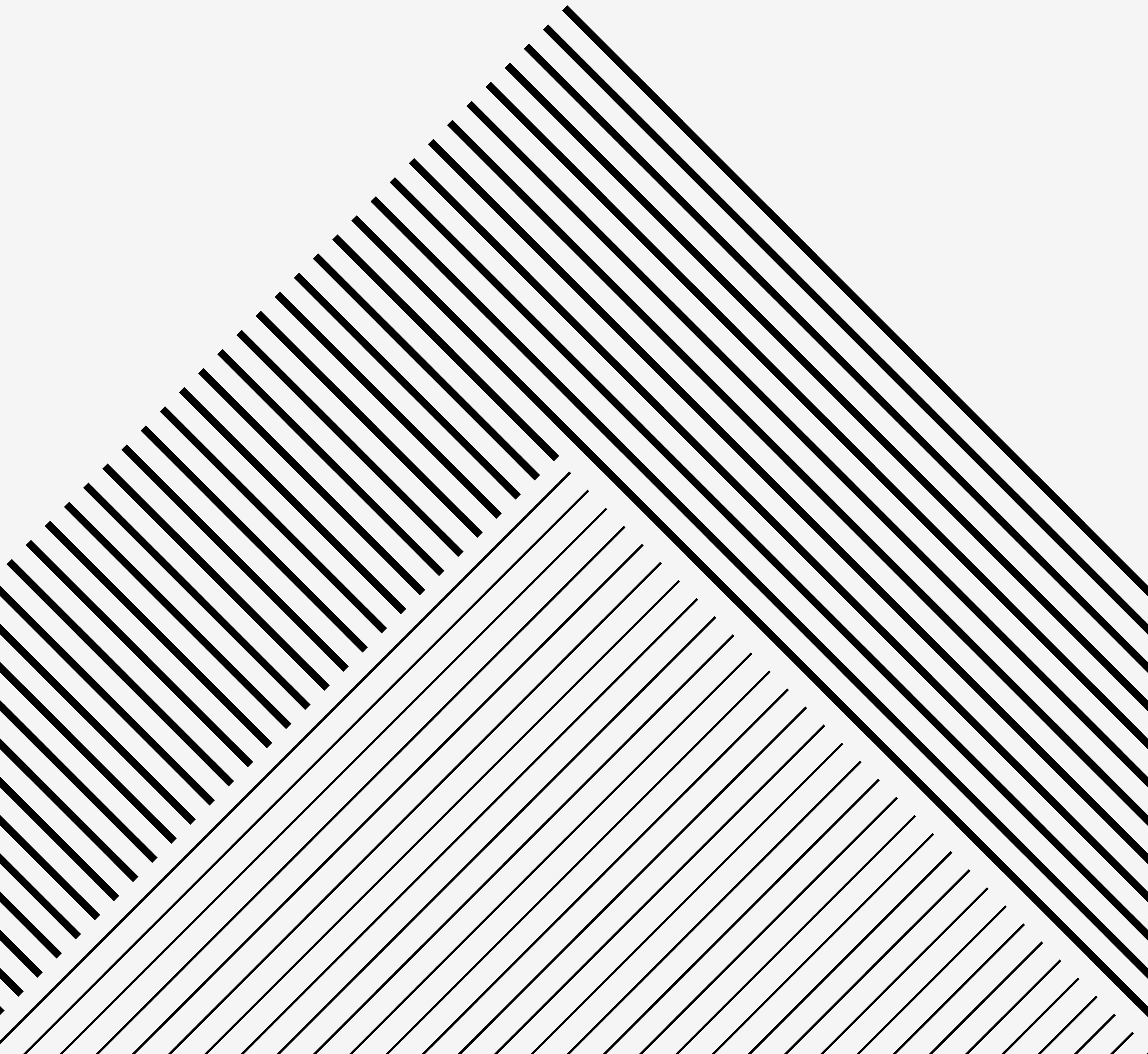

Sections 1.1, 1.2, 2.1

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linear algebra: always a line → $y = ax+b$ or $-ax_1 + x_2 = b$, everything else is nonlinear

For 2 equations w/ 2 unknowns: $3x+y=4$ and $2x-y=1$ — only 1 solution $(1,1)$

↳ $y=-3x+4$ ↳ $y=2x-1$

In general we have m equations and n unknowns : $x_1, x_2, x_3, \dots x_n$ where x_i "x sub i" where i is the index

E_1: $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

E_2: $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

⋮ ↳ b_i are known, a_{ij} are coefficients

E_m: $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

$$\left.\begin{vmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots\dots \\ a_m \dots & a_{mn}\end{vmatrix}\right\}\text{matrix of coefficients} \qquad \left.\begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{vmatrix}\right\}\text{vector of unknowns}$$

- 3 possibilities
  1) There is a unique solution intersect
  2) There are no solutions parallel
  3) There are infinitely many solutions same line

eg. if a linear system has solutions $\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right|$ and $\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right|$ then: it has to be infinitely many solutions

$m \times n \Rightarrow \boxed{A}\ \underset{x}{|} = \underset{b}{|}$ we go from equations → equations'. we want equivalent system - has the same set of solutions $Ax=b \rightarrow Ex=c$

eg. $\begin{vmatrix}3 & 1\\2 & -1\end{vmatrix}\begin{vmatrix}x_1\\x_2\end{vmatrix} = \begin{vmatrix}4\\1\end{vmatrix} \rightarrow \left|\begin{matrix}3 & 1\\2 & -1\end{matrix}\right|\left.\begin{matrix}4\\1\end{matrix}\right|$ our pivot (nonzero) augmented matrix now replace E_2 by $E_2 - \frac{2}{3}E_1$ $\left|\begin{matrix}3 & 1\\0 & -\frac{5}{3}\end{matrix}\right|\left.\begin{matrix}4\\-\frac{5}{3}\end{matrix}\right|$ we use a multiplier to get this to zero

we utilize a triangular system ◺| to solve and find a multiplier $(-\frac{a_{21}}{a_{11}})$ $-\frac{5}{3}x_2 = -\frac{5}{3} \Rightarrow x_2 = 1$ and $3x_1 + x_2 = 4 \Rightarrow x_1 = 1$

- elementary operations maintain the same set of solutions
- Gaussian Elimination: (a) forward elimination (b) backward substitution
  ↳ a finite sequence of elementary operations (3 kinds) and produces an equivalent system
  - ↳ (i) interchange row i w/ row j
  - (ii) replace E_i w/ a multiple of it, say αE_i, $\alpha \neq 0$
  - (iii) replace E_j w/ $E_j + \alpha E_i$

eg. E_1: $2x_1 + 1x_2 + 1x_3 = 1$

E_2: $6x_1 + 2x_2 + 1x_3 = -1$

E_3: $-2x_1 + 2x_2 + 1x_3 = 7$

$$\left|\begin{matrix}2 & 1 & 1\\6 & 2 & 1\\-2 & 2 & 1\end{matrix}\right|\left.\begin{matrix}1\\-1\\7\end{matrix}\right|$$

↳ make equal zero

we first concentrate on a pivot point, then replace E_2 w/ $E_2 - mE_1 \Rightarrow m=-3$

now w/ the same pivot, replace E_3 w/ $E_3 - mE_1 \Rightarrow m=1$

$$\left|\begin{matrix}2 & 1 & 1\\0 & -1 & -2\\0 & -3 & 2\end{matrix}\right|\left.\begin{matrix}1\\-4\\8\end{matrix}\right|$$

now we have a 2nd equivalent system and use a pivot in E_2 to eliminate in E_3. pivot: a_{11}, a_{22}, or a_{33}

replace E_3 w/ $E_3 - mE_2$ where $m=-3$ $\left|\begin{matrix}2 & 1 & 1\\0 & -1 & -2\\0 & 0 & -4\end{matrix}\right|\left.\begin{matrix}1\\-4\\-4\end{matrix}\right|$ now we back substitute

$-4x_3 = -4 \Rightarrow x_3 = 1$

$-x_2 - 2(1) = -4 \Rightarrow x_2 = 2$

$2x_1 + 2 + 1 = 1 \Rightarrow x_1 = -1$

$$x = \begin{vmatrix}x_1\\x_2\\x_3\end{vmatrix} = \begin{vmatrix}-1\\2\\1\end{vmatrix}$$

always plug back in and check the solution is true E_1 ✓ E_2 ✓ E_3 ✓

eg. $\left|\begin{matrix}2 & 1 & 1\\4 & 2 & 1\\-2 & 1 & 0\end{matrix}\right|\left.\begin{matrix}5\\8\\-5\end{matrix}\right| \rightarrow \left|\begin{matrix}2 & 1 & 1\\0 & 0 & -1\\0 & 2 & 1\end{matrix}\right|\left.\begin{matrix}6\\-2\\0\end{matrix}\right| \xrightarrow{E_2 \leftrightarrow E_3} \left|\begin{matrix}2 & 1 & 1\\0 & 2 & 1\\0 & 0 & -1\end{matrix}\right|\left.\begin{matrix}5\\0\\-2\end{matrix}\right| \rightarrow$ $x_3 = \frac{-2}{-1} = 2$, $x_2 = (0-1\cdot 2)/2 = -1$, $x_1 = (5-1\cdot(-1)-1\cdot 2)/2 = 2$ $\rightarrow x = \begin{vmatrix}2\\-1\\2\end{vmatrix} \rightarrow$ E_1 ✓ E_2 ✓ E_3 ✓

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- Row Echelon Form: similar to Gaussian (pivot + multiplier)
  1) if there is a 0 in pivot position, interchange rows have to have nonzero below in same column.
  2) if there are only zero's, move to next column.

eg.
$$A = \left|\begin{matrix}1 & 2 & 1 & 3 & 3\\2 & 4 & 0 & 4 & 4\\1 & 2 & 3 & 5 & 5\\2 & 4 & 0 & 4 & 7\end{matrix}\right| \rightarrow \left|\begin{matrix}1 & 2 & 1 & 3 & 3\\0 & 0 & -2 & -2 & -2\\0 & 0 & 2 & 2 & 2\\0 & 0 & -2 & -2 & 1\end{matrix}\right| \rightarrow \left|\begin{matrix}1 & 2 & 1 & 3 & 3\\0 & 0 & -2 & -2 & -2\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 3\end{matrix}\right| \rightarrow \left|\begin{matrix}1 & 2 & 1 & 3 & 3\\0 & 0 & -2 & -2 & -2\\0 & 0 & 0 & 0 & 3\\0 & 0 & 0 & 0 & 0\end{matrix}\right|$$

* Row Echelon form is unique

$m_{21} = -\frac{2}{1} = -2$ $m_{33} = \frac{2}{-2} = 1$

$m_{31} = -\frac{1}{1} = -1$ $m_{43} = -\frac{-2}{-2} = -1$

$m_{41} = -\frac{2}{1} = -2$

- Definition: for A matrix, rank of A is the # of pivots, or # of nonzero rows in E_A, or # of basic columns
  - ↳ Basic columns: columns corresponding to pivot column, other colums are nonbasic
  - ↳ Rank A < m (rows) iff E_A has a row of zeros (equal any other time)

Section 2.3 + 2.4

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- For rectangular matrix, 1 soln + infinitely many solns are combined.
  - ↳ 2 possibilities : i) A linear system is consistent if it has at least one soln
    - ii) A linear system is inconsistent if it has no soln.

eg. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 1 \\ 6 \\ 8 \end{vmatrix}$ $\quad |A \times | b| \rightarrow |E_A|c|$ $\quad \left|\begin{matrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 1 \\ 4 \\ 1 \end{matrix}\right|$ inconsistent

↳ $Ax = b$ is inconsistent if in $|E_A|c|$ we have a row of the form : $|0\,0 \ldots 0 / \alpha|$. $\quad 0x_1 + 0x_2 + 0x_3 = 0 \neq \alpha$

↳ $Ax = b$ is consistent iff : i) there is no row in $|E_A|c|$ of the form $|0\,0 \ldots 0|\alpha|$, $\alpha \neq 0$.
- ii) b is not a basic column of $|A|b|$
- iii) Rank A = Rank $|A|b|$
- iv) $b = \sum \alpha_j a_{*j}$ (b is the linear combination of basic columns , j is basic column)

- Variables corresponding to the nonbasic columns are free variables
  - ↳ for any values of the free variables, we can find a particular soln, x_p.

$\left|\begin{matrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 1 \\ 4 \\ 0 \end{matrix}\right|$ $\quad x_1 + 2x_2 + 3x_3 + 4x_4 = 1$, $\; 2x_3 + 2x_4 = 4$ $\quad$ set $x_2, x_4 = 0$ $\quad 2x_3 + 0 = 4 \Rightarrow x_3 = 2$ $\quad x_1 + 0 + 6 + 0 = 1$ $\quad x_1 = -5$ $\quad x_p = \begin{vmatrix} -5 \\ 0 \\ 2 \\ 0 \end{vmatrix}$

↳ We get the coefficients for the basic columns

$b = \begin{vmatrix} 1 \\ 6 \\ 7 \end{vmatrix} = \sum a_j a_{*j} = -5\begin{vmatrix} 1 \\ 2 \\ 7 \end{vmatrix} + 2\begin{vmatrix} 3 \\ 8 \\ 11 \end{vmatrix}$

- if $b = 0$ then the system is homogenous, if $b \neq 0$ then the system is non-homogenous.
  - ↳ $x = 0$ is always a solution of the homogenous system
  - ↳ all solutions of $Ax = b$ consists of all solutions of $Ax = 0 + x_p$

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- Definition : Let $x = \begin{vmatrix} x_1 \\ x_2 \\ \vdots \end{vmatrix}$, $y = \begin{vmatrix} y_1 \\ y_2 \\ \vdots \end{vmatrix}$ be column vectors (in $\mathbb{R}^n$), then given two scalars α, β, a linear combination of the vectors is $\alpha x + \beta y = \begin{vmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_n + \beta y_n \end{vmatrix}$ $\quad$ eg. $x = \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix}$ $y = \begin{vmatrix} -1 \\ 4 \\ 0 \end{vmatrix}$ then $3x + 2y = 3\begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix} + 2\begin{vmatrix} -1 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} 5 \\ 14 \\ 3 \end{vmatrix}$ this is a linear combo

- $Ax = 0$ is the homogenous system and is always consistent since $A \cdot 0 = 0$; $a_{11}0 + a_{12} + \cdots + a_{1n}0 = 0$
  - ↳ the unique solution is the trivial solution : If A is s.t. $Ax = b$ has always a unique solution $\forall b$ then the homogeneous system, $Ax = 0$, has only the trivial solution

- $Ax = 0$ has infinitely many solutions, so we use Echelon form to find all solutions

$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \rightarrow \left|\begin{matrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right|$ $\quad x_2, x_4$ are free variables

$r = 2$, $n - r = 4 - 2 = 2$ free variables

↳ Write basic variables in terms of the free variables

$2x_3 + 2x_4 = 0$, $\; x_3 = -x_4$ $\rightarrow$ $x_1 + 2x_2 + 3(-x_4) + 4x_4 = 0$, $\; x_1 = -2x_2 - x_4$

Write as linear combo

$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{vmatrix} = x_2 \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} -1 \\ 0 \\ -1 \\ 1 \end{vmatrix}$

↳ the general soln. of $Ax = 0$ is $x_h + a_{1h}\begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} + a_{2h}\begin{vmatrix} -1 \\ 0 \\ -1 \\ 1 \end{vmatrix}$ for all $a_{1h}, a_{2h} \in \mathbb{R}$

↳ a_1, a_2 are scalars, and note that $\alpha_1 = 0$ and $\alpha_2 = 0$ will give the trivial solution

$\left|\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right| \rightarrow \left|\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right| \rightarrow \left|\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right|$ $\quad R = 2$ $\quad$ 3 free variables : x_3, x_4, x_5

$x_2 = 0$

$x_1 + \cancel{x_2}^{0} + x_3 + x_4 + x_5 = 0$

$x_1 = -x_3 - x_4 - x_5$

$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{vmatrix} = \begin{vmatrix} -x_3 - x_4 - x_5 \\ 0 \\ x_3 \\ x_4 \\ x_5 \end{vmatrix} = x_3 \begin{vmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} + x_5 \begin{vmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$

* If there are $p = n - r$, free variables, we can find the nontrivial solutions to $Ax = 0$. All soln. are of the form $x_n = \alpha_1 x_{n_1} + \alpha_2 x_{n_2} + \cdots x_p x_{n_p}$ where $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_p \in \mathbb{R}$. that being linear combinations of x_{n_i}

Section 2.5

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• A linear system $Ax=b$ where $b\neq 0$, is called nonhomogeneous and x_p is the particular solution.

↳ we know that $Ax_p=b$ and $Ax_n=0$, so we can say $A(x_n+x_p)=b$

$\begin{vmatrix}1&1&1&1\\2&2&4&2\end{vmatrix}\begin{vmatrix}x_1\\x_2\\x_3\\x_4\end{vmatrix}=\begin{vmatrix}3\\4\end{vmatrix}$ note $b=\begin{vmatrix}3\\4\end{vmatrix}\neq 0$ $\quad \left|\begin{matrix}1&1&1&1\\0&0&2&0\end{matrix}\right|\begin{matrix}3\\-2\end{matrix}\Big|$ $r=2$, x_2+x_4 are free

$2x_3=-2 \quad x_3=-1$

$x_1+x_2+x_3+x_4=3$

$x_1=3-\cancel{x_2}^{0}+1-\cancel{x_4}^{0}=4$

$x_p=\begin{vmatrix}x_1\\x_2\\x_3\\x_4\end{vmatrix}=\begin{vmatrix}4\\0\\-1\\0\end{vmatrix}$

↳ putting zero in free variable give coefficients to basic columns.

$\begin{vmatrix}3\\4\end{vmatrix}=4\begin{vmatrix}1\\2\end{vmatrix}+(-1)\begin{vmatrix}1\\4\end{vmatrix}$

↳ For the homogeneous system $[E_c|0]$ we write basic variables in terms of free variables

$[E_c|0]=\left|\begin{matrix}1&1&1&1\\0&0&2&0\end{matrix}\right|\begin{matrix}0\\0\end{matrix}\Big|$ $\quad 2x_3=0 \Rightarrow x_3=0$, $x_1=-x_2-\cancel{x_3}^{0}-x_4$

$x_n=\begin{vmatrix}x_1\\x_2\\x_3\\x_4\end{vmatrix}=\begin{vmatrix}-x_2-x_4\\x_2\\0\\x_4\end{vmatrix}=x_2\begin{vmatrix}-1\\1\\0\\0\end{vmatrix}+x_4\begin{vmatrix}-1\\0\\0\\1\end{vmatrix}=\alpha_1\begin{vmatrix}-1\\1\\0\\0\end{vmatrix}+\alpha_2\begin{vmatrix}-1\\0\\0\\1\end{vmatrix}\quad \forall \alpha_1,\alpha_2\in\mathbb{R}.$

$A=\begin{vmatrix}2&2&2\\1&2&3\\4&0&1\end{vmatrix}\quad b=\begin{vmatrix}4\\1\\7\end{vmatrix}\rightarrow\left|\begin{matrix}2&2&2\\1&2&3\\4&0&1\end{matrix}\right|\begin{matrix}4\\1\\7\end{matrix}\Big|\rightarrow\left|\begin{matrix}2&2&2\\0&1&2\\0&-4&-3\end{matrix}\right|\begin{matrix}4\\-1\\-1\end{matrix}\Big|\rightarrow\left|\begin{matrix}2&2&2\\0&1&2\\0&0&5\end{matrix}\right|\begin{matrix}4\\-1\\-5\end{matrix}\Big|$

$m_{21}=\cdot\frac{1}{2}$, $m_{31}=-2$ $\quad m_{32}=4$ $\quad r=3$, no free variables

$5x_3=-5 \Rightarrow x_3=-1$

$x_2+2(-1)=-1 \Rightarrow x_2=1$

$2x_1+2(1)+2(-1)=4 \Rightarrow x_1=2$

$x_p=\begin{vmatrix}x_1\\x_2\\x_3\end{vmatrix}=\begin{vmatrix}2\\1\\-1\end{vmatrix}$

*in the case w/ no free variables, the homogeneous system only has the trivial solution, $x_n=0$.

