TEST 1

1.1 - PROPERTIES OF PROB.

Prob. measure:

(1) $P(A) \geq 0$

(2) $P(S) = 1$

(3) If $A_1, A_2 \ldots$ are events and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

If $A_1, A_2 \ldots A_K$ are events and $A_i \cap A_j = \emptyset$ for $i \neq j$ then $P\left(\bigcup_{i=1}^{K} A_i\right) = \sum_{i=1}^{K} P(A_i)$

Let A, B be events
$A \cup B$ = A or B occurs
$A \cap B$ = A and B occurs
A^c = A does not occur
$A \setminus B = A \cap B^c$ = A occur but not B

Theorems:

(1) $P(A^c) = 1 - P(A)$

(2) $P(\emptyset) = 0$

(3) If $A \subseteq B$, $P(A) \leq P(B)$

(4) $P(A) \leq 1$

(5) Inclusion-Exclusion $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(6) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

1.2 - ENUMERATION

Population size = n Sample size = r

	Sampling w/o re	Sampling w/ re
Ordered Permutation	$nPr = \frac{n!}{(n-r)!}$	n^r
Unordered Combination	$nCr = \frac{n!}{r!(n-r)!}$	$\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{r} \binom{n}{k} a^k b^{n-k}$$

Proof →

$$(a+b)^n = (a+b)(a+b)\dots(a+b) = \sum x_1 x_2 \dots x_n, \quad x_1 = a, b \quad x_2 = a, b \dots$$

$$= \sum_{k=0}^{n} \sum a^k b^{n-k} \text{ exactly } x\text{'s are } a\text{'s} = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Pascal Triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Multinomial Theorem

$$(a_1 + a_2 + \dots + a_s)^n = \binom{n}{n_1, n_2 \dots n_s}$$

$$0 \leq n_1, n_2 \dots n_s \leq n$$

Proof →

$$(a_1 + a_2 + \dots + a_s)^n = \sum_{n_1 + n_2 + \dots + n_s = n} \binom{n}{n_1} \binom{n-n_1}{n_2} \dots \binom{n - n_1 - \dots - n_{s-1}}{n_s} a_1^{n_1} a_2^{n_2} \dots a_s^{n_s}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} \times \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \times \dots \times \frac{\cancel{(n-n_1-\dots-n_{s-1})!}}{n_s!\,\underbrace{(n-n_1-\dots-n_s)!}_{0!=1}}$$

$$= \frac{n!}{n_1!\, n_2! \dots n_s!} = \binom{n}{n_1, n_2 \dots n_s}$$

1.3 – CONDITIONAL PROB.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \xrightarrow{\text{Multiplication Rule}} P(A \cap B) = P(A).\,P(B|A) = P(B).\,P(A|B)$$

↓

$$P(A \cap B \cap C) = P(A).\,P(B|A).\,P(C|A \cap B)$$

Total prob. Theorem

$$P(A) = P(B_1).\,P(A|B_1) + \dots + P(B_n).\,P(A|B_n)$$

$\{B_1, B_2 \dots B_n\}$ be a partition of A & mutually exclusive

1.4 – INDEPENDENCE

$$P(A \cap B) = P(A).P(B) \overset{\text{OR}}{=\!=} P(B|A) = P(B),\ P(A) \neq 0$$

↳ (A^c, B^c) & (A, B^c) & (A^c, B) also independent

3 events
$$P(A \cap B) = P(A).P(B)$$
$$P(A \cap C) = P(A).P(C) \quad \& \quad P(A \cap B \cap C) = P(A).P(B).P(C)$$
$$P(B \cap C) = P(B).P(C)$$

$A^* = A$ or A^c

n events $P(A_{i_1}^* \cap A_{i_2}^* \ldots \cap A_{i_k}^*) = P(A_{i_1}^*).P(A_{i_2}^*)\ldots P(A_{i_k}^*)$ where $k = 1, 2 \ldots n$

$1 \leq i_1 < i_2 \ldots < i_k \leq n$

1.5 - BAYES THEOREM

$\{B_1, B_2 \ldots B_n\}$ be a partition of A

posterior probs

prior probs

$$P(B_i | A) = \frac{P(B_i).P(A|B_i)}{\sum_{j=0}^{n} P(B_j).P(A|B_j)}$$

RANDOM VARIABLES

	Discrete	Continuous
Distribution	pmf $f_X(i) = P(X=i)$	pdf $P(X \in B) = \int_B f(x)dx$ cdf $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$
Mean	$\mu_X = \sum x f(x)$ $E[g(x)] = \sum g(x)f(x)$	$\mu_X = \int_{-\infty}^{\infty} x f(x)dx$ $E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$

Variance $\sigma_X^2 = E[(X-\mu)^2] = E[X^2] - \mu^2$

cdf: $P(a \leq X \leq b) = F_X(b) - F_X(a)$ $\quad P(X < b) = F_X(b-)$

$F_X(a+) = \lim_{x \to a^+} F_X(x)$ $\quad P(X = b) =$ jump size at b

Median $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$

$F_X(a-) = \lim_{x \to a^-} F_X(x)$

Quantiles $P(X \leq x) \geq q$ and $P(X \geq x) \geq 1-q$

DISTRIBUTIONS

	Ber(p)	Bin(n,p)	Geom(p)	Negbin(r,p)	HG(N_1, N_2, n)	Poisson(λ)
pmf	$\begin{array}{c\|c\|c} x & 0 & 1 \\ \hline f(x) & q & p \end{array}$	$\binom{n}{k}p^k q^{n-k}$	$q^{k-1}p$	$\binom{k-1}{r-1}p^r q^{k-r}$	$\frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}}$	$\frac{\lambda^k}{k!}e^{-\lambda}$
μ_X	p	np	$\frac{1}{p}$	$\frac{r}{p}$	$\frac{nN_1}{N}$	λ
σ_X^2	pq	npq	$\frac{q}{p^2}$	$\frac{rq}{p^2}$	$n\cdot\frac{N_1}{N}\cdot\frac{N_2}{N}\cdot\frac{N-n}{N-1}$	λ
γ	$\frac{1-2p}{\sqrt{pq}}$	$\frac{1-2p}{\sqrt{npq}}$	$\frac{2-p}{\sqrt{1-p}}$	$\frac{2-p}{\sqrt{r(1-p)}}$		$\frac{1}{\sqrt{\lambda}}$
$M_X(t)$	$q+pe^t$	$(q+pe^t)^n$	$\frac{pe^t}{1-qe^t}$	$\left(\frac{pe^t}{1-qe^t}\right)^r$		$e^{\lambda(e^t-1)}$

