§2.3 Linear Equations

Def: A 1st order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x) \qquad (1)$$

is said to be a linear equation in the variable y.

Sometimes (not in HW), one can solve a 1st order LDE immediately by integrating:

Recall Product Rule:

$$(fg)' = fg' + gf'$$

eg $$(4+x^2)\frac{dy}{dx} + 2xy = 4x$$

ie. $$\frac{d}{dx}\left[(4+x^2)\,y\right] = 4x$$ integrate Both sides

$\therefore$ $$(4+x^2)y = 2x^2 + c$$

$$y = \frac{2x^2}{4+x^2} + \frac{c}{4+x^2}$$ (general Solution)

However, normally it is NOT this easy.

To solve 1^{st} order linear equations, we need to rig Product Rule: (Use integrating factor)

1) Put the DE in standard form.

$$1\frac{dy}{dx} + P(x)\,y = f(x) \qquad (2)$$

2) Multiply each term by an integrating factor $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \underbrace{P(x)\mu(x)}_{\mu'}\,y = \mu(x)\,f(x)$$

i.e. $\mu y' + (P(x)\mu)y = \mu f(x)$

To rig Product Rule, we need $\mu' = P(x)\mu$

i.e. $\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$.

$$\frac{1}{\mu}d\mu = P(x)\,dx$$

$$\therefore \int \frac{1}{\mu}\,d\mu = \int P(x)\,dx$$

$$\ln(\mu) = \int P(x)\,dx$$

and so $e^{\ln(\mu)} = e^{\int P(x)\,dx + k}$

k any constant

So, $\boxed{\mu(x) = e^{\int P(x)\,dx}}$ Let $k = 0$.

3) Integrate both sides.

4) Divide by $\mu(x)$. $y = \frac{1}{\mu(x)} \int \mu(x) f(x)\,dx$

We multiply each term of a linear 1st-order DE (in standard form) by the appropriate integrating factor to proceed as in the first example.

eg $\frac{dy}{dx} + 5y = e^{x}$

$P(x) = 5$

$\mu(x) = e^{\int 5\,dx}$

$= e^{5x}$

$\left(e^{5x}\frac{dy}{dx} + 5e^{5x}y = e^{5x}\cdot e^{x}\right)$

$\frac{d}{dx}\left[e^{5x}y\right] = e^{6x}$

Integrate both side WRT x

$e^{5x}y = \frac{e^{6x}}{6} + C$

$y = \frac{1}{6}\frac{e^{6x}}{e^{5x}} + \frac{C}{e^{5x}}$

$y = \frac{1}{6}e^{x} + \underbrace{Ce^{-5x}}_{\text{Transient term}}$

A transient term $\to 0$ as $x \to \infty$.

§2.3 Linear Equations (cont.)

Standard Form $\frac{dy}{dx} + P(x)\, y = f(x)$

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

coefficient is a 1

eg $xy' + 2y = \sqrt{x^2+1}$, $x > 0$

$$y' + \underbrace{\frac{2}{x}}_{P(x)} y = \frac{\sqrt{x^2+1}}{x}, \quad x > 0$$

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2 \qquad x>0$$

$$\frac{d}{dx}\left[x^2 y\right] = \frac{\sqrt{x^2+1}}{x} \cdot x^2$$

$$\left(x^2 y' + \frac{2}{x} \cdot x^2 y = \frac{d}{dx}\left[x^2 y\right] \right)$$

$$\frac{d}{dx}\left[x^2 y\right] = x\sqrt{x^2+1}$$

$$x^2 y = \int x\sqrt{x^2+1}\,dx$$

$$x^2 y = \frac{1}{2}\int u^{1/2}\,du \qquad u = x^2+1$$

$$du = 2x\,dx$$

$$\frac{du}{2} = x\,dx$$

$$x^2 y = \frac{1}{2}\cdot\frac{2}{3} u^{3/2} + C$$

$$x^2 y = \frac{1}{3}(x^2+1)^{3/2} + C$$

$$y = \frac{(x^2+1)^{3/2}}{3x^2} + \frac{C}{x^2}$$

transient term (As $x \to \infty$, term $\to 0$)

eg. IVP

$y' - 3y = 3x, \quad y(0) = -1$

$\mu(x) = e^{-\int 3\,dx} = e^{-3x}$

$\underbrace{e^{-3x} y' - 3 \cdot e^{-3x} y} = 3x \cdot e^{-3x}$

$\int \frac{d}{dx}\left[e^{-3x} y\right] = \int 3x \cdot e^{-3x}\,dx$

u	dv
$3x$ $+$	e^{-3x}
3 $-$	$-\frac{1}{3}e^{-3x}$
0	$\frac{1}{9}e^{-3x}$

$e^{-3x} y = -xe^{-3x} - \frac{1}{3}e^{-3x} + C$

GS: $y = -x - \frac{1}{3} + Ce^{3x}$

$y(0) = -1$ $\quad -1 = 0 - \frac{1}{3} + Ce^{0}$

$-\frac{2}{3} = c$

PS: $y = -x - \frac{1}{3} - \frac{2}{3}e^{3x}$ (No transient terms).

Hint: Hw #33

$\int \ln x\,dx = x\ln x - x + C$

Memorize

$u = \ln x$
$du = \frac{1}{x}dx$
$dv = dx$
$v = x$

$x\ln x - \int 1\,dx$

$x\ln x - x + C$

Nume.

(T 1 1.1, 1.2, 2.1, 2.2, 2.3, 2.4, 2.6, 3.1)