$\left|\begin{matrix}1&1&1&1&1\\1&2&1&1&1\\2&4&2&2&2\end{matrix}\right|\begin{matrix}3\\2\\4\end{matrix}\Big|\rightarrow\left|\begin{matrix}1&1&1&1&1\\0&1&0&0&0\\0&2&0&0&0\end{matrix}\right|\begin{matrix}3\\-1\\-2\end{matrix}\Big|\rightarrow\left|\begin{matrix}1&1&1&1&1\\0&1&0&0&0\\0&0&0&0&0\end{matrix}\right|\begin{matrix}3\\-1\\0\end{matrix}\Big|$

$m_{21}=\cdot 1$, $m_{31}=-2$ $\quad m_{32}=-2$ $\quad$ homogeneous

$x_2=-1$

$x_1+\cancel{x_2}^{-1}+\cancel{x_3}^{0}+\cancel{x_4}^{0}+\cancel{x_5}^{0}=3$

$x_1=4$

$x_p=\begin{vmatrix}4\\-1\\0\\0\\0\end{vmatrix}$

$\left|\begin{matrix}1&1&1&1&1\\1&2&1&1&1\\2&4&2&2&2\end{matrix}\right|\begin{matrix}0\\0\\0\end{matrix}\Big|\rightarrow\left|\begin{matrix}1&1&1&1&1\\0&1&0&0&0\\0&2&0&0&0\end{matrix}\right|\begin{matrix}0\\0\\0\end{matrix}\Big|\rightarrow\left|\begin{matrix}1&1&1&1&1\\0&1&0&0&0\\0&0&0&0&0\end{matrix}\right|\begin{matrix}0\\0\\0\end{matrix}\Big|$

$X_n=\begin{vmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{vmatrix}=\begin{vmatrix}-x_3-x_4-x_5\\0\\x_3\\x_4\\x_5\end{vmatrix}=x_3\begin{vmatrix}-1\\0\\1\\0\\0\end{vmatrix}+x_4\begin{vmatrix}-1\\0\\0\\1\\0\end{vmatrix}+x_5\begin{vmatrix}-1\\0\\0\\0\\1\end{vmatrix}$

all soln $= x_p+x_n$

Section 3.2

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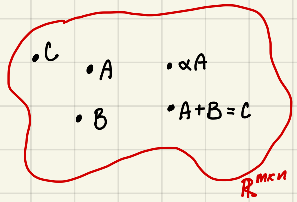
- $A = \boxed{}$ a 2-D array, $m \times n$. + $v = \begin{vmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{vmatrix}$ a $n \times 1$ matrix (or column vector) $\neq$ $|v_1\ v_2\ \cdots\ v_n|$. We shall see that, the latter is v_T (v transposed, $1 \times n$) $|1\ 2| = \begin{vmatrix} 1 \\ 2 \end{vmatrix}^T$ and $\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} = |1\ 2\ 3|^T$
- Algebra of matrices looks at operations $(+, -, \cdot, \div)$
  - ↳ the concept of $A = B$ means they have the same # of rows + columns and $a_{ij} = b_{ij}\ \forall i,j$, $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$
- Sum of $A+B$ (both have to have the same order) $\in \mathbb{R}^{m\times n}$

  $(A+B)_{ij} = A_{ij} + B_{ij}$ and $C = A+B$ means $c_{ij} = a_{ij} + b_{ij}$ and $A, B, C \in \mathbb{R}^{m \times n}$

  $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 0 & 7 & 2 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 9 & 5 \\ 1 & 5 & 8 \end{vmatrix}$ * set of $m \times n$ matrices, $\mathbb{R}^{m\times n}$, is closed under addition



- Most properties of matrix addition are inherited from the addition in $\mathbb{R}$
  - ↳ $\exists 0$ s.t. $a + 0 = a$ → $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix}$ *the zero matrix
  - ↳ associative property: $(A+B)+C = A+(B+C)$
  - ↳ commutative property: if $A+B = B+A$ $(A+B)_{ij} = A_{ij} + B_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = B+A$
  - ↳ $\exists -A$ s.t. $A + (-A) = 0$ and $(-A)_{ij} = -(A_{ij})$

  } can all be proven directly
- Scalar multiplication (# and matrix)
  - ↳ product of $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}^{m\times n}$ $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and $2A = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 6 \\ 2 & 2 & 2 \end{vmatrix}$ $(\alpha A)_{ij} = \alpha a_{ij}$
  - ↳ $\mathbb{R}^{m\times n}$ is closed under scalar multiplication.
- Properties:
  1) $\alpha(\beta A) = (\alpha\beta)A$
  2) $\alpha(A+B) = \alpha A + \alpha B$ $\alpha(A+B)_{ij} = \alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij} = (\alpha A)_{ij} + (\alpha B)_{ij}$. Hence, $\alpha(A+B) = \alpha A + \alpha B$
  3) $(\alpha+\beta)A = \alpha A + \beta A$ $[(\alpha+\beta)A]_{ij} = (\alpha+\beta)A_{ij} \overset{*}{=} (\alpha+\beta)a_{ij} = \alpha a_{ij} + \beta a_{ij} \overset{*}{=} \alpha A_{ij} + \beta A_{ij} = [\alpha A + \beta A]_{ij}$ by def of sum

     * by def of scalar
- $\mathbb{R}^{m\times n}$ is not closed under transposition unless $m = n$ (square matrix)
  - ↳ Properties:
    1) $(A^T)^T = A$ $(A^T)^T_{ij} = (A^T)_{ji} = a_{ij} = A_{ij}$
    2) $(A+B)^T = A^T + B^T$ $[(A+B)^T]_{ij} = (A+B)_{ji} = a_{ji} + b_{ji} = A^T_{ij} + B^T_{ij}$
    3) $(\alpha A)^T = \alpha A^T$ $(\alpha A)^T_{ij} = (\alpha A)_{ji} = \alpha a_{ji} = \alpha(A^T)_{ij}$
- A square matrix, A, is symmetric iff $A^T = A$, $a_{ij} = a_{ji}$, $i \neq j$ $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}^T = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$
  - ↳ A is skew symmetric if $A^T = -A$ $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}^T = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$
  - ↳ A is nonsymmetric if $A^T \neq A$.

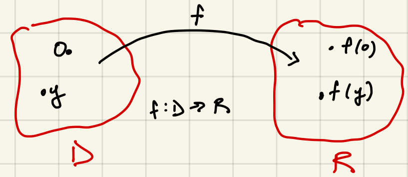
  * The only matrix that is both symmetric and skew symmetric is the zero matrix

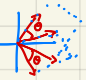
Section 3.3

- **Linear transformations**: transformations of maps (functions) from 1 set, D (domain), to another, R (range),
  - ↳ a function is linear if f of the sum = sum of the f: $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$
    - ↳ $f(\alpha x + y) = \alpha f(x) + f(y) \quad \forall x, y \in D$

$f: \mathbb{R}^3 \to \mathbb{R}^3$ and $x, y \in \mathbb{R}^3$ $f\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} x_1 - x_2 \\ x_3 + x_1 \\ x_3 \end{vmatrix}$ and $x+y = \begin{vmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{vmatrix}$

$$f(x+y) = \begin{vmatrix} (x_1+y_1)-(x_2+y_2) \\ (x_3+y_3)+(x_1+y_1) \\ x_3+y_3 \end{vmatrix} = \begin{vmatrix} x_1-x_2 \\ x_3+x_1 \\ x_3 \end{vmatrix} + \begin{vmatrix} y_1-y_2 \\ y_3+y_1 \\ y_3 \end{vmatrix} = f(x) + f(y)$$

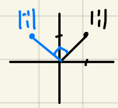


- Examples of linear transformations: 1) rotation by a fixed angle θ. 
  2) reflection
  3) projection

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{vmatrix}$

$$f(x+\alpha y) = f\begin{vmatrix} x_1+\alpha y_1 \\ x_2+\alpha y_2 \end{vmatrix} = \begin{vmatrix} 2(x_1+\alpha y_1) - (x_2+\alpha y_2) \\ 3(x_1+\alpha y_1) + 4(x_2+\alpha y_2) \end{vmatrix} = \begin{vmatrix} 2x_1 + 2\alpha y_1 - x_2 - \alpha y_2 \\ 3x_1 + 3\alpha y_1 + 4x_2 + 4\alpha y_2 \end{vmatrix} = \begin{vmatrix} (2x_1 - x_2) + (2\alpha y_1 - \alpha y_2) \\ (3x_1 + 4x_2) + (3\alpha y_1 + 4\alpha y_2) \end{vmatrix}$$

$$= \begin{vmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{vmatrix} + \alpha \begin{vmatrix} 2y_1 - y_2 \\ 3y_1 + 4y_2 \end{vmatrix} = f(x) + \alpha f(y)$$

rotation: $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ $\theta = \frac{\pi}{2}$ for $\begin{vmatrix} 1 \\ 1 \end{vmatrix}$: 

we get $\begin{vmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \boxed{\begin{vmatrix} -1 \\ 1 \end{vmatrix}}$

- Every application of a matrix-vector product is a linear transformation

$A: \mathbb{R}^n \to \mathbb{R}^m$ $A = \begin{vmatrix} 3 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{vmatrix}_{m \times n} \cdot x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = Ax$

- Every linear map can be expressed as a matrix-vector pdt: $\alpha x = \begin{vmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{vmatrix} \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{vmatrix}$
- Transposition is a linear transformation: $(A^T)_{ij} = A_{ji}$ and $(A + \alpha B)^T = A^T + \alpha B^T$
- Let A be a n×n matrix, $A = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 1 & -1 \\ 2 & 0 & 0 \end{vmatrix}$, $\text{trace } A = \sum_{i=1}^{n} a_{ii}$ eg. trace $A = 3 + 1 + 0 = 4$
  - ↳ trace (A+B) = trace A + trace B
  - ↳ trace (αA) = α trace A
  - ↳ $\text{trace}(\alpha A + B) = \sum_{i=1}^{n} \alpha a_{ii} + b_{ii} = \alpha \sum a_{ii} + \sum b_{ii} = \alpha \text{ trace } A + \text{trace } B$
- For every linear transformation we can define 2 important sets: $N(f) = \ker(f) = \{x \in D : f(x) = 0\}$
  - ↳ kernel f = null space
- **Range of f**: all elements in R which can be reached by f ⇒ $R(f) = \{y \in R : \exists x \in D, f(x) = y\}$

$A: \mathbb{R}^2 \to \mathbb{R}^2$ $A = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$ Rank = 1 1 free variable → $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ $x_h = \alpha \begin{vmatrix} 1 \\ -1 \end{vmatrix}$ $N(A) = \left\{\alpha \begin{vmatrix} 1 \\ -1 \end{vmatrix}\right\}$ and $R(A) = \{y \in \mathbb{R}^2 : \exists x \in \mathbb{R}^2, f(x) = y\}$

if $y = \begin{vmatrix} x_1 + x_2 \\ 2(x_1 + x_2) \end{vmatrix}$ then $R(A) = \left\{\beta \begin{vmatrix} 1 \\ 2 \end{vmatrix}\right\}$

- When there are no free variables, the null is the 0 matrix: $N(A) = \{0\}$ rank = 2 and n = 2
  - ↳ the range is the entire map: $R(A) = \mathbb{R}^2$
  - ↳ the inverse is true: if null space $N(A) = \{\mathbb{R}^2\}$ then the range $R(A) = \{0\}$ rank = 0 and n = 2

Section 3.4 + 3.5

- Composition of 2 functions is linear : $f \circ g(x) = f(g(x))$ where $f: D \to R$ and $g: P \to D$ so $f \circ g: P \to R$
  - ↳ $f \circ g(x + \alpha y) = f(g(x + \alpha y)) = f(g(x)) + \alpha f(g(y))$

  proof: $f(g(x+\alpha y)) = f(g(x) + g(\alpha y))$ since g is linear. But then $f(g(x)) + \alpha f(g(y))$ since f is also linear q.e.d.

- take $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} x_1 \\ \frac{1}{2}(x_1+x_2) \\ x_2 \end{vmatrix}$ then $\begin{vmatrix} 1 \\ 1 \end{vmatrix} \to \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ and $\begin{vmatrix} 1 \\ 2 \end{vmatrix} = \begin{vmatrix} 1 \\ 3/2 \\ 2 \end{vmatrix}$
  then $g: \mathbb{R}^3 \to R$, $g\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_2$ then $\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \to 1$ and $\begin{vmatrix} 1 \\ 3/2 \\ 2 \end{vmatrix} \to 3/2$
  - ↳ then $g \circ f\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = g\left(f\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}\right) = g\begin{vmatrix} x_1 \\ \frac{1}{2}(x_1+x_2) \\ x_2 \end{vmatrix} = \frac{1}{2}(x_1+x_2)$ so then $g \circ f: \mathbb{R}^2 \to R$

- Every linear transformation corresponds to a matrix multiplication

$$\begin{vmatrix} 4x_1 + 3x_2 + x_3 \\ 2x_1 - x_2 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} \longrightarrow \begin{vmatrix} 4 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} = \begin{vmatrix} 13 \\ 0 \end{vmatrix} = 1\begin{vmatrix} 4 \\ 2 \end{vmatrix} + 2\begin{vmatrix} 3 \\ -1 \end{vmatrix} + 3\begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

- Now what if we have 2 column vectors : $A \cdot |x \vdots y| = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & \vdots & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & \vdots & a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{vmatrix}$

$$\underset{2 \times 3}{\begin{vmatrix} 4 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix}} \underset{3 \times 2}{\begin{vmatrix} 1 & \vdots & 1 \\ 2 & \vdots & 1 \\ 3 & \vdots & 1 \end{vmatrix}} = \underset{\boxed{2 \times 2}}{\begin{vmatrix} 13 & \vdots & 8 \\ 0 & \vdots & 1 \end{vmatrix}}$$

- if $C = AB$ then $c_{ij} = \sum_{k=1}^{n}(a_{ik}b_{kj})$, note: $A \cdot B \neq B \cdot A$ but $\operatorname{trace}(AB) = \operatorname{trace}(BA)$

$$AB = \overset{A}{\begin{vmatrix} 4 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix}} \overset{B}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{vmatrix}} = \begin{vmatrix} 13 & 8 \\ 0 & 1 \end{vmatrix} \quad \text{and} \quad BA = \overset{B}{\begin{vmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{vmatrix}} \overset{A}{\begin{vmatrix} 4 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix}} = \begin{vmatrix} 6 & 2 & 1 \\ 10 & 5 & 2 \\ 14 & 8 & 3 \end{vmatrix}$$

$\operatorname{trace}(AB) = 14 = \operatorname{trace}(BA)$

- Matrix pdt is not commutative
- Note for any square matrix, A, $A + A^T$ is symmetric and $A - A^T$ is skew symmetric
  - ↳ Then for any $n \times n$ A, it can be written as a sum of a symmetric + skew symmetric system
    - ↳ $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ $\quad S + K$
  - ↳ If $A = S + K$ and $S^T = S$ and $K^T = -K$, then $S = \frac{A + A^T}{2}$ and $K = \frac{A - A^T}{2}$

Section 3.6-3.7

Properties of matrix product

- **Distributive property:** $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$

$$\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}1&1\\0&2\end{vmatrix}+\begin{vmatrix}1&0\\-1&0\end{vmatrix}\right)=\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}2&1\\-1&2\end{vmatrix}\right)=\begin{vmatrix}0&5\\2&11\end{vmatrix}$$

$$\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}1&1\\0&2\end{vmatrix}\right)+\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}1&0\\-1&0\end{vmatrix}\right)=\begin{vmatrix}1&5\\3&11\end{vmatrix}+\begin{vmatrix}-1&0\\-1&0\end{vmatrix}=\begin{vmatrix}0&5\\2&11\end{vmatrix} \quad \checkmark$$

- **Associative Property:** $A\cdot(BC) = (AB)\cdot C$ and $(\alpha B)\cdot C = \alpha\cdot(BC) = B\cdot(\alpha C)$

$$\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}1&1\\0&2\end{vmatrix}\cdot\begin{vmatrix}1&0\\-1&0\end{vmatrix}\right)=\begin{vmatrix}1&2\\3&4\end{vmatrix}\left(\begin{vmatrix}0&0\\-2&0\end{vmatrix}\right)=\begin{vmatrix}-4&0\\-8&0\end{vmatrix} \quad \checkmark$$

and $\left(2\cdot\begin{vmatrix}1&2\\3&4\end{vmatrix}\right)\cdot\begin{vmatrix}1&1\\0&2\end{vmatrix} = \begin{vmatrix}2&4\end{vmatrix}$

$$\left(\begin{vmatrix}1&2\\3&4\end{vmatrix}\cdot\begin{vmatrix}1&1\\0&2\end{vmatrix}\right)\begin{vmatrix}1&0\\-1&0\end{vmatrix}=\begin{vmatrix}1&5\\3&11\end{vmatrix}\begin{vmatrix}1&0\\-1&0\end{vmatrix}=\begin{vmatrix}-4&0\\-8&0\end{vmatrix}$$

- Note, matrix pdt is not communative: $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$
  - ↳ but associativity still exists: $A^k \cdot A^s = A^{k+s}$ and $(A^k)^s = A^{ks}$
- **The identity matrix:** diagonal matrix w/ ii entries $=1$ and ij entries $=0$ $(i \neq j)$ $I = \begin{vmatrix}1&0&\cdots&0\\0&1&&\\\vdots&&\ddots&\\0&&&1\end{vmatrix}$ s.t. $A\cdot I = A$ and $I \cdot A = A$
  - ↳ recall that $A^2 = A\cdot A$ and $A^3 = A\cdot A\cdot A = A^2 A = A\cdot A^2$. Thus $A^n = A\cdot A\cdots A$ } n times
  - ↳ then $A^0 = I$ the same way $a^0 = 1$ (I, A and A, I)
- $(AB)^T = B^T A^T$ note reversed order so a $\overset{A}{3\times5}$ times $\overset{B}{5\times2}$ gives 3×2 (A B = AB and AB^T = 2×3)
  - ↳ B^T is a 2×5 and A^T is a 5×3, so $B^T A^T$ gives a 2×3 same as $(AB)^T$
  - ↳ $(AA^T)^T = (A^T)^T\cdot(A)^T$ by definition $= A\cdot A^T \Rightarrow$ is always symmetric
  - ↳ $A^T A x = A^T b$ from $Ax = b$

$A = \begin{vmatrix}1&2&3\\1&1&1\end{vmatrix}$ $A^T = \begin{vmatrix}1&1\\2&1\\3&1\end{vmatrix}$ $A\cdot A^T = \begin{vmatrix}1&2&3\\1&1&1\end{vmatrix}\begin{vmatrix}1&1\\2&1\\3&1\end{vmatrix} = \begin{vmatrix}14&6\\6&3\end{vmatrix}$ ← symmetric

$A^T\cdot A = \begin{vmatrix}1&1\\2&1\\3&1\end{vmatrix}\begin{vmatrix}1&2&3\\1&1&1\end{vmatrix} = \begin{vmatrix}2&3&4\\3&5&\boxed{7}\\4&\boxed{7}&10\end{vmatrix}$ super symmetric!

- Recall that $\text{trace}(AB) = \text{trace}(BA)$ $14+3=17$ and $2+5+10=17$.
  - ↳ we can apply this to pdt of 3 matrices: $\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$ but not $\neq \text{trace}(BAC) \neq \text{trace}(CBA)$

Matrix Inversion

- In $\mathbb{R}$, for $\forall a \neq 0$, $\exists\, b \neq 0$ s.t. $a\cdot b = 1$ and $b \cdot a = 1$ namely $b = \frac{1}{a} = a^{-1}$
- Same w/ square matrices of full rank
  - ↳ If A is $n\times n$ and $\text{Rank}\,A = n$ $\exists X$, $n\times n$, of rank n so that $XA = AX = I$ (the identity matrix) and X is unique, we call it A^{-1}
  - ↳ we solve for X by aug matrix: $\left|A\right|\left|X\right| = \left|I\right|$ w/ Gaussian elimination
  - ↳ A has n pivots and no free variables
- Special cases:
  - a diagonal matrix, $A = \begin{vmatrix}2&0\\0&3\end{vmatrix}$ $A^{-1} = \begin{vmatrix}1/2&0\\0&1/3\end{vmatrix}$
  - a triangular matrix, $A = \begin{vmatrix}2&1\\0&3\end{vmatrix}$ $A^{-1} = \begin{vmatrix}1/2&-1/6\\0&1/3\end{vmatrix}$
- If A^{-1} exists, we say that A is invertible or non singular
  - ↳ A is nonsingular if:
    1) $\text{rank}\,A = n$
    2) there are n pivots
    3) there are no free variables
    4) there are n basic columns
    5) $Ax = 0 \Rightarrow x = 0$
    6) $\exists!$ solution to $Ax = b$ for $\forall b$
    7) $\exists A^{-1} \Rightarrow A\cdot A^{-1} = I$
    8) $\exists A^{-1} \Rightarrow A^{-1}\cdot A = I$

  } these statements are all equivalent

#5 proof: Let A be $n\times n$ w/ $\text{rank}\,A = n$ and $\exists X, Y$ s.t. $AX = I$ and $AY = I$ then $A(X-Y) = AX - AY = I - I = 0$ Hence $X - Y = 0$ and $X = Y$.

- Proposition: If A is an $n\times n$ nonsingular and $AX = I$ then $XA = I$

Proof: Let X be nonsingular, otherwise $\exists v \neq 0$ s.t. $Xv = 0$ Then $v = Iv = AXv = A\cdot 0 = 0$. Contradiction. Hence, X is nonsingular. Since, X nonsingular, $\exists X^{-1}$ s.t. $X\cdot X^{-1} = I$. But then $AX = I$ and $AX\cdot X^{-1} = I\cdot X^{-1}$. But $X\cdot X^{-1} = I$. So $A\cdot X\cdot X^{-1} = A\cdot I = A$ and $I\cdot X^{-1} = X^{-1}$. Then $A = X^{-1}$. Then, $XA = X(X^{-1}) = I$. q.e.d.

- Proposition: $(A^2)^{-1} = (A^{-1})^2$
  Proof: $A^2 \cdot (A^{-1})^2 = A^2 \cdot A^{-1} \cdot A^{-1} = A \cdot A \cdot A^{-1} \cdot A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$.
- Proposition: $(A^T)^{-1} = (A^{-1})^T$
  Proof: $A^T \cdot (A^{-1})^T = (A \cdot A^{-1})^T = I \quad \Rightarrow (AB)^{-T} = A^{-T} B^{-T}$
- Proposition: Let A, B be $n \times n$ nonsingular. Then AB nonsingular.
  Proof: $(AB)^{-1} = B^{-1}A^{-1}$. So $A \cdot B \cdot B^{-1} A^{-1} = A(B \cdot B^{-1}) \cdot A^{-1} = A I A^{-1} = A A^{-1} = I$ q.e.d.
  ↳ then we can also say that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Section 3.9 · 3.10

- **Elementary matrix**: $I - uv^T$, $u, v \in \mathbb{R}^n$, $v^T u \neq 1$ $I-uv^T$ is nonsingular bc $\neq 0$
- We can write the inverse of $A + uv^T$ if $1 + v^T A^{-1} u \neq 0$

Proposition: elementary matrix is nonsingular we prove if the inverse exist + inverse is elementary.

