Homework #12 Daniel B. Szyld

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1.

$$v_1 = \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix} \qquad v_2 = \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$$

$$v_1^T v_2 = 1 - 2 + 1 + 0 = 0 \qquad \cos\theta = 0 \qquad \theta = \frac{\pi}{2}$$

Vectors v_1 and v_2 are orthogonal.

To find a vector $\perp$ to both v_1 v_2 we take $v \notin \operatorname{span}\{v_1, v_2\}$ and do Gram Schmidt

For example $v = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}$.

First we normalize v_1 and v_2

$$\|v_1\| = \sqrt{1+4+1+1} = \sqrt{7}$$

$$\|v_2\| = \sqrt{1+1+1} = \sqrt{3}$$

$$q_1 = \frac{1}{\sqrt{7}} \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix}, \quad q_2 = \frac{1}{\sqrt{3}} \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$$

$$w_3 = v - \langle v, q_1 \rangle q_1 - \langle v, q_2 \rangle q_2$$

$$= \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \left(\frac{1}{\sqrt{7}} \cdot 3\right) \cdot \frac{1}{\sqrt{7}} \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix} - \left(\frac{1}{\sqrt{3}} \cdot (-1)\right) \cdot \frac{1}{\sqrt{3}} \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \frac{3}{7} \begin{vmatrix} 1 \\ 2 \\ -1 \\ 1 \end{vmatrix} + \frac{1}{3} \begin{vmatrix} 1 \\ -1 \\ -1 \\ 0 \end{vmatrix} = \frac{1}{21} \begin{vmatrix} 19 \\ -4 \\ 23 \\ 12 \end{vmatrix}$$

(It is immediate to check that $w_3^T q_1 = 0$, $w_3^T q_2 = 0$)

Since $1 - \frac{3}{7} + \frac{1}{3} = \frac{21-9+7}{21} = \frac{19}{21}$

$$1 - \frac{6}{7} - \frac{1}{3} = \frac{21-18-7}{21} = \frac{-4}{21}$$

$$1 + \frac{3}{7} - \frac{1}{3} = \frac{21+9-7}{21} = \frac{23}{21}$$

$$1 - \frac{3}{7} = \frac{21-9}{21} = \frac{12}{21}$$

For completeness, though not needed for this exercise, we normalize w_3. To compute $\|w_3\|$, we note that

$$19^2 + 4^2 + 23^2 + 12^2 = 361 + 16 + 529 + 144 = 1050 = 21 \times 50$$

so that $\|w_3\| = \sqrt{\frac{1}{21} \cdot \frac{1}{21} \cdot 21 \times 50} = \sqrt{\frac{50}{21}}$

$$r_3 = \|w_3\| \qquad q_3 = \frac{\sqrt{21}}{\sqrt{50}} \cdot \frac{1}{21} \begin{vmatrix} 19 \\ -4 \\ 23 \\ 12 \end{vmatrix} = \frac{1}{\sqrt{21}\sqrt{50}} \begin{vmatrix} 19 \\ -4 \\ 23 \\ 12 \end{vmatrix}$$

(2) (a)

$$v_1 = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} \qquad v_2 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix}$$ + again Gram Schmidt

$$\|v_1\| = \sqrt{1+4+9+1} = \sqrt{15} \qquad q_1 = \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}$$

$$w_2 = v_2 - \langle q_1, v_2 \rangle q_1 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix} - \frac{1}{\sqrt{15}}(3+2) \cdot \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 0 \\ 1 \\ 2 \end{vmatrix} - \frac{1}{3} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$$ and it checks $w_2^T v_1 = 0$

Normalize $\nu_2 = \|w_2\| = \frac{1}{3}\sqrt{1+4+25} = \frac{\sqrt{30}}{3}$

$$q_2 = \frac{1}{\nu_2} w_2 = \frac{3}{\sqrt{30}} \cdot \frac{1}{3} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix} = \frac{1}{\sqrt{30}} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$$

q_1, q_2 basis of $S = \text{Span}\{v_1, v_2\}$ and $q_1^T q_2 = 0$

$\|q_1\| = \|q_2\| = 1.$

(b) It will suffice to find two vectors $z_1, z_2 \notin S$ (linearly independent) and do Gram-Schmidt.