1 $X \sim$ Bin(n,p) $\approx$ Poisson(np) if $np^2 < 0.05$, $n > 25$

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk}\binom{n}{k}p^k q^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(pe^t)^k q^{n-k} = (q+pe^t)^n$$

$$g(t) = \ln M_X(t) = \ln(q+pe^t)^n = n\ln(q+pe^t)$$

$$\mu_X = g'(0) = \left.\frac{npe^t}{q+pe^t}\right|_{t=0} = \frac{np}{q+p} = np$$

$$\sigma_X^2 = g''(0) = n\left.\left(\frac{pe^t}{q+pe^t} - \frac{(pe^t)^2}{(q+pe^t)^2}\right)\right|_{t=0} = n\left(\frac{p}{q+p} - \frac{p^2}{(q+p)^2}\right) = npq$$

$$\gamma_X = \frac{g'''(0)}{\sigma_X^3} = \frac{np(1-p)(1-2p)}{(npq)^{3/2}} = \frac{npq(1-2p)}{(npq)^{3/2}} = \frac{1-2p}{\sqrt{npq}}$$

2 $X \sim$ Geom(p)

$$M_X(t) = E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} f(k) = \sum_{k=1}^{\infty} e^{tk}q^{k-1}p = pe^t\sum_{k=1}^{\infty} e^{t(k-1)}q^{k-1}$$

$$= pe^t\sum_{k=1}^{\infty}(e^t q)^{k-1} \overset{s=k-1}{=} pe^t\sum_{s=0}^{\infty}(e^t q)^s = \frac{pe^t}{1-qe^t}$$

$$g(t) = \ln M_X(t) = \ln\frac{pe^t}{1-qe^t} = \ln p + t - \ln(1-qe^t)$$

$$\mu_X = g'(0) = \left(1 - \frac{-qe^t}{1-qe^t}\right)\Big|_{t=0} = 1 + \frac{q}{1-q} = 1 + \frac{q}{p} = \frac{p+q}{p} = \frac{1}{p}$$

$$\sigma_X^2 = g''(0) = \frac{qe^t}{1-qe^t} + \frac{q^2e^{2t}}{(1-qe^t)^2}\Big|_{t=0} = \frac{q}{1-q} + \frac{q^2}{(1-q)^2} = \frac{q}{p} + \frac{q^2}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

$$\gamma_X = \frac{g'''(0)}{\sigma_X^3} = \frac{q(1+q)}{p^3} \times \left(\frac{q}{p^2}\right)^{-3/2} = \frac{1+q}{\sqrt{q}} = \frac{1+(1-q)}{\sqrt{1-q}} = \frac{2-q}{\sqrt{1-q}}$$

3 $X \sim \text{Negbin}(r, p) \longrightarrow Y \sim \text{Geom}(p)$

$$M_X(t) = \sum_{k=r}^{\infty} e^{tk}\binom{k-1}{r-1} p^r q^{k-r} = p^r \sum_{k=r}^{\infty} e^{(k-r)t} e^{rt} \binom{k-1}{r-1} q^{k-r}$$

$$= (pe^t)^r \sum_{k=r}^{\infty} \binom{k-1}{r-1} (qe^t)^{k-r} = (pe^t)^r \frac{1}{(1-qe^t)^r} = \left(\frac{pe^t}{1-qe^t}\right)^r$$

$$g_X(t) = \ln M_X(t) = r\ln\left(\frac{pe^t}{1-qe^t}\right) = r\ln M_Y(t) = r g_Y(t)$$

$$g_X'(0) = r g_Y'(0) \longrightarrow \mu_X = r\mu_Y = r \times \frac{1}{p} = \frac{r}{p}$$

$$g_X''(0) = r g_Y''(0) \longrightarrow \sigma_X^2 = r\sigma_Y^2 = r \times \frac{q}{p^2} = \frac{rq}{p^2}$$

$$g_X'''(0) = r g_Y'''(0) \longrightarrow \gamma_X = \frac{r g_Y'''(0)}{\sigma_X^3} = \frac{r g_Y'''(0)}{(r\sigma_Y)^{3/2}} = \frac{1}{\sqrt{r}}\gamma_Y$$

4 $X \sim HG(N_1, N_2, n)$

$$\mu_X = E[X] = \sum_{k=0}^{N_1} k \frac{\binom{N_1}{k}\binom{N_2}{n-k}}{\binom{N}{n}} = \sum_{k=1}^{N_1} \frac{\frac{N_1!}{(k-1)!(N_1-k)!} \times \frac{N_2!}{(n-k)!(N_2-n+k)!}}{\frac{N!}{n!(N-n)!}}$$

$s = k-1$

$k = s+1$

$$= N_1 \sum_{s=0}^{N_1-1} \frac{\frac{(N_1-1)!}{s!(N_1-1-s)!} \times \frac{N_2!}{(n-1-(k-1))!\,(N_2-(n-1)+(k-1))!}}{\frac{N!}{((N-1)-(n-1))!\,(n-1)!\,n}}$$

$$= \frac{nN_1}{N} \underbrace{\sum_{s=0}^{N_1-1} \frac{\frac{(N_1-1)!}{s!(N_1-1-s)!} \times \frac{N_2!}{(n-1-s)!\,(N_2-(n-1)+s)!}}{\frac{(N-1)!}{((N-1)-(n-1))!\,(n-1)!}}}_{HG(N_1-1,\, N_2,\, n-1)} = \frac{nN_1}{N}$$

$$E[X^2] = E[X(X-1)] + E[X]$$

$$E[X(X-1)] = \sum_{k=0}^{N_1} k(\cancel{k-1}) \frac{\frac{N_1!}{(N_1-k)!\cancel{k!}} \times \frac{N_2!}{(N_2-n+k)!(n-k)!}}{\frac{N!}{(N-n)!\,n!}}$$

$$= \sum_{k=2}^{N_1} \frac{\frac{N_1!}{(N_1-k)!(k-2)!} \times \frac{N_2!}{(N_2-n+k)!(n-k)!}}{\frac{N!}{(N-n)!\,n!}}$$