Proof: $(I - uv^T)^{-1} = I - \frac{uv^T}{v^Tu - 1} = I + \frac{uv^T}{1 - v^Tu}$

eg. $v = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $u = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $v^T u = 1$ so $vu^T = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ so $I - uv^T = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix}$ this is singular bc rank 1

$(I - uv^T)^{-1} = I - \left(\frac{1}{v^Tu - 1}\right) \cdot uv^T$ we can rewrite rank 1 updates

- Look at the elementary operations:
  i) interchange rows $\begin{vmatrix} 1 & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} \\ 0 & \boxed{1} & \boxed{0} \end{vmatrix} = I - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \to 1\,0\,{-1}\,1\,1 \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}$
  ii) multiply by scalar $\begin{vmatrix} 1 & & \\ & \alpha & \\ & & 1 \end{vmatrix} = I - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 0 \end{vmatrix} \to 1\,0\,1\,0\,1 \begin{vmatrix} 1 \\ 1-\alpha \\ 1 \end{vmatrix}$
  iii) add to a row a multiple of another $\begin{vmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I - \begin{vmatrix} 0 & 0 & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \to 1\,1\,0\,0\,1 \begin{vmatrix} 0 \\ \alpha \\ 0 \end{vmatrix}$

- Gaussian elimination $\equiv$ multiplying by elementary operations

eg. $A = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} \to \begin{vmatrix} 4 & 2 & 3 \\ 0 & 1/2 & 1/4 \\ 0 & 1 & 3/2 \end{vmatrix} \to \begin{vmatrix} 4 & 2 & 3 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{vmatrix} = U$ upper

$m_{21} = -1/4$
$m_{31} = -1/2$

$(E_{21} E_{31} A)$ $(E_{32} E_{31} E_{21} A) = U \Rightarrow A = (E_{32} E_{31} E_{21})^{-1} U = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$

$E_{21} = \begin{vmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $E_{31} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{vmatrix}$ $E_{32} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

$E_{21}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $E_{31}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{vmatrix}$ $E_{32}^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix}$ $\Big\}$ $\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2 & 1 \end{vmatrix}}_{L \text{ lower}} \cdot U = A$

- We just did **LU Factorization**: A $n \times n$ matrix, can be written as $PA = L \cdot U$ and for $\forall b$ $Ax = b \equiv Ly = b$ and $Ux = y$
  - ↳ L unit is lower triangular, U is upper triangular

eg. $A = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix}$ $b = \begin{vmatrix} -1 \\ -1 \\ -3 \end{vmatrix}$ $L = \begin{vmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2 & 1 \end{vmatrix}$ $U = \begin{vmatrix} 4 & 2 & 3 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{vmatrix}$

$Ly = b = \begin{vmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & 2 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} -1 \\ -1 \\ -3 \end{vmatrix}$ $y_1 = -1$ $\frac{1}{4} y_1 + y_2 = 1$, $y_2 = -1 + 1/4 = -\frac{3}{4}$ $y_3 = -3 + \frac{1}{2} + \frac{3}{2} = -1$ $y = \begin{vmatrix} -1 \\ -3/4 \\ -1 \end{vmatrix}$ $x = \begin{vmatrix} 1 \\ -1 \\ -1 \end{vmatrix}$

$Ux = y = \begin{vmatrix} 4 & 2 & 3 \\ 0 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -1 \\ -3/4 \\ -1 \end{vmatrix}$ $x_3 = -1$ $x_2 = (-3/4 - \frac{1}{4}) 2 = -1$ $x_3 = (-1 + 2 + 3)/4 = 1$

↳ L is always nonsingular, U is singular if there is a 0
  ↳ A is nonsingular iff U is nonsingular

- $E = I - uv^T \Rightarrow E^T = (I - uv^T)^T = I^T - (uv^T)^T = I - (v^T)^T (u)^T = I - vu^T$ this is also elementary just different
  - ↳ $(EA)^T = A^T E^T$ apply on left is equiv to apply on right transposed.
- For fixed P, let $PA = LU$ w/ L lower unit triangular + U upper triangular, then L + V unique

Proof: Let L_1, L_2 be lower unit Δ and U_1, V_2 ∇. $PA = L_1 U_1 = L_2 V_2$. Then $L_1^{-1} L_1 U_1 = L^{-1} L_2 V_2 = U_1$. But then $U_1 V_2^{-1} = L_1^{-1} L_2$. Since inverse of ∇ is also ∇, we have $\nabla \nabla = \Delta \Delta$ so we can say that $\nabla = \Delta$. This then must be a diagonal matrix (both upper + lower). Since L_1 and L_2 are lower <u>unit</u> Δ. The diagonal consists of all 1's. Then $U_1 U_2^{-1} = L_1^{-1} L_2 = I$. Then $L_1 = L_2$ and $U_1 = U_2$.

eg. $A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix}$ $E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 2 & -1 & -1 \end{vmatrix}$ $E_{31} E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{vmatrix}$

$m_{21} = -1$
$m_{31} = -1$
$m_{32} = -4$

$E_{32} E_{31} E_{21} A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{vmatrix} \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} = U$ $E_{32} E_{31} E_{21} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -4 & 1 \end{vmatrix}$

$\underset{A}{\begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix}} = \underset{L}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 & 0 \end{vmatrix}} \underset{U}{\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 9 \end{vmatrix}}$

$Ax = b$ $b = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}$

$Ly = b$ $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}$ $y_1 = 3$ $y_2 = -2 - (1 \cdot 3) = -2 - 3 = -5$ $y_3 = 1 - (4 \cdot -5) - (1 \cdot 3) = 18$

$Ux = y$ $\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 3 \\ -5 \\ 18 \end{vmatrix}$

Section 4.1 + 4.2

2/21

$\boxed{4.1}$

- Sets closed under addition + scalar multiplication: $\mathbb{R}, \mathbb{C}, \mathbb{R}^{m \times n}$
  - ↳ a vector space is a set which is closed under addition + scalar multiplication ; $V = \{$set of linear functions from $X \to Y\}$ $\mathbb{R}^2 + \mathbb{R}^3$
  - ↳ $\Pi_n = \{$polynomials of degree $\leq n\}$ isometric to $\mathbb{R}^{n+1}$
  - ↳ Conditions of vector spaces: - if $x, y \in V$, then $x+y \in V$.
    - if $x \in V$, then $\alpha x \in V$, $\forall \alpha \in F$.
    - if $x, y \in V$ and $\alpha \in F$, then $\alpha x + y \in V$.
    - $0 \in V$ such that $x + 0 = x$.
  - ↳ A vector is an element of a vector space.

must be linear

- Let $S \subseteq V$ and V is a vector space. S is called a subspace and is a vector space closed under addition and scalar multiplication: if $x, y \in S$, then $x+y \in S$, $\alpha x \in S$, and $\alpha x + y \in S$.
- $\{v_1, v_2, v_3, \ldots, v_n\} \in V$ we say span the subspace $S = \{V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n\}$

eg. Do $v_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$, $v_2 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$, $v_3 = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}$ span $\mathbb{R}^3$? can every vector be written as a linear combination of the plane?

(is the system consistent for all b) $A = |v_1 \; v_2 \; v_3|$ $x = \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{vmatrix}$ is there $Ax = b$ for $\forall b \in \mathbb{R}^3$?

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{rank} = 3$$

$\boxed{4.2}$

- Two very important subspaces: - $N(A) = \{x \in \mathbb{R}^n \text{ w/ } Ax = 0\}$ → Null space
  - $R(A) = \{y \,/\, \exists x \text{ s.t. } Ax = y\}$ → Range, image, column space
  - ↳ these are subspaces: $v, w \in R(A)$, $v = Ay$ + $w = Az$. then $\alpha v + w = \alpha Ay + Az = A(\alpha y + z)$

    $v, w \in N(A)$, $Av = 0$ + $Aw = 0$. then $A(\alpha v + w) = \alpha Av + Aw = 0 + 0 = 0$
    - ↳ we show that closed under (+) and ($\alpha \cdot$)

eg. $S = \{x \in \mathbb{R}^3 : 3x_1 + x_2 + x_3 = 0\}$ $v = \begin{vmatrix} 3 \\ 1 \\ 1 \end{vmatrix}$ $x^T v = 0$ and $v^T x = 0$

$S^{\perp} = \{x : x^T y = 0, y \in S\}$
this is a subspace

Section 4.3

- Let $v_1, v_2, \dots v_n \in V$, we say that they are **linearly dependent** if one can be written as a linear combination of the others.

  eg. $v_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$ $v_3 = \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix}$ $\Rightarrow v_3 = 2\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

  ↳ Equivalently: if $A = [v_1\ v_2 \cdots v_n]$ then $v_1, v_2, \dots v_n$ are linearly dependent if $Ax = 0$ has a nontrivial solution.

  eg. Determine if linearly independent: $\begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$ $\begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix}$ $\begin{vmatrix} 5 \\ 6 \\ 7 \end{vmatrix}$

  $$\begin{vmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 5 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 5 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{vmatrix} \leftarrow \text{the null space is nontrivial}$$

  $m_1 = -2$, $m_2 = -1$ $\qquad$ $m_3 = 1/2$

  $\boxed{\text{dependent}}$

  find coefficients $-2x_2 = 4x_3$ $\qquad$ $x_1 + x_2 + 5x_3 = 0$

  $x_2 = -2x_3$ $\qquad$ $x_1 - 2x_3 + 5x_3 = 0$ $\qquad$ $x_1 = -3x_3$ $\qquad$ $x = \begin{vmatrix} -3 \\ -2 \\ 1 \end{vmatrix}$

  ↳ Equivalently: $v_1, v_2, \dots v_n$ are **linearly independent** if none of them can be written as linear combinations of the other

  ↳ equivalent condition: if a linear combination exists and $\Sigma\, \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

  ↳ In a $\mathbb{R}^{n \times n}$ if columns are linearly independent then the matrix is nonsingular. (dependent $\Rightarrow$ singular).

  ↳ linear independent if linear combination $= 0$ and all coefficients are zero.

- For an $m \times n$ matrix, the following are equivalent:

  i) columns of $A^{m \times n}$ are linearly independent
  ii) $Ax = 0 \Rightarrow x = 0$
  iii) $N(A) = \{0\}$
  iv) $\text{Rank}(A) = n$

  a) columns of A are linearly dependent
  b) $Ax = 0$ for some $x \neq 0$
  c) $N(A) \supsetneq \{0\}$
  d) $\text{Rank}\, A < n$

- For an $n \times n$ (square) matrix, the following are equivalent:

  i) columns of A are linearly independent
  ii) A is nonsingular
  iii) $N(A) = \{0\}$ $[Ax = 0 \Rightarrow x = 0]$
  iv) rows of A are linearly independent
  v) $N(A^T) = \{0\}$ $[A^T x = 0 \Rightarrow x = 0]$
  vi) $\text{rank}\, A = n$
  vii) $\text{Range}(A) = R(A) = \mathbb{R}^n$
  viii) $R(A^T) = \mathbb{R}^n$

Section 4.4 + 4.5

- We saw **spanning sets** and also **linearly independent** sets $V = \text{Span}(v_1, v_2, \dots v_n) = \{x = \sum_{i=1}^{n} x_i v_i\}$
  - ↳ if $\{v_1, v_2 \cdots v_n\}$ is a linearly independent set, we say that $\{v_1, v_2 \dots v_n\}$ is a **basis** of V
  - ↳ we can also say the **dimension** of V is n.
    - ↳ # of elements = **cardinality** = dimension of the subspace.
- Let B basis of S and $\dim S = k$ where $B = \{v_1 \cdots v_k\}$
  - ↳ then B is a maximal set of linearly independent vectors + minimal spanning set.

$\boxed{\text{Quick Review}}$

- **Definition**: A vector space V is finite dimension (over a field F) if V admits a finite spanning set.
  - ↳ $\exists$ $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ s.t. every $v \in V$ can be written as $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \sum_{j=1}^{n} c_j \vec{v}_j$
    - ↳ otherwise V is infinite dim-l
- $\mathbb{R}^n$ is finite dim-l : $\left\{ \vec{e}_1 \begin{vmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix}, e_2 = \begin{vmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{vmatrix}, \dots, \vec{e}_n \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{vmatrix} \right.$ is finite + it is a spanning set
- Π_m is the vector space of polynomials of degree $\le m$ (Π_m is finite dim-l b/c $\{1, x, x^2, \dots x^m\}$ is a spanning set)
  - ↳ Π is the vector space of all poly-ls (in x)
    - ↳ Π does not admit a finite spanning set, Π is infinite dimensional if not defined

$\boxed{\text{4.4 con't}}$

- Definition: A subset B of a vector space V is called a **basis** if: i) B is linearly independent $(B = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\})$
  ii) B is a spanning
  - ↳ **linear independence** - $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0} \implies c_1 = c_2 = \dots = c_n = 0$

eg. $\mathbb{R}^n \to \left\{ \vec{e}_1 = \begin{vmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{vmatrix}, \vec{e}_2 = \begin{vmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{vmatrix}, \dots, \vec{e}_n = \begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{vmatrix} \right\}$ → this is std basis of $\mathbb{R}^n$

if $n = 3$ $\mathbb{R}^3$ has a basis $\left\{ \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \right\}$

Π_m has a basis $\{1, x, x^2, \dots, x^n\}$ if $C_0 1 + C_1 x + C_2 x^2 + \dots + C_m x^m = 0$ then $C_0 = C_1 = C_2 = \dots = C_m = 0$

eg. $A = \begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & -2 & -3 & -5 \\ 0 & 2 & 2 & 2 \end{vmatrix}$ $R(A)$ the column space of A is a subspace of $\mathbb{R}^3$

Find basis of $R(A)$: lets try the columns $\left\{ \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ -2 \\ 2 \end{vmatrix}, \begin{vmatrix} 0 \\ -3 \\ 2 \end{vmatrix}, \begin{vmatrix} -2 \\ -5 \\ 2 \end{vmatrix} \right\}$ is a spanning set but not linearly independent. $W_2 = W_1 + W_3$

We use row echelon form to solve: $\begin{matrix} \\ \\ \tfrac{1}{2}\times \end{matrix}\begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & -2 & -3 & -5 \\ 0 & 2 & 2 & 2 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & -2 & -3 & -5 \\ 0 & 1 & 1 & 1 \end{vmatrix} \to \begin{matrix} \\ -\tfrac{1}{3}\times \\ \\ \end{matrix}\begin{vmatrix} 1 & 1 & 0 & -2 \\ 0 & -3 & -3 & -3 \\ 0 & 1 & 1 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ in echelon form

basic

We use the basic columns as the answer: $\left\{ \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ -2 \\ 2 \end{vmatrix} \right\}$ is linearly independent + spans $R(A)$, thus is a basis

Find basis of $N(A) = \{x \in \mathbb{R}^4 : Ax = 0\}$ we know is a subspace in $\mathbb{R}^4$
↳ solution set of the homogeneous linear system

Change to reduced row echelon form → $\begin{vmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ this is unique $\quad x_1 = x_3 + 3x_4$
$x_2 = -x_3 - x_4$

soln set $= \left\{ \begin{vmatrix} x_3 + 3x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{vmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \left\{ x_3 \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} 3 \\ -1 \\ 0 \\ 1 \end{vmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{span}\left\{ \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 3 \\ -1 \\ 0 \\ 1 \end{vmatrix} \right\}$

$u_1 = \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix}$ $u_2 = \begin{vmatrix} 3 \\ -1 \\ 0 \\ 1 \end{vmatrix}$ $\{u_1, u_2\}$ is a spanning set + linearly independent ⟹ basis of $N(A)$

- **Proposition**: Let V be a finite dimensional vector space and $\{v_1, v_2 \dots v_n\}$ is a $\underbrace{\text{basis}}$ of V. Then every list $w_1, w_2, \dots, w_m$ of $m > n$ is linearly dependent.
  ↳ can even be just spanning set

Proof: need to show that $\exists$ $c_1, c_2 \cdots c_m \in \mathbb{R}$ not all zero s.t. $\sum_{j=1}^{m} c_j \vec{w}_j = \vec{0}$

$\forall\, 1 \le j \le m$, $\vec{w}_j \sum_{i=1}^{n} a_{ij} \vec{v}_i$ $a_{ij} \in \mathbb{R}$. $1 \le i \le n$ and $1 \le j \le m$. $A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix}$. Row reduction will give non-pivital columns

so $Ax = 0$ has a non-trivial soln.

$\vec{c} = \begin{vmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{vmatrix} \in \mathbb{R}^m$. Not all c_j are zero s.t. $A\vec{c} = \vec{0} \to \sum_{j=1}^{m} a_{ij} c_j = 0 \;\; \forall\, 1 \le i \le n$

$\sum_{j=1}^{m} c_j w_j = \sum_{j=1}^{m} c_j \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} c_j \right) \vec{v}_i = \vec{0}$

- **Corollary**: if B and $\tilde{B}$ are bases of a vector space V, then B and $\tilde{B}$ have same # of elements

Proof: B is a spanning set ⟹ every linearly independent set must have $\le \#B$ $(\#\tilde{B} \le \#B)$
$\tilde{B}$ is a spanning set ⟹ every linearly independent set must have $\le \#\tilde{B}$ $(\#B \le \#\tilde{B})$ $\Big\} \#B = \#\tilde{B}$

- Definition: the **dimension**, $\dim(V)$, of a vector space V is # of elements in any basis of V.

eg. $\dim(\mathbb{R}^n) = n$
$\dim(\Pi_m) = m + 1$ $(\{1, x, x^2, \dots x^m\})$

$A = \begin{vmatrix} 1 & 1 & 0 & -2 \\ 1 & -2 & -3 & -5 \\ 0 & 2 & 2 & 2 \end{vmatrix}$ → $R(A)$ has basis $\left\{ \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ -2 \\ 2 \end{vmatrix} \right\} \implies \dim(R(A)) = 2$
→ $N(A)$ has basis $\left\{ \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 3 \\ -1 \\ 0 \\ 1 \end{vmatrix} \right\} \implies \dim(N(A)) = 2$

Section 4.5 + 4.7

- Theorem: (Characterization of a basis) V is a finite dim-l vector space, $B \subseteq V$ where $B = \{v_1, v_2, \dots, v_n\}$, then the following are equiv:
  - i) B is a basis of V
  - ii) B is a minimal spanning set of V
  - iii) B is a maximal linearly independent subset of V

↳ Proof of equiv of (i) + (iii): given that B is a basis need to show B is max linearly indepen—t.

(i → iii) $B \subsetneq \tilde{B}$, from proposition $\tilde{B}$ is linearly dependent → B is maximal lin ind—pdnt

(iii → i) B is max l.i. subset of V need to show B is basis (spanning set)

$B = \{v_1, v_2, \cdots v_n\}$ $v \in V$: if $\vec{v} \in B$, $\vec{v} \in \text{span } B$. if $\vec{v} \notin B$, $B \subsetneq B \cup \{v\} = \{v_1, v_2, \cdots, v_n, v\}$ lin. dep.

$\exists\ c_1, c_2, \cdots c_n, c \in \mathbb{R}$ s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n + c\cdot\vec{v} = \vec{0}$ (c must $\neq 0$) $\vec{v} = -\frac{c_1}{c}\vec{v}_1 - \frac{c_2}{c}\vec{v}_2 - \dots - \frac{c_n}{c}\vec{v}_n \in \text{span}(B)$

eg. let V be an n-dimensional vector space + $\{v_1, v_2, \dots, v_n\}$ be lin. indep. Show that $\{v_1, v_2, \dots, v_n\}$ is a basis

Proof: need to show its a spanning set

$v \in V$ then $\{v_1, v_2, \cdots, v_n, v\}$ → by proposition it must be linearly dep. Contradition!

4.7

- A linear function: $f(\alpha x + y) = \alpha f(x) + f(y)$ or $T: V \to W$ on linear spaces

↳ Linear transformations on linear maps : $T(\alpha x + y) = \alpha T(x) + T(y)$ → $T\left(\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}\right) = \begin{vmatrix} x_1 + x_2 \\ x_3 \end{vmatrix}$ => $T = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

↳ If $S \subseteq V$, a subspace, then $T(S)$ is subspace of W

↳ Proof: take $v, w \in T(S)$, then $v = T(x)$ and $w = T(y)$ where $x, y \in S$. $\alpha v + w = \alpha T(x) + T(y)$, by def. By linearity of T we can say $\alpha v + w = T(\alpha x + y)$. Then $\alpha x + y \in S$ since it is a subspace. Therefore, $\alpha v + w \in T(S)$.

