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Due Tuesday 17 October 2023, 11 AM

Linear Algebra, Math 2101-003

Homework set for extra credit

(It is not mandatory to turn this in)

Consider a $n \times n$ projection P, i.e., $P^2 = P$. It projects onto $W = \mathcal{R}(P)$ along (or parallel to) $V = \mathcal{N}(P)$. That is, if $v \in V$, then $Pv = 0$ and if $w \in W$, $Pw = w$. Recall that for any vector space with a norm (here $\mathbb{R}^n$), we can define the matrix norm

$$\|P\| = \max_{\|x\|=1} \|Px\|.$$

1. (1 point). Show that if $P \neq 0$, $\|P\| \geq 1$.

2. (1 point). Show that if $P \neq 0$ and if $V \perp W$, that is $P^T = P$, then $\|P\| = 1$.

3. (4 points). Show that if $P \neq 0$, and $P \neq I$, then $\|I - P\| = \|P\|$.

① Given $P^2 = P$

If $P \neq 0$ then $0 \neq \|P\| = \|P^2\| \leqslant \|P\|^2$

$\therefore \|P\| \leqslant \|P\|^2$

$\therefore 1 \leqslant \|P\|$ (divide by $\|P\| \neq 0$) or $\|P\| \geqslant 1$ QED!

② Take a non-zero vector x such that $x = Ix = (P + I - P)x = Px + (I-P)x$

Consider $\langle Px, (I-P)x \rangle$

$= (Px)^T (I-P)x$ by definition of matrix norm

$= x^T P^T (I-P)x$ by property of transposition $(AB)^T = B^T A^T$

$= x^T P (I-P)x$ since $P^T = P$ (given)

$= x^T (PI - P^2)x$ by distributive law

$= x^T (P - P)x$ since $P^2 = P$ (given) & $PI = P$

$= x^T 0 x = 0$ $\therefore \langle Px, (I-P)x \rangle = 0$

$\therefore \|x\|^2 = \|Px\|^2 + \underbrace{\|(I-P)x\|^2}_{\geq 0} \geq \|Px\|^2$

$\therefore \|x\|^2 \geq \|Px\|^2$

$x^T x \geq (Px)^T Px$ by definition of matrix norm

$x^T x - x^T P^T Px \geq 0$ by property of transposition $(AB)^T = B^T A^T$

$x^T x (1 - P^T P) \geq 0$ by associative law

$x^T x (1 - \|P\|^2) \geq 0$ by definition of matrix norm

$\underbrace{\|x\|^2}_{> 0 \ \forall x \neq 0} (1 - \|P\|^2) \geq 0$

$\therefore 1 - \|P\|^2 \geq 0 \quad \therefore \|P\|^2 \leq 1 \quad \therefore \|P\| \leq 1$

but $\|P\| \geq 1$ (part (a))

$\therefore \|P\| = 1$

QED!

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