Section 5.9. Complementary Subspaces and Projections

Recall if $\mathcal{X}, \mathcal{Y}$ subspaces of V vector space

$$\mathcal{X} + \mathcal{Y} = \{v \in V \,/\, v = x + y,\ x \in \mathcal{X}, y \in \mathcal{Y}\}$$

is a subspace

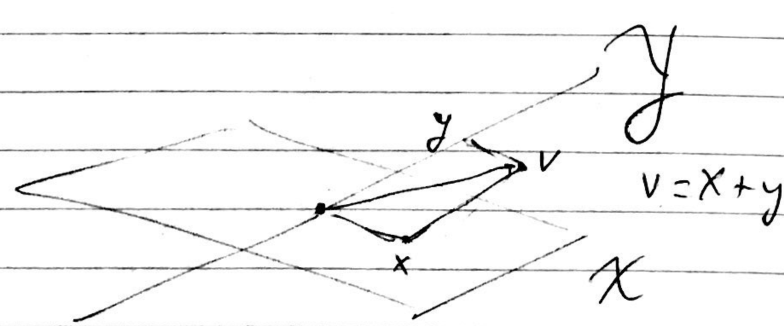
Now, if in addition

(i) $\mathcal{X} + \mathcal{Y} = V$ (the whole space)

and (ii) $\mathcal{X} \cap \mathcal{Y} = \{0\}$

Then we say that $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces, and denoted $\mathcal{X} \oplus \mathcal{Y}$.

Example $\mathcal{X}$ ~~line~~ plane in $\mathbb{R}^3$ and $\mathcal{Y}$ line (not contained in $\mathcal{X}$).



$$\left.\begin{array}{l}\mathcal{X} + \mathcal{Y} = V \\ \mathcal{X} \cap \mathcal{Y} = \{0\}\end{array}\right. \Rightarrow \dim \mathcal{X} + \dim \mathcal{Y} = \dim V \ (= n)$$

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Lemma. If $X, Y \subset V$ are complementary subspaces. Then every $v \in V$ can be written as $v = x + y$, $x \in X$, $y \in Y$ in a unique manner.

Proof. First. by definition of complementary subspaces

$$V = X + Y$$

Therefore every $v \in V$ can be written as

$$v = x + y$$
$$x \in X, \; y \in Y.$$

To show uniqueness, consider two such decompositions

$$v = x_1 + y_1 = x_2 + y_2$$

Thus $\underbrace{x_1 - x_2}_{\in X} = \underbrace{y_2 - y_1}_{\in Y}$

(subspaces)

so that $\begin{matrix} x_1 - x_2 \\ y_2 - y_1 \end{matrix} \in X \cap Y = \{0\}$.

$\Rightarrow \; x_1 = x_2, \quad y_2 = y_1$ q.e.d.

This is also equivalent to say

that if $B_x = \{v_1, v_2 \ldots v_k\}$

and $B_y = \{w_1, w_2 \ldots w_m\}$

then the bases do not have any vector in common, i.e. $B_x \cap B_y = \phi$

Thus $v \in V$, $v = \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j$

and α_i, β_j are unique.

Let X, Y be complementary subspaces.

given $v \in V$ $\quad v = x + y$ uniquely.

Define $P_x : V \to V \qquad P_x v = x$
$\qquad\quad P_y : V \to V \qquad P_y v = y = (I - P_x) y$

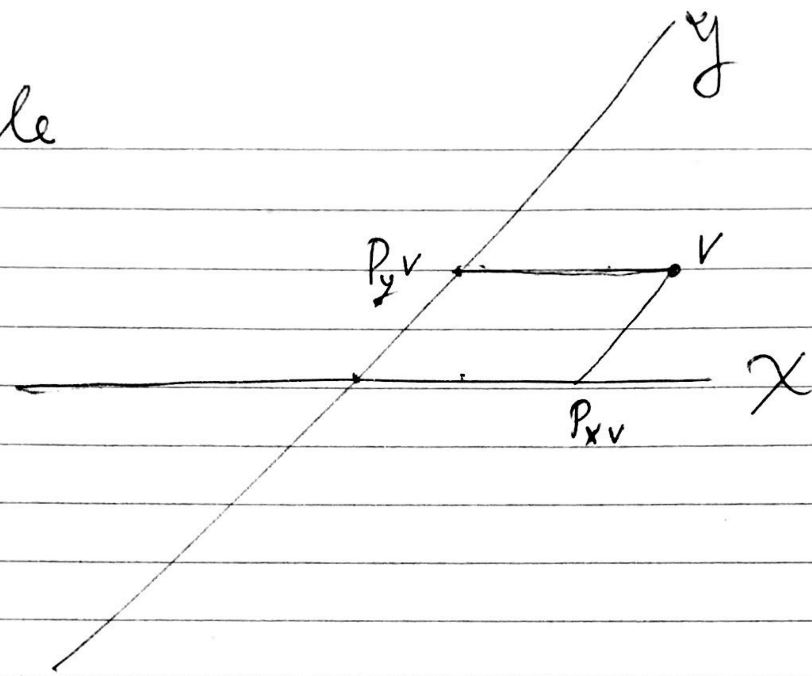
First of all P_x, P_y are well defined.

