{15}$$

2.6

Exercise 2.6. When Alice spends the day with the babysitter, there is a 0.6 probability that she turns on the TV and watches a show. Her little sister Betty cannot turn the TV on by herself. But once the TV is on, Betty watches with probability 0.8. Tomorrow the girls spend the day with the babysitter.

(a) What is the probability that both Alice and Betty watch TV tomorrow?
(b) What is the probability that Betty watches TV tomorrow?
(c) What is the probability that only Alice watches TV tomorrow?

Let A be the event that Alice watches TV tomorrow $\rightarrow P(A) = 0.6$

B be the event that Betty watches TV tomorrow

(a) $P(A \cap B) = P(A) \cdot P(B \mid A) = 0.6 \times 0.8 = 0.48$

(b) $P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^c) \cdot P(A^c)$ by law of total probability

$= 0.8 \times 0.6 + 0 \times (1 - 0.6) = 0.48$

(c) B^c be the event that Betty does not watch TV tomorrow

$P(A \cap B^c) = P(A) \cdot P(B^c \mid A) = 0.6 \times (1 - 0.8) = 0.6 \times 0.2 = 0.12$

2.7

Exercise 2.7.

(a) Use the definition of conditional probability and additivity of probability to show that $P(A^c \mid B) = 1 - P(A \mid B)$.
(b) Suppose $P(A \mid B) = 0.6$ and $P(B) = 0.5$. Find $P(A^c B)$.

(a) $P(A^c | B) = \frac{P(A^c \cap B)}{P(B)}$ & $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$(A^c \cap B) \cup (A \cap B) = (A^c \cup A) \cap B = \Omega \cap B = B$

↳ distribution law

$(A^c \cap B) \cap (A \cap B) = \emptyset \quad \therefore P(A^c \cap B) + P(A \cap B) = P(B)$

$\therefore P(A^c | B) + P(A|B) = \frac{P(A^c \cap B) + P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$\therefore P(A^c | B) = 1 - P(A|B)$ QED!

(b) $P(A^c \cap B) = P(A^c | B).\ P(B) = (1 - P(A|B)).\ P(B)$

$= (1 - 0.6) \times 0.5 = 0.4 \times 0.5 = 0.2$

$\boxed{2.8}$

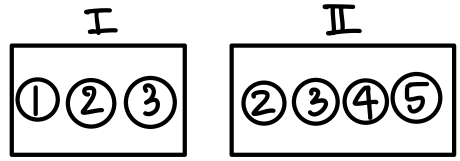
Exercise 2.8. We shuffle a deck of cards and deal three cards (without replacement). Find the probability that the first card is a queen, the second is a king and the third is an ace.

Let A be the event that 1^{st} card is queen
B be the event that 2^{nd} card is king
C be the event that 3^{rd} card is ace

$P(A \cap B \cap C) = P(A).\ P(B|A).\ P(C|A \cap B) = \frac{4}{52} \times \frac{4}{51} \times \frac{4}{50} = \frac{8}{16575}$

multiplication rule

≈ 0.0004827



Section 2.2

2.9 → 2.11

Exercise 2.5. We have two urns. The first urn contains three balls labeled 1, 2 and 3. The second urn contains four balls labeled 2, 3, 4 and 5. We choose one of the urns randomly, so that the probability of choosing the first one is $1/5$ and the probability of choosing the second is $4/5$. Then we sample one ball (uniformly at random) from the chosen urn. What is the probability that we picked a ball labeled 2?

2.9

Exercise 2.9. We return to the setup of Exercise 2.5. Suppose that ball 3 was chosen. What is the probability that it came from the second urn?

Let A be the event that ball 3 was chosen

B be the event that it came from 2^{nd} urn $\quad \therefore P(B) = 4/5$

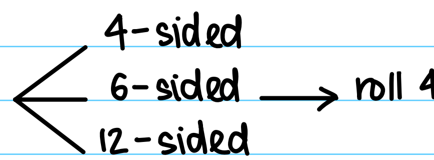
$\therefore B^C$ be the event that it came from 1^{st} urn $\quad \therefore P(B^C) = 1/5$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)}$$

$$= \frac{1/4 \times 4/5}{1/4 \times 4/5 + 1/3 \times 1/5} = \frac{1/5}{1/5 + 1/15} = \frac{3}{4}$$

2.10

Exercise 2.10. I have a bag with 3 fair dice. One is 4-sided, one is 6-sided, and one is 12-sided. I reach into the bag, pick one die at random and roll it. The outcome of the roll is 4. What is the probability that I pulled out the 6-sided die?



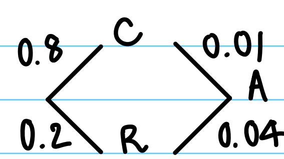
$P(6\text{-sided} \mid 4)$

$$= \frac{P(6\text{-sided})P(4|6\text{-sided})}{P(4\text{-sided})P(4|4\text{-sided}) + P(6\text{-sided})P(4|6\text{-sided}) + P(12\text{-sided})P(4|12\text{-sided})}$$

$$= \frac{1/3 \times 1/6}{1/3 \times 1/4 + 1/3 \times 1/6 + 1/3 \times 1/12} = \frac{1}{3}$$

2.11

Exercise 2.11. The Acme Insurance company has two types of customers, careful and reckless. A careful customer has an accident during the year with probability 0.01. A reckless customer has an accident during the year with probability 0.04. 80% of the customers are careful and 20% of the customers are reckless. Suppose a randomly chosen customer has an accident this year. What is the probability that this customer is one of the careful customers?



Let C denote the event that a customer is careful $\quad \therefore P(C) = 0.8$

R denote the event that a customer is reckless $\quad \therefore P(R) = 0.2$

A denote the event that a customer has an accident

$$P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{P(C)P(A|C)}{P(C)P(A|C) + P(R)P(A|R)} = \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.2 \times 0.04} = \frac{1}{2}$$

Section 2.3

2.12 → 2.19

2.12

Exercise 2.12. We choose a number from the set $\{1, 2, 3, \ldots, 100\}$ uniformly at random and denote this number by X. For each of the following choices decide whether the two events in question are independent or not.

(a) $A = \{X \text{ is even}\}$, $B = \{X \text{ is divisible by } 5\}$.
(b) $C = \{X \text{ has two digits}\}$, $D = \{X \text{ is divisible by } 3\}$.
(c) $E = \{X \text{ is a prime}\}$, $F = \{X \text{ has a digit } 5\}$. Note that 1 is not a prime number.

(a) $A = \{2, 4, \ldots 100\} \rightarrow \#A = 50 \quad \therefore P(A) = \frac{\#A}{\#\Omega} = \frac{50}{100} = \frac{1}{2}$

$B = \{5, 10, \ldots 100\} \rightarrow \#B = 20 \quad \therefore P(B) = \frac{\#B}{\#\Omega} = \frac{20}{100} = \frac{1}{5}$

$A \cap B = \{10, 20 \ldots 100\} \rightarrow \#(A \cap B) = 10 \quad \therefore P(A \cap B) = \frac{\#(A \cap B)}{\#\Omega} = \frac{10}{100} = \frac{1}{10}$

$P(A \cap B) = P(A)P(B) \rightarrow$ A & B are independent

(b) $C = \{10, 11 \ldots 99\} \rightarrow \#C = 90 \quad \therefore P(C) = \frac{\#C}{\#\Omega} = \frac{90}{100} = \frac{9}{10}$

$D = \{3, 6 \ldots 99\} \rightarrow \#D = 33 \quad \therefore P(D) = \frac{\#D}{\#\Omega} = \frac{33}{100}$

$C \cap D = \{12, 15 \ldots 99\} \rightarrow \#(C \cap D) = 30 \quad \therefore P(C \cap D) = \frac{\#(C \cap D)}{\#\Omega} = \frac{30}{100} = \frac{3}{10}$

$P(C \cap D) \neq P(C)P(D) \rightarrow$ C & D are not independent

(c) $E = \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

$\therefore \#E = 22 \quad \therefore P(E) = \frac{\#E}{\#\Omega} = \frac{22}{100} = \frac{11}{50}$

$F = \{5, 15, 25 \ldots 95\} \cup \{50, 51 \ldots 59\} \rightarrow \#F = 19 \quad \therefore P(F) = \frac{\#F}{\#\Omega} = \frac{19}{100}$

$E \cap F = \{5, 53, 59\} \rightarrow \#(E \cap F) = 3 \quad \therefore P(E \cap F) = \frac{\#(E \cap F)}{\#\Omega} = \frac{3}{100}$

$P(E \cap F) \neq P(E)P(F) \longrightarrow$ E & F are not independent

2.13

Exercise 2.13. Suppose that $P(A) = 1/3$, $P(B) = 1/3$ and $P(AB^c) = 2/9$. Decide whether A and B are independent or not.

$A = A \cap \Omega = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c)$

distribution law

$(A \cap B) \cap (A \cap B^c) = \emptyset \longrightarrow P(A) = P(A \cap B) + P(A \cap B^c)$

$\frac{1}{3} = P(A \cap B) + \frac{2}{9} \rightarrow P(A \cap B) = \frac{1}{9} = \frac{1}{3} \times \frac{1}{3} = P(A)\,P(B)$

$\therefore$ A & B are independent

2.14

Exercise 2.14. Let A and B be two disjoint events. Under what condition are they independent?

A, B are disjoint events $\longrightarrow A \cap B = \emptyset \longrightarrow P(A \cap B) = P(\emptyset) = 0$

A, B independent $\longrightarrow P(A \cap B) = P(A)\,P(B) = 0$

$\therefore$ Condition : $P(A) = 0$ or $P(B) = 0$

2.15

Exercise 2.15. Every morning Ramona misses her bus with probability $\frac{1}{10}$, independently of other mornings. What is the probability that next week she catches her bus on Monday, Tuesday and Thursday, but misses her bus on Wednesday and Friday?

