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Solutions to HW#5

1. (a).

In HW#1 (exercise 1.25) we showed that given $A = \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix}$, the solution to

$AX = I$ is $X = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix}$,

an thus $A^{-1} = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix}$.

(b) Check that $AA^{-1} = I$

$$\begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Since.

$4 \cdot 2 + (-8)4 + 5 \cdot 5 = 8 - 32 + 25 = 1$

$4 \cdot (-4) + (-8)(-7) + 5(-8) = -16 + 56 + 40 = 0$

$4 \cdot 3 + (-8) \cdot 4 + 5 \cdot 4 = 12 - 32 + 20 = 0$

$4 \times 2 + (-7) \cdot 4 + 4 \cdot 5 = 8 - 28 + 20 = 0$

$4 \cdot (-4) + (-7)(-7) + 4 \cdot (-8) = -16 + 49 - 32 = 1$

$4 \cdot 3 + (-7) \cdot 4 + 4 \cdot 4 = 12 - 28 + 16 = 0$

$3 \cdot 2 + (-4)(4) + 2 \cdot 5 = 6 - 16 + 10 = 0$

$3 \cdot (-4) + (-4) \cdot (-7) + 2(-8) = -12 + 28 - 16 = 0$

$3 \cdot 3 + (-4) \cdot 4 + 2 \cdot 4 = 9 - 16 + 8 = 1$

check $A^{-1}A = I$

$$\begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix} \begin{bmatrix} 4 & -8 & 5 \\ 4 & -7 & 4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ since

$2.4 + (-4).4 + 3.3 = 8 - 16 + 9 = 1$

$2.(-8) + (-4).(-7) + 3(-4) = -16 + 28 - 12 = 0$

$2.5 + (-4).4 + 3.2 = 10 - 16 + 6 = 0$

$4.4 + (-7).4 + 4.3 = 16 - 28 + 12 = 0$

$4.(-8) + (-7)(-7) + 4(-4) = -32 + 49 - 16 = 1$

$4.5 + (-7).4 + 4.2 = 20 - 28 + 8 = 0$

$5.4 + (-8).4 + 4.3 = 20 - 32 + 12 = 0$

$5.(-8) + (-8)(-7) + 4(-4) = -40 + 56 - 16 = 0$

$5.5 + (-8).4 + 4.2 = 25 - 32 + 8 = 1$

c) $$(A^{-1})^T = \begin{bmatrix} 2 & 4 & 5 \\ -4 & -7 & -8 \\ 3 & 4 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & 4 & 3 \\ -8 & -7 & -4 \\ 5 & 4 & 2 \end{bmatrix}$$

to check we multiply $A^T \cdot (A^{-1})^T$ and see if we obtain I.

or $(A^{-1})^T A^T$

$(A^{-1})^T \cdot A^T$

$$\begin{bmatrix} 2 & 4 & 5 \\ -4 & -7 & -8 \\ 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 & 4 & 3 \\ -8 & -7 & -4 \\ 5 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since

$2 \cdot 4 + 4 \cdot (-8) + 5 \cdot 5 = 8 - 32 + 25 = 1$

$2 \cdot 4 + 4 \cdot (-7) + 5 \cdot 4 = 8 - 28 + 20 = 0$

$2 \cdot 3 + 4 \cdot (-4) + 5 \cdot 2 = 6 - 16 + 10 = 0$

$(-4) \cdot 4 + (-7) \cdot (-8) + (-8) \cdot 5 = -16 + 56 - 40 = 0$

$(-4) \cdot 4 + (-7)(-7) + (-8) \cdot 4 = -16 + 49 - 32 = 1$

$(-4) \cdot 3 + (-7) \cdot (-4) + (-8) \cdot 2 = -12 + 28 - 16 = 0$

$3 \cdot 4 + 4 \cdot (-8) + 4 \cdot 5 = 12 - 32 + 20 = 0$

$3 \cdot 4 + 4 \cdot (-7) + 4 \cdot 4 = 12 - 28 + 16 = 0$

$3 \cdot 3 + 4 \cdot (-4) + 4 \cdot 2 = 9 - 16 + 8 = 1$

(3a).
$$\left[\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 4 & -7 & 4 & 1 \\ 3 & -4 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -7/4 & 1/4 \end{array}\right]$$

$m_{21} = 1 \quad m_{31} = \frac{3}{4}$

(just as in HW 1)

$$m_{32} = 2 \rightarrow \left[\begin{array}{ccc|c} 4 & -8 & 5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1/4 & 1/4 \end{array}\right]$$

thus $x_3 = 1$

$x_2 = 1$ $\quad X = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

$x_1 = 1$

Compute $A^{-1}b = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-4+3 \\ 4-7+4 \\ 5-8+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓

(b). Solution is unique because A is invertible, i.e. non singular, and thus full rank, there are no free variables, which means the homogeneous system has only the trivial solution.

since $x = x_p = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ is a solution

and I can only add $x_h = 0$ to it, x is the only solution.

(2)

Let $Q = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

(a) $Q^2 = Q \cdot Q = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} =$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$

(b). since $Q \cdot Q = I$ then $Q^{-1} = Q$