Solution: Apply Gaussian elimination to the augmented matrix $[\mathbf{A}|\mathbf{b}]$ as shown:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 4 & 3 & 1 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Because a row of the form $(0\ \ 0\ \ \cdots\ \ 0\ \mid\ \alpha)$ with $\alpha \neq 0$ never emerges, the system is consistent. We might also observe that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ so that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. Finally, by completely reducing $\mathbf{A}$ to $\mathbf{E_A}$, it is possible to verify that $\mathbf{b}$ is indeed a combination of the basic columns $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*5}\}$.

Exercises for section 2.3

2.3.1. Determine which of the following systems are consistent.

(a) $\begin{aligned} x + 2y + z &= 2, \\ 2x + 4y \quad\ &= 2, \\ 3x + 6y + z &= 4. \end{aligned}$ (b) $\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x + 2y + 5z &= 0, \\ 4x + 2y + 6z &= 0. \end{aligned}$

(c) $\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 4. \end{aligned}$ (d) $\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 2. \end{aligned}$

(e) $\begin{aligned} 2w + x + 3y + 5z &= 1, \\ 4w + \quad\ \ 4y + 8z &= 0, \\ w + x + 2y + 3z &= 0, \\ x + \ y + \ z &= 0. \end{aligned}$ (f) $\begin{aligned} 2w + x + 3y + 5z &= 7, \\ 4w + \quad\ \ 4y + 8z &= 8, \\ w + x + 2y + 3z &= 5, \\ x + \ y + \ z &= 3. \end{aligned}$

2.3.2. Construct a 3×4 matrix $\mathbf{A}$ and 3×1 columns $\mathbf{b}$ and $\mathbf{c}$ such that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix for an inconsistent system, but $[\mathbf{A}|\mathbf{c}]$ is the augmented matrix for a consistent system.

2.3.3. If $\mathbf{A}$ is an $m \times n$ matrix with $rank\,(\mathbf{A}) = m$, explain why the system $[\mathbf{A}|\mathbf{b}]$ must be consistent for every right-hand side $\mathbf{b}$.

2.3.4. Consider two consistent systems whose augmented matrices are of the form $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{A}|\mathbf{c}]$. That is, they differ only on the right-hand side. Is the system associated with $[\mathbf{A} \mid \mathbf{b}+\mathbf{c}]$ also consistent? Explain why.

2.3.5. Is it possible for a parabola whose equation has the form $y = \alpha+\beta x+\gamma x^2$ to pass through the four points $(0,1)$, $(1,3)$, $(2,15)$, and $(3,37)$? Why?

2.3.6. Consider using floating-point arithmetic (without scaling) to solve the following system:

$$\begin{aligned} .835x + .667y &= .168, \\ .333x + .266y &= .067. \end{aligned}$$

(a) Is the system consistent when 5-digit arithmetic is used?
(b) What happens when 6-digit arithmetic is used?

2.3.7. In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market— say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$. One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen. Determine whether or not it is possible to meet exactly the recommendation by applying some combination of the three brands of fertilizer.

2.3.8. Suppose that an augmented matrix $[\mathbf{A}|\mathbf{b}]$ is reduced by means of Gaussian elimination to a row echelon form $[\mathbf{E}|\mathbf{c}]$. If a row of the form

$$(0 \quad 0 \quad \cdots \quad 0 \mid \alpha), \quad \alpha \neq 0$$

does not appear in $[\mathbf{E}|\mathbf{c}]$, is it possible that rows of this form could have appeared at earlier stages in the reduction process? Why?