Section 1.1 & 1.2

V is a vectorspace if it is closed under addition & scalar multiplication

$x, y \in V \rightarrow \alpha x + y \in V$

$0 \in V$

(1) $\alpha = 0$
$\alpha . x = 0 . x = 0$
$\alpha x \in V, \; 0 \in V$

(2) $\alpha = -1$
$\alpha . x = -x \in V$
$-x + x = 0 \in V$

If $0 \notin S$, S is not a vectorspace

Unit vectors $(e_i)_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ $\quad x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$

$A = B$ if $A, B \in \mathbb{R}^{m \times n}$
$a_{ij} = b_{ij} \quad \forall \, i = 1 \dots m, \; j = 1 \dots n$

$(A^T)^T = A \qquad (A+B)^T = A^T + B^T$
$(\alpha A)^T = \alpha A^T$

Diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

Symmetric $\quad A^T = A$
skew-symmetric $\quad A^T = -A \rightarrow a_{ij} = -a_{ji}$
$a_{ii} = -a_{ii} \; (i = j) \longrightarrow \quad a_{ii} = 0$

Section 1.3

Subspace $S \subseteq V$
- trivial subspaces $\{0\}, V$
- non-trivial subspaces line, planes

Affine space $A = p + X = \{ v \in V \mid v = p + x, \; x \in X \}$

Linearly dependent if one of them can be written as linear combination
if $\exists x \neq 0 \ni Ax = 0$

Linearly independent $\sum \alpha_i v_i = 0 \rightarrow \alpha_i = 0 \ \forall i$

$A = [v_1 \ v_2 \dots v_n]$
$Ax = 0 \rightarrow x = 0$

Section 1.4 & 1.5

Norm $\|x\| = \left(\sum_{i=1}^{n} x^2 \right)^{1/2} = \sqrt{x^T x}$

(1) $\|x\| \geq 0$
(2) $\|x\| = 0 \rightarrow x = 0$
(3) $\|\alpha x\| = |\alpha| \|x\|$
(4) $\|x+y\| \leq \|x\| + \|y\|$ (Δ inequality)

$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle}$
$= \sqrt{\alpha^2 \langle x, x \rangle}$
$= |\alpha| \sqrt{\langle x, x \rangle}$
$= |\alpha| \|x\|$

Inner product $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = \mathrm{trace}(x^T y)$

(1) Bilinear $\langle \alpha x + y, v \rangle = \alpha \langle x, v \rangle + \langle y, v \rangle$ $\quad \langle \alpha u, \alpha v \rangle = \alpha^2 \langle u, v \rangle$
$\langle x, v + \beta w \rangle = \langle x, v \rangle + \beta \langle x, w \rangle$
(2) $\langle x, x \rangle \geq 0$
$\langle x, x \rangle = 0 \rightarrow x = 0$ $\quad \langle x, y \rangle = 0 \rightarrow x \perp y$
(3) Symmetric $\langle x, y \rangle = \langle y, x \rangle$

1-norm $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$
$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$

Proof $\rightarrow$ $\|x+y\|_1 = |x_1 + y_1| + |x_2 + y_2| + \dots + |x_n + y_n|$
$\leq |x_1| + |y_1| + |x_2| + |y_2| + \dots |x_n| + |y_n|$
$= \|x_1\|_1 + \|y_1\|_1$

∞-norm $\|x\|_\infty = \max_{i=1}^{n} |x_i|$

General $\|x\|_p = \left(\sum |x_i|^p \right)^{1/p}$ $\quad (1 \leq p \leq 2)$

CBS Inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$

Proof $\rightarrow$ If $x = 0$, $\|x\| = 0 \rightarrow \langle x, y \rangle = 0 \rightarrow$ inequality holds

