Section 3.2

$A =$ $\quad$ $m \times n$

2D array

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$ an $n \times 1$ matrix (or column vector)

$$\neq [v_1 \; v_2 \; \ldots \; v_n]$$

We shall see, the latter is v^T (v transpose)

(even though we are used to thinking of a point in the plane - or space

$$\begin{vmatrix} 1 \\ 2 \end{vmatrix} = [1,2]^T \qquad [1\;2\;3] = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}^T \;)$$

Algebra of Matrices

(that is, operations $+, -$, product, etc)

First of all $A = B$

same order (say 6×10) and

same entries $a_{ij} = b_{ij}$ $i = 1, \ldots, m$
$j = 1, \ldots, n$

[Two matrices are equal if they have the same order, and every entry ij is equal to the other entry in the same position ij]

SUM. $A + B$

First A, B need to have the same order.

Cannot do $\cancel{\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}}$

NOT possible

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(\text{or } a_{ij} + b_{ij})$$

$$C = A + B$$

$$c_{ij} = a_{ij} + b_{ij}$$

Example

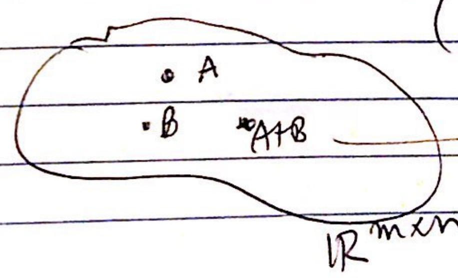
$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 7 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 5 \\ 1 & 5 & 8 \end{bmatrix}$$

Thus if $A, B \in \mathbb{R}^{m \times n}$ (order $m \times n$)

Then $C \in \mathbb{R}^{m \times n}$ (order $m \times n$)

$$\boxed{\text{Set of } m \times n \text{ matrices is } \underline{\text{closed}} \text{ under addition}}$$

(for each m, n)



inside the same set

Most properties of matrix addition are inherited from the addition in $\mathbb{R}$

"there exists"

$\exists\, 0 \ni \quad a+0=a$

$(a+b)+c = a+(b+c)$

"such that"

$\exists\, -a \ni \quad a+(-a)=0$

Same for matrices in $\mathbb{R}^{m\times n}$ (order $m \times n$)

- there exist a zero matrix

  O (each $ij = 0$)

  such that

  $A+O=A$

- Associative property

  $(A+B)+C = A+(B+C)$

- Commutative property $A+B=B+C$

- $-A$ exists so that

  $A+(-A)=O$

$$(-A)_{ij} = -(A_{ij})$$

$(\text{or } -a_{ij})$

The concept of a proof

It is a convincing argument based on definitions or previously shown results, to show a new result.

For example. $\exists\, O \ni A+O=A$.

proof consists of "exhibiting" the object O and showing that indeed $A+O=A$.

Proof (sometimes abbreviated Pf).

let O in $\mathbb{R}^{m\times n}$ such that each entry is 0 (zero in $\mathbb{R}$)

$$(A+O)_{ij} = a_{ij}+0 = a_{ij} = (A)_{ij}$$

by definition of addition $\Rightarrow (A+O)=A$

Similarly to show $\exists -A$

Proof.

$$(-A)_{ij} = -a_{ij}$$

Then $[A+(-A)]_{ij} = A_{ij} + (-A)_{ij} =$

definition of Matrix addition

$= a_{ij} + (-a_{ij}) = a_{ij} - a_{ij} = 0 = O_{ij}$

properties of $\mathbb{R}$

in $\mathbb{R}$

already shown

(as it happens, in previous page)

Commutativity

def. of Matrix +

Proof $(A+B)_{ij} = a_{ij} + b_{ij} =$

$= b_{ij} + a_{ij} = (B+A)_{ij}$

commutativity in $\mathbb{R}$

def of Matrix +

$$\Rightarrow A+B = B+A$$

definition of "$A=B$" (Matrix equality)

Associativity

in the same manner.

