* See Modules : Summary 4.1, 4.3, 4.4

§ 4.3 Homogeneous Linear Equations with Constant Coefficients

Second Order

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}$$

element of

Try $y = e^{mx}$ $\Rightarrow$ $y' = me^{mx}$ & $y'' = m^2 e^{mx}$

So $am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

$$e^{mx}\underbrace{(am^2 + bm + c)}_{\text{auxiliary equation}} = 0$$

$e^{mx} = 0$ ~~NO solution~~

$am^2 + bm + c = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 Cases : $b^2 - 4ac > 0$ two real roots

$b^2 - 4ac = 0$ one real roots

$b^2 - 4ac < 0$ conjugate complex numbers

Case 1 : Distinct Real Roots m_1, m_2

$$y_1 = e^{m_1 x} \text{ and } y = e^{m_2 x}$$

$m_1 \neq m_2$

$$W = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = \left| m_2 e^{m_1 x} e^{m_2 x} - m_1 e^{m_1 x} e^{m_2 x} \right|$$

$$= e^{(m_1 + m_2)x} (m_2 - m_1)$$

Since $m_1 \neq m_2$, $e^{(m_1+m_2)x} \neq 0 \ \forall x$,

$W \neq 0$ $\therefore$ So the two solution are linear independent.

Case 1 : m_1, m_2 real, $m_1 \neq m_2$

General solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2 : One real root with multiplicity 2

$$ay'' + by' + cy = 0$$

$$y'' + \frac{b}{a} y' + \frac{c}{a} = 0$$

$$m_1 = \frac{-b \pm \sqrt{\overset{0}{b^2 - 4ac}}}{2a} \qquad m_1 = \frac{-b}{2a}$$

$$y_1 = e^{m_1 x} \qquad \frac{b}{a} = -2m_1$$

$$y_2 = e^{m_1 x} \int \frac{e^{-\int (-2m_1) dx}}{\left(e^{m_1 x}\right)^2} dx \qquad \int 2m_1 dx = 2m_1 x$$

$$= e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int 1 dx$$

$$= x e^{m_1 x}$$

Case 2: m root multiplicity 2

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

Case 3: $b^2 - 4ac < 0$

2 complex roots that are conjugates of each other.

$$\left.\begin{aligned} m_1 &= \alpha + \beta i \\ m_2 &= \alpha - \beta i \end{aligned}\right\}$$

Proof in book.

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

Examples : Second order HLDE with Constant coefficients

eg $y'' - y' - 2y = 0$ $\qquad m_1 = 2$

AE : $m^2 - m - 2 = 0$ $\qquad m_2 = -1$

$(m-2)(m+1) = 0$

$$y = c_1 e^{2x} + c_2 e^{-x}$$

eg $y'' - 7y' = 0$

$m^2 - 7m = 0$ $\qquad y = c_1 + c_2 e^{7x}$

$m(m-7) = 0$

$m = 0$, $m = 7$

Case 2

eg $$y'' - y' + \frac{1}{4}y = 0$$

$$m^2 - m + \frac{1}{4} = 0$$

$$\left(m - \frac{1}{2}\right)^2 = 0$$

$$m = \frac{1}{2}$$

$$y = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$$

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eg Case 2 IVP

$$y'' - y' + \frac{1}{4}y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}$$

from previous example,

general solution:

$$y = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$$

$$y' = \frac{1}{2} c_1 e^{\frac{x}{2}} + c_2 \frac{x}{2} e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}}$$

$$2 = c_1 + 0 \qquad \frac{1}{3} = \frac{1}{2}c_1 + 0 + c_2$$

$$c_1 = 2 \qquad \frac{1}{2} = \frac{1}{2}(2) + c_2 \Rightarrow c_2 = -\frac{2}{3}$$

$$y = 2e^{x/2} - \frac{2}{3} x e^{\frac{x}{2}}$$

$\square$

Case 3 eg $y'' + 2y' + 2y = 0$

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$\alpha = -1, \beta = 1$

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

Higher-Order LDE with CC

Must factor the auxiliary equation & then apply principle of superposition to this.

eg $(m-1)(m-2)^2(m^2+m+1)=0$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

GS $y = C_1 e^{x} + C_2 e^{2x} + C_3 x e^{2x} + C_4 e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}x\right)$
$+ C_5 e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

For n^{th} DE, n factors.

n^{th} IVP, n initial conditions needed.

That is, need n equations in n unknowns.

Hints to Factoring auxiliary equations.

- Factor out any common factors.

  Hw 15, 23, 35

- Perfect squares

  $u^2 - 2uv + v^2 = (u-v)^2$

  $u^2 + 2uv + v^2 = (u+v)^2$

  u^2, v^2 perfect squares

  Hw 24, 25, (35)

- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

  $a^3 + b^3 = (a+b)(a^2-ab+b^2)$

- synthetic division

  Used to help find roots of linear Factors.

Use Rational Root theorem:

If $\frac{a}{b}$ is a root of a polynomial of degree n, then b divides leading coefficient and a divides the constant term.

eg $m^3 - 2m^2 - 5m + 6 = 0$

1 good

a

possibilities for a: Factors of 6: $\pm 1, \pm 2, \pm 3, \pm 6$.

Try 3.

$$\begin{array}{r|rrr|r}
3 & 1 & -2 & -5 & 6 \\
 & & \overset{1\times 3}{3} & \overset{1\times 3}{3} & \overset{-2\times 3}{-6} \\
\hline
 & 1 & 1 & -2 & 0
\end{array}$$

bring down leading coef.

add $-2+3$

add $-5+3$

remainder 0. So 3 is a root

and $m-3$ is a factor of the poly.

So $m^3 - 2m^2 - 5m + 6$

$= (m-3)(m^2 + m - 2)$

$= (m-3)(m+2)(m-1) \overset{\text{set}}{=} 0$

$m_1 = 3, \quad m_2 = -2, \quad m_3 = 1$

So $y = c_1 e^{3x} + c_2 e^{-2x} + c_3 e^{x}$

is the general solution to the DE