Section 1.1 & 1.2

V is a vectorspace if it is closed under addition & scalar multiplication

$x, y \in V \rightarrow \alpha x + y \in V$

$0 \in V$

(1) $\alpha = 0$

$\alpha . x = 0 . x = 0$

$\alpha x \in V,\ 0 \in V$

(2) $\alpha = -1$

$\alpha . x = -x \in V$

$-x + x = 0 \in V$

If $0 \notin S$, S is not a vectorspace

Unit vectors $(e_i)_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\quad x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$

$A = B$ if $A, B \in \mathbb{R}^{m \times n}$

$a_{ij} = b_{ij} \quad \forall\, i = 1 \ldots m$

$j = 1 \ldots n$

$(A^T)^T = A \qquad (A + B)^T = A^T + B^T$

$(\alpha A)^T = \alpha A^T$

Diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Symmetric $\quad A^T = A$

skew-symmetric $\quad A^T = -A \rightarrow a_{ij} = -a_{ji}$

$a_{ii} = -a_{ii} \ (i = j) \longrightarrow \quad a_{ii} = 0$

Section 1.3

Subspace $S \subseteq V$
- trivial subspaces $\{0\}$, V
- non-trivial subspaces line, planes

Affine space $A = p + X = \{ v \in V \mid v = p + x,\ x \in X \}$

Linearly dependent if one of them can be written as linear combination

if $\exists\, x \neq 0 \ni Ax = 0$

Linearly independent $\sum \alpha_i v_i = 0 \rightarrow \alpha_i = 0 \ \forall i$

$A = [v_1 \ v_2 \dots v_n]$
$Ax = 0 \rightarrow x = 0$

Section 1.4 & 1.5

Norm $\|x\| = \left(\sum_{i=1}^{n} x^2\right)^{1/2} = \sqrt{x^T x}$

(1) $\|x\| \geq 0$
(2) $\|x\| = 0 \rightarrow x = 0$
(3) $\|\alpha x\| = |\alpha| \|x\|$
(4) $\|x+y\| \leq \|x\| + \|y\|$ (Δ inequality)

$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle}$
$= \sqrt{\alpha^2 \langle x, x \rangle}$
$= |\alpha| \sqrt{\langle x, x \rangle}$
$= |\alpha| \|x\|$

Inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \text{trace}(x^T y)$

(1) Bilinear $\langle \alpha x + y, v \rangle = \alpha \langle x, v \rangle + \langle y, v \rangle$ $\quad \langle \alpha u, \alpha v \rangle = \alpha^2 \langle u, v \rangle$
$\langle x, v + \beta w \rangle = \langle x, v \rangle + \beta \langle x, w \rangle$
(2) $\langle x, x \rangle \geq 0$
$\langle x, x \rangle = 0 \rightarrow x = 0$ $\quad \langle x, y \rangle = 0 \rightarrow x \perp y$
(3) Symmetric $\langle x, y \rangle = \langle y, x \rangle$

1-norm $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$
$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

Proof $\rightarrow$ $\|x+y\|_1 = |x_1 + y_1| + |x_2 + y_2| + \dots + |x_n + y_n|$
$\leq |x_1| + |y_1| + |x_2| + |y_2| + \dots |x_n| + |y_n|$
$= \|x_1\|_1 + \|y_1\|_1$

∞-norm $\|x\|_\infty = \max_{i=1}^{n} |x_i|$

General $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$ $\quad (1 \leq p \leq 2)$

CBS Inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$

Proof $\rightarrow$ If $x = 0$, $\|x\| = 0 \rightarrow \langle x, y \rangle = 0 \rightarrow$ inequality holds