4.5

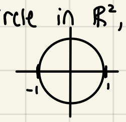
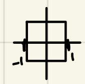
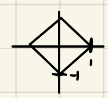
- $\text{Rank}(A \cdot B) = \text{Rank } B - \dim(N(A) \cap R(B)) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$ we can only decrease the Rank

Section 5.1 + 5.3

- A vector is an element of a vector space eg. a point in $\mathbb{R}^3$
- **Norm**: for $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^T x} = (x^T x)^{1/2}$ this number is considered the length
  - ↳ $\| \ \| : V \to \mathbb{R} = \{x \in \mathbb{R} : x \geq 0\}$ must satisfy the following conditions:
    1. $\|x\| \geq 0$
    2. $\|x\| = 0 \Rightarrow x = 0$ the only vector w/ zero length is the zero vector
    3. $\|\alpha x\| = |\alpha| \|x\|$
    4. $\|x+y\| \leq \|x\| + \|y\|$ triangular inequality
  - ↳ Euclidean norm: $V \in \mathbb{R}^2$ or 2-norm
- **Inner product** or dot product btwn 2 vectors $\langle v, w \rangle : V \times V \to \mathbb{R}$ eg. $\langle v, w \rangle = v^T w = \sum v_i w_i$
  - ↳ **Bilinear**: linear in each component ; $\langle \alpha x + y, v \rangle = \alpha \langle x, v \rangle + \langle y, v \rangle$
  - ↳ $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Rightarrow x = 0$
  - ↳ **Symmetric**: $\langle x, y \rangle = \langle y, x \rangle$
- For any inner product, there is a norm induced by it, but not vice versa.
  - ↳ $\|x\| = \sqrt{\langle x, x \rangle}$
  - ↳ $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle} = \sqrt{\alpha^2 \langle x, x \rangle} = \sqrt{\alpha^2}\sqrt{\langle x, x \rangle} = \|\alpha\| \cdot \|x\|$

eg. $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ prove is a norm by checking conditions.

eg. $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$, $1 \leq p \leq 2$

- **Unit sphere**: unit circle in $\mathbb{R}^2$, set of all vectors, v, w/ length, $\|v\| = 1$.
  - ↳ for $\|v\|_2 = 1$  $\begin{vmatrix} 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \end{vmatrix} \Rightarrow \begin{vmatrix} \sin\theta \\ \cos\theta \end{vmatrix}$ for $\|v\|_\infty = 1$  for $\|v\|_1 = 1$ 

- **Cauchy-Bunyakowski-Schwarz inequality (CBS)**: $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ where $\|x\| = \sqrt{\langle x, x \rangle}$
  - ↳ Proof: if $x = 0$ $\langle x, y \rangle = 0$ and $\|x\| = 0$. The inequality holds. For $x \neq 0$, let $\alpha = \langle x, y \rangle / \|x\|^2 = \langle x, y \rangle / \langle x, x \rangle$. Let $v = \alpha x - y$. Then $\langle x, v \rangle = \langle x, \alpha x - y \rangle = 0$ since $\langle x, \alpha x - y \rangle = \alpha \langle x, x \rangle - \langle x, y \rangle = 0$. Consider $\|v\|^2$. We know $\|v\|^2 \geq 0$.
    $\|v\|^2 = \|\alpha x - y\|^2 = \langle \alpha x - y, \alpha x - y \rangle = \alpha \langle x, \alpha x - y \rangle - \langle y, \alpha x - y \rangle = 0 - \langle y, \alpha x - y \rangle = \langle y, y \rangle - \alpha \langle y, x \rangle = \|y\|^2$
    $- \frac{\langle x, y \rangle \langle y, x \rangle}{\|x\|^2} = \|y\|^2 - |\langle x, y \rangle|^2 / \|x\|^2 \geq 0 \to |\langle x, y \rangle|^2 \geq \|y\|^2 \|x\|^2$
  - ↳ This is used to show that any induced norm satisfies Δ inequality.
    $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$
  - ↳ $p, q \in \Pi_n = V$ where $(p, q) = \int_{-\infty}^{\infty} p(x) q(x)\, dx \to$ induced norm: $\|p\| = (p, p) = \left(\int p^2(x)\, dx\right)^{1/2} \geq 0$
  - ↳ $V = \mathbb{R}^{m \times n}$ $\langle A, B \rangle = \operatorname{trace}(A^T B) \to \langle A, A \rangle = \operatorname{trace}(A^T A) = \sum a_{ij}^2 \geq 0 \to \|A\| = \left(\sum a_{ij}^2\right)^{1/2}$

Section 5.2 • 5.4

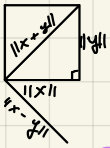
• We say that $x, y \in V$ are orthogonal if $\langle x, y\rangle = 0$ inner pdt is zero

↳ The standard inner pdt in $\mathbb{R}^n$ is $\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i = 0$ and it corresponds to $90°$

eg. $x = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $y = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $\langle x, y\rangle = x^T y = 0$

eg. $p(x) = 1$ and $q(x) = x - \alpha$ where $\langle p, q\rangle = \int_0^1 pq\,dx$ what value of α will make $\langle p, q\rangle = 0$, orthogonal

$\int_0^1 (x-\alpha)\,dx \rightarrow \frac{x^2}{2} - \alpha x \Big|_0^1 \rightarrow \frac{1}{2} - \alpha \rightarrow \alpha = \frac{1}{2}$



$\|x\|^2 + \|y\|^2 = \|x+y\|^2$ Pythagoras Thrm : in $\mathbb{R}^n$ for any V, if $x^T y = 0$ 2 are orthogonal then $\|x\|^2 + \|y\|^2 = \|x-y\|^2$

$\|x\|^2 + \|y\|^2 = \|x-y\|^2$ Proof : $\|x\|^2 + \|y\|^2 - \|x-y\|^2 = \langle x,x\rangle + \langle y,y\rangle - \langle x-y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle - \langle x,x\rangle - \langle y,y\rangle + 2\langle x,y\rangle$

eg. $x = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$ $y = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}$ $x^T y = 1 + 0 - 1 = 0$ eg. $x = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix}$ $y = \begin{vmatrix} 4 \\ 0 \\ 0 \\ -1 \end{vmatrix}$ $x^T y = 4 + 0 + 0 - 4 = 0$

• A set $\{v_1, v_2, \cdots, v_n\}$ is orthogonal if $\langle v_i, v_j\rangle = 0$ and $i \neq j$

↳ Thrm : Let $\{v_1, v_2, \dots, v_n\}$ be an orthogonal set. Then $\{v_1, v_2, \dots, v_n\}$ is linearly independent. The linear combo $= 0$ if $\alpha = 0$

Proof : Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Then $\langle v_j, \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n\rangle = \alpha_1\langle v_1, v_j\rangle + \alpha_2\langle v_2, v_j\rangle + \cdots + \alpha_j\langle v_j, v_j\rangle + \cdots + \alpha_n\langle v_n, v_j\rangle$

Then by def of orthogonal $\langle v_i, v_j\rangle = 0$ and only $\langle v_j, v_j\rangle > 0$. Then α_j must $= 0$. Thus, linear ind.

• Collary : Orthogonal vectors form a basis

• Angle btwn 2 vectors : $v, w \in V$, $v \neq 0$ and $w \neq 0$ => $\cos\theta = \frac{\langle v, w\rangle}{\|v\|\,\|w\|}$

↳ Recall CBS inequality : $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ so we can say $|\cos\theta| \leq 1$

eg. $v = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$ $w = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $\cos\theta = \frac{\langle v, w\rangle}{\|v\|\,\|w\|} = \frac{3}{\sqrt{5}\sqrt{2}} = \frac{3}{\sqrt{10}}$ eg. $v = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ $w = \begin{vmatrix} 3 \\ 0 \\ 4 \end{vmatrix}$ $\cos\theta = \frac{3}{5\sqrt{2}}$ eg. $v = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $w = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $\cos\theta = \frac{1}{\sqrt{2}\sqrt{1}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\theta = \frac{\pi}{4}$

eg. $\int_{-1}^{1} p(x)\,q(x)\,dx$ $p(x) = x$, $q(x) = x^2 - x$ $\langle p, q\rangle = \int_{-1}^{1} x^3 - x^2 = \frac{x^4}{4} - \frac{x^3}{3}\Big|_{-1}^{1}$ $\left(\frac{1}{4} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{-1}{3}\right) = -\frac{2}{3}$

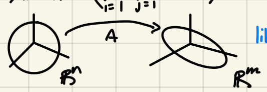
$\langle p, p\rangle = \left(\int_{-1}^{1} x^2\right)^{1/2} = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \left(\frac{2}{3}\right)^{1/2}$

$\langle q, q\rangle = \left(\int_{-1}^{1} (x^2 - x)^2\right)^{1/2} = \int x^4 - 2x^3 + x^2 = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} - \left(\frac{-1}{5} - \frac{1}{2} + \frac{-1}{3}\right) = \frac{2}{5} + \frac{2}{3} = \left(\frac{16}{15}\right)^{1/2}$

• Let U be a unit matrix w/ orthogonal columns. Then $U^T U = I$ and $u_i^T u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

• Matrix Norm : $\mathbb{R}^{m \times n}$ a vector space, $A \in \mathbb{R}^{m \times n}$, $\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$ and $\|AB\| \leq \|A\|\,\|B\|$ this condition + original 3

↳ Matrix norm as operator* from $\mathbb{R}^n$ to $\mathbb{R}^m$



like a map $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$ sup is maximum

*largest length of vector in the image of the unit sphere

• Orthogonal matrices : unitary matrices

↳ A has orthonormal columns where $A^T A = I$ bc $A^T A = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

↳ A maintains length and angles does not change

↳ $\|Qx\| = \|x\|$ bc $\|Q^k\| = 1$ always (that being it is orthogonal matrix)

Section 5.5

Gram-Schmidt Orthogonalization

- Given a set of vectors $v_1, v_2, v_3, \dots, v_n$. Find $q_1, q_2, q_3, \dots, q_n$ s.t. $\text{span}\{q_1\} = \text{span}\{v_1\}$, $\text{span}\{q_2\} = \text{span}\{v_2\}$ $\cdots$ $\text{span}\{q_1, q_2, \dots q_n\} = \text{span}\{v_1, v_2, \dots v_n\}$
  - ↳ $q_1, q_2, \dots q_n$ is the orthogonal set. $\left(\langle q_i, q_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}\right)$
  - ↳ Claim: if $w_2 = v_2 - \langle v_2, q_1\rangle q_1$ then $\langle w_2, q_1\rangle = 0$.
    - Proof: $\langle w_2, q_1\rangle = \langle (v_2 - \langle v_2, q_1\rangle q_1), q_1\rangle = \langle v_2, q_1\rangle - \langle v_2, q_1\rangle \overset{1}{\cancel{\langle q_1, q_1\rangle}} = 0$
  - ↳ To normalize q_2: $q_2 = \frac{w_2}{\|w_2\|}$
  - ↳ More generally: $w_j = v_j - \sum_{i=1}^{j-1} \langle v_j, q_i\rangle q_i = v_j - \langle v_j, q_1\rangle q_1 - \langle v_j, q_2\rangle q_2 - \cdots - \langle v_j, q_{j-1}\rangle q_{j-1}$
- Take an $n \times k$ matrix A w/ columns $a_1, a_2, \dots, a_k$. Find $Q = \{q_1, q_2, \dots, q_k\}$ s.t. $Q^TQ = I$ and $R(Q) = R(A)$
  - ↳ We can write $A = QR$ where R is upper triangular → QR factorization

eg. $v_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix}$ $v_3 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$

$q_1 = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$ $w_2 = v_2 - \langle v_2, q_1\rangle q_1 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} - \frac{1}{\sqrt{2}}(2)\frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} - \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix}$ → $\|w_2\| = 2$ → $q_2 = \frac{1}{2}\begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

$w_3 = v_3 - \langle v_3, q_1\rangle q_1 - \langle v_3, q_2\rangle q_2 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} - 1\begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1/2 \\ -1/2 \\ 0 \end{vmatrix}$ → $q_3 = \sqrt{2} \cdot \frac{1}{2}\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} = \frac{\sqrt{2}}{2}\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}$

- Observe that $A = Q \cdot R$, where R is upper triangular w/ positive diagonals :
  - ↳ $a_1 = q_1 \cdot \|a_1\|$ → $q_1 = a_1/\|a_1\|$ → $a_2 = \langle a_2, q_1\rangle q_1 + \|w_2\| q_2$ bc $w_2 = \|w_2\| q_2 = a_2 - \langle a_2, q_1\rangle q_1$
  - ↳ $R_{ij} = \langle a_j, q_i\rangle$ where $i < j$ and $R_{ij} = \|w_i\|$ when $i = j$

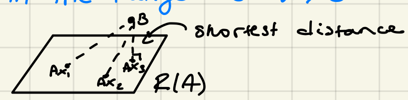
eg. $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = Q \cdot R = \begin{vmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} \sqrt{2} & 2/\sqrt{2} & 1/\sqrt{2} \\ 0 & 2 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{vmatrix}$ inverse of Q is Q^T

↳ we use QR factorization → $Ax = b$ → $QRx = b$ → $Qy = b$, $Rx = y$ → $y = Q^Tb$

- Least Squares Problem

$Ax = b$

eg. $A = \begin{vmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{vmatrix}$ $b = \begin{vmatrix} 2 \\ 3 \\ 4 \end{vmatrix}$ → $\left|\begin{matrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{matrix}\middle|\begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\right|$ → $\left|\begin{matrix} 1 & 1 \\ 0 & -1 \\ 0 & -1 \end{matrix}\middle|\begin{matrix} 2 \\ -1 \\ 0 \end{matrix}\right|$ → $\left|\begin{matrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 2 \\ -1 \\ 1 \end{matrix}\right|$ b is not in the Range bc $\alpha \neq 0$



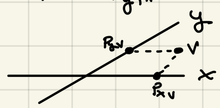
- ↳ there is no soln. How can we find the best possible soln.
- ↳ find x s.t. $\|b - Ax\|$ is small as possible $\{x \in \min_x \|b - Ax\|\}$
  - ↳ shortest distance is when $b \perp R(A)$
  - ↳ Use QR factorization

eg con't. $Q = \begin{vmatrix} 1/3 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \\ 2/3 & -\sqrt{2}/6 \end{vmatrix}$ $R = \begin{vmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{vmatrix}$ $\|Ax - b\| = \|QRx - b\|$ $Q^TQ = I$ $\|Q\| = 1 = \|Q^T\|$ so $\|Ax - b\| = \|Q^T(QRx) - b\|$

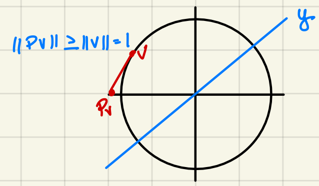
$= \|Rx - Q^Tb\|$

Section 5.9

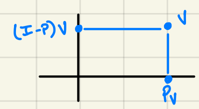
- Let $\vec{x}, \vec{y}$ be subspaces of $\vec{V}$. then $x+y = \{v = x+y \cdot x \in X \text{ and } y \in Y\}$
- X and Y are called complementary if : i) $\vec{x} + \vec{y} = \vec{V}$
  - ↳ $\dim(x+y) = \dim(V)$ ii) $\vec{x} \cap \vec{y} = \{0\}$
  - ↳ if complementary, any $v \in V$ can be written as $v = x+y$ in a <u>unique</u> manner.
    - ↳ Proof: suppose $v = x_1 + y_1 = x_2 + y_2$, then $x_1 - x_2 = y_2 - y_1$. But then $x_1 - x_2 \in X$ and $y_2 - y_1 \in Y$. But since $x_1 - x_2 = y_2 - y_1$ then $x_1 - x_2 \in Y$ and $y_2 - y_1 \in X$. But $X \cap Y = \{0\}$. so $x_1 - x_2 = 0 = y_2 - y_1$. Therefore $x_1 = x_2$ and $y_1 = y_2$.
- If X, Y are complementary, $P_x V = X$ projection x onto y and $P_y V = y$
  - ↳ $P_x + P_y = I \rightarrow v = P_x v + P_y v$ for all $v \in V \rightarrow P_{y,x} = I - P_{x,y} \rightarrow P_x^2 = P_x \rightarrow P_x P_y = 0 = P_y P_x$
  - ↳ P_x : projection onto X along Y



  - ↳ P_y : projection onto Y along X
  - ↳ $P^2 = P$; P a projection onto $X = R(P)$ along $Y = N(P)$
  - ↳ P is always singular as long as $P \neq I$. → bc the null space is not $\{0\}$
  - ↳ $\|P\| \geq 1$ never smaller than 1, P^2 will never go to 0
    - ↳ Proof: $P = P^2 = P \cdot P$. $\|P\| = \|P^2\| \leq \|P\| \cdot \|P\|$. so $1 \leq \|P\|$ divide both by $\|P\|$
- Recall: $\|A\| = \sup \|Ax\|$ where $\|x\| \neq 0$ when A is an operator and $\|A\| = \sup \|Ax\|$ if $\|x\| = 1$
  - ↳ if $\|Ax\| = \alpha \|x\|$ $\forall x$, then $\|A\| \leq \alpha$



- Recall: $P^2 = P$; projects onto $\overset{x}{R(P)}$ along $N(P) = \overset{y}{R(I-P)}$ → a complementary projection
  - ↳ $I - P$ projects onto $R(I-P) = N(P)$ along $N(I-P) = R(P)$
  - ↳ Note that $P(I-P) = 0$ and $P + (I-P) = I$
- What if projection is orthogonal? $X \perp Y$, $P^2 = P$
  - ↳ $(Pv) \perp (I-P)v$ $\forall v$



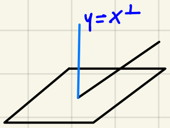
  - ↳ if $P^T = P$ then $X \perp Y$, $R(P) \perp N(P)$, and $\|P\| = 1$ symmetric is implied
    - ↳ Proof of orthogonality: Let $z \in R(P)$ and $z = Pw$ for some w. Also let $x \in N(P)$ such that $Px = 0$. Then $\langle z, x \rangle = \langle Pw, x \rangle$ $= w^T P^T x = w^T \overset{0}{\overbrace{Px}} = 0$.
  - ↳ By pythagoras: $\|P\| \leq 1$ in general, if $\|P\| = 1$, it is an orthogonal projection, if $\|P\| < 1$ it is an oblique projection
- Take an orthogonal matrix (not square) → $Q = \square$. Let $P = QQ^T = \square \, \square = \square$
  - ↳ Then $P^2 = Q \underbrace{Q^T Q}_{I} Q^T = QQ^T = P$. $R(P) = R(Q)$ This is a special case
- Gram-Schmidt is used to find an orthonormal basis of a subspace
- Alternative formula for P, Q : Let $P = U$ where col of U are basis of X, let there be a V where col of V are a basis of $Y^\perp$ find so $V^T U$ makes square
  - ↳ Recall that $N(P) = Y$. We also know that $R(P) = R(U)$ → Then $P = U(V^T U)^{-1} V^T$: the projection onto $R(P)$ along $N(P)$ This is generic
  - ↳ Check: $P^2 = U(V^TU)^{-1} \overbrace{V^T U (V^T U)^{-1}}^{I} V^T = U(V^TU)^{-1}V^T = P$ ✓

eg. $U = \begin{vmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix}$ $V = \begin{vmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{vmatrix}$ $V^TU = \begin{vmatrix} -3 & -1 \\ -2 & -2 \end{vmatrix}$ $(V^TU)^{-1} = \begin{vmatrix} 1 & -1/3 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -1/3 & 0 \\ 0 & -3/4 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -2/3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1/3 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -1/3 & 0 \\ 1/2 & -3/4 \end{vmatrix} = \begin{vmatrix} -1/2 & -1/4 \\ 1/2 & -3/4 \end{vmatrix}$

$P = U(V^TU)^{-1}V^T = \begin{vmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -1/2 & -1/4 \\ 1/2 & -3/4 \end{vmatrix} \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -3/4 & 1/2 & 1/4 \\ -1/4 & -1/2 & 3/4 \end{vmatrix} = \begin{vmatrix} 7/4 & 1/2 & -5/4 \\ -3/4 & 1/2 & 1/4 \\ 4/4 & -1/2 & 3/4 \end{vmatrix} = \frac{1}{4} \begin{vmatrix} 7 & 2 & -5 \\ -3 & 2 & 1 \\ 4 & -2 & 3 \end{vmatrix}$

most painful thing EVER

Section 5.11 + 5.12



- $Y = X^\perp$ is the orthoganol complement to X
- Let M be a set of V, a vector space, w/ an inner pdt. then:
  - $M \oplus M^\perp = V$
  - $\dim M = 1$, $M = \{\alpha v\}$
  - $M^\perp = \{x \in V : \langle x, v\rangle = 0\}$
  - $M^\perp = \{x \in V : \langle x, v\rangle = 0 \ \forall v \in M\}$

- To find the orthoganol complement: 1) given the basis of X, pick 2 vectors
  2) Do QR / Gran-Schmidt to get orthoganol basis of X

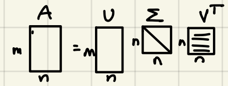
eg. $v = \begin{vmatrix}1\\2\\3\end{vmatrix} \to \|v\| = \sqrt{14} \to X = \left\{\alpha \begin{vmatrix}1\\2\\3\end{vmatrix}\right\}$

$q_1 = \frac{1}{\sqrt{14}}\begin{vmatrix}1\\2\\3\end{vmatrix}$ $\quad w_1 = \begin{vmatrix}1\\0\\0\end{vmatrix}$ $\quad w_2 = \begin{vmatrix}0\\1\\0\end{vmatrix}$ $\quad z_2 = w_1 - \langle w_1, q_1\rangle q_1 = \begin{vmatrix}1\\0\\0\end{vmatrix} - \frac{1}{\sqrt{14}}\frac{1}{\sqrt{14}}\begin{vmatrix}1\\2\\3\end{vmatrix} = \frac{1}{14}\begin{vmatrix}13\\-2\\-3\end{vmatrix}$ $\quad \|z_2\| = \frac{1}{14}\sqrt{13^2+4+9} = \frac{1}{14}\sqrt{182}$

$q_3 = \frac{14}{\sqrt{182}}\begin{vmatrix}13\\-2\\-3\end{vmatrix}$ $\quad z_3 = w_3 - \langle z_3, q_1\rangle q_1 - \langle z_3, q_2\rangle q_2 \to q_2 = \frac{z_3}{\|z_3\|}$

- Recall: $R(A) \perp N(A^T)$ and $\dim R(A) + \dim N(A^T) = n$ and $\dim R(A^T) + \dim N(A) = m$
  - ↳ we can say that $R(A) \oplus N(A^T) = \mathbb{R}^n$ and $R(A^T) \oplus N(A) = \mathbb{R}^m$

Singular Value Decomposition



- We know that $A = LU$ and $A = QR$ → New decomposition. $A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$
  - ↳ U, V are orthoganol matrices where $U^TU = I_m$ and $V^TV = I_n$
  - ↳ Σ (sigma) is the diagonal matrix where $\sigma_i \geq 0$

eg. $A = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{vmatrix}1&1&0\\0&0&\sqrt{2}\\1&-1&0\end{vmatrix}\begin{vmatrix}3&0&0\\0&2&0\\0&0&1\end{vmatrix}\begin{vmatrix}1&1&0\\1&-1&0\\0&0&\sqrt{2}\end{vmatrix} = \frac{1}{2}\begin{vmatrix}5&1&0\\0&0&2\\1&5&0\end{vmatrix} = 3u_1v_1^T + 2u_2v_2^T + u_3v_3^T$

