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Due Tueday 12 September 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #2

1. For each of the following four sets of vectors $\{u, v\}$, compute the (Euclidean) angle between them.

(a) $u = \begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix}$, $v = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$.

(b) $u = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$, $v = \begin{vmatrix} 1 \\ 1 \\ -2 \end{vmatrix}$.

(c) $u = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$, $v = \begin{vmatrix} 2 \\ 2 \end{vmatrix}$.

(d) $u = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$, $v = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$.

2. Let $\Pi_n = \{p(x)$ polynomials of degree $\leq n\}$ and define the inner product for this space as $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$. Let $p(x) = 1$, $q(x) = 3x^2 - 1$. Show that $\langle p, q\rangle = 0$.

3. Consider two lines in $\mathbb{R}^n$, that is fix $v, w \in \mathbb{R}^n$ and define

$$L_1 = \{x \in \mathbb{R}^n \mid x = \alpha v, \text{ for some } \alpha \in \mathbb{R}\},$$
$$L_2 = \{y \in \mathbb{R}^n \mid y = \beta w, \text{ for some } \beta \in \mathbb{R}\},$$

and let θ be the angle between v and w, that is,

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|}.$$

Prove that for all $x \in L_1$, $y \in L_2$ it holds that the angle betweem x amd y is also θ.

Two more exercises on the reverse.

4. Let S be a subspace of a vector spave V with an inner product. Define

$$S^{\perp} = \{x \in V \mid \langle x, v\rangle = 0, \forall v \in S\},$$

that is, the set of all vectors which are orthogonal to all vectors in S. Prove that $S^{\perp}$ is also a subspace of V.

5 (a). Let $S = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$, i.e., the horizontal plane. Show that

$$S^{\perp} = \left\{ \alpha \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \mid \alpha \in \mathbb{R} \right\},$$

i.e., the vertical axis is orthogonal to the horizontal plane.

(b). Show that if $S = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$, then

$$S^{\perp} = \left\{ \alpha \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \mid \alpha \in \mathbb{R} \right\}.$$

(c). What subspace is $V^{\perp}$, if V is the vector space?

(d). Show that for every S subspace of a vector space V,

$$\left(S^{\perp}\right)^{\perp} = S.$$

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① (a) $u = \begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix}$, $v = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

$\langle u, v \rangle = 0(0) + 1(0) + 1(1) = 0 + 0 + 1 = 1$

$\|u\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{0+1+1} = \sqrt{2}$

$\|v\| = \sqrt{0^2 + 0^2 + 1^2} = \sqrt{0+0+1} = 1$

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \frac{1}{\sqrt{2}(1)} = \frac{1}{\sqrt{2}} \longrightarrow \theta = 45^\circ$$

(b) $u = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$, $v = \begin{vmatrix} 1 \\ 1 \\ -2 \end{vmatrix}$

$\langle u, v \rangle = 1(1) + 1(1) + 1(-2) = 1 + 1 - 2 = 0$

$\hookrightarrow u \perp v \longrightarrow \theta = 90^\circ$

(c) $u = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$, $v = \begin{vmatrix} 2 \\ 2 \end{vmatrix}$

$\langle u, v \rangle = 1(2) + (-1)2 = 2 - 2 = 0$

$\hookrightarrow u \perp v \longrightarrow \theta = 90^\circ$

(d) $u = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$, $v = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$

$\langle u, v \rangle = 1(-1) + 0(1) = -1 + 0 = -1$

$\|u\| = \sqrt{1^2 + 0^2} = \sqrt{1+0} = 1$

$\|v\| = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|} = \frac{-1}{1(\sqrt{2})} = \frac{-1}{\sqrt{2}} \longrightarrow \theta = 135^\circ$$

② $$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\,dx = \int_{-1}^{1} 1(3x^2 - 1)\,dx = \int_{-1}^{1} (3x^2 - 1)\,dx = (x^3 - x)\Big|_{-1}^{1}$$

$$= (1^3 - 1) - [(-1)^3 - (-1)] = (1 - 1) - (-1 + 1) = 0 - 0 = 0$$

$\therefore \langle p, q \rangle = 0$ QED!

③ For all $x \in L_1$, $y \in L_2$: $x = \alpha v$, $y = \beta w$ for some $\alpha, \beta \in \mathbb{R}$

$\langle x, y \rangle = \langle \alpha v, \beta w \rangle = \alpha \langle v, \beta w \rangle = \alpha\beta \langle v, w \rangle$

Let φ denote the angle between x and y, need to show $\varphi \equiv \theta$

$$\cos\varphi = \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \overset{\text{by linearity of inner product}}{=} \frac{\alpha\beta\langle v, w\rangle}{\|\alpha v\|\,\|\beta w\|} = \frac{\alpha\beta\langle v, w\rangle}{|\alpha|\,\|v\|\,|\beta|\,\|w\|}$$ by property of inner product

Case 1 $\rightarrow$ α, β are both positive $\therefore |\alpha| = \alpha, |\beta| = \beta$

$$\cos\varphi = \frac{\alpha\beta\langle v, w\rangle}{\alpha\beta\|v\|\,\|w\|} = \frac{\langle v, w\rangle}{\|v\|\,\|w\|} = \cos\theta \rightarrow \varphi \equiv \theta$$ QED!

