§ 4.4 Supplement

Example: Using the Differential Operator with Products Involving an Exponential.

Method 1: without the operator

$y'' - 2y' - 3y = -3xe^{-x}$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, \ m = -1$

$y_c = C_1 e^{3x} + C_2 e^{-x}$

$y_p = \cancel{(Ax+B)}e^{-x}$

So

$y_p = (Ax^2 + Bx)e^{-x}$

$y_p = (Ax^2 + Bx)e^{-x}$

$y_p' = (Ax^2 + Bx)(-e^{-x}) + e^{-x}(2Ax + B)$

$= -e^{-x}(Ax^2 + Bx - 2Ax - B)$

$= -e^{-x}(Ax^2 + (B - 2A)x - B)$

$y_p'' = -e^{-x}(2Ax + B - 2A) + (Ax^2 + (B-2A)x - B)e^{-x}$

$= e^{-x}(-2Ax - B + 2A + Ax^2 + Bx - 2Ax - B)$

$= e^{-x}(Ax^2 + (-4A + B)x + (2A - 2B))$

$y'' - 2y' - 3y = -3xe^{-x}$; y_p from last page

Vertically

$$\begin{array}{lllll}
y'': & e^{-x}(Ax^2 & +(-4A+B)x & & +(2A-2B)) \\
-2y': & e^{-x}(2Ax^2 & +(2B-4A)x & & -2B) \\
-3y: & e^{-x}(-3Ax^2 & -3Bx & & +0) \\
\hline
& \cancel{e^{-x}}(0x^2 & -8Ax & +0Bx & +2A-4B)
\end{array}$$

$$\overset{\text{let}}{=} -3x\cancel{e^{-x}}$$

So $\quad -8A = -3 \quad$ and $\quad 2A - 4B = 0$

$A = \frac{3}{8}$

$-4B = -2\left(\frac{3}{8}\right)$

$B = -\frac{6}{8} \cdot -\frac{1}{4} = \frac{3}{16}$

Hence, $y_p = \frac{3}{8}x^2e^{-x} + \frac{3}{16}xe^{-x}$.

The general solution is

$y = c_1e^{3x} + c_2e^{-x} + \frac{3}{8}x^2e^{-x} + \frac{3}{16}xe^{-x}$.

Method 2: With the Differential Operator

$$y'' - 2y' - 3y = -3xe^{-x}$$

$$m^2 - 2m - 3 = 0$$

$$(m-3)(m+1) = 0$$

$$m = 3, \quad m = -1$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

$$\cancel{y_p = (Ax + B)e^{-x}}$$

$$y_p = (Ax^2 + Bx)e^{-x}$$

Recall: $D[e^{at} f] = ae^{ax} f + e^{ax} f'$

$$= e^{ax}(D + a)[f].$$

So $D[e^{-x}(Ax^2 + Bx)] = e^{-x}(D - 1)[Ax^2 + Bx]$,

and, by linearity,

$$(D^2 - 2D - 3)[e^{-x}(Ax^2 + Bx)]$$

shift derivative by -1.

$$= e^{-x}\left((D-1)^2 - 2(D-1) - 3\right)[Ax^2 + Bx]$$

$$= e^{-x}(D^2 - 4D)[Ax^2 + Bx]$$

$$= e^{-x}\left(2A - 4(2Ax + B)\right)$$

$$\begin{array}{r} D^2 - 2D + 1 \\ -2D + 2 \\ -3 \\ \hline D^2 - 4D + 0 \end{array}$$

$$(Ax^2 + Bx)' = 2Ax + B$$

$$(Ax^2 + Bx)'' = 2A$$

So $\cancel{e^{-x}}(2A - 8Ax - 4B) \overset{\text{set}}{=} -3x\cancel{e^{-x}}$

$-8A = -3$

$A = 3/8$

$2A - 4B = 0$

$4B = 2\left(\frac{3}{8}\right)$ and So $B = \frac{3}{16}$

Hence, $y = c_1 e^{3x} + c_2 e^{-x} + \frac{3}{8}x^2 e^{-x} + \frac{3}{16}x e^{-x}.$