For example $z_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$, $z_2 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$.

$$w_3 = z_1 - \langle z_1, q_1 \rangle q_1 - \langle z_1, q_2 \rangle q_2$$

$$= \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} - \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} - \left(-\frac{1}{\sqrt{30}}\right) \cdot \frac{1}{\sqrt{30}} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} - \frac{1}{15} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} + \frac{1}{30} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix} = \frac{1}{10} \begin{vmatrix} 9 \\ -2 \\ -2 \\ 1 \end{vmatrix}$$

$$1 - \frac{1}{15} - \frac{1}{30} = \frac{30-2-1}{30} = \frac{27}{30} = \frac{9}{10}$$

$$0 - \frac{2}{15} - \frac{2}{30} = \frac{-4-2}{30} = \frac{-6}{30} = \frac{-2}{10}$$

$$0 - \frac{3}{15} + 0 = -\frac{6}{30} = \frac{-2}{10}$$

$$0 - \frac{1}{15} + \frac{5}{30} = \frac{-2+5}{30} = \frac{3}{30} = \frac{1}{10}$$

And we can check that

$w_3 \perp q_1$

$w_3 \perp q_2$

Similarly

$$w_4 = z_2 - \langle z_2, q_1 \rangle q_1 - \langle z_2, q_2 \rangle q_2$$

$$= \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} - \frac{5}{\sqrt{30}} \cdot \frac{1}{\sqrt{30}} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{15} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} - \frac{5}{30} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix} = \frac{1}{10} \begin{vmatrix} 1 \\ 2 \\ -2 \\ 1 \end{vmatrix}$$

$$0 - \frac{1}{15} + \frac{5}{30} = \frac{-2+5}{30} = \frac{3}{30} = \frac{1}{10}$$

$$0 - \frac{2}{15} + \frac{10}{30} = \frac{-4+10}{30} = \frac{6}{30} = \frac{2}{10}$$

$$0 - \frac{3}{15} - 0 = -\frac{6}{30} = \frac{-2}{10}$$

$$1 - \frac{1}{15} - \frac{25}{30} = \frac{30-2-25}{30} = \frac{3}{30} = \frac{1}{10}$$

and we can check that $w_4 \perp q_1$, $w_4 \perp q_2$

Thus w_3 and w_4 are orthogonal to all linear combinations $\alpha q_1 + \beta q_2$, i.e. $w_3, w_4 \in S^\perp$ and since they are linear independent, they form a basis of $S^\perp$

(c) We can show that $w \notin S$ by looking either the original basis $\{v_1, v_2\}$ or the orthonormal basis $\{q_1, q_2\}$ and see if either $[v_1, v_2 | w]$ or $[q_1, q_2 | w]$ is inconsistent (if one is inconsistent so will the other). (8)

Alternatively I use the result that if $w \in S$ then $w = \langle w, q_1 \rangle q_1 + \langle w, q_2 \rangle q_2$

so I compute

$$\langle w, q_1 \rangle = \frac{1}{\sqrt{15}} (1+2+3+1) = \frac{7}{\sqrt{15}}$$

$$\langle w, q_2 \rangle = \frac{1}{\sqrt{30}} (-1-2+0+5) = \frac{2}{\sqrt{30}}$$

Thus $\langle w, q_1 \rangle q_1 + \langle w, q_2 \rangle q_2 = \frac{7}{\sqrt{15}} \cdot \frac{1}{\sqrt{15}} \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \end{vmatrix} + \frac{2}{\sqrt{30}} \frac{1}{\sqrt{30}} \begin{vmatrix} -1 \\ -2 \\ 0 \\ 5 \end{vmatrix}$

the third entry (for example) is $\frac{21}{15} \neq 1$, and this shows $w \notin S$.

To see if $w \in W = S^{\perp}$, I use the basis we just computed (or the same directions).

$$\left| \begin{array}{cc|c} 1 & 9 & 1 \\ 2 & -2 & 1 \\ -2 & -2 & 1 \\ 1 & 1 & 1 \end{array} \right| \rightarrow \left| \begin{array}{cc|c} 1 & 9 & 1 \\ 0 & -20 & -1 \\ 0 & 16 & 3 \\ 0 & -8 & 0 \end{array} \right| \rightarrow \left[\begin{array}{cc|c} 1 & 9 & 1 \\ 0 & -20 & -1 \\ 0 & 0 & 11/5 \\ 0 & 0 & 2/5 \end{array} \right] \text{ inconsistent!}$$

$m_{21} = 2$

$m_{31} = -2$

$m_{41} = 1$

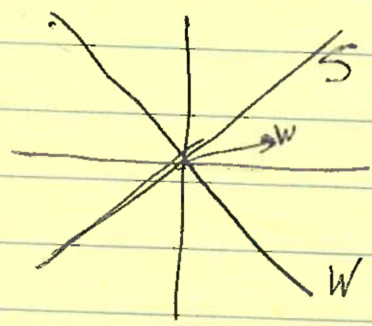
$m_{32} = +\frac{16}{20} = +\frac{4}{5}$

$m_{42} = +\frac{8}{20} = \frac{2}{5}$

Thus $w \notin W$

Explanation: S and W are two orthogonal planes in $\mathbb{R}^4$ there are many vectors not in those planes. (although all vectors in $\mathbb{R}^4$ are in $S+W$).

