Summary of § 5.1

$$m x''(t) + \beta x'(t) + k x = F(t)$$

$x(t)$: distance from equilibrium

— $F(t) = 0$ —

§ 5.1.1 and § 5.1.2 IVP with Homogeneous 2nd order linear equations with constant coefficients.

Auxiliary Equation Solutions:

$r_1, r_2 < 0$ — 2 distinct real roots: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ overdamped

$r < 0$ — 1 real root with mult. 2: $x(t) = (c_1 + c_2 t) e^{rt}$ Critically damped

for spring mass

Complex conjugate:

Case 1: $r = 0 \pm \omega i$ Undamped

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

Case 2: $r = a \pm bi$ Underdamped

$a < 0$ $$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

§5.1.1 Undamped Free Motion

$$mx''(t) + kx = 0$$

$\beta = 0$

$\mathcal{F}(t) = 0$

harmonic motion

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

→ imaginary roots

$r = \pm \omega i$

ω: circular frequency

$T = \dfrac{2\pi}{\omega}$ = period (length of one cycle)

Equil: $x(t) = 0$

Local max/min: $x'(t) = 0$

Compact Forms: $A = \sqrt{c_1^2 + c_2^2}$ amplitude

(6) $x(t) = A \sin(\omega t + \phi)$

where $\phi = \tan^{-1}\left(\dfrac{c_1}{c_2}\right)$ if (c_2, c_1) in QI or QIV

OR $\phi = \pi + \tan^{-1}\left(\dfrac{c_1}{c_2}\right)$ if (c_2, c_1) in QII OR QIII

(6)' $x(t) = A \cos(\omega t - \delta)$

$\delta = \dfrac{\pi}{2} - \phi$

OR (c_1, c_2) QI, QIV

$\delta = \tan^{-1}\left(\dfrac{c_2}{c_1}\right)$ OR

$\delta = \pi + \tan^{-1}\left(\dfrac{c_2}{c_1}\right)$ QII, QIII

§ 5.1.2 Damped Free Motion

$$mx'' + \beta x' + kx = 0 \quad , \quad m, \beta, k > 0$$

$$mr^2 + \beta r + k = 0$$

$$r_1, r_2 = \frac{-\beta \pm \sqrt{\beta^2 - 4km}}{2m}$$

$\beta^2 - 4km > 0$ $\quad x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r_1, r_2 < 0$

Overdamped (no oscillation)

$\beta^2 - 4km = 0$ $\quad x(t) = (c_1 + c_2 t) e^{rt}$

$r < 0$ critically damped

$\beta^2 - 4km < 0$ underdamped

$$x(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$a = -\frac{\beta}{2m}$ $\qquad$ b: quasi frequency

Quasi Period $= \frac{2\pi}{b}$.

Compact form : $x(t) = A e^{at} \sin(bt + \phi)$

§ 5.1-3 Forced Vibration

Now $f(t) \neq 0$

To solve, use § 4.4 or § 4.6 (Nonhomogeneous IVP)

$x = x_c + x_p$

I) Undamped

eg $f(t) = F_0 \cos \mu t$

Case 1: $\omega \neq \mu$.

No HW

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + A \cos \mu t + B \sin \mu t$$

using 4.4

Case 2: $\omega = \mu$

HW 41 + 42

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \underbrace{t(A \cos \omega t + B \sin \omega t)}_{\text{oscillate bet'l } t \text{ and } -t}$$

As $t \uparrow$, reach resonance and spring breaks.

HW 33 + 34 + 35

II) Damped

eg $f(t) = F_0 \cos \mu t$

$x(t) = x_c(t) + x_p(t)$

$$x(t) = \underbrace{c_1 x_1 + c_2 x_2}_{} + \underbrace{A \cos \mu t + B \sin \mu t}_{}$$

depends on
underdamped,
critically damped,
underdamped

↓

transient state

$x_c(t) \to 0$ as $t \to \infty$

Steady state

$x(t) \approx x_p(t)$
as t gets
very large.