$s = k-2$

$k = s+2$

$$= \sum_{s=0}^{N_1-2} \frac{\frac{N_1(N_1-1)(N_1-2)!}{(N_1-s-2)!\,s!} \times \frac{N_2!}{(N_2-(n-2)+s)!(n-2-s)!}}{\frac{N(N-1)(N-2)!}{(N-2-(n-2))!(n-2)!\,n(n-1)}}$$

$$= n(n-1)\frac{N_1(N_1-1)}{N(N-1)} \underbrace{\sum_{s=0}^{N_1-2} \frac{\binom{N_1-2}{s}\binom{N_2}{n-2-s}}{\binom{N-2}{n-2}}}_{HG(N_1-2,\, N_2,\, n-2)} = n(n-1)\frac{N_1(N_1-1)}{N(N-1)}$$

$$\sigma^2 = E[X^2] - \mu_X^2 = E[X(X-1)] + \mu_X - \mu_X^2$$

$$= n(n-1)\frac{N_1(N_1-1)}{N(N-1)} + n\frac{N_1}{N} - n^2\frac{N_1^2}{N^2}$$

$$= n\frac{N_1}{N}\left(\frac{(n-1)(N_1-1)}{N-1} + 1 - n\frac{N_1}{N}\right)$$

$$= n\frac{N_1}{N} \times \frac{N(n-1)(N_1-1) + N(N-1) - nN_1(N-1)}{N(N-1)}$$

$$= n\frac{N_1}{N} \times \frac{\cancel{nN_1N} - nN - N_1N + \cancel{N} + N^2 - \cancel{N} - \cancel{nN_1N} + nN_1}{N(N-1)}$$

$$= n\frac{N_1}{N} \times \frac{1}{N(N-1)}\left(\underbrace{n(N_1-N)}_{-nN_2} + \underbrace{N(N-N_1)}_{NN_2}\right) = \boxed{n\frac{N_1}{N}\frac{N_2}{N}\cdot\frac{N-n}{N-1}}$$

$$-nN_2 + NN_2 = N_2(N-n)$$

$n=1 : \sigma^2 = npq$

$n=n : \sigma^2 = 0$

5 $X \sim \text{Poisson}(\lambda) \longrightarrow$ Poisson process $N_I \sim \text{Poisson}(\lambda . |I|)$

$$M_X(t) = E[e^{tX}] = \sum_{K=1}^{\infty} e^{tK} \frac{\lambda^K}{K!} e^{-\lambda} = \left(\sum_{K=1}^{\infty} \frac{(e^t\lambda)^K}{K!}\right) e^{-\lambda} = e^{e^t\lambda} \cdot e^{-\lambda} = e^{\lambda(e^t-1)}$$

$$\sum_{K=0}^{\infty} \frac{x^K}{K!} = e^x$$

$$g(t) = \ln M_X(t) = \lambda(e^t - 1)$$

$$g'(t) = \lambda e^t \longrightarrow \mu = g'(0) = \lambda$$
$$g''(t) = \lambda e^t \longrightarrow \sigma^2 = g''(0) = \lambda \longrightarrow \sigma = \sqrt{\lambda}$$
$$g'''(t) = \lambda e^t \longrightarrow E[(X-\mu)^3] = g'''(0) = \lambda \longrightarrow \gamma = \frac{g'''(0)}{\sigma^3} = \frac{\lambda}{(\sqrt{\lambda})^3} = \frac{1}{\sqrt{\lambda}}$$

PROPERTIES

Moment Generating Function

$$M_X(t) = E[e^{tX}] \longrightarrow E[X^n] = M_X^{(n)}(0)$$

Proof $\rightarrow$

① $M_X'(t) = \frac{d}{dt} E[e^{tX}] = E[Xe^{tX}] \longrightarrow M_X''(t) = E[X^2 e^{tX}]$

$\therefore M_X^{(n)}(t) = E[X^n e^{tX}] \quad \therefore M_X^{(n)}(0) = E[X^n]$

Taylor Expansion

② $M_X(t) = E[e^{tX}] = E\left[\sum_{n=0}^{\infty} \frac{(tX)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n$

$\therefore E[X^n] = M_X^{(n)}(0)$ by Taylor series expansion

Skewness

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3} \longrightarrow = E[X^3] - 3\mu\sigma^2 - \mu^3$$

Newton's Binomial Theorem

$|w| < 1$

① $\frac{1}{(1-w)^r} = \sum_{K=0}^{\infty} \binom{r+K-1}{r-1} w^K$ ② $\frac{1}{(1-w)^r} = \sum_{s=r}^{\infty} \binom{s-1}{r-1} w^{s-r}$

$$(b+1)^n = \sum_{K=0}^{n} \binom{n}{K} b^K = \sum_{K=0}^{n} \frac{n!}{K!(n-K)!} b^K = \sum_{K=0}^{n} \frac{n(n-1)\dots(n-K+1)}{K!} b^K$$

Proof $\rightarrow$

$$\therefore (1-w)^{-r} = \sum_{K=0}^{n} \frac{(-r)(-r-1)\dots(-r-K+1)}{K!} (-w)^K$$
$$= (-1)^K (-1)^K \sum_{K=0}^{n} \frac{r(r+1)\dots(r+K-1)}{K!} w^K = \sum_{K=0}^{n} \frac{(r+K-1)!}{(r-1)!\,K!} w^K$$

$$= \sum_{k=0}^{n} \binom{r+k-1}{r-1} w^k \overset{s=k+r}{=\!=} \sum_{s=r}^{\infty} \binom{s-1}{r-1} w^{s-r}$$

Poisson Theorem

$S_n \sim Bin(n,p)$ & $p = \frac{\lambda}{n} \longrightarrow S_n \longrightarrow X \sim Poisson(\lambda),\ n \to \infty$

Proof $\rightarrow$

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \cdot \lambda^k \cdot \frac{1}{n^k} \cdot \left(1-\frac{\lambda}{n}\right)^n \cdot \frac{1}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n \cdot \cancelto{1}{\frac{n}{n}} \times \cancelto{1}{\frac{n-1}{n}} \times \dots \times \cancelto{1}{\frac{n-k+1}{n}} \cdot \cancelto{1}{\frac{1}{\left(1-\frac{\lambda}{n}\right)^k}} \overset{n\to\infty}{=} \frac{\lambda^k}{k!} e^{-\lambda},\ k = 0, 1, 2 \dots$$

TEST 2

3.2-EXPONENTIAL, GAMMA, CHI-SQUARE

Gamma(α, λ) $\boxed{\dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}}$ $\xrightarrow{\theta = 1/\lambda}$ Gamma(α, θ) $\boxed{\dfrac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)}}$, $x \geq 0$

Properties $\rightarrow$

(1) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

(2) $\Gamma(n) = (n-1)!$

(3) $\Gamma(1) = 1$

(4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof $\rightarrow$

Step 1 : cdf of T_n

$$F_{T_n}(t) = P(T_n \leq t) = 1 - P(T_n > t) = 1 - P(N_{[0,t]} \leq n-1) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Step 2 : pdf of T_n

$$f_{T_n}(t) = \frac{d}{dt} F_{T_n}(t) = -\sum_{k=0}^{n-1} \frac{k}{k!} (\lambda t)^{k-1} \lambda e^{-\lambda t} - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} (-\lambda) e^{-\lambda t}$$

$$= \sum_{k=0}^{n-1} \frac{t^k \lambda^{k+1}}{k!} e^{-\lambda t} - \sum_{k=1}^{n-1} \frac{t^{k-1} \lambda^k}{(k-1)!} e^{-\lambda t}$$

$$= \sum_{k=0}^{n-1} \frac{t^k \lambda^{k+1}}{k!} e^{-\lambda t} - \sum_{s=0}^{n-2} \frac{t^s \lambda^{s+1}}{s!} e^{-\lambda t} = \frac{t^{n-1} \lambda^n}{(n-1)!} e^{-\lambda t} \rightarrow \text{Gamma}(n, \lambda)$$

