Linear Algebra Math 2101 D. B. Szyld

Exam #1 ANSWERS

1. (a) The linear system $Ax=b$ (A $m \times n$) is consistent, i.e. it has at least one solution, or equivalently:

(1) Rank $[A|b] =$ Rank A

(2) b is not a basic column of $[A|b]$

If $[A|b]$ is equivalent to $[E|c]$ with E upper Echelon form

(3) There is no row in $[E|c]$ of the form

$$[0\,0\,0 \cdots 0 | \alpha] \qquad \alpha \neq 0$$

(4) Rank $[E|c] =$ Rank E

(5) b can be written as a linear combination of the basic columns of A.

(only four statements were required)

(b) (3) $\Rightarrow$ (4) If there are no rows of the form $[0\,0\,0\,0 \ldots 0|\alpha]$, $\alpha \neq 0$ in $[E|c]$, then

all rows are either all zeros, or

of the form $[0 0 \cdots 0 \, e_1 \, e_2 \cdots e_n | \alpha]$

where $e_1 \neq 0$ is a pivot in some position in the row. Since the number of pivots is the rank, then

$$\text{Rank}[E|c] = \text{Rank}\, E.$$

(c) Conversely, if $\text{Rank}[E|c] = \text{Rank}\, E$,

then both $[E|c]$ and E have the same number of pivots. This implies that no row of $[E|c]$ can have the form $[0 0 \cdots 0 0 | \alpha]$, $\alpha \neq 0$.

2.(a) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = \sin(x_1 + x_2)$$

A function is linear if $f(x+y) = f(x) + f(y)$ and $f(\alpha x) = \alpha f(x)$.

This function is not linear.
For example let $x = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

then $f\left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}\right) = \sin(\pi + 0) = 0$

let $\alpha = \frac{1}{2}$ $\quad f\left(\alpha \begin{bmatrix} \pi \\ 0 \end{bmatrix}\right) = \sin\left(\frac{\pi}{2} + 0\right) = \sin\frac{\pi}{2} = 1$

$$\neq \frac{1}{2} \cdot 0$$

2.(b) $A = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$ $\qquad f(x) = Ax$

$N(f) = \{x \,/\, f(x) = 0\} = \{x \,/\, Ax = 0\} =$

= solutions of the homogeneous system $Ax = 0$

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix}$$

$m_{31} = -1$ $\qquad\qquad$ $m_{32} = -2$

three pivots, no free variables, only solution is the trivial solution $x = 0$

$N(f) = \{0\}$.

3. $v = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$ $\quad w = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

(a) $v^T w = 1 - 1 + 2 = 2$ $\qquad v w^T = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix} [1 \; 1 \; 1] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

(b) $A = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix}$ $\qquad A \cdot v = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix} = \begin{bmatrix} 2+0+0 \\ 1-2-2 \\ 1+1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$

$w^T A v = [1 \; 1 \; 1] \begin{vmatrix} 2 \\ -3 \\ 0 \end{vmatrix} = 2 - 3 + 0 = -1$

$$v^T A^T w = (Av)^T w = w^T A v = -1$$

(c)
$$A^2 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 2+0+0 & 0+0+0 & 0+0+0 \\ 1\cdot 2+2\cdot 1+(-1)\cdot 1 & 1\cdot 0+2\cdot 2+(-1)^2 & 1\cdot 0+2\cdot(-1)+(-1)^2 \\ 1\cdot 2+(-1)1+(-1)1 & 1\cdot 0+(-1)2+(-1)^2 & 1\cdot 0+(-1)^2+(-1)^2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -2 \\ 1 & 5 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$ symmetric

$$A A^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$ symmetric

(d) They are both symmetric because.

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

$$(A A^T)^T = (A^T)^T A^T = A A^T$$

(4)
(a) For example $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ $A^T = -A$

(b) $A^T = -A$

$$(A^T)_{ij} = a_{ji} = -a_{ij}$$

$\Rightarrow$ for $i = j$

$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$, $a_{ii} = 0$

all diagonals entries are zero.

4.(c) $A^T = -A$

$$(\alpha A)^T = \alpha A^T = \alpha(-A) = -(\alpha A)$$

5. $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -1/2 \end{bmatrix}$

$$\underset{3\times 2}{A} \cdot \underset{2\times 3}{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ 1 & -1/2 & -1/2 \end{bmatrix}$$

$$\underset{2\times 3}{B} \cdot \underset{3\times 2}{A} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

(b) $\underset{2\times 3}{B} \cdot \underset{3\times 2}{A} = I$ does not say $A = B^{-1}$

In fact for B rectangular we can not talk of an inverse.

(6.)

(a) $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -12 \end{bmatrix} \rightarrow$

$m_{21} = -1$

$m_{31} = -1$

$m_{32} = -4$

$\rightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

2 pivots 1 free variable

Rank $A = 2$

(b) A singular because:

(1) Rank $A < n$
(2) $\exists$ free variables
(3) number of pivots < # of columns

(4) $\exists\ v \neq 0 \ni Av = 0$
(a non trivial solution of the homogeneous system)

7 (a) $A = \begin{vmatrix} -1/2 & 3/2 \\ 1 & -1 \end{vmatrix}$ $\quad B = \begin{vmatrix} 1 & 3/2 \\ a & 1/2 \end{vmatrix}$

$$A \cdot B = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}a & 0 \\ 1-a & 1 \end{bmatrix}$$

thus for $1 - a = 0$ $\quad a = 1$

we also see that $-\frac{1}{2} + \frac{3}{2} \cdot 1 = 1$ ✓

thus $B = A^{-1} = \begin{vmatrix} 1 & 3/2 \\ 1 & 1/2 \end{vmatrix}$

(b) $B^{-1} = A = \begin{vmatrix} -1/2 & 3/2 \\ 1 & -1 \end{vmatrix}$

(8) wish to show $(AB) = (BA)^T$

will show $(BA)^T = AB$

$$(BA)^T \underset{\substack{\uparrow \\ \text{transpose} \\ \text{of product}}}{=} A^T B^T \underset{\substack{\uparrow \\ A,B \\ \text{symmetric}}}{=} A B$$ qed.