1 Tues 8/29

Linear equations $y = ax + b$

$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$$ represents 2 eqns, 2 unknowns

row, col

e.g.: $$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 4 \\ 2 \end{vmatrix} \rightarrow \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} -1 \\ 3 \end{vmatrix}$$

Scalar multiplications

↳ number $\alpha \in \mathbb{F} \{\mathbb{R}, \mathbb{C}\}$

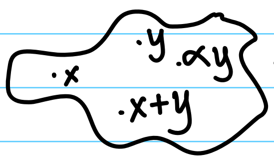
e.g.: $\alpha = 2 \quad x = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} \quad \alpha x = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$

$$V = \begin{vmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{vmatrix} \in \mathbb{R}^n \qquad \boxed{(\alpha V)_i = \alpha V_i} \qquad \boxed{(\alpha A)_{ij} = \alpha A_{ij}}$$

e.g.: $$4 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 4 \\ 4 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix} + \begin{vmatrix} 2 \\ -1 \\ 0 \end{vmatrix} = \begin{vmatrix} 3 \\ 1 \\ 3 \end{vmatrix}$$

We can add & multiply vectors by a scalar $\boxed{(x+y)_i = x_i + y_i}$
same length

Def → V a vector space if it is closed under addition & scalar multi



$\mathbb{R}^{m \times n}$ rectangular matrices

$\boxed{(A+B)_{ij} = A_{ij} + B_{ij}}$ fn $[a,b]$ cont. fn on $[a,b]$

that is: if $x, y \in V$, $\alpha \in \mathbb{R}$ $\quad x+y \in V$, $\alpha x \in V$ or for short $\alpha x + y \in V$

! Prove vectorspace $\rightarrow$ $\alpha x + y \in V \quad \forall x, y \in V$

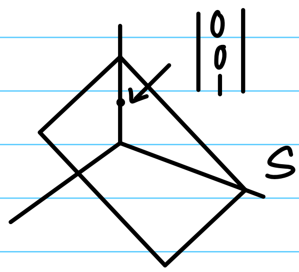
Note $\rightarrow$ As a consequence: V vectorspace then $0 \in V$
$\exists$ (such that) $x + 0 = x$

Proof 1 $\rightarrow$ Let $\alpha = 0$: $\alpha . x = 0 . x = 0$ since $\alpha x \in V$, $0 \in V$

Proof 2 $\rightarrow$ Let $\alpha = -1$ then $\alpha . x = -x \in V$, $-x + x = 0 \in V$

$\therefore$ If $0 \notin S$, S is not a vectorspace

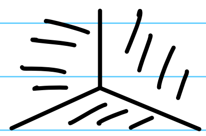
e.g: $S = \{ x \in \mathbb{R}^3 , x_1 + x_2 + x_3 = 1 \}$
↳ not vectorspace bc $0 \notin S$



$P_n = \{$set of polynomials of degree $= n\}$ $\rightarrow$ not vectorspace bc if must apply for all
$P(x) = a_0 + a_1 x + \ldots + a_n x^n$

but $P_n = \{$polynomials of degree $\leq n\}$ $\rightarrow$ vectorspace bc $0, 1 \ldots n$

$\mathbb{R}^3 = \{ x \in \mathbb{R}^3 \mid x_i \geq 0 \}$



Col. vectors

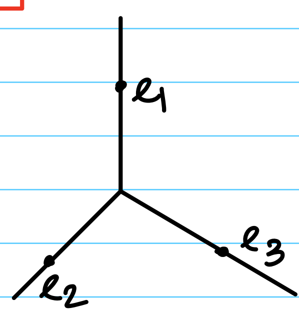
$v = \begin{vmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{vmatrix}$ $\quad v^T = [v_1 \ v_2 \ \ldots \ v_n]$ $\rightarrow$ Transpose $\boxed{(A^T)_{ij} = A_{ji}}$

$\boxed{\begin{matrix} A \in \mathbb{R}^{m \times n} \\ (A^*)_{ij} = \overline{a_{ji}} \end{matrix}}$

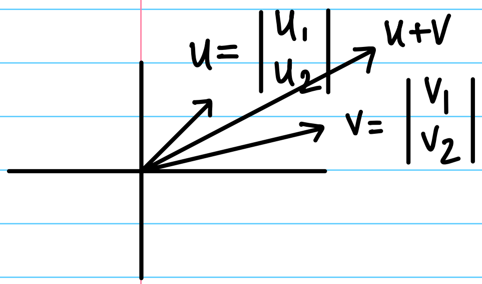
e.g: $A = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix}$ $\mathbb{R}^{2 \times 3}$ $\quad A^T = \begin{vmatrix} 2 & 1 \\ 2 & 0 \\ 1 & -1 \end{vmatrix}$ $\mathbb{R}^{3 \times 2}$

Euclidian Vector

$e_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{vmatrix} \quad e_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{vmatrix} \quad e_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \cdots \end{vmatrix} \quad \ldots \quad e_n = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 1 \end{vmatrix}$



$$x = x_1 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + x_2 \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} + x_3 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$$



$$u + v = \begin{vmatrix} u_1 + v_1 \\ u_2 + v_2 \end{vmatrix}$$

$$A = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} \xrightarrow[\mathbb{R}^3 \to \mathbb{R}^2]{\text{transformation}} Ax = \begin{vmatrix} 2x_1 + 2x_2 + x_3 \\ x_1 - x_3 \end{vmatrix} \in \mathbb{R}^2$$

$$A^T = \begin{vmatrix} 2 & 1 \\ 2 & 0 \\ 1 & -1 \end{vmatrix} = B \longrightarrow Bv = \begin{vmatrix} 2v_1 + v_2 \\ 2v_1 \\ v_1 - v_2 \end{vmatrix}$$

Algebra of matrices (operations $+, -, \dots$)

Some space $\mathbb{R}^{m \times n}$

$A = B$ if $A, B \in \mathbb{R}^{m \times n}$ — same order

$a_{ij} = b_{ij}$ $\forall i = 1 \dots m$ — same entries

$j = 1 \dots n$ — in the same positions (i,j)

$\hookrightarrow C = A + B \in \mathbb{R}^{m \times n}$

! Properties:

there exists

(1) $\exists\, 0 \ni A + 0 = A \quad \forall A \in \mathbb{R}^{m \times n}$ $\qquad 0_{ij} = 0$

$a + 0 = a \in \mathbb{R}$

(2) Associativity $(A + B) + C = A + (B + C)$

(3) Commutativity $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$

$A + B = B + A$

$a_{ij} + b_{ij} = b_{ij} + a_{ij}$

(4) $\exists -A \ni A + (-A) = 0$ $\qquad (-A)_{ij} = -A_{ij}$ or $-a_{ij}$

Proof (2) $\rightarrow$ $((A+B)+C)_{ij} = (a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$

$\hookrightarrow$ def of ij $\qquad \hookrightarrow$ associativity in $\mathbb{R}$

$= [A+(B+C)]_{ij} \longrightarrow (A+B)+C = A+(B+C)$

↳ def of ij

$$\boxed{\begin{aligned} &\text{Properties: } (1)\ \alpha(\beta A) = (\alpha\beta)A \\ &(2)\ \alpha(A+B) = \alpha A + \alpha B \\ &(3)\ (\alpha+\beta)A = \alpha A + \beta A \end{aligned}}$$

Proof (2) → $[\alpha(A+B)]_{ij} = \alpha(A+B)_{ij} = \alpha(a_{ij}+b_{ij}) = \alpha a_{ij} + \alpha b_{ij} = (\alpha A)_{ij} + (\alpha B)_{ij}$

def of scalar product $(\alpha A)_{ij} = \alpha a_{ij}$ — def of $A+B$ — distributative property in $\mathbb{R}$

QED!