As analogy let S, W in $\mathbb{R}^2$ be lines like $S = \left\{ \alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix} \right\}$ $W = \left\{ \beta \begin{vmatrix} 1 \\ -1 \end{vmatrix} \right\}$



and $w = \begin{vmatrix} 3 \\ 1 \end{vmatrix} \notin S$ $w \notin W$.

but $w \in S + W = \mathbb{R}^2$

(d) If U has orthonormal columns and they are a basis of S, then $P = UU^T$

Thus taking q_1, q_2 from part (a)

$$U = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \frac{1}{\sqrt{15}} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & -1 \\ 2\sqrt{2} & -2 \\ 3\sqrt{2} & 0 \\ \sqrt{2} & 5 \end{bmatrix}$$

$$U^T = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & 3\sqrt{2} & \sqrt{2} \\ -1 & -2 & 0 & 5 \end{bmatrix}$$

$$P = UU^T = \frac{1}{30}\begin{bmatrix} 3 & 6 & 6 & -3 \\ 6 & 12 & 12 & -6 \\ 6 & 12 & 18 & 6 \\ -3 & -6 & 6 & 27 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 4 & 6 & 2 \\ -1 & -2 & 2 & 9 \end{bmatrix}$$

It checks that $P^T = P$, $P^2 = P$, an orthogonal projector

$$Q = I - P = \frac{1}{10}\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 4 & 6 & 2 \\ -1 & -2 & 2 & 9 \end{bmatrix} =$$

$$= \frac{1}{10}\begin{bmatrix} 9 & -2 & -2 & 1 \\ -2 & 6 & -4 & 2 \\ -2 & -4 & 4 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix}$$ and it checks that $Q^T = Q$, $Q^2 = Q$.

$$QP = \frac{1}{10}\begin{bmatrix} 9 & -2 & -2 & 1 \\ -2 & 6 & -4 & 2 \\ -2 & -4 & 4 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix} \cdot \frac{1}{10}\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 4 & 6 & 2 \\ -1 & -2 & 2 & 9 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \checkmark$$

$$PQ = \frac{1}{100}\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 4 & -2 \\ 2 & 4 & 6 & 2 \\ -1 & -2 & 2 & 9 \end{bmatrix}\begin{bmatrix} 9 & -2 & -2 & 1 \\ -2 & 6 & -4 & 2 \\ -2 & -4 & 4 & -2 \\ 1 & 2 & -2 & 1 \end{bmatrix} = 0 \quad \checkmark$$

(e) $$Pw = \frac{1}{10}\begin{bmatrix} 4 \\ 8 \\ 14 \\ 8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ 4 \\ 7 \\ 4 \end{bmatrix}$$

$$Qw = \frac{1}{10}\begin{bmatrix} 6 \\ 2 \\ -4 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

3. $(Pw)^T Qw = \frac{1}{25}\,[6 + 4 - 14 + 4] = 0$

4. $Pw + Qw = \frac{1}{5}\begin{vmatrix}2\\4\\7\\4\end{vmatrix} + \frac{1}{5}\begin{vmatrix}3\\1\\-2\\1\end{vmatrix} = \frac{1}{5}\begin{vmatrix}5\\5\\5\\5\end{vmatrix} = \begin{vmatrix}1\\1\\1\\1\end{vmatrix} = w$

1. Can I write Pw as $\alpha v_1 + \beta v_2$?
or in other words is $Pw \in R([v_1\ v_2])$?
or for that matter $5Pw \in R([v_1\ v_2])$? (subspace)

or, is $[v_1\ v_2 \,|\, 5Pw]$ consistent ?

$$\left[\begin{array}{cc|c} v_1 & v_2 & 5Pw \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 3 & 1 & 7 \\ 1 & 2 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Consistent and $\alpha = 2,\ \beta = 1$

2. Similarly

$$\left[\begin{array}{cc|c} 10w_4 & 10w_3 & 5Qw \end{array}\right] = \left[\begin{array}{cc|c} 1 & 9 & 3 \\ 2 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{cc|c} 1 & 9 & 3 \\ 0 & -20 & -5 \\ 0 & 16 & 4 \\ 0 & -8 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 9 & 3 \\ 0 & -20 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

consistent and

$\beta = \frac{1}{4} \quad \alpha = \frac{3}{4}$

$$\alpha\, 10 w_4 + \beta\, 10 w_3 = 5\, Qw$$

$$Qw = \frac{2}{4} w_3 + \frac{6}{4} w_4$$