Second $\quad v = P_x v + P_y v$ for all $v \in V$
so that $P_x + P_y = I$, i.e. $P_y = I - P_x$
and $P_x = I - P_y$

these are called <u>projections</u>.

P_x P_y <u>complementary projections</u>

Example



$$\mathcal{X} = \left\{ v \in V \;/\; v = \alpha \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \; \alpha \in \mathbb{R} \right\}$$

$$\mathcal{Y} = \left\{ v \in V \;/\; v = \alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \; \alpha \in \mathbb{R} \right\}$$

$P_{\mathcal{X}}$ projection onto $\mathcal{X}$ along $\mathcal{Y}$

$P_{\mathcal{Y}}$ projection onto $\mathcal{Y}$ along $\mathcal{X}$

In general if X has colums with a basis of $\mathcal{X}$ and Y has colums with a basis of $\mathcal{Y}$

then $P = P_X = [X \mid 0][X \mid Y]^{-1}$

Why does this inverse exist?

and $Q = I - P = I - [X\ 0][X\ Y]^{-1}$

$= [X\ Y][X\ Y]^{-1} - [X\ 0][X\ Y]^{-1} =$

$= ([X\ Y] - [X\ 0])[X\ Y]^{-1}$

$= [0 \mid Y][X \mid Y]^{-1}$

Note $\quad R(P_X) = \mathcal{X} \qquad R(P_Y) = \mathcal{Y}$

Also: $\quad P_X(P_X v) = P_X v \quad \forall v \in V$

$\Rightarrow P^2 = P$

$P_X y = 0 \quad \forall y \in \mathcal{Y}.$

$P^2 = P$, P is a projection onto $R(P)\ (= \mathcal{X})$ along $N(P)\ (= \mathcal{Y})$

$Q = I - P$ a projection onto $R(Q) = N(P)$ along $N(Q) = R(P)$

This implies

$$Q^2 = (I-P)(I-P) = I - 2P + P^2$$
$$= I - 2P + P = I$$

$$Q = I - P$$
$$\Rightarrow \quad PQ = P - P^2 = 0 \qquad PQ = 0$$

$$Q^2 = Q = Q - QP \Rightarrow QP = 0$$

For the example $X = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $Y = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$$[X\,Y] = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \qquad [X\,Y]^{-1} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$P = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$P \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} v_1 - v_2 \\ 0 \end{vmatrix} \in \mathcal{X} = \alpha \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$Q = I - P = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \qquad Q \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} v_2 \\ v_2 \end{vmatrix} \in \mathcal{Y}$$

Note $P \begin{vmatrix} 1 \\ 1 \end{vmatrix} = 0 \quad P \begin{vmatrix} \alpha \\ \alpha \end{vmatrix} = 0$

$$P \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \qquad P \begin{vmatrix} \alpha \\ 0 \end{vmatrix} = \begin{vmatrix} \alpha \\ 0 \end{vmatrix}$$

$$Q \begin{vmatrix} \alpha \\ \alpha \end{vmatrix} = \begin{vmatrix} \alpha \\ \alpha \end{vmatrix} \qquad Q \begin{vmatrix} \alpha \\ 0 \end{vmatrix} = 0$$

$$P^2 = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = P$$

then $P = P_x = [X \,|\, 0]\,[X \,|\, Y]^{-1}$

Why does this inverse exist?

