Example 1.2.1

Problem: Solve the following system using Gaussian elimination with back substitution:

$$\begin{aligned} v - w &= 3, \\ -2u + 4v - w &= 1, \\ -2u + 5v - 4w &= -2. \end{aligned}$$

Solution: The associated augmented matrix is

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ -2 & 4 & -1 & 1 \\ -2 & 5 & -4 & -2 \end{array}\right).$$

Since the first pivotal position contains 0, interchange rows one and two before eliminating below the first pivot:

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ -2 & 4 & -1 & 1 \\ -2 & 5 & -4 & -2 \end{array}\right) \xrightarrow{\textit{Interchange } R_1 \textit{ and } R_2} \left(\begin{array}{ccc|c} -2 & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ -2 & 5 & -4 & -2 \end{array}\right) R_3 - R_1$$

$$\longrightarrow \left(\begin{array}{ccc|c} -2 & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -3 & -3 \end{array}\right) R_3 - R_2 \longrightarrow \left(\begin{array}{ccc|c} -2 & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & -6 \end{array}\right).$$

Back substitution yields

$$\begin{aligned} w &= \frac{-6}{-2} = 3, \\ v &= 3 + w = 3 + 3 = 6, \\ u &= \frac{1}{-2}(1 - 4v + w) = \frac{1}{-2}(1 - 24 + 3) = 10. \end{aligned}$$

Exercises for section 1.2

1.2.1. Use Gaussian elimination with back substitution to solve the following system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 + 2x_2 + 2x_3 &= 1, \\ x_1 + 2x_2 + 3x_3 &= 1. \end{aligned}$$

1.2.2. Apply Gaussian elimination with back substitution to the following system:

$$\begin{aligned} 2x_1 - x_2 \phantom{{}- x_3} &= 0, \\ -x_1 + 2x_2 - x_3 &= 0, \\ -x_2 + x_3 &= 1. \end{aligned}$$

1.2.3. Use Gaussian elimination with back substitution to solve the following system:

$$\begin{aligned} 4x_2 - 3x_3 &= 3, \\ -x_1 + 7x_2 - 5x_3 &= 4, \\ -x_1 + 8x_2 - 6x_3 &= 5. \end{aligned}$$

1.2.4. Solve the following system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1, \\ x_1 + x_2 + 3x_3 + 3x_4 &= 3, \\ x_1 + x_2 + 2x_3 + 3x_4 &= 3, \\ x_1 + 3x_2 + 3x_3 + 3x_4 &= 4. \end{aligned}$$

1.2.5. Consider the following three systems where the coefficients are the same for each system, but the right-hand sides are different (this situation occurs frequently):

$$\begin{aligned} 4x - 8y + 5z &= 1 \mid 0 \mid 0, \\ 4x - 7y + 4z &= 0 \mid 1 \mid 0, \\ 3x - 4y + 2z &= 0 \mid 0 \mid 1. \end{aligned}$$

Solve all three systems at one time by performing Gaussian elimination on an augmented matrix of the form

$$[\mathbf{A} \mid \mathbf{b_1} \mid \mathbf{b_2} \mid \mathbf{b_3}].$$

1.2.6. Suppose that matrix $\mathbf{B}$ is obtained by performing a sequence of row operations on matrix $\mathbf{A}$. Explain why $\mathbf{A}$ can be obtained by performing row operations on $\mathbf{B}$.

1.2.7. Find angles α, β, and γ such that

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3, \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2, \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9, \end{aligned}$$

where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

1.2.8. The following system has no solution:

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 1, \\ -x_1 + 4x_2 - 3x_3 &= 0, \\ -x_1 + 5x_2 - 4x_3 &= 0. \end{aligned}$$

Attempt to solve this system using Gaussian elimination and explain what occurs to indicate that the system is impossible to solve.

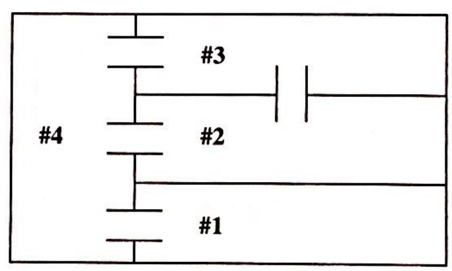
1.2.9. Attempt to solve the system

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 4, \\ -x_1 + 4x_2 - 3x_3 &= 5, \\ -x_1 + 5x_2 - 4x_3 &= 6, \end{aligned}$$

using Gaussian elimination and explain why this system must have infinitely many solutions.

1.2.10. By solving a 3×3 system, find the coefficients in the equation of the parabola $y = \alpha + \beta x + \gamma x^2$ that passes through the points $(1, 1)$, $(2, 2)$, and $(3, 0)$.

1.2.11. Suppose that 100 insects are distributed in an enclosure consisting of four chambers with passageways between them as shown below.



At the end of one minute, the insects have redistributed themselves. Assume that a minute is not enough time for an insect to visit more than one chamber and that at the end of a minute 40% of the insects in each chamber have not left the chamber they occupied at the beginning of the minute. The insects that leave a chamber disperse uniformly among the chambers that are directly accessible from the one they initially occupied—e.g., from #3, half move to #2 and half move to #4.

(a) If at the end of one minute there are 12, 25, 26, and 37 insects in chambers #1, #2, #3, and #4, respectively, determine what the initial distribution had to be.

(b) If the initial distribution is 20, 20, 20, 40, what is the distribution at the end of one minute?

1.2.12. Show that the three types of elementary row operations discussed on p. 8 are not independent by showing that the interchange operation (1.2.7) can be accomplished by a sequence of the other two types of row operations given in (1.2.8) and (1.2.9).

1.2.13. Suppose that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix associated with a linear system. You know that performing row operations on $[\mathbf{A}|\mathbf{b}]$ does not change the solution of the system. However, no mention of *column operations* was ever made because column operations can alter the solution.

(a) Describe the effect on the solution of a linear system when columns $\mathbf{A}_{*j}$ and $\mathbf{A}_{*k}$ are interchanged.

(b) Describe the effect when column $\mathbf{A}_{*j}$ is replaced by $\alpha\mathbf{A}_{*j}$ for $\alpha \neq 0$.

(c) Describe the effect when $\mathbf{A}_{*j}$ is replaced by $\mathbf{A}_{*j} + \alpha\mathbf{A}_{*k}$.

Hint: Experiment with a 2×2 or 3×3 system.

1.2.14. Consider the $n \times n$ ***Hilbert matrix*** defined by

$$\mathbf{H} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}.$$

Express the individual entries h_{ij} in terms of i and j.

1.2.15. Verify that the operation counts given in the text for Gaussian elimination with back substitution are correct for a general 3×3 system. If you are up to the challenge, try to verify these counts for a general $n \times n$ system.

1.2.16. Explain why a linear system can never have exactly two different solutions. Extend your argument to explain the fact that if a system has more than one solution, then it must have infinitely many different solutions.