D. Szyld

Example of oblique projection

Let $\mathcal{X} = \{ x \,/\, x_1 + 2x_2 + x_3 = 0 \}$

and $\mathcal{Y} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

A basis for $\mathcal{X}$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

A basis for $\mathcal{Y}^\perp$ is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Let $U = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$

The projection onto $\mathcal{X}$ along $\mathcal{Y}$ is

$$P = U (V^T U)^{-1} V^T$$

$$V^T U = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -2 & -2 \end{bmatrix}$$

Let us compute $(V^T U)^{-1}$

$$\begin{bmatrix} -3 & -1 \\ -2 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & -1 \\ 0 & -4/3 \end{bmatrix} = \mathcal{U}$$

$$m_{12} = -\frac{2}{3} \qquad -2 + \left(-\frac{2}{3}\right)(-1) = -2 + \frac{2}{3} = \frac{-4}{3}$$

$$\mathcal{L} = \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix} \qquad \mathcal{L}^{-1} = \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix}$$

$$\mathcal{U} = \begin{bmatrix} -3 & 0 \\ 0 & -4/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{U}^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 0 \\ 0 & -\frac{3}{4} \end{bmatrix} = \begin{bmatrix} -1/3 & 1/4 \\ 0 & -3/4 \end{bmatrix}$$

$$(V^T U)^{-1} = \mathcal{U}^{-1} \mathcal{L}^{-1} = \begin{bmatrix} -1/3 & 1/4 \\ 0 & -3/4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & -3/4 \end{bmatrix}$$

$$-\frac{1}{3} - \frac{2}{4 \cdot 3} = -1/3 - 1/6 = -1/2$$

$$P = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & -3/4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} =$$

$$P = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/4 & 1/2 & -1/4 \\ -1/4 & -1/2 & 3/4 \end{bmatrix}$$

$-\frac{1}{2}+\frac{1}{4}=-\frac{1}{4}$ $\quad \frac{1}{2}-\frac{3}{4}=-\frac{1}{4}$ $\quad 2/4-3/4=-1/4$

$$P = \begin{bmatrix} 3/4 & -1/2 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/2 & 3/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

Note this P is <u>not</u> symmetric, corresponding to an <u>oblique</u> projection

check 1. $P^2 = P$?

$$P^2 = \frac{1}{16}\begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & 3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 12 & -8 & -4 \\ -4 & 8 & -4 \\ -4 & -8 & 12 \end{bmatrix} = P$$

check 2. $P \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ i.e. $N(P) = \mathcal{Y}$

check 3. ~~Th~~ every colum of P is in $\mathcal{X}$ so that $R(P) = \mathcal{X}$.