Linear Algebra 2101.

HW 8

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TRUE or FALSE ?

(a) If A and B are nonsingular $AB = BA$ (i.e. they commute).

FALSE. for example $A = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}$, $B = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$

Easy to see that both A and B are rank 2 (nonsingular)

$$AB = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 4 & 4 \end{vmatrix}.$$

$$BA = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ -1 & 3 \end{vmatrix}.$$ thus $AB \neq BA$

(b) $(AB)^{-1} = B^{-1}A^{-1}$ TRUE

It suffices to show that $(AB)(B^{-1}A^{-1}) = I$

$$(AB)(B^{-1}A^{-1}) = A\ B B^{-1} A^{-1} = A(BB^{-1})A^{-1} =$$
$$= A \cdot I A^{-1} = AA^{-1} = I \quad ✓$$

(c) A nonsingular, implies $(-A)$ non singular

TRUE. It suffices to exhibit $(-A)^{-1}$

but $(-A)^{-1} = -A^{-1}$ since $(-A)(-A)^{-1} = (-A)-A^{-1} =$
$$= A \cdot A^{-1} = I$$

(d) A, B nonsingular implies $A+B$ nonsingular

FALSE For example $\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ -2 & 0 \end{vmatrix}$
which is singular.

(e) $A = LU$. U nonsingular $\Rightarrow$ A nonsingular.

TRUE. L always nonsingular.
since product of nonsingular matrices is nonsingular. $A = L.U$ is nonsingular.
Another proof: $A^{-1} = U^{-1}L^{-1}$, thus A^{-1} exists.
(if U is upper triangular)
yet another proof: U nonsingular if and only if its diagonal entries are nonzero, that is all diagonal entries are pivots. We have n pivots rank $A = n$, nonsingular

(f) the inverse of a lower triangular is upper triangular.

FALSE, for example let $L = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$

$L^{-1} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}$ is not upper triangular

In fact, the inverse of a lower triangular matrix is always lower triangular, and the inverse of an upper triangular matrix is always upper triangular.

3.10.1

let $A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{bmatrix}$

(a) Determine the LU factors of A

$m_{21} = \frac{4}{1} = 1$ $\quad E_{31} E_{21} A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 4 & 15 \end{bmatrix}$

$m_{31} = \frac{3}{1} = 3$

$m_{32} = \frac{4}{2} = 2$ $\quad E_{32} E_{31} E_{21} A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = U$

$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ - check that $L.U = A$ ✓

(b) Solve $Ax = b$ for $b^{(1)} = \begin{vmatrix} 6 \\ 0 \\ -6 \end{vmatrix}$ and $b^{(2)} = \begin{vmatrix} 6 \\ 6 \\ 12 \end{vmatrix}$

for $b = \begin{vmatrix} 6 \\ 0 \\ -6 \end{vmatrix}$ first solve $Ly = b$

$y_1 = 6$

$4.6 + y_2 = 0 \Rightarrow y_2 = -24$

$3.6 + 2.(-24) + y_3 = -6$, $18 - 48 + y_3 = -6 \Rightarrow y_3 = 24$

$y = \begin{vmatrix} 6 \\ -24 \\ 24 \end{vmatrix}$, solve $Ux = y$, $x_3 = 24/3 = 8$

$2x_2 + 6.8 = -24$, $2x_2 = -24 - 48 = -72$ $\quad x_2 = -\frac{72}{2}$

$x_2 = -36$

$x_1 + 4.(-36) + 5.8 = 6$ $\quad x_1 = 6 - 40 + 144 = 110$

Thus $x = \begin{vmatrix} 110 \\ -36 \\ 8 \end{vmatrix}$. check $A \cdot x = b$. ✓

