Section 3.7 Matrix Inversion

In real numbers

$a \in \mathbb{R}$ $\quad a \neq 0$ $\quad \exists\, b,\ b \neq 0$

$\ni \quad a \cdot b = 1 \quad b \cdot a = 1$

Indeed $b = \frac{1}{a} = a^{-1}$

Same with square matrices of full rank

If A is $n \times n$ and rank $A = n$

$\exists$ X $n \times n$ of rank n so that

$$XA = AX = I \qquad I \ \ n \times n \text{ identity}$$

First we look into finding X so that $AX = I$

Consider Augmented Matrix $[A|I]$

do Gaussian elimination $\rightarrow [U/Z]$

Since A is full rank $U = \nabla$ upper triangular n pivots

Obtain X by back substitution $UX = Z$

$AX = I$

n pivots. no free variables

Always consistent

We do this one column at a time

Example $A = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$

$$[A \mid I] = \left|\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right| \longrightarrow \left|\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right|$$

$$m_{21} = -\frac{-1}{1} = 1$$

Let us call $b' = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $b^2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

First x corresponding to b'

$x_2 = -1$

$x_1 - (-1) = 1$ $\quad x_1 = 0$ $\quad X = \begin{vmatrix} 0 \\ -1 \end{vmatrix}$

First column

Second. x corresponding to b^2

$x_2 = -1$ $\quad X = \begin{vmatrix} -1 \\ -1 \end{vmatrix}$ second column

$x_1 - (-1) = 0$ $\quad x_1 = -1$

$$X = \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}$$

check $A \cdot X = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ ✓

We call this matrix "A inverse"
and denote it A^{-1} (just as a^{-1})

In this way
$$A A^{-1} = A^{1-1} = A^{0} = I$$

If A^{-1} exists we say that

A is <u>invertible</u> or nonsingular

Thus. A is nonsingular
$\Leftrightarrow$ rank $A = n$
$\Leftrightarrow$ n pivots
$\Leftrightarrow$ no free variables
$\Leftrightarrow$ $Ax = 0 \Rightarrow x = 0$
$\Leftrightarrow$ $Ax = b$ has a unique solution

<u>Proposition</u>. If A^{-1} exists then, it is <u>unique</u>

<u>Proof</u>. let A be $n \times n$ of rank $A = n$
let X Y be such that
$$AX = I \quad , \quad AY = I$$

Then $A(X - Y) = AX - AY = 0$

but since $Ax = 0 \Rightarrow x = 0$

Applying this to each col of $X-Y$, then $X - Y = 0$, i.e. $X = Y$ qed

~~Lemma~~

Proposition. A $n \times n$ nonsingular

if $AX = I$ then $XA = I$

Proof. First note that X must be nonsingular

otherwise $\exists\ v \neq 0$ with $Xv = 0$

but $v = Iv = AXv = A0 = 0$

$v = 0$ contradiction.

X nonsingular, then $\exists X^{-1}$ so that $XX^{-1} = I$

$$AX = I \Rightarrow AXX^{-1} = IX^{-1} = X^{-1}$$

$$AI = X^{-1} \qquad A = X^{-1}$$

multiply by $X \Rightarrow XA = XX^{-1} = I$

$XA = I$ q.e.d

Note, we just showed $(\underbrace{A^{-1}}_{X^{-1}})^{-1} = A$

Also Note $(A^2)^{-1} = (A^{-1})^2$

and $(A^{-1})^T = (A^T)^{-1}$

<u>Proposition</u> Let A, B be $n \times n$ nonsingular, then $A \cdot B$ nonsingular.

<u>Proof</u>. It suffices to exhibit $(AB)^{-1}$ so that $AB\,(AB)^{-1} = I$

Well $(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{aligned} AB\,B^{-1}A^{-1} &= A\,(B\,B^{-1})\,A^{-1} \\ &= A\,I\,A^{-1} = AA^{-1} = I \end{aligned}$$

qed.

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$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$$

$A_{n\times n}$

If A is not invertible (nonsingular)

we say that A is <u>singular</u>

$\iff \exists\, x \neq 0 \ni Ax = 0$

$\iff$ rank $A < n$

$\iff \exists$ free variables

Example $A = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix}$$

In general triangular matrix with some diagonal entry $= 0$, singular

In particular diagonal matrix $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

or $\begin{vmatrix} 1 & & \\ & 2 & \\ & & 0 \end{vmatrix}$.

Not $I^{-1} = I$ since $I \cdot I = I$