§ 5.1.2 Spring/Mass Systems: Free Damped Motion

A mass in a perfect vacuum is unrealistic.

Now assume there is a damping force acting on the mass proportional to the velocity

$$m x'' = -kx - \beta x'$$

So $\quad m x'' + \beta x' + kx = 0$

AE: $\quad m r^2 + \beta r + k = 0$

$$r_1, r_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

Book's way too complicated

$r_1, r_2 = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

$2\lambda = \beta/m$

$\omega^2 = \dfrac{k}{m}$

Real roots must be negative exponential decay.

IF

- $\beta^2 - 4km > 0$, $\quad x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $\quad r_1, r_2 < 0$

  OVERDAMPED (No oscillation)

- $\beta^2 - 4km = 0$, $\quad x(t) = (c_1 + c_2 t) e^{-\beta t/2m}$, $\quad r = -\dfrac{\beta}{2m} < 0$

  Critically Damped No oscillation

- $\beta^2 - 4km < 0$, $\quad x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$, $\quad r = a \pm bi, \ a < 0$

  Underdamped

Case 1: $\beta^2 - 4km > 0$ overdamped. HW #27

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

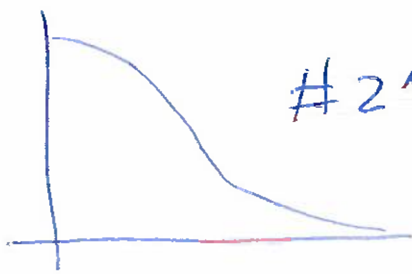
r_1, r_2 must be negative.

Dies exponentially.

May or may not have a local max or min depends on c_1 and c_2.

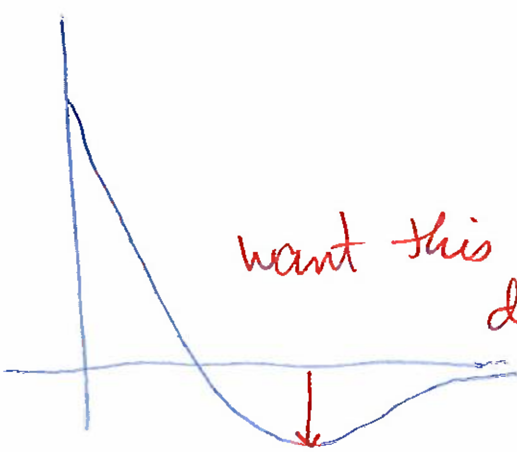
If there is no local max or min, then suprema is at the initial value.

eg



#27a

#27b



want this for extreme displacement after equilibrium.

* Equilibrium: $x(t) \stackrel{set}{=} 0$

To find locals: $x'(t) \stackrel{set}{=} 0$. *

Case 2 : Critically Damped

HW 25

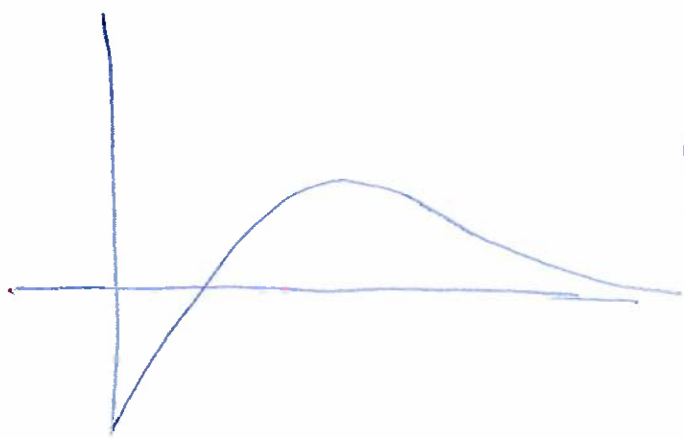
$$\beta^2 - 4km = 0$$

$$x(t) = (c_1 + c_2 t)e^{rt} \quad , \quad r < 0 .$$

Dies exponentially ..

May or may not have a local max / min.

eg #25



Extreme displacement after equilibrium.

