Undamped Forced Motion

5.1.3 Example – Undamped (Case 1) – no Homework problems: A 128-lb weight stretches a spring 2 ft. The weight is started in motion with no initial velocity by displacing it 6 inches above the equilibrium position and by simultaneously applying to the weight an external force of F(t) = 8 sin(2t). Assuming no air resistance, find the subsequent motion of the weight.

$$mg = 128 \text{ pounds} \;\Rightarrow\; m = \frac{128}{32} = 4 \text{ slugs}$$

$$k = \frac{mg}{s} = \frac{128}{2} = 64 \text{ pounds/foot} \qquad\qquad 4x'' + 64x = 8\sin 2t$$

$$x'' + 16x = 2\sin 2t, \qquad x(0) = -\frac{1}{2},\; x'(0) = 0$$

Associated homogeneous equation: $r^2 + 16 = 0 \;\Rightarrow\; r = \pm 4i$

$$x_c(t) = c_1\cos 4t + c_2\sin 4t$$

$$x_p = A\cos 2t + B\sin 2t \qquad x_p' = -2A\sin 2t + 2B\cos 2t \qquad x_p'' = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t + 16A\cos 2t + 16B\sin 2t \overset{set}{=} 2\sin 2t$$

$$12A\cos 2t + 12B\sin 2t \overset{set}{=} 2\sin 2t$$

$$12A = 0 \;\Rightarrow\; A = 0 \qquad\qquad 12B = 2 \;\Rightarrow\; B = \frac{1}{6}$$

$$x_p = \frac{1}{6}\sin 2t$$

$$x(t) = c_1\cos 4t + c_2\sin 4t + \frac{1}{6}\sin 2t$$

$$x(0) = -\frac{1}{2} \;\Rightarrow\; -\frac{1}{2} = c_1$$

$$x'(t) = -4c_1\sin 4t + 4c_2\cos 4t + \frac{1}{3}\cos 2t$$

$$x'(0) = 0 \;\Rightarrow\; 0 = 4c_2 + \frac{1}{3} \;\Rightarrow c_2 = -\frac{1}{3}\cdot\frac{1}{4} = -\frac{1}{12}$$

$$x(t) = -\frac{1}{2}\cos 4t - \frac{1}{12}\sin 4t + \frac{1}{6}\sin 2t$$

5.1.3 Example – Undamped (Case 2): Now assume the above but suppose F(t) = 8 sin(4t).

$$x'' + 16x = 2\sin 4t, \qquad x(0) = -\frac{1}{2},\ x'(0) = 0$$

$$x_c(t) = c_1\cos 4t + c_2\sin 4t$$

Now, we must adjust since we need linear independence.

$$x_p = t\left(A\cos 4t + B\sin 4t\right)$$

$$x_p' = A\cos 4t + B\sin 4t + t\left(-4A\sin 4t + 4B\cos 4t\right)$$

$$x_p'' = -4A\sin 4t + 4B\cos 4t + -4A\sin 4t + 4B\cos 4t + t\left(-16A\cos 4t - 16B\sin 4t\right)$$
$$= 8B\cos 4t - 8A\sin 4t + t\left(-16A\cos 4t - 16B\sin 4t\right)$$

Everything with a factor of *t* must cancel once you plug x_p into the differential equation and it does. Otherwise, you made a mistake (only 2 sin4t on the right and so on the left only multiples of sin 4t and cos 4t should remain.) This is how undetermined coefficient method works.

$$8B\cos 4t - 8A\sin 4t \overset{set}{=} 2\sin 4t$$

$$8B = 0 \quad \Rightarrow \quad B = 0 \qquad\qquad -8A = 2 \quad \Rightarrow \quad A = -\frac{1}{4}$$

$$x_p = -\frac{1}{4}t\cos 4t$$

$$x(t) = c_1\cos 4t + c_2\sin 4t - \frac{1}{4}t\cos 4t$$

$$x(0) = -\frac{1}{2} \qquad \Rightarrow \qquad -\frac{1}{2} = c_1$$

$$x'(t) = -4c_1\sin 4t + 4c_2\cos 4t - \frac{1}{4}\cos 4t + t\sin 4t$$

$$x'(0) = 0 \qquad \Rightarrow \qquad 0 = 4c_2 - \frac{1}{4} \qquad \Rightarrow c_2 = \frac{1}{16}$$

$$x(t) = -\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t - \frac{1}{4}t\cos 4t$$

Note $-\frac{1}{4}t\cos 4t$ oscillates between -*t* and *t*. The amplitude becomes bigger and bigger and eventually the spring with break. This phenomenon is called **resonance** – when the external force has the same circular frequency as the circular frequency of the associated undamped free system.