If $x \neq 0$, $\alpha = \frac{\langle x,y\rangle}{\|x\|^2}$ $\quad V = \alpha x - y$

$$\langle x, V\rangle = \langle x, \alpha x - y\rangle = \alpha\langle x, x\rangle - \langle x, y\rangle = \frac{\langle x,y\rangle}{\|x\|^2}\|x\|^2 - \langle x,y\rangle = 0$$

$$\begin{aligned} 0 \leq \|v\|^2 = \langle v, v\rangle &= \langle \alpha x - y, \alpha x - y\rangle \\ &= \alpha\langle x, \alpha x - y\rangle - \langle y, \alpha x - y\rangle \\ &= \alpha(0) \qquad - \alpha\langle y, x\rangle + \langle y, y\rangle \\ &= \|y\|^2 - \frac{\langle x,y\rangle}{\|x\|^2}\langle x, y\rangle \\ &= \frac{\|y\|^2\|x\|^2 - \langle x,y\rangle^2}{\|x\|^2} \end{aligned}$$

$\therefore \|x\|^2\|y\|^2 \geqslant \langle x,y\rangle^2 \qquad \therefore |\langle x,y\rangle| \leq \|x\|\,\|y\|$

Proof Δ inequality $\rightarrow$

$$\begin{aligned} \|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \qquad a \leq |a| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

$\therefore \|x+y\| \leq \|x\| + \|y\|$

Backward Δ inequality $\big|\|x\| - \|y\|\big| \leq \|x-y\|$

Proof $\rightarrow$

$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\| \qquad \therefore \|x\| - \|y\| \leq \|x-y\|$

$\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\| = \|x-y\| + \|x\| \quad \therefore \|y\| - \|x\| \leq \|x-y\|$

$\therefore \big|\|x\| - \|y\|\big| \leq \|x-y\|$

Section 1.6

Orthogonality $\langle x,y\rangle = 0 \rightarrow \|x\|^2 + \|y\|^2 = \|x-y\|^2$

Angle $\quad \cos\theta = \frac{\langle x,y\rangle}{\|x\|\,\|y\|}$

Orthogonal set $\{u_1, u_2 \dots u_n\}$

$\langle u_i, u_j\rangle = 0 \quad (i \neq j)$

$\rightarrow$ linearly independent

Orthonormal $\langle u_i, u_j\rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

$\|u_i\| = 1$

$\rightarrow$ orthogonal matrix

(use $\frac{u}{\|u\|}$)

Section 1.7

Linear transformation $f(\alpha x + y) = \alpha f(x) + f(y)$

If y fixed: $f_y(x) = \langle x, y \rangle$

$f_y(x+u) = \langle x+u, y \rangle = \langle x, y \rangle + \langle u, y \rangle$

If x fixed: $f_x(u+v) = \langle u+v, y \rangle = \langle u, y \rangle + \langle v, y \rangle = f_x(u) + f_x(v)$

Matrix times vector

1^{st} coeff x 1^{st} col $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = y_1 x_1 + y_2 x_2 + y_3 x_3$

Nullspace kernel $\mathcal{N}(f) = \{ x \in V \mid f(x) = 0 \}$ $\qquad f(0) = 0$

Range $R(f) = \{ y \in \mathbb{R} \mid \exists x \in \mathcal{D}, f(x) = y \}$ $\qquad 0 \in R(f)$

Composition of linear maps is a linear map

Proof $\rightarrow$

$(f \circ g)(\alpha x + y) = f(g(\alpha x + y)) = f(\alpha g(x) + g(y))$ linear g

$\uparrow$ composition

$= \alpha f(g(x)) + f(g(y))$ linear f

$= \alpha (f \circ g)(x) + (f \circ g)(y)$

Matrix times matrix $c_{ij} = \sum_{k=1}^{n_2} a_{ik} b_{kj}$

matrix x each col $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ & $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

$Ax = b$ consistent if $\exists$ at least 1 sln (b is linear combination of A's cols)

Triangular $\rightarrow$ lower $a_{ij} = 0 \ (i < j)$

$\searrow$ upper $a_{ij} = 0 \ (i > j)$

$\therefore$

A triangular $\longrightarrow A^{-1}$ also triangular $\quad \text{diag}(A^{-1}) = \text{diag}\left(\frac{1}{a_{ii}}\right)$