! (1) V w/ $+, -$ (scalar multiplication, field $\mathbb{F}$ - usually $\mathbb{R}$) is a vectorspace if it is closed under addittion & scalar multiplication

(2) That is, if $u, v \in V$, $\alpha \in \mathbb{R}$ then $\alpha u + v \in V$

(3) Examples $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^{3\times5}, \mathbb{R}^{3\times3}, \mathbb{R}^{n\times n}, \mathbb{R}^{m\times n}$

(4) $A \in \mathbb{R}^{m\times n} \longrightarrow A^T \in \mathbb{R}^{n\times m}$

(5) $(A^T)_{ij} = A_{ji}$

(6) $1.A = A$ $\quad 0.A = 0$

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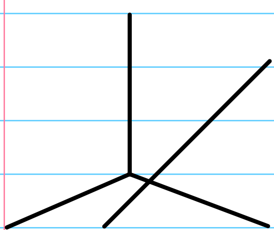
In general, $\mathbb{R}^{m\times n}$ not closed under addition

$\boxed{(A^T)^T = A} \xrightarrow{\text{Proof}} (A^T)^T_{ij} = (A^T)_{ji} = A_{ij}$

$\boxed{(A+B)^T = A^T + B^T} \xrightarrow{\text{Proof}} (A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T + B^T)_{ij}$

def transpose — def A + B — def transpose

$\alpha \in \mathbb{R}$ $\quad \boxed{(\alpha A)^T = \alpha A^T} \xrightarrow{\text{Proof}} (\alpha A)^T = (\alpha A)_{ji} = \alpha A_{ji} = \alpha(A^T)_{ij}$



$L \subseteq V$ if it is a vectorspace in its own right

↑ subspace (line only)

Take a subset S of a vectorspace if S is a vectorspace in its own right then it is a subspace of V $(S \subseteq V)$

Note → $\{0\} \subseteq V$

! If $A^T = A$, A is symmetric $\quad a_{ij} = a_{ji} \ \forall i,j = 1 \dots n$

$H^{n\times n} = \{A \in \mathbb{R}^{n\times n} \mid A^T = A\}$ is a subspace ! Prove

! If $A^T = -A$, A is skew-symmetric (anti-symmetric)

$a_{ij} = -a_{ji}\ \forall i,j = 1 \dots n \longrightarrow$ when $i=j$: $a_{ii} = -a_{ii} \longrightarrow 2a_{ii} = 0$

$\boxed{a_{ii} = 0 \ \forall i}$

e.g: $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^T = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$ $\quad \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{vmatrix}$

fixed $v \in V$

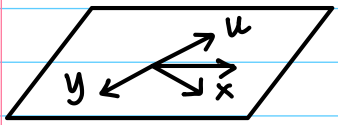
Line $L = \{\alpha v, \alpha \in \mathbb{R}\}$ is a subspace = span$\{v\}$

Take $x = \alpha v$ $\quad \gamma x + y = \gamma\alpha v + \beta = (\gamma\alpha + \beta)v \in V$

$y = \beta v$ $\quad$ ↳ subspace

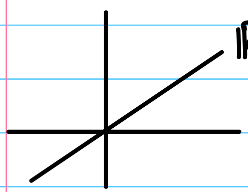
except for 0 bc span$\{0\}$ = subspace of 0

Plane $P = \{\alpha v + \beta w,\ v, w \in V,\ \alpha, \beta \in \mathbb{R}\} = \text{span}\{v, w\}$



Note → $\{0\}$, V trivial subspaces

$S \subsetneq V$ non-trivial subspace



$\mathbb{R}^2 \to x = \begin{vmatrix} \alpha \\ \alpha \end{vmatrix}$

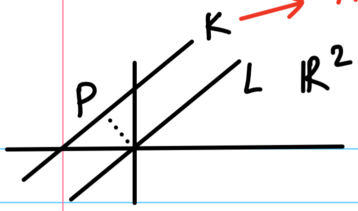
$L = \{\alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \alpha = \mathbb{R}\} = \{x \in \mathbb{R}^2, x_1 - x_2 = 0\}$

$L = \{a_1 x_1 + a_2 x_2 = 0\}$ in $\mathbb{R}^2$

$a_1 \neq 0 \to x_1 = \frac{a_2}{a_1} x_2 \to \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} a_2/a_1\, x_2 \\ x_2 \end{vmatrix} = \alpha \begin{vmatrix} a_2/a_1 \\ 1 \end{vmatrix} = \beta \begin{vmatrix} a_2 \\ a_1 \end{vmatrix}$

e.g: $\{3x_1 + 2x_2 - x_3 = 0\} \to x_1 = -\frac{2}{3}x_2 + \frac{1}{3}x_3$

$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x_2 \begin{vmatrix} -2/3 \\ 1 \\ 0 \end{vmatrix} + x_3 \begin{vmatrix} 1/3 \\ 0 \\ 1 \end{vmatrix} \to \alpha \begin{vmatrix} -2/3 \\ 1 \\ 0 \end{vmatrix} + \beta \begin{vmatrix} 1/3 \\ 0 \\ 1 \end{vmatrix} \to$ plane



Affine space $X = p + S$

! Linear combination is $\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k}_{\text{span}}$ — a point in the subspace

$v_1 \ldots v_k$ **linearly dependent** if one of them can be written as a linear combination of the others $\exists j \ni v_j = \sum_{\substack{i \neq j \\ i = 1 \ldots k}} \alpha_i v_i$

linearly independent if none of them can be written as a linear combination of the others : $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0 \rightarrow \alpha_i = 0$

$Ax = 0 \rightarrow x = 0$

e.g: $v_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}, v_2 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}, v_3 = \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix} = 2v_2 - v_1 = 2\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$

linearly dependent

$v_1 = 2v_2 - v_3$

$v_2 = \frac{1}{2}(v_3 + v_1) = \frac{1}{2}v_3 + \frac{1}{2}v_1$

Equivantly, $v_1 \ldots v_k$ are linearly dependent iff there exists scalars $\alpha_i \neq 0 \ni \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0$

$v_1 - 2v_2 + v_3 = 0$ $\qquad A = [v_1 \ v_2 \ \ldots \ v_k] \rightarrow A\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$

scalars $1 \quad -2 \quad 1$

↳ can find $x \neq 0 \ni Ax = 0$

e.g: $\underset{v_1}{\begin{vmatrix} 2 \\ 0 \\ 1 \end{vmatrix}}, \underset{v_2}{\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}}, \underset{v_3}{\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}}$

$v_1 = v_2 + v_3$

$v_1 - v_2 - v_3 = 0$ so $\begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$

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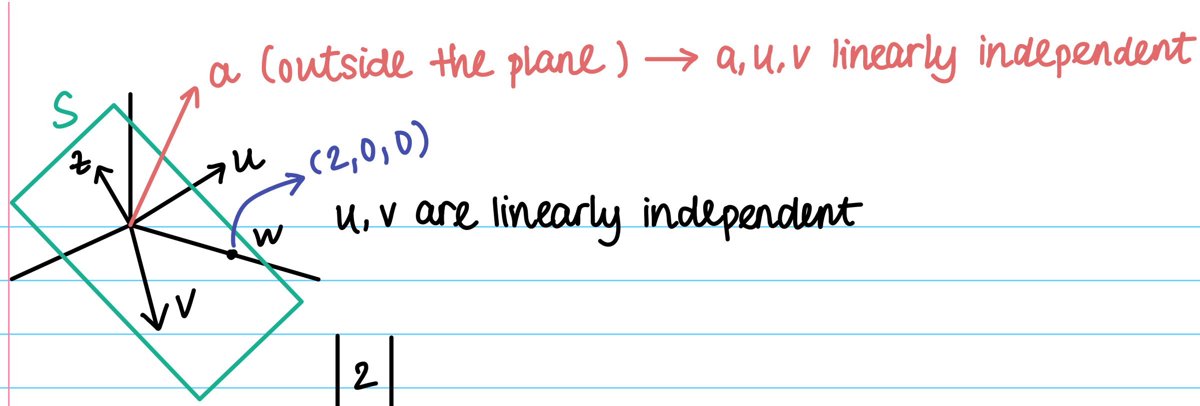
Let $v = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, u = \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}$ $S = \text{span}\{u, v\} = \{x \in \mathbb{R}^3, x = \alpha v + \beta u, \alpha, \beta \in \mathbb{R}\}$