and $Q = I - P = I - [X\; 0]\,[X\; Y]^{-1}$

$= [X\; Y]\,[X\; Y]^{-1} - [X\; 0]\,[X\; Y]^{-1} =$

$= ([X\; Y] - [X\; 0])\,[X\; Y]^{-1}$

$= [0 \,|\, Y]\,[X \,|\, Y]^{-1}$

<u>Note</u> $R(P_x) = \mathcal{X}$ $\quad R(P_y) = \mathcal{Y}$

<u>Also:</u> $P_x(P_x v) = P_x v \quad \forall v \in V$

$\Rightarrow P^2 = P$

$P_x y = 0 \quad \forall y \in \mathcal{Y}.$

$P^2 = P$, P is a projection onto

$R(P)\; (= \mathcal{X})$ along $N(P)\; (= \mathcal{Y})$

$Q = I - P$ a projection onto

$R(Q) = N(P)$ along $N(Q) = R(P)$

This implies

$$Q^2 = (I-P)(I-P) = I - 2P + P^2$$
$$= I - 2P + P = I$$

$$Q = I - P$$
$$\Rightarrow \quad PQ = P - P^2 = 0 \qquad PQ = 0$$

$$Q^2 = Q = Q - QP \Rightarrow QP = 0$$

For the example $X = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $Y = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$$[X\,Y] = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \qquad [X\,Y]^{-1} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$P = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$P\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} v_1 - v_2 \\ 0 \end{vmatrix} \in \mathcal{X} = \alpha\begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$Q = I - P = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \qquad Q\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} v_2 \\ v_2 \end{vmatrix} \in \mathcal{Y}$$

Note $P\begin{vmatrix} 1 \\ 1 \end{vmatrix} = 0 \quad P\begin{vmatrix} \alpha \\ \alpha \end{vmatrix} = 0$

$$P\begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \qquad P\begin{vmatrix} \alpha \\ 0 \end{vmatrix} = \begin{vmatrix} \alpha \\ 0 \end{vmatrix}$$

$$Q\begin{vmatrix} \alpha \\ \alpha \end{vmatrix} = \begin{vmatrix} \alpha \\ \alpha \end{vmatrix} \qquad Q\begin{vmatrix} \alpha \\ 0 \end{vmatrix} = 0$$

$$P^2 = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = P$$

$$Q^2 = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = Q$$

$$PQ = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

$$QP = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

Note. Since $P^2 = P$

$$\|P\| = \|P^2\| = \|P\,P\| \leq \|P\|\,\|P\|$$

$\Rightarrow$ if $P \neq 0$ (ie. if P is a projection onto a proper space)

then $\|P\| \geq 1$.

Similarly $\|I - P\| = \|Q\| \geq 1$

Alternative formula for P, Q.

Let $U = X$ matrix whose columns are B_X

Let V (of the same order as U) be a matrix whose columns are a basis of $Y^\perp$

Then $P = U(V^T U)^{-1} V^T$

Note $P^2 = U(V^T U)^{-1} \underbrace{V^T U (V^T U)^{-1}}_{I} V^T = P$

& $R(P) = R(U) = X$

if $y \in Y$. $V^T y = 0$ $\quad Py = 0$

if $x \in X$ $\quad x = Uw$ for some w

$Px = PUw = U \underbrace{(V^T U)^{-1} V^T U}_{I} w = Uw$
$= x$

in our example $U = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $\quad Y = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $\quad V = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

$V^T U = 1 \quad (V^T U)^{-1} = 1$

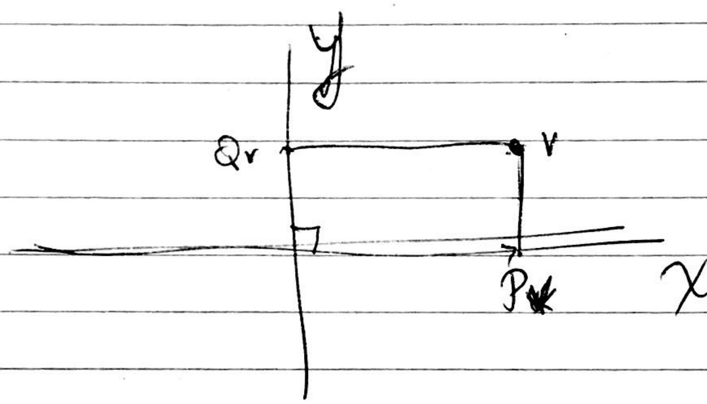
$P = UV^T = \begin{vmatrix} 1 \\ 0 \end{vmatrix} (1 \ -1) = \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$ ✓

<u>Special case.</u> if $X \perp Y$

then $V = U$

$P = U(U^T U)^{-1} U^T$ <u>symmetric</u>

In this case the projection is called an <u>orthogonal projection</u>



if $X \not\perp Y$. P is an oblique projection

Because of Pythagoras.

if $X \perp Y$. $\|Pv\| \le \|v\|$

$$\Rightarrow \quad \|P\| \le 1$$

$$\Rightarrow \quad \|P\| = 1$$

P orthogonal projection $\Leftrightarrow P^T = P \Leftrightarrow \|P\| = 1$.
(see Exercise 5.9.8)

Repeat. $\|P\| \leq 1$ in general

$\|P\| = 1$ orthogonal projection

$\|P\| < 1$ oblique projection.

Beatiful result. If $P \neq 0$ $P \neq I$

then $\|I - P\|_2 = \|P\|_2$

this is exercise 5.9.9 (homework 13)