EXAM 2B - Linear Algebra (1)

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1.(a)

$$A = \begin{vmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{vmatrix} \rightarrow$$

$m_{21} = \frac{-4}{2} = -2$

$m_{32} = -\frac{12}{3} = -4$

$m_{31} = \frac{-6}{2} = -3$

$$\rightarrow \begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} = U$$

$$L = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix}$$

check $L \cdot U = A$ ✓

(b) Rank $A = 3$ (3 pivots)

thus $R(A) = \mathbb{R}^3$

basis $\left\{ \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}, \begin{vmatrix} 2 \\ 7 \\ 18 \end{vmatrix}, \begin{vmatrix} 2 \\ 7 \\ 22 \end{vmatrix} \right\}$ the basic columns

or any other basis of $\mathbb{R}^3$, e.g. $\left\{ \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \right\}$

(c) $N(A) = \{0\}$ no free variables

(d) solve $LUx = b$

$Ly = b \quad Ux = y$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 4 \\ 10 \end{vmatrix}$$

$y_1 = 0$

$y_2 = 4$

$y_3 = (10 - 4 \cdot 4 - 3 \cdot 0)/1 = -6$

$$y = \begin{vmatrix} 0 \\ 4 \\ -6 \end{vmatrix}$$

. check $Ly = b$ ✓

$$\begin{vmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 4 \\ -6 \end{vmatrix}$$

$x_3 = -\frac{6}{4} = -\frac{3}{2}$

$x_2 = \left(4 - 3 \cdot \left(-\frac{3}{2}\right)\right)/3 = \left(\frac{8+9}{2}\right)/3 = \frac{17}{6}$

$x_1 = \left(0 - 2\frac{17}{6} - 2\left(-\frac{3}{2}\right)\right)/2 = \left(-\frac{17}{6} + \frac{3}{2}\right)$

$= \frac{-17+9}{6} = \frac{-8}{6}$

$$x = \begin{vmatrix} -8/6 \\ 17/6 \\ -3/2 \end{vmatrix} = \begin{vmatrix} -8/6 \\ 17/6 \\ -9/6 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} -8 \\ 17 \\ -9 \end{vmatrix}$$

check $Ax = b$ ✓

(2) We know $A = LU$

$$A^T = U^T L^T$$

where U^T is lower triangular and L^T is upper triangular.

to be an LU factorization we need the diagonals of the lower triangular to be ones.

write $U^T = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix}$

i.e. divide each column by the diagonal element and factor out a diagonal matrix

Thus

$$A^T = U^T L^T = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 12 \\ 0 & 0 & 4 \end{vmatrix}$$

where the new Upper triangular matrix is the product of diagonal times L^T (each ~~column~~ row is multiplied.)

(3) A is $m \times n$, A maps $\mathbb{R}^n$ into $\mathbb{R}^m$

$R(A) = \{ y \in \mathbb{R}^m / \exists x \in \mathbb{R}^n$ with $Ax = y \}$

[or $R(A) = \{$ linear combination of the columns of $A \}$ = column space]

take $v, w \in R(A)$ generic, that is, there exist x, z so that

$$Ax = v \quad \text{and} \quad Az = w$$

Consider now $\alpha \in \mathbb{R}$ and

$\alpha v + w$, does it belong to $R(A)$?

well

$$\alpha v + w = \alpha Ax + Az = A(\alpha x + z)$$

Thus $\alpha v + w \in R(A)$., $R(A)$ subspace

Similarly $N(A) = \{ x \in \mathbb{R}^n / Ax = 0 \}$

take $v, w \in N(A)$, that is $Av = 0$ $Aw = 0$

what about $\alpha v + w$?

$$A(\alpha v + w) = \alpha Av + Aw = \alpha \cdot 0 + 0 = 0$$

$\Rightarrow \alpha v + w \in N(A)$. subspace.