Properties →

$$M_X(t)=E[e^{tX}]=\int_0^\infty e^{tx}\frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha}e^{\frac{-x}{\theta}}dx=\frac{\theta'^\alpha}{\theta^\alpha}\int_0^\infty\frac{x^{\alpha-1}}{\Gamma(\alpha)\theta'^\alpha}e^{-\boxed{\left(\frac{1}{\theta}-t\right)}x}dx \qquad \frac{1}{\theta'} = \left(\frac{1}{\theta}-t\right)$$

$$=\left(\frac{\theta'}{\theta}\right)^\alpha\int_0^\infty \text{Gamma}(\alpha,\theta')dx=\left(\frac{1}{\frac{1}{\theta}-t}\times\frac{1}{\theta}\right)^\alpha=\frac{1}{(1-\theta t)^\alpha},\ t<\frac{1}{\theta}$$

$$g(t)=\ln M_X(t)=\alpha\ln\left(\frac{1}{1-\theta t}\right)=-\alpha\ln(1-\theta t)$$

$$g'(t)=-\alpha\frac{-\theta}{1-\theta t}=\alpha\theta(1-\theta t)^{-1}\rightarrow \mu_X=g'(0)=\alpha\theta$$

$$g''(t)=\alpha\theta^2(1-\theta t)^{-2} \rightarrow \sigma_X^2=g''(0)=\alpha\theta^2$$

$$g'''(t)=2\alpha\theta^3(1-\theta t)^{-3} \rightarrow \gamma_X=\frac{g'''(0)}{\sigma_X^3}=\frac{2\alpha\theta^3}{(\alpha\theta^2)^{3/2}}=\frac{2}{\sqrt{\alpha}}>0$$

Exp(λ) $\lambda e^{-\lambda x}$ $M_X(t)=\frac{\lambda}{\lambda-t}$, $\mu_X=\frac{1}{\lambda}$, $\sigma_X^2=\frac{1}{\lambda^2}$

Exp(θ) $\frac{1}{\theta}e^{-x/\theta}$ $M_X(t)=\frac{1}{1-\theta t}$, $\mu_X=\theta$, $\sigma_X^2=\theta^2$

Chi-Square $\chi^2(r)=\text{Gamma}(\frac{r}{2},\theta=2)$

$\mu_X=\alpha\theta=r$ $\sigma_X^2=\alpha\theta^2=2r$ $\gamma_X=\frac{2\sqrt{2}}{\sqrt{r}}$ $M_X(t)=\frac{1}{(1-2t)^{r/2}}$

Theorem → If $X\sim \text{Gamma}(\alpha,\theta)$ then $Y=\frac{2X}{\theta}\sim\chi^2(2r)$

Proof →

$$M_Y(t)=E[e^{tY}]=E[e^{\frac{t2X}{\theta}}]=E[e^{\left(\frac{2t}{\theta}\right)X}]=\frac{1}{(1-\theta.\frac{2t}{\theta})^\alpha}=\frac{1}{(1-2t)^\alpha}$$

$$=\frac{1}{(1-2t)^{\underset{\text{new } r}{\underbrace{2\alpha}}/2}}\sim\chi^2(2\alpha)$$

Q.E.D!

3.3 - NORMAL DISTRIBUTION

$X\sim N(\mu,\sigma^2)$ $\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $M_X(t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$, $\gamma=0$

Proof

$$M_X(t)=\int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

$u=\frac{x-\mu}{\sigma}$

$$=\int_{-\infty}^{\infty} e^{t(\mu+\sigma u)}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}u^2}du$$

$du=\frac{dx}{\sigma}$

1/2 times squared

$x=\mu+\sigma u$

$$=e^{t\mu}\int_{-\infty}^{\infty}\underbrace{e^{\boxed{t\sigma}u}\frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}u^2}}_{N(0,1)}du=e^{t\mu}e^{\frac{1}{2}t^2\sigma^2}=e^{\mu t+\frac{1}{2}\sigma^2t^2},\ t\in\mathbb{R}$$

Q.E.D!

Theorem $Z\sim N(0,1)\rightarrow E[Z^n]=\begin{cases}0 & \text{if } n \text{ odd}\\ (2k-1)!! & \text{if } n=2k \text{ even}\end{cases}$

Proof

$$M_Z(t)=e^{1/2t^2}=\sum_{k=0}^{\infty}\frac{(\frac{1}{2}t^2)^k}{k!}=\sum_{k=0}^{\infty}\boxed{\frac{t^{2k}}{k!\,2^k}}$$

also

$$=\sum_{n=0}^{\infty}\boxed{\frac{E[Z^n]}{n!}}t^n$$

compare

$\therefore E[Z^n]=0$ if n odd

$$\therefore \frac{E[Z^{2k}]}{(2k)!}=\frac{1}{k!\,2^k}\longrightarrow E[Z^{2k}]=\frac{(2k)!}{k!\,2^k}=\frac{2^k k!\,(2k-1)!!}{k!\,2^k}=(2k-1)!!$$

Q.E.D!

Theorem $Z\sim N(0,1)$ then $V=Z^2\sim\chi^2(1)$

Proof (1) Find cdf of V

$$F_V(v)=P(V\le v)=P(Z^2\le v)=P(-\sqrt{v}\le Z\le\sqrt{v})=F_Z(\sqrt{v})-F_Z(-\sqrt{v})$$

(2) Find pdf of V

$$f_V(v)=\frac{d}{dv}F_V(v)=\frac{d}{dv}\left(F_Z(\sqrt{v})-F_Z(-\sqrt{v})\right)$$

$$=\frac{1}{\sqrt{2\pi}}e^{-1/2v}\cdot\frac{1}{2\sqrt{v}}-\frac{1}{\sqrt{2\pi}}e^{-1/2v}\cdot\frac{-1}{2\sqrt{v}}$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{v}}e^{-1/2v},\ v>0=\frac{v^{1/2-1}}{\underbrace{\Gamma(\frac{1}{2})}_{\sqrt{\pi}}2^{1/2}}e^{-v/2},\ v>0\sim\chi^2(1)$$

Q.E.D!

