Math 2041: Summary 4.1, 4.3, 4.4

Section 4.1: Linear Homogeneous DE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

Uniqueness and Existence (restated) : Let $a_2(x), a_1(x)$, and $a_0(x)$ be continuous functions on an open interval *I* that contains the point x_0, with $a_2(x) \neq 0$ for all x in I. Then the initial value problem $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_0'$ has exactly one solution that exists throughout *I*.

Principle of Superposition: If y_1 and y_2 are two solutions to the Linear Homogenous Differential Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

Theorems: Consider the initial value problem

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_0'.$$

Then it is possible to find the constants c_1 and c_2 such that $y = c_1y_1 + c_2y_2$ is the solution to the initial value problem for any y_0 and y_0' if and only if the Wronskian $W = y_1y_2' - y_1'y_2 \neq 0$ at x_0. This implies that on an interval that contains x_0 on which the Existence and Uniqueness theorem holds, every solution to the differential equation is of the form $y = c_1y_1 + c_2y_2$.

This pair, y_1 and y_2, is called a fundamental set of solutions to $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$.

Section 4.3: Linear homogeneous equations with constant coefficients:

$$ay'' + by' + cy = 0$$

Auxiliary equation: $am^2 + bm + c = 0$

Case 1: The solutions of the auxiliary equation are two real, distinct roots, m_1 and m_2. Then the general solution of the differential equation is given by

$$y = c_1e^{m_1x} + c_2e^{m_2x}.$$

Case 2: The solution of the auxiliary equation is given by one root with multiplicity two, $m_1 = m$. Then the general solution of the differential equation is given by

$$y = c_1e^{mx} + c_2xe^{mx}.$$

Case 3: The solutions of the auxiliary equation are two complex roots, $m = \alpha \pm \beta i$. Then the general solution of the differential equation is given by

$$y = c_1e^{\alpha x}\cos(\beta x) + c_2e^{\alpha x}\sin(\beta x).$$

Section 4.4: Linear nonhomogeneous equations with constant coefficients:

$$ay'' + by' + cy = g(x).$$

Solution: $y = y_h + y_p$, where y_h is the general solution to the homogeneous differential equation $ay'' + by' + cy = 0$ and y_p is a specific solution to the nonhomogeneous differential equation $ay'' + by' + cy = g(x)$.

Method of Undetermined coefficients

$g(x)$	Form of y_p (See modification – last line)
$P_n(x)$, a polynomial of degree n	$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots A_1 x + A_0$
$ke^{\alpha x}$	$y_p = Ae^{\alpha x}$
$k_1 \sin \beta x + k_2 \cos \beta x$	$y_p = A \sin \beta x + B \cos \beta x$
Sums of the above e.g. $g(x) = e^{2x} + x$	Use superposition of particular solutions. e.g. $y_{p_1} = Ae^{2x}$; $y_{p_2} = Bx + C$ $y_p = y_{p_1} + y_{p_2}$
Products of the above e.g. $g(x) = e^{\alpha x} P_n(x) \sin \beta x$	$y_p = P$roducts of the above, combining constants. e.g. $y_p = e^{\alpha x} \sin \beta x \left(A_n x^n + \cdots A_1 x + A_0 \right) +$ $e^{\alpha x} \cos \beta x \left(B_n x^n + \cdots B_1 x + B_0 \right)$
Modification: If part of y_p is the same as y_c (so do not have linear independence), then multiply y_p by x^r for the r that does give linear independence. For Second order equations, $r \le 2$.	

Differential Operator:

$$D\left[e^{ax} y\right] = e^{ax} y' + ae^{ax} y = e^{ax}\left(y' + ay\right)$$
$$= e^{ax}\left(D + a\right)[y]$$

$$P(D)\left[e^{ax} y\right] = e^{ax} P(D + a)[y],$$

where $P(D)$ is a polynomial in D,