For $b = \begin{vmatrix} 6 \\ 6 \\ 12 \end{vmatrix}$, solve $Ly = b$

$y_1 = 6$

$4 \cdot 6 + y_2 = 6 \qquad y_2 = 6 - 24 = -18$

$3 \cdot 6 + 2 \cdot (-18) + y_3 = 12$

$18 - 36 + y_3 = 12$

$y_3 = 12 + 18 = 30 \qquad y = \begin{vmatrix} 6 \\ -18 \\ 30 \end{vmatrix}$

Solve $Ux = y \qquad x_3 = 30/3 = 10$

$2x_2 + 6 \cdot 10 = -18$

$2x_2 = -78, \quad x_2 = -39$

$x_1 + 4 \cdot (-39) + 5 \cdot 10 = 6$

$x_1 = 6 + 156 - 50 = 112$

$x = \begin{vmatrix} 112 \\ -39 \\ 10 \end{vmatrix}$. Check $Ax = b$ ✓

(c) Use the LU factors to determine A^{-1}.

We solve $AX = I$, i.e. 3 right hand sides.

We start with $b = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$, solve $Ly = b$

$y_1 = 1$

$4 \cdot 1 + y_2 = 0 \qquad y_2 = -4$

$3 \cdot 1 + 2 \cdot (-4) + y_3 = 0 \qquad y_3 = 8 - 3 = 5$

$$y = \begin{vmatrix} 1 \\ -4 \\ 5 \end{vmatrix}$$

Solve $Ux = y$, $\quad x_3 = 5/3$

$2x_2 + 6 \cdot \frac{5}{3} = -4$

$x_2 = (-4 - 10)/2 = -7$

$x_1 + 4 \cdot (-7) + 5 \cdot \frac{5}{3} = 1$

$$x_1 = \frac{3 - 25 + 28 \cdot 3}{3} = \frac{62}{3}$$

First column of A^{-1} is $x = \begin{vmatrix} 62/3 \\ -7 \\ \frac{5}{3} \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 62 \\ -21 \\ 5 \end{vmatrix}$

For second column consider $b = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$, Solve $Ly = b$

$y_1 = 0$

$2 \cdot 0 + y_2 = 1 \qquad y_2 = 1$

$3 \cdot 0 + 2 \cdot 1 + 1 \cdot y_3 = 0, \quad y_3 = -2$

Solve $Ux = y = \begin{vmatrix} 0 \\ 1 \\ -2 \end{vmatrix} \qquad x_3 = -2/3$

$2x_2 + 6 \cdot \left(-\frac{2}{3}\right) = 1 \Rightarrow x_2 = \frac{5}{2}$

$x_1 + 4 \cdot \frac{5}{2} + 5 \cdot \left(-\frac{2}{3}\right) = 0$

$x_1 + 10 + \left(-\frac{10}{3}\right) = 0 \qquad x_1 + \frac{30 - 10}{3} = 0 \qquad x_1 = -\frac{20}{3}$

Second column of A^{-1} is $x = \begin{vmatrix} -\frac{20}{3} \\ 5/2 \\ -2/3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} -40 \\ 15 \\ -4 \end{vmatrix}$

For the third column, let $b = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

solve $Ly = b$ $\quad y_1 = 0$

$y_2 = 0 \qquad y_3 = 1$

$Ux = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$ $\qquad x_3 = 1/3$

$2x_2 + 6 \cdot \frac{1}{3} = 0 \quad , \quad x_2 = -1$

$x_1 + 4(-1) + 5 \cdot \frac{1}{3} = 0$

$x_1 = 4 - \frac{5}{3} = \frac{7}{3}$

third column of A^{-1} is $x = \begin{vmatrix} 7/3 \\ -1 \\ 1/3 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 7 \\ -3 \\ 1 \end{vmatrix}$

thus $A^{-1} = \frac{1}{6}\begin{bmatrix} 10 & -40 & 14 \\ -42 & 15 & -6 \\ 124 & -4 & 2 \end{bmatrix}$.