All linear combinations of u, v is a subspace of $\mathbb{R}^3$

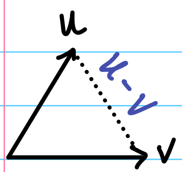
(in fact, horizontal plane) → bc $z = 0$



u, v are linearly independent

Take $w = u + v = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$ → u, v, w linearly dependent
any other vector z in S → z, u, v linearly dependent

e.g: $u = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$, $v = \begin{vmatrix} 2 \\ -1 \\ 2 \end{vmatrix}$ → u, v linearly independent bc not multiples of the other



NORM (length/distance)

$v \in \mathbb{R}^n$ $\boxed{\|v\| = (\sum x^2)^{1/2}}$ Euclidean norm

e.g: $v = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ $x = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$ $w = \begin{vmatrix} 3 \\ -4 \end{vmatrix}$

$\|v\| = \sqrt{1^2+0^2} = 1$ $\|x\| = \sqrt{1^2+1^2} = \sqrt{2}$ $\|w\| = \sqrt{3^2+(-4)^2} = 5$

$\|w - x\| = \left\| \begin{matrix} 2 \\ -5 \end{matrix} \right\| = \sqrt{4+25} = \sqrt{29}$ $z = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ $\|z\| = \sqrt{2}$

!

Norm : $\|.\| : V \to \mathbb{R}^+$ $\forall x, y \in V$

(1) $\|x\| \geq 0$

$\|x\| = 0$ if $x = 0$

(2) $\|\alpha x\| = |\alpha| \, \|x\|$

(3) $\|x + y\| \leq \|x\| + \|y\|$ → Δ inequality

$\boxed{\|x\|_2 = \sqrt{\langle x, y \rangle}}$

Inner product $\langle . , . \rangle : V \times V \to \mathbb{R}$ $\boxed{\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i}$

e.g: $x \begin{vmatrix} 1 \\ 2 \end{vmatrix}$ $y \begin{vmatrix} -1 \\ 1 \end{vmatrix}$ → $\langle x, y \rangle = 1(-1) + 2(1) = -1 + 2 = 1$

$\langle x+y, v\rangle = \langle x, v\rangle + \langle y, v\rangle$

product of sum w/ v — product of each

Notation: $\langle x, y\rangle$ (x,y) $\langle x|y\rangle$

(i) Bilinear (linear in each argument)

$\langle \alpha x + y, v\rangle = \alpha\langle x, v\rangle + \langle y, v\rangle$ Note: $\langle \alpha u, \alpha v\rangle = \alpha^2\langle u, v\rangle$

$\langle x, v + \beta w\rangle = \langle x, v\rangle + \beta\langle x, w\rangle$

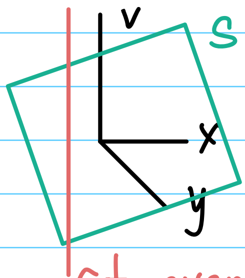
(ii) $(x,x) \geq 0$

$(x,x) = 0 \rightarrow x = 0$

(iii) Symmetric $(x,y) = (y,x)$

If $\langle x, y\rangle = 0$, we say $x \perp y$ (x is orthogonal to y)

e.g: $\langle \alpha x + \beta y, v\rangle = \alpha\langle x, v\rangle + \beta\langle y, v\rangle = 0 + 0 = 0$



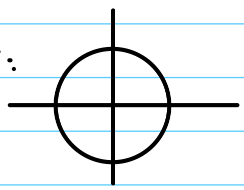
$S^\perp$ everything perpendicular to S

Every inner product produces a norm $\|x\| = \sqrt{\langle x, x\rangle}$

Def A unit vector is a vector of norm = 1

Unit sphere $S = \{x \in V \mid \|x\| = 1\}$

In $\mathbb{R}^2$:

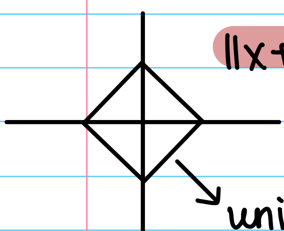


$v = \begin{vmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{vmatrix}$ $\|v\| = \frac{2}{4} + \frac{2}{4} = 1$

$\begin{vmatrix} \sin\theta \\ \cos\theta \end{vmatrix} \longrightarrow \sin^2\theta + \cos^2\theta = 1$

1- norm:

$x \in \mathbb{R}^n$: $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$ e.g: $x = \begin{vmatrix} 1 \\ 2 \end{vmatrix}$ $\|x\|_1 = 1 + 2 = 3$, $\|x\|_\infty = 2$



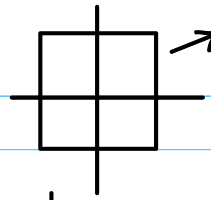
$\|x + y\|_1 = |x_1 + y_1| + |x_2 + y_2| + \ldots + |x_n + y_n|$

$\leq |x_1| + |y_1| + |x_2| + |y_2| + \ldots + |x_n| + |y_n|$

$= \|x\|_1 + \|y\|_1$

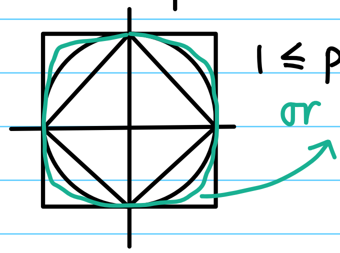
unit sphere for 1-norm

$\boxed{\infty\text{-norm}: \ \|x\|_\infty = \max_{i=1}^{n} |x_i|}$



→ unit sphere for ∞-norm

General → $\|x\|_p = \left(\sum |x_i^p|\right)^{1/p}$



$1 \leq p \leq 2$

or $p > 2$

CBS Inequality → $\boxed{|\langle x, y\rangle| \leq \underbrace{\|x\|}_{} \|y\|}$

induced by linear product $\langle x, x\rangle$

Proof → If $x = 0$, $\|x\| = 0 \longrightarrow \langle x, y\rangle = 0 \longrightarrow$ inequality holds

If $x \neq 0$, $\alpha = \dfrac{\langle x, y\rangle}{\|x\|^2}$

Take $v = \alpha x - y$: $\langle x, v\rangle = \langle x, \alpha x - y\rangle = \alpha\langle x, x\rangle - \langle x, y\rangle$

$$= \frac{\langle x, y\rangle}{\|x\|^2}\|x\|^2 - \langle x, y\rangle = \langle x, y\rangle - \langle x, y\rangle = 0$$

$\therefore \langle x, v\rangle = 0$

by property (1) of norms

$$0 \leq \|v\|^2 = \|\alpha x - y\|^2 = \langle \alpha x - y, \alpha x - y\rangle$$

$$= \alpha\underbrace{\langle x, \alpha x - y\rangle}_{0} - \langle y, \alpha x - y\rangle$$

$$= \langle y, y\rangle - \alpha\langle x, y\rangle$$

$$= \|y\|^2 - \frac{\langle x, y\rangle}{\|x\|^2}\langle x, y\rangle = \frac{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}{\|x\|^2}$$

↳ $\langle x, y\rangle^2 \leq \|x\|^2\|y\|^2 \longrightarrow |\langle x, y\rangle| \leq \|x\|\,\|y\|$ QED

Note → $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

$\forall a \in \mathbb{R}$, $a \leq |a|$

$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

$\longrightarrow \|x+y\| \leq \|x\| + \|y\|$ QED

Thursday 9/7

$|\langle x,y\rangle| \leq \|x\|\,\|y\|$ $\qquad$ $\|x+y\| \leq \|x\| + \|y\|$

Backward $\triangle$ inequality $\boxed{\forall x,y \in V: \ \big|\|x\| - \|y\|\big| \leq \|x-y\|}$

Proof $\rightarrow$ $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$ $\quad$ $\triangle$ inequality