$\text{diag}(AB) = \text{diag}(A) \cdot \text{diag}(B)$

$(AB)_{ii} = A_{ii} \cdot B_{ii}$

Section 1.8

Diagonal matrix $ij = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$ $\quad IA = A$, $AI = A$

$A^0 = I$

$A^x \cdot A^y = A^{x+y}$

$(A^x)^y = A^{x \cdot y}$

$(AB)^T = B^T A^T$

$\underbrace{(A^TA)^T}_{\text{always symmetric}} = A^T(A^T)^T = A^TA$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Inverse $A \cdot B = I \rightarrow B = A^{-1}$

exists if cols of A are linearly independent

trace(AB) = trace(BA)

Proof →

$$\text{trace}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} A_{i*}B_{*i} = \sum_{i=1}^{n}\sum_{k=1}^{m} a_{ik}b_{ki}$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{m} b_{ki}a_{ik} = \sum_{k=1}^{m}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{m} B_{k*}A_{*k} = \sum_{k=1}^{m} (BA)_{kk} = \text{trace}(BA)$$

$\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$

Nonsingular (linearly independent) $Ax = 0 \rightarrow x = 0$

$Ax = b$ has unique sln

$AX = I \rightarrow XA = I$

Section 1.9 & 1.10

Matrix norm $\|AB\| \leq \|A\| \cdot \|B\|$

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

Orthogonal matrix U

(1) maintain length

$$\|Ux\|^2 = \langle Ux, Ux \rangle = (Ux)^T Ux = x^T U^T U x = x^T I x = x^T x = \|x\|^2 \checkmark$$

(2) maintain angle

$$\cos(Ux, Uy) = \frac{\langle Ux, Uy \rangle}{\|Ux\|\|Uy\|} = \frac{(Ux)^T Uy}{\|x\|\|y\|} = \frac{x^Ty}{\|x\|\|y\|} = \frac{\langle x, y \rangle}{\|x\|\|y\|} = \cos\theta \checkmark$$

A, B orthogonal $\longrightarrow$ show $A^TA = I$

$$\|A^{-1}\| = \max_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \max_{y \neq 0} \frac{\|y\|}{\|Ay\|} = \frac{1}{\min\limits_{y \neq 0} \frac{\|Ay\|}{\|y\|}}$$

$$\rightarrow \min_{\|x\| \neq 0} \|Ax\| = \frac{1}{\|A^{-1}\|}$$

Reflections $R = I - \dfrac{2uu^T}{u^Tu} \xrightarrow[\|u\|=1]{u^Tu=1} R = I - uu^T$

(1) $R^T = R$

Proof $\rightarrow$ $R^T = I^T - 2(uu^T)^T = I - 2(uu^T) = R$

(2) $RR = I$

Proof $\rightarrow$ $RR = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4u\underset{\sim}{u^Tu}u^T$
$= I - 4uu^T + 4uu^T = I$

(3) $Ru = -u$

Proof $\rightarrow$ $Ru = (I - 2uu^T)u = u - 2u\underset{\sim}{u^Tu} = u - 2u = -u$

if $u^Tw = 0 \longrightarrow (I - 2uu^T)w = w - 2u\underset{\sim}{u^Tw} = w$

(4) $\therefore Rw = w$

Orthogonal projection $P = I - uu^T$

(1) $P^2 = P$ (general projection)

Proof $\rightarrow$ $P^2 = (I - uu^T)^2 = I - 2uu^T + u\underset{\sim}{u^Tu}u^T = I - 2uu^T + uu^T = I - uu^T = P$

(2) $P^T = P$ (orthogonal projection)

Proof $\rightarrow$ $P^T = I - (uu^T)^T = I - uu^T = P$

(3) if $w \perp u$ $Pw = w$

Proof $\rightarrow$ $P(u + w) = Pu + Pw = 0 + w = w$

END TEST I

Section 2.1

Gaussian Elimination → row interchanges
→ replace a row by a multiple of itself
→ replace a row by adding a multiple of another row

