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Section 4.4, Basis and Dimension

We saw spanning sets
and also linearly independent sets.

Let $V = \text{span}\{v_1, v_2, \ldots v_r\} = \{v = \sum_{i=1}^{r} \alpha_i v_i\}$

If $\{v_1\ v_2 \ldots v_r\}$ is a linearly independent set

we say that $\{v_1\ v_2 \ldots v_r\}$ is a basis of V.

and the dimension of V is r.

You can have more than one basis of a vector space (or subspace) but the number of elements of the bases does not change, the dimension.

Example let $v_1\ v_2$ $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$S = \text{span}\{v_1, v_2\} = \{x \in \mathbb{R}^3 / x_3 = 0\}$

$\{v_1\ v_2\}$ a basis (since they are l. i.)

Another basis of S $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

<u>Standard basis of $\mathbb{R}^n$</u>
<u>or Euclidian basis</u>

$e_1\ e_2 \dots e_n$

$$e_i = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} \leftarrow \text{ith position}$$

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \end{vmatrix} \quad \mathbb{R}^2 \qquad \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \quad \mathbb{R}^3$$

columns of the identity.

<u>Note</u>. $B = \{v_1, v_2 \dots v_n\}$ basis of S

$\Leftrightarrow$ B is <u>minimal spanning set</u> of S

$\Leftrightarrow$ B is <u>maximal linearly independent set</u> of S

Let $S = \{x \,/\, x = \alpha u \,,\ \alpha \in \mathbb{R}\}$

u spans S $\qquad$ $\{u\}$ basis of S

$\dim S = 1$ $\qquad$ a "line".

$\{v_1\ v_2\}$ linearly independent

$$\Rightarrow \quad v_1 \neq \alpha v_2$$

$$S = \text{span}\{v_1, v_2\} = \{x \mid x = \alpha_1 v_1 + \alpha_2 v_2\}$$

$\alpha_1\ \alpha_2 \in \mathbb{R}$

$\dim S = 2$ a "plane"

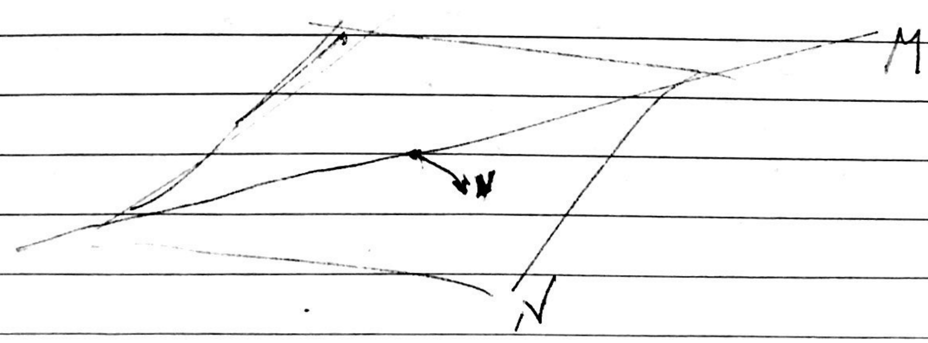
Consider two subspaces of V

If $M \subset N$, then $\dim M \leq \dim N$

and if $\dim M = \dim N \Rightarrow M = N$.

If $M \subseteq N$ and $\exists\, v \in N$, $v \notin M$

then $\dim M < \dim N$.



Back to the 4 important subspaces.

A $m \times n$

Recall $R(A)$ spanned by the columns of A

$$\dim R(A) = r = \text{rank}(A)$$

$$= \text{rank}(A^T) = \dim R(A^T)$$

spanned by rows of A.

