TEST 1

1.1 – SAMPLE SPACES

Def → The set of all possible outcomes of a random experiment

A subset of Ω is called an event

Prob. measure → $0 \le P(A) \le 1$ $P(\Omega) = 1$ $P(\emptyset) = 0$

If $A_1, A_2 \dots A_n$ is a sequence of pairwise disjoint events then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

APPENDIX B – SET NOTATIONS & OPS

Def →

$A \subseteq B$ if every element of A is an element of B

$A = B$ if $B \subseteq A$ and $A \subseteq B$

$A \cap B = \{x \mid x \in A$ and $x \in B\}$

$A \cup B = \{x \mid x \in A$ or $x \in B\}$

De Morgan Law → $\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c$ $\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$

Distribution Law → $A \cap \left(\bigcup_i^{\infty} B_i\right) = \bigcup_i^{\infty} (A \cap B_i)$ $A \cup \left(\bigcap_i^{\infty} B_i\right) = \bigcap_i^{\infty} (A \cup B_i)$

1.2 – RANDOM SAMPLING

Def →

Size n
Sample k

	w/o replacement	w/ replacement
Permutations ordered	${}_nP_k = \frac{n!}{(n-k)!}$	n^k
Combinations unordered	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$

Notation → ordered $\Omega = \{(s_1, s_2, s_3)\}$ $s_i \neq s_j$ for all $i \neq j$
unordered $\Omega = \{\{s_1, s_2, s_3\}\}$

1.3 – INFINITELY MANY OUTCOMES

$$1 = P(\infty) + \underbrace{\sum_{k=1}^{\infty} \frac{1}{a^k}}_{\frac{a}{1-r}} = P(\infty) + \frac{1/a}{1 - 1/a}$$

$a \rightarrow$ 1st term

1.4 – RULES OF PROB.

$A = A_1 \cup A_2 \cup \ldots A_n$
$A_i \cap A_j = \emptyset \ (i \neq j)$ $P(A) = P(A_1) + P(A_2) + \ldots$

Inclusion Exclusion → $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
if $A \cap B = \emptyset \longrightarrow P(A \cup B) = P(A) + P(B)$

$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

General → $P(A_1 \cup \ldots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < i_k \leq n} P(A_1 \cap A_2 \ldots \cap A_k)$

Complements → $P(A) + P(A^c) = 1$

Monotonicity → If $A \subseteq B$, $P(A) \leq P(B)$

1.5 – RANDOM VARIABLES

Def → $\mathcal{B}$ set of all subsets of $\mathbb{R}$
μ_X prob. measure on $\mathcal{B}$

2.1 – CONDITIONAL PROB.

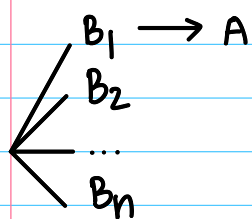
$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \xrightarrow[\text{Rule}]{\text{Multiplication}} P(A \cap B) = P(A).\ P(B \mid A)$

$P(A \cap B \cap C) = P(A).\ P(B \mid A).\ P(C \mid A \cap B)$

General → $P(A_1 \cap A_2 \dots A_n) = P(A_1) . P(A_2 | A_1) . P(A_3 | A_1 \cap A_2) \dots P(A_n | A_1 \cap A_2 \dots A_{n-1})$

Total prob → $B_1, B_2 \dots B_n$ be partition of Ω $P(A) = \sum_{j=1}^{n} P(B_j) . P(A | B_j)$

2.2 – BAYES' FORMULA



$$P(B_i | A) = \frac{P(B_i) . P(A | B_i)}{\sum_{j=1}^{n} P(B_j) . P(A | B_j)} \quad \forall i = 1, 2 \dots n$$

OR $$P(B_i | A) = \frac{P(B_i) . P(A | B_i)}{P(A)}$$

2.3 – INDEPENDENCE

A, B are independent if $P(A \cap B) = P(A) . P(B)$

↳ $P(A | B) = P(A)$

↓ A, B disjoint ($A \cap B = \emptyset$)

$P(A \cap B) = P(\emptyset) = 0 = P(A) . P(B) \longrightarrow P(A) = 0$ or $P(B) = 0$

Theorem → A, B independent then $\left.\begin{array}{l} A^c \,\&\, B \\ A \,\&\, B^c \\ A^c \,\&\, B^c \end{array}\right\}$ are independent

