Sections 5.1 and 5.3 - Norms, inner products, orthogonality

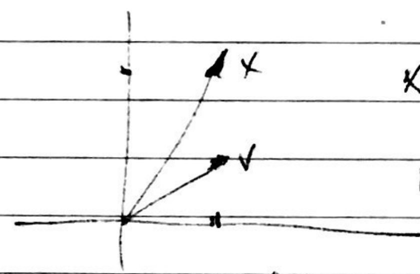
We want to consider length and angles.

for length in $\mathbb{R}^n$ for example

$$\|x\| = \sqrt{x^T x} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$v = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$ $\quad$ $x = \begin{vmatrix} 2 \\ 3 \end{vmatrix}$

$\|v\| = \sqrt{2^2+1^2} = \sqrt{5}$ $\quad$ $\|x\| = \sqrt{2^2+3^2} = \sqrt{13}$



this is one way of measuring length.
It is the Euclidean norm (or 2-norm)

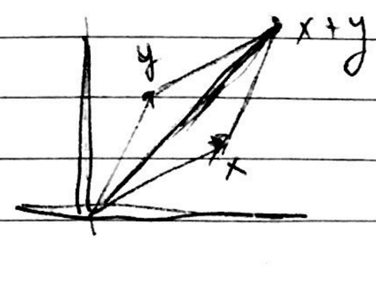
General Definition of a Norm for a vector space V

$\|\ \|: V \to \mathbb{R}$ satisfying

(1) $\|x\| \geq 0$ $\quad$ (2) $\|x\| = 0 \Rightarrow x = 0$

(3) $\|\alpha x\| = |\alpha|\,\|x\|$

(4) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)



This is a norm for any vector space

($\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, Π_n, $\mathcal{C}([a,b])$) ...

Other examples of norms in $\mathbb{R}^n$

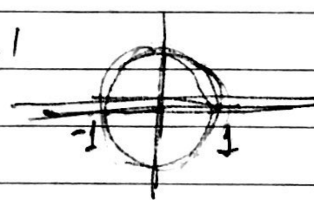
$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p} \qquad 1 \le p \le 2$$

Unit circle (in $\mathbb{R}^2$), or unit sphere
set of all vectors v with $\|v\| = 1$

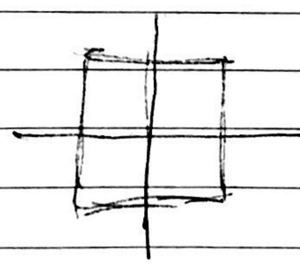
for $\|v\|_2 = 1$



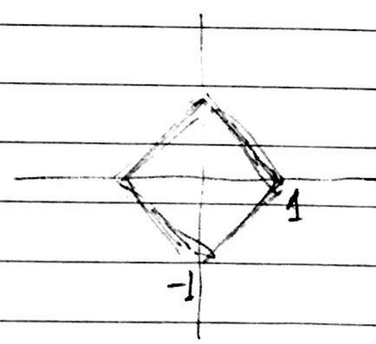
$\begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $\begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $\begin{vmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{vmatrix}$

$\begin{vmatrix} \sin\theta \\ \cos\theta \end{vmatrix}$

$\|v\|_\infty = 1$



$\|v\|_1 = 1$



Inner product between two vectors $x, y \in V$

$(x,y) : V \times V \to \mathbb{R}$ | in book $\langle x | y \rangle$

(i) bilinear (linear in each argument)

(ii) $(x,x) \geq 0$ and $(x,x) = 0 \Rightarrow x = 0$

(iii) Symmetric $(x,y) = (y,x)$

(i) means $(u + \alpha v, y) = (u,y) + \alpha (v,y)$

Standard inner product

$$(x,y) = x^T y$$

so $\|x\|_2 = \sqrt{x^T x} = \sqrt{(x,x)}$

For any inner product there is a norm induced by it.

$$\|x\| = \sqrt{(x,x)}$$

Not $\|\alpha x\| = \sqrt{(\alpha x, \alpha x)} = \sqrt{\alpha^2 (x,x)} = |\alpha| \sqrt{(x,x)}$
$= |\alpha| \, \|x\|$

Cauchy - Bunyakovskii - Schwarz inequality
aka Cauchy - Schwarz inequality (or CBS ineq.)

$$|(x,y)| \leq \|x\| \, \|y\|$$

(where $\| \ \|$ are the norms induced by the inner product)

Proof If $x=0$ $(x,y)=0$ $\|x\|=0$
inequality holds

For $x \neq 0$ define $\alpha = (x,y)/\|x\|^2 = (x,y)/(x,x)$

and consider $v = \alpha x - y$

Note that $(x,v) = (x, \alpha x - y) = 0$

since $(x, \alpha x - y) = \alpha(x,x) - (x,y) = 0$

Now compute $\|v\|^2 \geq 0$

$$0 \leq \|\alpha x - y\|^2 = (\alpha x - y, \alpha x - y)$$

$$= \underbrace{\alpha(x, \alpha x - y)}_{=0} - (y, \alpha x - y) =$$

$$= (y,y) - \alpha(y,x) = \|y\|^2 - \frac{(x,y)(y,x)}{\|x\|^2}$$

Using the symmetry and taking common denominator, we have

$$0 \le \frac{\|x\|^2\|y\|^2 - (x,y)^2}{\|x\|^2} \Rightarrow |(x,y)| \le \|x\| \, \|y\|$$

q.e.d.

We can use CBS ineq. to show that the Euclidean norm (or any induced norm) satisfies Δ inequality
triangle

$$\|x+y\|^2 = (x+y, x+y) =$$

$$(x,x) + 2(x,y) + (y,y)$$

$$\le \|x\|^2 + 2|(x,y)| + \|y\|^2$$

$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

qed

More examples. $V = \Pi_n$

$$(p, q) = \int_{-\infty}^{\infty} p(x)\, q(x)\, dx$$

(or some other limit in the integral).

induced Norm. $$\|p\|_2 = \left(\int p^2(x)\, dx \right)^{1/2}$$

$$V = \mathbb{R}^{m \times n}$$

$$(A, B) = \operatorname{trace}(A^T B)$$

$$\|A\|_F = \sqrt{\operatorname{trace}(A^T A)} = \left(\sum_{i,l} a_{il}^2 \right)^{1/2}$$

$$(A^T A)_{ij} = \sum_{l=1}^{m} a_{li}\, a_{lj}\, , \quad (A^T A)_{ii} = \sum_{l=1}^{m} a_{li}^2$$

$$\operatorname{trace}(A^T A) = \sum_{i=1}^{n} \sum_{l=1}^{m} a_{li}^2$$

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{l=1}^{m} a_{li}^2 \right)^{1/2}$$