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Linear Algebra, Math 2101-003
Homework set #9

1. Consider a vector space with inner product $\langle .,. \rangle$ (e.g., $\mathbb{R}^n$ with the Euclidean inner product). Consider the norm induced by the inner product, that is, $\|x\| = \sqrt{\langle x, x \rangle}$.
Prove that if two vectors x, y have the same norm, i.e., $\|x\| = \|y\|$, then their sum is orthogonal to their difference, i.e., $x + y \perp x - y$.

2. Consider the subspace $\mathcal{S}$ spanned by the following three vectors in $\mathbb{R}^4$,

$$x_1 = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \end{vmatrix}, \ x_2 = \begin{vmatrix} -1 \\ -2 \\ 0 \\ -1 \end{vmatrix}, \ x_3 = \begin{vmatrix} -3 \\ -1 \\ -1 \\ 1 \end{vmatrix}.$$

Construct an orthonormal basis of $\mathcal{S}$.

3. Given an $n \times n$ matrix A whose column sums are all one. Show that one is an eigenvalue of A

1 Need to show that $\langle x+y, x-y \rangle = 0$

Consider $\langle x+y, x-y \rangle$

$= \langle x, x-y \rangle + \langle y, x-y \rangle$ by linearity of 1^{st} component

$= \langle x, x \rangle - \langle x, y \rangle + \langle y, x \rangle - \langle y, y \rangle$ by linearity of 2^{nd} component

$= \|x\|^2 - \langle x, y \rangle + \langle x, y \rangle - \|y\|^2$ by commutative property & definition of norm

$= \|x\|^2 - \|y\|^2 = 0$ $\quad \|x\| = \|y\| \rightarrow \|x\|^2 = \|y\|^2$

$\therefore \langle x+y, x-y \rangle = 0 \quad \therefore x+y \perp x-y$ QED!

2 $x_1 = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \end{vmatrix}, \ x_2 = \begin{vmatrix} -1 \\ -2 \\ 0 \\ -1 \end{vmatrix}, \ x_3 = \begin{vmatrix} -3 \\ -1 \\ -1 \\ 1 \end{vmatrix} \rightarrow u_1 = \dfrac{x_1}{\|x_1\|} = \dfrac{1}{\sqrt{2}} \begin{vmatrix} -1 \\ 0 \\ 0 \\ 1 \end{vmatrix}$

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$\langle x_2, u_1 \rangle = (-1)\left(\frac{-1}{\sqrt{2}}\right) + 0 + 0 + (-1)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$$w_2 = x_2 - \langle x_2, u_1\rangle u_1 = \begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix} - 0u_1 = \begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix} \rightarrow u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix}$$

$$\langle x_3, u_1\rangle = (-3)\left(\frac{-1}{\sqrt{2}}\right) + 0 + 0 + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\langle x_3, u_2\rangle = (-3)\left(\frac{-1}{\sqrt{6}}\right) + (-1)\left(\frac{-2}{\sqrt{6}}\right) + 0 - \frac{1}{\sqrt{6}} = \frac{3}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} = \frac{4}{\sqrt{6}}$$

$$w_3 = x_3 - \langle x_3, u_1\rangle u_1 - \langle x_3, u_2\rangle u_2$$

$$= \begin{bmatrix}-3\\-1\\-1\\1\end{bmatrix} - 2\sqrt{2}\,\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} - \frac{4}{\sqrt{6}}\,\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix} = \begin{bmatrix}-3\\-1\\-1\\1\end{bmatrix} - \begin{bmatrix}-2\\0\\0\\2\end{bmatrix} - \begin{bmatrix}-2/3\\-4/3\\0\\-2/3\end{bmatrix} = \begin{bmatrix}-1/3\\1/3\\-1\\-1/3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1\\1\\-3\\-1\end{bmatrix}$$

$$\therefore u_3 = \frac{w_3}{\|w_3\|} = \frac{\sqrt{3}}{6}\begin{bmatrix}-1\\1\\-3\\-1\end{bmatrix} \qquad \therefore Q = [u_1 \ u_2 \ u_3] = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6\\ 0 & -2/\sqrt{6} & \sqrt{3}/6\\ 0 & 0 & -\sqrt{3}/2\\ 1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6\end{bmatrix}$$

$$R_{11} = \|w_1\| = \sqrt{2} \qquad R_{12} = \langle x_2, u_1\rangle = 0$$
$$R_{22} = \|w_2\| = \sqrt{6} \qquad R_{13} = \langle x_3, u_1\rangle = 2\sqrt{2} \qquad \therefore R = \begin{bmatrix}\sqrt{2} & 0 & 2\sqrt{2}\\ 0 & \sqrt{6} & 4/\sqrt{6}\\ 0 & 0 & \sqrt{3}\end{bmatrix}$$
$$R_{33} = \|w_3\| = 2/\sqrt{3} \qquad R_{23} = \langle x_3, u_2\rangle = \frac{4}{\sqrt{6}}$$

$$\text{Verify: } Q.R = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6\\ 0 & -2/\sqrt{6} & \sqrt{3}/6\\ 0 & 0 & -\sqrt{3}/2\\ 1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 & 2\sqrt{2}\\ 0 & \sqrt{6} & 4/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix} = \begin{bmatrix}-1 & -1 & -3\\ 0 & -2 & -1\\ 0 & 0 & -1\\ 1 & -1 & 1\end{bmatrix} = A \checkmark$$

3 Let $A = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$ where $\sum_{i=1}^{n} a_{i*} = 1 \therefore A^T = \begin{bmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{bmatrix}$

Consider $v = \begin{vmatrix} 1 \\ 1 \\ \dots \\ 1 \end{vmatrix}$ $\therefore A^T v = \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ \dots \\ 1 \end{vmatrix} = \begin{vmatrix} a_{11} + a_{21} + \dots + a_{n1} \\ a_{12} + a_{22} + \dots + a_{n2} \\ \dots \\ a_{1n} + a_{2n} + \dots + a_{nn} \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ \dots \\ 1 \end{vmatrix}$

$\therefore A^T v = v = 1.v$ $\quad \therefore \lambda = 1$ is an eigenvalue of A^T

Consider $\det(A - \lambda I) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$ by $\det A = \det A^T$

$\therefore A$ & A^T have the same eigenvalue

$\therefore \lambda = 1$ is an eigenvalue of A QED!