Underdamped

HW 29, 30

<u>Case 3</u> : $\beta^2 - 4km < 0$

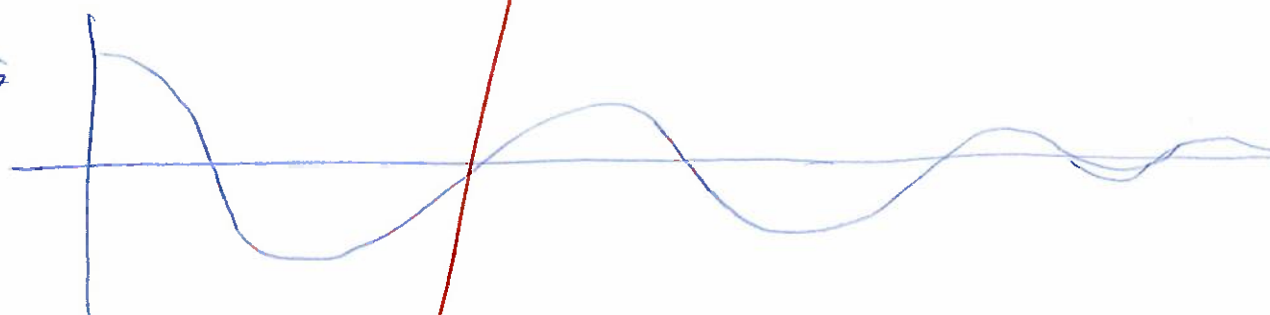
$r = a \pm bi$

$$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$a < 0$

Still dies exponentially.

eg



b quasi frequency

$\frac{2\pi}{b}$ = quasi period.

Compact form : $x(t) = Ae^{at}(\sin(bt + \phi))$

Same as before.

(Book:

$b = \sqrt{\omega^2 - \lambda^2}$

$\omega^2 = \frac{k}{m}$ | $2\lambda = \beta/m$)

Example: A mass of a ¼ slug is attached to a spring with spring constant of 2 lb/ft. The mass is started in motion by initially displacing it 2 feet in the downward direction and giving it an initial velocity of 2 ft/s in the upward direction.

a) Find the subsequent motion of the mass, if the damping constant due to air resistance is one times the velocity.

$\downarrow +$

$x(0) = 2$

$x'(0) = -2$

$m = \frac{1}{4}, \quad \beta = 1, \quad k = 2, \quad f(t) = 0$

$$m x'' + \beta x' + k x = 0$$

$$\frac{1}{4} x'' + x' + 2x = 0$$

AE: $\frac{1}{4} r^2 + r + 2 = 0$

$$\Rightarrow r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(8)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 32}}{2}$$

$$= \frac{-4 \pm 4i}{2}$$

$$= \underbrace{-2}_{a} \pm \underbrace{2}_{b} i$$

$$x(t) = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t$$

$$x(0) = 2 \quad \Rightarrow \quad 2 = c_1$$

$$x'(t) = -2c_1 e^{-2t}\cos 2t - 2c_1 e^{-2t}\sin 2t$$
$$-2c_2 e^{-2t}\sin 2t + 2c_2 e^{-2t}\cos 2t$$

$$x'(0) = -2 \quad \Rightarrow \quad -2 = -2c_1 + 2c_2$$
$$2c_2 = -2 + 2(2)$$
$$2c_2 = 2 \quad \Rightarrow c_2 = \underline{1}$$

So $x(t) = 2e^{-2t}\cos 2t + e^{-2t}\sin 2t$

---

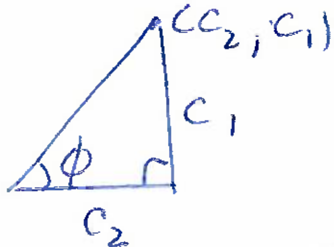
b) Write in the compact form $x(t) = Ae^{at}\sin(bt + \phi)$

(23) modified

$$A = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$(c_2, c_1) = (1, 2)$ QI

no adjustment



$$\tan\phi = \frac{c_1}{c_2}$$

$$\phi = \tan^{-1}\left(\frac{2}{1}\right)$$

$$x(t) = \sqrt{5}\, e^{-2t}\sin\left(2t + \underbrace{\tan^{-1}(2)}_{\approx}\right)$$

$$= \sqrt{5}\, e^{-2t}\sin(2t + 1.107)$$

c) What is the quasi frequency and the quasi period?

Quasi frequency : $b = 2$

Quasi Period $:= \frac{2\pi}{2} = \pi$

d) When will the object first return to equilibrium?

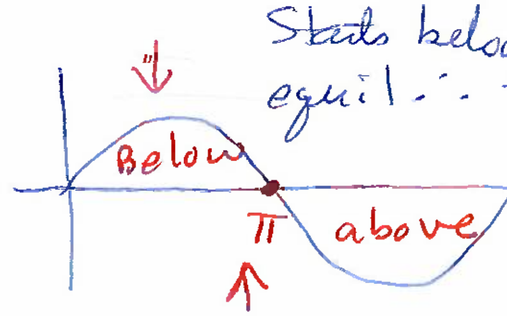
$x(t) \overset{set}{=} 0$

Need compact form to answer this.

$$\sqrt{5}e^{-2t} \sin(2t + 1.107) \overset{set}{=} 0$$

$$\sin(\underbrace{2t + 1.107}_{\theta}) = 0$$

Starts below equil ∴ +



$$2t + 1.107 = \pi$$

$$2t = \pi - 1.107$$

$$t = \frac{\pi - 1.107}{2} \approx 1.017 \text{ seconds}$$