Scalar Multiplication.

product of $\alpha \in \mathbb{R}$ $A \in \mathbb{R}^{m\times n}$

(number times matrix)

like $2A$, twice A

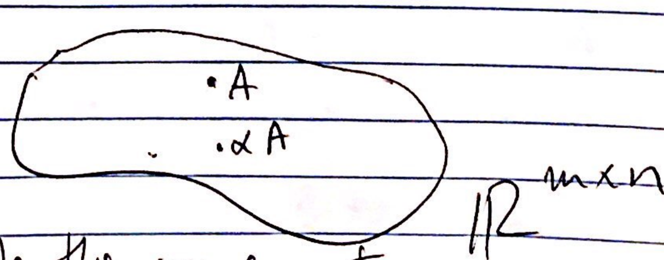
$$(2A)_{ij} = 2a_{ij}$$

$$2\begin{bmatrix} 3 & 5 & 1 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 10 & 2 \\ 4 & -2 & 0 \end{bmatrix}$$

In general $(\alpha A)_{ij} = \alpha a_{ij}$

α negative, or non integer - the same

$\boxed{\mathbb{R}^{m\times n} \text{ is closed under scalar multiplication}}$



stays inside the same set

Properties.

one associative

two distributive

1 - $\alpha(\beta A) = (\alpha\beta)A$

2 - $\alpha(A+B) = \alpha A + \alpha B$

3 - $(\alpha+\beta)A = \alpha A + \beta A$

Proof of 2.

$$[\alpha(A+B)]_{ij} = \alpha(A+B)_{ij}$$

def. of scalar product

$$= \alpha[a_{ij}+b_{ij}] = \alpha a_{ij} + \alpha b_{ij} =$$

def. of Matrix +

distributive prop. in $\mathbb{R}$

$$= (\alpha A)_{ij} + (\alpha B)_{ij}$$

def of scalar prod. (twice)

$\Rightarrow \alpha(A+B) = \alpha A + \alpha B$

by definition of Matrix =

Remark When we say in a proof that we use convincing arguments, this means every step has to be justified, either by a definition or a result previously shown

Note $1.A = A$
and $0.A = 0$

Matrix transposition

denoted A^T. $(A^T)_{ij} = a_{ji}$

If A is $m \times n$ A^T is $n \times m$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 5 & 0 \end{bmatrix}$$

2×3 3×2

Thus - In general $\mathbb{R}^{m \times n}$ NOT closed under transposition.

Except when $m = n$ - square matrices

Properties

- $(A^T)^T = A$.

Proof $\left(A^T\right)^T_{ij} = \left(A^T\right)_{ji} = a_{ij}$

$= A_{ij}$

- $(A+B)^T = A^T + B^T$

Pf. $(A+B)^T_{ij} = (A+B)_{ji} =$

$= A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij}$

def of +

$\Rightarrow (A+B)^T = A^T + B^T$

- $(\alpha A)^T = \alpha A^T$

$(\alpha A)^T_{ij} = (\alpha A)_{ji} = \alpha\, A_{ji} = \alpha (A^T)_{ij}$

Recall $v^T = [v_1, \dots v_n]$

$\begin{vmatrix} 1 \\ 3 \\ 2 \end{vmatrix}^T = [1\ 3\ 2]$

Def. A square Matrix A is symmetric if $A^T = A$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}$$

(rows = columns).

In particular a diagonal Matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & & \\ & & \ddots & & \\ & & & & \lambda_n \end{bmatrix}$$

is symmetric

Diagonal if $a_{ij} = 0 \quad i \neq j$

A matrix is non symmetric if it is not symmetric

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Definition. A matrix A is skew-symmetric (or anti simmetric) if $A^T = -A$

example $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

or $\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$

Lemma (or Proposition): If $A^T = -A \in \mathbb{R}^{n \times n}$

Then $a_{ii} = 0 \quad i = 1, \dots, n.$

Proof. $(A^T)_{ii} \underset{\uparrow}{=} (-A)_{ii}$ (for any i)

by skew-symm,

$\Rightarrow \quad a_{ii} = -a_{ii}$

$\Rightarrow \quad 2a_{ii} = 0 \quad \Rightarrow a_{ii} = 0$

q.e.d.

Do all exercises of section 2.3