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5 October 2023

Due Thursday 12 October 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #6

1. (4.5 points).
For each of the following 3×3 matrices, compute $rank(A)$, determine the basic columns (show them), and determine if A is singular or nonsingular.

(a). $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -10 \end{bmatrix}$.

(b). $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$.

(c). $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -11 \end{bmatrix}$.

2. (2.5 points). Let A be the matrix of exercise 1 (c). Compute the solution x to $Ax = b$, with $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, using forward elimination and back substitution.
Hint, use the information you have from exercise 1 (c).

3. (3 points).
(a). Write down explicitly the three elementary matrices that you implicitly used in exercise 1 (c), that is, the matrices E_1, E_2, E_3 such that $E_3 E_2 E_1 A = U$, U upper triangular.
(b). Find L, lower unit triangular such that $A = LU$.

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① (a) $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -10 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -12 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 4R_2} \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

pivots = 2

$\therefore$ rank(A) = 2 < n = 3

$\therefore$ A is singular

Basic columns $\begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix}$ & $\begin{vmatrix} 3 \\ 2 \\ -1 \end{vmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

pivot = 1

$\therefore$ rank(A) = 1 < n = 3

$\therefore$ A is singular

Basic column $\begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$

(c) $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -11 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -13 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 4R_2} \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

pivots = 3

$\therefore$ rank(A) = 3 = n

$\therefore$ A is nonsingular

Basic columns $\begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix}, \begin{vmatrix} 3 \\ 2 \\ -1 \end{vmatrix}, \begin{vmatrix} 2 \\ -1 \\ -11 \end{vmatrix}$

② $\underset{A}{\begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -11 \end{bmatrix}} \underset{x}{x} = \underset{b}{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}} \rightarrow \left[\begin{array}{ccc:c} 2 & 3 & 2 & 1 \\ 2 & 2 & -1 & 0 \\ 2 & -1 & -11 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc:c} 2 & 3 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & -4 & -13 & 2 \end{array}\right]$

$\xrightarrow{R_3 \leftarrow R_3 - 4R_2} \left[\begin{array}{ccc:c} 2 & 3 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & -1 & 6 \end{array}\right] \rightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$

$-x_3 = 6 \rightarrow x_3 = -6$

$-x_2 - 3x_3 = -1 \longrightarrow x_2 + 3x_3 = 1 \longrightarrow x_2 = 1 - 3x_3 = 1 - 3(-6) = 19$

$2x_1 + 3x_2 + 2x_3 = 1$

$2x_1 + 3(19) + 2(-6) = 1$

$2x_1 + 57 - 12 = 1 \rightarrow 2x_1 = -44 \rightarrow x_1 = -22 \quad \therefore x = \begin{bmatrix} -22 \\ 19 \\ -6 \end{bmatrix}$

3 (a) $m_{21} = \dfrac{-a_{21}}{a_{11}} = \dfrac{-2}{2} = -1 \longrightarrow E_{21} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = E_1$

$m_{31} = \dfrac{-a_{31}}{a_{11}} = \dfrac{-2}{2} = -1 \longrightarrow E_{31} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = E_2$

$m_{32} = \dfrac{-a_{32}}{a_{22}} = \dfrac{-(-4)}{-1} = \dfrac{4}{-1} = -4 \rightarrow E_{32} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{vmatrix} = E_3$

Verify: $E_3 E_2 E_1 A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -11 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{vmatrix}$

$= U \checkmark$

(b) $E_3 E_2 E_1 A = U \longrightarrow A = (E_3 E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} E_3^{-1} U = LU$

$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}$

Verify: $LU = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -11 \end{vmatrix} = A \checkmark$