$mx'' + \beta x' + kx = F(t)$, $x = x(t)$

§5.1.1 Free Undamped Motion.

$\beta = 0$, $F(t) = 0$

no damping medium.

$x(0) = x_0$
distance from the origin at $t = 0$.
$x'(0) = v_0$

$$mx'' + kx = 0$$

AE: Use r instead of m.

$$mr^2 + k = 0$$

$$r^2 = -\frac{k}{m} \Rightarrow r = \pm\sqrt{\frac{k}{m}}\, i$$

Hence, the equation of motion is

$$x(t) = c_1 \cos\sqrt{\frac{k}{m}}\, t + c_2 \sin\sqrt{\frac{k}{m}}\, t.$$

Let $\omega = \sqrt{\frac{k}{m}}$, we get

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Harmonic Motion.

ω: circular frequency.

T Period $= \frac{2\pi}{\omega}$

Frequency: $f = \left(\frac{1}{T}\right) = \frac{\omega}{2\pi}$ number of cycles completed per second.

Amplitude $= \sqrt{c_1^2 + c_2^2}$

Be careful. Since $\downarrow$ is $+$,
max of $x(t)$ is a positive displacement corresponding to the object attaining its greatest distance below the equilibrium position.

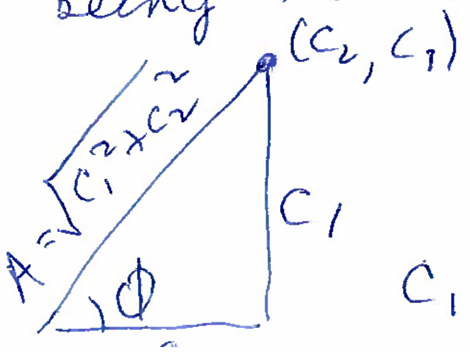
v.s. min of $x(t)$ " negative "
" " above equilibrium.

We need an alternate form to solve some questions.

Alternative Forms of $x(t)$ Compact forms.

(6) $x(t) = A \sin(\omega t + \phi)$
Book

Pf: $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$
Let (c_2, c_1) be in a point in $\mathbb{R}^2$ with ϕ being the angle formed from the origin.



$\sin \phi = \frac{c_1}{A}$ $\tan \phi = \frac{c_1}{c_2}$

$\cos \phi = \frac{c_2}{A}$

$c_1 = A \sin \phi$; $c_2 = A \cos \phi$

$$\tan\phi = \frac{c_1}{c_2}$$

$$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right) \text{ if } Q\,I \text{ or } Q\,4$$

OR $$\phi = \pi + \tan^{-1}\left(\frac{c_1}{c_2}\right) \text{ if } Q\,II \text{ or } Q\,III.$$

Recall $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$x(t) = c_1\cos\omega t + c_2\sin\omega t$$
$$= A\sin\phi\cos\omega t + A\cos\phi\sin\omega t$$

So $x(t) = A\sin(\omega t + \phi)$

where $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude

and ϕ is phase angle.

$x(t) = A\sin(\omega t + \phi)$ from

$x(t) = c_1\cos\omega t + c_2\sin\omega t$

$$A = \sqrt{c_1^2 + c_2^2}$$

(c_2, c_1)

$$\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right) \text{ if } Q\,I \text{ or } Q\,IV$$

OR $$\phi = \pi + \tan^{-1}\left(\frac{c_2}{c_1}\right) \text{ if } Q\,II \text{ or } Q\,III.$$

$(6)'$ $\quad x(t) = A\cos(\omega t - \delta)$

Pf: similar to (6) except now use the point (c_1, c_2).

Book uses ϕ again, but not same angle as (6).

$\delta = \frac{\pi}{2} - \phi$

complimentary angle

OR

Use cofunction identity & even identity

$$x(t) = A\sin(\omega t + \phi)$$

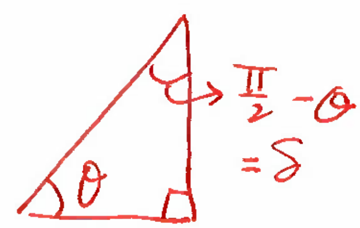
$$= A\cos\left(\tfrac{\pi}{2} - (\omega t + \phi)\right)$$

$$= A\cos\left(\tfrac{\pi}{2} - \omega t - \phi\right)$$

$$= A\cos\left(\omega t - \left(\tfrac{\pi}{2} - \phi\right)\right)$$

$$= A\cos(\omega t - \delta)$$

$\sin\theta = \cos(\frac{\pi}{2} - \phi)$



even identity

$\cos(-\theta) = \cos\theta$

$\tan\delta = \dfrac{c_2}{c_1}$.

