Definition. Let $x = \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}$, $y = \begin{vmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{vmatrix}$

two (column) vectors of the same length

$\alpha x + \beta y$ is a linear combination

of them, where α, β are scalars (numbers).

Example. $x = \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix}$ $y = \begin{vmatrix} -1 \\ 4 \\ 0 \end{vmatrix}$

then $3x + 2y = 3 \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix} + 2 \begin{vmatrix} -1 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} 5 \\ 14 \\ 3 \end{vmatrix}$

is a linear combination.

Another: $-2x + y = -2 \begin{vmatrix} 3 \\ 2 \\ 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 \\ 4 \\ 0 \end{vmatrix} = \begin{vmatrix} -7 \\ 0 \\ -2 \end{vmatrix}$

The same definition applies to rows.

When performing Gaussian elimination

we replace a row of a matrix with a

linear combination of two rows,

the "current" row, and the pivot row.

For example if we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$E_2 - 1 . E_1$$

the second row is replaced by $1 . E_2 + (-1) E_1$

Section 2.3, additional example.

Let the augmented system be as follows

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 4 & 2 & 2 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & -2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent.

Rank 2

basic columns 1, 2: $\begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} \quad \begin{vmatrix} 1 \\ 2 \\ 4 \end{vmatrix}$

* Free variables $n - r = 5 - 2 = 3$

$x_3 \quad x_4 \quad x_5$

right hand side b, a linear combination of the basic columns

For this example, we find a particular solution.

Set $x_3 = x_4 = x_5 = 0$

solve for x_1, x_2

In second equation $x_2 = -1$

First equation $x_1 = (3 - 1x_2)/1$
$= (3+1)/1 = 4$

$$x_{part} = \begin{vmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \text{and} \quad b = \begin{vmatrix} 3 \\ 2 \\ 4 \end{vmatrix} = 4 \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 \\ 2 \\ 4 \end{vmatrix}.$$

check!

In essence, row 3 here has no additional information.

Row 3 is a linear combination of Row 2 and row 1

Row 3 = 2 row 2 + 0. row 1

(here just a multiple of another row).

If we go back to example on p. 23.

Row 3 = 1. Row 2 + 1. Row 1

—

This is the general case.

a row of zeros in the upper Echelon form of an augmented system indicates that on row is a linear combination of the other rows.

Say. $E_3'' = E_3' - 2E_2'$

$$= (E_3 - 4E_1) - 2(E_2 + 2E_1)$$

$$0 = E_3 - 4E_1 - 2E_2 - 4E_1$$

$$E_3 = 8E_1 + 2E_2$$

This is exercise 2.1.3. (d).

—.—