Case 2 $\rightarrow$ α, β are both negative $\therefore |\alpha| = -\alpha, |\beta| = -\beta$

$$\cos\varphi = \frac{\alpha\beta\langle v, w\rangle}{(-\alpha)(-\beta)\|v\|\,\|w\|} = \frac{\alpha\beta\langle v, w\rangle}{\alpha\beta\|v\|\,\|w\|} = \frac{\langle v, w\rangle}{\|v\|\,\|w\|} = \cos\theta \rightarrow \varphi \equiv \theta$$ QED!

Case 3 $\rightarrow$ α is negative, β is positive $\therefore |\alpha| = -\alpha, |\beta| = \beta$

$$\cos\varphi = \frac{\alpha\beta\langle v, w\rangle}{(-\alpha)\beta\|v\|\,\|w\|} = \frac{-\langle v, w\rangle}{\|v\|\,\|w\|} = -\cos\theta \rightarrow \varphi = 180^\circ - \theta \rightarrow \varphi \not\equiv \theta$$

Case 4 $\rightarrow$ α is positive, β is negative $\therefore |\alpha| = \alpha, |\beta| = -\beta$

$$\cos\varphi = \frac{\alpha\beta\langle v, w\rangle}{\alpha(-\beta)\|v\|\,\|w\|} = \frac{-\langle v, w\rangle}{\|v\|\,\|w\|} = -\cos\theta \rightarrow \varphi = 180^\circ - \theta \rightarrow \varphi \not\equiv \theta$$

$\therefore$ The proof does not apply when α, β have opposite signs
$\therefore$ The proof applies for some $\alpha, \beta \in \mathbb{R}$ that have the same signs QED!

(4) Take $y, w \in S^\perp$, $\alpha \in \mathbb{R}$; wish to show $(\alpha y + w) \in S^\perp$
$\forall v \in S$; $y, w \in S^\perp$ then $\langle y, v\rangle = \langle w, v\rangle = 0$
Consider $\langle \alpha y + w, v\rangle = \alpha\langle y, v\rangle + \langle w, v\rangle = \alpha(0) + 0 = 0 + 0 = 0$
by linearity of inner product

$\therefore \langle \alpha y + w, v\rangle = 0$
$\therefore (\alpha y + w) \in S^\perp$ $\therefore S^\perp$ is a subspace of V QED!

(5) (a) Take $v \in S$: $v = \begin{vmatrix} v_1 \\ v_2 \\ 0 \end{vmatrix}$ $\forall\, v_1, v_2 \in \mathbb{R}$

Take $w \in S^{\perp}$: $w = \alpha \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \alpha \end{vmatrix} \quad \forall \alpha \in \mathbb{R}$

To show that the vertical axis is orthogonal to the horizontal plane, wish to show that $\langle v, w \rangle = 0$

Consider $\langle v, w \rangle = v_1(0) + v_2(0) + 0(\alpha) = 0 + 0 + 0 = 0 \quad \therefore \langle v, w \rangle = 0$

definition of inner product

QED!

(b) Take $v \in S$: $v_1 + v_2 + v_3 = 0$ where $v = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix}$

Take $w \in S^{\perp}$: $w = \alpha \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} \alpha \\ \alpha \\ \alpha \end{vmatrix} \quad \forall \alpha \in \mathbb{R}$

Need to show that $S^{\perp}$ is the orthogonal complement of S or $\langle v, w \rangle = 0$

Consider $\langle v, w \rangle = (v_1)\alpha + (v_2)\alpha + (v_3)\alpha$ definition of inner product

$= (v_1 + v_2 + v_3)\alpha$

$= 0(\alpha) = 0 \quad \therefore \langle v, w \rangle = 0$ QED!

(c) If V is the vectorspace, $V^{\perp}$ consists of all vectors that are orthogonal to every vector in V. Since every vector in V is orthogonal to itself, the only vector satisfying this is the zero vector (0).

$\therefore V^{\perp} = \{0\}$

(d) To show that $(S^{\perp})^{\perp} = S$, need to show 2 inclusions:

(i) $S \subseteq (S^{\perp})^{\perp}$

By definition, $S^{\perp} = \{x \in V \mid \langle x, v \rangle = 0, \forall v \in S\}$

$\therefore (S^{\perp})^{\perp} = \{y \in V \mid \langle y, w \rangle = 0, \forall w \in S^{\perp}\}$

i.e: $(S^{\perp})^{\perp}$ denotes the set of all vectors in vector space V that are orthogonal to every vector in orthogonal complement of S

Let $a \in S$, $b \in S^{\perp}$: $a \perp b$

$\therefore a \in (S^{\perp})^{\perp}$

$\therefore a \in S$ and $a \in (S^{\perp})^{\perp} \qquad \therefore S \subseteq (S^{\perp})^{\perp}$

(ii) $(S^{\perp})^{\perp} \subseteq S$

Let $c \in (S^{\perp})^{\perp}$: $b \perp c$ with $b \in S^{\perp}$

but $b \perp a$ with $a \in S$ (from (i))

$\therefore$ $c \in S$, otherwise c would not be orthogonal to all of $S^{\perp}$

$\therefore$ $c \in S$ and $c \in (S^{\perp})^{\perp}$ $\quad\therefore (S^{\perp})^{\perp} \subseteq S$

Since (i) & (ii) are satisfied, $S = (S^{\perp})^{\perp}$ QED!