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Linear Algebra 2101 1

Exam 2 - Responses

1. $S = \{ x \in \mathbb{R}^3 \mid 2x_1 + 3x_2 + 2x_3 = 0 \}$

a. S subspace. Let $v \in S$, $w \in S$
wish to show that $\alpha v + w \in S$

$v \in S \Rightarrow 2v_1 + 3v_2 + 2v_3 = 0$

$w \in S \Rightarrow 2w_1 + 3w_2 + 2w_3 = 0$

Consider $\alpha v + w$

$$2(\alpha v + w)_1 + 3(\alpha v + w)_2 + 2(\alpha v + w)_3 =$$

$$= \alpha 2 v_1 + 2w_1 + \alpha 3 v_2 + 3w_2 + \alpha 2 v_3 + 2w_3$$

$$= \alpha(2v_1 + 3v_2 + 2v_3) + (2w_1 + 3w_2 + 2w_3)$$

$$= \alpha . 0 + 0 = 0 \quad \checkmark$$

$$\Rightarrow \alpha v + w \in S.$$

b. It suffices to show a vector $z \in \mathbb{R}^3$ $z \notin S$
take For example $z = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

then $2z_1 + 3z_2 + 2z_3 = 2.1 + 3.1 + 2.1 = 7 \neq 0$
$z \in S$.

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2. S a subspace. Let $S^\perp = \{ x / x^T y = 0 \ \forall y \in S \}$

take $v, w \in S^\perp$, what about $\alpha v + w$?

$(\alpha v + w)^T y = \alpha v^T y + w^T y = \alpha . 0 + 0 = 0$

$\Rightarrow \alpha v + w \in S^\perp$, $S^\perp$ subspace.

3. (a) $A + A = 2A$.

A nonsingular, $\exists A^{-1}$

$$(2A)\left(\tfrac{1}{2}A^{-1}\right) = I$$

$2A$ nonsingular. TRUE

b) $(AB)^T = B^T A^T$ TRUE

(c) $A = LU$ $Ux = 0$ $Ax = LUx = L.0 = 0$

TRUE

(d) $A = LU$. $Ax = 0$ since L is nonsingular (lower unit triangular) then L^{-1} exists.

$L^{-1}Ax = Ux = L^{-1}0 = 0$ TRUE

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4 (a)

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -10 \end{vmatrix} \to \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -12 \end{vmatrix} \to \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{vmatrix}$$

$m_{21} = -1$ $m_{32} = -4$

$m_{31} = -1$

$$L = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix} \quad U = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{vmatrix}$$

(b) two basic columns 1st and 2nd

$$R(A) = \text{span}\left\{ \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix} \begin{vmatrix} 3 \\ 2 \\ -1 \end{vmatrix} \right\}$$

4.(c) x_3 free variable

$$x_2 = -3x_3$$

$$2x_1 + 3(-3x_3) + 2x_3 = 0$$

$$2x_1 = 7x_3$$

$$x_1 = \frac{7}{2}x_3$$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 7/2\, x_3 \\ -3\, x_3 \\ x_3 \end{vmatrix} = x_3 \begin{vmatrix} 7/2 \\ -3 \\ 1 \end{vmatrix}$$

check $Ax = 0$ ✓

(d)

$c \in R(A)$ if

$$c = \alpha_1 \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix} + \alpha_2 \begin{vmatrix} 3 \\ 2 \\ -1 \end{vmatrix}$$

i.e. $\begin{vmatrix} 2 & 3 \\ 2 & 2 \\ 2 & -1 \end{vmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{vmatrix} 1 \\ -1 \\ \alpha \end{vmatrix}$

$\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} \to \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = U \qquad L = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$m = -1 \qquad Ly = \begin{vmatrix} 1 \\ -1 \end{vmatrix} \qquad y_1 = 1$

$y_2 = -2$

$Ux = \begin{vmatrix} 1 \\ -2 \end{vmatrix} \qquad x_2 = 2 \quad x_1 = (1 - 3.2)/2 = \frac{-5}{2}$

check ✓

(4 d) cont.

then $2 \cdot \frac{-5}{2} + (-1) \cdot 2 = \alpha$

$\alpha = -7$.

