Section 3.9 Elementary Matrices

Elementary Matrices are of the form

$$I - uv^T \qquad u, v \in \mathbb{R}^n \qquad v^Tu \neq 1$$

Proposition $(I - uv^T)^{-1} = I - \frac{uv^T}{v^Tu - 1}$ if $v^Tu \neq 1$

[$\Rightarrow$ the inverse of an elementary matrix is an elementary matrix]

Recall $v^Tu \in \mathbb{R}$ $\quad v^Tu - 1$ a number.

Proof $(I - uv^T)\left(I - \frac{1}{v^Tu-1} \cdot uv^T\right) =$

$$= I - \frac{1}{v^Tu-1} \cdot uv^T - uv^T + \frac{1}{v^Tu-1} \cdot (u\underbrace{v^Tu}v^T)$$

$$= I - \underbrace{\left(\frac{1}{v^Tu-1} + 1\right)}_{\frac{1+v^Tu-1}{v^Tu-1} = \frac{v^Tu}{v^Tu-1}} uv^T + \frac{v^Tu}{v^Tu-1}(uv^T) = I$$

qed

Corollary. $\boxed{\text{Elementary matrices are nonsingular}}$

~~Three~~

Note EA affect the Rows of A

AE affect the colums of A

three important examples (for row operations on the left)

Type I. Row interchanges e.g. $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Type II Multiply a row by α e.g. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Type III Add a muliple of row i to row j

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{vmatrix}$$

Note $E = I - uv^T$ $\qquad E = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}$

$E^T = I - vu^T$ $\qquad E^T = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$(EA)^T = A^T E^T$$

1. Transpose of Elementary matrix is elementary
2. Again, mult on the right act on columns

Section 3.10 LU Factorization

Recall $A \to U$ upper Echelon
or upper triangular

We can replicate this process by multiplying on the left ~~with~~ by elementary matrices.

- Row interchanges
- Add a multiple of the pivot row to the current row

Example
$$A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix}$$

$$m_{21} = -\frac{2}{2} = -1 \qquad m_{31} = -\frac{2}{2} = -1$$

$$E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 2 & -1 & -1 \end{vmatrix}$$

$$E_{31} E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{vmatrix}$$

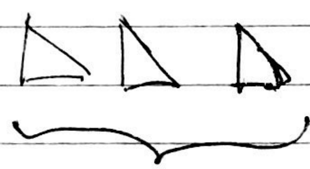
$$m_{32} = -\frac{-4}{-1} = -4$$

$$E_{32} E_{31} E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} = U$$

Repeat $E_{32}E_{31}E_{21}A = U$

$$A = (E_{32}E_{31}E_{21})^{-1}U$$

$$= E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\,U$$



L

$A = LU$ $\qquad$ LU factorization

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}$$

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}}_{L}\underbrace{\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix}}_{U} = LU$$

Every matrix (possibly with row interchanges) has an LU factorization

L ~~lower~~ unit lower triangular. L upper triangular

Extremely useful. Fundamental

Want to solve $Ax = b$

write $LUx = b$

First solve $Ly = b$ $\quad \Delta | = |$

(super easy unit lower triangular)

then solve $Ux = y$ $\quad \nabla | = |$

back substitution

Example $b = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}$ $\quad Ax = b \quad LUx = b$

$Ly = b$

$$\begin{vmatrix} 1 & & \\ 1 & 1 & \\ 1 & 4 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}$$

$y_1 = 3$

$3 + y_2 = -2$

$y_2 = -2 - 3 = -5$

$1 * 3 + 4 * (-5) + y_3 = 1$

$y_3 = 1 + 17 = 18$

$$y = \begin{vmatrix} 3 \\ -5 \\ 18 \end{vmatrix} \qquad Ux = y$$

$$\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 3 \\ -5 \\ 18 \end{vmatrix} \qquad x_3 = \frac{18}{9} = 2$$

$$x_2 = (-5 + 3 \cdot 2) / -1 = -1$$

$$x_1 = (3 - 3 \cdot 1 - 2 \cdot 2) / 2 = 2$$

$$x = \begin{vmatrix} 2 \\ -1 \\ 2 \end{vmatrix}$$

check.

$$A \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \qquad E_{21} = \begin{vmatrix} 1 & \\ 1/2 & 1 \end{vmatrix} \qquad L = E_{21}^{-1} = \begin{vmatrix} 1 & 0 \\ -1/2 & 1 \end{vmatrix}$$

$$m_{21} = -\frac{-1}{2} = \frac{1}{2}$$

$$U = E_{21} A = \begin{vmatrix} 1 & 0 \\ 1/2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 5/2 \end{vmatrix}$$

$$b = \begin{vmatrix} 0 \\ 5 \end{vmatrix} \qquad Ly = b \qquad \begin{matrix} y_1 = 0 \\ y_2 = 5 \end{matrix} \qquad y = \begin{vmatrix} 0 \\ 5 \end{vmatrix}$$

$Ux = y \qquad \begin{vmatrix} 2 & 1 \\ 0 & 5/2 \end{vmatrix} \begin{vmatrix} \cancel{y_1} x_1 \\ \cancel{y_2} x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 5 \end{vmatrix}$

$\frac{5}{2} y_2 = 5 \quad \cancel{y_2} x_2 = 2$

$x_1 = (0 - 1 \cdot 2)/2 = -1$

$x = \begin{vmatrix} -1 \\ 2 \end{vmatrix}$. ✓

Can Find A^{-1} by solving $AX = I$

or $(LU)X = I$

one column at a time

$Ly = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \qquad y_1 = 1$

$L = \begin{vmatrix} 1 & 0 \\ -1/2 & 1 \end{vmatrix} \qquad -\frac{1}{2} + y_2 = 0$

$y_2 = +1/2$

$y = \begin{vmatrix} 1 \\ -1/2 \end{vmatrix}$

$Ux = \begin{vmatrix} 1 \\ +1/2 \end{vmatrix} \qquad \begin{vmatrix} 2 & 1 \\ 0 & 5/2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 1 \\ +1/2 \end{vmatrix}$

$\frac{5}{2} x_2 = +\frac{1}{2} \quad x_2 = +\frac{1}{5}$

$x_1 = (1 - 1 \cdot (-\frac{1}{5}))/2 = \frac{2}{5}$

Thus First column of A^{-1} is $\begin{vmatrix} 2/5 \\ 1/5 \end{vmatrix}$

Second column $Ax = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$Ly = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $\quad L = \begin{vmatrix} 1 & 0 \\ -1/2 & 1 \end{vmatrix}$ $\quad y_1 = 0$

$y_2 = 1$

$Ux = y \quad \begin{vmatrix} 2 & 1 \\ 0 & 5/2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $\quad x_2 = 2/5$

$2x_1 + 2/5 = 0$

$x_1 = -1/5$

$$A^{-1} = \begin{vmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{vmatrix} = \frac{1}{5} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

check $AA^{-1} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \cdot \frac{1}{5} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$