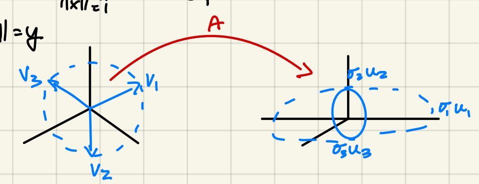
↳ lets find a B of Rank = 2

↳ $B = 3u_1v_1^T + 2u_2v_2^T = \frac{1}{2}\begin{vmatrix}1&1\\0&0\\1&-1\end{vmatrix}\begin{vmatrix}3&0\\0&2\end{vmatrix}\begin{vmatrix}1&1&0\\1&-1&0\end{vmatrix} = \frac{1}{2}\begin{vmatrix}5&1&0\\0&0&0\\1&5&0\end{vmatrix}$

$A - B = \frac{1}{2}\begin{vmatrix}0&0&0\\0&0&2\\0&0&0\end{vmatrix} = \begin{vmatrix}0&0&0\\0&0&1\\0&0&0\end{vmatrix} \to \|A-B\| = 1 = \sigma_3$

- A map $\mathbb{R}^n \to \mathbb{R}^m$ has $v = \langle v_1, v_2, v_3 \ldots v_n\rangle$ which is an orthogonal basis of $\mathbb{R}^n$, so $A = \sum_{i=1}^{n}\sigma_i u_i v_i^T$ then $Av_j = \sum_{i=1}^{n}\sigma_i u_i v_i^T v_j = \sigma_j u_j$ $\quad v_i^Tv_j = 0$ if $i \neq j$, $v_i^Tv_j = 1$ if $i = j$
  - ↳ u_j has length 1
  - ↳ σ_j has length of image of v_j, when A is applied → so σ_j is maximum stretch $= \|A\| = \max_{\|x\|=1}\|Ax\| = \sigma_1$
    - ↳ proof: recall U and V are unitary orthoganol so that $\|Ux\| = \|x\|$ and $\|Vx\| = x$ or $\|V^Ty\| = y$

$$\|A\| = \max_{\|x\|=1}\|U\Sigma V^Tx\| = \max_{\|V^Tx\|=1}\|U\Sigma(V^Tx)\| = \max_{\|y\|=1}\|\Sigma(V^Tx)\| = \max_{\|y\|=1}\|\Sigma y\| = \sigma_1$$



- Note if A is square and nonsingular then $A^{-1} = (U\Sigma V^T)^{-1} = V^{-T}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$
  - ↳ the $\max\|A^{-1}\|_2 = \frac{1}{\sigma_n}$
- SVD exists for any $m \times n$ matrix, but is not unique U, V unique up to 1st sign Σ is always positive

eg. If given A, find σ_1, u_1, v_1. we can find orth. basis of $\mathbb{R}^n$ s.t. $\{v_1, v_2, \ldots v_n\}$ and $\mathbb{R}^m$ s.t. $\{u_1, u_2, \ldots u_m\}$

Then $U_1^T A V_1 = \begin{vmatrix}\sigma_1 & w \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{vmatrix} = T_1 \to \|T\| = \|A\| = \sigma_1$ repeat process for σ_2 and σ_3 $\quad w \in \mathbb{R}^{m-1}$ if we show $w = 0$ then $\sigma_2 = \|B\|$

Study: norms + matrix norms
- inner pdts
- complementary subspaces
- Orthoganol complements $\quad$ orthonormal → norm = 1, Q gives orthonormal
- Singular value decomp.
- Projection $P^2 = P$

Section 6.1 • 6.2

- A $n \times n$ matrix is singular iff $\det A = 0$. If $\det A$ is 2×2 then $\det A = a_{11}a_{22} - a_{12}a_{21}$
  - ↳ in general: $\det A = \sum_p \sigma_p a_{1p} a_{2p_2} \cdots a_{np_n}$ sum of all permutations

eg. $A = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} \Rightarrow \det A = 1(3-4) - 0(2-4) + 1(2-3) = -2$ $\quad B = \begin{vmatrix} 3 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix}$ $\quad \det B = 3(-1-4) - 0(\quad) + 1(2-2) = -15$

- ↳ Cramer's Rule is used to solve linear systems : $x_i = \dfrac{\det A_i}{\det A}$ where column i is replaced w/ b not best method

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$\boxed{\text{Eigen value}}$

- **Eigenvectors**: in a $n\times n$ matrix, v is an eigenvector if there is a $\lambda \in \mathbb{R}$ or $\mathbb{C}$ s.t. $Av = \lambda v$ privelegded directions
  - ↳ Note: if there is a zero eigenvalue, A is singular there is no zero if nonsingular

eg. $\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \Rightarrow$ nonsingular $\quad \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 4 \\ 4 \end{vmatrix} = 4\begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $\quad + \quad \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 1 \\ -1 \end{vmatrix} = \begin{vmatrix} 2 \\ -2 \end{vmatrix} = 2\begin{vmatrix} 1 \\ -1 \end{vmatrix}$

  - ↳ $N(A) = \{$ set of eigenvectors for $\lambda = 0 \}$
  - ↳ $Av = \lambda v \iff (A - \lambda I)v = 0$; we have eigenvector $v \neq 0$ w/ eigen value λ iff $A - \lambda I$ is singular iff $\det(A - \lambda I) = 0$
    - ↳ $\det(A - \lambda I)$ is a polynomial of deg n, $p(\lambda) \Rightarrow$ the eigen values are characteristic polynomial of A

eg. $A - \lambda I = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \Rightarrow \det(A - \lambda I) = (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$

- $\sigma(A)$, the spectrum of A, $= \{$ set of all eigenvalues of A $\} = \{$ roots of char. poly. $\}$

eg. $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow p(\lambda) = (1-\lambda)(-1-\lambda) - 1 = -1 + \lambda^2 - 1 = \lambda^2 - 2 \Rightarrow \lambda = \pm\sqrt{-2} = \pm\sqrt{2}\,i$

- How do we find eigenvector after finding eigen values? Find the nullspace

eg. $\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = -x_1 + x_2 = 0 \quad v = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ and $\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow v = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

- For a fixed λ, the set of eigenvectors corresponding to $\lambda = N(A - \lambda I)$ aka the eigenspace $\Rightarrow$ it is a subspace!
  - ↳ $\dim N(A - \lambda I)$ is called the geometric multiplicity of λ cannot be larger than multiplicity of eigen values
  - ↳ the algebraic multiplicity of $\lambda \geq$ geometric multiplicity of λ.
    - ↳ if geo $\lneq$ algebraic, then λ is called defective and if A has one or more defective eigen value(s), A is defective
  - ↳ take any $v \in N(A - \lambda I)$ then $Av = \lambda v \Rightarrow$ an invariant subspace: $v \in S$ and $Av \in S$

eg. $A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$ $\det(A - \lambda I) = (2-\lambda)\det\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \rightarrow \sigma(A) = \{2, 4\}$ 2 has mult. 2 and 4 has mult. 1 $\Rightarrow$ $Eig = \left\{\alpha\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + \alpha\begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}\right\}$

- If A is diagonal or triangular, $\sigma(A) = \{a_{11}, \dots, a_{nn}\}$, the set of eigenvalues are the diagonal entries.

Power method

$A = V \Lambda V^{-1}$

$e^A = \sum \frac{1}{k!} A^k$ converge if eigenvalues are neg $\Big\}$ 7.2

study 6.1, 7.1, 7.2

Review

AB nonsing if both A + B nonsing

$A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{vmatrix}$

$\det(AB) = \det A \cdot \det B$ $\quad \det A = \sum_{i \text{ or } j} (-1)^{i+j} a_{ij} \det A_{ij}$ $\quad \det(A - \lambda I) = (2-\lambda) \cdot \det \begin{vmatrix} x & x \\ x & x \end{vmatrix}$

Λ is the complete set of eigenvectors $\rightarrow$ $AV = V\Lambda$ and $A = V\Lambda V^{-1} \rightarrow$ power method

A is diagonalizable if $\exists$ nonsing matrix P w/ $P^{-1}AP =$ Diagonal.

To prove linear independence, prove that for a lin combo : $\alpha_1 v_1 + \alpha_2 v_2 \cdots$, all $\alpha = 0$

$A = U\Sigma V^T \rightarrow A^T = V\Sigma U^T \rightarrow A^TA = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T$ and $AA^T = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$ u + v are orthonormal

$\sigma_i(A) = \sqrt{\sigma_i(A^TA)}$

eigenvectors are nullspace of $(A - \lambda I)$

The spectral radius, ρ, = $\{\max |\lambda|, \lambda \in \sigma(A)\}$

Homework #13 due 4/25

Mercedes Nguyen

1) $A = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{vmatrix}$ $\quad \det(\lambda I_n - A) = 0$

$$\lambda I_3 - A = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-43/25 & 24/25 \\ 0 & 24/25 & \lambda-57/25 \end{vmatrix} \begin{matrix} \lambda-2 & 0 \\ 0 & \lambda-43/25 \\ 0 & 24/25 \end{matrix}$$

$$\Rightarrow (\lambda-2)(\lambda-43/25)(\lambda-57/25) + 0 + 0 - 0 - (\lambda-2)(24/25)(24/25) - 0 = 0$$

$$(\lambda-2)\left[(\lambda-43/25)(\lambda-57/25) - (24/25)^2\right] = 0$$

$$(\lambda-2)\left[\lambda^2 - 43/25\lambda - 57/25\lambda + \tfrac{2451}{625} - \tfrac{576}{625}\right] = 0$$

$$(\lambda-2)(\lambda^2 - 4\cancel{100/25}\lambda + 3) = 0$$

$$(\lambda-2)(\lambda-3)(\lambda-1) = 0 \Rightarrow \lambda = 1, 2, 3$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 18/25 & -24/25 \\ 0 & -24/25 & 32/25 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 18/25 & -24/25 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0$$

$x_1 = 0$, $x_2 = 4/3 x_3$ free, $x_3 = 1$ $\quad V_1 = \begin{vmatrix} 0 \\ 4/3 \\ 1 \end{vmatrix}$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{vmatrix} - (2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & -7/25 & -24/25 \\ 0 & -24/25 & 7/25 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 24/7 \\ 0 & -27/7 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0 \quad V_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{vmatrix} - (3)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -32/25 & -24/25 \\ 0 & -24/25 & -18/25 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & 0 & 0 \\ 0 & -24/25 & -18/25 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0 \quad V_3 = \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix}$$

Check: $AV_2 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix} \qquad \lambda_2 V_2 = (2)\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$ ✓

$\lambda_1 V_1 = \begin{vmatrix} 0 \\ 4/3 \\ 1 \end{vmatrix} \qquad \lambda_2 V_2 = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix} \Rightarrow V_1^T V_2 = \begin{vmatrix} 0 & 4/3 & 1 \end{vmatrix} \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix} = (0)(2) + (4/3)(0) + (1)(0) = 0$ ✓

2) If $A = A^T$, then A is symmetric by definition. Suppose A has 2 distinct eigenvalues λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$. Then $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. But then, $(Av_1)^T = v_1^T A^T = Av_1^T = \lambda_1 v_1^T$, similarly for v_2. But then, $v_2(Av_1^T) = \lambda_1 v_1^T v_2$, since $v_2 A = Av_2 = \lambda_2 v_2$, then $\lambda_2 v_2 v_1^T = \lambda_1 v_1^T v_2$. Then $\lambda_2 v_2 v_1^T - \lambda_1 v_1^T v_2 = (\lambda_2 - \lambda_1)(v_1^T v_2) = 0$. But $\lambda_1 \neq \lambda_2$ so $(\lambda_2 - \lambda_1) \neq 0$. Then $(v_1^T v_2) = 0$. Thus they are orthoganol.

Homework #12 due 4/13

Mercedes Nguyen

1) $v_1 = \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix}$ $v_2 = \begin{vmatrix} -1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$ find orthoganol vector

From previous hw: We know that v_1 and v_2 are orthoganol. Another orthoganol vector would be $\{0\}$; $v_3 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$

2) $v_1 = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}$ $v_2 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix}$

$q_j = \frac{w_j}{\|w_j\|}$ $\quad w_j = v_j - \sum_{i=1}^{j-1} \langle v_j, q_i \rangle q_i$ $\quad a_1 = q_1 \|a_1\|$ $\quad a_j = \langle a_j, q_i \rangle q_i + \|w_j\| q_j$

$R_{ij} = \langle a_j, q_i \rangle$

a) construct an orthonormal basis $S = \text{span}\{v_1, v_2\}$

$\|v_1\| = \sqrt{1^2 + 2^2 + 3^2 + 1^2} = \sqrt{15}$ $\quad w_1 = v_1 - 0 = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}$ $\quad q_1 = \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}$ $\quad q_2 = \frac{1}{\sqrt{30}} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$

$w_2 = v_2 - \langle v_2, q_1 \rangle q_1 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix} - \frac{1}{\sqrt{15}}(5)\left(\frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}\right) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix} - \frac{1}{3}\begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} = \begin{vmatrix} -1/3 \\ -2/3 \\ 0 \\ 5/3 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$ $\quad \|w_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{25}{9}} = \sqrt{\frac{30}{9}} = \frac{\sqrt{30}}{3}$

Basis: $\left\{ \begin{vmatrix} 1/\sqrt{15} \\ 2/\sqrt{15} \\ 3/\sqrt{15} \\ 1/\sqrt{15} \end{vmatrix}, \begin{vmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ 0 \\ 5/\sqrt{30} \end{vmatrix} \right\}$

b) find basis of W $\quad W = S^\perp$

$A = \left| \begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array} \right| \to x_2, x_4$ are free variables $\quad x_1 + 2x_2 + 3x_3 + x_4 = 0 \Rightarrow x_1 + 2x_2 + 3(-2x_4) + x_4 = 0$

$x_3 + 2x_4 = 0 \Rightarrow x_3 = -2x_4 \quad \hookrightarrow x_1 = 5x_4 - 2x_2$

$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 5x_4 - 2x_2 \\ x_2 \\ -2x_4 \\ x_4 \end{vmatrix} = x_2 \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} 5 \\ 0 \\ -2 \\ 1 \end{vmatrix} \longrightarrow W = S^\perp = \left\{ \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 5 \\ 0 \\ -2 \\ 1 \end{vmatrix} \right\}$

c) show w does not lie in S nor W $\quad w = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$

There is no such α or β such that $\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \alpha \begin{vmatrix} 1/\sqrt{15} \\ 2/\sqrt{15} \\ 3/\sqrt{15} \\ 1/\sqrt{15} \end{vmatrix} + \beta \begin{vmatrix} -1/\sqrt{30} \\ -2/\sqrt{30} \\ 0 \\ 5/\sqrt{30} \end{vmatrix}$, thus $w \notin S$. Similarly,

there is no such α or β such that $\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \alpha \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} + \beta \begin{vmatrix} 5 \\ 0 \\ -2 \\ 1 \end{vmatrix}$, thus $w \notin W$. This is possible because

$S \cup S^\perp \neq \mathbb{R}^4$.

d) find P onto S and Q onto W $\quad P = U(V^T U)^{-1} V^T$ $\quad$ U is basis of X and V is basis of $Y^\perp$

$= QQ^T$

$P = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \to Q = I - P = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \to$ Check: $PQ = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = QP$

e) $Pw = \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}$ $\quad Qw = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 1 \end{vmatrix}$ $\quad (Pw)^T(Qw) = \begin{vmatrix} 1 & 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \\ 1 \end{vmatrix} = 0$ $\quad Pw + Qw = \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$

Homework #11 due 4/11

Mercedes Nguyen

1) (5.9.1) $B_x = \left\{ \begin{vmatrix} 1\\1\\1 \end{vmatrix}, \begin{vmatrix} 1\\2\\2 \end{vmatrix} \right\}$ $B_y = \left\{ \begin{vmatrix} 1\\2\\3 \end{vmatrix} \right\} \rightarrow X + Y$ basis of $\mathbb{R}^3$

a) $\text{Rank}(B) = \text{Rank}(X|Y) = \text{Rank}\left(\begin{vmatrix} 1\\1\\1 \end{vmatrix} \begin{vmatrix} 1\\2\\2 \end{vmatrix} \begin{vmatrix} 1\\2\\3 \end{vmatrix} \right) = 3$ since all columns are linearly independent.

↳ This implies $B_x \cup B_y$ is a basis of $\mathbb{R}^3$ which is equivalent to saying that X and Y are complementary.

b) $Q = I - P$ and $P = U(V^TU)^{-1}V^T = [X|0][X|Y]^{-1}$ from txtbook

$$P = \begin{vmatrix} 1&1&0\\1&2&0\\1&2&0 \end{vmatrix} \begin{vmatrix} 1&1&1\\1&2&2\\1&2&3 \end{vmatrix}^{-1} = \begin{vmatrix} 1&1&0\\1&2&0\\1&2&0 \end{vmatrix} \begin{vmatrix} 2&-1&0\\-1&2&-1\\0&-1&1 \end{vmatrix} = \begin{vmatrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{vmatrix} \rightarrow Q = \begin{vmatrix} 1&0&0\\0&1&0\\0&0&1 \end{vmatrix} - \begin{vmatrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{vmatrix} = \begin{vmatrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{vmatrix}$$

c) $v = \begin{vmatrix} 2\\-1\\1 \end{vmatrix}$ (Y along X) $\rightarrow Q$

$$Qv = \begin{vmatrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{vmatrix} \begin{vmatrix} 2\\-1\\1 \end{vmatrix} = \begin{vmatrix} 2\\4\\6 \end{vmatrix}$$

d) $P^2 = \begin{vmatrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{vmatrix} \begin{vmatrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{vmatrix} = \begin{vmatrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{vmatrix} = P$ ✓ $\quad Q^2 = \begin{vmatrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{vmatrix} \begin{vmatrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{vmatrix} = \begin{vmatrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{vmatrix} = Q$ ✓

e) $R(P)$ = basic columns of $P \rightarrow \begin{vmatrix} 1&1&-1\\0&3&-2\\0&0&0 \end{vmatrix} \rightarrow R(P) = \left\{ \begin{vmatrix} 1\\0\\0 \end{vmatrix}, \begin{vmatrix} 1\\3\\3 \end{vmatrix} \right\}$. Observe that $\begin{vmatrix} 1\\0\\0 \end{vmatrix} = c_1 \begin{vmatrix} 1\\1\\1 \end{vmatrix} + c_2 \begin{vmatrix} 1\\2\\2 \end{vmatrix}$ where $c_1 = 2$ and $c_2 = -1$ and we can also observe that $\begin{vmatrix} 1\\3\\3 \end{vmatrix} = d_1 \begin{vmatrix} 1\\1\\1 \end{vmatrix} + d_2 \begin{vmatrix} 1\\2\\2 \end{vmatrix}$ where $d_1 = -1$ and $d_2 = 2$.
Hence, $R(P)$ spans the same space as X

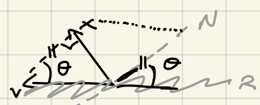
$N(Q) = \left| \begin{matrix} 0&-1&1\\0&-2&2\\0&-3&3 \end{matrix} \middle| \begin{matrix} 0\\0\\0 \end{matrix} \right| \rightarrow \left| \begin{matrix} 0&-1&1\\0&0&0\\0&0&0 \end{matrix} \middle| \begin{matrix} 0\\0\\0 \end{matrix} \right| \rightarrow$ free variables: $x_1, x_3 \rightarrow \begin{matrix} x_1 = x_1 \\ x_2 = x_3 \\ x_3 = x_3 \end{matrix} \rightarrow x = x_1 \begin{vmatrix} 1\\0\\0 \end{vmatrix} + x_3 \begin{vmatrix} 0\\1\\1 \end{vmatrix}$

$= \left\{ \begin{vmatrix} 1\\0\\0 \end{vmatrix}, \begin{vmatrix} 0\\1\\1 \end{vmatrix} \right\}$. Observe that $\begin{vmatrix} 1\\0\\0 \end{vmatrix} = m_1 \begin{vmatrix} 1\\1\\1 \end{vmatrix} + m_2 \begin{vmatrix} 1\\2\\2 \end{vmatrix}$ where $m_1 = 2$ and $m_2 = -1$, also observe that $\begin{vmatrix} 0\\1\\1 \end{vmatrix} = n_1 \begin{vmatrix} 1\\1\\1 \end{vmatrix} + n_2 \begin{vmatrix} 1\\2\\2 \end{vmatrix}$ where $n_1 = -1$ and $n_2 = 1$. Hence, $N(Q)$ spans the same space as X. Thus, $R(P) = X = N(Q)$

$N(P) = \left| \begin{matrix} 1&1&-1\\0&3&-2\\0&3&-2 \end{matrix} \middle| \begin{matrix} 0\\0\\0 \end{matrix} \right| \rightarrow \left| \begin{matrix} 1&1&-1\\0&3&-2\\0&0&0 \end{matrix} \middle| \begin{matrix} 0\\0\\0 \end{matrix} \right| \rightarrow$ free variable: $x_3 \rightarrow \begin{matrix} x_1 = x_3 - x_2 = x_3 - (\frac{2}{3}x_3) = \frac{1}{3}x_3 \\ x_2 = 2x_3/3 \\ x_3 = x_3 \end{matrix} \rightarrow x = x_3 \begin{vmatrix} 1/3\\2/3\\1 \end{vmatrix}$

Observe that $\begin{vmatrix} 1/3\\2/3\\1 \end{vmatrix} = \alpha \begin{vmatrix} 1\\2\\3 \end{vmatrix}$ where $\alpha = 1/3$. Hence, $N(P)$ spans the same space as Y.