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5. As before $m_{21} = -1$ $m_{31} = -1$ $m_{32} = -4$

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} \to \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -1 \end{vmatrix} \to \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 11 \end{vmatrix}$$

a) Rank $A = 3$, 3 pivots, 3 nonzero in diag(U)

(b) A nonsingular because
(i) A is 3×3 - 3 pivots.
(ii) No free variables
(iii) $N(A) = \{0\}$, i.e. $Ax = 0 \Rightarrow x = 0$
(iv) $\forall b \in \mathbb{R}^3$ $Ax = b$ has a unique solution

(c) $Ly = b$ $L = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}$ $b = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$

$y_1 = 1$
$y_2 = -2$
$y_3 = 8$

5(c) cont.

$$Ux = y \qquad U = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 11 \end{vmatrix} \qquad y = \begin{vmatrix} 1 \\ -2 \\ 8 \end{vmatrix}$$

$$x_3 = 8/11$$

$$x_2 = \left(-2 - (-3) * \frac{8}{11}\right) / (-1) = -2/11$$

$$x_1 = \left(1 - 2 * \frac{8}{11} - 3\left(-\frac{2}{11}\right)\right) / 2 = 1/22$$

$$x = \frac{1}{22} \begin{vmatrix} 1 \\ -4 \\ 16 \end{vmatrix} \qquad \text{check } A \cdot x = b \checkmark$$

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(6)

$$B = \begin{vmatrix} 1 & 3/2 \\ a & 1/2 \end{vmatrix} \to \begin{vmatrix} 1 & 3/2 \\ 0 & 1/2 - 3/2\, a \end{vmatrix}$$

$$m_{21} = -a$$

for B singular $\quad \frac{1}{2} - \frac{3}{2} a = 0$

$$\Rightarrow a = \frac{1}{3}$$

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7. (a) $S = \dfrac{A + A^T}{2}$

$$S^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = S$$

$$K = \frac{A - A^T}{2}$$

$$K^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -K$$

$$K + S = \frac{1}{2}(A + A^T + A - A^T) = \frac{1}{2} 2A = A$$

(b) $A = S_1 + K_1 \quad S_1 = S_1^T \quad K_1^T = -K_1$

$$A^T = S_1^T + K_1^T = S_1 - K_1$$

$$A + A^T = 2S_1 \Rightarrow S_1 = (A + A^T)/2 = S$$

$$A - A^T = 2K_1 \Rightarrow K_1 = (A - A^T)/2 = K$$

8 (a). A has rank 1 so that

$A = uv^T$ for some $u, v \in \mathbb{R}^n$

$$A^T = vu^T = A = uv^T$$

If $v \neq u$ $A^T \neq A$ thus $v = u$

$A = uu^T$

(b) $A = uu^T \quad a_{ii} = (u_i)^2 \geq 0$

and if $u_i = 0$ $\forall i$ $A = 0$ rank 0.

9. if A has rank 1

$$A = u v^T$$
$$A^T = v u^T = - u v^T \quad \text{(skew sym.)}$$

thus $a_{ij} = u_i v_j = -a_{ji}$
$= -u_j v_i$

$\Rightarrow u_i v_i = 0$ for all i

If $u_i = 0$ then the ith row of A is a zero row

and since $A^T = -A$

the ith column is a zero column

that is $v_i = 0$.

this is for all $i \Rightarrow A = 0$

rank $A \neq 1$.

10. $P \neq I \quad P^2 = P \Rightarrow P$ singular

Proof #1. Assume P nonsingular (and $P \neq I$)

Then $P^{-1} P^2 = P^{-1} P$ i.e. $P = I$

contradiction

$\Rightarrow P$ singular

Proof #2 $\quad P^2 = P \Rightarrow P^2 - P = P(I - P) = 0$

since $I \neq P$ $\exists$ a vector $\neq 0$

so that $Pv = 0 \Rightarrow P$ singular