Let $X_i = 1$ if Ramona catches her bus on day i $\therefore P(X_i = 1) = 9/10$

$X_i = 0$ if Ramona misses her bus on day i $\therefore P(X_i = 0) = 1/10$

$P(X_{Mon} = 1 \cap X_{Tue} = 1 \cap X_{Wed} = 0 \cap X_{Thur} = 1 \cap X_{Fri} = 0)$

$= P(X_{Mon} = 1) \times P(X_{Tue} = 1) \times P(X_{Wed} = 0) \times P(X_{Thur} = 1) \times P(X_{Fri} = 0)$

$= \frac{9}{10} \times \frac{9}{10} \times \frac{1}{10} \times \frac{9}{10} \times \frac{1}{10} = \frac{729}{10^5}$

2.16

Exercise 2.16. We flip a fair coin three times. For $i = 1, 2, 3$, let A_i be the event that among the first i coin flips we have an odd number of heads. Check whether the events A_1, A_2, A_3 are independent or not.

$P(A_1) = P(\{H,H,H\},\{H,T,T\},\{H,T,H\},\{H,H,T\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$P(A_2) = P(\{H,T,H\},\{H,T,T\},\{T,H,H\},\{T,H,T\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$P(A_3) = P(\{H,H,H\},\{H,T,T\},\{T,H,T\},\{T,T,H\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$P(A_1 \cap A_2) = P(\{H,T,H\},\{H,T,T\}) = \frac{2}{8} = \frac{1}{4} = P(A_1)\,P(A_2)$

$P(A_1 \cap A_3) = P(\{H,H,H\},\{H,T,T\}) = \frac{2}{8} = \frac{1}{4} = P(A_1)\,P(A_3)$

$P(A_2 \cap A_3) = P(\{H,T,T\},\{T,H,T\}) = \frac{2}{8} = \frac{1}{4} = P(A_2)\,P(A_3)$

$P(A_1 \cap A_2 \cap A_3) = P(\{H,T,T\}) = \frac{1}{8} = P(A_1)\,P(A_2)\,P(A_3)$ $\therefore$ A_1, A_2, A_3 are independent

2.17

Exercise 2.17. Suppose that the events A, B and C are mutually independent with $P(A) = 1/2$, $P(B) = 1/3$ and $P(C) = 1/4$. Compute $P(AB \cup C)$.

$P(A \cap B \cup C) = P(A \cap B) + P(C) - P(A \cap B \cap C)$ by Inclusion-Exclusion

$= P(A)P(B) + P(C) - P(A)P(B)P(C)$ by Independence Theorem

$= \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{3}{8}$

Exercise 2.18. We choose a number from the set $\{10, 11, 12, \ldots, 99\}$ uniformly at random.

2.18

(a) Let X be the first digit and Y the second digit of the chosen number. Show that X and Y are independent random variables.

(b) Let X be the first digit of the chosen number and Z the sum of the two digits. Show that X and Z are not independent.

(a) Let k be the possible values of chosen digits $\longrightarrow k = \{0, 1, 2, \ldots 9\}$

$P(X = k) = \frac{1}{9}$ & $P(Y = k) = \frac{1}{10}$

$P(X = k \cap Y = k) = \frac{1 \times 1}{9 \times 10} = \frac{1}{90} = P(X = k)P(Y = k) \longrightarrow$ X, Y are independent

(b) $P(X = 2) = \frac{1}{10}$ & $P(Z = 1) = \frac{1}{90}$

$P(X = 2, Z = 1) = P(\{20, 21 \ldots 29\} \cap \{10\}) = P(\emptyset) = 0 \neq P(X = 2)P(Z = 1)$

$\therefore$ X, Z are not independent

Exercise 2.19. We have an urn with balls labeled $1, \ldots, 7$. Two balls are drawn. Let X_1 be the number of the first ball drawn and X_2 the number of the second ball drawn. By counting favorable outcomes, compute the probabilities $P(X_1 = 4)$, $P(X_2 = 5)$, and $P(X_1 = 4, X_2 = 5)$ in cases (a) and (b) below.

(a) The balls are drawn *with* replacement.

(b) The balls are drawn *without* replacement.

(c) Does the answer to either (a) or (b) *prove* something about the independence of the random variables X_1 and X_2?

2.19

(a) $P(X_1 = 4) = \frac{1 \times 7}{7 \times 7} = \frac{1}{7}$, $P(X_2 = 5) = \frac{7 \times 1}{7 \times 7} = \frac{1}{7}$, $P(X_1 = 4, X_2 = 5) = \frac{1 \times 1}{7 \times 7} = \frac{1}{49}$

(b) $P(X_1 = 4) = \frac{1 \times 6}{7 \times 6} = \frac{1}{7}$, $P(X_2 = 5) = \frac{6 \times 1}{7 \times 6} = \frac{1}{7}$, $P(X_1 = 4, X_2 = 5) = \frac{1 \times 1}{7 \times 6} = \frac{1}{42}$

(c) Part (a): $P(X_1 = 4, X_2 = 5) = P(X_1 = 4)P(X_2 = 5)$

$\therefore$ X_1, X_2 are independent with replacement

Part (b): $P(X_1 = 4, X_2 = 5) \neq P(X_1 = 4)P(X_2 = 5)$

$\therefore$ X_1, X_2 are not independent without replacement

Section 2.4 — 2.20 → 2.23

$P(N>k) = q^k \longrightarrow$ Geometric distribution

2.20

Exercise 2.20. A fair die is rolled repeatedly. Use precise notation of probabilities of events and random variables for the solutions to the questions below.

(a) Write down a precise sum expression for the probability that the first five rolls give a three at most two times.
(b) Calculate the probability that the first three does not appear before the fifth roll.
(c) Calculate the probability that the first three appears before the twentieth roll but not before the fifth roll.

(a) $p = P(\text{roll a } 3) = \frac{1}{6}$, $q = 1-p = \frac{5}{6}$ $\qquad S_5 \sim \text{Bin}(5, p=\frac{1}{6})$

$$P(S_5 \le 2) = \sum_{k=0}^{2} \binom{5}{k} p^k q^{5-k} = \sum_{k=0}^{2} \binom{5}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{5-k}$$

or $P(N \geq 5) = P(N>4) = \left(\frac{5}{6}\right)^4 \longrightarrow$ fail 4 times

(b) Let N be the number of rolls to see the first 3

$$P(N>4) = \sum_{k=5}^{\infty} \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) = \frac{\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)}{1-\frac{5}{6}} = \frac{\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)}{\frac{1}{6}} = \left(\frac{5}{6}\right)^4$$

(c) $$P(4<N<20) = P(N>4) - P(N \geq 20) = P(N>4) - P(N>19) = \left(\frac{5}{6}\right)^4 - \left(\frac{5}{6}\right)^{19}$$

2.21

Exercise 2.21. Jane must get at least three of the four problems on the exam correct to get an A. She has been able to do 80% of the problems on old exams, so she assumes that the probability she gets any problem correct is 0.8. She also assumes that the results on different problems are independent.

(a) What is the probability she gets an A?
(b) If she gets the first problem correct, what is the probability she gets an A?

(a) Let N be the number of correct answers $\longrightarrow p = 0.8$ & $q = 0.2$

$$P(N \geq 3) = P(N=3) + P(N=4) = \binom{4}{3}(0.8)^3(0.2) + 0.8^4 = 0.8192$$

(b) $$P(N \geq 3 \mid X_1 = 1) = \frac{P(N \geq 3 \cap X_1 = 1)}{P(X_1=1)} = \frac{P(N \geq 2, X_1 = 1)}{P(X_1 = 1)} = \frac{P(N \geq 2)P(X_1=1)}{P(X_1=1)}$$

since N & X_i are independent

$$= P(N \geq 2)$$
$$= P(X_2, X_3 = 1) + P(X_2, X_3, X_4 = 1)$$
$$= \binom{3}{2}(0.8)^2(0.2) + 0.8^3 = 0.896$$

Exercise 2.22. Ann and Bill play rock-paper-scissors. Each has a strategy of choosing uniformly at random out of the three possibilities every round (independently of the other player and the previous choices).

(a) What is the probability that Ann wins the first round? (Remember that the round could end in a tie.)
(b) What is the probability that Ann's first win happens in the fourth round?
(c) What is the probability that Ann's first win comes after the fourth round?

$\boxed{2.22}$

(a) $P(\text{Ann wins } 1^{st} \text{ round}) = P(A_R B_S) + P(A_S B_P) + P(A_P B_R)$

$= P(A_R)P(B_S) + P(A_S)P(B_P) + P(A_P B_R) = \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{3} = \frac{1}{3}$

→ since each choice is independent of other players & previous choices

(b) $P(\text{Ann wins } 4^{th} \text{ round})$

$= P(\text{Ann lost } 1^{st} \text{ round}) \times P(\text{Ann lost } 2^{nd} \text{ round}) \times P(\text{Ann lost } 3^{rd} \text{ round}) \times P(\text{Ann wins } 4^{th} \text{ round})$

$= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3} = \frac{8}{81}$

(c) Let N be the number of rounds to see Ann's first win

$$P(N \geq 5) = \sum_{k=5}^{\infty}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right) = \frac{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)}{1-\frac{2}{3}} = \frac{\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)}{\frac{1}{3}} = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

Exercise 2.23. The probability that there is no accident at a certain busy intersection is 95% on any given day, independently of the other days.

(a) Find the probability that there will be no accidents at this intersection during the next 7 days.
(b) Find the probability that next September there will be exactly 2 days with accidents.
(c) Today was accident free. Find the probability that there is no accident during the next 4 days, but there is at least one by the end of the 10th day.