($-\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t$ is simply bounded oscillation).

Damped Forced Motion

5.1.3 Example – Damped: A 10 kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from equilibrium with an initial velocity of 1 m/s in the upward direction and with an applied external force f(t) = 5 sin t. Find the subsequent motion of the mass if the air resistance constant is 90. Find the transient state and the steady state.

$$m=10 \text{ kg}, \quad k=140 \text{ N/m}, \quad \beta=90, \quad f(t)=5\sin t$$

$$10x''+90x'+140x=5\sin t$$

$$x''+9x'+14x=\frac{1}{2}\sin t, \qquad x(0)=0,\ x'(0)=-1$$

Associated homogeneous equation:

$$r^2+9r+14=0 \;\Rightarrow\; (r+2)(r+7)=0 \;\Rightarrow\; r_1=-2,\ r_2=-7$$

$$x_c(t)=c_1e^{-2t}+c_2e^{-7t}$$

$$x_p=A\cos t+B\sin t \qquad x_p'=-A\sin t+B\cos t \qquad x_p''=-A\cos t-B\sin t$$

$$-A\cos t-B\sin t+9(-A\sin t+B\cos t)+14(A\cos t+B\sin t)\overset{set}{=}\frac{1}{2}\sin t$$

$$(13A+9B)\cos t+(-9A+13B)\sin t\overset{set}{=}\frac{1}{2}\sin t$$

$$13A+9B=0$$

$$-9A+13B=\frac{1}{2}$$ Using Cramer's Rule,

$$\text{A}=\frac{\begin{vmatrix} 13 & \frac{1}{2} \\ 9 & 0 \end{vmatrix}}{\begin{vmatrix} 13 & -9 \\ 9 & 13 \end{vmatrix}}=\frac{-\frac{9}{2}}{169+81}=-\frac{9}{500} \qquad \text{and} \qquad \text{B}=\frac{\begin{vmatrix} 1/2 & -9 \\ 0 & 13 \end{vmatrix}}{\begin{vmatrix} 13 & -9 \\ 9 & 13 \end{vmatrix}}=\frac{\frac{13}{2}}{169+81}=\frac{13}{500}$$

$$x_p=-\frac{9}{500}\cos t+\frac{13}{500}\sin t$$

$$x(t)=c_1e^{-2t}+c_2e^{-7t}-\frac{9}{500}\cos t+\frac{13}{500}\sin t$$

$$x(0)=0 \quad \Rightarrow \quad 0=c_1+c_2-\frac{9}{500}$$

$$x'(t) = -2c_1e^{-2t} - 7c_2e^{-7t} + \frac{9}{500}\sin t + \frac{13}{500}\cos t$$

$$x'(0) = -1 \qquad \Rightarrow \qquad -1 = -2c_1 - 7c_2 + \frac{13}{500}$$

$$c_1 + c_2 = \frac{9}{500} \qquad 2c_1 + 2c_2 = \frac{18}{500} \qquad -5c_2 = \frac{505}{500} \qquad c_1 = -\frac{9}{50}$$

$$-2c_1 - 7c_2 = -\frac{513}{500} \qquad -2c_1 - 7c_2 = -\frac{513}{500} \qquad c_2 = \frac{99}{500}$$

Solution: $x(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} - \frac{9}{500}\cos t + \frac{13}{500}\sin t$

Steady State: $-\frac{9}{500}\cos t + \frac{13}{500}\sin t$

Transient State: $-\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t}$

(Notice $\text{Transient State} \to 0$ as $t \to \infty$.)