Linear Algebra 2101-02 Daniel Szyd

Answers to HW 3.

1) $A = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$ $B = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

a) $C = A \cdot B = \begin{vmatrix} (3-2) & (2+0) & (1+2) \\ (-3+0) & (-2) & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \end{vmatrix}$

b) $A^T = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$ $B^T = \begin{vmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{vmatrix}$

c) $C^T = \begin{vmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{vmatrix}$

$$B^T \cdot A^T = \begin{vmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} (3-2) & -3+0 \\ 2+0 & -2+0 \\ 1+2 & -1+0 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{vmatrix}$$

Indeed $C^T = B^T A^T$ ✓

2) $v = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix}$ $w = \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix}$ (a) $w^T v = 0 \cdot 1 + 1 \cdot 2 + 2 \cdot (-1) = 0$

b) $v^T w = 1 \cdot 0 + 2 \cdot 1 + (-1) \cdot 2 = 0$

c) $v \cdot w^T = \begin{vmatrix} 1 \\ 2 \\ -1 \end{vmatrix} [0\ 1\ 2] = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{bmatrix}$

d) $w \cdot v^T = \begin{vmatrix} 0 \\ 1 \\ 2 \end{vmatrix} [1\ \ 2\ \ -1] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$

e) $v^T v = 1^2 + 2^2 + (-1)^2 = 6$

f) $v \cdot v^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [1\ \ 2\ \ -1] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

3, a) $\alpha = v^T v = 6$

$$P = I - \frac{1}{\alpha} v \cdot v^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\frac{1}{6} & -2/6 & 1/6 \\ -2/6 & 1-4/6 & 2/6 \\ 1/6 & 2/6 & 1-1/6 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Note that P is symmetric, i.e., $P^T = P$

b). $P \cdot v = \frac{1}{6} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5-4-1 \\ -2+4-2 \\ 1+4-5 \end{bmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} = 0$

3.b) (cont.)

$$P \cdot w = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0-2-2 \\ 0+2+4 \\ 0+2+10 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

c)

$$P^2 = P \cdot P. = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} =$$

$$= \frac{1}{36}\begin{bmatrix} 5^2+2^2+1^2 & 5(-2)+(-2)\cdot 2+1\cdot 2 & 5\cdot 1+(-2)\cdot 2+1\cdot 5 \\ (-2)\cdot 5+2\cdot(-2)+2\cdot 1 & (-2)^2+2^2+2^2 & (-2)\cdot 1+2\cdot 2+2\cdot 5 \\ 1\cdot 5+2\cdot(-2)+5\cdot 1 & 1\cdot(-2)+2\cdot 2+5\cdot 2 & 1^2+2^2+5^2 \end{bmatrix} =$$

associativity of product
and definition of scalar times matrix

$$= \frac{1}{36}\begin{bmatrix} 30 & -12 & 6 \\ -12 & 12 & 12 \\ 6 & 12 & 30 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} = P$$

(4.a) $P = I - \frac{1}{\alpha} v v^T$, where $\alpha = v^T v$

$$P^T = \left(I - \frac{1}{\alpha} v v^T\right)^T = I^T - \left(\frac{1}{\alpha} v v^T\right)^T =$$

$$= I - \frac{1}{\alpha}(v v^T)^T = I - \frac{1}{\alpha}(v^T)^T v^T =$$

$$= I - \frac{1}{\alpha} v v^T = P$$

(b) $P^2 = P \cdot P = \left(I - \frac{1}{\alpha} v v^T\right)\left(I - \frac{1}{\alpha} v v^T\right) =$

$$I \cdot I - I\left(\frac{1}{\alpha} v v^T\right) - \frac{1}{\alpha} v \cdot v^T \cdot I + \frac{1}{\alpha} \cdot \frac{1}{\alpha} v \cdot v^T v \cdot v^T$$

$$= I - \frac{1}{\alpha} v v^T - \frac{1}{\alpha} v v^T + \frac{1}{\alpha} \cdot \frac{1}{\alpha} v \underbrace{(v^T v)}_{\alpha} v^T =$$

$$= I \underbrace{- \frac{1}{\alpha} v v^T - \frac{1}{\alpha} v v^T + \frac{1}{\alpha} \cancel{\frac{1}{\alpha}} \cdot \cancel{\alpha} (v v^T)}_{=0} = P$$

(c) $P \cdot v = \left(I - \frac{1}{\alpha} v v^T\right) v \underset{\text{associativity}}{=} v - \frac{1}{\alpha} v \cdot (v^T v) =$

$$= v - \frac{1}{\alpha} \cdot v \cdot \alpha = v - \frac{1}{\alpha} \alpha v = v - v = 0$$

d) if $v^T w = 0$ Then $Pw = w$

$$Pw = \left(I - \frac{1}{\alpha} v v^T\right) w = w - \frac{1}{\alpha} v \cdot \underbrace{v^T w}_{=0} = w$$