§7.2 Inverse Laplace Transforms

I) This section presents another way of solving IVP of LDE with constant coefficients.

All in one step.

Use PFD (partial fraction decomposition) for proper fractions (degree of numerator < degree of denom) of Rational functions (poly/poly).

eg What is the form of the PFD of the following:

$$\frac{\text{poly of degree} < 8}{(s+1)(s-2)^3(s^2+a)(s^2+b)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} + \frac{Es+F}{s^2+a} + \frac{Gs+H}{s^2+b}$$

degree 0

degree 1

degree 1

degree 2

$(s+1)$: mult. 1, one fraction

$(s-2)^3$: mult. 3, three fractions

II) Laplace Transforms and Inverse

Laplace Transforms. (1-1 reverse processes)

eg $\mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$

So $\mathcal{L}^{-1}\left\{\dfrac{1}{s-2}\right\} = e^{2t}$.

T. 7.2.1 Some Inverse Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1)	1	$\dfrac{1}{s}$, $s > 0$
2)	e^{at}	$\dfrac{1}{s-a}$, $s > a$
3)	t^n	$\dfrac{n!}{s^{n+1}}$, $s > 0$
4)	$\sin(kt)$	$\dfrac{k}{s^2+k^2}$, $s > 0$
5)	$\cos(kt)$	$\dfrac{s}{s^2+k^2}$, $s > 0$

eg <u>Examples 1 & 2</u>

eg $$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4!}\cdot\frac{4!}{s^{4+1}}\right\}$$

$n+1=5$

$$= \frac{1}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = \frac{1}{24}t^4$$

eg $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \mathcal{L}\left\{\frac{1}{s^2+(\sqrt{7})^2}\right\}$$

$$= \mathcal{L}\left\{\frac{1}{\sqrt{7}}\cdot\frac{\sqrt{7}}{s^2+(\sqrt{7})^2}\right\}$$

$$= \frac{1}{\sqrt{7}}\sin(\sqrt{7}t)$$

eg $$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{2s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$$

$$= -2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos(2t) + 3\sin(2t)$$

eg $\mathcal{L}^{-1}\left\{\dfrac{3s}{s^2-s-6}\right\}$

$$\frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$3s = A(s+2) + B(s-3)$$

$s=3$: $9 = 5A \Rightarrow A = \frac{9}{5}$

$s=-2$: $-6 = -5B \Rightarrow B = \frac{6}{5}$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{9}{5}}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{6}{5}}{s+2}\right\}$$

$$= \frac{9}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}$$

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eg $\mathcal{L}^{-1}\left\{\dfrac{8s^2-4s+12}{s(s^2+4)}\right\}$

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$8s^2-4s+12 = A(s^2+4) + (Bs+C)s$$

$s=0$: $12 = 4A \Rightarrow A = 3$

s^2 : $8 = A + B$

$8 = 3 + B \Rightarrow B = 5$

s : $-4 = C$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5s-4}{s^2+4}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 3 + 5\cos(2t) - 2\sin(2t)$$

§ 7.2.2 Transforms of Derivatives

T 7.2.2 If $f, f', \ldots, f^{(n-1)}$, are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Suppose $\mathcal{L}\{y(t)\} = Y(s)$, $y(0)$ $y'(0)$.

Then $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

and $\mathcal{L}\{y''(t)\} = s^2 Y(s) - s\,y(0) - y'(0)$

$$
\begin{tikzcd}
\text{IVP} \arrow[r, "\text{Laplace Transform}"] & \text{Alg. Expression in } Y(s) \arrow[d] \\
\text{Solution of IVP} \arrow[u] & \text{Solve for } Y(s) \arrow[l, "\text{Inverse Laplace Transforms}"']
\end{tikzcd}
$$

eg $\frac{dy}{dt} + 3y = 13 \sin 2t, \quad y(0) = 6$

Assume
$\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\,\mathcal{L}\{\sin 2t\}$$

$$sY(s) - 6 + 3Y(s) = 13\left(\frac{2}{s^2+4}\right)$$

$sY(s) - y(0)$

$$Y(s)(s+3) = 6 + \frac{26}{s^2+4}$$

But do NOT forget $\frac{6}{s+3}$.

Solve

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)}$$

DO NOT combine as book does. Easier to do separately,

$$\frac{26}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$26 = A(s^2+4) + (Bs+C)(s+3)$$

$s=-3 \quad 26 = 13A \Rightarrow A = 2$

$s^2: \quad 0 = A + B \Rightarrow B = -2$

$$s: \quad 0 = C + 3B$$

$$0 = C + 3(-2)$$

$$C = 6$$

$$Y(s) = \frac{6}{s+3} + \frac{2}{s+3} + \frac{-2s}{s^2+4} + \frac{6}{s^2+4}$$

$$Y(s) = \frac{8}{s+3} - 2 \cdot \frac{s}{s^2+4} + 3 \cdot \frac{2}{s^2+4}$$

$$y(t) = 8e^{-3t} - 2\cos(2t) + 3\sin(2t)$$

Example : $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$

$$s^2 Y(s) - s \underbrace{y(0)}_{2} - \underbrace{y'(0)}_{1} + Y(s) = \frac{2}{s^2+4}$$

$$Y(s) \underbrace{(s^2+1)}_{\text{AE in } s} - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}$$ Already decomposed

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$s^3 : 0 = A + C$

$s^2 : 0 = B + D$

$s : 0 = A + 4C$

Constant : $2 = B + 4D$

$\Rightarrow B = 2 - 4D$

$0 = 2 - 4D + D$

$3D = 2$

$D = \frac{2}{3}$

$B = 2 - 4\left(\frac{2}{3}\right)$

$= 2 - \frac{8}{3} = \frac{-2}{3}$

$0 = A + 4(-A)$

$A = 0$

$C = 0$

$\Rightarrow$

$$Y(s) = \frac{-\frac{2}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$= -\frac{1}{3} \cdot \frac{2}{s^2+4} + \frac{5}{3} \cdot \frac{1}{s^2+1} + 2 \cdot \frac{s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= -\frac{1}{3} \sin 2t + \frac{5}{3} \sin t + 2 \cos t$$