If $x \neq 0$, $\alpha = \dfrac{\langle x, y\rangle}{\|x\|^2}$ $\qquad V = \alpha x - y$

$$\langle x, v\rangle = \langle x, \alpha x - y\rangle = \alpha\langle x, x\rangle - \langle x, y\rangle = \frac{\langle x, y\rangle}{\|x\|^2}\|x\|^2 - \langle x, y\rangle = 0$$

$$\begin{aligned} 0 \leq \|v\|^2 = \langle v, v\rangle &= \langle \alpha x - y, \alpha x - y\rangle \\ &= \alpha\langle x, \alpha x - y\rangle - \langle y, \alpha x - y\rangle \\ &= \alpha(0) \qquad - \alpha\langle y, x\rangle + \langle y, y\rangle \\ &= \|y\|^2 - \frac{\langle x, y\rangle}{\|x\|^2}\langle x, y\rangle \\ &= \frac{\|y\|^2\|x\|^2 - \langle x, y\rangle^2}{\|x\|^2} \end{aligned}$$

$\therefore \|x\|^2\|y\|^2 \geq \langle x, y\rangle^2 \qquad \therefore |\langle x, y\rangle| \leq \|x\|\,\|y\|$

Proof Δ inequality $\rightarrow$

$$\begin{aligned} \|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \qquad a \leq |a| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$\therefore \|x+y\| \leq \|x\| + \|y\|$

Backward Δ inequality $\big|\|x\| - \|y\|\big| \leq \|x-y\|$

Proof $\rightarrow$

$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\| \qquad \therefore \|x\| - \|y\| \leq \|x-y\|$

$\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\| = \|x-y\| + \|x\| \quad \therefore \|y-x\| \leq \|x-y\|$

$\therefore \big|\|x\| - \|y\|\big| \leq \|x-y\|$

Section 1.6

Orthogonality $\langle x, y\rangle = 0 \rightarrow \|x\|^2 + \|y\|^2 = \|x-y\|^2$

Angle $\qquad \cos\theta = \dfrac{\langle x, y\rangle}{\|x\|\,\|y\|}$

Orthogonal set $\{u_1, u_2 \ldots u_n\}$

$\langle u_i, u_j\rangle = 0 \quad (i \neq j)$

$\rightarrow$ linearly independent

Orthonormal $\langle u_i, u_j\rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

$\|u_i\| = 1$

$\rightarrow$ orthogonal matrix

(use $\dfrac{u}{\|u\|}$)

Section 1.7

Linear transformation $f(\alpha x + y) = \alpha f(x) + f(y)$

If y fixed: $f_y(x) = \langle x, y \rangle$

$f_y(x+u) = \langle x+u, y \rangle = \langle x, y \rangle + \langle u, y \rangle$

If x fixed: $f_x(u+v) = \langle u+v, y \rangle = \langle u, y \rangle + \langle v, y \rangle = f_x(u) + f_x(v)$

Matrix times vector

1st coeff x 1st col $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = y_1 x_1 + y_2 x_2 + y_3 x_3$

Nullspace kernel $\mathcal{N}(f) = \{ x \in V \mid f(x) = 0 \}$ $\quad f(0) = 0$

Range $R(f) = \{ y \in \mathbb{R} \mid \exists x \in \mathcal{D}, f(x) = y \}$ $\quad 0 \in R(f)$

Composition of linear maps is a linear map

Proof $\rightarrow$ $(f \circ g)(\alpha x + y) = f(g(\alpha x + y)) = f(\alpha g(x) + g(y))$ linear g

composition $\uparrow$

$= \alpha f(g(x)) + f(g(y))$ linear f

$= \alpha (f \circ g)(x) + (f \circ g)(y)$

Matrix times matrix $c_{ij} = \sum_{k=1}^{n_2} a_{ik} b_{kj}$

matrix x each col $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ & $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

$Ax = b$ consistent if $\exists$ at least 1 sln (b is linear combination of A's cols)

Triangular $\rightarrow$ lower $a_{ij} = 0 \ (i < j)$

$\searrow$ upper $a_{ij} = 0 \ (i > j)$

$\therefore$. =

. =

A triangular $\longrightarrow A^{-1}$ also triangular $\quad \text{diag}(A^{-1}) = \text{diag}\left(\frac{1}{a_{ii}}\right)$