4.1 - BIVARIATE DISTRIBUTION

Def $\rightarrow$ $f(x,y) = P(X=x, Y=y)$

Marginal $\rightarrow$ $f_X(x) = \sum_y f(x,y)$ $\quad$ $f_Y(y) = \sum_x f(x,y)$

4.2 - CORRELATION COEFFICIENT

$$\boxed{E[u(X,Y)] = \sum_{x,y} u(x,y) f(x,y)} \qquad \boxed{\mu_X = E[X] = \sum_{x,y} x f(x,y) = \sum_x x f_X(x)}$$

Proof $\rightarrow$

$$\mu_X = E[X] = \sum_{x,y} x f(x,y) = \sum_x \sum_y x f(x,y) = \sum_x x \sum_y y f(x,y) = \sum_x x f_X(x)$$

$$\mu_Y = E[Y] = \sum_{x,y} y f(x,y) = \sum_y y f_Y(y)$$

$$\sigma_X^2 = E[(X-\mu_X)^2] = E[X^2] - \mu_X^2 = \sum_x x^2 f_X(x) - \mu_X^2$$

$$\sigma_Y^2 = \sum_y y^2 f_Y(y) - \mu_Y^2$$

Covariance $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y = \boxed{\sum_{x,y} xy f(x,y) - \mu_X\mu_Y}$

Covariance Coefficient $\boxed{\rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}}$

(a) $Cov(X,Y) = E[XY] - \mu_X\mu_Y = \rho\sigma_X\sigma_Y$

(b) $E[XY] = \mu_X\mu_Y + \rho\sigma_X\sigma_Y$

① Find best-fit line

$$\sum_i (y - y_i)^2 f(x_i, y_i) = \sum_i [(ax_i + b) - y_i)^2] f(x_i, y_i) = E[(aX + b - Y)^2] \overset{set}{=} K(a,b)$$

$$0 = \frac{\partial K(a,b)}{\partial a} = E[2(aX + b - Y)X] = E[(aX + b - Y)X]$$

$$0 = \frac{\partial K(a,b)}{\partial b} = E[2(aX + b - Y)] = E[(aX + b - Y)]$$

$\therefore 0 = aE[X^2] + b\mu_X - E[XY]$

$0 = a\mu_X + b - \mu_Y = a\mu_X^2 + b\mu_X - \mu_X\mu_Y$ subtract

μ_X

$\therefore 0 = a\sigma_X^2 - C_{XY} \rightarrow a = \frac{C_{XY}}{\sigma_X^2} \therefore 0 = \frac{C_{XY}}{\sigma_X^2}\mu_X + b - \mu_Y \rightarrow b = \mu_Y - \frac{C_{XY}}{\sigma_X^2}\mu_X$

$\therefore y = ax + b = \frac{C_{XY}}{\sigma_X^2}x + \mu_Y - \frac{C_{XY}}{\sigma_X^2}\mu_X = \frac{C_{XY}}{\sigma_X^2}(x - \mu_X) + \mu_Y = \boxed{\rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) + \mu_Y}$

2 K(a,b) at minimum

$K(a,b) = E[(aX + b - Y)^2] = E\left\{\left[\rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) + \mu_Y - Y\right]^2\right\}$

$= \sigma_Y^2 E\left\{\left[\rho\left(\frac{X - \mu_X}{\sigma_X}\right) - \left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right]^2\right\}$

$\tilde{X}$ $\tilde{Y}$ normalized $\rightarrow N(0,1)$

$= \sigma_Y^2 E[(\rho\tilde{X} - \tilde{Y})^2]$

$= \sigma_Y^2 E[\rho^2\tilde{X}^2 - 2\rho\tilde{X}\tilde{Y} + \tilde{Y}^2] = \sigma_Y^2(\rho^2 \cdot 1 - 2\rho \cdot \rho + 1) = \boxed{\sigma_Y^2(1 - \rho^2)}$

Properties $\rightarrow$ $E\left[\frac{X - \mu_X}{\sigma_X}\right] = 0$ $\quad E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^2\right] = 1$ $\quad E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right] = \rho$

(a) If X, Y independent then $\rho = 0$
(b) $-1 \leq \rho \leq 1$

Proof $\rightarrow$ (a) $\rho = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X\sigma_Y} = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$

$E[XY] = \sum_{x,y} xy\, f(x,y) \overset{ind}{=} \sum_{x,y} xy\, f_X(x) f_Y(y) = \sum_x x f_X(x) \sum_y y f_Y(y) = \mu_X\mu_Y$

$\therefore C_{XY} = 0 \quad \therefore \rho = 0$

(b) Step 1 : Show that $|E[XY]| \leq \sqrt{E[X^2]}\sqrt{E[Y^2]}$ Cauchy-Schwarz

$0 \leq E[(\alpha X + Y)^2] = E[\alpha^2 X^2 + 2\alpha XY + Y^2] = \alpha^2 E[X^2] + 2\alpha E[XY] + E[Y^2]$

or $0 \leq a\alpha^2 + b\alpha + c$ for all $\alpha \in \mathbb{R}$ either 1 or no solution

$\hookrightarrow b^2 - 4ac = 4E[XY]^2 - 4E[X^2]E[Y^2] \leq 0$

$E[XY]^2 \leq E[X^2]E[Y^2]$

$|E[XY]| \leq \sqrt{E[X^2]}\sqrt{E[Y^2]}$

Step 2: Apply to $X-\mu_X$ & $Y-\mu_Y$

$$\therefore \left|E[(X-\mu_X)(Y-\mu_Y)]\right| \le \sqrt{E[(X-\mu_X)^2]}\sqrt{E[(Y-\mu_Y)^2]}$$

$$|Cov(X,Y)| \le \sigma_X \sigma_Y$$

$$\therefore |\rho| = \left|\frac{Cov(X,Y)}{\sigma_X\sigma_Y}\right| \le 1 \longrightarrow -1 \le \rho \le 1$$ Q.E.D!

4.3 - CONDITIONAL DISTRIBUTION

$$\boxed{g(x|y) = \frac{f(x,y)}{f_Y(y)}} = \frac{P(X=x, Y=y)}{P(Y=y)} = P(X=x \mid Y=y)$$

fn of x, y fixed

$$\boxed{h(y|x) = \frac{f(x,y)}{f_X(x)}} = \frac{P(Y=y, X=x)}{P(X=x)} = P(Y=y \mid X=x)$$

fn of y, y fixed

(1) $\mu_{X|Y} = E[X \mid Y=y] = \sum_x x\,g(x|y)$

(2) $\sigma^2_{X|Y} = \sum_x (x-\mu_{X|Y})^2 g(x|y) = \sum_x x^2 g(x|y) - \mu^2_{X|Y} = E[X^2 \mid Y=y] - (E[X \mid Y=y])^2$

Theorem $\rightarrow$ Suppose $\mu_{Y|X} = ax+b$ then $\mu_{Y|X} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$

