Linear Algebra - Math 2101

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Exercise 1.2.3 Apply Gaussian elimination with back substitution to the following system:

$$\begin{aligned} 4x_2 - 3x_3 &= 3 \\ -x_1 + 7x_2 - 5x_3 &= 4 \\ -x_1 + 8x_2 - 6x_3 &= 5 \end{aligned}$$

$$[A|b] = \left[\begin{array}{ccc|c} 0 & 4 & -3 & 3 \\ -1 & 7 & -5 & 4 \\ -1 & 8 & -6 & 5 \end{array}\right] \xrightarrow{\text{row interchange}}$$

$$\left[\begin{array}{ccc|c} -1 & 7 & -5 & 4 \\ 0 & 4 & -3 & 3 \\ -1 & 8 & -6 & 5 \end{array}\right] \xrightarrow{A'_{3*} \leftarrow A_{3*} - A_{1*}} \left[\begin{array}{ccc|c} -1 & 7 & -5 & 4 \\ 0 & 4 & -3 & 3 \\ 0 & 1 & -1 & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow{A''_{3*} \leftarrow A'_{3*} - \frac{1}{4}A'_{2*}} \left[\begin{array}{ccc|c} -1 & 7 & -5 & 4 \\ 0 & 4 & -3 & 3 \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{array}\right] = [T \mid c]$$

$x_3 = (1/4)/(-1/4) = -1$; $x_2 = [3 - (-3)\times(-1)]/4 = 0$

$x_1 = (4 - 7\times 0 - (-5)\times(-1))/(-1) = 1$

Thus $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Check: $4\times 0 - 3\times(-1) = 3$

$-1\times 1 + 7\times 0 - 5\times(-1) = -1+5 = 4$

$-1\times 1 + 8\times 0 - 6\times(-1) = -1+6 = 5$ ✓

Exercise 1.2.5. Three Linear Systems

$$\begin{array}{l|l|l} 4x - 8y + 5z = 1 & 0 & 0 \\ 4x - 7y + 4z = 0 & 1 & 0 \\ 3x - 4y + 2z = 0 & 0 & 1 \end{array}$$

Solve all three systems at one time using ~~the~~ an augmented system of the form $[A \mid b_1 \mid b_2 \mid b_3]$

$$\left[\begin{array}{ccc|c|c|c} 4 & -8 & 5 & 1 & 0 & 0 \\ 4 & -7 & 4 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} m_{21} = -1 \\ m_{31} = -3/4 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c|c|c} 4 & -8 & 5 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -7/4 & -3/4 & 0 & 1 \end{array}\right] \qquad m_{32} = -2$$

$$\rightarrow \left[\begin{array}{ccc|c|c|c} 4 & -8 & 5 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1/4 & 5/4 & -2 & 1 \end{array}\right]$$

First linear system

$$x_3 = (5/4)/(1/4) = 5$$

$$x_2 = (-1 - (-1) \times 5)/1 = 4$$

$$x_3 = (1 - (-8) \times 4) - 5 \times 5) / 4 = (1 + 32 - 25)/4 = 2$$

So, $x = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

check, e.g. $4 \times 2 - 8 \times 4 + 5 \times 5 = 8 - 32 + 25 = 1$

$4 \times 2 - 7 \times 4 + 4 \times 5 = 8 - 28 + 20 = 0$

Second linear system

$$x_3 = -2/(1/4) = -8$$

$$x_2 = (1 - (-1) \times (-8))/1 = -7$$

$$x_1 = (0 - (-8) \times (-7) - 5 \times (-8))/4 = (-56 + 40)/4 = -16/4 = -4$$

$$x = \begin{bmatrix} -4 \\ -7 \\ -8 \end{bmatrix}$$

check. e.g. $4 \times (-4) - 8 \times (-7) + 5 \times (-8)$

$= -16 + 56 - 40 = 0$

$4 \times (-4) - 7 \times (-7) + 4 \times (-8) = -16 + 49 - 32 = 1$

third linear system

$$x_3 = 1/(1/4) = 4$$

$$x_2 = \left(0 - (-1) \times 4\right)/1 = +4$$

$$x_1 = \left(0 - (-8) \times 4 - 5 \times 4\right)/4 =$$
$$(32 - 20)/4 = 12/4 = 3$$

$$x = \begin{vmatrix} 3 \\ 4 \\ 4 \end{vmatrix}$$

check. e.g.