(1) Non-zero pivots → eliminate below it
(2) zero in pivot position
↳ below → non-zero : row interchanges
→ zero : move to next col

Rank = # pivots = # basic cols → nonsingular rank(A) = # rows
→ singular $\begin{cases} \text{rank}(A) < \#\text{ rows} \\ A \text{ has zero rows} \end{cases}$

Rank(MA) = Rank(A) for M n×n nonsingular

Proof → $MAx = 0 \longrightarrow M^{-1}MAx = 0 \longrightarrow Ax = 0 \longrightarrow x = 0$ (A nonsingular)
$\therefore$ MA nonsingular $\therefore \text{rank}(MA) = \text{rank}(A)$ QED!

Section 2.2

left multiply $EA \rightarrow$ row operation
right multiply $AE \rightarrow$ col operation

LU factorization $E_3E_2E_1A = U \rightarrow A = (E_3E_2E_1)^{-1}U = \overbrace{E_1^{-1}E_2^{-1}E_3^{-1}}^{L}U$

Rank(A^T) = Rank(A)

Proof → $\text{rank}(A^T) = \text{rank}\begin{pmatrix} C_{r\times m} \\ 0 \end{pmatrix} = \text{rank}(C) = r = \text{rank}(A)$ QED!

Section 2.4

Rank(A^TA) = Rank(AA^T) = Rank(A)

Proof → Wish to show $Ax = 0$ iff $A^TAx = 0$
Suppose $Ax = 0 \rightarrow A^TAx = 0$
Suppose $A^TAx = 0 \rightarrow x^TA^TAx = (Ax)^TAx = \|Ax\|^2 = 0 \rightarrow Ax = 0$

$$\left.\begin{aligned} &\therefore \text{rank}(A^TA) = \text{rank}(A) \\ &\therefore \text{rank}((A^T)^TA^T) = \text{rank}(AA^T) = \text{rank}(A^T) \\ &\& \ \text{rank}(A) = \text{rank}(A^T) \end{aligned}\right\} \text{rank}(A^TA) = \text{rank}(AA^T) = \text{rank}(A)$$

$$\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(\underset{m\times n}{A}), \mathrm{rank}(\underset{n\times p}{B})\}$$

Proof $\rightarrow$

$\mathrm{rank}(A) = r$

$PA = E_A = \begin{bmatrix} C_{r\times n} \\ 0 \end{bmatrix}$

$\mathrm{rank}(AB) = \mathrm{rank}(PAB) = \mathrm{rank}(E_A B) = \mathrm{rank}\begin{bmatrix} C_{r\times n}B \\ 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} (CB)_{r\times p} \\ 0 \end{bmatrix} \leq r = \mathrm{rank}(A)$

$\mathrm{rank}(AB) = \mathrm{rank}((AB)^T) = \mathrm{rank}(B^T A^T) \leq \mathrm{rank}(B^T) = \mathrm{rank}(B)$ QED!

Full rank factorization $A = B_{m\times r} \cdot C_{r\times n}$ ($\mathrm{rank}\, B = \mathrm{rank}\, C = r$, nonunique)

B: basic cols

C: coeffs that produce all cols; non-zero rows in E_A

Section 2.5

	Consistent	Inconsistent
$Ax=b$	≥ 1 sln, $b \in R(A)$	no sln, $b \notin R(A)$
$[E\|c]$	no row of $[0\ 0 \dots 0 \| \alpha]$ $(\alpha \neq 0)$	1 row of $[0\ 0 \dots 0 \| \alpha]$ $(\alpha \neq 0)$
$\mathrm{rank}([A\|b])$	$= \mathrm{rank}(A)$	$> \mathrm{rank}(A)$

General sln $= x_p + x_h \xrightarrow{\alpha_1 = \alpha_2 = 0}$ trivial sln $(n = r)$

x_p: set free variables $= 0$

x_h: write basic variables in terms of free variables

$n - r = \#$ free variables

$x_h = \alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \dots + \alpha_p x_{h_p} \longrightarrow$ linear combination of x_{h_i}