Proof $\rightarrow$ $E[Y \mid X=x] = \sum_y y\,h(y|x) = \sum_y y\frac{f(x,y)}{f_X(x)} = ax+b$

$$\therefore \sum_y y f(x,y) = (ax+b) f_X(x)$$

sum x

$$\therefore \sum_x \sum_y y f(x,y) = \sum_x (ax+b) f_X(x)$$

$$\therefore \sum_y y \sum_x f(x,y) = \sum_x a x f_X(x) + \sum_x b f_X(x)$$

$$\therefore \sum_y y f_Y(y) = a\mu_X + b \longrightarrow \mu_Y = a\mu_X + b \longrightarrow b = \mu_Y - a\mu_X$$

x

$$\therefore \sum_x \sum_y x y f(x,y) = \sum_x (ax^2 + bx) f_X(x)$$

$\therefore\ E[XY] = a\sum_x x^2 f_X(x) + b\sum_x x f_X(x)$

$\therefore\ E[XY] = aE[X^2] + b\mu_X$

$\therefore\ \mu_X\mu_Y + \rho\sigma_X\sigma_Y = aE[X^2] + b\mu_X = aE[X^2] + (\mu_Y - a\mu_X)\mu_X$

$= aE[X^2] + \mu_X\mu_Y - a\mu_X^2$

$= a\sigma_X^2 + \mu_X\mu_Y$

$$\left.\begin{aligned} \therefore a &= \frac{\rho\sigma_X\sigma_Y}{\sigma_X^2} = \frac{\rho\sigma_Y}{\sigma_X} \\ b &= \mu_Y - \frac{\rho\sigma_Y}{\sigma_X}\mu_X = \mu_Y - \mu_X\frac{\rho\sigma_Y}{\sigma_X} \end{aligned}\right\} ax+b = \frac{\rho\sigma_Y}{\sigma_X}x + \mu_Y - \mu_X\frac{\rho\sigma_Y}{\sigma_X}$$

$= \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X)$ Q.E.D!

Known

$$\boxed{\begin{aligned} &(1)\ E[u(X,Y)\mid X=x] = \sum_y u(x,y)\,h(y\mid x) \\ &(2)\ E[u(X,Y)\mid Y=y] = \sum_x u(x,y)\,g(x\mid y) \end{aligned}}$$

Unknown

$$\boxed{\begin{aligned} &(1)\ E[\underbrace{E[u(X,Y)\mid X]}_{\text{fn of } X}] = E[u(X,Y)] \text{ or } E[\underbrace{E[u(X,Y)\mid Y]}_{\text{fn of } Y}] = E[u(X,Y)] \\ &(2)\ E[u(X)v(Y)\mid X] = u(X)E[v(Y)\mid X] \quad \therefore E[E[Y\mid X]] = E[Y] \end{aligned}}$$

Proof $\to$ (a) $E[E[u(X,Y)\mid X]] = E\left[\sum_y u(X,y)\,h(y\mid X)\right]$

$= \sum_x \sum_y u(x,y)\,h(y\mid x) f_X(x)$

$= \sum_x \sum_y u(x,y)\frac{f(x,y)}{\cancel{f_X(x)}}\cancel{f_X(x)} = E[u(X,Y)]$ Q.E.D!

(b) Subtask: Prove that $E[u(X)v(Y)\mid X] = u(X)E[v(Y)\mid X]$ on the set that $X=x$

$\therefore$ On the set that $X=x$:

given x value, u(x) becomes a const

LHS $= E[u(X)v(Y)\mid X] = E[u(x)v(Y)\mid X=x]$ $\therefore$ could factor out

$= u(x)E[v(Y)\mid X=x] =$ RHS at $X=x$ Q.E.D!

$V \quad = \quad EV + VE$

$\boxed{E[Y] = E[E[Y|X]]}$ $\boxed{Var(Y) = E[Var(Y|X)] + Var(E[Y|X])}$

Proof $\rightarrow$ $RHS = E[Var(Y|X)] + Var(E[Y|X])$

$= E[E[Y^2|X] - \cancel{E[Y|X]^2}] + E[\cancel{E[Y|X]^2}] - E[E[Y|X]]^2$

$= E[Y^2] - E[Y]^2 = Var(Y)$ Q.E.D!

4.4 - BIVARIATE DISTRIBUTION

Marginal $\rightarrow$ $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ $\quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

4.5 - BIVARIATE NORMAL DISTRIBUTION

$$\boxed{f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left\{\underbrace{\left(\frac{x-\mu_X}{\sigma_X}\right)}_{\tilde{x}}^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \underbrace{\left(\frac{y-\mu_Y}{\sigma_Y}\right)}_{\tilde{y}}^2\right\}}}$$

$\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$, $-1 < \rho < 1$ $\qquad -\infty < x, y < \infty$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\{\tilde{x}^2 - 2\rho\tilde{x}\tilde{y} + \tilde{y}^2\}}$$

Properties $\rightarrow$

(a) $f_X(x) \sim N(\mu_X, \sigma_X^2) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-1}{2\sigma_X^2}(x-\mu_X)^2}$, $x \in \mathbb{R}$

(b) $f_Y(y) \sim N(\mu_Y, \sigma_Y^2)$

(c) $f_{Y|X}(y|x) \sim N(\mu_{Y|X}, \sigma_{Y|X}^2) \rightarrow \mu_{Y|X} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$\sigma_{Y|X}^2 = \sigma_Y^2(1-\rho^2)$ $\qquad \rho_{XY} = \rho$

Proof $\rightarrow$ (a) $f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\{\tilde{x}^2 - 2\rho\tilde{x}\tilde{y} + \tilde{y}^2\}}$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2}\cdot\frac{1}{1-\rho^2}\left\{(\tilde{y}-\rho\tilde{x})^2-\rho^2\tilde{x}^2+\tilde{x}^2\right\}}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2}\cdot\frac{1}{1-\rho^2}\left\{\left(\frac{y-\mu_Y}{\sigma_Y}-\rho\frac{x-\mu_X}{\sigma_X}\cdot\frac{\sigma_Y}{\sigma_Y}\right)^2+(1-\rho^2)\tilde{x}^2\right\}}$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2}\cdot\frac{1}{1-\rho^2}\cdot\frac{1}{\sigma_Y^2}\left\{\underbrace{y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)}_{m\,=\,\text{regression line}}\right\}^2-\frac{1}{2}\cdot\frac{(1-\rho^2)\tilde{x}^2}{1-\rho^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-1}{2}\tilde{x}^2} \underbrace{\frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2}\cdot\frac{1}{1-\rho^2}\cdot\frac{1}{\sigma_Y^2}(y-m)^2}}_{Y\sim N(m,\,\sigma_Y^2(1-\rho^2))}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-1}{2}\tilde{x}^2}$$

$$\therefore f_X(x) = \int_y f(x,y)\,dy = \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-1}{2}\tilde{x}^2} = \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \sim N(\mu_X, \sigma_X^2)$$ Q.E.D!

