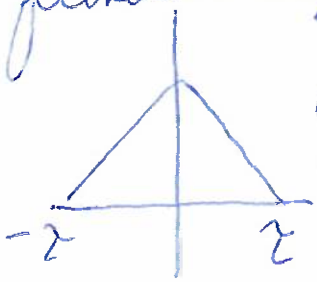
§ 7.5 The Dirac Delta Function

Impulse Function

Let $\tau > 0$, τ small. Then consider the function $d_\tau(t)$ (Book $\delta(t-\tau)$).



$y = d_\tau(t)$

$$\int_{-\tau}^{\tau} d_\tau(t)\,dt = 1$$
$$\| $$
$$\int_{-\infty}^{\infty} d_\tau(t)\,dt$$

impulse over infinitesimal period of time where.

(Above, continuous. In book, discont. rectangles. Doesn't matter.)

Def: $\lim_{\tau \to 0} d_\tau(t)$ is called the Dirac Delta "function" and is denoted by $\delta(t - t_0)$

→ NOT really a function, but a distribution.

Impulse of total size 1 that is concentrated at $t = t_0$. (Not in nature, but if something occurs in very short amt of time, can

$\delta(t-t_0)$ has the following property:

when $t_0 = 0$, $\mathcal{L}\{\delta(t)\} = 1$

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write now

T. 7.5.1 Transform of Dirac Delta Function

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}.$$

(Corollary: When $t_0 = 0$, this become $e^0 = 1$.)

Aside: NOT necessary

[In old book, harder problems:

$$\mathcal{L}\{f(t)\,\delta(t-t_0)\} = e^{-st_0} f(t_0)$$].

Now we will be able to solve IVP of the form

$$ay'' + by' + cy = \underbrace{f(t)}\,\underbrace{\delta(t-t_0)}, \quad y(0) = y_0,\ y'(0) = y_0'$$

where $f(t)$ a constant function

impulse function concentrated at t_0

eg $y'' - y' - 2y = 3\delta(t-1)$; $y(0)=0, y'(0)=1$

$$(s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1}) - (sY(s) - \underbrace{y(0)}_{0}) - 2Y(s) = 3e^{-s}$$

$$Y(s)(s^2 - s - 2) = 3e^{-s} + 1$$

AE in s.
Check.

$$Y(s) = \frac{3e^{-s}}{(s+1)(s-2)} + \frac{1}{(s+1)(s-2)}$$

$$\frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s+1)$$

$s=2$: $1 = 3B \Rightarrow B = 1/3$

$s=-1$: $1 = -3A \Rightarrow A = -1/3$

$$Y(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1} + e^{-s}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

the 3's cancel.

$$y(t) = \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} + u(t-1)\left(e^{2(t-1)} - e^{-(t-1)}\right)$$

$a=-1$
Add a.

eg $y'' + y = u(t - \frac{\pi}{2}) + 3\delta(t - \frac{3\pi}{2}) - u(t - 2\pi)$,

$y(0) = 0$

$y'(0) = 0$

$$Y(s)(s^2+1) = \frac{e^{-\frac{\pi}{2}s}}{s} + 3e^{-\frac{3\pi}{2}s} - \frac{e^{-2\pi s}}{s}$$

$$Y(s) = \frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)} + \frac{3e^{-\frac{3\pi}{2}s}}{s^2+1} - \frac{e^{-2\pi s}}{s(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs + C)s$$

$s = 0$: $1 = A + 0 \Rightarrow A = 1$

s^2: $0 = A + B \Rightarrow B = -1$

s: $0 = C$

$$Y(s) = e^{-\frac{\pi}{2}s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 3e^{-\frac{3\pi}{2}s}\left(\frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$y(t) = u(t - \tfrac{\pi}{2})\left(1 - \cos(t - \tfrac{\pi}{2})\right)$$

$f(t) = 1 = f(t - \frac{\pi}{2})$

Constant

$$+ 3u(t - \tfrac{3\pi}{2})\left(\sin(t - \tfrac{3\pi}{2})\right) - u(t - 2\pi)\left(1 - \cos(t - 2\pi)\right)$$

$$y(t) = u(t-\tfrac{\pi}{2})(1-\sin t) + 3u(t-\tfrac{3\pi}{2})\cos t$$
$$- u(t-2\pi)\underbrace{(1-\cos t)}.$$

the only one
that you would
need to do

$\cos(t-\frac{\pi}{2}) = \sin t$

$\sin(t-\frac{3\pi}{2}) = \cos t$

Cofunction identity
& even "

OR $\sin t \underbrace{\cos\frac{3\pi}{2}}_{0} - \cos t \underbrace{\sin\frac{3\pi}{2}}_{-1}$

$\sin(t+2\pi n) = \sin t$

$\cos(t+2\pi n) = \cos t$

periodicity
* must use

$\left(\sin(t-\pi) = -\sin t \quad \text{useful}\right)$