Adjust for Q2 + Q3.

$$\overset{c_1}{\underset{\|}{}}\qquad\qquad\overset{c_2}{\underset{\|}{}}$$

eg Suppose $x(t) = 4\cos 2t - 5\sin 2t$

a) Rewrite in form (6), $x(t) = A\sin(\omega t + \phi)$.

$\omega = 2$ $\qquad\qquad$ $(c_2, c_1) = (-5, 4)$

$\tan\phi = \frac{4}{-5} = -\frac{4}{5}$ $\qquad$ Q II need adjustment

$\phi = \tan^{-1}\left(-\frac{4}{5}\right) + \pi \approx 2.467$

$A = \sqrt{4^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$

$x(t) = \sqrt{41}\,\sin(2t + 2.467)$

b) Rewrite in form (6)'. $\qquad$ Q IV No adjustment $\quad (c_1, c_2) = (4, -5)$

$x(t) = A\cos(\omega t - \delta$

$x(t) = \sqrt{41}\cos\left(2t - \underbrace{\tan^{-1}\left(-\frac{5}{4}\right)}_{\approx -.896}\right)$

Check:

$\delta = \frac{\pi}{2} - \phi \approx \frac{\pi}{2} - 2.467 \approx$ ✓

Example: A mass of 2 kg is suspended from a spring with spring constant of 10 N/m and allowed to come to rest. It is then set into motion by giving it an initial velocity (downward) of 150 cm/s.

a) Find an expression for the motion of the mass, assuming there is no air resistance.
b) Determine the circular frequency, the period, and the amplitude for the motion.

$$mx'' + \beta x' + kx = F(t)$$

a) $\beta = 0, \quad F(t) = 0$

$m = 2$ kg, $\quad k = 10$ N/m

$$2x'' + 10x = 0, \quad x(0) = 0 \text{ (m)}$$
$$x'(0) = 150 \text{ cm/s}$$
$$* \ = 1.5 \text{ m/s}$$

AE: $2r^2 + 10 = 0$

$r^2 = -5$

$r = \pm\sqrt{5}\, i$

$$x(t) = c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t$$

$x(0) = 0 \Rightarrow 0 = c_1 \cos(0) + 0$

$c_1 = 0$

$$x'(t) = -\sqrt{5}\, c_1 \sin(\sqrt{5}t) + \sqrt{5}\, c_2 \cos(\sqrt{5}t)$$

$x'(0) = 1.5 \quad 1.5 = 0 + \sqrt{5}\, c_2 \cos(0)$

$$c_2 = \frac{1.5}{\sqrt{5}} = 0.6708$$

So $x(t) = 0.6708 \sin\sqrt{5}t$

b) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{2}} = \sqrt{5}$ rad/s

OR $\quad x(t) = 0.6708 \sin(\underset{\uparrow \atop \omega}{\sqrt{5}}\, t)$

$T = \frac{2\pi}{\sqrt{5}} \approx 2.81$ s

$\downarrow$

Period

Amplitude $= \sqrt{0^2 + \left(\frac{1.5}{\sqrt{5}}\right)^2} = \frac{1.5}{\sqrt{5}}$

$= 0.6708$

Hints for HW - next page

Hints for HW:

In many of the HW problems, weight is given.

Note: $m = \dfrac{\text{weight}}{g}$

SI: $g = 9.8\ m/s^2$

Eng: $g = 32\ ft/s^2$

In many problems, you need to find k:

Since $mg = ks$, $k = \dfrac{\text{weight}}{s}$

In one problem, you will need to find k with the extra info.

Equilibrium: $x(t) \overset{set}{=} 0$

Max. vertical distance
Min " "

$\Bigg\}$ $\pm$ Amplitude for harmonic motion. 5.1.1

Otherwise, $x'(t) \overset{set}{=} 0$.