$$4 \times 3 \quad -7 \times 4 + 4 \times 4 = 12 - 28 + 16 = 0$$
$$3 \times 3 \quad -4 \times 4 + 2 \times 4 = 9 - 16 + 8 = 1$$

Ex 2.1.1. Reduce each of the following matrices to row echelon form, determine the rank, and identify the basic columns

(a)
$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix} \xrightarrow[m_{31} = -2]{m_{21} = -2} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow$$

row interchange
$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Rank is 3

basic colums 1, 2, 4

i.e.. A_{*1}, A_{*2}, A_{*4}

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{bmatrix} \xrightarrow[\substack{m_{21}=-2 \\ m_{31}=-2 \\ m_{41}=-1 \\ m_{51}=-3}]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & -6 \\ 0 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$

$$\xrightarrow[\substack{m_{32}=1 \\ \cancel{m_{42}=1} \\ m_{52}=1}]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 3 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow[m_{45} = -\frac{-5}{-8} = \frac{-5}{8}]{m_{43} = \frac{-2}{-8} = +1/4} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 3 basic columns 1, 2, 3

(c)

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{bmatrix}$$

$$\begin{array}{r} \\ m=-1 \\ \\ m=-3 \\ \\ m=-4 \end{array} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 0 & 0 & 2 & -2 & 1 & -3 & 3 \\ 0 & 0 & 2 & -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 & -3 & 2 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 0 & 0 & -2 & 2 & 1 & -3 & -1 \end{bmatrix}$$

$$
\begin{array}{r}
\\
\\
\rightarrow \\
m=-1/2 \\
m=-3/2 \\
m=+2
\end{array}
\begin{bmatrix}
2 & 1 & 1 & 3 & 0 & 4 & 1 \\
0 & 0 & 2 & -2 & 1 & -3 & 3 \\
0 & 0 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 1/2 & -3/2 & 1/2 \\
0 & 0 & 0 & 0 & -3/2 & 9/2 & -3/2 \\
0 & 0 & 0 & 0 & 2 & -6 & 2
\end{bmatrix}
$$

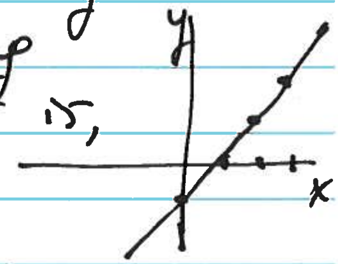
$$
\begin{array}{r}
\\
\rightarrow \\
\\
m=+1/2 \\
m=-3/2 \\
m=+2
\end{array}
\begin{bmatrix}
2 & 1 & 1 & 3 & 0 & 4 & 1 \\
0 & 0 & 2 & -2 & 1 & -3 & 3 \\
0 & 0 & 0 & 0 & -1 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Rank $A = 3$

basic columns A_{*1}, A_{*3}, A_{*5}

4. 2×2 system with solutions $(2,1)$ and $(3,2)$

This means we have infinitely many solutions, so the whole line passing through $(2,1)$ and $(3,2)$, that is, the line $y = x - 1$



So a possible linear system can be

$$x_1 - x_2 = 1$$
$$2x_1 - 2x_2 = 2$$

or
$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Of course there are many other linear systems with these two solutions. (but all equations are multiples of these).

(There is also an algebraic way to find the six unknowns a_{11} a_{12} a_{21} a_{22} b_1 b_2. We will discuss this later on).