$\boxed{3.10.3}$. $\quad A = \begin{bmatrix} \xi & 2 & 0 \\ 1 & \xi & 1 \\ 0 & 1 & \xi \end{bmatrix}$. Let us compute

the LU factorization, and set conditions on ξ

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1/\xi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and we need $\xi \neq 0$

$$E_{21} A = \begin{bmatrix} \xi & 2 & 0 \\ 0 & \xi - \frac{2}{\xi} & 1 \\ 0 & 1 & \xi \end{bmatrix}$$ and we need

$\xi - \frac{2}{\xi} \neq 0$ that is $\xi^2 - 2 \neq 0$, i.e. $\xi \neq \pm\sqrt{2}$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{\xi - \frac{2}{\xi}} & 1 \end{bmatrix}$$

$$E_{32} E_{21} A = \begin{bmatrix} \xi & 2 & 0 \\ 0 & \xi - \frac{2}{\xi} & 1 \\ 0 & 0 & \xi - \frac{1}{\xi - \frac{2}{\xi}} \end{bmatrix} = U$$

Thus, for the existence of the LU factors (without row interchanges) we need $\xi \neq 0$ and $\xi \neq \pm\sqrt{2}$.

If, in addition, we want U (and thus A) to be nonsingular we need

$$\xi - \frac{1}{\xi - \frac{2}{\xi}} = \xi - \frac{1}{\frac{\xi^2 - 2}{\xi}} = \xi - \frac{\xi}{\xi^2 - 2} =$$

$$= \frac{\xi^3 - 2\xi - \xi}{\xi^2 - 2} = \frac{\xi^3 - 3\xi}{\xi^2 - 2} \neq 0$$

that is $(\xi^2 - 3)\xi \neq 0$ or $\xi \neq \pm\sqrt{3}$.

Thus for $\xi = 0$, $\xi = \pm\sqrt{2}$, A fails to have an LU factorization (without pivoting), and for $\xi = \pm\sqrt{3}$, A is singular.

3.10.6 (a) Let us just multiply $L \cdot U$

$$L \cdot U = \begin{bmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \frac{\alpha_1\gamma_1}{\pi_1} + \pi_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \frac{\alpha_2\gamma_2}{\pi_2} + \pi_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \frac{\alpha_3\gamma_3}{\pi_3} + \pi_4 \end{bmatrix}$$

comparing with A we have $\pi_1 = \beta_1$ and

$$\frac{\alpha_i\gamma_i}{\pi_i} + \pi_{i+1} = \beta_i \qquad i = 2, 3, 4$$

So $\pi_{i+1} = \beta_i - \frac{\alpha_i\gamma_i}{\pi_i}$ as desired.

(b) We apply this recursion to T, that is to the case $\beta_i = 2$ $i = 1, 2, 3$ $\beta_4 = 1$
$\alpha_i = \gamma_i = -1$ (so that $\alpha_i \cdot \gamma_i = 1$)

and we obtain $\pi_1 = 2$

$$\pi_2 = 2 - \frac{1}{2} = 3/2$$

$$\pi_3 = 2 - \frac{1}{3/2} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\pi_4 = 1 - \frac{1}{4/3} = 1 - \frac{3}{4} = \frac{1}{4}$$

So that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

check that $T = L \cdot U$. ✓

[Note that one could use this recursion for the matrix in exercise 3.10.3]

3.10.7) We consider a 5×5 matrix with a bandwith $w=2$, that is with 5 diagonals. Thus it has the form

$$\begin{bmatrix} x & x & x & 0 & 0 \\ x & x & x & x & 0 \\ x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \end{bmatrix},$$

where x stands for a nonzero

we will then have $m_{21}\neq 0$ $m_{31}\neq 0$

but $m_{41}=m_{51}=0$, and similarly

$m_{32}\neq 0$ $m_{42}\neq 0$ but $m_{52}=0$

So $L = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ 0 & m_{42} & m_{43} & 1 & \\ 0 & 0 & m_{53} & m_{54} & 1 \end{bmatrix}$

the First row of U is the same as that of A. When we do the elimination the 2,5 entry becomes $0 - m_{21}\cdot 0 = 0$.
So U has also the same band structure.