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19 September 2023
Due Tuesday 26 September 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #4

1. (3.5 points).
Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, and $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Compute the following

(a) $x^T y$.

(b) xy^T.

(c) AA^T, and confirm that the result is symmetric.

(d) $A^T A$, and confirm that the result is symmetric.

(e) MA.

(f) $(MA)^T$ and $A^T M^T$, and confirm that $(MA)^T = A^T M^T$.

2. (4.5 +1 points).
Let $M = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$, and $A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$. Compute the following,

(a) M^2.

(b) $M^T M$, and confirm that the result is symmetric.

(c) A^2, A^4, A^{16}.

(d) (extra credit) $\lim_{k\to\infty} A^k$.

3. (2 points).
Let A be and $m \times n$ matrix, and let the two linear systems $Ax = b$ and $Ax = c$ be consistent. Prove that $Ax = (b + c)$ is also consistent.

4. (extra credit 2 points) Let Π_3, Π_2 be the set of polynomials of degree at most 3 and 2, respectively. Consider the derivative as the linear map

$$\frac{d}{dx} : \ \Pi_3 \to \Pi_2, \quad \frac{d}{dx}p(x) = q(x).$$

Consider $p(x) \in \Pi_3$ as $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, and $q(x) \in \Pi_2$ as $q(x) = b_0 + b_1 x + b_2 x^2$. Write a matrix A mapping

$$a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ onto } \ b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix},$$

so that A represent the linear map $\frac{d}{dx}$.
Write explicitly the matrix A with its numerical values.

Name: Elle Nguyen HW #4

1) $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(a) $x^T y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(2) + 1(3) \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$

(b) $xy^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) & 2(1) & 3(1) \\ 1(1) & 2(1) & 3(1) \\ 1(1) & 2(1) & 3(1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1) & 1(1)+1(2) & 1(1)+1(3) \\ 1(1)+2(1) & 1(1)+2(2) & 1(1)+2(3) \\ 1(1)+3(1) & 1(1)+3(2) & 1(1)+3(3) \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 10 \end{bmatrix}$ is symmetric since $a_{ij} = a_{ji}$ $\forall\, i,j = 1, 2, 3$ $(i \neq j)$

(d) $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1)+1(1) & 1(1)+1(2)+1(3) \\ 1(1)+2(1)+3(1) & 1(1)+2(2)+3(3) \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$ is symmetric since $a_{ij} = a_{ji}$ $\forall\, i,j = 1, 2$; $i \neq j$

(e) $MA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1)+1(1) & 1(1)+1(2)+1(3) \\ 0(1)+1(1)+1(1) & 0(1)+1(2)+1(3) \\ 1(1)+0(1)+1(1) & 1(1)+0(2)+1(3) \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 2 & 4 \end{bmatrix}$

(f) $(MA)^T = \left(\begin{bmatrix} 3 & 6 \\ 2 & 5 \\ 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 3 & 2 & 2 \\ 6 & 5 & 4 \end{bmatrix}$

match

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$A^T M^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(1)+1(1)+1(1) & 1(0)+1(1)+1(1) & 1(1)+1(0)+1(1) \\ 1(1)+2(1)+3(1) & 1(0)+2(1)+3(1) & 1(1)+2(0)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 2 \\ 6 & 5 & 4 \end{bmatrix} \qquad \therefore (MA)^T = A^T M^T$$

2 $M = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(a) $M^2 = M.M = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)+1/2(1/2)+0(0) & 1(1/2)+1/2(1)+0(0) & 1(0)+1/2(1/2)+0(1) \\ 1/2(1)+1(1/2)+1/2(0) & 1/2(1/2)+1(1)+1/2(0) & 1/2(0)+1(1/2)+1/2(1) \\ 0(1)+0(1/2)+1(0) & 0(1/2)+0(1)+1(0) & 0(0)+0(1/2)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 5/4 & 1 & 1/4 \\ 1 & 5/4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $M^T M = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)+\frac{1}{2}(\frac{1}{2})+0(0) & 1(\frac{1}{2})+\frac{1}{2}(1)+0(0) & 0(1)+\frac{1}{2}(\frac{1}{2})+1(0) \\ \frac{1}{2}(1)+1(\frac{1}{2})+0(0) & \frac{1}{2}(\frac{1}{2})+1(1)+0(0) & 0(\frac{1}{2})+\frac{1}{2}(1)+1(0) \\ 0(1)+\frac{1}{2}(\frac{1}{2})+1(0) & 0(\frac{1}{2})+\frac{1}{2}(1)+1(0) & 0(0)+\frac{1}{2}(\frac{1}{2})+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 5/4 & 1 & 1/4 \\ 1 & 5/4 & 1/2 \\ 1/4 & 1/2 & 5/4 \end{bmatrix}$$ is symmetric since $a_{ij} = a_{ji} \ \forall \ i,j = 1,2,3 \ ; \ i \neq j$