$\hookrightarrow \|x\| - \|y\| \leq \|x-y\|$

$\|a\| = \|-a\|$

$\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\| = \|x-y\| + \|x\|$

$\hookrightarrow \|y\| - \|x\| \leq \|x-y\|$

$\therefore \big|\|x\| - \|y\|\big| \leq \|x-y\|$ $\quad$ QED

e.g: $\langle p,q\rangle = \int_a^b p(x)q(x)\,dx$ $\qquad$ $\|p\| = \left(\int_a^b p^2(x)\,dx\right)^{1/2}$

$$\|\alpha p\| = \left(\int_a^b \alpha^2 p^2(x)\,dx\right)^{1/2} = |\alpha| \int_a^b p^2(x)\,dx = |\alpha|\,\|p\|$$

ORTHORGONALITY

e.g: $p(x) = 1,\ q(x) = x - \frac{1}{2}$. $\langle p,q\rangle = \int_0^1 p(x)q(x)\,dx = \int_0^1 (x - \frac{1}{2})\,dx$

$$= \left(\frac{x^2}{2} - \frac{1}{2}x\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{2} = 0$$

$\therefore p \perp q$

Pythagoras theorem $\boxed{\forall x,y: \langle x,y\rangle = 0 \text{ then } \|x\|^2 + \|y\|^2 = \|x+y\|^2}$

Proof $\rightarrow$ $\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 - \underbrace{2\langle x,y\rangle}_{0} + \|y\|^2 = \|x\|^2 + \|y\|^2$ QED

$\forall x,y \in V$, θ between x,y: $\boxed{\cos\theta = \frac{\langle x,y\rangle}{\|x\|\,\|y\|}}$ $\qquad$ $\cos\theta = 0 \rightarrow \theta = \pi/2$

$\{u_1, u_2 \ldots u_n\}$ orthogonal set: $\langle u_i, u_j\rangle = 0$ for $i,j = 1 \ldots n$ $(i \neq j)$

e.g: $\underset{u_1}{\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}}, \underset{u_2}{\begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix}}, \underset{u_3}{\begin{vmatrix} 1 \\ 1 \\ -2 \end{vmatrix}}$ $\qquad$ $u_1 \perp u_2$, $u_1 \perp u_3$, $u_2 \perp u_3$

Orthonormal set: $\langle u_i, u_j \rangle = 0$ and $\|u_i\| = 1$

If $U \in \mathbb{R}^{n \times n}$, $U = [u_1, u_2 \dots u_n]$ orthonormal columns
say U orthogonal matrix

e.g: $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I$ identity matrix $\quad Ix = x$, $P = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$ $\quad P \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} x_2 \\ x_3 \\ x_1 \end{vmatrix}$

$\rightarrow$ I permutated

If $u_1, u_2 \dots u_n$ orthogonal set $\rightarrow$ is a linearly independent set

Tuesday 9/12

$u = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$, $v = \begin{vmatrix} 2 \\ 2 \end{vmatrix}$

want to construct orthogonal matrix: $\|u\| = \sqrt{2}$, $\|v\| = \sqrt{8}$

$\frac{u}{\|u\|} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix}$, $\frac{v}{\|v\|} = \frac{1}{\sqrt{8}} \begin{vmatrix} 2 \\ 2 \end{vmatrix}$ $\quad \searrow \quad \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$

Lemma $\rightarrow$ Let $u_1, u_2 \dots u_n$ be an orthogonal set ($\langle u_i, u_j \rangle = 0, i \neq j$) then they are <u>linearly independent</u>.

$$\sum_{i=1}^{K} \alpha_i u_i = 0 \text{ then } \alpha_i = 0 \; (i = 1 \dots K)$$

Proof $\rightarrow$ Pick j: $0 = \langle \sum \alpha_i u_i, u_j \rangle = \sum \alpha_i \underbrace{\langle u_i, u_j \rangle}_{=0 \; i \neq j} = \alpha_j \underbrace{\langle u_j, u_j \rangle}_{>0} \quad \therefore \alpha_j = 0$

QED!

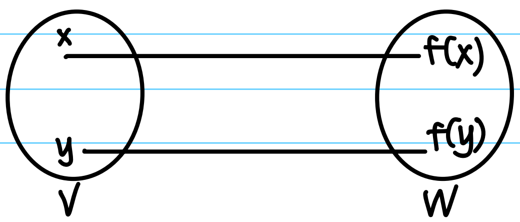
Section 1.7

Linear transformations $\rightarrow$ fns (maps) from 1 vectorspace to another

$f: V \rightarrow W$
$\mathcal{D}$ domain of the fn

$f(x+y) = f(x) + f(y)$
$f(\alpha x) = \alpha f(x)$
$f(\alpha x + y) = \alpha f(x) + f(y)$

} not applied to product of components



$\therefore f(0) = 0$
$\therefore f(x+y) = f(x) + f(y)$

e.g $V=\mathbb{R}^{5\times 2} \longrightarrow W=\mathbb{R}^{2\times 5}$

$f(A)=A^T$

$f(A+B)=(A+B)^T=A^T+B^T=f(A)+f(B)$

$f(\alpha A)=(\alpha A)^T=\alpha A^T=\alpha f(A)$

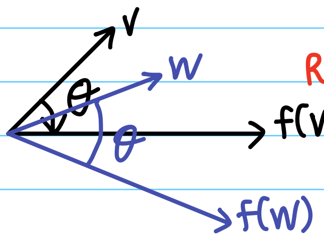
$$\boxed{f: \mathbb{R}^{n\times n} \longrightarrow \mathbb{R} \qquad \mathrm{trace}(A)=\sum_{i=1}^{n} a_{ii}}$$

e.g: $\mathrm{trace}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 3$ $\qquad \mathrm{trace}(A+B)=\sum(a_{ii}+b_{ii})=\sum a_{ii}+\sum b_{ii}$

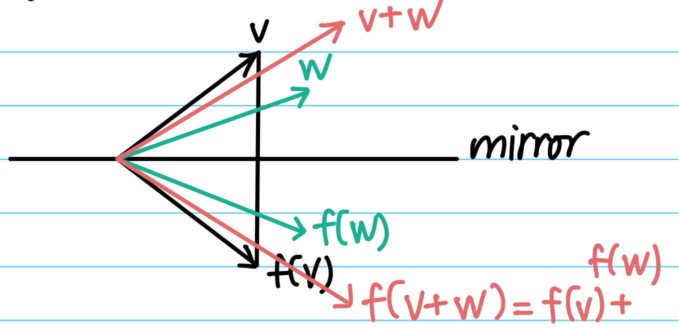
$=\mathrm{trace}(A)+\mathrm{trace}(B)$

trace(skew-symmetric matrix) $=0$

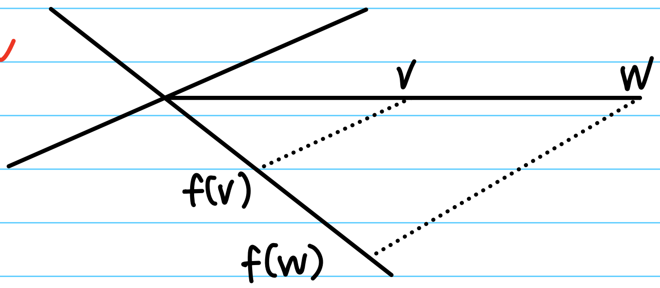
Rotation



Reflection



Projection



e.g: $\langle .,. \rangle: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $\qquad f_y=\mathbb{R}^n \longrightarrow \mathbb{R}$

$f_y(x)=\langle x,y\rangle$

$f_y(x+u)=\langle x+u,y\rangle=\langle x,y\rangle+\langle u,y\rangle$

$f_x(u+v)=\langle x,u+v\rangle=\langle x,u\rangle+\langle x,v\rangle=f_x(u)+f_x(v)$

Thursday 9/14

System of linear equations:

$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$

row

a_{ij} col

Linear transformation $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11}x_1+a_{12}x_2+a_{13}x_3 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3 \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= x_1 A_{*1} + x_2 A_{*2} + A_{*3}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

e.g: $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$f: \mathbb{R}^3 \to \mathbb{R}^2 \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix}$$

$$A_{m \times n} \quad f: \mathbb{R}^n \to \mathbb{R}^m \quad f(x) = \sum_{j=1}^{n} x_j A_{*j}$$

$$f(x+y) = \sum_{j=1}^{n} (x_j + y_j) A_{*j} = \sum_{j=1}^{n} x_j A_{*j} + \sum_{j=1}^{n} y_j A_{*j} = f(x) + f(y)$$

MATRIX & VECTOR MULTIPLICATION

e.g: $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

first coeff times first column ← $\boxed{(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j}$

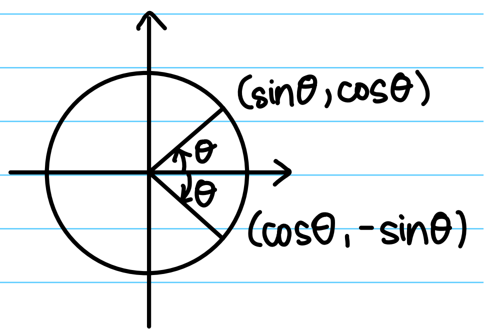
$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 x_1 + y_2 x_2 + y_3 x_3$$

$x^T y$

$$\boxed{\text{Linear mapping: } \begin{aligned} A(x+y) &= Ax + Ay \\ A(\alpha x) &= \alpha A x \end{aligned}}$$

e.g: $f: \mathbb{R}^2 \to \mathbb{R}^3 \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ rotation by θ $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$



$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

e.g: Reflection across $\alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\quad A = \frac{1}{5} \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}$

$$A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

nullspace kernel

$$f: V \to W \qquad \mathcal{N}(f) = \{x \in V \mid f(v) = 0\} \neq \emptyset$$

$$R(V) = \{y \in W \mid \exists x \to y = f(x)\}$$

range

Proof $\mathcal{N}(f)$ is a subspace:

$y_1, y_2 \in R(f) \to$ need to show $\alpha y_1 + y_2 \in R(f)$

$$\alpha y_1 + y_2 = \alpha f(x_1) + f(x_2) = f(\alpha x_1 + x_2)$$

$\therefore\ 0 \in R(f) \qquad \therefore\ R(f) \neq \emptyset$

<u>e.g</u>: $f: \mathbb{R}^3 \to \mathbb{R}^2$ linear mapping: $\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$$f\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} x_1 + x_2 \\ x_1 - x_2 \end{vmatrix} \qquad \mathcal{N}(f) = \left\{ \alpha \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \right\} \text{ since}$$

this product gives 0

$v_1, v_2 \dots v_n$ are linearly independant if $A = [v_1 \ v_2 \dots v_n]$

$Ax = 0 \to x = 0 \qquad \sum x_i v_i = 0 \to x_i = 0 \ \forall i$

but:

linearly dependant if $\exists x \neq 0 \ni Ax = 0$

$V \xrightarrow{g} W \xrightarrow{f} Z \qquad V, W, Z$ vectorspaces

f, g linear maps

! Composition of linear maps is a linear map, but not commutative

$x \in V$

$g(x) \in W \qquad f(g(x)) = f \circ g(x) \qquad f \circ g : V \to Z$

$f(g(x)) \in Z$

<u>e.g</u>: $V = \mathbb{R}^2$, $W = \mathbb{R}^3$, $Z = \mathbb{R}^3$

$$g\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{vmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \qquad f\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} y_1 + y_3 \\ 3y_2 - y_3 \\ y_1 - 2y_2 \end{vmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$(f \circ g)\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = f(g(x)) = f\begin{vmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{vmatrix} = \begin{vmatrix} x_1 + x_2 + 3x_2 \\ 3(x_1 - x_2) - 3x_2 \\ x_1 + x_2 - 2(x_1 - x_2) \end{vmatrix} = \begin{vmatrix} x_1 + 4x_2 \\ 3x_1 - 6x_2 \\ -x_1 + 3x_2 \end{vmatrix}$$

$$(f\circ g)(\alpha x+y) = f(g(\alpha x+y)) \overset{\uparrow \text{ g linear}}{=} f(g(\alpha x)+g(y)) \overset{\uparrow \text{ f linear}}{=} \alpha f(g(x)) + f(g(y))$$
$$= \alpha(f\circ g)(x) + (f\circ g)(y)$$ QED!

PRODUCT OF 2 MATRICES

e.g.:
$$\underset{m\times n}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix}} \underset{n\times k}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}} = \underset{m\times k}{\begin{bmatrix} 1 & 4 \\ 3 & -6 \\ -1 & 3 \end{bmatrix}}$$

$$\boxed{A.B = [A_{*1}, A_{*2}, A_{*3}][B_{x_1} \mid B_{x_2}] = [AB_{x_1} \mid AB_{x_2}]}$$

OR product of matrix w/ each col entry

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{vmatrix} 1 \\ 3 \\ -1 \end{vmatrix} \;\&\; \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{vmatrix} 4 \\ -6 \\ 3 \end{vmatrix}$$

$$C = A.B \rightarrow C_{ij} = (A.B)_{ij} = \sum_{k=1}^{n} a_{in} b_{nj}$$

Tuesday 9/19

Recall: Matrix times vector $Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$

$$= x_1\begin{vmatrix} a_{11} \\ a_{21} \end{vmatrix} + x_2\begin{vmatrix} a_{12} \\ a_{22} \end{vmatrix} + x_3\begin{vmatrix} a_{13} \\ a_{23} \end{vmatrix}$$

Matrix times matrix

$$A[x \mid y] = \left[\begin{array}{c|c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{array}\right]$$

$$\underset{n_1\times n_2}{A} \;.\; \underset{n_2\times n_3}{B} = \underset{n_1\times n_3}{C} \qquad C_{ij} = \sum_{k=1}^{n_2} A_{ik}B_{kj}$$

n_2 same

$AB \neq BA$ Matrix product is not commutative

row x column

e.g.:
$$\underset{3\times 2}{B} \;.\; \underset{2\times 3}{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$

$A = [a_1 \mid a_2 \dots \mid a_n]$ $\quad$ $Av = \sum_{j=1}^{n} v_j a_j = \sum_{j=1}^{n} v_j A_{*j}$

(i) $\left.\begin{matrix} Ax = 0 \\ x \neq 0 \end{matrix}\right\}$ columns of $A \rightarrow$ linearly dependent

(ii) $\left.\begin{matrix} Ax = 0 \\ x = 0 \end{matrix}\right\}$ columns of $A \rightarrow$ linearly independent

Note: $A.B = 0 \xrightarrow[\text{mean}]{\text{does not}} A = 0$ or $B = 0$ ✗

e.g: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

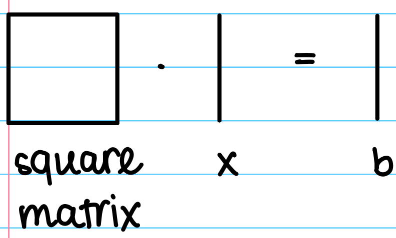
Similarly $\begin{matrix} AB = C \\ AD = C \end{matrix} \xrightarrow[\text{mean}]{\text{does not}} B = D$ ✗

e.g: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$

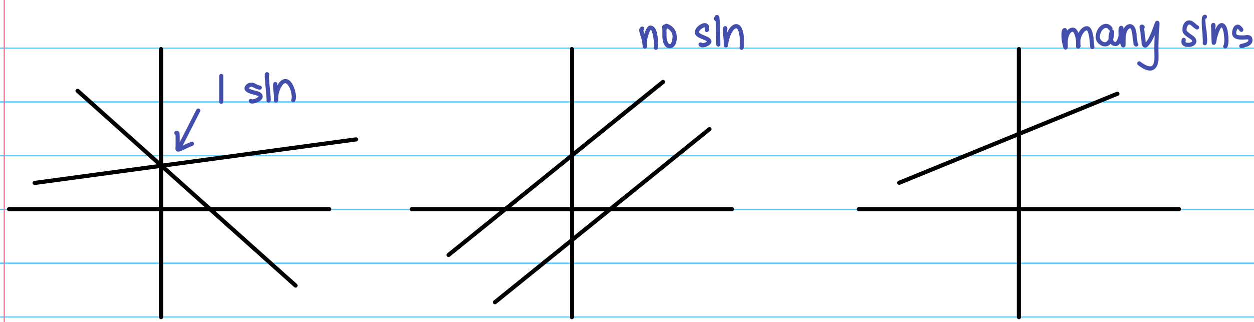
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$