$\boxed{2.23}$

(a) $p = P(\text{no accident}) = 0.95 \longrightarrow q = 1-p = 0.05$

$P(\text{no accidents in 7 days}) = P(\text{no accident in D1}) \dots P(\text{no accident in D7})$
$= 0.95 \times \dots \times 0.95 = 0.95^7 = 0.6983$

(b) Let N be the number of days in September with accidents

$$P(N=2) = \binom{30}{2} q^2 p^{28} = \binom{30}{2}(0.05)^2(0.95)^{28} \approx 0.2586$$

(c) $P(\text{no accidents in 4 days} \cap \text{at least 1 accident in the next 6 days})$
$= P(\text{no accidents in 4 days})\, P(\text{at least 1 accident in the next 6 days})$
$= p^4(1-p^6) = 0.95^4(1-0.95^6) \approx 0.2158$

Section 2.5 2.24 → 2.28

2.24

Exercise 2.24. A team of three is chosen randomly from an office with 2 men and 4 women. Let X be the number of women on the team.

(a) Identify the probability distribution of X by name.
(b) Give the probability mass function of X.

(a) $X \sim$ Hypergeom(6, 4, 3)

(b) $X = \{0, 1, 2, 3\}$

k	0	1	2	3
$P_X(k)$	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

$P(X=0) = P(3 \text{ men}, 0 \text{ women}) = 0$

$$P(X=1) = P(2 \text{ men}, 1 \text{ women}) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$$

$$P(X=2) = P(1 \text{ man}, 2 \text{ women}) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$$

$$P(X=3) = P(0 \text{ man}, 3 \text{ women}) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$$

2.25

Exercise 2.25. I have a bag with 3 fair dice: a 4-sided die, a 6-sided die, and a 12-sided die. I reach into the bag, pick one die at random and roll it *twice*. The first roll is a 3 and the second roll is a 4. What is the probability that I pulled out the 6-sided die?

Hint. The rolls of a given die are independent.

Let A be the event that first roll is 3
B be the event that second roll is 4

law of total probabilities

$$P(A \cap B) = P(\text{die}) . P(A \cap B \mid \text{die}) = P(\text{die}) . P(A \mid \text{die}) . P(B \mid \text{die})$$
$$= P(\text{die } 4) . P(A \mid \text{die } 4) . P(B \mid \text{die } 4)$$
$$+ P(\text{die } 6) . P(A \mid \text{die } 6) . P(B \mid \text{die } 6) + P(\text{die } 12) . P(A \mid \text{die } 12) . P(B \mid \text{die } 12)$$
$$= \frac{1}{3} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{3} \cdot \left(\frac{1}{6}\right)^2 + \frac{1}{3} \cdot \left(\frac{1}{12}\right)^2 = \frac{7}{216}$$

Bayes' theorem

$$P(\text{die } 6 \mid A \cap B) = \frac{P(\text{die } 6) . P(A \cap B \mid \text{die } 6)}{P(A \cap B)} = \frac{\frac{1}{3} \cdot \left(\frac{1}{6}\right)^2}{\frac{7}{216}} = \frac{2}{7}$$

2.26

Exercise 2.26. Suppose events A, B, C, D are mutually independent. Show that events AB and CD are independent. Justify each step from the definition of mutual independence.

$$P(A \cap B \cap C \cap D) = P(A)P(B)P(C)P(D) = P(AB)P(CD) = P((A \cap B) \cap (C \cap D))$$

∴ AB, CD are independent QED!

2.27

Exercise 2.27. We have two urns. Urn I has 1 green and 2 red balls. Urn II has 2 green and 1 yellow ball.

(a) Pick an urn uniformly at random, and then sample one ball from this urn. What is the probability that the ball is green?

(b) After sampling the ball in part (a) and recording its color, put it back into the same urn. Then repeat the entire experiment: choose one of the urns uniformly at random and sample one ball from this urn. What is the probability that we picked a green ball in both the first and the second experiment?

(c) Pick an urn uniformly at random, and then sample two balls with replacement from this same urn. What is the probability that both picks are green?

(d) Sample one ball from each urn. What is the probability that both picks are green?

I	II
1g 2r	2g 1y

(a) $P(\text{green}) = P(\text{I})\,P(\text{green} \mid \text{I}) + P(\text{II})\,P(\text{green} \mid \text{II}) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{1}{2}$

(b) Each experiment is independent

$P(\text{green} \cap \text{green}) = P(\text{green}) \cdot P(\text{green}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(c) $P(\text{green}) = P(\text{I})\,P(\text{green} \mid \text{I}) + P(\text{II})\,P(\text{green} \mid \text{II}) = \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} \times \frac{2}{3} = \frac{5}{18}$

(d) Each pick is independent

$P(\text{green}) = P(\text{green from I}) \cdot P(\text{green from II}) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

2.28

Exercise 2.28. We play a card game where we receive 13 cards at the beginning out of the deck of 52. We play 50 games one evening. For each of the following random variables identify the name and the parameters of the distribution.

(a) The number of aces I get in the first game.

(b) The number of games in which I receive at least one ace during the evening.

(c) The number of games in which all my cards are from the same suit.

(d) The number of spades I receive in the 5th game.

(a) $X \sim \text{Hypergeom}(52, 4, 13)$

(b) $X \sim \text{Bin}\left(50, 1 - \frac{\binom{48}{13}\binom{4}{0}}{\binom{52}{13}}\right)$

(c) $X \sim \text{Bin}\left(50, \frac{\binom{4}{1}\binom{13}{13}}{\binom{52}{13}}\right)$

(d) $X \sim \text{Hypergeom}(52, 13, 13)$

Section 3.1

3.1 → 3.4

3.1

Exercise 3.1. Let X have possible values $\{1, 2, 3, 4, 5\}$ and probability mass function

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

(a) Calculate $P(X \le 3)$.
(b) Calculate $P(X < 3)$.
(c) Calculate $P(X < 4.12 \mid X > 1.638)$.

(a) $P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{6}{14} = \frac{3}{7}$

(b) $P(X < 3) = P(X=1) + P(X=2) = \frac{1}{7} + \frac{1}{14} = \frac{3}{14}$

(c) $P(X < 4.12 \mid X > 1.638) = \frac{P(1.638 < X < 4.12)}{P(X > 1.638)} = \frac{P(2 \le X \le 4)}{P(X \geq 2)}$

$$= \frac{P(X=2) + P(X=3) + P(X=4)}{P(X=2) + P(X=3) + P(X=4) + P(X=5)}$$

$$= \frac{\frac{1}{14} + \frac{3}{14} + \frac{2}{7}}{\frac{1}{14} + \frac{3}{14} + \frac{2}{7} + \frac{2}{7}} = \frac{2}{3}$$

3.2

Exercise 3.2. Suppose the random variable X has possible values $\{1, 2, 3, 4, 5, 6\}$ and probability mass function of the form $p(k) = ck$.

(a) Find c.
(b) Find the probability that X is odd.

(a) $\sum_{k=1}^{6} p(k) = \sum_{k=1}^{6} ck = c(1+2+3+4+5+6) = 21c = 1 \rightarrow c = \frac{1}{21}$

(b) $P(X = 1, 3, 5) = p(1) + p(3) + p(5) = c + 3c + 5c = 9c = 9 \times \frac{1}{21} = \frac{3}{7}$

3.3

Exercise 3.3. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

(a) Verify that f is a density function.
(b) Calculate $P(-1 < X < 1)$.
(c) Calculate $P(X < 5)$.
(d) Calculate $P(2 < X < 4 \mid X < 5)$.

(a) $\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty} \int_0^t 3e^{-3x}dx = \lim_{t\to\infty} -\int_0^{-3t} e^u du = \lim_{t\to\infty} e^u \Big|_{-3t}^{0} = \lim_{t\to\infty} (1 - e^{-3t})$

$u = -3x \longrightarrow du = -3dx \longrightarrow 3dx = -du$ $\quad = 1 - 0 = 1$ ✓

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \& \quad f(x) \geq 0 \text{ for all } x \in \mathbb{R} \longrightarrow f \text{ is a density function} \quad \text{QED!}$$

(b) $$P(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx = \int_{0}^{1} 3e^{-3x}\,dx = -\int_{0}^{-3} e^{u}\,du = e^{u}\Big|_{-3}^{0} = 1 - e^{-3}$$

(c) $$P(X < 5) = \int_{0}^{5} f(x)\,dx = \int_{0}^{5} 3e^{-3x}\,dx = -\int_{0}^{-15} e^{u}\,du = e^{u}\Big|_{-15}^{0} = 1 - e^{-15}$$

(d) $$P(2 < X < 4 \mid X < 5) = \frac{P(2 < X < 4 \cap X < 5)}{P(X < 5)} = \frac{P(2 < X < 4)}{P(X < 5)}$$

$$P(2 < X < 4) = \int_{2}^{4} f(x)\,dx = \int_{2}^{4} 3e^{-3x}\,dx = -\int_{-6}^{-12} e^{u}\,du = e^{u}\Big|_{-12}^{-6} = e^{-6} - e^{-12}$$

$$\therefore P(2 < X < 4 \mid X < 5) = \frac{e^{-6} - e^{-12}}{1 - e^{-15}} \approx 0.002473$$

Exercise 3.4. Let $X \sim \text{Unif}[4, 10]$.