(c) $$A^2 = A.A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1(1)+1(0)+0(1) & 1(1)+1(1)+0(0) & 1(0)+1(1)+0(1) \\ 0(1)+1(0)+1(1) & 0(1)+1(1)+1(0) & 0(0)+1(1)+1(1) \\ 1(1)+0(0)+1(1) & 1(1)+0(1)+1(0) & 1(0)+0(1)+1(1) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$A^4 = A^2.A^2 = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 1(1)+2(1)+1(2) & 1(2)+2(1)+1(1) & 1(1)+2(2)+1(1) \\ 1(1)+1(1)+2(2) & 1(2)+1(1)+2(1) & 1(1)+1(2)+2(1) \\ 2(1)+1(1)+1(2) & 2(2)+1(1)+1(1) & 2(1)+1(2)+1(1) \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 5 & 5 & 6 \\ 6 & 5 & 5 \\ 5 & 6 & 5 \end{bmatrix}$$

$$A^8 = A^4.A^4 = \frac{1}{16}\begin{bmatrix} 5 & 5 & 6 \\ 6 & 5 & 5 \\ 5 & 6 & 5 \end{bmatrix} \frac{1}{16}\begin{bmatrix} 5 & 5 & 6 \\ 6 & 5 & 5 \\ 5 & 6 & 5 \end{bmatrix} = \frac{1}{256}\begin{bmatrix} 5 & 5 & 6 \\ 6 & 5 & 5 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 5 & 5 & 6 \\ 6 & 5 & 5 \\ 5 & 6 & 5 \end{bmatrix}$$

$$= \frac{1}{256}\begin{bmatrix} 5(5)+5(6)+6(5) & 5(5)+5(5)+6(6) & 5(6)+5(5)+6(5) \\ 6(5)+5(6)+5(5) & 6(5)+5(5)+5(6) & 6(6)+5(5)+5(5) \\ 5(5)+6(6)+5(5) & 5(5)+6(5)+5(6) & 5(6)+6(5)+5(5) \end{bmatrix}$$

$$= \frac{1}{256}\begin{bmatrix} 85 & 86 & 85 \\ 85 & 85 & 86 \\ 86 & 85 & 85 \end{bmatrix}$$

$$A^{16} = A^8 . A^8 = \frac{1}{256}\begin{bmatrix} 85 & 86 & 85 \\ 85 & 85 & 86 \\ 86 & 85 & 85 \end{bmatrix} \frac{1}{256}\begin{bmatrix} 85 & 86 & 85 \\ 85 & 85 & 86 \\ 86 & 85 & 85 \end{bmatrix}$$

$$= \frac{1}{65536}\begin{bmatrix} 85^2+2(86)(85) & 85^2+2(86)(85) & 2(85)^2+86^2 \\ 2(85)^2+86^2 & 85^2+2(86)(85) & 85^2+2(86)(85) \\ 85^2+2(86)(85) & 2(85)^2+86^2 & 85^2+2(86)(85) \end{bmatrix}$$

$$= \frac{1}{65536}\begin{bmatrix} 21845 & 21845 & 21846 \\ 21846 & 21845 & 21845 \\ 21845 & 21846 & 21845 \end{bmatrix}$$

(d) Each entry in A^{16} is approximately 0.3333 which is around $1/3$

$$\therefore \lim_{k\to\infty} A^k = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3 Let A be $m \times n$ matrix

Since $Ax = b$ is consistent, b is a linear combination of columns of A

$Ax = c$ is consistent, c is a linear combination of columns of A

i.e: $b = \sum_{i=1}^{n} \beta_i A_{*b_i}$ and $c = \sum_{i=1}^{n} \alpha_i A_{*b_i}$ where A_{*b_i} are basic columns

$$\therefore b + c = \sum_{i=1}^{n} \beta_i A_{*b_i} + \sum_{i=1}^{n} \alpha_i A_{*b_i} = \sum_{i=1}^{n} (\beta_i + \alpha_i) A_{*b_i} = \sum_{i=1}^{n} \gamma_i A_{*b_i}$$

$\therefore$ By definitions, $(b+c)$ is a linear combination of columns of A

$\therefore$ $Ax = (b+c)$ is also consistent QED!

4

For $p(x) \in \Pi_3$ — basis elements $\{1, x, x^2, x^3\}$
— coefficients $\{a_0, a_1, a_2, a_3\}$

For $q(x) \in \Pi_2$ — basis elements $\{1, x, x^2\}$
— coefficients $\{b_0, b_1, b_2\}$

To express linear combination of basis elements of Π_2, need to compute derivative of each basis element of Π_3 w.r.t x

$\frac{d}{dx} p_1(x) = \frac{d}{dx}(1) = 0$ $\quad \therefore q_1(x) = 0(1) + 0(x) + 0(x^2)$ $\quad \therefore A_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore (b_0, b_1, b_2) = (0, 0, 0)$

$\frac{d}{dx} p_2(x) = \frac{d}{dx}(x) = 1$ $\quad \therefore q_2(x) = 1(1) + 0(x) + 0(x^2)$ $\quad \therefore A_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore (b_0, b_1, b_2) = (1, 0, 0)$

$\frac{d}{dx} p_3(x) = \frac{d}{dx}(x^2) = 2x$ $\quad \therefore q_3(x) = 0(1) + 2(x) + 0(x^2)$ $\quad \therefore A_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$\therefore (b_0, b_1, b_2) = (0, 2, 0)$

$\frac{d}{dx} p_4(x) = \frac{d}{dx}(x^3) = 3x^2$ $\quad \therefore q_4(x) = 0(1) + 0(x) + 3(x^2)$ $\quad \therefore A_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$\therefore (b_0, b_1, b_2) = (0, 0, 3)$

$\therefore A_{3\times 4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ which aligns with mapping $a_{4\times 1}$ onto $b_{3\times 1}$ to get a 3×4 matrix