Solve $Ax = b$ $\rightarrow$ $Ly = b$, $Ux = y$

$N(A) = N(AA^T)$

$N(A^T) = N(A^T A)$

$N(A) = \{x \mid Ax = 0\} = x_h$

END TEST 2

Section 9

A singular $\rightarrow \det A = 0$

nonsingular $\rightarrow \det A \neq 0$

diagonal / triangular $\rightarrow \det A = \prod_i a_{ii}$

$\det(\alpha A) = \alpha^n \det A$ $\quad \det A = \det A^T$ $\quad \det A^{-1} = \dfrac{1}{\det A}$

$\det(AB) = \det A . \det B$

(1) LU factorization $A = LU \longrightarrow \det A = \underbrace{\det L}_{1} . \det U = \det U$

(2) Permutations

$$\begin{array}{cccccc}
 & a_{11} & a_{12} & a_{13} & \\
 & a_{21} & a_{22} & a_{23} & \\
- & a_{31} & a_{32} & a_{33} & + \\
- & a_{11} & a_{12} & a_{13} & + \\
- & a_{21} & a_{22} & a_{23} & + \\
 & a_{31} & a_{32} & a_{33} &
\end{array}$$

$\det A = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$

(3) Cofactor expansion $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathring{A}_{ij}$ matrix after eliminating row i, col j

Section 4.1 & 4.2

$\{v_1, v_2 \dots v_n\}$ linearly independent set $\rightarrow$ a basis of V

$\rightarrow \dim V = n$

Hyperplane in $\mathbb{R}^n \longrightarrow$ dimension $(n-1)$

2 subspaces of V

- if $M \subseteq N$, $\dim M \leq \dim N$
- if $\dim M = \dim N$, $M = N$

$B = \{v_1, v_2 \dots v_n\}$ is a basis of S $\iff$ B is a minimal spanning set of S

$\iff$ B is a maximal linearly independent set

$\downarrow$

$v = \sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \longrightarrow$ unique form, α_i unique

coordinates of v w.r.t the basis

Proof $\rightarrow$ If there is another set of scalars $\rightarrow \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i \rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) v_i = 0$

$\therefore \alpha_i = \beta_i$

$\dim N(A) = \#\text{free variables} = n - r$
$\dim R(A) = \text{rank}(A) = r$

$\dim N(A) + \dim R(A) = n$ (col space)
$\dim N(A^T) + \dim R(A^T) = m$ (row space)

$\{v_1, v_2 \dots v_r\}$ basis of X
$\{w_1, w_2 \dots w_k\}$ basis of Y

if $X \cap Y = \{0\} \rightarrow \{v_1, v_2 \dots w_1 \dots w_k\}$ basis $X + Y$

$X \cap Y \neq \{0\}$

$\dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$

Section 3.1

$Au = \lambda u$
- u → eigenvector
- λ → eigenvalue

$A(\alpha u) = \alpha A u = \alpha \lambda u = \lambda(\alpha u)$

$\therefore$ Multiple of eigenvector has the same eigenvalue

$A^k u = \lambda^k u$

A singular
- eigenvalue $= 0$
- eigenvector $\in N(A)$

$\sigma(A^T) = \sigma(A) \rightarrow$ same eigenvalues

Spectrum $\sigma(A)$ = set of eigenvalues $\rightarrow \dim N(A - \lambda I) \geq 1$ if $\lambda \in \sigma(A)$

Characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$
- $\leq n$ roots
- $\leq n$ eigenvalues

n odd $\rightarrow \geq 1$ real root

Algebraic multiplicity = multiplicity of λ as a root of $p_A(\lambda)$
$\geq$ Geometric multiplicity $= \dim N(A - \lambda I)$

If geometric < algebraic $\rightarrow A, \lambda$ defective
geometric = algebraic $\rightarrow \forall \lambda \in \sigma(A) \rightarrow A$ diagonalizable