(b) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)} \sim N(m, \sigma_Y^2(1-\rho^2)),\ m = \mu_Y + \rho\dfrac{\sigma_Y}{\sigma_X}(x-\mu_X)$ Q.E.D!

(c) $E[XY] = E[E[XY|X]] = E[XE[Y|X]]$

$$= E\left\{X\left[\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)\right]\right\}$$

$$= \mu_Y\mu_X + \rho\frac{\sigma_Y}{\sigma_X}(\sigma_X^2 + \cancel{\mu_X^2} - \cancel{\mu_X^2}) = \mu_Y\mu_X + \rho\sigma_Y\sigma_X$$

$\therefore C_{X,Y} = E[XY] - \mu_X\mu_Y = \rho\sigma_Y\sigma_X \quad \therefore \rho_{X,Y} = \dfrac{C_{X,Y}}{\sigma_X\sigma_Y} = \dfrac{\rho\sigma_X\sigma_Y}{\sigma_X\sigma_Y} = \rho$ Q.E.D!

Theorem →

> Let X, Y have a bivariate normal distribution then
> X, Y are independent $\leftrightarrow \rho = 0$

Def →

> Let X, Y be any random variables
> X, Y are uncorrelated if $\rho_{XY} = 0$

X, Y independent $\longrightarrow$ X, Y uncorrelated

$\not\leftarrow$ not other way

5.1 - FUNTION OF RANDOM VARIABLE

cdf → $F_Y(y) = P(Y \le y) = \ldots \rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \ldots$

Change of variable → Suppose $y = g(x) \rightarrow x = g^{-1}(y)$

Given $X \sim f_X(x)$. Let $Y = g(X)$ then $\boxed{f_Y(y) = \sum_{x: g(x) = y} f_X(x) \frac{1}{|g'(x)|} \Bigg|_{x = g^{-1}(y)}}$

* 5.2 - JACOBIAN

$$f_Y(y_1, y_2) = f_X(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right|$$

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \det \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

need $x_1 = \ldots$ + range of Y_1, Y_2

$x_2 = \ldots$

5.3 - INDEPENDENT R.V

$$Y = \sum_{i=1}^{n} a_i X_i$$

(a) $\mu_Y = \sum_{i=1}^{n} a_i \mu_i$

Proof → $\mu_Y = E[Y] = E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu_i$ Q.E.D!

(b) If $X_1 \ldots X_n$ are inde, $\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Proof → $\sigma_Y^2 = E[(Y - \mu_Y)^2] = E[(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \mu_i)^2]$

$= E[(\sum_{i=1}^{n} a_i (X_i - \mu_i))^2] = \sum_{i,j=1}^{n} a_i a_j E[(X_i - \mu_i)(X_j - \mu_j)] = \sum_{i,j=1}^{n} a_i a_j C_{ij}$

$= \sum_{i=1}^{n} a_i^2 \sigma_i^2 + \sum_{i \neq j} a_i a_j C_{ij} \xrightarrow[C_{ij} = 0]{\text{if } i,j \text{ inde}} \sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$ Q.E.D!

same as $E[(X_i - \mu_i)^2] = \sigma_i^2$

Def → Sample mean $\bar{X} = \frac{X_1 + \ldots + X_n}{n}$ Sample variance $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

Properties → (a) $\mu_{\bar{X}} = \mu_X$ (b) $\sigma_{\bar{X}}^2 = \frac{\sigma_X^2}{n}$ (c) $E[S^2] = \sigma_X^2$

Proof → (a) $\mu_{\bar{X}} = E[\bar{X}] = E\left[\frac{X_1 + \ldots + X_n}{n}\right] = \frac{1}{n}\left(E[X_1] + \ldots + E[X_n]\right) = \frac{\mu_{X_1} + \ldots + \mu_{X_n}}{n} = \mu_X$

(b) $\sigma_{\bar{X}}^2 = E[(\bar{X} - \mu_{\bar{X}})^2] = E\left[\left(\frac{X_1 + \ldots + X_n}{n} - \mu_X\right)^2\right]$

(a)

$= E\left[\left(\frac{(X_1 - \mu_X) + \ldots + (X_n - \mu_X)}{n}\right)^2\right] = \frac{1}{n^2}\sum_{i,j=1}^{n} E[(X_i - \mu_X)(X_j - \mu_X)]$

$= \frac{1}{n^2}\sum_{i,j=1}^{n} E[(X_i - \mu_X)^2] = \frac{1}{n^2}(n\sigma_X^2) = \frac{\sigma_X^2}{n}$

(c) Show that $E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \sigma_X^2$

$n\sigma_X^2 = E\left[\sum_{i=1}^{n}(X_i - \mu_X)^2\right] = E\left[\sum_{i=1}^{n}(X_i - \bar{X} + \bar{X} - \mu_X)^2\right]$

$= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + 2\sum_{i=1}^{n}(X_i - \bar{X})(\bar{X} - \mu_X) + \sum_{i=1}^{n}(\bar{X} - \mu_X)^2\right]$

$= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + 2(\bar{X} - \mu_X)\sum_{i=1}^{n}(X_i - \bar{X}) + \sum_{i=1}^{n}(\bar{X} - \mu_X)^2\right]$

$= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + 2(\bar{X} - \mu_X)\left[\left(\sum_{i=1}^{n} X_i\right) - n\bar{X}\right]\right] + nE[(\bar{X} - \mu_X)^2]$

$= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + 2(\bar{X} - \mu_X)\underbrace{\left(X_1 + X_2 + \ldots + X_n - nX\frac{X_1 + \ldots + X_n}{n}\right)}_{0}\right] + n\frac{\sigma_X^2}{n}$

$= E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] + \sigma_X^2 \longrightarrow E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = n\sigma_X^2 - \sigma_X^2 = (n-1)\sigma_X^2$

$\therefore E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{\cancel{(n-1)}\sigma_X^2}{\cancel{n-1}} = \sigma_X^2 = E[S^2]$ Q.E.D!

*5.4 - M.G.F TECHNIQUE

Theorem → Let $Y = \sum_{i=1}^{n} a_i X_i$ then $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$

Proof $\rightarrow$ $M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} a_i X_i}] \overset{\text{inde.}}{=} E[\prod_{i=1}^{n} e^{t a_i X_i}] = \prod_{i=1}^{n} E[e^{(a_i t) X_i}]$

$= \prod_{i=1}^{n} M_{X_i}(a_i t)$ Q.E.D!