A, B, C → A, B, C are mutually independent if

(1) $P(A \cap B \cap C) = P(A) . P(B) . P(C)$

(2) $P(A \cap B) = P(A) . P(B)$

(3) $P(A \cap C) = P(A) . P(C)$

(4) $P(B \cap C) = P(B) . P(C)$

General → $A_1, A_2 \dots A_n$ are mutually independent if

$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \dots P(A_{i_k})$ for all $k = 2, 3 \dots n$

$1 \leq i_1 < i_2 < \dots < i_k \leq n$

Theorem → $A_1, A_2 \dots A_n$ are independent then

$P(A_{i_1}^* \cap \dots \cap A_{i_k}^*) = P(A_{i_1}^*) \dots P(A_{i_k}^*)$ for all $k = 1, 2 \dots n$

$A^* = A$ or A^c $\quad 1 \le i_1 < i_2 < \dots < i_k \le n$

e.g: A, B, C are independent then

$P(A \cap B^c) = P(A) . P(B^c)$

$P(A^c \cap B^c) = P(A^c) . P(B^c)$

$P(A \cap B^c \cap C) = P(A) . P(B^c) . P(C)$

Def → $X_1, X_2 \dots X_n$ are independent if

$P(X_1 \in B_1, X_2 \in B_2 \dots X_n \in B_n) = P(X_1 \in B_1) . P(X_2 \in B_2) \dots P(X_n \in B_n)$

Theorem → $X_1, X_2 \dots X_n$ are independent if

$P(X_1 = x_1, X_2 = x_2 \dots X_n = x_n) = P(X_1 = x_1) . P(X_2 = x_2) \dots P(X_n = x_n)$

for all possible values $x_1, x_2, \dots x_n$ of the random variables.

TEST 2

2.4 - INDEPENDENT TRIALS

(1) Bernoulli trial (success/failure) $Ber(p)$

(2) Binomial distribution $Bin(n, p) = \binom{n}{k} p^k q^{n-k}$

(3) Geometric distribution $Geom(p) = q^{k-1} p$ — $P(X > k) = q^k$, $P(X \le k) = 1 - q^k$

(4) Hypergeometric distribution $Hypergeom(N, N_A, n) = \dfrac{\binom{N_A}{k}\binom{N - N_A}{n-k}}{\binom{N}{n}}$

n → sample size

3.1 - PROB. DISTRIBUTIONS

(1) pmf discrete random variables

(2) pdf cont. random variables $P(X \in B) = \int_B f_X(x) dx$

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)dx = P(X \in \mathbb{R}) = 1$

(iii) $\int_a^b f_X(x)dx = P(a < X < b)$
$= P(a \leq X \leq b)$

(iv) $P(X = a) = \int_a^a f_X(x)dx = 0$

X~ std normal distribution   pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $P(X > 0) = \frac{1}{2}$

X~ Unif[a, b]   pdf $f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0 \end{cases}$

(3) Infinitesimal method

$P(a < X < a + \varepsilon) \approx f_X(a).\varepsilon \longrightarrow f_X(a) = \lim_{\varepsilon \to 0} \frac{P(a < X < a + \varepsilon)}{\varepsilon}$

3.2 - CDF

cdf $F_X(s) = P(X \leq s)$

(1) $F_X(s)$ nondecreasing

(2) $\lim_{s \to \infty} F_X(s) = 1$ $\lim_{s \to -\infty} F_X(s) = 0$

(3) $F_X(s)$ right-cont : $\lim_{s \to a^+} F_X(s) = F_X(a)$

(4) $P(a \leq X \leq b) = F_X(b) - F_X(a)$

(5) $P(X < b) = F_X(b-)$

(6) $P(X = b) =$ Jump size at b

(7) $F_X(a+) = \lim_{s \to a^+} F_X(s)$ $F_X(a-) = \lim_{s \to a^-} F_X(s)$

3.3 - EXPECTATION

3.4 - VARIANCE

	Discrete	Continuous
Mean	$E[X] = \sum_k k P(X = k)$	$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
Variance	$Var(X) = \sum_k (k - \mu)^2 P(X = k)$	$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x)dx$

