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Due Tueday 19 September 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #3

1. (4 points).
Show that each of the following maps are linear transformations.

(a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = x_1 + x_2 + x_3$.

(b) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(\mathbf{x}) = \begin{vmatrix} x_1 + x_2 \\ x_1 - x_3 \end{vmatrix}$.

(c) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(\mathbf{x}) = \begin{vmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 \end{vmatrix}$.

(d) Let $\Pi_n = \{p(x)$ polynomial of degree $\leq n\}$. $f : \Pi_n \to \Pi_{n-1}$, $f(p) = \dfrac{dp(x)}{dx}$.

2. (2 points).
Explain why the following are not linear transformations.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

(b) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = \begin{vmatrix} x_1 \\ x_2 \\ x_1 x_2 \end{vmatrix}$.

3. (4 points).
Let V, W be two vector spaces, and let $f : V \to W$ be a linear transformation.
Let $\mathcal{N}(f) = \{\mathbf{x} \in V \mid f(\mathbf{x}) = 0\}$ and $\mathcal{R}(f) = \{\mathbf{y} \in W \mid \exists \mathbf{x} \in V \text{ with } f(\mathbf{x}) = \mathbf{y}\}$.

(a) Prove that $\mathcal{N}(f)$ is a subspace of V.

(b) Show that $\mathcal{R}(f)$ is a non-empty set.

Name: Elle Nguyen HW #3

1 (a) Take $x, y \in \mathbb{R}^3$; $\alpha \in \mathbb{R}$ → wish to show $f(\alpha x + y) = \alpha f(x) + f(y)$

Consider $f(\alpha x + y) = (\alpha x + y)_1 + (\alpha x + y)_2 + (\alpha x + y)_3$

$$= \alpha x_1 + y_1 + \alpha x_2 + y_2 + \alpha x_3 + y_3 \quad \text{distributive law}$$
$$= (\alpha x_1 + \alpha x_2 + \alpha x_3) + (y_1 + y_2 + y_3) \quad \text{associative law}$$
$$= \alpha(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)$$
$$= \alpha f(x) + f(y)$$

∴ The map is linear transformation QED!

(b) Take $x, y \in \mathbb{R}^3$; $\alpha \in \mathbb{R}$ → wish to show $f(\alpha x + y) = \alpha f(x) + f(y)$

Consider:

associative law

$$f(\alpha x + y) = \begin{vmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{vmatrix} = \begin{vmatrix} \alpha x_1 + y_1 + \alpha x_2 + y_2 \\ \alpha x_1 + y_1 - \alpha x_3 - y_3 \end{vmatrix} = \begin{vmatrix} (\alpha x_1 + \alpha x_2) + (y_1 + y_2) \\ (\alpha x_1 - \alpha x_3) + (y_1 - y_3) \end{vmatrix}$$

$$= \begin{vmatrix} \alpha(x_1 + x_2) + (y_1 + y_2) \\ \alpha(x_1 - x_3) + (y_1 - y_3) \end{vmatrix} = \alpha \begin{vmatrix} x_1 + x_2 \\ x_1 - x_3 \end{vmatrix} + \begin{vmatrix} y_1 + y_2 \\ y_1 - y_3 \end{vmatrix} \quad \text{by definitions of sum of matrices}$$
$$= \alpha f(x) + f(y)$$

∴ The map is linear transformation QED!

(c) Take $x, y \in \mathbb{R}^2$; $\alpha \in \mathbb{R}$ → wish to show $f(\alpha x + y) = \alpha f(x) + f(y)$

Consider:

$$f(\alpha x + y) = \begin{vmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \end{vmatrix} = \begin{vmatrix} (\alpha x_1 + y_1) - (\alpha x_2 + y_2) \\ (\alpha x_1 + y_1) + (\alpha x_2 + y_2) \\ 2(\alpha x_1 + y_1) \end{vmatrix}$$

associative law

distributive law

$$= \begin{vmatrix} \alpha x_1 + y_1 - \alpha x_2 - y_2 \\ \alpha x_1 + y_1 + \alpha x_2 + y_2 \\ 2\alpha x_1 + 2y_1 \end{vmatrix} = \begin{vmatrix} (\alpha x_1 - \alpha x_2) + (y_1 - y_2) \\ (\alpha x_1 + \alpha x_2) + (y_1 + y_2) \\ 2\alpha x_1 + 2y_1 \end{vmatrix} = \begin{vmatrix} \alpha(x_1 - x_2) + (y_1 - y_2) \\ \alpha(x_1 + x_2) + (y_1 + y_2) \\ 2\alpha x_1 + 2y_1 \end{vmatrix}$$

by definitions of sum of matrices

$$= \alpha \begin{vmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 \end{vmatrix} + \begin{vmatrix} y_1 - y_2 \\ y_1 + y_2 \\ 2y_1 \end{vmatrix} = \alpha f(x) + f(y)$$

∴ The map is linear transformation QED!

