§ 3.1 Linear Modeling

(1st order - Some separable, some linear).

1) Newton's Law of Cooling/warming
   Thursday

2) Dilution

3) Falling Bodies } do on your own

Dilution

Mass of a chemical in a solution = Volume of Solution × concentration

SI System

$$g = L \cdot \frac{g}{L}$$

Let units help you.

OR weight = Volume × Concentration

$$lbs = gal \times \frac{lbs}{gal}$$

If divide both side by g - gravity constant, then have mass on left.

$$\text{Concentration} = \frac{\text{mass}}{\text{volume}}$$

Let $A(t)$ = amount of chemical (salt) at time t.

Let $A_0 = A(0)$. → sometimes this is given explicitly or sometimes you need to use V_0 & initial concentration to find $A(0)$.

$$\frac{dA}{dt} = \underbrace{\text{rate in}}_{\text{rate at which Chemical comes in}} - \underbrace{\text{rate out}}_{\text{rate at which the chemical flows out.}}$$

: input rate - Output rate of chemical

$$= \text{concentration of ingress} \times \overset{e}{\text{rate of change of volume}} - \text{conc. of egress} \times \overset{f}{\text{rate of change of volume}}$$

well mixed

$$\text{Conc.} = \frac{\text{mass}}{\text{volume}}$$

$$\frac{A \to A(t)}{\underbrace{V_0 + et - ft}_{V(t)}}$$

Example 1: A tank initially holds 100 gal of a brine solution containing 20 lbs of salt. At $t=0$, fresh water is poured into the tank at a rate of 5 gal/min, while the well-mixed mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time t.

$V(0) = V_0 = 100\,g$ $\qquad A(t) = ?$

$e = 5\text{ gal/min} = f$ $\qquad A_0 = 20\text{ lbs}$

$$\frac{dA}{dt} = \frac{0\text{ lbs}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} - \frac{A(t)}{100+5t-5t}\frac{\text{lbs}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}}$$

$$\frac{dA}{dt} = \underbrace{0\,\frac{\text{lbs}}{\cancel{\text{gal}}} \cdot 5\,\frac{\cancel{\text{gal}}}{\text{min}}}_{\text{lbs/min}} - \underbrace{\frac{A(t)}{100+5t-5t}\,\frac{\text{lbs}}{\text{gal}} \cdot \frac{5\,\text{gal}}{\text{min}}}_{\text{lbs/min}}$$

$$\frac{dA}{dt} = -\frac{5\,A(t)}{100}$$

$$\frac{dA}{dt} = -\frac{1}{20}A(t)$$

$$\frac{dA}{dt} + \frac{1}{20}A = 0$$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{20}\,dt} = e^{\frac{1}{20}t}$$

$$\frac{d}{dt}\left[e^{\frac{t}{20}}A\right] = 0$$

$$e^{\frac{t}{20}}A = 0 + C$$

$$A(t) = \frac{C}{e^{t/20}}$$

$$A(t) = Ce^{-t/20}$$

A(0) = 20
20=Ce^0
C=20
A(t) = 20 e^(-t/20)
particular solution

continued - next class

Example 2 : See modules

Given : $V(0) = 100$ gals

$A(0) = 9$ lb.

$$\frac{dA}{dt} = \underbrace{1 \frac{lb}{gal} \cdot 3 \frac{\cancel{gal}}{min}}_{\text{rate in}} - \underbrace{\frac{A(t)}{100 + 3t - 3t} \frac{lb}{gal} \cdot \frac{3\,gal}{min}}_{\text{rate out}}$$

$\frac{lb}{min}$

$$V(t) = V_0 + et - ft$$

$$\frac{dA}{dt} = 3 - \frac{3}{100} A(t)$$

$$\mu(t) = e^{\int .03\,dt} = e^{.03t}$$

$$\frac{dA}{dt} + 0.03 A = 3$$

$$\frac{d}{dt}\left[e^{.03t} A\right] = 3e^{.03t}$$

$$e^{.03t} A = \int 3e^{.03t}\,dt$$

$$e^{.03t} A = 3 \frac{e^{.03t}}{.03} + C$$

$$\frac{3}{\frac{3}{100}} = 3 \times \frac{100}{3} = 100$$

$$e^{.03t} A = 100 e^{.03t} + C$$

(1) GS $\quad A(t) = 100 + Ce^{-.03t}$

② solve C

$A(0) = 1$

So, $1 = 100 + Ce^{0}$

$-99 = C$

③ ps $A(t) = 100 - 99e^{-.03t}$

b) $2 = -99e^{-.03t} + 100$ $t = ?$ when $A = 2$ lbs

$e^{-.03t} = \frac{98}{99}$

$t = -\frac{1}{0.03} \ln\left(\frac{98}{99}\right) \approx 0.338$ min

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Example 3 :

$V(0) = 10$ gal

$A(0) = 0$ lbs

$e = 4$ gal / min

$f = 2$ gal / min

$$\frac{dA}{dt} = 1 \frac{lb}{gal} \cdot 4 \frac{gal}{min} - \frac{A(t)}{10 + 4t - 2t} \frac{lb}{gal} \cdot 2 \frac{gal}{min}, \quad A(0) = 0$$

$$\frac{dA}{dt} = 4 - \frac{2}{10 + 2t} A$$

$$\frac{dA}{dt} + \frac{2}{10 + 2t} A = 4$$

$$\mu(t) = e^{\int \frac{2}{10+2t} dt} = e^{\frac{2}{2}\int \frac{1}{10+2t} dt} = e^{\ln|10+2t|}$$

$$= e^{\ln(10+2t)}$$
$$= 10+2t$$

$$\frac{d}{dt}\left[(10+2t)A\right] = 4(10+2t)$$
$$= 40+8t$$

$$(10+2t)A = 40t + 4t^2 + C$$

$$A(t) = \frac{40t+4t^2+C}{10+2t} \qquad A(0)=0$$

$$0 = \frac{40(0)+4(0)^2}{10+2(0)} + C$$

$$C = 0$$

a) $$A(t) = \frac{40t+4t^2}{10+2t}$$

b) $$V(t) = 10+2t$$
$$50 = 10+2t$$
$$40 = 2t$$
$t = 20$ min for overflow

c) $$A(20) = \frac{40(20)+4(20)^2}{\underbrace{10+2(20)}_{50\text{ gal}}} = 48 \text{ lb.}$$

Newton's Law of Cooling / Warming

p. 24 (§ 1.3) The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium.