$E[c] = c$
$E[aX] = aE[X]$

std deviation $\uparrow$ $\quad$ $E[X]$ $\uparrow$

$$\sigma_X^2 = Var(X) = E[(X-\mu)^2]$$

$$Var(aX+b) = a^2 Var(X)$$
$$Var(X) = E[X^2] - E[X]^2$$

	$Ber(p)$	$Bin(n,p)$	$Geom(p)$	$Unif[a,b]$	$Exp(\lambda)$
Distribution	$p^k q^{1-k}$	$\binom{n}{k} p^k q^{n-k}$	$q^{k-1}p$	$\frac{1}{b-a},\ a \le x \le b$	$\lambda e^{-\lambda x},\ x \ge 0$
Mean	p	np	$\frac{1}{p}$	$\frac{a+b}{2}$	$\frac{1}{\lambda}$
Variance	pq	npq	$\frac{q}{p^2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{\lambda^2}$

Median $\quad P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$

Quantiles $\quad P(X \le x) \ge q$ and $P(X \ge x) \ge 1-q$

3.5 - GAUSSIAN DISTRIBUTION

std normal distribution $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \longrightarrow Z \sim N(\mu, \sigma^2) = N(0,1)$

random variable $X \longrightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-1}{2\sigma^2}(X-\mu)^2}$

(1) $\left.\begin{array}{l} X \sim N(\mu_X, \sigma_X^2) \\ Y = aX+b \end{array}\right\}$ $Y \sim N(a\mu_X + b, a^2\sigma_X^2)$

Proof $\rightarrow$

$$F_Y(y) = P(Y \le y) = P(aX+b \le y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right),\ a \ge 0$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \left(\frac{y-b}{a}\right)' f_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}\,\sigma_X} e^{\frac{-1}{2\sigma_X^2}\left(\frac{y-b}{a}-\mu_X\right)^2} = \frac{1}{\sqrt{2\pi}\,a\sigma_X} e^{\frac{-[y-(b+a\mu_X)]^2}{2a^2\sigma_X^2}}$$

$\therefore Y \sim N(a\mu_X + b, a^2\sigma_X^2)$

(2) $X \sim N(\mu_X, \sigma_X^2) \rightarrow Z = \frac{X-\mu_X}{\sigma_X} \sim N(0,1)$

Proof $\rightarrow$

$$Z = \frac{1}{\sigma_X}X - \frac{\mu_X}{\sigma_X} \longrightarrow a = \frac{1}{\sigma_X},\ b = \frac{-\mu_X}{\sigma_X} \qquad \sigma_Z^2 = a^2\sigma_X^2 = \frac{1}{\sigma_X^2}\sigma_X^2 = 1$$

$$\mu_Z = a\mu_X + b = \frac{\mu_X}{\sigma_X} - \frac{\mu_X}{\sigma_X} = 0 \qquad \therefore Z \sim N(0,1)$$

(3) $\Phi(-x) = 1 - \Phi(x)$

4.1 - NORMAL APPROXIMATION

(1) Central limit theorem

$$S_n \sim Bin(n,p) \rightarrow \lim_{n\to\infty} P\left(a \leq \frac{S_n - np}{\sqrt{npq}} \leq b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = P(a \leq Z \leq b)$$

(2) Distribution $X_n \to X$ in distribution if $P(a \leq X_n \leq b) \to P(a \leq X \leq b),\ n\to\infty$

$$P(X=a) = P(X=b) = 0$$

(3) Application $S_n \sim Bin(n,p)$ $\frac{S_n - np}{\sqrt{npq}} \to Z$ in distribution, $Z \sim N(0,1)$

(4) Continuity correction $P(S_n = k) = P(k - 0.5 \leq S_n \leq k + 0.5)$

4.2 - LAW OF LARGE NUMBERS

$S_n \sim Bin(n,p)$ $\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1$ ← weak law

$P\left(\frac{S_n}{n} \to p\right) = 1$ ← strong law

Proof →

$$1 \geq \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = \lim_{n\to\infty} P\left(-\varepsilon < \frac{S_n}{n} - p < \varepsilon\right)$$

$$= \lim_{n\to\infty} P\left(\frac{-\varepsilon\sqrt{n}}{\sqrt{npq}} < \frac{S_n - np}{\sqrt{npq}} < \frac{\varepsilon\sqrt{n}}{\sqrt{npq}}\right) \geq \lim_{n\to\infty} P(-c < Z < c) \text{ for any } c$$

$$= \Phi(c) - \Phi(-c) \text{ for any } c \xrightarrow{\text{Let } c\to\infty} 1 - 0 = 1$$

$$\therefore \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) = 1$$ QED!

$$\lim_{n\to\infty} P\left(\frac{S_n}{n} \in I\right) = \begin{cases} 0 & \text{if } p \notin I \\ 1 & \text{if } p \in I \end{cases}$$