Properties $\rightarrow$ (a) $M_Y(t) = [M_X(t)]^n$ (b) $M_{\bar{X}}(t) = [M_X(\frac{t}{n})]^n$

Theorem $\rightarrow$ $X_1, X_2 \dots X_n$ be independent random variables & $Y = X_1 + X_2 + \dots + X_n$

(a) $X_i \sim Gamma(\alpha_i) \longrightarrow Y \sim Gamma(\alpha_1 + \alpha_2 + \dots + \alpha_n)$

(b) $X_i \sim \chi^2(r_i) \longrightarrow Y \sim \chi^2(r_1 + r_2 + \dots + r_n)$

(c) $X_i \sim Ber(p) \longrightarrow Y \sim Bin(n, p)$

(d) $X_i \sim N(\mu_i, \sigma_i^2) \longrightarrow W = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2(n)$

(e) $W = Z_1^2 + Z_2^2 + \dots + Z_n^2 \longrightarrow W \sim \chi^2(n)$

*5.5 - RANDOM NORMAL DISTRIBUTION

Theorem $\rightarrow$ $X_i \sim N(\mu_i, \sigma_i^2)$ $Y = \sum_{i=1}^{n} c_i X_i$ then

$\boxed{Y \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)}$ $\bar{X} \sim N(\mu_X, \frac{\sigma_X^2}{n})$

Proof $\rightarrow$ $M_Y(t) = E[e^{tY}] = E[e^{t\sum_{i=1}^{n} c_i X_i}] = E[\prod_{i=1}^{n} e^{c_i t X_i}] \overset{\text{inde}}{=} \prod_{i=1}^{n} E[e^{c_i t X_i}]$

$= \prod_{i=1}^{n} M_{X_i}(c_i t) = \prod_{i=1}^{n} e^{c_i \mu_i t + \frac{1}{2} c_i^2 \sigma_i^2 t^2}$ Q.E.D!

$= e^{\left(\sum_{i=1}^{n} c_i X_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} c_i^2 \sigma_i^2\right) t^2} \rightarrow Y \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right)$

Properties $\rightarrow$ (a) $\bar{X}$ and S^2 are independent (b) $\frac{(n-1)S^2}{\sigma_X^2} \sim \chi^2(n-1)$

(b) $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \rightarrow$ Recall: $\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2$

$$\therefore \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu_X}{\sigma_X}\right)^2}_{A\sim\chi^2(n)} = \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma_X}\right)^2}_{B} + \underbrace{\left(\frac{\overline{X}-\mu_X}{\sigma_X/\sqrt{n}}\right)^2}_{C\sim\chi^2(1)}$$

B, C independent

$$M_A(t) = M_B(t) . M_C(t) \rightarrow \frac{1}{(1-2t)^{n/2}} = M_B(t) . \frac{1}{(1-2t)^{1/2}}$$

$$\therefore M_B(t) = \frac{1}{(1-2t)^{n/2 - 1/2}} = \frac{1}{(1-2t)^{n-1/2}} \rightarrow B \sim \chi^2(n-1)$$

$$\therefore (n-1)\frac{S^2}{\sigma_X{}^2} \sim \chi^2(n-1)$$ Q.E.D!

Theorem $X \sim N(\mu_X, \sigma_X{}^2)$

(1) $W = \frac{(n-1)S^2}{\sigma_X{}^2} = \sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma_X}\right)^2 \sim \chi^2(n-1)$

(2) $\overline{X} \sim N(\mu_X, \frac{\sigma_X{}^2}{n})$

(3) $\left.\begin{array}{l} Z \sim N(0,1) \\ U \sim \chi^2(r) \end{array}\right\}$ $T = \frac{Z}{\sqrt{U/r}} \sim t(r)$ t-distribution

(4) $$\frac{\overline{X}-\mu_X}{S/\sqrt{n}} = \frac{\frac{\overline{X}-\mu_X}{\sigma_X/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma_X{}^2}/n-1}} = \frac{N(0,1)}{\chi^2(n-1)} \sim t(n-1)$$

5.6 - CENTRAL LIMIT THEOREM

adjust ± 0.5 for continuity correction

$$\overline{X} \sim N(\mu_X, \frac{\sigma_X{}^2}{n}) \rightarrow P(a < \frac{\overline{X}-\mu_X}{\sigma_X/\sqrt{n}} < b) = P(a < Z < b)$$

OR $$Y \sim N(n\mu_X, n\sigma^2) \rightarrow P(a < \frac{Y-n\mu}{\sigma\sqrt{n}} < b) = P(a < Z < b)$$

5.7 - APPROXIMATIONS

$$X_i \sim Ber(n,p) \rightarrow Y = X_1 + X_2 + \dots + X_n \sim Bin(n,p) \rightarrow Y \sim N(np, npq)$$

7.1 - CONFIDENCE INTERVAL

(1) μ unknown, σ^2 known $\quad CI = \left[\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right]$

Proof $\rightarrow$

$$P\left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = P\left(|\bar{X} - \mu| \le z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$$

$$= P\left(\frac{|\bar{X} - \mu|}{\frac{\sigma}{\sqrt{n}}} \le z_{\frac{\alpha}{2}}\right) = P\left(|Z| \le z_{\frac{\alpha}{2}}\right) \geq 1 - \alpha$$ Q.E.D!

(2) μ, σ^2 unknown

$X \sim N(\mu, \sigma^2)$ $\quad CI = \left[\bar{X} - t_{\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}, \bar{X} + t_{\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}\right]$

Proof $\rightarrow$

$$P\left(\bar{X} - t_{\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}\right) = P\left(|\bar{X} - \mu| \le t_{\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}\right)$$

$$= P\left(\frac{|\bar{X} - \mu|}{\frac{S}{\sqrt{n}}} \le t_{\frac{\alpha}{2}}(n-1)\right) = P\left(|T| \le t_{\frac{\alpha}{2}}(n-1)\right) = P\left(-t_{\frac{\alpha}{2}}(n-1) \le T \le t_{\frac{\alpha}{2}}(n-1)\right)$$

$$\geq 1 - \alpha^2$$ Q.E.D!

Any distr. $\quad CI = \left[\bar{X} - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right]$ if $n \geq 30$

Proof $\rightarrow$ For any dist. X w/ finite μ, σ^2: $\dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \rightarrow Z \sim N(0,1)$ if $n \rightarrow \infty$

(3) One-sided CI $= \left[\bar{X} - z_{\alpha}\frac{S}{\sqrt{n}}, \infty\right)$ if $n \geq 30$ $\xrightarrow[\text{known}]{\text{if } \sigma^2}$ $\left[\bar{X} - z_{\alpha}\frac{\sigma}{\sqrt{n}}, \infty\right)$

Tips $\rightarrow$

$n \geq 30 \longrightarrow$ use $z_{\alpha}, z_{\alpha/2}$

$n < 30$ $\rightarrow$ if normal population: use $t_{\alpha}, t_{\alpha/2}$
$\searrow$ otherwise : use $z_{\alpha}, z_{\alpha/2}$

? F dist.