$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i \qquad \det(A) = \prod_{i=1}^{n} \lambda_i$

A symmetric
- $\rightarrow$ left eigenvectors of A^T = right eigenvectors of A
- distinct λs $\rightarrow$ orthogonal eigenvectors

Spectral radius $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ $\qquad \rho(A) \leq \|A\|$

Proof $\rightarrow$ $Ax = \lambda x \rightarrow \|Ax\| = \|\lambda x\| = |\lambda| \|x\| \rightarrow |\lambda| \leq \frac{\|Ax\|}{\|x\|} \leq \|A\|$

Section 3.2

A, B are similar $A \sim B$ if $\exists$ P nonsingular $\ni$ $B = P^{-1}AP$

$B \sim A$ $\quad A = P^{-1}BP$

(1) $\lambda \in \sigma(A) \rightarrow \lambda \in \sigma(B)$

$Av = \lambda v \rightarrow P^{-1}BPv = \lambda v \rightarrow BPv = P\lambda v = \lambda(Pv) \rightarrow \lambda \in \sigma(B)$

(2) $p_A(\lambda) = p_B(\lambda)$

$$p_A(\lambda) = \det(A - \lambda I) = \det(P^{-1}BP - \lambda P^{-1}P)$$
$$= \det P^{-1}(B - \lambda I)P$$
$$= \det P^{-1} . \det(B - \lambda I) . \det P = \det(B - \lambda I) = p_B(\lambda)$$

n linearly independent eigenvectors

A has complete set of eigenvectors $\rightarrow A \sim \Lambda = \mathrm{diag}(\lambda_1 \ \lambda_2 \dots \lambda_n)$

$\{v_1, v_2 \dots v_n\}$ basis of $\mathbb{R}^n \rightarrow \alpha_i$ unique

$V = [v_1 \ v_2 \dots v_n]$ Spectral decomposition $A = V\Lambda V^{-1} \rightarrow \Lambda = V^{-1}AV$

A symmetric $\longrightarrow A = V\Lambda V^T$

$\|A\| = |\lambda_{max}|$ $\qquad \frac{1}{\|A^{-1}\|} = |\lambda_{min}|$

Section 3.5

$A = U\Sigma V^T$ — $U_{m\times n}, V^T{}_{n\times n}$ orthogonal

$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2 \dots \sigma_n)$ in decreasing order

if $\sigma_n = \sigma_{r+1} = 0$, $\sigma_r \neq 0 \longrightarrow \mathrm{rank}A = r$

$R(A) = R(U)$ $\qquad Ax = U\Sigma V^T x = Uw$

$N(A) = N(V^T)$ $\qquad Ax = 0 \rightarrow U\Sigma V^T x = 0 \rightarrow V^T x = 0$

$R(A^T) = R(V)$ $\qquad A = U\Sigma V^T \rightarrow A^T = V\Sigma^T U^T = V\Sigma U^T$

$N(A^T) = N(U^T)$ $\qquad A^T x = 0 \rightarrow V\Sigma U^T x = 0 \rightarrow U^T x = 0$

$\|A\|_2 = \max_{\|x\|=1} \|Ax\| = \sigma_{max} = \sigma_1$ $\qquad \|A^{-1}\|_2 = \frac{1}{\sigma_{min}} = \frac{1}{\sigma_n}$

Note: if $Q^TQ = I \xrightarrow{SVD} Q = QII$

$B = \sum_{i=1}^{k} \sigma_i u_i v_i^T$

Find matrix B of rank k $\ni \|A - B\|$ smallest $\qquad \|B - A\| = \sigma_{k+1}$

$A = U\Sigma V^T$ $\Big\{$ $AA^T = U\Lambda U^T \rightarrow N(AA^T - \lambda I)$ give cols of U $\Big\}$ normalize
$A^T = V\Sigma U^T$ $\Big\}$ $A^TA = V\Lambda V^T \rightarrow N(A^TA - \lambda I)$ give rows of V^T

