Name: Elle Nguyen

Daniel B. Szyld

7 November 2023

Due Tuesday 14 November 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #7

Consider the following singular value decompositions

$$A = \begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{2\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{vmatrix}.$$

(a) (4pts.) Exhibit the matrix B of rank 1 which is closest to A in 2-norm.
(b) (3pts.) What is the value of $\|B - A\|_2$? Explain.
(c) (3 pts.) Exhibit a different singular value decomposition of A, i.e., where some of the factors are not exactly the same.

(a) $$A = \frac{\sqrt{2}}{2} \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \frac{\sqrt{6}}{6} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{vmatrix}$$

$$B = \sigma_1 u_1 v_1^T = \frac{\sqrt{2}}{2} \times \frac{\sqrt{6}}{6} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \begin{vmatrix} 3 \end{vmatrix} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \end{vmatrix} = \frac{\sqrt{3}}{2} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \end{vmatrix}$$

$$= \frac{\sqrt{3}}{2} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & \sqrt{3} & 1 \\ 0 & 0 & 0 \end{vmatrix} = \frac{\sqrt{3}}{6} \begin{vmatrix} 3\sqrt{2} & 3\sqrt{3} & 3 \\ 3\sqrt{2} & 3\sqrt{3} & 3 \\ 0 & 0 & 0 \end{vmatrix}$$

(b) $$A = \frac{\sqrt{2}}{2} \times \frac{\sqrt{6}}{6} \begin{vmatrix} 3 & -1 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{vmatrix} = \frac{\sqrt{3}}{6} \begin{vmatrix} 2\sqrt{2} & 3\sqrt{3} & 5 \\ 4\sqrt{2} & 3\sqrt{3} & 1 \\ 2 & -\sqrt{6} & \sqrt{2} \end{vmatrix}$$

$$B - A = \frac{\sqrt{3}}{6} \begin{vmatrix} \sqrt{2} & 0 & -2 \\ -\sqrt{2} & 0 & 2 \\ -2 & \sqrt{6} & -\sqrt{2} \end{vmatrix} = \frac{-\sqrt{6}}{6} \begin{vmatrix} -1 & 0 & \sqrt{2} \\ 1 & 0 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{3} & 1 \end{vmatrix}$$

$$B = \sum_{i=1}^{k} \sigma_i u_i v_i^T \rightarrow \|B - A\|_2 = \left\| \sum_{i=k+1}^{k} \sigma_i u_i v_i^T \right\| = \sigma_{max} = \sigma_{k+1} = \sigma_{1+1}$$
$$= \sigma_2 = 1$$

$\therefore$ The norm of $(B-A)$ is the largest singular value at the $(k+1)$th order

$\therefore \|B - A\|_2 = \sigma_{k+1} = \sigma_{1+1} = \sigma_2 = 1 \qquad (\text{rank} B = k = 1)$

(c) $\sigma_2 = \sigma_3 = 1$

SVD of A can be constructed by switching col 2 and col 3 of matrix U & row 2 and row 3 of matrix V^T

$$A = \frac{\sqrt{2}}{2} \begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{vmatrix} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \frac{\sqrt{6}}{6} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{vmatrix}$$

$$= \frac{\sqrt{3}}{6} \begin{vmatrix} 3 & 0 & -1 \\ 3 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{vmatrix} \begin{vmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{vmatrix} = \frac{\sqrt{3}}{6} \begin{vmatrix} 2\sqrt{2} & 3\sqrt{3} & 5 \\ 4\sqrt{2} & 3\sqrt{3} & 1 \\ 2 & -\sqrt{6} & \sqrt{2} \end{vmatrix} = A \text{ (part b)} \checkmark$$

$$\therefore A = \begin{vmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & 0 & -\sqrt{6}/3 \end{vmatrix}$$