A "<u>vector space</u>". is a set which is closed under addition and scalar mult.

Note - "scalar" $\Rightarrow$ field. F

set V, field F.

$\Rightarrow$ $0 \in V$. $x, y \in V \Rightarrow x+y \in V$

$x \in V \Rightarrow \alpha x \in V$ $\forall \alpha \in F$

0 is such that $x + 0 = x$

"typical" example $\mathbb{R}^n$, $\mathbb{R}$

but also $\mathbb{R}^{n \times m}$, $\mathbb{R}$

or $\mathbb{C}^n$, $\mathbb{C}$

also e.g. $\Pi_n = \{$ polynomials of deg $\leq n\}$

or : $C[0,1]$ continuous functions on $[0,1]$.

<u>Very important concept</u> subspaces

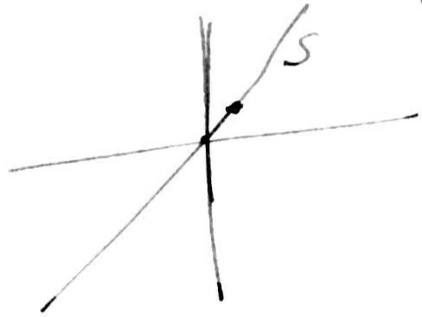
$S \subset V$ is a subspace

if S. is a vector space

i.e. closed under addition and scalar mult.

example

$V = \mathbb{R}^2$ $\quad S = \{x \,/\, x = \alpha u \,.\, \alpha \in \mathbb{R}\}$



line is a subspace.

but line through origin only!

no "curved lines"! either

Note $\{0\}$ subspace

$\{0\}, V$ are the trivial subspaces

In $\mathbb{R}^3$. subspaces $\{0\}$, lines, planes, $\mathbb{R}^3$

plane $\Pi = \{x \,/\, x = \alpha u_1 + \beta u_2 ,\ \alpha, \beta \in \mathbb{R}\}$

We say that the plane Π is <u>spanned</u> by u_1, u_2

i.e. $u_1\, u_2$ "generate" the whole subspace

In general given $\{v_1\ v_2 \dots v_n\}$

$$\text{Span}\{v_1\ v_2 \dots v_n\} = \{x \,/\, x = \sum_{i=1}^{n} \alpha_i v_i\}, \alpha_i \in \mathbb{R}\}$$

$$= \{x \,/\, x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n,\ \alpha_i \in \mathbb{R},\ i=1 \dots n\}$$

this is a subspace! check!

example e_i Euclidean vector span $\mathbb{R}^n$

do $\mathbb{R}^2$ $\mathbb{R}^3$ ...

the set $\{1, x, x^2, \dots x^n\}$ span $\mathbb{P}_n\mathbb{R}$.

If I ask: do $v_1 = \begin{vmatrix}1\\1\\1\end{vmatrix}$ $v_2 = \begin{vmatrix}1\\2\\2\end{vmatrix}$ $v_3 = \begin{vmatrix}1\\2\\3\end{vmatrix}$

span $\mathbb{R}^3$?

i.e. can I write any $b \in \mathbb{R}^3$ as $b = \sum_{i=1}^{3} \alpha_i v_i$

let $x = \begin{vmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{vmatrix}$ this is the same as

$Ax = b$ $A = \begin{vmatrix}v_1 & v_2 & v_3\end{vmatrix}$

i.e. is $Ax = b$ consistent $\forall b \in \mathbb{R}^3$?

(97) 4.1 p.4
4.2 p.1

Answer yes iff. Rank $A = 3$

so $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ rank 3

---

what about $\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix}$. no

they span a plane not the whole of $\mathbb{R}^3$

In fact

$\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix}$ span $\mathbb{R}^3$ as well.

reviewed exercise 4.1. 1, 2, 4, 6, 8!, 11, 12

Two very important subspaces

$N(A) = \{x / Ax = 0\}$ Null space

$R(A) = \{y / \exists x$ so that $Ax = y\}$ Range

or image space

or colum space

= subspace spaned by colums of A

$= \{$set $Ax\}$

these concepts are general for functions

$N(f) = \{ x / f(x) = 0 \}$

$R(f) = \{ \text{set of } f(x) \}$

so $R(A)$ $N(A)$ correspond to $f(x) = Ax$

Example

$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$ $R(A) = \{ x = \alpha \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} + \beta \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \}$

$N(A)$ all solution of homogeneous system

if only trivial solution $N(A) = \{0\}$

for this A $A \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$

$A = LU$
$N(A) = N(U)$
$Ux = 0 \Rightarrow Ax = 0$
$Ax = 0 \Rightarrow L^{-1}Ax = 0$
$Ux = 0$

$x_2 + 2x_3 = 0$ $x_2 = -2x_3$

$x_1 + x_2 + x_3 = 0$

$\cancel{x_1 = 0}$

$\cancel{N(A) = \{ x = \alpha \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix} \}}$

$x_1 + x_3 = 0$
$x_1 = -x_3$

$N(A) \{ \begin{vmatrix} -x_3 \\ -2x_3 \\ x_3 \end{vmatrix} = \alpha \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix}$

$x_1 + x_3 = 0$

$N(A) = \{ \cancel{x = \alpha \begin{vmatrix} 1 \\ -2 \\ 1 \end{vmatrix}} \; x = \alpha \begin{vmatrix} 1 \\ -2 \\ 1 \end{vmatrix}$

Of course we can define Range and Nullspace for A^T

title of section: 4 fundamental subspaces

$R(A^T)$ = column space of A^T

= row space of A = span rows of A

$N(A^T)$ is sometimes called left (hand) Nullspace $x \in N(A^T)$

since $A^T x = 0 \Rightarrow x^T A = 0$.

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recommended exercises. 4.2: 1, 2, 3, 4!,

5!, 8, 10, 11, 12, 13.