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Linear Algebra 2101 (1)

Homework # 8

Exercise 4.2.5. A $n \times n$

(a) If $R(A) = \mathbb{R}^n$, explain why A must be nonsingular.

$R(A) \{y \,/\, Ax = y\} = \mathbb{R}^n$

this means that for every right hand side b

$Ax = b$ has a solution

(it is always consistent)

Rank $A = n$ (n pivots)

$\Rightarrow A$ nonsingular

$Ax = b$ has unique solution

In particular $Ax = 0$ has only $x = 0$ solution

i.e. $N(A) = \{0\}$

(b) Four fundamental Subspaces

$R(A) = \mathbb{R}^n$ $N(A) = \{0\}$, A nonsingular, A^T also

A^T nonsingular

$A^T x = b$ has always a unique solution

$\Rightarrow R(A^T) = \mathbb{R}^n$

$N(A^T) = \{0\}$ since $A^T x = 0$
$\Rightarrow x = 0$

2.(a) Prove that $R(AB) \subseteq R(A)$

$R(A) = \{ y / \exists x \text{ with } Ax = y \}$

$R(AB) = \{ y / \exists w \text{ with } ABw = y \}$

Thus if $y \in R(AB)$, $\exists w$ with $ABw = y$

let $x = Bw$, then $Ax = ABw = y$
$\Rightarrow y \in R(A)$

so that $R(AB) \subseteq R(A)$.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$R(A)$ = linear combination of the columns
= linear combination of basic columns

LU or Gaussian elimination

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two basic columns (Rank 2)

basic colums $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$R(A) = \left\{ x = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \alpha_1\ \alpha_2 \in \mathbb{R}$$

for $R(AB)$. Compute

$$A \cdot B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 2 \\ 0 & 6 \end{bmatrix}$$

$$R(AB) = \left\{ \gamma \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix},\ \gamma \in \mathbb{R} \right\}$$

$$\begin{vmatrix} 4 \\ 2 \\ 6 \end{vmatrix} = 2 \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} + 2 \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

so that $R(AB) \subsetneq R(A)$

$R(AB)$ subspace of $R(A)$

(in this example a proper subspace)