$\sigma_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)}$ if A symmetric: $\sigma_i(A) = |\lambda_i(A)|$

Section 4.7

2 subspaces $X + Y = \mathbb{R}^n$, $X \cap Y = \{0\}$ $\Big>$ X, Y complementary subspaces $X \oplus Y$ $\dim X + \dim Y = \dim V = n$

$\exists!\ x \in X, y \in Y \ni v = x + y$

Proof $\rightarrow$ $v \in V$

$v = x_1 + y_1 = x_2 + y_2 \rightarrow \underbrace{x_1 - x_2}_{\in X} = \underbrace{y_2 - y_1}_{\in Y}$

$x_1 - x_2,\ y_2 - y_1 \in X \cap Y = \{0\} \rightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \rightarrow$ decomposition is unique

QED!

$R(A) \oplus N(A^T) = \mathbb{R}^m$ $R(A^T) \oplus N(A) = \mathbb{R}^n$

Section 4.3

$v = x + y$ projections $x = P_X v$, $y = P_Y v = (I - P_X)v$ $\Big\}$ P_X, P_Y complementary projections

$P = U(V^TU)^{-1}V^T$ $\begin{cases} U \text{ cols in } X \\ V \text{ cols in } Y^\perp \\ P^2 = P \end{cases}$ $R(P) = R(U) = X$, $N(P) = Y$

If $X \perp Y$: $U = V \rightarrow$ cols are orthonormal $P = UU^T$
$P = U(U^TU)^{-1}U^T = UI^{-1}U^T = UU^T$ $\|P\| = 1$

$Q = I - P$ $\Big<$
$Q^2 = (I-P)(I-P) = I - 2P + P^2 = I - 2P + P = I - P$
$R(Q) = N(P)$ $R(P) = N(Q)$
$Q^2 = Q = Q(I-P) = Q - QP$ $\therefore QP = PQ = 0$

$M^\perp + M = V$, $M \cap M^\perp = \{0\}$ $\Big>$ $M \oplus M^\perp$ $(M^\perp)^\perp = M$

$R(A^T) = N(A)^\perp$
$R(A) = N(A^T)^\perp$

$x \in R(A) \rightarrow x = Aw$
$y \in N(A^T) \rightarrow A^Ty = 0$
$\Big\{$ $\langle y, x\rangle = y^TAw = (A^Ty)^Tw = 0^Tw = 0$ $\therefore R(A) \perp N(A^T)$ QED!

Section 5.1

Find orthonormal basis of a subspace

(1) $u_1 = \frac{x_1}{\|x_1\|} \xrightarrow{(2)} \langle x_2, u_1 \rangle$

(3) $w_2 = x_2 - \langle x_2, u_1 \rangle u_1 \xrightarrow{(4)} u_2 = \frac{w_2}{\|w_2\|}$

(5) Orthogonal basis $Q = | u_1 \ \ u_2 |$

Q - R factorization

orthogonal basis

upper triangular

$R_{ii} = \|w_i\|$

$R_{ij} = \langle x_j, u_i \rangle$

unique $A = Q_1 R = Q_2 R_2 \quad \therefore R_1 R_2^{-1} = Q_1^T Q_2$ orthogonal

$R(A) = R(Q)$

$= R(P)$ matrix P of rank $1 \ni P = QQ^T$, $P^2 = P$, $P^T = P$

(orthogonal projection onto $R(A)$)

Find $x \ni \|b - Ax\|$ min

(1) QR factorization

(2) expand Q → choose linearly independent vector → obtain $\tilde{Q}$

→ testing $\langle x_3, u_1 \rangle$ & $\langle x_3, u_2 \rangle \neq 0$

(3) $\tilde{Q}^T b$

(4) $\tilde{R} x = \tilde{Q}^T b$ (take 2 entries)

(5) Solve for x

(6) Verify $\|b - Ax\|$ (should match 3^{rd} entry of $\tilde{Q}^T b$)

END FINAL