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Section 2.4

Homogeneous linear systems are linear systems $Ax=b$, where $b=0$

If $b \neq 0$ this is an nonhomogeneous linear system. (next section)

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So, we want to solve $Ax=0$

Note $A.0=0$ always!

$$a_{11}0+a_{12}0+\cdots+a_{1n}0=0$$
$$a_{21}0\cdots$$
$$\vdots$$
$$a_{m1}0+a_{m2}0+\cdots+a_{mn}0=0$$

This is called the trivial solution to the homogeneous system.

Example. $\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$$2.0+1.0=0$$
$$-1.0+3.0=0$$

If $Ax=0$ has a unique solution then it must be the trivial solution.

Repeat. $Ax=0$ cannot have no solutions ($x=0$ always a solution)

If $Ax=0$ has infinitely many solutions, we want to find them all.

How? Use the Echelon form (equivalent system - same solutions).

Take the 3×4 example in p. 20 (section 2.3 Notes)

and make it homogeneous

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We already did $[A|b] \to [E|c]$

and here $[A|0] \to [E|0]$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$2x_3 + 2x_4 = 0$$

Write the basic variables x_1, x_3 in terms of the free variables.

[Now you see they are called free variables since they can take <u>any</u> value].

Second equation says $2x_3 + 2x_4 = 0$

that is $x_3 = -x_4$

First equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$x_1 = -2x_2 - 3(-x_4) - 4x_4$$

$$x_1 = -2x_2 - x_4$$

solution
$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{vmatrix} = x_2 \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} + x_4 \begin{vmatrix} -1 \\ 0 \\ -1 \\ 1 \end{vmatrix}$$

$$x_{h_1} = \begin{vmatrix} -2 \\ 1 \\ 0 \\ 0 \end{vmatrix} \qquad x_{h_2} = \begin{vmatrix} -1 \\ 0 \\ -1 \\ 1 \end{vmatrix}$$

one solution for each free variable

<u>check!</u>

Do not confuse with the particular solution - Free variables = 0. solve for x_p.

~~All~~ All solutions of the homogeneous system are linear combinations of these (two in this case, as many as free variables)

$$x_h = \alpha_1 x_{h_1} + \alpha_2 x_{h_2}$$

for all values of α_1 α_2 scalars i.e. arbitrary numbers.

Note in particular $\alpha_1 = 0$ $\alpha_2 = 0$ gives you the trivial solution

How many x_{h_i} ? $n - r$, r rank A

= number of free variables.

If $r = n$, only the trivial solution

like in the example of p 29

$$\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 7/2 \end{bmatrix}$$

$E_2 + \frac{1}{2} E_1$

$3 + \frac{1}{2} = \frac{7}{2}$

2 pivots rank 2. two colums.

~~wi~~ $n - r = 2 - 2 = 0$

And you can even solve
back substitution

$7/2 \, x_2 = 0$

$x_2 = 0$

$2x_1 + 1 \cdot 0 = 0 \qquad x_1 = 0/2 = 0$

$$X = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}.$$

Let us consider again the additional example of section 2.3 (p.22). (Also p. 26). Now homogeneous.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 2 & 4 & 2 & 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2 basic colums
3 free variables
$x_3 \; x_4 \; x_5$

Second equation $\quad x_2 = 0$

First equation

$$x_1 + \underset{\uparrow = 0}{x_2} + x_3 + x_4 + x_5 = 0$$

$$x_1 = -x_3 - x_4 - x_5$$

solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 - x_5 \\ 0 \\ x_3 \\ x_4 \\ \cancel{x_3} x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

3 free variables

$$x_{h_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad x_{h_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad x_{h_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

check!

All solutions of $Ax=0$ are of the form

$$x_h = \alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \alpha_3 x_{h_3} \qquad \alpha_1\ \alpha_2\ \alpha_3 \in \mathbb{R}$$

In general.

If there are $p = n - r$ free variables we can find $x_{h_1}\ x_{h_2} \dots x_{h_p}$ nontrivial solutions to $Ax=0$ (the homogeneous system).

All solutions of $Ax=0$ are of the form $x_h = \alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \dots + \alpha_p x_{h_p}$

$\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_p \in \mathbb{R}$ (real numbers),

that is all linear combinations of the x_{h_i} $i = 1, \dots p$.