Recall. n columns

$r = \text{rank}$

$n - r$ free variables

$$\dim N(A) = n - r$$

$N(A)$ spanned by $x_{h_1}\, x_{h_2} \cdots x_{h_k}$

$k = n - r$

Thus, we have an important result

$$\dim R(A) + \dim N(A) = n \quad (\#\text{ colums})$$

similarly

$$\dim R(A^T) + \dim N(A^T) = m$$

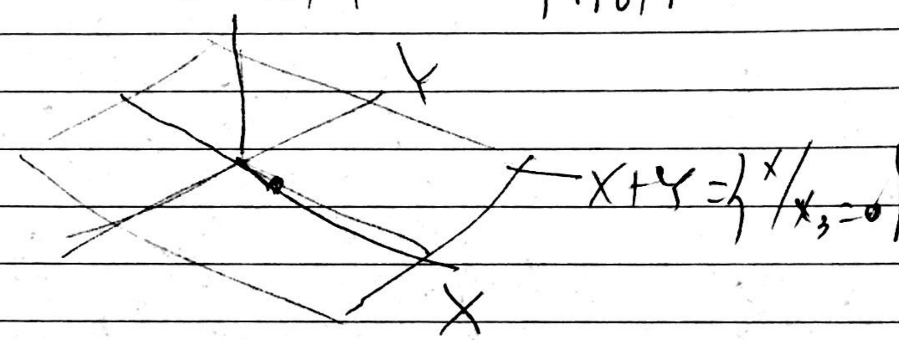
Define the "sum" of subspaces

X, Y subspaces of V

$$X + Y = \{ x + y \ / \ x \in X, \ y \in Y \}$$

Prove that $X+Y$ is a subspace

example $X = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $Y = \left\{ \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$



Another ex. $X = \{ x \in \mathbb{R}^3 / \ x_2 - x_3 = 0 \}$

$Y = \{ x \in \mathbb{R}^3 / \ x_1 + x_2 + x_3 = 0 \}$

$X = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$Y = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ $X + Y = \mathbb{R}^3$

If $\{v_1, v_2 \ldots v_r\}$ is a basis of X

$\{w_1, w_2 \ldots w_k\}$ is a basis of Y

How can I build a basis of $X+Y$?

Answer. If $X \cap Y = \{0\}$

then $\{v_1, v_2 \ldots v_r, w_1 \ldots w_k\}$ basis of $X+Y$

(as in first example)

If, on the other hand $X \cap Y \supsetneq \{0\}$

Then, we have to discard the duplicates.

$$\Rightarrow \boxed{\dim (X+Y) = \dim X + \dim Y - \dim (X \cap Y)}$$

In Example 2.

$X \cap Y$

$x_2 = x_3$

$x_1 + 2x_2 = 0 \qquad x_1 = -2x_3$

~~$x_1 + 2x_3 = 0$~~

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -2x_3 \\ x_3 \\ x_3 \end{vmatrix} = x_3 \begin{vmatrix} -2 \\ 1 \\ 1 \end{vmatrix} = \alpha \begin{vmatrix} -2 \\ 1 \\ 1 \end{vmatrix}$$

$\dim X \cap Y = 1$

$\dim X+Y = 3 = 2 + 2 - 1$

Section 4.5

$$\text{rank}(AB) = \text{rank}(B) - \dim\left(N(A) \cap R(B)\right)$$

$$\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$$

i.e. rank cannot increase by product,
only decrease.

Recall $\text{rank}(A) = \text{rank}(A^T)$

Special case. $\text{rank}(A^T A) = \text{rank}(A) = \text{rank}(A A^T)$
rank is not reduced

In fact,
$$R(A^T A) = R(A) \quad R(A A^T) = R(A)$$
$$N(A^T A) = N(A) \quad N(A A^T) = N(A^T)$$

In general $R(AB) \subseteq R(A)$
$N(AB) \supseteq N(B)$
(or $N(B) \subseteq N(AB)$)

Example 4.4.8 p. 206

Section 4.7

We have seen linear functions

Recall $f(\alpha x+y)=\alpha f(x)+f(y)$

Now think about linear functions between vector spaces.

$$T: V \to W$$

example 1 $\quad V=\mathbb{R}^3 \quad W=\mathbb{R}^2$

$$T\left(\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}\right) = \begin{vmatrix} x_1+x_2 \\ x_3 \end{vmatrix}$$

Example 2. V - differentiable functions $\to W$

$$T(f)=\frac{df}{dx} \qquad \text{linear.}$$

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these are called linear transformations

(or linear maps).

Let $S \subset V$ subspace

then $T(S) \subset W$ also subspace -

In Example 1.

For example $S = \{ x \in \mathbb{R}^3 / x_1 = -x_2 \}$

$T(S) = \{ v \in \mathbb{R}^2 / v_1 = 0 \}$

S plane in $\mathbb{R}^3$

T(S) line in $\mathbb{R}^2$

Once you fix a basis for V
and another for W.

You can represent T by
a Matrix.

For example 1 with Euclidean bases

$T(x) = A x$

$$A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$