$R(Q)$ = basic columns of $Q = \left\{ \begin{vmatrix} -1\\-2\\-3 \end{vmatrix} \right\}$. Observe that $\begin{vmatrix} -1\\-2\\-3 \end{vmatrix} = \beta \begin{vmatrix} 1\\2\\3 \end{vmatrix}$ where $\beta = -1$. Hence, $R(Q)$ spans the same space as Y. Therefore, $N(P) = Y = R(Q)$.



2) (5.9.9) $\|I-P\|_2 = \|P\|_2$ for all $P \neq 0 \neq I$ and $P^2 = P$

We know that in $\mathbb{R}^2$, $\max \|v\|_2 = 1$. We also know that $\|P\|_2 = \max_{\|x\|_2=1} \|Px\|_2 = 1$. Let $N(P) = N$ and $R(P) = P$. Suppose v is on R and x is $\perp$ to N. Then the projection of x onto v is P. Since a right Δ is formed w/ v, x, and P, we can relate the two by saying $\sin\theta = \frac{\|x\|_2}{\|v\|_2}$. But $\|P\|_2 = \max_{\|x\|_2} \|Px\|_2 = 1 = \|v\|_2$. Then $\sin\theta = \frac{1}{\|v\|_2} = \frac{1}{\|P\|_2}$. But P is $\parallel$ to N, so θ is also the angle between N and R. Recall than $N(P) = R(I-P)$ and $R(P) = N(I-P)$. So, θ is also the angle between $R(I-P)$ and $N(I-P)$. Thus $\sin\theta = \frac{1}{\|I-P\|_2} = \frac{1}{\|P\|_2}$. Therefore, $\|I-P\|_2 = \|P\|_2$. ∴

Homework #10 due 4/4

Mercedes Nguyen

1) $A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$ $b = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$ $q_j = \frac{w_j}{\|w_j\|}$ $w_j = v_j - \sum_{i=1}^{j-1} \langle v_j, q_i \rangle q_i$ $a_1 = q_1 \|a_1\|$ $a_j = \langle a_j, q_1 \rangle q_1 + \|w_j\| q_j$

$R_{ij} = \langle a_j, q_i \rangle$

a) $v_1 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ $v_3 = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix}$

$w_1 = v_1 - 0 \Rightarrow w_1 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \Rightarrow \|w_1\| = \sqrt{1+4+4} = \sqrt{9} = 3 \Rightarrow q_1 = \frac{1}{3}\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$

$\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1 & 2 & 2 \end{vmatrix}$

$w_2 = v_2 - (\langle v_2, q_1 \rangle q_1) = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} - \left(\frac{1}{3}\right)(5)\left(\frac{1}{3}\right)\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} - \frac{5}{9}\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} - \begin{vmatrix} 5/9 \\ 10/9 \\ 10/9 \end{vmatrix} = \begin{vmatrix} 4/9 \\ -1/9 \\ -1/9 \end{vmatrix} = \frac{1}{9}\begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix}$

$\|w_2\| = \sqrt{\frac{16}{81} + \frac{1}{81} + \frac{1}{81}} = \sqrt{\frac{18}{81}} = \frac{\sqrt{2}}{3} \Rightarrow q_2 = \frac{\cancel{3}^{1}}{\sqrt{2}} \cdot \frac{1}{\cancel{9}_{3}}\begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix} = \frac{1}{3\sqrt{2}}\begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix} = \begin{vmatrix} 2\sqrt{2}/3 \\ -\sqrt{2}/6 \\ -\sqrt{2}/6 \end{vmatrix}$

$w_3 = v_3 - \langle v_3, q_1 \rangle q_1 + \langle v_3, q_2 \rangle q_2 = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} - \left(\frac{1}{3}\right)(14)\left(\frac{1}{3}\right)\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + \left(\frac{1}{3\sqrt{2}}\right)(2)\left(\frac{1}{3\sqrt{2}}\right)\begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix} = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} - \frac{14}{9}\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix}$

$= \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} - \begin{vmatrix} 14/9 \\ 28/9 \\ 28/9 \end{vmatrix} + \begin{vmatrix} 4/9 \\ -1/9 \\ -1/9 \end{vmatrix} = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} - \begin{vmatrix} 2 \\ 3 \\ 3 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix}$ $\|w_3\| = \sqrt{1+1} = \sqrt{2} \Rightarrow q_3 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix}$

$R_{12} = \langle a_2, q_1 \rangle = \left(\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \begin{vmatrix} 1/3 & 2/3 & 2/3 \end{vmatrix}\right) = 5/3$

$R_{13} = \langle a_3, q_1 \rangle = \left(\begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} \begin{vmatrix} 1/3 & 2/3 & 2/3 \end{vmatrix}\right) = 14/3$

$R_{23} = \langle a_3, q_2 \rangle = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} \begin{vmatrix} \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} \end{vmatrix} = 4\frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$

$$\boxed{Q = \begin{vmatrix} 1/3 & 2\sqrt{2}/3 & 0 \\ 2/3 & -\sqrt{2}/6 & -1/\sqrt{2} \\ 2/3 & -\sqrt{2}/6 & 1/\sqrt{2} \end{vmatrix} \qquad R = \begin{vmatrix} 3 & 5/3 & 14/3 \\ 0 & \sqrt{2}/3 & \sqrt{2}/3 \\ 0 & 0 & \sqrt{2} \end{vmatrix}}$$

b) $A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{vmatrix} \to$ $\boxed{\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix} = U \qquad L = (I E_{12} E_{13} E_{23})^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}}$

$m_{12} = -2$
$m_{13} = -2$
$m_{23} = -1$

c) i) $Qy = b + Rx = y$ (hint: $y = Q^T b$) $b = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$

$Q^T b = \begin{vmatrix} 1/3 & 2/3 & 2/3 \\ 2\sqrt{2}/3 & -\sqrt{2}/6 & -\sqrt{2}/6 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} \times b = y = Rx$

$\begin{vmatrix} 1/3 & 2/3 & 2/3 \\ 2\sqrt{2}/3 & -\sqrt{2}/6 & -\sqrt{2}/6 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix} = \begin{vmatrix} 3 & 5/3 & 14/3 \\ 0 & \sqrt{2}/3 & \sqrt{2}/3 \\ 0 & 0 & \sqrt{2} \end{vmatrix} \begin{vmatrix} x \end{vmatrix}$

$\begin{vmatrix} 2/3 + 8/3 + 12/3 \\ 4\sqrt{2}/3 - \frac{\cancel{4}2\sqrt{2}}{\cancel{6}3} - \frac{\cancel{6}3\sqrt{2}}{\cancel{6}3} \\ 0 - 4/\sqrt{2} + 6/\sqrt{2} \end{vmatrix} = \begin{vmatrix} 3 & 5/3 & 14/3 \\ 0 & \sqrt{2}/3 & \sqrt{2}/3 \\ 0 & 0 & \sqrt{2} \end{vmatrix} \begin{vmatrix} x \end{vmatrix} = \begin{vmatrix} 22/3 \\ -\sqrt{2}/3 \\ 2/\sqrt{2} \end{vmatrix} \Rightarrow$

$-10/3$ $\quad 14/3 \quad 22-4 = 18/3 = 6$

$3x_1 + 5/3 x_2 + 14/3 x_3 = 22/3 \Rightarrow x_1 = 2$

$\sqrt{2}/3 x_2 + \sqrt{2}/3 x_3 = -\sqrt{2}/3 \Rightarrow x_2 = -2$

$\sqrt{2} x_3 = 2/\sqrt{2} \Rightarrow x_3 = 1$

$\boxed{x = \begin{vmatrix} 2 \\ -2 \\ 1 \end{vmatrix}}$ ✓

ii) $Ly = b + Ux = y$

$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} \begin{vmatrix} y \end{vmatrix} = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$

$y_1 = 2$
$2y_1 + y_2 = 4 \Rightarrow y_2 = 0$
$2y_1 + y_2 + y_3 = 6 \Rightarrow y_3 = 2$

$\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix} \begin{vmatrix} x \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ 2 \end{vmatrix}$

$x_1 + x_2 + 2x_3 = 2 \Rightarrow x_1 = 2$
$-x_2 - 2x_3 = 0 \Rightarrow x_2 = -2$
$2x_3 = 2 \Rightarrow x_3 = 1$

$\Rightarrow \boxed{x = \begin{vmatrix} 2 \\ -2 \\ 1 \end{vmatrix}}$ ✓

2) (5.6.5) U and V are $n \times n$ unitary (orthoganol) matrices

By definition $U^*U = I = UU^*$ (same for V)

a) Consider the following: $(UV)^*(UV) = V^*U^*UV = V^*V = I$; thus, UV must be unitary.

b) Suppose $U = I$ and $V = -I$. Then $U + V = I + (-I) = 0 \neq I$. Hence, not unitary.

c) $\begin{vmatrix} U & 0 \\ 0 & V \end{vmatrix} \cdot \left(\begin{vmatrix} U & 0 \\ 0 & V \end{vmatrix}\right)^* = \begin{vmatrix} UU^* & 0 \\ 0 & VV^* \end{vmatrix} = \begin{vmatrix} I & 0 \\ 0 & I \end{vmatrix}$; thus $\begin{vmatrix} U & 0 \\ 0 & V \end{vmatrix}$ is unitary

Homework # 9 due 3/23

Mercedes Nguyen

1) a) $v_1 = \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ $\cos\theta = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$\langle v_1, v_2 \rangle = v_1^T v_2 = |-1\ 1\ 0| \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = -1 + 0 + 0 = -1$

$\boxed{\theta = \frac{3\pi}{4} = 135°}$

$\|v_1\| = \sqrt{v_1^T v_1} = \left(|-1\ 1\ 0| \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix}\right)^{1/2} = \sqrt{1+1+0} = \sqrt{2}$

$\|v_2\| = \sqrt{v_2^T v_2} = \left(|1\ 0\ 0| \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}\right)^{1/2} = \sqrt{1} = 1$

b) orthoganol to both (x) : $\langle x, v_1 \rangle = 0 = \langle x, v_2 \rangle$ $(\theta = 90°)$

$\boxed{x = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}}$ $\left.\begin{array}{l} \Rightarrow \langle x, v_1 \rangle = x^T v_1 = |0\ 0\ 0| \begin{vmatrix} -1 \\ 1 \\ 0 \end{vmatrix} = 0 + 0 + 0 = 0 \\ \Rightarrow \langle x, v_2 \rangle = x^T v_2 = |0\ 0\ 0| \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = 0 + 0 + 0 = 0 \end{array}\right\}$ orthoganol to both

2) $v_1 = \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$ $\cos\theta = \frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|}$

$\langle v_1, v_2 \rangle = v_1^T v_2 = |1\ 2\ -1\ 1| \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix} = 1 + (-2) + 1 + 0 = 0 \rightarrow$ Bc inner product 0, v_1 and v_2 are orthoganol. Hence, $\boxed{\theta = 90°}$

3) (5.4.9) If A is "normal", then $R(A) \perp N(A)$.

Proof: By definition, $A^T A = A A^T$. By product identity of vector spaces, $R(A) = R(AA^T)$ and $N(A) = N(A^T A)$. Then let $r \in R(A)$ and we can say $r = AA^T x$ for some x. Similarly, let $n \in N(A)$. We can say that $0 = A^T A n$. But then, $\langle r, n \rangle = r^T n = (AA^T x)^T (n) = x^T A A^T A n$ ... $= x^T A^T A n = x^T (0) = 0$. Hence, $r \perp n$. Therefore, $R(A) \perp N(A)$. q.e.d.

Homework #8 due 3/16

Mercedes Nguyen

1) (4.2.5)

a) Let A be an $n \times n$ matrix and $R(A) = \mathbb{R}^n$. Observe that $R(I_n) = \mathbb{R}^n$ as well. Then we can say that $R(A) = R(I_n)$ and, by equal ranges, $A \overset{col}{\sim} I_n$. Then $\text{Rank}(A) = \text{Rank}(I_n) = n$. Hence, A is nonsingular.

b) If A is nonsingular then: i) $R(A) = \mathbb{R}^n$

ii) $R(A^T) = \mathbb{R}^n$

iii) $N(A) = \{0\}$

iv) $N(A^T) = \{0\}$

2) a) Let $b \in R(AB)$. Then $\exists x$ such that $b = ABx = A(Bx)$. Observe that b is the image of Bx. Hence, $b \in R(A)$. Therefore, $R(AB) \subseteq R(A)$.

b) $A = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$ $B = \begin{vmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{vmatrix}$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow R(A) = \left\{ \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix}, \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \right\}, \quad \text{Rank}(A) = 2$$

$m_{21} = -1$ $\quad m_{32} = -1$

$m_{31} = -2$

$$AB = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 0 & 2 \\ 0 & 6 \end{vmatrix} \longrightarrow \begin{vmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{vmatrix} \Rightarrow R(AB) = \left\{ \begin{vmatrix} 4 \\ 2 \\ 6 \end{vmatrix} \right\}, \quad \text{Rank}(AB) = 1$$

$m_{21} = -2$

$m_{31} = -\frac{3}{2}$

Observe that $\begin{vmatrix} 4 \\ 2 \\ 6 \end{vmatrix} = x_1 \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} + x_2 \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ where $x_1 = 2$ and $x_2 = 2$. Also $\text{Rank}(AB) < \text{Rank}(A)$ so $\dim R(AB) < \dim R(A)$. Therefore, $R(AB) \subseteq R(A)$.

Homework #7 due 2/23

Mercedes Nguyen

1) $S, T \subseteq V$ (2 subspaces of vector space)

a) $S \cap T = \{x \in V : x \in S \text{ and } x \in T\}$ (prove is a subspace)

Suppose S & T are subspaces and $x, y \in S \cap T$. Then $x, y \in S$ and $x, y \in T$. Since they are subspaces, we can say that $x+y \in S$ and $x+y \in T$ by closure property of addition of a subspace. But then $x+y \in S \cap T$. Hence, $S \cap T$ is closed under addition. Similarly, let $\alpha \in F$, the field V is over, then $\alpha x + y \in S$ and $\alpha x + y \in T$ by closure property of scalar multiplication. But then $\alpha x + y \in S \cap T$ and $S \cap T$ is closed under scalar multiplication. Therefore, by definition, $S \cap T$ is a subspace.

b) Suppose $S = \{ \begin{vmatrix} x \\ x \end{vmatrix} : x \in \mathbb{R} \}$ and $T = \{ \begin{vmatrix} y \\ 0 \end{vmatrix} : y \in \mathbb{R} \}$. Observe that both are subspaces since both contain the zero vector and both are closed under addition and scalar multiplication. $S \cup T$ is not a subspace. For example, take $u = \begin{vmatrix} 3 \\ 3 \end{vmatrix} \in S$ and $v = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \in T$. $u+v = \begin{vmatrix} 3 \\ 3 \end{vmatrix} + \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 4 \\ 3 \end{vmatrix}$. But $\begin{vmatrix} 4 \\ 3 \end{vmatrix} \notin S$ and $\begin{vmatrix} 4 \\ 3 \end{vmatrix} \notin T$. Hence, $u+v \notin S \cup T$ and $S \cup T$ is not closed under addition and therefore, not a subspace.

2) $p_1(x) = 1 + x \quad p_2(x) = 1 - x \quad p_3(x) = (1+x)(1-x) = 1 - x^2$

$$\Pi_2 = ax^2 + bx + c = \alpha p_1(x) + \beta p_2(x) + \gamma p_3(x)$$
$$ax^2 + bx + c = \alpha(1+x) + \beta(1-x) + \gamma(1-x^2)$$
$$ax^2 + bx + c(x^0) = \alpha + \alpha x + \beta - \beta x + \gamma - \gamma x^2$$
$$= -\gamma x^2 + \alpha x - \beta x + \alpha + \beta + \gamma$$
$$= (-\gamma)x^2 + (\alpha - \beta)x + (\alpha + \beta + \gamma)x^0$$

$a = -\gamma$
$b = \alpha - \beta$
$c = \alpha + \beta + \gamma$
$\beta = \alpha - b$
$\alpha = b + \beta$
$\alpha = c - \beta - \gamma$
$2\alpha = b + c - \gamma = b + c - (-a)$
$\beta = c - \alpha - \gamma$
$\beta = \alpha - b$
$2\beta = c - b - \gamma = c - b - (-a)$

$\rightarrow$ $\gamma = -a$
$\beta = \frac{a-b+c}{2}$
$\alpha = \frac{a+b+c}{2}$

After substituting p_1, p_2, and p_3 into Π_2, the coefficients of a 2nd degree polynomial were found which shows that any polynomial of the 2nd degree can be written as a linear combination of p_1, p_2, and p_3. Therefore, they span Π_2.

Homework #6 due 2/21

Mercedes Nguyen

1) a) <u>False</u>, let $A = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}$ which are both nonsingular since Rank $A = 2 = n_A$ and Rank $B = 2 = n_B$.
$AB = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 0 & 2 \end{vmatrix}$ and $BA = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 7 \\ 0 & 2 \end{vmatrix} \neq \begin{vmatrix} 3 & 5 \\ 0 & 2 \end{vmatrix}$. Hence, $AB \neq BA$.

b) <u>True</u>, let A and B be nonsingular. Then by definition, $\exists A^{-1}$ such that $A \cdot A^{-1} = I = A^{-1} \cdot A$ + $\exists B^{-1}$ such that $B \cdot B^{-1} = I = B^{-1} \cdot B$. Then $AB(AB)^{-1} = I$. But then $A^{-1}AB(AB)^{-1} = I \cdot A^{-1}$. But $A^{-1}A = I$ so $(I)B(AB)^{-1} = A^{-1}$. But then $B^{-1}B(AB^{-1}) = B^{-1}A^{-1}$ and since $B^{-1}B = I$, $(AB)^{-1} = B^{-1}A^{-1}$.

c) <u>True</u>, let A be nonsingular. Then Rank $A = n$. But $-A = (-1)A = (-1)\,\forall a_{ij}$, $i,j \in \mathbb{N}$. Then Rank $(-A) = n$. Hence, $-A$ is nonsingular.

d) <u>False</u>, let $A = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ and $B = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$, which are both nonsingular since Rank $A = n_A$ and Rank $B = n_B$. But $A + B =$ $\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$ and Rank $(A+B) = 0$, and $A+B$ is singular.

e) <u>True</u>, since L is a lower Δ with all diagonal entries equal to 1 which is always nonsingular since Rank $L = n_L$. Suppose U is nonsingular and $A = LU$ Then $L \cdot L^{-1} = I = L^{-1} \cdot L$ and $UU^{-1} = I = U^{-1}U$. But then $(LU)^{-1} = U^{-1}L^{-1}$ since proved in part (b). But then $LU(LU)^{-1} = LUU^{-1}L^{-1} = L(I)L^{-1} = I$. Then LU is nonsingular. But $A = LU$, so A must be nonsingular if U is nonsingular.

f) <u>False</u>, let $L = \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}$. Then $L^{-1} = \begin{vmatrix} 1/2 & 0 \\ -1/6 & 1/3 \end{vmatrix}$. Observe that $L \cdot L^{-1} = \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix}\begin{vmatrix} 1/2 & 0 \\ -1/6 & 1/3 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. But L^{-1} is also lower Δ.

