§ 7.1 Laplace Transform ($t \geq 0$)

I) **Definition**: Suppose $f(t)$ is piecewise continuous on $[0, \infty)$. Then the Laplace transform of $f(t)$ is

$$\underset{\displaystyle F(s)}{\underset{\displaystyle ||}{\mathcal{L}\{f(t)\}}} = \int_0^\infty \underbrace{e^{-st}}_{\text{Kernel}} f(t)\,dt,$$

provided this limit exists.

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eg. Suppose $\int_0^\infty t^2 e^{-(s+1)t}\,dt = \mathcal{L}\{f(t)\}$ for some function $f(t)$. Identify $f(t)$.

$$\int_0^\infty t^2 e^{-(s+1)t}\,dt = \int_0^\infty \underbrace{e^{-st}}_{\text{Kernel}} \cdot \underbrace{t^2 \cdot e^{-t}}_{f(t)}\,dt.$$

So $f(t) = t^2 e^{-t}$.

Recall $\lim_{x\to\infty} e^{-ax} = 0$ only if $a > 0$.

exponential decay.

$$\left(\lim_{x\to\infty} e^{-ax} = \lim_{x\to\infty} \frac{1}{\underbrace{e^{ax}}_{\to\infty,\ a>0.}} = 0 \right)$$

Using the definition, find the Laplace transform of the following function and for which s?

eg $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{T\to\infty} \int_0^T e^{-st}\, dt$$

$$= \lim_{T\to\infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \lim_{T\to\infty} \left(\frac{e^{-sT}}{-s} - \left(-\frac{1}{s}\right) \right)$$

$$= \frac{1}{s}, \quad s > 0$$

$$\left(e^{-sT} \to 0 \text{ as } T \to \infty \text{ if } s > 0 \right)$$

So $\mathcal{L}\{1\} = \frac{1}{s}$.

eg $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\,dt$$

$$= \lim_{T\to\infty} \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^T$$

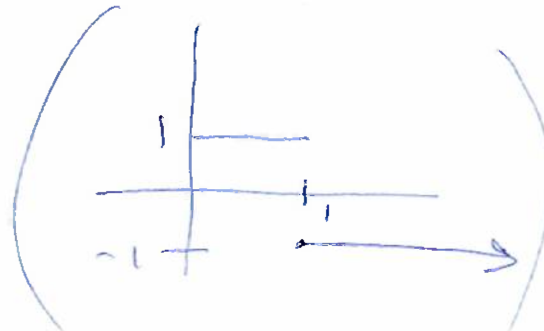
$$= \lim_{T\to\infty} \left[\underbrace{-\frac{e^{-(s-a)T}}{s-a}}_{\to 0} + \frac{1}{s-a} \right] = \frac{1}{s-a}$$

when $s - a > 0$, i.e. $s > a$.

So, for $s > a$, $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$.

eg $\mathcal{L}\{e^{3t}\} = \dfrac{1}{s-3}$ for $s > 3$.

eg $f(t) = \begin{cases} 1 & , \ 0 \le t < 1 \\ -1 & , \ t \ge 1 \end{cases}$



$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st}\, dt + \int_1^\infty -e^{-st}\, dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^1 - \left.\frac{e^{-st}}{-s}\right]_1^\infty$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} - \Big(\underset{s>0}{0} - \frac{e^{-s}}{-s}\Big)$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s} - \frac{2e^{-s}}{s} \quad , \quad s > 0$$

eg $f(t) = \begin{cases} t & , 0 \le t < 2 \\ 0 & , t \ge 2 \end{cases}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 t e^{-st}\,dt + \underbrace{\int_2^\infty 0\,dt}_{0}$$

$$= \frac{te^{-st}}{-s}\Bigg]_0^2 - \int_0^2 \frac{e^{-st}}{-s}\,dt$$

$u = t$
$du = dt$
$dv = e^{-st}\,dt$
$V = \frac{e^{-st}}{-s}$

$$uv\Big]_a^b - \int_a^b v\,du$$

$$= \frac{2e^{-2s}}{-s} - 0 - \frac{e^{-st}}{(-s)^2}\Bigg]_0^2$$

$$= -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$$

We will learn an (easier) way of finding the Laplace Transform of a piecewise continuous function in 7.3.2 and 7.4.

$$\mathcal{L}\{t^n\}$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$u = t$
$du = dt$
$dv = e^{-st}\,dt$
$v = \frac{e^{-st}}{-s}$

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$$

$$= -\frac{t e^{-st}}{s}\Big]_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt$$

$$= 0 + 0 + \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}}$$

$e^{st} \to \infty$ much faster than any power of t.

$$= \frac{1}{s}\cdot\frac{1}{s}$$

$$= \frac{1}{s^2}$$

$u = t^2$
$du = 2t\,dt$
$dv = e^{-st}\,dt$
$v = -\frac{e^{-st}}{s}$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$$

$$= -\frac{t^2 e^{-st}}{s}\Big]_0^\infty + \int_0^\infty 2t\,\frac{e^{-st}}{s}\,dt$$

$$= 0 + \frac{2}{s}\int_0^\infty t e^{-st}\,dt$$

$$= \frac{2}{s}\cdot\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$$

$u = t^n$

$du = nt^{n-1}\,dt$

$dv = e^{-st}\,dt$

$v = \dfrac{e^{-st}}{-s}$

$$= \left.\frac{t^n e^{-st}}{-s}\right]_0^\infty + \int_0^\infty nt^{n-1}\frac{e^{-st}}{s}\,dt$$

$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

$$= \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

Pattern:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^3\} = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{6}{s^4} = \frac{3!}{s^4}$$

$$\mathcal{L}\{t^4\} = \frac{4}{s}\mathcal{L}\{t^3\} = \frac{4}{s}\cdot\frac{3!}{s^4} = \frac{4!}{s^5}$$

$\vdots$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

II) The Laplace transform is linear.

Assume $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$.

Then
$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\}$$
$$= \int_0^\infty e^{-st}[c_1 f(t) + c_2 g(t)]\,dt$$
$$= \int_0^\infty e^{-st} c_1 f(t)\,dt + \int_0^\infty e^{-st} c_2 g(t)\,dt$$
$$= c_1 \int_0^\infty e^{-st} f(t)\,dt + c_2 \int_0^\infty e^{-st} g(t)\,dt$$
$= c_1 F(s) + c_2 G(s)$ by the properties of the definite integral and the limit laws.

Laplace formulas: learn for Test 2

a) $\mathcal{L}\{1\} = \frac{1}{s}$

b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, \dots$

c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

d) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

e) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

eg $\mathcal{L}\{t^2 - 2t + 5\}$

$$= \mathcal{L}\{t^2\} - 2\mathcal{L}\{t\} + 5$$

$$= \frac{2}{s^3} - \frac{2}{s^2} + \frac{5}{s}$$

eg $f(t) = t^3 - e^{-9t} + 5 + \cos 2t + \sin t$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{3!}{s^4} - \frac{1}{s+9} + \frac{5}{s} + \frac{s}{s^2+25} + \frac{1}{s^2+1}$$

$$F(s) = \frac{6}{s^4} - \frac{1}{s+9} + \frac{5}{s} + \frac{s}{s^2+25} + \frac{1}{s^2+1}$$