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Homework #1

Math 2101-02 (1)

Exercise 5.9.1. Let $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^3$ with bases

$$B_{\mathcal{X}} = \left\{ \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}, \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \right\}, \quad B_{\mathcal{Y}} = \left\{ \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \right\}$$

(a) $\mathcal{X}$ and $\mathcal{Y}$ are complementary since

$$\dim \mathcal{X} + \dim \mathcal{Y} = 2 + 1 = 3$$

and $B_{\mathcal{X}} \cup B_{\mathcal{Y}}$ has linearly independent vectors, as shown in the following

Let $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$. Do elimination (i.e., LU)

$$A \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}, \text{ rank } 3.$$

(b) Determine P projector onto $\mathcal{X}$ along $\mathcal{Y}$ and Q onto $\mathcal{Y}$ along $\mathcal{X}$

$$P = U\left(V^T U\right)^{-1} V^T$$

We need V with 2 linearly independent columns of $\mathcal{Y}^\perp$. For example $V = \begin{vmatrix} 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{vmatrix}$

Let us compute $V^T U =$

$$\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{vmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Since this matrix is unit lower triangular we know $(V^T U)^{-1} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix}$

$$P = U(V^T U)^{-1} V^T = \begin{vmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{vmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{bmatrix}$$

You can check that $P^2 = P$, that $PU = U$ and for $y = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}$, $Py = 0$.

$$Q = I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix}$$

Check that $Q^2 = Q$, $R(Q) = y$, clearly, and $QU = 0$.

(c) projection of $v = \begin{vmatrix} 2 \\ -1 \\ 1 \end{vmatrix}$ onto $\mathcal{Y}$ along $\mathcal{X}$.

$$Q\, v = \begin{vmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{vmatrix} \begin{vmatrix} 2 \\ -1 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$$

(d) verify. $P^2 \stackrel{?}{=} P$ $\quad Q^2 = Q$. (see above)

(e) $R(P) = \mathcal{X}$ since $PU = U$

$N(Q) = \mathcal{X}$ since $QU = 0$

$N(P) = \mathcal{Y}$ since $P\mathcal{Y} = 0$

$R(Q) = \mathcal{Y}$ since all colums of Q are multiples of the vector in $B_{\mathcal{Y}}$.

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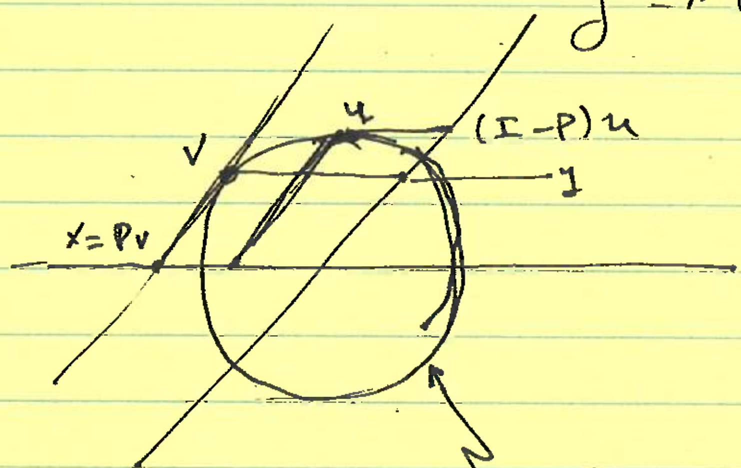
2. $P^2 = P \quad P \neq I \quad P \neq 0$

$\|I - P\| = \|P\|$ (2-norm)

$\mathcal{Y} = N(P) = R(I-P)$



$\mathcal{X} = R(P)$

unit circle $S_1 = \{\|x\|_2 = 1\}$

$\|P\| = \|Pv\|$ for some $v \in S_1$, i.e. $\|v\| = 1$

$v = x + y \quad x \in \mathcal{X},\ y \in \mathcal{Y}$ unique decomposition

so that $x = Pv$, i.e.,

$\|Pv\| = \|Px\| = \|x\|$

let us consider the unit vectors in the direction of x, y,

i.e. let $z = \frac{x}{\|x\|}$, $w = \frac{y}{\|y\|}$

so that $v = \|x\| z + \|y\| w$

I am looking for $u \ni \|(I-P)u\| = \|P\| = \|x\|$

Consider $u = \|y\| z + \|x\| w$

since $(I-P)z = 0$, $(I-P)w = w$

then $(I-P)u = \|x\| (I-P) w = \|x\| . w$

Now $\|(I-P)u\| = \|x\| = \|P\|$

since $\|w\|=1$

Thus, if $\|u\|=1$

we conclude that

$$\|I-P\| = \max_{\|s\|=1} \|(I-P)s\| \geq \|(I-P)u\| = \|P\|$$

Thus $\|I-P\| \geq \|P\|$

Similarly, exchanging $\mathcal{X}$ with $\mathcal{Y}$,

we obtain $\|P\| \geq \|I-P\|$

so that $\|I-P\| = \|P\|$.

All we need is to show that $\|u\|=1$.

Well, we know $\|v\|=1$

$$\|v\|^2 = \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle z,w\rangle \|x\|\,\|y\|$$

$$\|u\|^2 = \|(\|y\|z + \|y\|w)\|^2 = \|y\|^2 + \|y\|^2 + 2\langle w,z\rangle \|x\|\,\|y\|$$

so that $\|u\|^2 = \|v\|^2 = 1$ q.e.d.