3.4

(a) Calculate $P(X < 6)$.
(b) Calculate $P(|X - 7| > 1)$.
(c) For $4 \leq t \leq 6$, calculate $P(X < t \mid X < 6)$.

$$X \sim \text{Unif}[4, 10] \longrightarrow f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{6}, & 4 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

(a) $$P(X < 6) = \int_{4}^{6} f(x)\,dx = \int_{4}^{6} \frac{1}{6}\,dx = \frac{1}{6}x\Big|_{4}^{6} = \frac{1}{6}(6-4) = \frac{2}{6} = \frac{1}{3}$$

(b) $$|X - 7| > 1 \longrightarrow X - 7 > 1 \text{ or } X - 7 < -1$$
$$X > 8 \text{ or } X < 6$$

$$P(|X-7| > 1) = P(X > 8) + P(X < 6) = \int_{8}^{10} f(x)\,dx + \frac{1}{3} = \frac{1}{6}(2) + \frac{1}{3} = \frac{2}{3}$$

(c) $$P(X < t \mid X < 6) = \frac{P(X < t \cap X < 6)}{P(X < 6)} = \frac{P(X < t)}{1/3} = 3 \cdot \frac{t-4}{6} = \frac{t-4}{2}$$

Section 3.2

3.5 → 3.7

3.5

Exercise 3.5. Suppose that the discrete random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1/3, & \text{if } 1 \le x < \frac{4}{3} \\ 1/2, & \text{if } \frac{4}{3} \le x < \frac{3}{2} \\ 3/4, & \text{if } \frac{3}{2} \le x < \frac{9}{5} \\ 1, & \text{if } x \ge \frac{9}{5}. \end{cases}$$

Find the possible values and the probability mass function X.

$X = 1, \frac{4}{3}, \frac{3}{2}, \frac{9}{5}$

$P_X(k) =$ jump size at k

$P_X(1) = \frac{1}{3} - 0 = \frac{1}{3}$ $\quad P_X\left(\frac{3}{2}\right) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$P_X\left(\frac{4}{3}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ $\quad P_X\left(\frac{9}{5}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

k	1	$4/3$	$3/2$	$9/5$
$P_X(k)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$

3.6

Exercise 3.6. Find the cumulative distribution function of the random variable X from both Exercise 3.1 and 3.3.

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

(i)

$$F(s) = P(X \le s) = \begin{cases} 0, & s < 1 \\ \frac{1}{7}, & 1 \le s < 2 \\ \frac{1}{7} + \frac{1}{14} = \frac{3}{14}, & 2 \le s < 3 \\ \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{6}{14}, & 3 \le s < 4 \\ \frac{1}{7} + \frac{1}{14} + \frac{3}{14} + \frac{2}{7} = \frac{10}{14}, & 4 \le s < 5 \\ 1, & s \ge 5 \end{cases}$$

(ii)

$$F(s) = P(X \le s) = \int_0^s f(x)dx = \int_0^s 3e^{-3x} dx = e^{-3x}\Big|_s^0 = 1 - e^{-3s}$$

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

3.7

Exercise 3.7. Suppose that the continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & \text{if } x < \sqrt{2} \\ x^2 - 2, & \text{if } \sqrt{2} \le x < \sqrt{3} \\ 1, & \text{if } \sqrt{3} \le x. \end{cases}$$

(a) Find the smallest interval $[a, b]$ such that of $P(a \le X \le b) = 1$.
(b) Find $P(X = 1.6)$.
(c) Find $P(1 \le X \le \frac{3}{2})$.
(d) Find the probability density function of X.

(a) $P(a \le X \le b) = F_X(b) - F_X(a)$

If $a > \sqrt{2} \rightarrow F_X(a) > 0$

$b < \sqrt{3} \rightarrow F_X(b) < 1$

$\therefore F_X(b) - F(a) < 1 \;\therefore\; [a,b] = [\sqrt{2}, \sqrt{3}]$

(b) $P(X = 1.6) = F_X(1.6) - F_X(1.6-) = F_X(1.6) - F_X(1.6) = 0$

↑ $F(x)$ is continuous

(c) $P(1 \le X \le \frac{3}{2}) = F_X\left(\frac{3}{2}\right) - F_X(1) = \left(\frac{3}{2}\right)^2 - 2 - 0 = \frac{9}{4} - 2 = \frac{1}{4}$

(d) pdf: $f(x) = F'(x) = \begin{cases} 0, & x < \sqrt{2} \\ 2x, & \sqrt{2} \le x \le \sqrt{3} \\ 0, & x \ge \sqrt{3} \end{cases} = \begin{cases} 2x, & \sqrt{2} \le x \le \sqrt{3} \\ 0, & \text{otherwise} \end{cases}$

Section 3.3

3.8 → 3.13

Exercise 3.8. Let X be the random variable from Exercise 3.1.

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

(a) Compute the mean of X.
(b) Compute $E[\,|X-2|\,]$.

3.8

(a) $E[X] = \sum_k k.P_X(k) = 1 \times \frac{1}{7} + 2 \times \frac{1}{14} + 3 \times \frac{3}{14} + 4 \times \frac{2}{7} + 5 \times \frac{2}{7} = \frac{7}{2}$

(b) $E[|X-2|] = \sum_k |k-2| P_X(k) = 1 \times \frac{1}{7} + 0 \times \frac{1}{14} + 1 \times \frac{3}{14} + 2 \times \frac{2}{7} + 3 \times \frac{2}{7} = \frac{25}{14}$

Exercise 3.9. Let X be the random variable from Exercise 3.3.

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

(a) Find the mean of X.
(b) Compute $E[e^{2X}]$.

3.9

$u = 3x \qquad dv = e^{-3x}dx$

$du = 3dx \qquad v = \frac{e^{-3x}}{-3}$

(a) $E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} 3x e^{-3x} dx = \lim_{t\to\infty} \int_0^t 3x e^{-3x} dx$

$= \lim_{t\to\infty} \left(-xe^{-3x}\Big|_0^t + \int_0^t e^{-3x}dx \right) = \lim_{t\to\infty} \left(-te^{-3t} - \frac{1}{3}e^{-3x}\Big|_0^t \right)$

$= \lim_{t\to\infty} \left(-te^{-3t} - \cancel{\frac{1}{3}e^{-3t}} + \frac{1}{3} \right) \overset{LH}{=} \lim_{t\to\infty} \cancel{\frac{-1}{3e^{3t}}} + \frac{1}{3} = \frac{1}{3}$

0 0

(b) $E[e^{2X}] = \int_{-\infty}^{\infty} e^{2x} f(x) dx = \int_0^{\infty} 3e^{2x} e^{-3x} dx = \lim_{t\to\infty} \int_0^t 3e^{-x} dx = \lim_{t\to\infty} 3e^{-x}\Big|_t^0$

$= \lim_{t\to\infty} (3 - \cancel{3e^{-t}}) = 3$

0

Exercise 3.10. Let X have probability mass function

$$P(X=-1) = \tfrac{1}{2}, \quad P(X=0) = \tfrac{1}{3}, \quad \text{and} \quad P(X=1) = \tfrac{1}{6}.$$

Calculate $E[|X|]$ using the approaches in (a) and (b) below.

(a) First find the probability mass function of the random variable $Y = |X|$ and using that compute $E[|X|]$.
(b) Apply formula (3.24) with $g(x) = |x|$.

3.10

(a) $P(|X|=0) = P(X=0) = \frac{1}{3} \qquad P(|X|=1) = P(X=1) + P(X=-1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

$E[|X|] = \sum_k |k| P_X(k) = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$

(b) $E[g(X)] = \sum_k g(k) P_X(k) = |-1| \times \frac{1}{2} + |0| \times \frac{1}{3} + |1| \times \frac{1}{6} = \frac{1}{2} + 0 + \frac{1}{6} = \frac{2}{3}$

Exercise 3.11. Let Y be a random variable with density function $f(x) = \frac{2}{3}x$ for $x \in [1,2]$ and $f(x) = 0$ otherwise. Compute $E[(Y-1)^2]$.

3.11

$f(x) = \begin{cases} \frac{2}{3}x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

$$E[(Y-1)^2] = \int_{-\infty}^{\infty}(x-1)^2 f(x)\,dx = \int_1^2 (x-1)^2 \frac{2}{3}x\,dx = \frac{2}{3}\int_1^2 x(x^2-2x+1)\,dx$$

$$= \frac{2}{3}\int_1^2 (x^3-2x^2+x)\,dx = \frac{2}{3}\left(\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}\right)\Big|_1^2 = \frac{2}{3}\left(4-\frac{16}{3}+2-\frac{1}{4}+\frac{2}{3}-\frac{1}{2}\right) = \frac{7}{18}$$

Exercise 3.12. Suppose that X is a random variable taking values in $\{1, 2, 3, \ldots\}$ with probability mass function

3.12

$$p_X(n) = \frac{6}{\pi^2} \cdot \frac{1}{n^2}.$$

Show that $E[X] = \infty$.

$$E[X] = \sum_{k=1}^{\infty} k\,P_X(k) = \sum_{k=1}^{\infty} k \cdot \frac{6}{\pi^2} \cdot \frac{1}{k^2} = \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k} = \frac{6}{\pi^2}\cdot\infty = \infty$$ QED!

$f(x) = \frac{1}{x}$ is decreasing, positive, continuous on $(1,\infty)$

$$\int_1^{\infty}\frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \cancel{\ln t}^{\,\infty} - 0 = \infty \text{ diverges}$$

$\therefore \sum_{k=1}^{\infty}\frac{1}{k}$ diverges by the Integral test

Exercise 3.13. Compute the following.

3.13

(a) The median of the random variable X from both Exercise 3.1 and 3.3.
(b) The 0.9th quantile of the random variable X from Exercise 3.3.

