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Section 2.1

Row Echelon Form and Rank of a rectangular matrix

The goal is to use the Gaussian Elimination process to reduce the rectangular system (using elementary transformations) to something "like ~~triangular~~" triangular, that is with the pivots leading the rows (and zeros before them).

Process: - If (nonzero) pivot. Eliminate below

- If zero in pivot position and below there is some nonzero (in same column). Do row interchanges
- If below there are all zeros, move to the next column (repeat if necessary)

Example

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{bmatrix}$$

$E_2 - 2E_1$

$E_3 - E_1$

$E_4 - 2E_1$

First pivot in (1,1) position

Second pivot in (2,3) position (of 5')

[Note value is -2, which is not $a_{23} = 0$]

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E = E_A$$

$E_3' - E_2'$

$E_4' - E_2'$

exchange row 3 and row 4

Row Echelon Form of A

<u>Note.</u> The Echelon form of a matrix is <u>unique</u> in the sense that the number and location of the pivots is fixed.

<u>Definition</u> <u>Rank A</u> = # of pivots in E_A

= # of nonzeros rows in E_A

<u>Basic columns</u> of A, are the columns where the pivots are.

In example columns 1, 3, 5, that is,

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$$

Rank of a matrix = # of basic columns

The other columns are non-basic columns.
In example these are colums 2, 4.

Another example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & 1 & 2 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -3 & -9 & -8 \\ 0 & -5 & -7 & -8 & -14 \end{bmatrix}$$

$E_2 - 2E_1$

$E_3 - 3E_1$

$$\to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -3 & -9 & -8 \\ 0 & 0 & -2 & 7 & \frac{-2}{3} \end{bmatrix}$$

$$-7 + \frac{5}{3}3 = -2$$

$E_3' - \frac{5}{3}E_2'$

$$-8 - \left(\frac{-5}{3}\right)(-9) = -8 + 15 = 7$$

$$-14 - \left(\frac{-5}{3}\right)(-8) = -\frac{14 \times 3 + 40}{3} = \frac{-2}{3}$$

Rank 3. basic colums 1, 2, 3

that is $\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \quad \begin{vmatrix} 2 \\ 1 \\ 1 \end{vmatrix} \quad \begin{vmatrix} 3 \\ 3 \\ 2 \end{vmatrix}$

"last" example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \longrightarrow$$

$E_3 - E_1$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$E_3' - E_2'$

Rank is $\underline{2}$

basic colums 1, 2. $\begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ $\begin{vmatrix} 2 \\ 2 \\ 0 \end{vmatrix}$

Matrix is $m \times n$

if Rank $A < m$, we have zero columns in E_A

this statement is $\Longleftrightarrow$
i.e. implication in both directions
"if and only if".

Can do all exercises of section 2.1 (p. 46)

2.1.1 part of homework 1