§1.2 Initial-Value Problem

When obtaining a solution of a 1^{st} order DE, we obtain a single constant (parameter).

arbitrary constant.

eg $y' = 2x$

$y = x^2 + C$ one-parameter family of solutions.

For an n-th order DE, we seek an n-parameter family of solution.

A solution to a DE that free of parameters is called a particular solution.

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n-th order Initial-Value Problem (IVP):

Solve $$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

(1) Subject to n initial conditions:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots\ y^{(n-1)}(x_0) = y_{n-1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants.

To solve (1):

Step 1: Find an n-parameter family of solution to the DE. (general solution)

Step 2: Use the initial condition(s) at x_0 to determine the n constants in the family.

Step 3: State the particular solution.

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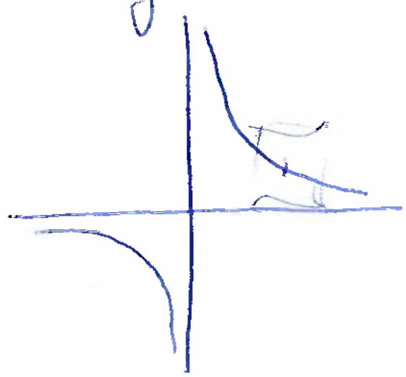
* The particular solution is defined on some interval I containing the number x_0.

eg Function

$y = \frac{1}{x}$



$D_{\frac{1}{x}} = (-\infty, 0) \cup (0, \infty)$

$x y' + x = 0$

$y = \frac{1}{x}$ is a solution of this DE on any interval not containing zero.

Use largest such interval:

$(-\infty, 0)$ OR $(0, \infty)$

Now suppose IVP $x y' + x = 0, y(2) = \frac{1}{2}$

Then $I = (0, \infty)$ since $2 \in (0, \infty)$

in

~~Example~~ 1.2.4

Example 1

$y = ce^x$ is a one-parameter family of solutions of the first order DE

$$y' = y.$$

(All solutions are defined on $(-\infty, \infty)$).

(a) Suppose $y(0) = 3$. Then

$$y = ce^x$$
$$3 = Ce^0$$
$$3 = C$$

So the particular solution is $y = 3e^x$.

(b) Suppose the curve passes thru $(1, -2)$.
i.e. $y(1) = -2$

$$-2 = Ce^1$$
$$-2 = Ce$$
$$C = -2e^{-1}$$

Then $y = -2e^{-1}e^x$

$y = -2e^{x-1}$ is the particular solution.

Example 3

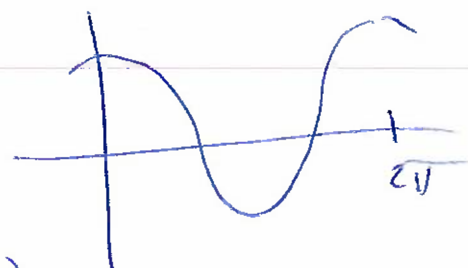
$x = c_1 \cos(4t) + c_2 \sin(4t)$ is a 2-parameter family of solutions to

$$x'' + 16x = 0 \qquad \text{(verify)}.$$

Find the solution of the initial value problem

$$x'' + 16x = 0, \quad x\left(\tfrac{\pi}{2}\right) = -2, \quad x'\left(\tfrac{\pi}{2}\right) = 1.$$

$$-2 = c_1 \cos\Big(\underbrace{4 \cdot \tfrac{\pi}{2}}_{2\pi}\Big) + c_2 \sin(2\pi)$$

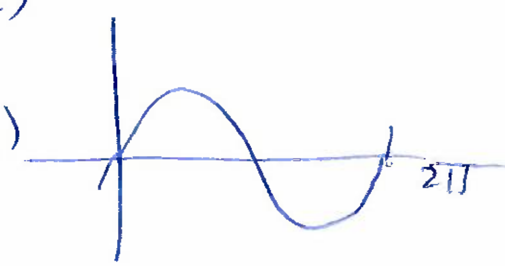


$$-2 = c_1(1) + 0$$

$$c_1 = -2$$

$$x'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

$$1 = -4c_1 \sin(2\pi) + 4c_2 \cos(2\pi)$$



$$1 = 0 + 4c_2$$

$$c_2 = \frac{1}{4}$$

Particular Solution:

$$x = -2\cos 4t + \tfrac{1}{4}\sin 4t$$

Existence & Uniqueness (refer to textbook)

Questions?

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(Example 4 - on next page)

Theorem 1.2.1 Existence of a Unique Solution

see texbook

Example 4:

$\frac{dy}{dx} = xy^{1/2}$ has two solutions that pass through the point $(0,0)$:

$$y = 0, \quad y = \frac{1}{16}x^4 \quad (\text{verify})$$

Why does this not contradict T. 1.2.1?

$f(x,y) = xy^{1/2}$ Continuous at $(0,0)$

$\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$ NOT continuous at $(0,0)$. $y \neq 0$.

So the theorem does NOT apply.

HW 18 Determine a region of the xy-plane for which the given DE would have a unique solution whose graph passes thru (x_0, y_0) in the region.

$\frac{dy}{dx} = \sqrt{xy} \implies$ $f(x,y) = \sqrt{xy}$, $\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{xy}}$ $\Big\rangle$ $xy > 0$

$xy > 0 \implies$ $x > 0$ and $y > 0$

OR $x < 0$ and $y < 0$