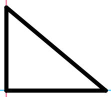
$Ax = b$ is consistent if $\exists$ at least 1 solution x
$\iff$ b is a linear combination of the columns of A

e.g:

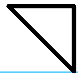


$\rightarrow x_i = \frac{b_i}{a_{ii}}$



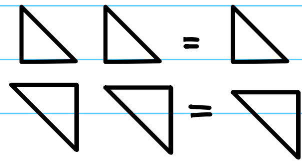
 lower triangular

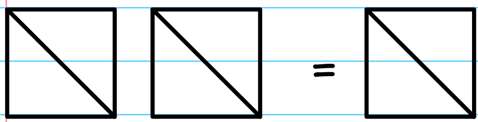
$a_{ij} = 0 \ (i < j)$

 upper triangular

$a_{ij} = 0 \ (i > j)$

Theorem → Let A, B be upper / lower triangular then A.B is also upper / lower triangular





A B

diag (AB) = $a_{ii} b_{ii}$

Proof →

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{vmatrix} b_{1j} \\ \dots \\ b_{nj} \end{vmatrix}$$

2 upper triangulars give upper

$$= \begin{bmatrix} 0 & 0 & \dots & 0 \,|\, a_{ii} & \dots & a_{in} \end{bmatrix} \begin{vmatrix} b_{1j} \\ \vdots \\ b_{jj} \\ 0 \\ 0 \\ 0 \end{vmatrix} = 0 \text{ if } i > j$$ → upper triangular

! Properties of matrix product :

(1) Distributive property $A(B+C) = AB + AC$

$(D+E)F = DF + EF$

$$\text{e.g: } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 11 \end{bmatrix}$$

$$\text{OR } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 11 \end{bmatrix}$$

(2) Associative property $A(BC) = (AB)C$

$$\text{e.g: } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}$$

$$\text{OR } \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}$$

$\alpha BC = (\alpha B)C = B(\alpha C)$

$(\alpha B)(\alpha C) = \alpha^2 BC$

Def → Identity matrix $n \times n$ I

$ii = 1$

$ij = 0 \ (i \neq j)$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\boxed{Ix = x}$ $\quad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{vmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{vmatrix} = \begin{vmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{vmatrix}$

$\boxed{x^T I = x^T}$ $\quad [x_1 \; x_2 \ldots x_n] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = [x_1 \; x_2 \ldots x_n]$

$\underset{n\times n}{I} . \underset{n\times k}{A} = \underset{n\times k}{A}$ $\qquad$ $\underset{n\times n}{A} . \underset{n\times n}{I} = \underset{n\times n}{A}$

$A^2 = A . A$
$A^3 = A . A . A = A^2 . A = A . A^2$
$A^4 = (A^2)^2$
$(A^2)^3 = (A^3)^2 = A^6 = A^4 . A^2$

$A^0 = I$
$A^x . A^y = A^{x+y}$
$(A^x)^y = A^{x.y}$

$A . A^{-1} = I$ if $B_{m\times n}$ exists $\ni A.B = I \quad \therefore B = A^{-1}$ (A inverse)
$B.A = I$

B exists if columns of A are linearly independent
↳ A is called **nonsingular**

$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

$\boxed{(AB)^T = B^T A^T}$ $\qquad$ $\boxed{(A+B)^T = A^T + B^T}$

Thursday 9/21

all n columns are linearly independent
(1) $n \times n$ nonsingular iff $Ax = 0 \rightarrow x = 0$

(2) I identity $n\times n \rightarrow Iv = v \quad \forall v \in \mathbb{R}^n$

(3) Corollary A $n\times n$ nonsingular if $Ax = b$ has unique sln

Proof →
Let x, y be 2 slns of $Ax = b$
$\left.\begin{array}{l} Ax = b \\ Ay = b \end{array}\right\} \begin{array}{l} Ax - Ay = 0 \\ A(x-y) = 0 \rightarrow x = y \end{array}$ QED!

(4) Corollary A $n\times n$ nonsingular $\exists$ unique $X \ni AX = I$

Proof →
Solve $Ax = e_i$ get unique sln
↳ ith column of I

We call X : A^{-1} (A inverse)

(5) If $AX = I$ & A nonsingular then $XA = I$

Proof → X nonsingular

otherwise, $\exists\ v \neq 0 \ni Xv = 0$

$v = Iv = AXv = A.0 = 0$ (contradiction to $v \neq 0$)

$\therefore X$ nonsingular $\ni X.X^{-1} = I$

$AX = I \longrightarrow AXX^{-1} = X^{-1}$ QED!

$A\underbrace{I} = X^{-1}$ or $A = X^{-1} \longrightarrow XA = XX^{-1} \rightarrow XA = I$

(6) A diagonal, $a_{ii} \neq 0$

$$A = \begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} \longrightarrow A^{-1} = \begin{vmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & \\ & & \ddots & \\ & & & 1/a_{nn} \end{vmatrix}$$

(7) A triangular & A^{-1} also triangular $\quad diag(A^{-1}) = diag\left(\frac{1}{a_{ii}}\right)$

e.g: $A = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} \quad A^{-1} = \begin{vmatrix} 1 & 0 \\ -3/2 & 1/2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

(8) $(A^{-1})^{-1} = A$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ → reverse order

(9) $(AB)^{-1} = B^{-1}A^{-1}$ → product of nonsingular matrices is nonsingular

Proof → $A\underbrace{BB^{-1}}A^{-1} = A\underbrace{IA^{-1}} = AA^{-1} = I$

(10) $(A^{T})^{-1} = (A^{-1})^{T} = A^{-T}$

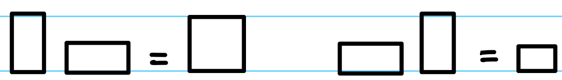
e.g $A = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \rightarrow A^{-1} = \frac{1}{2}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \xrightarrow{\text{confirm}} AA^{-1} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = I$

$A^{T} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$ ← inverse match →

Proof → $A^{T}[(A^{-1})^{T}] = (A^{-1}A)^{T} = I^{T} = I$ QED!

