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Linear Algebra

Section 2.3 . Consistency of linear systems

Recall $A = \begin{bmatrix} a_{11} & \text{---} & a_{1n} \\ a_{21} & \text{---} & a_{2n} \\ \vdots & & \\ a_{m1} & \text{---} & a_{mn} \end{bmatrix}$ $m \times n$

$x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{vmatrix}$ $b = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{vmatrix}$

$Ax = b$ linear system $m \times n$

Example 3×4

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$$

Definition. A linear system is consistent if it posseses at least one solution.

Otherwise, it is incosistent.

We saw examples in first class.

If no solution, inconsistent.

If one or infinitely many, consistent.

We want a systematic way of deciding when $[A|b]$ is consistent. (or not)

The idea is to use Gaussian Elimination to reduce the system to upper echelon form (recall, same solution set, equivalent systems).

$$[A|b] \longrightarrow [E|c]$$

In the new system consistency (or inconsistency) is apparent (obvious)

Inconsistent. iff (if and only if) you have a row of the form

$$[0\ 0\ 0 \cdots 0 \mid \alpha] \qquad \alpha \neq 0$$

i.e. representing $0x_1 + 0x_2 + \cdots + 0x_n = \alpha$

remark

If system is consistent we will see how to find one and all solutions of $Ax = b$

Back to 3×4 example

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 8 & 10 & 6 \\ 3 & 6 & 11 & 14 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 2 & 2 & 5 \end{array}\right] \rightarrow$$

$E_2 - 2E_1$ (or $A_{2*} - 2A_{1*}$)

$E_3 - 3E_1$ (or $A_{3*} - 3A_{1*}$)

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = [E \mid c]$$

$E_3' - E_2'$ $\left(M_{3*}' - M_{2*}'\right)$

Rank A = Rank $E = 2$

system is inconsistent.

Note Rank $[A|b]$ = Rank $[E|c] = 3$

Linear System is consistent ($Ax = b$) if either of the following hold:

1. there is no row that in $[E|c]$ is $[0\,0\,0\,0\,0 \mid \alpha]$ $\alpha \neq$
2. b is a non-basic column of $[A|b]$
3. rank $[A|b]$ = rank A
4. $b = \sum\limits_{\substack{\text{basic} \\ \text{columns } j}} \alpha_j A_{*j}$

Let us redo the same A with a different right hand side

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 8 & 10 & 6 \\ 3 & 6 & 11 & 14 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 2 & 2 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank $= 2$ $\quad$ b non-basic column $[A|b]$

basic columns 1, 3

$$\begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \qquad \begin{vmatrix} 3 \\ 8 \\ 11 \end{vmatrix}$$

We say that x_i is a <u>Free Variable</u> if i is a non-basic column.

Here x_2, x_4 are free variables.

We can find a <u>particular solution</u> of a consistent linear system by setting the free variables to zero and do back substitution

In the last example set $x_2 = 0$ $x_4 = 0$

Equation 2 is then $2x_3 + 4x_4 = 4$

$$2x_3 = 4 \qquad \boxed{x_3 = 2}$$

First equation: $x_1 + 2\cdot 0 + 3.2 + 4.0 = 1$

$$x_1 = 1 - 6 = -5$$

particular solution $x = x_p = \begin{vmatrix} -5 \\ 0 \\ 2 \\ 0 \end{vmatrix}$

$$-5 \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} + 2 \begin{vmatrix} 3 \\ 8 \\ 11 \end{vmatrix} = \begin{vmatrix} -5+6 \\ -10+16 \\ -15+22 \end{vmatrix} = \begin{vmatrix} 1 \\ 6 \\ 7 \end{vmatrix} = b$$

basic columns