2) (3.10.1) $A = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{vmatrix}$

a) $\begin{vmatrix} 1 & 4 & 5 \\ 4 & 18 & 26 \\ 3 & 16 & 30 \end{vmatrix} \to \begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 4 & 15 \end{vmatrix} \to \begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix} = U$ and $L = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$

$m_{21} = -4$, $m_{31} = -3$; $m_{32} = -2$

b) $b_1 = \begin{vmatrix} 6 \\ 0 \\ -6 \end{vmatrix}$ $Ly = b$ $Ux = y$

$\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 6 \\ 0 \\ -6 \end{vmatrix}$ $y_1 = 6$ $4y_1 + y_2 = 0$, $4(6) + y_2 = 0 \Rightarrow y_2 = -24$ $3y_1 + 2y_2 + y_3 = -6$, $3(6) + 2(-24) + y_3 = -6 \Rightarrow y_3 = 24$

$\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 6 \\ -24 \\ 24 \end{vmatrix}$ $3x_3 = 24$, $x_3 = 8$ $2x_2 + 6x_3 = -24$, $2x_2 + 6(8) = -24 \Rightarrow x_2 = -36$ $x_1 + 4x_2 + 5x_3 = 6$, $x_1 + 4(-36) + 5(8) = 6 \Rightarrow x_1 = 110$

$$\boxed{X_1 = \begin{vmatrix} 110 \\ -36 \\ 8 \end{vmatrix}}$$

$b_2 = \begin{vmatrix} 6 \\ 6 \\ 12 \end{vmatrix}$

$\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 6 \\ 6 \\ 12 \end{vmatrix}$ $x_1 = 6$ $4x_1 + x_2 = 6$, $4(6) + x_2 = 6 \Rightarrow x_2 = -18$ $3x_1 + 2x_2 + x_3 = 12$, $3(6) + 2(-18) + x_3 = 12 \Rightarrow x_3 = 30$

$\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 6 \\ -18 \\ 30 \end{vmatrix}$ $x_3 = 10$ $2x_2 + 6x_3 = -18$, $2x_2 + 6(10) = -18$ $x_2 = -39$ $x_1 + 4x_2 + 5x_3 = 6$, $x_1 + 4(-39) + 5(10) = 6 \Rightarrow x_1 = 112$

$$\boxed{X_2 = \begin{vmatrix} 112 \\ -39 \\ 10 \end{vmatrix}}$$

c) $\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}\begin{vmatrix} y \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $y_1 = \begin{vmatrix} 1 & 0 & 0 \end{vmatrix}$ $4y_1 + y_2 = \begin{vmatrix} 0 & 1 & 0 \end{vmatrix}$, $y_2 = \begin{vmatrix} -4 & 1 & 0 \end{vmatrix}$ $3y_1 + 2y_2 + y_3 = \begin{vmatrix} 0 & 0 & 1 \end{vmatrix}$, $y_3 = \begin{vmatrix} 5 & -2 & 1 \end{vmatrix}$

$\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}\begin{vmatrix} X \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 5 & -2 & 1 \end{vmatrix}$ $x_3 = \begin{vmatrix} \frac{5}{3} & -\frac{2}{3} & \frac{1}{3} \end{vmatrix}$ $2x_2 + 6x_3 = \begin{vmatrix} -4 & 1 & 0 \end{vmatrix}$, $x_2 = \begin{vmatrix} -7 & \frac{5}{2} & -1 \end{vmatrix}$ $x_1 + 4x_2 + 5x_3 = \begin{vmatrix} 1 & 0 & 0 \end{vmatrix}$, $x_3 = \begin{vmatrix} \frac{62}{3} & -\frac{20}{3} & \frac{7}{3} \end{vmatrix}$

$$A^{-1} = \begin{vmatrix} 62/3 & -20/3 & 7/3 \\ -7 & 5/2 & -1 \\ 5/3 & -2/3 & 1/3 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 124 & -40 & 14 \\ -42 & 15 & -6 \\ 10 & -4 & 2 \end{vmatrix}$$

3) (3.10.3) $A = \begin{vmatrix} \varepsilon & 2 & 0 \\ 1 & \varepsilon & 1 \\ 0 & 1 & \varepsilon \end{vmatrix} \rightarrow \begin{vmatrix} \varepsilon & 2 & 0 \\ 0 & \frac{\varepsilon^2-2}{\varepsilon} & 1 \\ 0 & 1 & \varepsilon \end{vmatrix} \rightarrow \begin{vmatrix} \varepsilon & 2 & 0 \\ 0 & \left(\frac{\varepsilon^2-2}{\varepsilon}\right) & 1 \\ 0 & 0 & \left(\frac{\varepsilon^3-3\varepsilon}{\varepsilon^2-2}\right) \end{vmatrix}$

$m_{21} = -1/\varepsilon$ $\qquad m_{32} = -\frac{\varepsilon}{\varepsilon^2-2}$

$m_{32} = 0$

$$\varepsilon, \frac{\varepsilon^2-2}{\varepsilon}, \frac{\varepsilon^3-3\varepsilon}{\varepsilon^2-2} \neq 0 \Rightarrow \varepsilon \neq 0, \pm\sqrt{2}, \pm\sqrt{3}$$

4) (3.10.6) $T = \begin{vmatrix} \beta_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \beta_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{vmatrix}$

a) $LU = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \alpha_1/\pi_1 & 1 & 0 & 0 \\ 0 & \alpha_2/\pi_2 & 1 & 0 \\ 0 & 0 & \alpha_3/\pi_3 & 1 \end{vmatrix} \begin{vmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ 0 & \pi_2 & \gamma_2 & 0 \\ 0 & 0 & \pi_3 & \gamma_3 \\ 0 & 0 & 0 & \pi_4 \end{vmatrix} = \begin{vmatrix} \pi_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \frac{\gamma_1\alpha_1}{\pi_1}+\pi_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \frac{\gamma_2\alpha_2}{\pi_2}+\pi_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \frac{\gamma_3\alpha_3}{\pi_3}+\pi_4 \end{vmatrix}$

$$\left.\begin{aligned} &\frac{\gamma_1\alpha_1}{\pi_1} + \pi_{1+1} = \beta_{1+1} = \beta_2 \\ &\frac{\gamma_2\alpha_2}{\pi_2} + \pi_{2+1} = \beta_{2+1} = \beta_3 \qquad \pi_1 = \beta_1 \\ &\frac{\gamma_3\alpha_3}{\pi_3} + \pi_{3+1} = \beta_{3+1} = \beta_4 \end{aligned}\right\} \text{recursion formula}$$

$LU = \begin{vmatrix} \beta_1 & \gamma_1 & 0 & 0 \\ \alpha_1 & \beta_2 & \gamma_2 & 0 \\ 0 & \alpha_2 & \beta_3 & \gamma_3 \\ 0 & 0 & \alpha_3 & \beta_4 \end{vmatrix} = T$ ✓

b) $T = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$

$\beta_1 = \pi_1 = 2 = \beta_2 = \beta_3$, $\beta_4 = 1$

$\gamma_1 = -1 = \gamma_2 = \gamma_3$

$\alpha_1 = -1 = \alpha_2 = \alpha_3$

$$\pi_2 = \beta_2 - \frac{\alpha_1\gamma_1}{\pi_1} = 2 - \frac{(-1)(-1)}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\pi_3 = \beta_3 - \frac{\alpha_2\gamma_2}{\pi_2} = 2 - \frac{(-1)(-1)}{\frac{3}{2}} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\pi_4 = \beta_4 - \frac{\alpha_3\gamma_3}{\pi_3} = 1 - \frac{(-1)(-1)}{4/3} = 1 - \frac{3}{4} = \frac{1}{4}$$

$L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 1/4 \end{vmatrix}$

Extra Credit (3.10.7)

$A = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{53} & \alpha_{54} & \alpha_{55} \end{vmatrix}$ has $w = 2$

$E_{21}E_{31}A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{53} & \alpha_{54} & \alpha_{55} \end{vmatrix} \rightarrow E_{21}E_{31}E_{32}E_{42}A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 & 0 \\ 0 & m_{42} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ 0 & {}^*\alpha_{22} & {}^*\alpha_{23} & {}^*\alpha_{24} & 0 \\ 0 & {}^*\alpha_{32} & {}^*\alpha_{33} & {}^*\alpha_{34} & {}^*\alpha_{35} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} \\ 0 & 0 & \alpha_{53} & \alpha_{54} & \alpha_{55} \end{vmatrix}$

* indicate after applying multiplier

$E_{21}E_{31}E_{32}E_{42}E_{43}E_{53}A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 & 0 \\ 0 & m_{42} & m_{43} & 1 & 0 \\ 0 & 0 & m_{53} & 0 & 1 \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ 0 & \alpha^*_{22} & \alpha^*_{23} & \alpha^*_{24} & 0 \\ 0 & 0 & \alpha^*_{33} & \alpha^*_{34} & \alpha^*_{35} \\ 0 & 0 & \alpha^*_{43} & \alpha^*_{44} & \alpha^*_{45} \\ 0 & 0 & \alpha_{53} & \alpha_{54} & \alpha_{55} \end{vmatrix}$

$E_{21}E_{31}E_{32}E_{42}E_{43}E_{53}E_{54}A = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 & 0 \\ 0 & m_{42} & m_{43} & 1 & 0 \\ 0 & 0 & m_{53} & m_{54} & 1 \end{vmatrix} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ 0 & \alpha^*_{22} & \alpha^*_{23} & \alpha^*_{24} & 0 \\ 0 & 0 & \alpha^*_{33} & \alpha^*_{34} & \alpha^*_{35} \\ 0 & 0 & 0 & \alpha^*_{44} & \alpha^*_{45} \\ 0 & 0 & 0 & \alpha^*_{54} & \alpha^*_{55} \end{vmatrix} \rightarrow \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ 0 & \alpha^*_{22} & \alpha^*_{23} & \alpha^*_{24} & 0 \\ 0 & 0 & \alpha^*_{33} & \alpha^*_{34} & \alpha^*_{35} \\ 0 & 0 & 0 & \alpha^*_{44} & \alpha^*_{45} \\ 0 & 0 & 0 & 0 & \alpha^*_{55} \end{vmatrix} = U$

$L = E_{21}E_{31}E_{32}E_{42}E_{43}E_{53}E_{54} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 & 0 \\ 0 & m_{42} & m_{43} & 1 & 0 \\ 0 & 0 & m_{53} & m_{54} & 1 \end{vmatrix}$

Homework #5 due 2/14

Mercedes Nguyen

1) a) $A^{-1} = \begin{vmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -9 & 4 \end{vmatrix}$

b) $A \cdot A^{-1} = \begin{vmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{vmatrix} \begin{vmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I \quad A^{-1} \cdot A = \begin{vmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{vmatrix} \begin{vmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I$

Hence, $AA^{-1} = A^{-1}A = I$.

c) $A^T = \begin{vmatrix} 4 & 4 & 3 \\ -8 & -7 & -4 \\ 5 & 4 & 2 \end{vmatrix}$ $\quad (A^T)^{-1} = \left|\begin{array}{ccc|c|c|c} 4 & 4 & 3 & 1 & 0 & 0 \\ -8 & -7 & -4 & 0 & 1 & 0 \\ 5 & 4 & 2 & 0 & 0 & 1 \end{array}\right| \rightarrow \left|\begin{array}{ccc|c|c|c} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & -\frac{7}{4} & -\frac{5}{4} & 0 & 1 \end{array}\right| \rightarrow \left|\begin{array}{ccc|c|c|c} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 1 & 1 \end{array}\right|$

$m_{21} = 2$
$m_{31} = -\frac{5}{4}$
$m_{32} = 1$

back sub: $\frac{1}{4}x_{31} = \frac{3}{4}$
$x_{31} = 3$
$\frac{1}{4}x_{32} = 1$
$x_{32} = 4$
$\frac{1}{4}x_{33} = 1$
$x_{33} = 4$

$x_{21} + 2x_{31} = 2$
$x_{21} + 2(3) = 2$
$x_{21} + 6 = 2$
$x_{21} = -4$
$x_{22} + 2x_{32} = 1$
$x_{22} + 2(4) = 1$
$x_{22} + 8 = 1$
$x_{22} = -7$
$x_{23} + 2x_{33} = 0$
$x_{23} + 2(4) = 0$
$x_{23} + 8 = 0$
$x_{23} = -8$

$4x_{11} + 4x_{21} + 3x_{31} = 1$
$4x_{11} + 4(-4) + 3(3) = 1$
$4x_{11} - 16 + 9 = 1$
$4x_{11} = 8$
$x_{11} = 2$
$4x_{12} + 4x_{22} + 3x_{32} = 0$
$4x_{12} + 4(-7) + 3(4) = 0$
$4x_{12} - 28 + 12 = 0$
$4x_{12} = 16$
$x_{12} = 4$
$4x_{13} + 4x_{23} + 3x_{33} = 0$
$4x_{13} + 4(-8) + 3(4) = 0$
$4x_{13} - 32 + 12 = 0$
$4x_{13} = 20$
$x_{13} = 5$

$(A^T)^{-1} = \begin{vmatrix} 2 & 4 & 5 \\ -4 & -7 & -8 \\ 3 & 4 & 4 \end{vmatrix}$

$(A^{-1})^T = \begin{vmatrix} 2 & 4 & 5 \\ -4 & -7 & -8 \\ 3 & 4 & 4 \end{vmatrix} = (A^T)^{-1}$ ✓

2) $Q = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

a) $Q^2 = \left(\frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\right) = \frac{1}{2}\left(\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}\right) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

b) $Q^{-1} = \left|\begin{array}{cc|c|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{array}\right| \rightarrow \left|\begin{array}{cc|c|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & -\frac{2}{\sqrt{2}} & -1 & 1 \end{array}\right|$

$m_{21} = -1$

back sub: $-\frac{2}{\sqrt{2}}x_{21} = -1$
$x_{21} = \frac{\sqrt{2}}{2}$
$-\frac{2}{\sqrt{2}}x_{22} = 1$
$x_{22} = -\frac{\sqrt{2}}{2}$

$\frac{1}{\sqrt{2}}x_{11} + \frac{1}{\sqrt{2}}x_{21} = 1$
$\frac{1}{\sqrt{2}}x_{11} + \cancel{\frac{1}{\sqrt{2}}}\left(\frac{\cancel{\sqrt{2}}}{2}\right) = 1$
$\frac{1}{\sqrt{2}}x_{11} = \frac{1}{2}$
$x_{11} = \frac{\sqrt{2}}{2}$
$\frac{1}{\sqrt{2}}x_{12} + \frac{1}{\sqrt{2}}x_{22} = 0$
$\frac{1}{\sqrt{2}}x_{12} + \cancel{\frac{1}{\sqrt{2}}}\left(-\frac{\cancel{\sqrt{2}}}{2}\right) = 0$
$\frac{1}{\sqrt{2}}x_{12} = \frac{1}{2}$
$x_{12} = \frac{\sqrt{2}}{2}$

$Q^{-1} = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{2}}{2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

3)

a) $Ax = b$

Method 1: $\left|\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 4 & -7 & 4 & 1 \\ 3 & -4 & 2 & 1 \end{array}\right| \rightarrow \left|\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -\frac{7}{4} & \frac{1}{4} \end{array}\right| \rightarrow \left|\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{array}\right|$

$m_{21} = -1$
$m_{31} = -\frac{3}{4}$
$m_{32} = -2$

back sub: $\frac{1}{4}x_3 = \frac{1}{4}$
$x_3 = 1$

$x_2 - x_3 = 0$
$x_2 - 1 = 0$
$x_2 = 1$

$4x_1 - 8x_2 + 5x_3 = 1$
$4x_1 - 8(1) + 5(1) = 1$
$4x_1 = 4$
$x_1 = 1$

$\boxed{x_p = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}}$

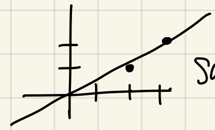
Method 2: $A^{-1}b = \begin{vmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -9 & 4 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

b) The solution is unique because A is nonsingular since $\exists A^{-1}$ such that $A \cdot A^{-1} = I = A^{-1}A$. By definition, since A is nonsingular, then $\exists!$ solution to $Ax = b$, $\forall b$.

4) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$

$(x_1, x_2) = (3,2)$ or $(2,1)$

$X = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$ or $\begin{vmatrix} 3 \\ 2 \end{vmatrix}$



same line

$y = x - 1$

$-x + y = -1$

$-2x + 2y = -2$

$A = \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} \quad b = \begin{vmatrix} -1 \\ -2 \end{vmatrix}$

Homework #4 due 2/7

Mercedes Nguyen

1) a) Let A and B be matrices that are symmetric and commute. Then by definition of symmetric, $A^T = A$ and $B^T = B$. Also, by commutativity $AB = BA$. But then, $(AB)^T = B^T A^T$ by definition of matrix product. Since A and B are symmetric thus, $(AB)^T = B^T A^T = (B)(A) = BA = AB$. Hence, by definition AB is symmetric.

b) Let $A = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ and $B = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}$. Both are symmetric but do not commute. Observe that $AB = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 4 & 5 \end{vmatrix}$ and $BA = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix}$. Hence, $AB \neq BA$. Also, observe that neither AB nor BA are symmetric.

2) It is not true since matrix product is not always commutative. A will only commute with A^T iff it is diagonal, symmetric, or skew symmetric. Let $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$. Then $A^T = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$. Observe that $A \cdot A^T = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 11 \\ 11 & 25 \end{vmatrix}$ and $A^T A = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 10 & 14 \\ 14 & 20 \end{vmatrix}$. Thus $A \cdot A^T \neq A^T \cdot A$. We can only claim that trace(AB) = trace(BA) and that both are symmetric.

Homework #3 due 2/7

Mercedes Nguyen

1) $A = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$ $B = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

a) $C = AB$

$$C = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{vmatrix}$$

b) A^T and B^T

$$A^T = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

$$B^T = \begin{vmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{vmatrix}$$

c) $C^T = \begin{vmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{vmatrix}$

$$B^T \cdot A^T = \begin{vmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} (3-2) & (-3+0) \\ (2+0) & (-2+0) \\ (1+2) & (-1+0) \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{vmatrix} = C^T$$

2) $v = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$ $w = \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix}$

a) $w^T \cdot v$

$w^T = \begin{vmatrix} 0 & 1 & 2 \end{vmatrix}$

$w^T \cdot v = \begin{vmatrix} 0 & 1 & 2 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} = [0]$

b) $v^T \cdot w$

$v^T = \begin{vmatrix} 1 & 2 & -1 \end{vmatrix}$

$v^T \cdot w = \begin{vmatrix} 1 & 2 & -1 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix} = [0]$

c) $v \cdot w^T$

$$\begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} \begin{vmatrix} 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

d) $w \cdot v^T$

$$\begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix} \begin{vmatrix} 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

e) $v^T \cdot v$

$$\begin{vmatrix} 1 & 2 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} = [6]$$

f) $v \cdot v^T$

$$\begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} \begin{vmatrix} 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

3) $\alpha = v^T \cdot v = 6$, $I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, $v \cdot v^T = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{vmatrix}$

a) $$P = I - \frac{1}{\alpha} v \cdot v^T = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \left(\frac{1}{6} \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{vmatrix} \right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{vmatrix} = \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix}$$

b) $$P \cdot v = \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} = \begin{vmatrix} (5/6 - \frac{2}{3} - \frac{1}{6}) \\ (-\frac{1}{3} + \frac{2}{3} - \frac{1}{3}) \\ (\frac{1}{6} + \frac{2}{3} - \frac{5}{6}) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$P \cdot w = \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix} = \begin{vmatrix} (0 - \frac{1}{3} + \frac{1}{3}) \\ (0 + \frac{1}{3} + \frac{2}{3}) \\ (0 + \frac{1}{3} + \frac{5}{3}) \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix}$$

c) $$P^2 = P \cdot P = \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix} \cdot \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix} = \begin{vmatrix} (\frac{25}{36} + \frac{1}{9} + \frac{1}{36}) & (-\frac{5}{18} - \frac{1}{9} + \frac{1}{18}) & (\frac{5}{36} - \frac{1}{9} + \frac{5}{36}) \\ (-\frac{5}{18} - \frac{1}{9} + \frac{1}{18}) & (\frac{1}{9} + \frac{1}{9} + \frac{1}{9}) & (-\frac{1}{18} + \frac{1}{9} + \frac{5}{18}) \\ (\frac{5}{36} - \frac{1}{9} + \frac{5}{36}) & (-\frac{1}{18} + \frac{1}{9} + \frac{5}{18}) & (\frac{1}{36} + \frac{1}{9} + \frac{25}{36}) \end{vmatrix} = \begin{vmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{vmatrix}$$

4) a) $P^T = (I - \frac{1}{\alpha} v \cdot v^T)^T = I^T - (\frac{1}{\alpha} v \cdot v^T)^T$ by property of transposition. But $I^T = I$ since symmetric and $(\frac{1}{\alpha} v \cdot v^T)^T = \frac{1}{\alpha}(v \cdot v^T)^T$ by property of transposition. But $(v \cdot v^T)^T = (v^T)^T \cdot (v)^T = v \cdot v^T$ by definition of matrix product. Thus $P^T = I^T - (\frac{1}{\alpha} v \cdot v^T)^T = I - \frac{1}{\alpha}(v \cdot v^T) = P$. q.e.d.

b) $P^2 = P \cdot P = (I - \frac{1}{\alpha} v \cdot v^T)(I - \frac{1}{\alpha} v \cdot v^T) = I^2 - (\frac{1}{\alpha} v \cdot v^T) I - I(\frac{1}{\alpha} v \cdot v^T) + (\frac{1}{\alpha} v \cdot v^T)^2 = I - \frac{2}{\alpha}(v \cdot v^T \cdot I) + (\frac{1}{\alpha} v \cdot v^T)^2$ by property of the identity matrix and commutativity of scalar multiplication. $= I - \frac{2}{\alpha}(v \cdot v^T) + (\frac{1}{\alpha} v \cdot v^T)^2$ by property of the identity matrix. $= I - \frac{2}{\alpha}(v \cdot v^T) + \frac{1}{\alpha^2}(v \cdot v^T)^2 = I - \frac{2}{\alpha}(v \cdot v^T) + \frac{1}{\alpha^2}(v \cdot v^T \cdot v \cdot v^T)$ by associativity of matrix multiplication. $= I - \frac{2}{\alpha}(v \cdot v^T) + \frac{1}{\alpha^2}(v(\alpha) v^T)$ since $\alpha = v^T \cdot v$. $= I - \frac{2}{\alpha}(v \cdot v^T) + \frac{1}{\alpha}(v \cdot v^T)$ by associativity of scalar multiplication. Then $P^2 = I - \frac{1}{\alpha}(v \cdot v^T) = P$. q.e.d.

c) $P \cdot v = (I - \frac{1}{\alpha} v \cdot v^T) \cdot v = I \cdot v - (\frac{1}{\alpha} v \cdot v^T) \cdot v$ by distributive property. But $I \cdot v = v$ by definition of identity matrix and $\frac{1}{\alpha} v \cdot v^T \cdot v = \frac{1}{\alpha} v(\alpha) = v$ by associative property. Then $I \cdot v - (\frac{1}{\alpha} v \cdot v^T) v = (v) - (v) = 0$. q.e.d.

d) Let $\alpha = v^T \cdot v$ and $v^T \cdot w = 0$. Then $P \cdot w = (I - \frac{1}{\alpha} v \cdot v^T) \cdot w = I \cdot w - (\frac{1}{\alpha} v \cdot v^T) \cdot w$ by distributive property. But $I \cdot w = w$ by definition of identity matrix and $\frac{1}{\alpha} v \cdot v^T \cdot w = \frac{1}{\alpha} v \cdot 0 = 0$ by initial conditional statement and definition of zero matrix. Then $P \cdot w = I \cdot w - (\frac{1}{\alpha} v \cdot v^T) \cdot w = (w) - (0) = w$ by definition of zero matrix. q.e.d.