$\text{diag}(AB) = \text{diag}(A) \cdot \text{diag}(B)$

$(AB)_{ii} = A_{ii} \cdot B_{ii}$

Section 1.8

Diagonal matrix $ij = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$ $IA = A$, $AI = A$

$A^0 = I$
$A^x \cdot A^y = A^{x+y}$
$(A^x)^y = A^{x \cdot y}$

$(AB)^T = B^T A^T$
$\underbrace{(A^TA)^T}_{\text{always symmetric}} = A^T (A^T)^T = A^TA$

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

Inverse $A \cdot B = I \to B = A^{-1}$
→ exists if cols of A are linearly independent

$\text{trace}(AB) = \text{trace}(BA)$

Proof →

$$\text{trace}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} A_{i*} B_{*i} = \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} b_{ki}$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ki} a_{ik} = \sum_{k=1}^{m} \sum_{i=1}^{n} b_{ki} a_{ik} = \sum_{k=1}^{m} B_{k*} A_{*k} = \sum_{k=1}^{m} (BA)_{kk} = \text{trace}(BA)$$

$\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$

Nonsingular (linearly independent) $Ax = 0 \to x = 0$
$Ax = b$ has unique sln
$AX = I \to XA = I$

Section 1.9 & 1.10

Matrix norm $\|AB\| \leq \|A\| \cdot \|B\|$

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

Orthogonal matrix U

(1) maintain length

$$\|Ux\|^2 = \langle Ux, Ux \rangle = (Ux)^T Ux$$
$$= x^T U^T U x = x^T I x$$
$$= x^T x = \|x\|^2 \checkmark$$

(2) maintain angle

$$\cos(Ux, Uy) = \frac{\langle Ux, Uy \rangle}{\|Ux\| \|Uy\|}$$
$$= \frac{(Ux)^T Uy}{\|x\| \|y\|} = \frac{x^T y}{\|x\| \|y\|} = \frac{\langle x, y \rangle}{\|x\| \|y\|} = \cos\theta \checkmark$$

A, B orthogonal $\longrightarrow$ show $A^TA = I$

$$\|A^{-1}\| = \max_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \max_{y \neq 0} \frac{\|y\|}{\|Ay\|} = \frac{1}{\min_{y \neq 0} \frac{\|Ay\|}{\|y\|}}$$

$$\rightarrow \min_{\|x\| \neq 0} \|Ax\| = \frac{1}{\|A^{-1}\|}$$

Reflections $R = I - \dfrac{2uu^T}{u^Tu} \xrightarrow[\|u\|=1]{u^Tu=1} R = I - uu^T$

(1) $R^T = R$

Proof $\rightarrow$ $R^T = I^T - 2(uu^T)^T = I - 2(uu^T) = R$

(2) $RR = I$

Proof $\rightarrow$ $RR = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4u\underbrace{u^Tu}u^T$

$= I - 4uu^T + 4uu^T = I$

(3) $Ru = -u$

Proof $\rightarrow$ $Ru = (I - 2uu^T)u = u - 2u\underbrace{u^Tu} = u - 2u = -u$

if $u^Tw = 0 \longrightarrow (I - 2uu^T)w = w - 2u\underbrace{u^Tw} = w$

(4) $\therefore Rw = w$

Orthogonal projection $P = I - uu^T$

(1) $P^2 = P$ (general projection)

Proof $\rightarrow$ $P^2 = (I - uu^T)^2 = I - 2uu^T + u\underbrace{u^Tu}u^T = I - 2uu^T + uu^T = I - uu^T = P$

(2) $P^T = P$ (orthogonal projection)

Proof $\rightarrow$ $P^T = I - (uu^T)^T = I - uu^T = P$

(3) if $w \perp u$ $Pw = w$

Proof $\rightarrow$ $P(u + w) = Pu + Pw = 0 + w = w$