(a) (i)

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

$P(X \le m) \ge \frac{1}{2}$ & $P(X \ge m) \ge \frac{1}{2}$

$m=3$: $P(X\le 3) = P(X=1)+P(X=2)+P(X=3) = \frac{1}{7}+\frac{1}{14}+\frac{3}{14} = \frac{3}{7}$

$m=4$: $P(X\le 4) = P(X\le 3)+P(X=4) = \frac{3}{7}+\frac{2}{7} = \frac{5}{7} \ge \frac{1}{2}$ ✓

$P(X\ge 4) = P(X=4)+P(X=5) = \frac{2}{7}+\frac{2}{7} = \frac{4}{7} \ge \frac{1}{2}$ ✓

$m=5$: $P(X\le 5) = P(X\le 4)+P(X=5) = \frac{5}{7}+\frac{2}{7} = 1 \ge \frac{1}{2}$ ✓

$P(X\ge 5) = P(X=5) = \frac{2}{7}$

$\therefore m = 4$

(ii) $P(X\le m) = \int_0^m f(x)\,dx = \int_0^m 3e^{-3x}\,dx = e^{-3x}\Big|_m^0 = 1-e^{-3m} = \frac{1}{2}$

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

$\therefore e^{-3m} = \frac{1}{2} \rightarrow -3m = \ln\left(\frac{1}{2}\right) = -\ln 2 \rightarrow \boxed{m = \frac{\ln 2}{3}}$

(b) $P(X \leq q) \geq 0.9$ and $P(X \geq q) \geq 0.1$
X is continuous $\rightarrow P(X \leq q) + P(X \geq q) = 1$

$\therefore P(X \leq q) = 0.9$
$P(X \geq q) = 0.1$

$\therefore 1 - e^{-3m} = 0.9$

$e^{-3m} = 0.1 \rightarrow -3m = \ln 0.1 = -\ln 10 \rightarrow \boxed{m = \frac{\ln 10}{3}}$

Section 3.4

3.14 → 3.16

3.14

Exercise 3.14. Find the variance of the random variable X from both Exercise 3.1 and 3.3.

x	1	2	3	4	5
$p_X(x)$	1/7	1/14	3/14	2/7	2/7

(i) $E[X] = 1 \times \frac{1}{7} + 2 \times \frac{1}{14} + 3 \times \frac{3}{14} + 4 \times \frac{2}{7} + 5 \times \frac{2}{7} = \frac{7}{2}$

$E[X^2] = 1^2 \times \frac{1}{7} + 2^2 \times \frac{1}{14} + 3^2 \times \frac{3}{14} + 4^2 \times \frac{2}{7} + 5^2 \times \frac{2}{7} = \frac{197}{14}$

$\therefore \mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{197}{14} - \left(\frac{7}{2}\right)^2 = \frac{51}{28}$

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

(ii) $E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty 3x e^{-3x}\,dx = \frac{1}{3}$

$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty 3x^2 e^{-3x}\,dx$

$u = 3x^2 \qquad dv = e^{-3x}dx$

$du = 6x\,dx \qquad v = \frac{e^{-3x}}{-3}$

$= -x^2 e^{-3x}\Big|_0^\infty + 2\int_0^\infty x e^{-3x}\,dx = \frac{2}{3}\int_0^\infty 3x e^{-3x}\,dx = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

$\therefore \mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

3.15

Exercise 3.15. Suppose that the random variable X has expected value $E[X] = 3$ and variance $\mathrm{Var}(X) = 4$. Compute the following quantities.

(a) $E[3X + 2]$
(b) $E[X^2]$
(c) $E[(2X + 3)^2]$
(d) $\mathrm{Var}(4X - 2)$

(a) $E[3X+2] = 3E[X] + 2 = 3(3) + 2 = 11$

(b) $\mathrm{Var}(X) = E[X^2] - E[X]^2 \rightarrow E[X^2] = \mathrm{Var}(X) + E[X]^2 = 4 + 3^2 = 13$

(c) $E[(2X+3)^2] = E[4X^2 + 12X + 9] = 4E[X^2] + 12E[X] + 9$
$= 4(13) + 12(3) + 9 = 97$

(d) $\mathrm{Var}(4X-2) = 4^2\,\mathrm{Var}(X) = 16 \times 4 = 64$

3.16

Exercise 3.16. Let Z have the following density function

$$f_Z(z) = \begin{cases} \frac{1}{7}, & 1 \le z \le 2 \\ \frac{3}{7}, & 5 \le z \le 7 \\ 0, & \text{otherwise.} \end{cases}$$

Compute both $E[Z]$ and $\mathrm{Var}(Z)$.

$E[Z] = \int_1^2 z f_Z(z)\,dz + \int_5^7 z f_Z(z)\,dz$

$= \int_1^2 \frac{1}{7} z\,dz + \int_5^7 \frac{3}{7} z\,dz = \frac{3}{14} + \frac{72}{14} = \frac{75}{14}$

$$E[Z^2] = \int_1^2 z^2 f_Z(z)\,dz + \int_5^7 z^2 f_Z(z)\,dz = \int_1^2 \frac{1}{7} z^2\,dz + \int_5^7 \frac{3}{7} z^2\,dz = \frac{661}{21}$$

$$\therefore \mathrm{Var}(Z) = E[Z^2] - E[Z]^2 = \frac{661}{21} - \left(\frac{75}{14}\right)^2 = \frac{1633}{588}$$

Section 3.5

3.17 → 3.18

3.17

Exercise 3.17. Let X be a normal random variable with mean $\mu = -2$ and variance $\sigma^2 = 7$. Find the following probabilities using the table in Appendix E.

(a) $P(X > 3.5)$
(b) $P(-2.1 < X < -1.9)$
(c) $P(X < 2)$
(d) $P(X < -10)$
(e) $P(X > 4)$

$X \sim N(-2, 7)$

$$\therefore Z = \frac{X-\mu_X}{\sigma_X} = \frac{X+2}{\sqrt{7}}$$

(a) $P(X > 3.5) = P\left(\frac{X-\mu_X}{\sigma_X} > \frac{3.5-\mu_X}{\sigma_X}\right) = P\left(Z > \frac{3.5-(-2)}{\sqrt{7}}\right) = P\left(Z > \frac{5.5}{\sqrt{7}}\right)$

$= 1 - \Phi\left(\frac{5.5}{\sqrt{7}}\right) \approx 1 - \Phi(2.08) = 1 - 0.9812 = 0.0188$

(b) $P(-2.1 < X < -1.9) = P\left(\frac{-2.1-\mu_X}{\sigma_X} < \frac{X-\mu_X}{\sigma_X} < \frac{-1.9-\mu_X}{\sigma_X}\right)$

$= P\left(\frac{-2.1+2}{\sqrt{7}} < Z < \frac{-1.9+2}{\sqrt{7}}\right) = P\left(\frac{-0.1}{\sqrt{7}} < Z < \frac{0.1}{\sqrt{7}}\right) = \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \Phi\left(\frac{-0.1}{\sqrt{7}}\right)$

$= \Phi\left(\frac{0.1}{\sqrt{7}}\right) - \left[1 - \Phi\left(\frac{0.1}{\sqrt{7}}\right)\right] = 2\Phi\left(\frac{0.1}{\sqrt{7}}\right) - 1 \approx 2\Phi(0.04) - 1 = 2(0.5160) - 1$

$= 0.032$

(c) $P(X < 2) = P\left(\frac{X-\mu_X}{\sigma_X} < \frac{2-\mu_X}{\sigma_X}\right) = P\left(Z < \frac{2+2}{\sqrt{7}}\right) = P\left(Z < \frac{4}{\sqrt{7}}\right) \approx \Phi(1.51)$

$= 0.9345$

(d) $P(X < -10) = P\left(\frac{X-\mu_X}{\sigma_X} < \frac{-10-\mu_X}{\sigma_X}\right) = P\left(Z < \frac{-10+2}{\sqrt{7}}\right) = P\left(Z < \frac{-8}{\sqrt{7}}\right) = \Phi\left(\frac{-8}{\sqrt{7}}\right)$

$= 1 - \Phi\left(\frac{8}{\sqrt{7}}\right) \approx 1 - \Phi(3.02) = 1 - 0.9987 = 0.0013$

(e) $P(X > 4) = P\left(\frac{X-\mu_X}{\sigma_X} > \frac{4-\mu_X}{\sigma_X}\right) = P\left(Z > \frac{4+2}{\sqrt{7}}\right) = P\left(Z > \frac{6}{\sqrt{7}}\right) = 1 - \Phi\left(\frac{6}{\sqrt{7}}\right)$

$\approx 1 - \Phi(2.27) = 1 - 0.9884 = 0.0116$

3.18

Exercise 3.18. Let X be a normal random variable with mean 3 and variance 4.

(a) Find the probability $P(2 < X < 6)$.
(b) Find the value c such that $P(X > c) = 0.33$.
(c) Find $E(X^2)$.

Hint. You can integrate with the density function, but it is quicker to relate $E(X^2)$ to the mean and variance.

$X \sim N(3, 4)$

$$\therefore Z = \frac{X-\mu_X}{\sigma_X} = \frac{X-3}{2}$$

(a) $P(2 < X < 6) = P\left(\frac{2-\mu_X}{\sigma_X} < \frac{X-\mu_X}{\sigma_X} < \frac{6-\mu_X}{\sigma_X}\right) = P\left(\frac{2-3}{2} < Z < \frac{6-3}{2}\right)$

$= P\left(\frac{-1}{2} < Z < \frac{3}{2}\right) = \Phi\left(\frac{3}{2}\right) - \Phi\left(\frac{-1}{2}\right) = \Phi(1.5) - [1 - \Phi(0.5)]$

$= 0.9332 + 0.6915 - 1 = 0.6247$

(b) $P(X > c) = P\left(\frac{X-\mu_X}{\sigma_X} > \frac{c-\mu_X}{\sigma_X}\right) = P\left(Z > \frac{c-3}{2}\right) = 1 - \Phi\left(\frac{c-3}{2}\right) = 0.33$

$\therefore \Phi\left(\frac{c-3}{2}\right) = 0.67 \quad \therefore \frac{c-3}{2} = 0.44 \rightarrow c-3 = 0.88 \rightarrow c = 3.88$

(c) $Var(X) = E[X^2] - E[X]^2 \rightarrow E[X^2] = Var(X) + E[X]^2 = 4 + 3^2 = 13$

Section 4.1

4.1 → 4.4

4.1

Exercise 4.1. In a high school there are 1200 students. Estimate the probability that more than 130 students were born in January under each of the following assumptions. You do not have to use the continuity correction.

(a) Months are equally likely to contain birthdays.
(b) Days are equally likely to be birthdays.