Homework #2

due 1/31

Mercedes Nguyen

a) $A = \begin{vmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 7 & 7 \\ 3 & 6 & 6 & 6 \end{vmatrix}$

$$Ax = 0 = \left|\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 2 & 5 & 7 & 7 & 0 \\ 3 & 6 & 6 & 6 & 0 \end{array}\right| \rightarrow \left|\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

$m_{21} = -2$

$m_{31} = -3$

$R = 2$ + 2 free variables

$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$

$x_1 = -2x_2 - 2x_3 - 2x_4$

$x_2 + 3x_3 + 3x_4 = 0$

$x_2 = -3x_3 - 3x_4$

$x_1 = -2(-3x_3 - 3x_4) - 2x_3 - 2x_4 = 4x_3 + 4x_4$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 4x_3 + 4x_4 \\ -3x_3 - 3x_4 \\ x_3 \\ x_4 \end{vmatrix} = \boxed{x_3 \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix}}$$

linear combination

b)

$$Ax = b = \left|\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 2 & 5 & 7 & 7 & 7 \\ 3 & 6 & 6 & 6 & 9 \end{array}\right| \rightarrow \left|\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

$m_{21} = -2$

$m_{31} = -3$

$x_1 + 2x_2 + 2x_3 + 2x_4 = 3$

$x_2 + 3x_3 + 3x_4 = 1$

$\Rightarrow$ $x_2 = 1 - 3x_3 - 3x_4$

$x_1 = 3 - 2(1 - 3x_3 - 3x_4) - 2x_3 - 2x_4$

$x_1 = 3 - 2 + 6x_3 + 6x_4 - 2x_3 - 2x_4$

$= 1 + 4x_3 + 4x_4$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} 1 + 4x_3 + 4x_4 \\ 1 - 3x_3 - 3x_4 \\ x_3 \\ x_4 \end{vmatrix} = \boxed{\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + x_3 \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix}}$$

c) $x_3 = 0 \quad x_4 = 0$

$$\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + (0) \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + (0) \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix} = \boxed{\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}}$$

Check: $(1) + 2(1) + 2(0) + 2(0) = 3$ ✓

$2(1) + 5(1) + 7(0) + 7(0) = 7$ ✓

$3(1) + 6(1) + 6(0) + 6(0) = 9$ ✓

$x_3 = 1 \quad x_4 = 1$

$$\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + (1) \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + (1) \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix} = \boxed{\begin{vmatrix} 9 \\ -5 \\ 1 \\ 1 \end{vmatrix}}$$

Check: $(9) + 2(-5) + 2(1) + 2(1) = 9 - 10 + 4 = 3$ ✓

$2(9) + 5(-5) + 7(1) + 7(1) = 18 - 25 + 14 = 7$ ✓

$3(9) + 6(-5) + 6(1) + 6(1) = 27 - 30 + 12 = 9$ ✓

$x_3 = 0 \quad x_4 = 1$

$$\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + (0) \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + (1) \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix} = \boxed{\begin{vmatrix} 5 \\ -2 \\ 0 \\ 1 \end{vmatrix}}$$

Check: $(5) + 2(-2) + 2(0) + 2(1) = 5 - 4 + 2 = 3$ ✓

$2(5) + 5(-2) + 7(0) + 7(1) = 10 - 10 + 7 = 7$ ✓

$3(5) + 6(-2) + 6(0) + 6(1) = 15 - 12 + 6 = 9$ ✓

$x_3 = 1 \quad x_4 = 0$

$$\begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} + (1) \begin{vmatrix} 4 \\ -3 \\ 1 \\ 0 \end{vmatrix} + (0) \begin{vmatrix} 4 \\ -3 \\ 0 \\ 1 \end{vmatrix} = \boxed{\begin{vmatrix} 5 \\ -2 \\ 1 \\ 0 \end{vmatrix}}$$

Check: $(5) + 2(-2) + 2(1) + 2(0) = 5 - 4 + 2 = 3$ ✓

$2(5) + 5(-2) + 7(1) + 7(0) = 10 - 10 + 7 = 7$ ✓

$3(5) + 6(-2) + 6(1) + 6(0) = 15 - 12 + 6 = 9$ ✓

d) $\text{Rank}_A = 2$

Basic columns : 1, 2

free variables : $n - R = 4 - 2 = 2$ free variables (true ; x_3 and x_4 are free variables)

Homework #1 due 1/24

Mercedes Nguyen

1.2.3)
$$\begin{aligned}4x_2-3x_3&=3\\-x_1+7x_2-5x_3&=4\\-x_1+8x_2-6x_3&=5\end{aligned} \rightarrow \left|\begin{array}{ccc|c}0&4&-3&3\\-1&7&-5&4\\-1&8&-6&5\end{array}\right| \rightarrow \left|\begin{array}{ccc|c}-1&7&-5&4\\0&4&-3&3\\-1&8&-6&5\end{array}\right| \rightarrow \left|\begin{array}{ccc|c}-1&7&-5&4\\0&4&-3&3\\0&1&-1&1\end{array}\right| \rightarrow \left|\begin{array}{ccc|c}-1&7&-5&4\\0&4&-3&3\\0&0&-\frac14&\frac14\end{array}\right|$$

$m_{31}=-\frac{-1}{-1}=-1$, E_3+mE_1

$m_{32}=-\frac14$, E_3+mE_2

back sub: $-\frac14x_3=\frac14 \Rightarrow x_3=-1$

$4x_2-3x_3=3$

$4x_2-3(-1)=3$

$4x_2+3=3$

$4x_2=0 \Rightarrow x_2=0$

$-x_1+7x_2-5x_3=4$

$-x_1+7(0)-5(-1)=4$

$-x_1+0+5=4$

$-x_1=-1 \Rightarrow x_1=1$

$$\boxed{X=\left|\begin{array}{c}1\\0\\-1\end{array}\right|}$$

check: $-(1)+8(0)-6(-1)=5$

$-1+0+6=5$ ✓

$-(1)+7(0)-5(-1)=4$

$-1+0+5=4$ ✓

$4(0)-3(-1)=3$

$0+3=3$ ✓

1.2.5)
$$\begin{aligned}4x_1-8x_2+5x_3&=1\\4x_1-7x_2+4x_3&=0\\3x_1-4x_2+2x_3&=0\end{aligned}\left|\begin{array}{c|c}0&0\\1&0\\0&1\end{array}\right| \rightarrow \left|\begin{array}{ccc|c|c|c}4&-8&5&1&0&0\\4&-7&4&0&1&0\\3&-4&2&0&0&1\end{array}\right| \rightarrow \left|\begin{array}{ccc|c|c|c}4&-8&5&1&0&0\\0&1&-1&-1&1&0\\0&2&-\frac74&-\frac34&0&1\end{array}\right|$$

$m_{21}=-\frac44=-1$

$m_{31}=-\frac34$

$m_{32}=-\frac21=-2$

$$\rightarrow \left|\begin{array}{ccc|c|c|c}4&-8&5&1&0&0\\0&1&-1&-1&1&0\\0&0&\frac14&\frac54&-2&1\end{array}\right|$$

Back sub: $\frac14x_3=\frac54\,|-2\,|\,1$

$x_3=5\,|-8\,|\,4$

$x_2-x_3=-1\,|\,1\,|\,0$

$x_2-(5\,|-8\,|\,4)=-1\,|\,1\,|\,0$

$x_2=4\,|-7\,|\,4$

$4x_1-8x_2+5x_3=1\,|\,0\,|\,0$

$4x_1-8(4\,|-7\,|\,4)+5(5\,|-8\,|\,4)=1\,|\,0\,|\,0$

$x_1=2\,|-4\,|\,3$

$$\boxed{X=\left|\begin{array}{c|c|c}2&-4&3\\4&-7&4\\5&-8&-4\end{array}\right|}$$

Check: $3(2)-4(4)+2(5)=0$

$6-16+10=0$ ✓

$4(-4)-7(-7)+4(-8)=1$

$-16+49-32=1$ ✓

$4(3)-8(4)+5(-4)=0$

$12-32-20=0$ ✓

2.1.1)

a)
$$\left|\begin{array}{cccc}1&2&3&3\\2&4&6&9\\2&6&7&6\end{array}\right|$$

$m_{21}=-\frac21=-2$

$m_{31}=-\frac21=-2$

$$\left|\begin{array}{cccc}1&2&3&3\\0&0&0&3\\0&2&1&0\end{array}\right| \quad \left|\begin{array}{cccc}1&2&3&3\\0&2&1&0\\0&0&0&3\end{array}\right|$$

Rank: 3

Basic columns: 1, 2, 4

b)
$$\left|\begin{array}{ccc}1&2&3\\2&6&8\\2&6&0\\1&2&5\\3&8&6\end{array}\right|$$

$m_{21}=-\frac21=-2$

$m_{31}=-\frac21=-2$

$m_{41}=-\frac11=-1$

$m_{51}=-\frac31=-3$

$$\left|\begin{array}{ccc}1&2&3\\0&2&2\\0&2&-6\\0&0&2\\0&2&-3\end{array}\right|$$

$m_{32}=-1$

$m_{42}=0$

$m_{52}=-1$

$$\left|\begin{array}{ccc}1&2&3\\0&2&2\\0&0&-8\\0&0&2\\0&0&-5\end{array}\right|$$

$m_{43}=-\frac{2}{-8}=\frac14$

$m_{53}=-\frac{-5}{-8}=\frac58$

$$\left|\begin{array}{ccc}1&2&3\\0&2&2\\0&0&-8\\0&0&0\\0&0&0\end{array}\right|$$

Rank: 3

Basic columns: 1, 2, 3

c)
$$\left|\begin{array}{ccccccc}2&1&1&3&0&4&1\\4&2&4&4&1&5&5\\2&1&3&1&0&4&3\\6&3&4&8&1&9&5\\0&0&3&-3&0&0&3\\8&4&2&14&1&13&3\end{array}\right|$$

$m=-2$

$m=-1$

$m=-3$

$m=0$

$m=-4$

$$\left|\begin{array}{ccccccc}2&1&1&3&0&4&1\\0&0&2&-2&1&-3&3\\0&0&2&-2&0&0&2\\0&0&1&-1&1&-3&2\\0&0&3&-3&0&0&3\\0&0&-2&2&1&-3&-1\end{array}\right|$$

$m=-1$

$m=-1/2$

$m=-3/2$

$m=1$

$$\left|\begin{array}{ccccccc}2&1&1&3&0&4&1\\0&0&2&-2&1&-3&3\\0&0&0&0&-1&3&-1\\0&0&0&0&\frac12&-\frac32&\frac12\\0&0&0&0&-\frac32&\frac92&-\frac32\\0&0&0&0&2&-6&2\end{array}\right|$$

$m=\frac12$

$m=-\frac32$

$m=2$

$$\left|\begin{array}{ccccccc}2&1&1&3&0&4&1\\0&0&2&-2&1&-3&3\\0&0&0&0&-1&3&-1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{array}\right|$$

Rank: 3

Basic columns: 1, 3, 5

Review

$\underset{n\times k}{A}\cdot\underset{k\times m}{B} = \underset{n\times m}{C}$: $\underset{1\times n}{\rule{2em}{0.4pt}} \cdot \underset{n\times 1}{\Big|} = \#$ or $\underset{n\times 1}{\Big|} \cdot \underset{1\times n}{\rule{2em}{0.4pt}} = \underset{n\times n}{\square}$

$A\cdot A = A^2$, $A\cdot I = A$ and $I\cdot A = A$ $I = \begin{vmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \cdots & 1 \end{vmatrix}$, $BA \neq AB$

$\exists A^{-1}$ s.t. $A^{-1}\cdot A = I = A\cdot A^{-1}$

if v + w are $n\times 1$, then $v\cdot w^T$ or $w\cdot v^T$ always give a rank 1 $n\times n$ matrix.

Daniel B. Szyld 11 February 2021

Linear Algebra, Math 2101-002
Exam #1

Write your name on your response sheet. This is a closed book and notes exam. Explain your answers in as much detail as possible. Check your examples and results. You need to explain every step of your logic, development and calculation. No credit will be given without the intermediate steps or the explanations.

b is not a basic column
There is at least 1 soln
$\exists$ row s.t. $0\,0\,0 \ldots 0 \,|\, \alpha$, $\alpha \neq 0$
Rank A = Rank $A|b$

1. (25 pts. + 10 extra credit)
Let A be an $m \times n$ matrix.
(a) Give **four** statements equivalent to: "The linear system $Ax = b$ is consistent."
(b) Pick two of your equivalent statements, and show that one implies the other.
(c) (extra credit) Show the converse of (b), that is, if you showed that the first statement implies the second, show now that the second implies the first.

2. (25 pts.)
(a) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = \sin(x_1 + x_2)$. Show that this function is a linear map, or explain why it is not a linea map.
(b) Let $A = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$ and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ define as $f(x) = Ax$. Compute and completely describe the nullspace of f, $\mathcal{N}(f)$.

3. (30 pts. + 10 pts. extra credit)
Let $A = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix}$ and let $v = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$ and $w = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$.
(a) Compute $v^T w$, and vw^T.
(b) Compute Av, and $w^T Av$. Give the value of $v^T A^T w$ without actually computing it. Explain how you came out with this response.
(c) Compute A^2, $A^T A$ and AA^T.
(d) (extra credit) Check and explain why the latter two are symmetric matrices.

4. (20 pts.)
(a) Give an example of a 3×3 skew-symmetric matrix.
(b) Prove that for any $n \times n$ skew-symmetric matrix the diagonal entries are all zeros.

1) a) "$Ax = b$ is consistent"

* ★ • Rank A = Rank $A|b$
* ★ • There does not exist a row such that $000\ldots0|\alpha$ where $\alpha \neq 0$
* • b is not a basic column
* • $b = \sum \alpha_j a_j$

b) If there doesn't exist a row such that $000\ldots0|\alpha$ where $\alpha \neq 0$, then $\alpha = 0$ and the row consists of all zeros. Then if there are m rows, Rank $A|b$ would only be able to have a maximum of $m-1$ rows. Similarly, Rank A would also only be able to have a maximum of $m-1$ rows. If Rank $A = x \leq m-1$ then Rank $A|b = x \leq m-1$ since no $\alpha \in \mathbb{R}$ can exist in b. Hence, Rank A = Rank $A|b$.

c) If Rank A = Rank $A|b$, then column b is a nonbasic column. Then by definition there must be a zero in the last entry of b. This implies in the final row we have the form $|E_A|c| \in |000\ldots0|0|$.

2) a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = \sin(x_1 + x_2)$ we need to show $f(x+y) = f(x) + f(y)$

$$f(x_1) + f(x_2) = \sin(x_1) + \sin(x_2) = 2\sin\left(\frac{x_1+x_2}{2}\right)\cos\left(\frac{x_1-x_2}{2}\right) \neq \sin(x_1+x_2)$$. Hence, not linear.

b) $A = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$, $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = Ax$ compute + define nullspace $N(f) = \{x \in D : f(x) = 0\}$

$$\left|\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right| \to \left|\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right| \to \left|\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right| \quad x_n = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \quad N(f) = \left\{ \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \right\}$$

$m_{31} = -1$ $m_{32} = -2$

there is only 1 solution $x_n = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$. The null space, $N(f) = \left\{ \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \right\}$

3) $A = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix}$ $v = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$ $w = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

a) $v^T \cdot w = \underset{1\times3}{\begin{vmatrix} 1 & -1 & 2 \end{vmatrix}} \underset{3\times1}{\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}} = [2]$ $v \cdot w^T = \underset{3\times1}{\begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}} \underset{1\times3}{\begin{vmatrix} 1 & 1 & 1 \end{vmatrix}} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{vmatrix}$

b) $Av = \underset{3\times3}{\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix}} \underset{3\times1}{\begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}} = \begin{vmatrix} 2 \\ -3 \\ 0 \end{vmatrix}$ $w^T \cdot Av = \begin{vmatrix} 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 \\ -3 \\ 0 \end{vmatrix} = [-1]$

$v^T A^T w$ will yield a 1×1 matrix. This implies that the transpose of the product yields the same 1×1 matrix, so $v^T A^T w = (v^T A^T w)^T = (w)^T \cdot (A^T)^T \cdot (v^T)^T = w^T A v = [-1]$.

c)

4) a) 3×3 skew sym $A^T = -A$ $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \to \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$

$$\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix}$$

b) Let A be an $n \times n$ matrix that is skew symmetric. Then by definition, $A^T = -A$. By definition of transposition, $(a_{ij})^T = a_{ji}$. But then when $i = j$, $(a_{ij})^T = a_{ji} = a_{ij}$. Then by definition of skew symmetric $a_{ji} = -a_{ij}$. But $a_{ji} = a_{ij}$, so by transitive property in $\mathbb{R}$, $a_{ij} = -a_{ij}$. $\exists! \alpha \in \mathbb{R}$ such that $\alpha = -\alpha$, namely $\alpha = 0$. Hence, $\forall a_{ij} \in A$ such that $i = j$ must be 0.

Exam #2

4) a) A nonsingular, implies $A^{-1}\cdot A = A\cdot A^{-1} = I$. Then A^T is also nonsingular. Then $(A^{-1})^T \cdot A^T = (A^{-1}\cdot A)^T = I$
Hence $(A^{-1})^T = (A^T)^{-1}$.

b) $(AB)^{-T} \cdot (AB)^T = I$. But $(AB)^T = B^T A^T$ and since both nonsing. Then $\exists (B^T A^T)^{-1} \cdot B^T A^T = I$
since $(AB^T) = B^T A^T$. then $(AB)^{-T} = (B^T A^T)^{-1} = B^{-T} A^{-T}. = A^{-T} B^{-T}$ bc nonsing.

5) $S = \frac{A + A^T}{2}$ and $K = \frac{A - A^T}{2}$ are sym + skew (confirm $A = S + K$) we need to show $S = S^T$ and $K^T = -K$

$S^T = \left(\frac{A + A^T}{2}\right)^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T)$ bc closed under T bc $m = n$ (sq matrix) $= \frac{1}{2}(A + A^T) = S$
$K^T = \left(\frac{A - A^T}{2}\right)^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -K$.
$S + K = \frac{A + A^T}{2} + \frac{A - A^T}{2} = \frac{A + A^T + A - A^T}{2} = \frac{\cancel{2}A}{\cancel{2}} = A$.

b) Suppose $A = S + K$ and S is sym and K is skew. Then $S^T = S$ and $K^T = -K$ by def. Then $A^T = (S+K)^T = S^T + K^T$
But then $A^T = S - K$, then $S = A^T + K$ and $K = S - A^T$. But since $A = S + K$ we can say that
$S = A^T + (A - S)$ and $K = (A - K) - A^T$. Then $S + S = A^T + A \Rightarrow 2S = A^T + A \Rightarrow S = \frac{A^T + A}{2}$ and $K + K = A - A^T$
...... $K = \frac{A - A^T}{2}$

6) $P = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \updownarrow = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$

$\left|\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right|$ if $P^{-1} = P$ then $P^2 = P \cdot P = P(P^{-1}) = I$

$x_1 = 0, 1$
$x_2 = 1, 0$

b) $A = \left|\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}\right| \updownarrow \left|\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}\right|$ rearrange to an upper $\triangle$ so $A^{-1} = \left|\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}\right|$ not the best proof

7) $P^2 = P$ and $P \neq I$ suppose P is nonsing. Then $\exists P^{-1}$ s.t. $P \cdot P^{-1} = P^{-1} \cdot P = I$
since $P^2 = P$ then $P \cdot P^{-1} = (P^2) \cdot P^{-1} = P(P \cdot P^{-1}) = P(I) = P$. Then $P = I$. Contradiction! thus
P is sing.

Exam #3

1) $A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{vmatrix}$ $\quad b = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times A \left(E_{ij} \ldots\right)$

$m_{21} = -1$
$m_{31} = -1$

$\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{vmatrix}$ $\quad m_{32} = -4$

$\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} = U \quad (E_{21}E_{31}E_{32})^{-1} = \left(\begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -4 & 1 \end{vmatrix}\right)^{-1} = L = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}$

$Ly = b$
$Ux = y$

$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix} \begin{vmatrix} y \end{vmatrix} = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$ $\quad y_1 = 1 \quad y_2 = (-1 - y_1)/1 = -2 \quad y_3 = 2 - y_1 - 4y_2/1 = 2 - 1 - (-8) = 9$

$y = \begin{vmatrix} 1 \\ -2 \\ 9 \end{vmatrix}$

$\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} \begin{vmatrix} x \end{vmatrix} = \begin{vmatrix} 1 \\ -2 \\ 9 \end{vmatrix}$ $\quad x_3 = 1 \quad x_2 = (-2 - (-3)x_3)/-1 = 2 - 3 = -1 \quad x_1 = (1 - 2x_3 - 3x_2)/2$
$(1 - 2 + 3)/2 = 1$

$x = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$ ✓

2) Linear dependence?

$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{vmatrix}$ ✓ independent
trivial solution
$Ax = 0 \rightarrow x = 0$

$m_{21} = -2$
$m_{31} = -3$
$m_{32} = -2$

Span $\mathbb{R}^3$? yes, bc Rank = 3.

b) Skipping

3) show closed under add + scalar mult. $S = \{x : x^T v = 0\}$ $\quad v = \begin{vmatrix} 2 \\ 3 \\ 1 \\ 1 \end{vmatrix}$ → dimension of $V = 5$

Suppose $y \in S$, then $y^T v = 0$

a) $x + y = x^T v + y^T v = 0 + 0 = 0$ ✓ $\quad \alpha x = \alpha x^T v = \alpha(0) = 0$ ✓

b) skipping

4) $U = \begin{vmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 2 \end{vmatrix}$ $\quad V = \begin{vmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{vmatrix}$

a) $N(U) = \{x : Ux = 0\}$ → subspace of $\mathbb{R}^2$ $\quad \mathbb{R}^n$ since $U = \overset{3\times 2}{m \times n}$

$\begin{vmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 2 \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 3 \\ 0 & -2 \\ 0 & -4 \end{vmatrix}$ $\quad \begin{vmatrix} 1 & 3 \\ 0 & -2 \\ 0 & 0 \end{vmatrix}$ $\quad$ Rank $2 = n \neq m$ $\quad$ cannot give w/o computing

$m_{21} = -1$
$m_{31} = -2$
$m_{32} = -2$

b) $UV^T = 3\times 2 \cdot 2\times 3 \rightarrow 3\times 3$

$V^T = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$U = \begin{vmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 2 \end{vmatrix}$