(5)

(a)(b). $A = m \times n$

It holds that

$\dim R(A) + \dim N(A) = \#$ columns $= n$

so (b) true

$\dim R(A) = \#$ pivots
$\dim N(A) = \#$ free variables
} sum $= n$

(a) is false

For example $A = [2\ 2]$ it is 1×2

$n = 2$ columns $m = 1$

1 pivot $\dim R(A) = 1$
1 free variable $\dim N(A) = 1$

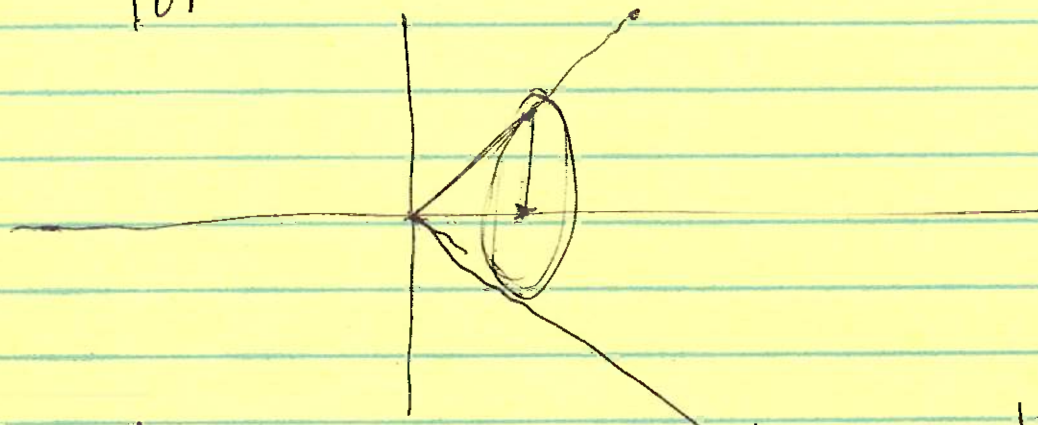
[in fact $N(A) = \left\{ \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$]

so that $\dim R(A) + \dim N(A) = 2 \neq 1.$

(c) False

consider for example in $\mathbb{R}^3$

$v = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ and $\theta = \pi/4$ 45°



the set is a "cone" with axis $\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$

and if you add two you fall outside the set.

For example $\begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$ and $\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}$

(d) True $S = \{x \in V / \langle x, v \rangle = 0\} = \{\alpha v\}^\perp$

take $z, w \in S$

$$\langle \alpha z + w, v \rangle = \alpha \langle z, v \rangle + \langle w, w \rangle$$
$$= \alpha \cdot 0 + 0 = 0$$

(e) False $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ y A singular

$A = L \quad U$

5. (a) A non singular $(n \times n)$

(i) Rank $A = n$

(ii) columns of A are linearly independent

(iii) rows of A are linearly independent

(iv) $N(A) = \{0\}$

(v) $Ax = 0 \Rightarrow x = 0$

(vi) No free variables

(vii) n pivots

(viii) for all b, $Ax = b$ has a unique solution

(b) let A $n \times n$ $A = [a_1, a_2, - a_n]$
and $a_i^T a_j = 0$

Orthogonal sets are linearly independent
thus by (ii) A non singular

(6) Observe if $n=2$

We have two 2×2 matrices of rank 2, both are nonsingular, and their product must be nonsingular

i.e. for $n=2$

if rank $A=2$ rank $B=2$
rank $AB=2$

So we need at least $n=3$

~~Again if~~ We need the rank AB to be "reduced".

For that to be the case we need some element in $R(B)$ to be in $N(A)$. For an example we can take a column of B in $N(A)$.

For example

$$A=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad B=\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

both have rank 2. last column of B is in $N(A)$

and $A \cdot B = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$ of rank 1.

(b) We explained some above, but in addition, we know

$$\text{Rank}(AB) = \text{Rank}(B) - \dim\left(R(A) \cap N(A)\right)$$

in this case $\text{Rank}(B) = 2$

$$\dim\left(R(A) \cap N(A)\right) = 1$$