4.3 - APPLICATIONS OF THE NORMAL APPROXIMATION

$\varepsilon = \frac{1}{2\sqrt{n}} z_{\frac{1+\alpha}{2}}$ $\quad \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - p\right| < \varepsilon\right) \geq \alpha$

100α % confidence interval for p is $(\hat{p} - \varepsilon, \hat{p} + \varepsilon)$, $\hat{p} = \frac{S_n}{n}$

4.4 - POISSON APPROXIMATION

$X \sim \text{Poisson}(\lambda)$ $\quad \frac{\lambda^k}{k!} e^{-\lambda}$ $\quad E[X] = \text{Var}(X) = \lambda$

$S_n \sim \text{Bin}(n, \frac{\lambda}{n})$ $\quad \lim_{n\to\infty} P(S_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

Proof $\rightarrow$

$$\lim_{n\to\infty} P(S_n = k) = \lim_{n\to\infty} \binom{n}{k} p^k q^{n-k} = \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\dots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^n \cdot \frac{1}{\left(1 - \frac{\lambda}{n}\right)^k}$$

$$= \lim_{n\to\infty} \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \cdot \underbrace{\frac{1}{\left(1 - \frac{\lambda}{n}\right)^k}}_{1} \cdot \underbrace{\frac{n}{n}}_{1} \cdot \underbrace{\frac{n-1}{n}}_{1} \cdots \underbrace{\frac{n-k+1}{n}}_{1} = \frac{\lambda^k}{k!} e^{-\lambda}$$ QED!

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1 + \frac{x}{n}\right)^n} = \lim_{n\to\infty} e^{n \ln\left(1 + \frac{x}{n}\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(1 + \frac{x}{n}\right)}{1/n}}$$

$y = \frac{1}{n}$

$$= \lim_{y\to 0} e^{\frac{\ln(1 + xy)}{y}} \stackrel{LH}{=} \lim_{y\to 0} e^{\frac{x}{1+xy}} = e^x \quad \therefore \left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

! Hypergeometric approximation to Binomial distribution

$X \sim \text{Hypergeometric}(N, N_A, n)$

Suppose $\frac{N_A}{N} \to p$ $\quad P(X = k) \to \text{Bin}(n, p) \approx N(np, npq)$ as $N \to \infty$

Proof $\rightarrow$

$$P(X = k) = \frac{\binom{N_A}{k}\binom{N_B}{n-k}}{\binom{N}{n}} = \frac{N_A!}{k!(N_A - k)!} \cdot \frac{N_B!}{(n-k)!(N_B - n + k)!} \cdot \frac{n!(N-n)!}{N!}$$

$$= \frac{n!}{k!(n-k)!} \cdot \overbrace{\frac{\cancel{N_A}^{p}}{\cancel{N}} \cdot \frac{\cancel{N_A-1}^{p}}{\cancel{N-1}} \cdots \frac{\cancel{N_A-k+1}^{p}}{\cancel{N-k+1}}}^{k} \cdot \overbrace{\frac{\cancel{N_B}^{q}}{\cancel{N-k}} \cdot \frac{\cancel{N_B-1}^{q}}{\cancel{N-k-1}} \cdots \frac{\cancel{N_B-n+k+1}^{q}}{\cancel{N-n+1}}}^{n-k}$$

$$\rightarrow \binom{n}{k} p^k q^{n-k} = Bin(n,p)$$ QED!

4.5 - EXPONENTIAL DISTRIBUTION

$X \sim Exp(\lambda) \longrightarrow$ pdf $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $E[X] = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$

(1) Application waiting time problems

(2) cdf $F_X(t) = P(X \leq t) = \int_0^t \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_0^t = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

(3) Properties $P(X > t) = \begin{cases} e^{-\lambda t}, & t \geq 0 \\ 1, & t < 0 \end{cases}$

(4) Memoryless property $P(X > t+s \mid X > t) = P(X > s) \quad t > 0$

Proof $\rightarrow$

$$P(X > t+s \mid X > t) = \frac{P(X > t+s \cap X > t)}{P(X > t)} = \frac{P(X > t+s)}{P(X > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = \frac{e^{-\lambda t} e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$ QED!

4.6 - POISSON PROCESS

(1) Time interval $N_I \sim Poisson(\lambda \underbrace{|I|}_{\text{length of interval}})$

(2) If $I_1, I_2 \ldots I_n$ are nonoverlapping then $N_{I_1}, N_{I_2} \ldots N_{I_n}$ are mutually ind.

(3) Intensity λ → arrival time of phone calls → what is the distribution?