(d) Take $v(x), w(x) \in \Pi_n$; $\alpha \in \mathbb{R} \rightarrow$ wish to show $f(\alpha v(x) + w(x))$
$= \alpha f(v(x)) + f(w(x))$

Consider $f(\alpha v(x) + w(x)) = \dfrac{d(\alpha v(x) + w(x))}{dx} = \dfrac{d(\alpha v(x)) + d(w(x))}{dx}$ distributive law

$= \dfrac{d(\alpha v(x))}{dx} + \dfrac{d(w(x))}{dx} = \alpha \dfrac{d(v(x))}{dx} + \dfrac{d(w(x))}{dx} = \alpha f(v(x)) + f(w(x))$

by definitions of sum of derivatives

$\therefore$ The map is linear transformation QED!

2 (a) Take $x, y \in \mathbb{R}$; $\alpha \in \mathbb{R} \rightarrow$ wish to show $f(\alpha x + y) \neq \alpha f(x) + f(y)$

Consider $f(\alpha x + y) = (\alpha x + y)^2$

$= (\alpha x)^2 + 2(\alpha x)y + y^2$ by $(a+b)^2 = a^2 + 2ab + b^2$

$= \alpha^2 x^2 + 2(\alpha xy) + y^2$ by commutative law

$\neq \alpha x^2 + y^2 = \alpha f(x) + f(y)$

$\therefore$ The map is not linear transformation QED!

(b) Proof by counterexample:

Take $v_1 = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \longrightarrow f(v_1) = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$

$v_2 = \begin{vmatrix} 4 \\ 5 \\ 6 \end{vmatrix} \longrightarrow f(v_2) = \begin{vmatrix} 4 \\ 5 \\ 20 \end{vmatrix}$

$f(v_1) + f(v_2) = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + \begin{vmatrix} 4 \\ 5 \\ 20 \end{vmatrix} = \begin{vmatrix} 5 \\ 7 \\ 22 \end{vmatrix}$

$v_1 + v_2 = \begin{vmatrix} 5 \\ 7 \\ 9 \end{vmatrix} \rightarrow f(v_1 + v_2) = \begin{vmatrix} 5 \\ 7 \\ 35 \end{vmatrix} \neq \begin{vmatrix} 5 \\ 7 \\ 22 \end{vmatrix} = f(v_1) + f(v_2)$

$\therefore$ The map is not linear transformation QED!

3 (a) Take $v, w \in \mathcal{N}(f)$; $\alpha \in \mathbb{R} \rightarrow$ wish to show $(\alpha v + w) \in \mathcal{N}(f)$

Since $v, w \in \mathcal{N}(f)$: $f(v) = f(w) = 0$

Consider:

property of linearity ... from hypothesis above

$f(\alpha v + w) = f(\alpha v) + f(w) = \alpha f(v) + f(w) = \alpha(0) + 0 = 0 + 0 = 0$

$\therefore f(\alpha v + w) = 0$

$\therefore (\alpha v + w) \in \mathcal{N}(f)$ or $\mathcal{N}(f)$ is a subspace of V QED!

(b) Take $x_1, x_2 \in V$; $\alpha \in \mathbb{R}$

$\therefore \exists\, v, w \in W \ni f(x_1) = v$ and $f(x_2) = w$

Wish to show $(\alpha v + w) \in \mathcal{R}(f)$

Consider $\alpha v + w = \alpha f(x_1) + f(x_2) = f(\alpha x_1) + f(x_2)$ property of linearity

hypothesis $\qquad = f(\alpha x_1 + x_2)$

$\therefore (\alpha v + w) \in \mathcal{R}(f)$

$\therefore \mathcal{R}(f)$ is a subspace or $0 \in \mathcal{R}(f)$

$\therefore \mathcal{R}(f)$ is non-empty QED!