↳ $(AB)^{T} = B^{T}A^{T}$

! AA^{T} & $A^{T}A$ is symmetric



Proof → $A \; m\times n \rightarrow A^T \; n\times m$

$$\left.\begin{aligned}(AA^T)^T &= (A^T)^T A^T = AA^T \\ (A^TA)^T &= (A^T)(A^T)^T = A^TA\end{aligned}\right\} \text{QED!}$$

(11) $AB \neq BA$ but $\operatorname{trace}(AB) = \operatorname{trace}(BA)$

Proof → $\operatorname{trace}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} A_{i*} B_{*i} = \sum_{i=1}^{n} \sum_{k=1}^{m} a_{ik} b_{ki}$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ki} a_{ik} = \sum_{k=1}^{m} \sum_{i=1}^{n} b_{ki} a_{ik} = \sum_{k=1}^{m} B_{*i} A_{i*} = \sum_{k=1}^{m} (BA)_{kk} = \operatorname{trace}(BA)$$

QED!

e.g: $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$ $B = \begin{vmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 1 \end{vmatrix}$ $\rightarrow AB = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix}$ $\operatorname{trace}(AB) = 0 + 2 = 2$

match

$BA = \begin{vmatrix} 3 & 5 & 7 \\ -1 & -2 & -3 \\ 1 & 1 & 1 \end{vmatrix}$ $\operatorname{trace}(BA) = 3 - 2 + 1 = 2$

(12) $\operatorname{trace}(ABC) = \operatorname{trace}(CAB) = \operatorname{trace}(BCA)$
$\neq \operatorname{trace}(BAC)$

CHAPTER 1.9

Recall →

Vectorspace w/ a norm V

- $\|v\| \geq 0$
- $\|v\| = 0 \rightarrow v = 0$
- $\|\alpha v\| = |\alpha| \|v\|$
- $\|v + w\| \leq \|v\| + \|w\|$

Matrix norm $V = \mathbb{R}^{m\times n}$

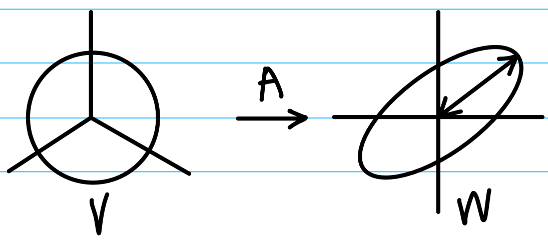
- $\|A\| \geq 0$
- $\|A\| = 0 \rightarrow A = 0$
- $\|\alpha A\| = |\alpha| \|A\|$
- $\|A + B\| \leq \|A\| + \|B\|$
- $\|AB\| \leq \|A\| . \|B\|$

$$\|A\|_F = \left(\sum_i \sum_j a_{ij}^2\right)^{1/2}$$

$\langle A, B\rangle = \operatorname{trace}(A^TB)$

$\langle A, A\rangle = \operatorname{trace}(A^TA)$

CBS $\langle A, B\rangle \leq \|A\|_F \|B\|_F$ $\therefore \|A\| = \sqrt{\operatorname{trace}(A^TA)}$



matrix norm induced by vector norm in V, W

$$\boxed{\sup_{\|v\|_V = 1} \|Av\|_W = \|A\|_{V,W}}$$

If 2 norm in V, W: $\boxed{\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}}$

$v \rightarrow \frac{v}{\|v\|} = \frac{1}{\|v\|} v$ has norm 1 $\forall\, v \neq 0$

U orthogonal matrix (n x n) if columns are orthonormal & $\|U_v\| = 1$

$U^T U = I$

$\langle u_i, u_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

maintain length & angles

length $\|Uv\|_2^2 = \langle Uv, Uv \rangle = (Uv)^T Uv = v^T U^T Uv = v^T I v = v^T v = \|v\|^2$

angle $\measuredangle (Ux, Uy) = \frac{\langle Ux, Uy \rangle}{\|Ux\| \|Uy\|} = \frac{(Ux)^T Uy}{\|Ux\| \|Uy\|} = \frac{x^T U^T Uy}{\|Ux\| \|Uy\|} = \frac{x^T y}{\|x\| \|y\|} = \measuredangle (x, y)$

$V^T V = I \rightarrow V^{-1} = V^T$

$\|V\|_2 = \sup_{x \neq 0} \frac{\|Vx\|}{\|x\|} = \sup_{x \neq 0} \frac{\|x\|}{\|x\|} = 1$

TEST 1 REVIEW

Tuesday 9/26

1. Vectorspace V

If $x, y \in V \rightarrow x + y \in V$, $\alpha v \in V \;\; \forall \alpha \in \mathbb{R}$ $\xrightarrow{\text{V vectorspace}}$ then $0 \in V$

2. subspace $S \subseteq V$ is a vectorspace $\xrightarrow{\text{S subspace}}$ $0 \in S$

e.g.:  $\left\{ x \mid x = \alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix} \right\} = S$

3. Linearly independent $v_1, v_2, \dots v_n \in V$

$\sum \alpha_i v_i = 0 \rightarrow \alpha_i = 0 \;\; \forall i$

$A = [v_1 \; v_2 \dots v_n]$

$Ax = 0 \rightarrow x = 0$

④ Linearly dependent $v_1, v_2 \dots v_n$ if $\exists x \neq 0$ w/ $Ax = 0$

⑤ Matrix A $n \times n$

A is non-singular: $Ax = 0 \longrightarrow x = 0$

$\exists A^{-1} \ni AA^{-1} = I \longrightarrow A^{-1}A = I$

e.g: $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

trace = sum of diagonals

symmetric $A^T = A$ if $a_{ij} = a_{ji}$

skew-symmetric $A^T = -A \longrightarrow a_{ii} = 0$

$(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1}$

⑥ To show X is the inverse of M $MX = I$

$AB = \boxed{} \longrightarrow$ each column of product is a linear combination of columns of A

---

(1) A is orthogonal $\rightarrow$ every pair of column is orthogonal to each other
$\searrow$ each column has norm 1

$\therefore \|A\|_2 = 1$

(2) $A^T A = 1$ $\langle Ax, Ay \rangle = \langle x, y \rangle$ maintain angle

(3) $\|A\| = \sup\limits_{\|x\|=1} \|Ax\| = \sup\limits_{\|x\| \neq 0} \dfrac{\|Ax\|}{\|x\|}$

(4) $\|AB\| \leq \|A\| \|B\|$

(5) If A, B orthogonal ($A^T A = I$, $B^T B = I$) then AB is orthogonal

Proof $\rightarrow$ Need to show $(AB)^T AB = I$

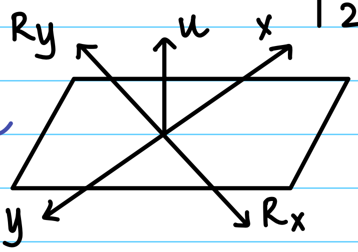
$(AB)^T AB = B^T \underset{\sim}{A^T A} B = B^T \underset{\sim}{I B} = B^T B = I$ QED!

(6) If A non-singular $\rightarrow \|A^{-1}\| = \max\limits_{x \neq 0} \dfrac{\|A^{-1}x\|}{\|x\|} = \max\limits_{y \neq 0} \dfrac{\|y\|}{\|Ay\|} = \dfrac{1}{\min\limits_{y \neq 0} \dfrac{\|Ay\|}{\|y\|}}$

A non-singular $\left.\begin{matrix} \\ x \neq 0 \end{matrix}\right\}$ $Ax \neq 0$ $\qquad Ay = x \rightarrow y = A^{-1}x$

e.g.: $\{x \in \mathbb{R}^3 \mid 3x_1 - x_2 + 2x_3 = 0\} \rightarrow u = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ or $\left\{ \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

Reflection on hyperplane



$H_u = \{x \in V \mid \langle x, u \rangle = 0\}$

$Ru = -u$

If $x \in H_u$ then $R_x = x$ $\qquad \boxed{R = I - 2\dfrac{uu^T}{u^Tu}}$ $\qquad R^T = R$

$R^TR = R^2 = I$

If $\|u\| = 1$ then $R = I - 2uu^T$

e.g.: $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \dfrac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$ with $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u^Tu = 3$

$$R^TR = RR = \frac{1}{3}\left(\frac{1}{3}\right)\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I \ \checkmark$$

$$Ru = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -u \ \checkmark$$

Take $w^Tu = 0$ or $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Rw = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = w \ \checkmark$$

Tuesday 10/3

Reflector across hyperplane $H = \{x \mid \langle x, u \rangle = 0\}$ orthogonal to u

$$R = I - 2\frac{uu^T}{u^Tu} = I - \frac{2}{u^Tu}uu^T \xrightarrow[u^Tu = 1]{\|u\| = 1} R = I - 2uu^T$$

$R^T = R \longrightarrow R^T R = I$
$Ru = -u$
if w is such that $\langle w, u \rangle = 0$ $(w \in H) \longrightarrow Rw = w$