Step 1 - Find cdf of T_n

$$F_{T_n}(t) = P(T_n \leq t) = 1 - P(T_n > t) = 1 - P(\underbrace{N_{[0,t]}}_{\sim Poisson(\lambda t)} \leq n-1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Step 2 - Find pdf of T_n

$$f_{T_n}(t) = \frac{d}{dt} F_{T_n}(t) = -\frac{d}{dt}\left(\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}\right)$$

$$= \lambda e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} - e^{-\lambda t} \sum_{k=1}^{n-1} \frac{\lambda k (\lambda t)^{k-1}}{k!}$$

$$= e^{-\lambda t}\left(\sum_{k=0}^{n-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{k=1}^{n-1} \frac{\lambda^k t^{k-1}}{(k-1)!}\right) = e^{-\lambda t}\left(\sum_{k=0}^{n-1} \frac{\lambda^{k+1} t^k}{k!} - \sum_{s=0}^{n-2} \frac{\lambda^{s+1} t^s}{s!}\right)$$

$$= \begin{cases} e^{-\lambda t} \cdot \frac{\lambda^n t^{n-1}}{(n-1)!}, & t > 0 \\ 0, & t \leq 0 \end{cases} \leftarrow T_n \sim Gamma(n, \lambda)$$

Def $\rightarrow$ $X \sim Gamma(r, \lambda) \rightarrow$ pdf $f_X(x) = \begin{cases} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$\Gamma(r) \longrightarrow = \int_0^\infty x^{r-1} e^{-x} dx$, $r > 0$

* Properties: $\Gamma(n) = (n-1)!$ $\quad \Gamma(1) = 1$

$\Gamma(r) = (r-1)\Gamma(r-1), \ r > 1$ $\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$E[X] = \frac{r}{\lambda}$ $\quad Var(X) = \frac{r}{\lambda^2}$

Proof $\rightarrow$ $E[X] = \int_0^\infty x \cdot \frac{\lambda^r \cdot x^{r-1} \cdot e^{-\lambda x}}{\Gamma(r)} dx = \int_0^\infty \frac{\lambda^r \cdot x^r \cdot e^{-\lambda x}}{\Gamma(r)} dx$

5.1 - MOMENT GENERATING FUNCTION

$$M(t) = E[e^{tX}] \longrightarrow E[X^n] = M_X^{(n)}(0)$$ ← n^{th} derivative

Proof → (1) $M(t) = E[e^{tX}] \longrightarrow M'_X(t) = E[Xe^{tX}] \longrightarrow M''_X(t) = E[X^2 e^{tX}]$

$\therefore M_X^{(n)}(t) = E[X^n e^{tX}] \longrightarrow M_X^{(n)}(0) = E[X^n e^0] = E[X^n]$ QED!

(2) $M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n$

$\therefore E[X^n] = M_X^{(n)}(0)$ by Taylor series expansion QED!

Summary

	Distribution	Moment Generating Function	
(1)	Poisson(λ)	$e^{\lambda(e^t-1)}$	$-\infty < t < \infty$
(2)	Geom(p)	$\dfrac{pe^t}{1-qe^t}$	$qe^t < 1 \rightarrow e^t < \frac{1}{q} \rightarrow t < -\ln q$
(3)	Unif(a,b)	$\begin{cases} \dfrac{e^{bt}-e^{at}}{(b-a)t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$	
(4)	Ber(p)	$q + pe^t$	$-\infty < t < \infty$
(5)	Bin(n,p)	$(q+pe^t)^n$	$-\infty < t < \infty$
(6)	Exp(λ)	$\begin{cases} \dfrac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda \end{cases}$	
(7)	$N(\mu, \sigma^2)$	$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$	

Equal in distribution if $P(X \in B) = P(Y \in B) \longrightarrow X \overset{d}{=} Y$

If $M_X(t) = M_Y(t)$ $\forall t$ in interval containing $0 \longrightarrow X \overset{d}{=} Y$

5.2 – DISTRIBUTION OF A FUNCTION

(1) Discrete random variables $\longrightarrow$ build new pmf

(2) Continuous random variables

cdf method → find cdf of Y $\quad F_Y(y) = P(Y \leq y) \ldots$

↘ find pdf of Y $\quad f_Y(y) = \frac{d}{dy} F_Y(y) \ldots$

Transformation method $\quad f_Y(y) = \sum f_X(x) \dfrac{1}{|g'(x)|} \Bigg|_{x = g^{-1}(y)}$