§ 7.4 Operational Properties, II

Not on final OR Q10, but need for 2 WebAssign problems

T.7.4.1 Derivatives of a Transform

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$,

then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Pf: $n = 1$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

uniform convergence

$$= \int_0^\infty \frac{d}{ds}\left[e^{-st} f(t)\right] dt$$

$$\frac{d}{ds}\left(e^{-st} f(t)\right) = -te^{-st} f(t)$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -\mathcal{L}\{t f(t)\}$$

By induction, $\mathcal{L}\{t^n f(t)\} = \underbrace{\left(-\frac{d}{ds}\right)^n}_{\text{operator}} F(s)\Big]$

$$= (-1)^n \frac{d^n}{ds^n} F(s). \quad \square$$

eg $\mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$

$$= -\frac{(s^2+9)(1) - s(2s)}{(s^2+9)^2} = -\frac{-s^2+9}{(s^2+9)^2}$$

$$= \frac{s^2-9}{(s^2+9)^2}$$

In general, $\mathcal{L}\{t\cos kt\} = \frac{s^2-k^2}{(s^2+k^2)^2}$

and $\mathcal{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}$

book proves

Using above,

eg $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+25)^2}\right\}$

$$= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{10s}{(s^2+25)^2}\right\}$$

$$= \frac{1}{10}t\sin 5t$$

Need $2ks$

ie. $2(5)s = 10s$

in numerator

§ 7.4.2 Transforms of Integrals

Convolutions → on final

Def: The convolution of two functions $f(t)$ and $g(t)$ is

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

(Note: if $f(t) = k$, $f(t-\tau) = k$)

While you will NOT have to compute this, you will have to recognize a convolution when given.

Note $\boxed{f * g = g * f}$

i.e. $\int_0^t f(t-\tau)\, g(\tau)\, d\tau$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Pf: Let $u = t - \tau$. $du = -d\tau$, $\tau = t - u$, when $\tau = t$, $u = 0$. when $\tau = 0$, $u = t$

$$-\int_t^0 f(u) g(t-u)\, du = \int_0^t f(u)\, g(t-u)\, du.$$

Convolution Theorem * * *

T.7.4.2 If f and g are piecewise continuous on $[0,\infty)$ and of exponential order (see p. 282 - grows at most exponentially),

then
$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$
$$= F(s)\,G(s)$$

(Pf in book)

---

eg $\mathcal{L}\left\{\int_0^t e^{\tau}\sin\tau \, d\tau\right\}$

No $t-\tau$. So $f(t)=1$ and $g(t)=e^{t}\sin t$.

$= \mathcal{L}\{f*g\}$

$= \mathcal{L}\{1\}\,\mathcal{L}\{e^{t}\sin t\}$

$\mathcal{L}\{\sin t\}\Big|_{s \to s-1}$

$= \dfrac{1}{s}\cdot\dfrac{1}{(s-1)^2+1} = \dfrac{1}{s\left((s-1)^2+1\right)}$

Example 3 (Be careful - NOT same as last example.)

$$\mathcal{L}\{e^t * \sin t\} = \mathcal{L}\{e^t\}\,\mathcal{L}\{\sin t\}$$
$$= \frac{1}{s-1}\cdot\frac{1}{s^2+1}$$
$$= \frac{1}{(s-1)(s^2+1)}$$

(Here, $f(t) = e^t$, $g(t) = \sin t$.

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\})$$

For Problem 38, there is NO need to use formula 8.
The procedure below is easier.

eg

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s-2)^2}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\left.\frac{1}{s^2}\right|_{s\to s-2}\right\}$$
$$= \underbrace{1}_{f} * \underbrace{te^{2t}}_{g}$$

= Next page

Can also use PFD

(Also derivatives: $\mathcal{L}\{-\frac{d}{ds}(\frac{1}{s-2})\} = te^{2t}$)

$$1 * te^{2t}$$

$$= \int_0^t \underbrace{f(t-\tau)}_{=1} g(\tau)\, d\tau$$

$$= \int_0^t 1 \cdot \tau e^{2\tau}\, d\tau$$

$$= \frac{\tau}{2} e^{2\tau} - \frac{1}{4} e^{2\tau} \Big]_0^t$$

$$= \left(\frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t}\right) - \left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t} + \frac{1}{4}$$

---

summary

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$