$T(t)$ temperature of the body at time t

T_m : temperature of surrounding medium

k constant of proportionality

↳ $k < 0$

$$\frac{dT}{dt} = k(T - T_m)$$ Separable

Example 4 (§ 3.1) Cooling of a Cake

Temperature of a cake when it leaves an oven is $300^\circ F$. Three minutes later, it is $200^\circ F$. How long will it take for the cake to cool off to a room temperature of $70^\circ F$? ↳ need to find k first.

$T_m = 70^\circ F$,

$\frac{dT}{dt} = k(T-70)$, $T(0) = 300^\circ F$

$\int \frac{1}{T-70}\, dT = \int k\, dt$

$\ln|T-70| = kt + c_1$

$|T-70| = e^{kt+c_1}$

$|T-70| = e^{kt} \cdot e^{c_1}$

$T-70 = \pm e^{c_1} e^{kt}$ $\qquad C = \pm e^{c_1}$

$T-70 = Ce^{kt}$

GS: $T(t) = Ce^{kt} + 70$ $\qquad$ Solve IVP

Solve for C: $\qquad T(0) = 300$

$300 = Ce^{k(0)} + 70$

$230 = Ce^{0} \Rightarrow C = 230$

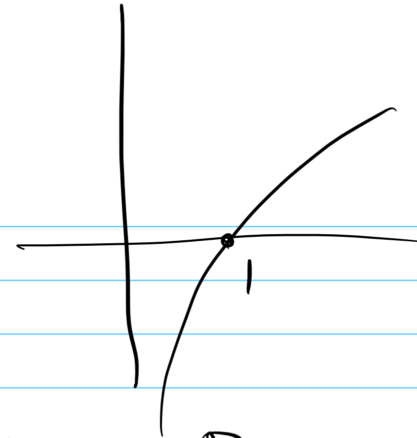
PS: $T(t) = 70 + 230e^{kt}$

$T(3) = 200$ $\qquad$ Find k.

$200 = 70 + 230e^{3k}$

$130 = 230e^{3k}$

$$e^{3k} = \frac{130}{230}$$



$$3k = \ln\left(\frac{13}{23}\right)$$

$$k = \frac{1}{3}\ln\left(\frac{13}{23}\right) \approx -0.19018$$

So $T(t) = 70 + 230e^{-.19018t}$

How long to cool to room temperature?

$t = ?$ when $T = 70$?

$$70 = 70 + 230e^{-0.19018t}$$

$$0 = 230e^{-0.19018t}$$

No solution

But $\lim_{t \to \infty} T(t) = 70$

~ $\frac{1}{2}$ hour to get close to 70°.

See table in book.

p. 26-27 in §1.3

③ Falling Bodies

A) No air Resistance

$$F = ma$$
$$ma = F$$
$$m\frac{dv}{dt} = mg$$

↑+

Newton's 1st Law

Net force acting on body is 0. Not moving.

$$F = \sum F_k = 0$$

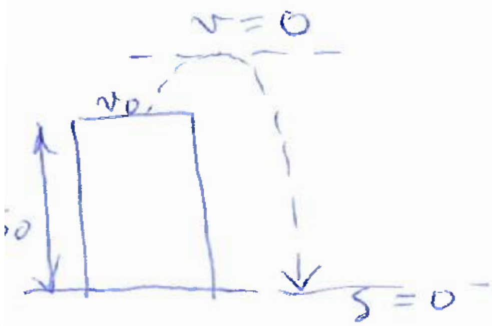
Newton's 2nd law -

$$F = ma$$

Net force acting on body is NOT 0.

So $\frac{dv}{dt} = -g$.

ie. $\frac{d^2s}{dt^2} = -g$, $s(0) = s_0$
$s'(0) = v_0$.



solving IVP:

$$\frac{ds}{dt} = -gt + C$$

$$v_0 = -g(0) + C \Rightarrow C = v_0$$

So $v = \frac{ds}{dt} = -gt + v_0$.

$s(t) = -\frac{1}{2}gt^2 + v_0 t + C$. $\quad s(0) = s_0$

$$s_0 = 0 + 0 + c \Rightarrow c = s_0$$

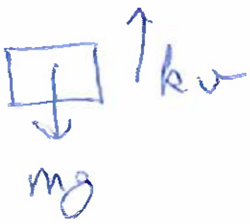
So, $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$

Galileo's Law in elementary physics.

Hw: 36 TRY

B) with Air Resistance

HW 36 & 37
(Skip 38)



Heavier bodies fall faster. Why?

Because of air resistance.

If $\downarrow +$, $F = mg - \underbrace{kv}_{\text{viscous damping}}$

positive constant of proportion — depends on medium.

Since $F=ma$, get

$$m\frac{dv}{dt} = mg - kv$$

$$\therefore \quad m\frac{d^2s}{dt^2} = mg - k\frac{ds}{dt}$$

OR

$$m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg$$