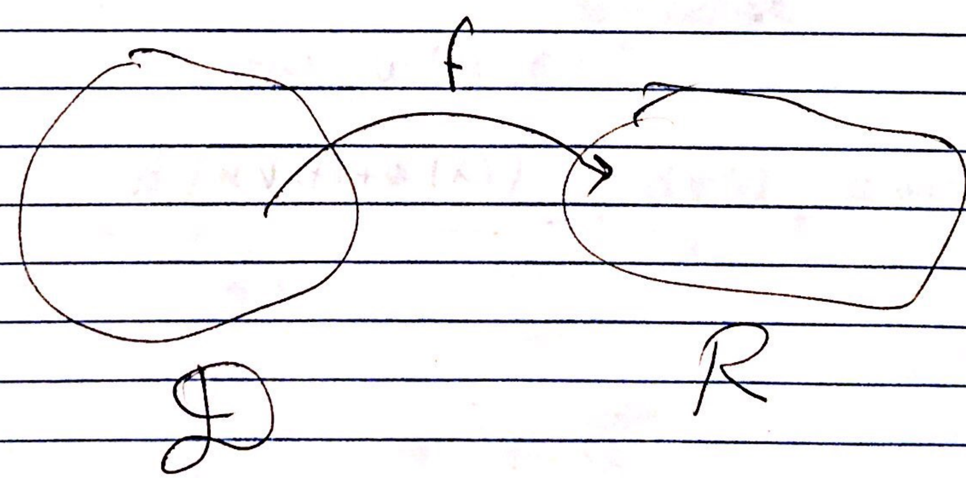
Section 3.3 Linear Transformations

Transformations or maps (or functions) from one set $\mathcal{D}$ (the domain) to another, $\mathcal{R}$.

These sets must allow for addition of its elements as well as scalar multiplication.

examples of such sets: $\mathbb{R}^2$, $\mathbb{R}$, $\mathbb{R}^3$, $\mathbb{R}^{5\times 2}$, $\mathcal{C} = \{\text{continuous functions}\}$, $\mathcal{C}^1 = \{\text{functions with continuous first derivatives}\}$,

$\Pi_n = \{\text{set of polynomials of degree} \leq n\}$.



A function is linear if

f of the sum = sum of the f

$$f(x+y) = f(x) + f(y)$$

and $$f(\alpha x) = \alpha f(x).$$

or in one statement

$$f(\alpha x + y) = \alpha f(x) + f(y)$$

$$\forall x, y \in D$$

↑ for all

Example. $f: C^1 \to C$

derivative

let $v(x), w(x) \in C'$

$$\frac{d\left(\alpha v(x) + w(x)\right)}{dx} = \alpha \frac{dv(x)}{dx} + \frac{dw(x)}{d(x)}$$

linear ✓

$f: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f\begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} a+b \\ b-c \\ 2a \end{vmatrix}$

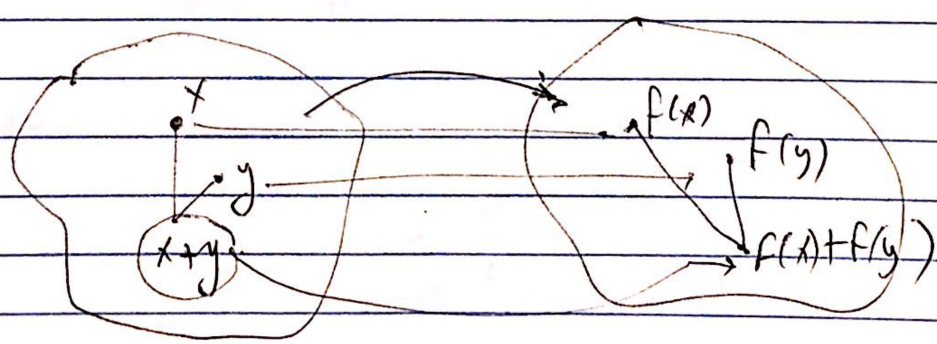
Take $x, y \in \mathbb{R}^3$

$$f(\alpha x + y) = f\begin{bmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha x_1 + y_1) + (\alpha x_2 + y_2) \\ (\alpha x_2 + y_2) - (\alpha x_3 + y_3) \\ 2(\alpha x_1 + y_1) \end{bmatrix} =$$

$$= \alpha \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 - y_3 \\ 2y_1 \end{bmatrix} = \alpha f(x) + f(y)$$

Linear ✓



Nice examples of linear transformations

$\mathbb{R}^2 \to \mathbb{R}^2 \quad (\text{or } \mathbb{R}^n \to \mathbb{R}^n$

1. rotation by a fixed angle θ
2. Reflection
3. Projection

Examples of Linear Map on Matrices

$$f(A) = A^T$$

$$(\alpha A + B)^T = \alpha A^T + B^T$$

$A_{n \times n}$

$$\text{trace } A = \sum_{i=1}^{n} a_{ii} \quad (\text{sum of diag entries})$$

$$\text{trace}: \mathbb{R}^{n \times n} \to \mathbb{R}$$

$$\text{trace}(\alpha A + B) = \sum_i \alpha a_{ii} + b_{ii} =$$

$$= \alpha \sum a_{ii} + \sum b_{ii}$$

$$= \alpha \text{ trace}(A) + \text{trace}(B)$$

Read book!

For every linear transformation f We can define two important sets

$$N(f) = Ker(f) = \{x \in D \,/\, f(x) = 0\}$$

Null space of f, or Kernel of f.

All the elements of D which are mapped onto zero.

Note that $f(0) = 0$ always!

Many proofs. e.g. let $\alpha = 0$

$$f(0) = f(0 \cdot x) = 0 \cdot f(x) = 0$$

or let $0 = x - x$

$$f(0) = f(x) - f(x) = 0.$$

$N(f)$ never empty!

Range of f $\quad R(f) = \{y \in R \,/\, \exists x \in D,\ f(x) = y\}$

all the elements in R which can be reached by f. Note again $0 \in R(f)$ always.

We know how to multiply a matrix $A \in \mathbb{R}^{m \times n}$ times a vector $x \in \mathbb{R}^n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 2 \cdot (-1) + 1 \cdot 0 \\ 1 \cdot 2 + (-1) \cdot (-1) + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$A \cdot x$$

Matrix-vector multiplication is a linear transformation

$$A(\alpha x + y) = \alpha A x + A y$$

In fact every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix.

For example. Matrix for f in p.(62)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b-c \\ 2a \end{bmatrix}$$

or $$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \\ 2x_1 \end{bmatrix}$$

So, for example

Rotation by θ

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$