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Linear Algebra, Math 2101-002
Homework set #5

1. (3 points).
(a). Give an example of two 2×2 singular matrices A, B, such that $A + B$ is nonsingular.
(b). Give an example of two 2×2 nonsingular matrices A, B, such that $A + B$ is singular.

2. (3 points).
(a). Prove that if Q_1 and Q_2 are orthogonal matrices, then, their product Q_1Q_2 is also an orthogonal matrix.
(b) Prove that for any finite set of orthogonal matrices $Q_1, Q_2, \ldots, Q_k$, their product $Q_1Q_2 \cdots Q_k$ is also an orthogonal matrix.

3. (4 points).
Let P be an orthogonal projection, i.e., $P^2 = P$, and $P^T = P$. Let $Q = I - P$.
(a). Show that Q is also an orthogonal projection.
(b). Show that $PQ = 0$ and $QP = 0$.

1 (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is singular because A_1 & A_2 are multiples of each other ($A_1 = A_2$)

$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is singular because B_1 & B_2 are multiples of each other ($B_1 = -B_2$)

$\therefore A + B = \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I$ is nonsingular

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are nonsingular since neither columns are multiples of each other

$\therefore A + B = \begin{bmatrix} 1+0 & 0+1 \\ 0+1 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is singular (like part (a))

2 (a) To see if Q_1Q_2 is an orthogonal matrix, need to show $(Q_1Q_2)^TQ_1Q_2 = I$

Consider $(Q_1Q_2)^TQ_1Q_2$

$= Q_2^TQ_1^TQ_1Q_2$ by property of transposition $(AB)^T = B^TA^T$

$= Q_2^TIQ_2$ Q_1 is orthogonal matrix so $Q_1^TQ_1 = I$

$= Q_2^TQ_2$ by property of identity matrix $IA = A$

$= I$ Q_2 is orthogonal matrix so $Q_2^TQ_2 = I$

$\therefore (Q_1Q_2)^TQ_1Q_2 = I$

$\therefore Q_1Q_2$ is an orthogonal matrix by definition QED!

(b) Assume the product $P = Q_1Q_2 \dots Q_k$

To show P is an orthogonal matrix, need to show $P^TP = I$

Consider P^TP

$= (Q_1Q_2\dots Q_k)^T(Q_1Q_2\dots Q_k)$

$= Q_k^T\dots Q_2^T\underbrace{Q_1^TQ_1}Q_2\dots Q_k$ by property of transposition $(AB)^T = B^TA^T$

$= Q_k^T\dots Q_2^T\underbrace{I\ Q_2}\dots Q_k$ Q_1 is orthogonal matrix so $Q_1^TQ_1 = I$

$= Q_k^T\dots Q_2^TQ_2\dots Q_k$ by property of identity matrix $IA = A$

$\vdots$

Since $Q_1, Q_2\dots Q_k$ are arbitrary orthogonal matrices, the above steps can be performed repeatedly to achieve

$P^TP = Q_k^TQ_k = I$

$\hookrightarrow$ Q_k is orthogonal matrix so $Q_k^TQ_k = I$

$\therefore P^TP = I$

$\therefore (Q_1Q_2\dots Q_k)^T(Q_1Q_2\dots Q_k) = I$

$\therefore Q_1Q_2\dots Q_k$ is an orthogonal matrix by definition QED!

3 (a) To show that Q is an orthogonal projection, wish to show:

(i) $Q^2 = Q$

Consider $Q^2 = Q.Q = (I-P)(I-P)$ since given $Q = I - P$

$= I^2 - IP - PI + P^2$ by distributive law

$= I - P - P + P$ by property of identity matrix $(IP = PI = P)$ & $P^2 = P$

$= I - P = Q$

$\therefore Q^2 = Q$ QED!

(ii) $Q^T = Q$

Consider $Q^T = (I - P)^T = I^T - P^T$ by property of transposition $(A-B)^T = A^T - B^T$

$= I - P$ by property of identity matrix $I^T = I$

$= Q$ & $P^T = P$ (given)

$\therefore Q^T = Q$

Since (i) & (ii) are satisfied, Q is an orthogonal projection QED!

(b) Consider $PQ = P(I - P) = PI - P^2$ by distributive law

$= P - P$ by property of identity matrix

$= 0$ $(PI = P)$ & $P^2 = P$ (given)

$\therefore PQ = 0$ QED!

Consider $QP = (I - P)P = IP - P^2$ by distributive law

$= P - P$ by property of identity matrix

$= 0$ $(IP = P)$ & $P^2 = P$ (given)

$\therefore QP = 0$ QED!