S_n = # students born in January

(a) $S_n \sim \text{Bin}(1200, \frac{1}{12})$, $q = 1-p = \frac{11}{12}$

$$P(S_n > 130) = P\left(\frac{S_n - np}{\sqrt{npq}} > \frac{130-np}{\sqrt{npq}}\right) \approx P\left(Z > \frac{130 - 1200\times\frac{1}{12}}{\sqrt{1200\times\frac{1}{12}\times\frac{11}{12}}}\right) = P(Z > 3.13)$$

$$= 1 - P(Z < 3.13) = 1 - \Phi(3.13) \approx 1 - 0.9991 = 0.0009$$

(b) $S_n \sim \text{Bin}(1200, \frac{31}{365})$, $q = 1-p = \frac{334}{365}$

$$P(S_n > 130) = P\left(\frac{S_n - np}{\sqrt{npq}} > \frac{130-np}{\sqrt{npq}}\right) \approx P\left(Z > \frac{130 - 1200\times\frac{31}{365}}{\sqrt{1200\times\frac{31}{365}\times\frac{334}{365}}}\right) \approx P(Z > 2.91)$$

$$= 1 - P(Z < 2.91) = 1 - \Phi(2.91) \approx 1 - 0.9982 = 0.0018$$

4.2

Exercise 4.2. The probability of getting a single pair in a poker hand of 5 cards is approximately 0.42. Find the approximate probability that out of 1000 poker hands there will be at least 450 with a single pair.

S_n = # single pairs → $S_n \sim \text{Bin}(n,p) = \text{Bin}(1000, 0.42)$ → $q = 1-p = 0.58$

$$P(S_n \geq 450) = P\left(\frac{S_n - np}{\sqrt{npq}} \geq \frac{450-np}{\sqrt{npq}}\right) \approx P\left(Z \geq \frac{450 - 1000\times 0.42}{\sqrt{1000\times 0.42\times 0.58}}\right)$$

$$\approx P(Z \geq 1.92) = 1 - P(Z < 1.92) = 1 - \Phi(1.92) \approx 1 - 0.9726 = 0.0274$$

4.3

Exercise 4.3. Approximate the probability that out of 300 die rolls we get exactly 100 numbers that are multiples of 3.

S_n = # numbers that are multiples of 3 → $S_n \sim \text{Bin}(n,p) = \text{Bin}(300, \frac{1}{3})$

$$P(S_n = 100) = P(99.5 \leq S_n \leq 100.5) = P\left(\frac{99.5 - np}{\sqrt{npq}} \leq \frac{S_n - np}{\sqrt{npq}} \leq \frac{100.5-np}{\sqrt{npq}}\right)$$

$$\approx P\left(\frac{99.5 - 300\times\frac{1}{3}}{\sqrt{300\times\frac{1}{3}\times\frac{2}{3}}} \leq Z \leq \frac{100.5 - 300\times\frac{1}{3}}{\sqrt{300\times\frac{1}{3}\times\frac{2}{3}}}\right) \approx P(-0.06 \leq Z \leq 0.06)$$

$$= \Phi(0.06) - \Phi(-0.06) = \Phi(0.06) - 1 + \Phi(0.06) = 2\times 0.5239 - 1 = 0.0478$$

$\boxed{4.4}$

Exercise 4.4. Liz is standing on the real number line at position 0. She rolls a die repeatedly. If the roll is 1 or 2, she takes *one* step to the right (in the positive direction). If the roll is 3, 4, 5 or 6, she takes *two* steps to the right. Let X_n be Liz's position after n rolls of the die. Estimate the probability that X_{90} is at least 160.

X_n = # steps taken up to & including the n^{th} roll

S_n = # rolls 3, 4, 5, 6 $\rightarrow X_n = 2S_n + n - S_n = n + S_n$

$\rightarrow S_n \sim Bin(n, p) = Bin(n, \frac{2}{3})$, $p = \frac{4}{6} = \frac{2}{3}$, $q = \frac{1}{3}$

$$P(X_{90} \geq 160) = P(90 + S_{90} \geq 160) = P(S_{90} \geq 70) = P\left(\frac{S_{90} - np}{\sqrt{npq}} \geq \frac{70 - np}{\sqrt{npq}}\right)$$

$$\approx P\left(Z \geq \frac{70 - 90 \times \frac{2}{3}}{\sqrt{90 \times \frac{2}{3} \times \frac{1}{3}}}\right) \approx P(Z \geq 2.24) = 1 - P(Z < 2.24)$$

$$= 1 - \Phi(2.24) = 1 - 0.9875 = 0.0125$$

Correction

Let S_n = # successes in n trials (getting 1 or 2) $\rightarrow S_{90} \sim Bin(90, \frac{2}{6})$

$$X_n = S_n + 2\underbrace{(n - S_n)}_{\text{failure}} = 2n - S_n$$

1 step

2 steps

$$P(X_{90} \geq 160) = P(2(90) - S_{90} \geq 160) = P(180 - S_{90} \geq 160) = P(S_{90} \leq 20)$$

$$= P\left(\frac{S_{90} - np}{\sqrt{npq}} \leq \frac{20 - (90) \times \frac{2}{6}}{\sqrt{90 \times \frac{2}{6} \times \frac{4}{6}}}\right) = P(Z \leq \ldots) = \Phi(\ldots) = \ldots$$

Section 4.2 4.5

4.5

Exercise 4.5. Consider the setup of Exercise 4.4. Find the limits below and explain your answer.

(a) Find $\lim_{n\to\infty} P(X_n > 1.6n)$.
(b) Find $\lim_{n\to\infty} P(X_n > 1.7n)$.

X_n = # steps taken up to & including the n^{th} roll

S_n = # rolls 3, 4, 5, 6 $\to X_n = 2S_n + n - S_n = n + S_n$

$\to S_n \sim Bin(n, p) = Bin(n, \frac{2}{3})$, $p = \frac{4}{6} = \frac{2}{3}$, $q = \frac{1}{3}$

(a) $\lim_{n\to\infty} P(X_n > 1.6n) = \lim_{n\to\infty} P(n + S_n > 1.6n) = \lim_{n\to\infty} P(S_n > 0.6n)$

$= \lim_{n\to\infty} P\left(\frac{S_n}{n} - p > \frac{0.6n}{n} - p\right) = \lim_{n\to\infty} P\left(\frac{S_n}{n} - p > 0.6 - \frac{2}{3}\right) \geq \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < 0.07\right) = 1$

(b) $\lim_{n\to\infty} P(X_n > 1.7n) = \lim_{n\to\infty} P(n + S_n > 1.7n) = \lim_{n\to\infty} P(S_n > 0.7n)$

$= \lim_{n\to\infty} P\left(\frac{S_n}{n} - p > \frac{0.7n}{n} - p\right) = \lim_{n\to\infty} P\left(\frac{S_n}{n} - p > 0.7 - \frac{2}{3}\right) \leq \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| > 0.03\right) = 0$

Correction

(a) $\lim_{n\to\infty} P(X_n > 1.6n) = \lim_{n\to\infty} P(2n - S_n > 1.6n) = \lim_{n\to\infty} P(S_n < 0.4n)$

$= \lim_{n\to\infty} P\left(\frac{S_n}{n} < 0.4\right) = 1$ → always true

$p = \frac{2}{6} = \frac{1}{3}$

(b) $\lim_{n\to\infty} P(X_n > 1.7n) = \lim_{n\to\infty} P(2n - S_n > 1.7n) = \lim_{n\to\infty} P(S_n < 0.3n)$

$= \lim_{n\to\infty} P\left(\frac{S_n}{n} < 0.3\right) = 0$ → wrong inequality

$p = \frac{2}{6} = \frac{1}{3}$

$\boxed{\text{Section 4.3}}$ 4.6 → 4.8

$\boxed{4.6}$

Exercise 4.6. A pollster would like to estimate the fraction p of people in a population who intend to vote for a particular candidate. How large must a random sample be in order to be at least 95% certain that the fraction $\hat{p}$ of positive answers in the sample is within 0.02 of the true p?

$\alpha = 0.95, \ \varepsilon = 0.02$

$$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}} \to \sqrt{n} = \frac{1}{2\varepsilon} z_{\frac{1+\alpha}{2}} = \frac{1}{2(0.02)} z_{\frac{1+0.95}{2}} = \frac{z_{0.975}}{0.04} = \frac{1.96}{0.04}$$

$$\to n = \left(\frac{1.96}{0.04}\right)^2 = 2401 \to \text{Sample size at least } 2401$$

$\boxed{4.7}$

Exercise 4.7. A political interest group wants to determine what fraction $p \in (0, 1)$ of the population intends to vote for candidate A in the next election. 1000 randomly chosen individuals are polled. 457 of these indicate that they intend to vote for candidate A. Find the 95% confidence interval for the true fraction p.

$n = 1000, \ \alpha = 0.95, \ \hat{p} = \frac{S_n}{n} = \frac{457}{1000} = 0.457$

$$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}} = \frac{1}{2\sqrt{1000}} z_{\frac{1+0.95}{2}} = \frac{z_{0.975}}{2\sqrt{1000}} = \frac{1.96}{2\sqrt{1000}} = 0.03099$$

Confidence interval is $(\hat{p} - \varepsilon, \hat{p} + \varepsilon) = (0.457 - 0.03099, 0.457 + 0.03099)$
$= (0.42301, 0.48799)$

$\boxed{4.8}$

Exercise 4.8. In a million rolls of a biased die the number 6 shows up 180,000 times. Find a 99.9% confidence interval for the unknown probability that the die rolls 6.

