# Proof of Case 3 in §4.3.

Recall • $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$,

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

• $i^2 = -1$

So, $i^{2m} = (-1)^m$ and

$i^{2m+1} = i(i^{2m}) = (-1)^m i$

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Euler's formula : $e^{ix} = \cos x + i \sin x$

Pf: $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(ix)^{2m+1}}{(2m+1)!}$

Let $n = 2m$.

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^{2m} x^{2m+1}}{(2m+1)!}$$

$$= \cos x + i \sin x$$

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So, $y = e^{(\alpha + \beta i)x}$

$= e^{\alpha x} e^{\beta i x}$

$= e^{\alpha x}(\cos \beta x + i \sin \beta x)$.

Similar statement for $y = e^{(\alpha - \beta i)x}$

Now, $ay'' + by' + cy = 0$

AE: $am^2 + bm + c = 0$

If the square roots are complex (conjugate) :

complex

$$m_1 = \alpha + \beta i$$
$$m_2 = \alpha - \beta i$$

So $y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha - \beta i)x}$

However, Complex solution with complex coefficients. We want real solution

Let $c_1 = 1$ and $c_2 = 1$. Then

$$y = e^{\alpha x} e^{\beta i x} + e^{\alpha x} e^{-\beta i x}$$
$$= e^{\alpha x}(\cos\beta x + i \sin\beta x)$$
$$+ e^{\alpha x}(\cos(-\beta x) + i \sin(-\beta x))$$

$\cos(-\beta x) = \cos(\beta x)$ even fn

$\sin(-\beta x) = -\sin(\beta x)$ odd fn

$$= e^{\alpha x}(\cos\beta x + i \sin\beta x) + e^{\alpha x}(\cos\beta x - i \sin\beta x)$$
$$= 2e^{\alpha x}\cos(\beta x).$$

So $y_1 = e^{\alpha x}\cos(\beta x)$ is a real solution to the DE.

Letting $c_1 = i$, $c_2 = -i$, get $y_2 = e^{\alpha x}\sin(\beta x)$.

So $y = c_3 e^{\alpha x}\cos\beta x + c_4 e^{\alpha x}\sin(\beta x)$.