§ 4.6 Variation of Parameters

Review: Cramer's Rule

Coefficient matrix

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

column matrix of variables

column matrix of constants

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - c_2 b_1$$

$$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1 c_2 - a_2 c_1$$

Cramer's Rule:

$$x = \frac{\Delta_x}{\Delta} \quad ; \quad y = \frac{\Delta_y}{\Delta}$$

as long as $\Delta \neq 0$.

2^{nd} order linear DE:

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x)$$

Put in standard position:

$$y'' + \underbrace{P(x)}_{\frac{a_1(x)}{a_2(x)}}\,y' + \underbrace{Q(x)}_{\frac{a_0(x)}{a_2(x)}}\,y = \underbrace{f(x)}_{\frac{g(x)}{a_2(x)}}$$

Assume $y_h = c_1 y_1 + c_2 y_2$. Replace c_i with u_i. ($u_i(x)$)

$$y = u_1 y_1 + u_2 y_2 .$$

Now assume

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}}$$

Then

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_1}{W} = \frac{-y_2 f(x)}{W}$$

W - wronskian of fund'l set.

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$$

Then integrate u_1' and u_2' to find u_1 and u_2.

Replace u_1 and u_2 into formula,

(If keep constants of integration, you will get y instead of y_p).

See Examples.

eg $y'' - 2y' + y = \frac{e^x}{x}$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$ $\qquad y_h = c_1 e^x + c_2 x e^x$.

$m = 1$, mult. 2

Assume $y = u_1 e^x + u_2 x e^x$ and that $u_1' e^x + u_2' x e^x = 0$. Then

$$u_1' e^x + u_2' (\underbrace{xe^x + x}_{y_2'}) = \frac{e^x}{x}$$

$$\begin{bmatrix} e^x & xe^x \\ e^x & xe^x + x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$-y_2 \cdot f(x)$

$$u_1' = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & xe^x + e^x \end{vmatrix}}{e^{2x}} = \frac{-xe^x \cdot \frac{e^x}{x}}{e^{2x}} = \frac{-e^{2x}}{e^{2x}} = -1$$

$y_1 \cdot f(x)$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}}{W} = \frac{e^x \cdot \frac{e^x}{x}}{e^{2x}} = \frac{e^{2x}}{xe^{2x}} = \frac{1}{x}$$

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$$u_1' = -1 \quad \Rightarrow \quad u_1 = -x + c_1$$

$$u_2' = \frac{1}{x} \quad \Rightarrow \quad u_2 = \ln|x| + c_2$$

$$y = (-x + c_1)e^x + (\ln|x| + c_2)\,x e^x$$

$$= c_1 e^x + \underbrace{c_2 x e^x - x e^x} + x e^x \ln|x|$$

$$y = c_1 e^x + c_3 x e^x + x e^x \ln|x|$$

(where $c_3 = c_2 - 1$).

OR

$$y_p = -x e^x + \ln|x|\, x e^x$$

$$y = y_h + y_p = c_1 e^x + c_2 x e^x - x e^x + x e^x \ln|x|$$

$$= c_1 e^x + c_3 x e^x + x e^x \ln|x|$$

Some problems can be done with either the Method of Undetermined Coefficients or Variation of Parameters.

eg $y'' - 2y' - 3y = -3xe^{-x}$

Method 1 : M of U C (with diff. Operator)

$m^2 - 2m - 3 = 0$

$(m+1)(m-3) = 0$

$m_1 = 3, \quad m_2 = -1$

$y_h = C_1 e^{3x} + C_2 e^{-x}$

Guess

$y_p = xe^{-x}(Ax + B)$

$\uparrow$ modification needed

So

$y_p = e^{-x}(Ax^2 + Bx)$

$(D^2 - 2D - 3)\left[e^{-x}(Ax^2 + Bx)\right]$

$= e^{-x}\left((D-1)^2 - 2(D-1) - 3\right)\left[Ax^2 + Bx\right]$

$= e^{-x}(D^2 - 4D)\left[Ax^2 + Bx\right]$

$= e^{-x}\left(2A - 4(2Ax + B)\right)$

$= \cancel{e^{-x}}(-8Ax + 2A - 4B) \overset{\text{set}}{=} -3x\cancel{e^{-x}}$

$D^2 - 2D + 1$
$-2D + 2$
-3
$D^2 - 4D$

$(Ax^2 + Bx)' = 2Ax + B$

$(Ax^2 + Bx)'' = 2A$

So $-8A = -3$ and $2A - 4B = 0$

$A = \frac{3}{8}$ $\qquad -4B = -2\left(\frac{3}{8}\right)$

$B = \frac{3}{16}$

$y = \underbrace{C_1 e^{3x} + C_2 e^{-x}}_{y_h} + \underbrace{\frac{3}{8}x^2 e^{-x} + \frac{3}{16}xe^{-x}}_{y_p}$

Method 2: Variation of Parameters

$$y'' - 2y' - 3y = -3xe^{-x} \qquad y_h = c_1 e^{3x} + c_2 e^{-x}$$

Assume $y = u_1 e^{3x} + u_2 e^{-x}$

and $u_1' e^{3x} + u_2' e^{-x} = 0$.

Then $3u_1' e^{3x} - u_2' e^{-x} = -3xe^{x}$.

$$\begin{bmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ -3xe^{-x} \end{bmatrix}$$

$$W = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = -e^{2x} - 3e^{2x} = -4e^{2x}$$

$$u_1' = \frac{-e^{x} \cdot -3xe^{-x}}{-4e^{2x}} = -\frac{3}{4} x e^{-4x}$$

$$u_2' = \frac{e^{3x} \cdot -3xe^{-x}}{-4e^{2x}} = \frac{-3xe^{2x}}{-4e^{2x}} = \frac{3}{4} x$$

IBP

u	dV
$-\frac{3}{4}x$ $+$	e^{-4x}
$-\frac{3}{4}$ $-$	$-\frac{1}{4}e^{-4x}$
0	$\frac{1}{16}e^{-4x}$

$u_1' = -\frac{3}{4} x e^{-4x}$

So $u_1 = \int -\frac{3}{4} x e^{-4x}\, dx$

$= \frac{3}{16} x e^{-3x} + \frac{3}{64} e^{-4x} + c_1$

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$$u_2' = \frac{3}{4}x$$

$$u_2 = \frac{3}{8}x^2 + C_2$$

$$y = \left(\frac{3}{16}xe^{-4x} + \frac{3}{64}e^{-4x} + c_1\right)e^{3x} + \left(\frac{3}{8}x^2 + C_2\right)e^{-x}$$

$$= C_1 e^{3x} + C_3 e^{-x} + \frac{3}{16}xe^{-x} + \frac{3}{8}x^2 e^{-x}$$

where $C_3 = C_2 + \frac{3}{64}$

(Different particular solutions but same general solution.)