$n = 10^6, \ \alpha = 0.999, \ \hat{p} = \frac{S_n}{n} = \frac{180000}{10^6} = 0.18$

$$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}} = \frac{1}{2\sqrt{10^6}} z_{\frac{1+0.999}{2}} = \frac{z_{0.9995}}{2000} = \frac{3.3}{2000} = 0.00165$$

Confidence interval for p is $(\hat{p} - \varepsilon, \hat{p} + \varepsilon) = (0.18 - 0.00165, 0.18 + 0.00165)$
$= (0.17835, 0.18165)$

Section 4.4 4.9 → 4.11

4.9

Exercise 4.9. Let $X \sim \text{Poisson}(10)$.

(a) Find $P(X \geq 7)$.
(b) Find $P(X \leq 13 \mid X \geq 7)$.

$$X \sim \text{Poisson}(10) \rightarrow P_X(k) = \frac{10^k}{k!} e^{-10}$$

(a) $$P(X \geq 7) = 1 - P(X < 7) = 1 - P(X \leq 6) = 1 - \sum_{k=0}^{6} P_X(k) = 1 - \sum_{k=0}^{6} \frac{10^k}{k!} e^{-10} = 0.8699$$

(b) $$P(X \leq 13 \mid X \geq 7) = \frac{P(X \leq 13 \cap X \geq 7)}{P(X \geq 7)} = \frac{P(7 \leq X \leq 13)}{P(X \geq 7)}$$

$$= \frac{\sum_{k=7}^{13} P_X(k)}{P(X \geq 7)} = \frac{\sum_{k=7}^{13} \frac{10^k}{k!} e^{-10}}{0.8699} = 0.8441$$

4.10

Exercise 4.10. A hockey player scores at least one goal in roughly half of his games. How would you estimate the percentage of games where he scores a hat-trick (three goals)?

S_n = #goals in a game → $S_n \approx \text{Poisson}(\lambda)$

$$0.5 \approx P(S_n \geq 1) = 1 - P(S_n = 0) = 1 - \frac{\lambda^0}{0!} e^{-\lambda} = 1 - e^{-\lambda}$$

$$\therefore e^{-\lambda} = 0.5 \rightarrow -\lambda = \ln 0.5 \rightarrow \lambda = -\ln 0.5 = \ln 2$$

$$P(S_n = 3) = \frac{\lambda^3}{3!} e^{-\lambda} = \frac{(\ln 2)^3}{6} e^{-\ln 2} = 0.02775 \xrightarrow{\times 100} 2.775\%$$

≥ 3 goals is also a HT

$$P(S_n \geq 3) = 1 - P(S_n = 0) - P(S_n = 1) - P(S_n = 2)$$

$$= 1 - \left(\frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} \right) = 1 - e^{-\ln 2}\left(1 + \ln 2 + \frac{(\ln 2)^2}{2}\right) = 0.0333$$

4.11

Exercise 4.11. On the first 300 pages of a book, you notice that there are, on average, 6 typos per page. What is the probability that there will be at least 4 typos on page 301? State clearly the assumptions you are making.

S_n = #typos on a page = # typos on p. 301

$S_n \sim \text{Poisson}(\lambda)$ $\mu_S = E[S_n] = \lambda = 6$

$$P(S_n \geq 4) = 1 - P(S_n < 4) = 1 - P(S_n \leq 3) = 1 - \sum_{k=0}^{3} P_X(k) = 1 - \sum_{k=0}^{3} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= 1 - \sum_{k=0}^{3} \frac{6^k}{k!} e^{-6} = 0.8488$$

$\boxed{\text{Section 4.5}}$ 4.12 → 4.14

$\boxed{4.12}$ **Exercise 4.12.** Let $T \sim \mathrm{Exp}(\lambda)$. Compute $E[T^3]$.

$$T \sim \mathrm{Exp}(\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$E[T^3] = \int_0^\infty x^3 \lambda e^{-\lambda x}\,dx = \lambda \lim_{t\to\infty} \int_0^t x^3 e^{-\lambda x}\,dx \qquad \begin{array}{ll} u = x^3 & dv = e^{-\lambda x}dx \\ du = 3x^2dx & v = \frac{e^{-\lambda x}}{-\lambda} \end{array}$$

$$= \lambda \lim_{t\to\infty} \left(-\frac{x^3}{\lambda} e^{-\lambda x} \Big|_0^t + \frac{3}{\lambda} \int_0^t x^2 e^{-\lambda x}\,dx \right) \qquad \begin{array}{ll} u = x^2 & dv = e^{-\lambda x}dx \\ du = 2xdx & v = \frac{e^{-\lambda x}}{-\lambda} \end{array}$$

$$= \lim_{t\to\infty} \left(-x^3 e^{-\lambda x} \Big|_0^t - \frac{3}{\lambda} x^2 e^{-\lambda x} \Big|_0^t + \frac{6}{\lambda} \int_0^t x e^{-\lambda x}\,dx \right) \qquad \begin{array}{ll} u = x & dv = e^{-\lambda x}dx \\ du = dx & v = \frac{e^{-\lambda x}}{-\lambda} \end{array}$$

$$= \lim_{t\to\infty} \left(\cancelto{0}{\frac{-t^3}{e^{\lambda t}}} - \cancelto{0}{\frac{3}{\lambda} \frac{t^2}{e^{\lambda t}}} - \frac{6}{\lambda^2} x e^{-\lambda x} \Big|_0^t + \frac{6}{\lambda^2} \int_0^t e^{-\lambda x}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-\cancelto{0}{\frac{6t}{\lambda^2 e^{\lambda t}}} - \frac{6}{\lambda^3} e^{-\lambda x} \Big|_0^t \right) = \frac{-6}{\lambda^3} \lim_{t\to\infty} \left(\cancelto{0}{e^{-\lambda t}} - 1 \right) = \frac{6}{\lambda^3}$$

$\boxed{4.13}$ **Exercise 4.13.** Let $T \sim \mathrm{Exp}(1/3)$.

(a) Find $P(T > 3)$.
(b) Find $P(1 \leq T < 8)$.
(c) Find $P(T > 4 \mid T > 1)$.

$$\lambda = \frac{1}{3} \to T \sim \mathrm{Exp}\left(\tfrac{1}{3}\right) = \begin{cases} \frac{1}{3} e^{-x/3}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(a) $P(T > 3) = \int_3^\infty \frac{1}{3} e^{-x/3}\,dx = e^{-x/3} \Big|_\infty^3 = e^{-1} - \cancelto{0}{e^{-\infty}} = \frac{1}{e}$

(b) $P(1 \leq T < 8) = \int_1^8 \frac{1}{3} e^{-x/3}\,dx = e^{-x/3} \Big|_8^1 = e^{-1/3} - e^{-8/3}$

(c) $P(T > 4 \mid T > 1) = \dfrac{P(T > 4 \cap T > 1)}{P(T > 1)} = \dfrac{P(T>4)}{P(T>1)} = \dfrac{e^{-4/3}}{e^{-1/3}} = e^{-1} = \dfrac{1}{e}$

Exercise 4.14. The lifetime of a lightbulb can be modeled with an exponential random variable with an expected lifetime of 1000 days.

(a) Find the probability that the lightbulb will function for more than 2000 days.

(b) Find the probability that the lightbulb will function for more than 2000 days, given that it is still functional after 500 days.

$\boxed{4.14}$

T = lifetime

$$E[T] = \frac{1}{\lambda} = 1000 \to \lambda = \frac{1}{1000} \to T \sim Exp(\lambda) = Exp\left(\frac{1}{1000}\right)$$

(a) $P(T > 2000) = 1 - P(T \leq 2000) = F_T(2000) = e^{-2000 \cdot 1/1000} = e^{-2}$

(b) $P(T > 2000 \mid T > 500) = P(T > 1500) = 1 - P(T \leq 1500) = F_T(1500)$

$$= e^{-1500 \cdot \frac{1}{1000}} = e^{-1.5}$$

Section 4.6

4.15

4.15

Exercise 4.15. Suppose that a class of students is star-gazing on top of the local mathematics building from the hours of 11 PM through 3 AM. Suppose further that meteors arrive (i.e. they are seen) according to a Poisson process with intensity $\lambda = 4$ per hour. Find the following.

(a) The probability that the students see more than 2 meteors in the first hour.
(b) The probability they see zero meteors in the first hour, but at least 10 meteors in the final three hours (midnight to 3 AM).
(c) Given that there were 13 meteors seen all night, what is the probability there were no meteors seen in the first hour?

Poisson process of arrival times (11 pm − 3am) $\rightarrow N_I \sim \text{Poisson}(\lambda . |I|)$

(a) $N_{[11,12]} \sim \text{Poisson}(4.1) = \text{Poisson}(4)$

$$P(N_{[11,12]} > 2) = 1 - P(N_{[11,12]} \leq 2) = 1 - \frac{4^0}{0!}e^{-4} - \frac{4^1}{1!}e^{-4} - \frac{4^2}{2!}e^{-4} = 0.7619$$

(b) $N_{[12,3]} \sim \text{Poisson}(4.3) = \text{Poisson}(12)$

$$P(N_{[11,12]} = 0 \cap N_{[12,3]} \geq 10) \overset{\text{inde}}{=\!=} P(N_{[11,12]} = 0) . P(N_{[12,3]} \geq 10)$$

$$= \frac{4^0}{0!}e^{-4} . (1 - P(N_{[12,3]} \leq 9)) = e^{-4}\left(1 - e^{-12}\left(\frac{12^0}{0!} + \ldots + \frac{12^9}{9!}\right)\right) = 0.01388$$

(c) $$P(N_{[11,12]} = 0 \mid N_{[11,3]} = 13) = \frac{P(N_{[11,12]} = 0 \cap N_{[11,3]} = 13)}{P(N_{[11,3]} = 13)}$$

$$= \frac{P(N_{[11,12]} = 0 \cap N_{[12,3]} = 13)}{P(N_{[11,3]} = 13)} \overset{\text{inde}}{=\!=} \frac{P(N_{[11,12]} = 0) . P(N_{[12,3]} = 13)}{P(N_{[11,3]} = 13)}$$

$$= \frac{\cancel{e^{-4}}\left(\frac{12^{13}}{\cancel{13!}}\cancel{e^{-12}}\right)}{\frac{16^{13}}{\cancel{13!}}\cancel{e^{-16}}} = \left(\frac{12}{16}\right)^{13} = 0.02376$$

Section 5.1

5.1 → 5.5

5.1

Exercise 5.1. Let X be a discrete random variable with probability mass function

$$P(X=-6)=\tfrac{4}{9},\quad P(X=-2)=\tfrac{1}{9},\quad P(X=0)=\tfrac{2}{9},\quad \text{and}\quad P(X=3)=\tfrac{2}{9}.$$

Find the moment generating function of X.

$$M_X(t)=E[e^{tX}]=\sum e^{tK}\cdot P(X=K)=\frac{4}{9}e^{-6t}+\frac{1}{9}e^{-2t}+\frac{2}{9}e^{0t}+\frac{2}{9}e^{3t}$$

$$=\frac{4}{9}e^{-6t}+\frac{1}{9}e^{-2t}+\frac{2}{9}e^{3t}+\frac{2}{9}$$

Exercise 5.2. Suppose that X has moment generating function

$$M_X(t)=\tfrac{1}{2}+\tfrac{1}{3}e^{-4t}+\tfrac{1}{6}e^{5t}.$$

5.2

(a) Find the mean and variance of X by differentiating the moment generating function to find moments.

(b) Find the probability mass function of X. Use the probability mass function to check your answer for part (a).

(a) $$M_X'(t)=\frac{-4}{3}e^{-4t}+\frac{5}{6}e^{5t}\longrightarrow E[X]=M_X'(0)=\frac{-4}{3}+\frac{5}{6}=\frac{-3}{6}=\frac{-1}{2}$$

$$M_X''(t)=\frac{16}{3}e^{-4t}+\frac{25}{6}e^{5t}\longrightarrow E[X^2]=M_X''(0)=\frac{16}{3}+\frac{25}{6}=\frac{57}{6}=\frac{19}{2}$$

$$\therefore Var(X)=E[X^2]-E[X]^2=\frac{19}{2}-\left(\frac{-1}{2}\right)^2=\frac{19}{2}-\frac{1}{4}=\frac{37}{4}$$

(b) pmf:

K	0	-4	5
$P_X(K)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$E[X]=0.\frac{1}{2}-\frac{4}{3}+\frac{5}{6}=\frac{-1}{2}$$

$$E[X^2]=0.\frac{1}{2}+\frac{16}{3}+\frac{25}{6}=\frac{19}{2}$$

$$\therefore Var(X)=E[X^2]-E[X]^2=\frac{19}{2}-\left(\frac{-1}{2}\right)^2=\frac{19}{2}-\frac{1}{4}=\frac{37}{4}$$

Exercise 5.3. Let $X\sim$ Unif[0, 1]. Find the moment generating function $M(t)$ of X. Note that the calculation of $M(t)$ for $t\neq 0$ puts a t in the denominator, hence the value $M(0)$ has to be calculated separately.

5.3

$$\text{pdf Unif}[0,1]:\ f(x)=\begin{cases}0 & ,\ x\notin[0,1]\\ \frac{1}{b-a}=\frac{1}{1-0}=1 & ,\ x\in[0,1]\end{cases}$$

$$M_X(t)=E[e^{tX}]=\int_0^1 e^{tX}.f(x)dx=\int_0^1 e^{tX}dx=\frac{e^{tX}}{t}\Big|_0^1=\frac{e^t-1}{t},\quad t\neq 0$$

$$M_X(t)=E[e^{tX}]=\int_0^1 e^{tX}.f(x)dx=\int_0^1 e^0.1dx=\int_0^1 1dx=1\quad ,\ t=0$$

5.4

Exercise 5.4. In parts (a)–(d) below, either use the information given to determine the distribution of the random variable, or show that the information given is not sufficient by describing at least two different random variables that satisfy the given condition.

(a) X is a random variable such that $M_X(t) = e^{6t^2}$ when $|t| < 2$.
(b) Y is a random variable such that $M_Y(t) = \frac{2}{2-t}$ for $t < 0.5$.
(c) Z is a random variable such that $M_Z(t) = \infty$ for $t \geq 5$.
(d) W is a random variable such that $M_W(2) = 2$.

(a) $\mu_X = 0$ $\quad \frac{1}{2}\sigma_X^2 = 6 \rightarrow \sigma_X^2 = 12$ $\quad \therefore X \sim N(\mu_X, \sigma_X^2) = N(0, 12)$

(b) $M_Y(t) = \frac{2}{2-t} = \frac{\lambda}{\lambda - t}$ $\quad \therefore \lambda = 2$ $\quad \therefore Y \sim Exp(\lambda) = Exp(2)$

(c) Cannot identify the distribution $\longrightarrow Y \sim Exp(\lambda) : M_Y(t) = \infty$ if

(d) Cannot identify the distribution

$W_1 \sim N(0, \frac{\ln 2}{2}) : M_{W_1}(t) = e^{\frac{1}{2}\sigma^2 t^2} = e^{\frac{\ln 2 \times 2^2}{4}} = e^{\ln 2} = 2$

$W_2 \sim \text{Poisson}\left(\frac{\ln 2}{e^2 - 1}\right) : M_{W_2}(t) = e^{\lambda(e^t - 1)} = e^{\frac{\ln 2}{e^2-1}(e^2 - 1)} = e^{\ln 2} = 2$

5.5

Exercise 5.5. The moment generating function of the random variable X is $M_X(t) = e^{3(e^t - 1)}$. Find $P(X = 4)$.

Hint. Look at Example 5.16.

$M_X(t) = e^{3(e^t - 1)} = e^{\lambda(e^t - 1)} \rightarrow \lambda = 3$, $X \sim \text{Poisson}(\lambda) = \text{Poisson}(3)$

$P(X = 4) = \frac{3^4}{4!} e^{-3} = \frac{27}{8} e^{-3} \approx 0.168$

Section 5.2

$5.6 \to 5.8$

5.6

Exercise 5.6. Suppose that X is a discrete random variable with probability mass function

$$P(X=-1)=1/7, \quad P(X=0)=1/14, \quad P(X=2)=3/14, \quad P(X=4)=4/7.$$

Compute the probability mass function of $(X-1)^2$.

Let $Y=(X-1)^2 \to Y=\{1,4,9\}$

$P(Y=1)=P((X-1)^2=1)=P(X=0)+P(X=2)=\frac{1}{14}+\frac{3}{14}=\frac{4}{14}=\frac{2}{7}$

$P(Y=4)=P((X-1)^2=4)=P(X=-1)=\frac{1}{7}$

$P(Y=9)=P((X-1)^2=9)=P(X=4)=\frac{4}{7}$

pmf:

k	1	4	9
$P_Y(k)$	$2/7$	$1/7$	$4/7$

5.7

Exercise 5.7. Suppose $X \sim \text{Exp}(\lambda)$ and $Y=\ln X$. Find the probability density function of Y.

pdf $\text{Exp}(\lambda)$: $f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0, & x<0\end{cases} \longrightarrow$ cdf $F_X(x)=\begin{cases}1-e^{-\lambda x}, & x\geq 0\\ 0, & x<0\end{cases}$

cdf: $F_Y(y)=P(Y\leq y)=P(\ln X\leq y)=P(X\leq e^y)=1-e^{-\lambda e^y}$, $e^y>0$ or $y\in\mathbb{R}$

pdf: $f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{d}{dy}(1-e^{-\lambda e^y})=\lambda e^y e^{-\lambda e^y}=\lambda e^{y-\lambda e^y}$, $y\in\mathbb{R}$

5.8

Exercise 5.8. Let $X \sim \text{Unif}[-1,2]$. Find the probability density function of the random variable $Y=X^2$.

pdf $\text{Unif}[-1,2]$: $f_X(x)=\begin{cases}\frac{1}{b-a}=\frac{1}{2-(-1)}=\frac{1}{3}, & x\in[-1,2]\\ 0, & x\notin[-1,2]\end{cases}$

$x\in[-1,2]\to y=x^2\in[0,4]$

cdf: $F_Y(y)=P(Y\leq y)=P(X^2\leq y)=P(-\sqrt{y}\leq X\leq\sqrt{y})=F_X(\sqrt{y})-F_X(-\sqrt{y})$

pdf: $f_Y(y)=\frac{d}{dy}F_Y(y)=\frac{1}{2\sqrt{y}}f_X(\sqrt{y})+\frac{1}{2\sqrt{y}}f_X(-\sqrt{y})$

(1) $0\leq\sqrt{y}\leq 1\to 0\leq y\leq 1\to f_Y(y)=\frac{1}{2\sqrt{y}}\cdot\frac{1}{3}+\frac{1}{2\sqrt{y}}\cdot\frac{1}{3}=\frac{1}{3\sqrt{y}}$, $y\neq 0$

(2) $1\leq\sqrt{y}\leq 2\to 1\leq y\leq 4\to f_Y(y)=\frac{1}{2\sqrt{y}}\cdot\frac{1}{3}+\frac{1}{2\sqrt{y}}\cdot 0=\frac{1}{6\sqrt{y}}$

Hence, pdf $f_Y(y) = \begin{cases} \dfrac{1}{3\sqrt{y}}, & 0 < y \leq 1 \\ \dfrac{1}{6\sqrt{y}}, & 1 \leq y \leq 4 \\ 0, & \text{otherwise} \end{cases}$

$\sqrt{y} < 2 \rightarrow y < 4$ & $-1 < -\sqrt{y} \rightarrow \sqrt{y} < 1 \rightarrow y < 1$

$$= \begin{cases} \dfrac{1}{3} \cdot \dfrac{1}{2\sqrt{y}}, & 1 \leq y \leq 4 \\ \dfrac{1}{3} \cdot \dfrac{1}{2\sqrt{y}} \cdot 2, & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$