Proof $\rightarrow$ $R^T = (I - 2uu^T)^T = I^T - 2(uu^T)^T = I - 2uu^T = R$ ✓

$Ru = (I - 2uu^T)u = Iu - 2u\underbrace{u^T u}_{I} = Iu - 2uI = u - 2u = -u$ ✓

$Rw = (I - 2uu^T)w = Iw - 2u\underbrace{u^T w}_{0} = w$ ✓

$0 \longrightarrow \langle w, u \rangle = 0$ or $u^T w = 0$

$R^T R = RR = (I - 2uu^T)^2 = I^2 - 4Iuu^T + 4u\underbrace{u^T u}u^T$
$= I - 4uu^T + 4uu^T = I$ ✓

CHAPTER 1.10

Recall $\rightarrow$ A $n \times n$ nonsingular if $\exists\, X \ni AX = I \longrightarrow$ we call $X = A^{-1}$
$AX = I \longrightarrow XA = I$

(1) It has an inverse $A^{-1} \ni AA^{-1} = I$, $A^{-1}A = I$
(2) $Ax = 0 \longrightarrow x = 0$
(3) All columns are linearly independent
(4) $Ax = b$ has a unique sln $\forall b$

Let Q $n \times n \longrightarrow Q^T Q = I \longrightarrow Q^{-1} = Q^T$
Special case R reflection : $R^T R = I$, $R^T = R$ $>$ $R^{-1} = R^T = R$

ORTHOGONAL PROJECTIONS

$P^2 = P$ $\quad \boxed{P = I - uu^T}$ $\quad$ if $\langle w, u \rangle = 0$ $\quad Pw = w$
$P^T = P$

Proof $\rightarrow$ $P^2 = P.P = (I - uu^T)^2 = I - 2Iuu^T + u\underbrace{u^T u}_{I}u^T = I - 2uu^T + uu^T = I - uu^T$
$= P$ ✓

$Pw = (I - uu^T)w = Iw - u\underbrace{u^T w}_{0} = w - 0 = w$ ✓
0 since $\langle w, u \rangle = 0$ or $u^T w = 0$

e.g: $u = \frac{1}{3}\begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} \longrightarrow \|u\| = \frac{1}{3}\sqrt{1+4+4} = \frac{1}{3}\sqrt{9} = \frac{1}{3}(3) = 1$

$$uu^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$P = I - uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

$$P^2 = P.P = \frac{1}{81}\begin{bmatrix} 72 & -18 & -18 \\ -18 & 45 & -36 \\ -18 & -36 & 45 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} = P \checkmark$$

$$Pu = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \checkmark$$

$$w^T u = 0. \text{ Take } w = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \longrightarrow Pw = \frac{1}{9}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -18 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$= w \checkmark$$

e.g: $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \longrightarrow uu^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$P = I - uu^T = I - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$R = I - 2uu^T = I - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

CHAPTER 2.1

$Ax = b$ is consistent iff $\longrightarrow$ 2 possible cases — unique sln / infinitely many slns

(1) $\exists$ at least 1 sln

(2) b is a linear combination of the columns of A

$Ax = b$ is inconsistent iff there is no sln e.g: $\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 3 \\ 4 \end{vmatrix}$ no sln

$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \end{vmatrix}$ one sln $x = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$

$\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 3 \\ 3 \end{vmatrix}$ many slns $\begin{vmatrix} 1 \\ 1 \end{vmatrix}, \begin{vmatrix} 3 \\ 0 \end{vmatrix}, \dots$

GAUSSIAN ELIMINATION

$Ax = b \longrightarrow Ex = c$

find an equivalent system (same set of sln)

Thursday 10/5

Elementary row operations

(1) Row interchanges (operations row i w/ row j)

(2) Replace a row by a multiple of itself

(3) Replace a row by adding a multiple of another row

Forward elimination

e.g: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} x = \begin{vmatrix} -1 \\ -2 \\ -1 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 3 & -2 \\ 3 & 1 & 2 & -1 \end{vmatrix} \xrightarrow[R_3' \leftarrow R_3 - 3R_1]{R_2' \leftarrow R_2 - 2R_1} \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 0 \\ 0 & -5 & -7 & 2 \end{vmatrix}$

[A | b]

$m_{21} = \dfrac{-a_{21}}{a_{11}} = \dfrac{-2}{1} = -2 \qquad m_{31} = \dfrac{-a_{31}}{a_{11}} = \dfrac{-3}{1} = -3$

$\xrightarrow{R_3'' \leftarrow R_3' - \frac{5}{3}R_1'} \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & -2 & 2 \end{vmatrix} \qquad m_{32} = \dfrac{-a'_{32}}{a'_{22}} = \dfrac{-(-5)}{-3} = -\dfrac{5}{3}$

[U | c]

Back substitution

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 2 \end{vmatrix}$

$x_3 = \dfrac{2}{-2} = -1$

$-3x_2 - 3x_3 = 0 \rightarrow -3x_2 = -3 \rightarrow x_2 = 1$

$x_1 + 2x_2 + 3x_3 = -1 \rightarrow x_1 = 0$

$\therefore x = \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}$

Verify $1(0) + 2(1) + 3(-1) = -1$ ✓

$$Ux = \begin{vmatrix} u_{11} & u_{12} & \cdots & u_{nn} \\ 0 & u_{22} & \cdots & u_{nn} \\ & & \ddots & \\ & & & u_{nn} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{vmatrix} = \begin{vmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{vmatrix}$$

$$x_n = \frac{c_n}{u_{nn}} \qquad x_k = \left(c_k - \sum_{j=k}^{n} u_{ij} x_j\right) \Big/ u_{kk} \qquad (k = n-1, n-2 \ldots 2, 1)$$

Rank of matrix = # pivots

$\left.\begin{array}{l} A \ n \times n \\ A \text{ nonsingular} \end{array}\right\}$ Rank $A = n$ (full rank) $\qquad \left.\begin{array}{l} A \ n \times n \\ A \text{ singular} \end{array}\right\}$ Rank $A < n$

Exam question: Give 5 conditions of A nonsingular & prove implications

How to use elementary row operations?

(1) If non-zero pivots $\longrightarrow$ eliminate below it

(2) If zero in pivot position

(i) if non-zero below $\longrightarrow$ row interchanges (put non-zero in pivot position)

(ii) if all zeros below $\longrightarrow$ move to next column (repeat if necessary)

e.g.: $A = \begin{vmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{vmatrix} \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - 2R_1}} \begin{vmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{vmatrix}$ move to next col

$\xrightarrow{\substack{R_3 \leftarrow R_3 + R_2 \\ R_4 \leftarrow R_4 - R_2}} \begin{vmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} \xrightarrow[R_3 \,\&\, R_4]{\text{interchange}} \begin{vmatrix} \boxed{1} & 2 & \boxed{1} & 3 & \boxed{3} \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = E_A$

3 pivots $\longrightarrow$ 3 linearly independent cols (or 3 basic cols)

↳ 1st non-zero in each row of upper echlon form

Basic cols are the pivot cols & are linearly independent

$$m_{31} = \frac{-a_{31}}{a_{11}} = \frac{-1}{1} = -1 \qquad m_{32} = \frac{-a_{32}}{a_{22}} = \frac{-(-2)}{2} = 1$$

e.g.: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{R_3 \uparrow R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank 2, basic cols #1 & #2

e.g.: Rank 1 matrix: $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -3 & 6 & 9 \\ 4 & -4 & 8 & 12 \end{vmatrix}$ or $\begin{vmatrix} 1 \\ 2 \end{vmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$

cols are linearly dependent

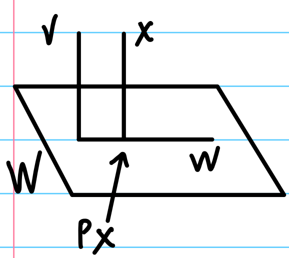
$\therefore uv^T$ is singular

Elementary matrix $(n \times n)$ of the form $E = I - \boxed{uv^T} \, (u^T v \neq 1)$

$$|u_1 \ u_2| |v_1 \ v_2| = u_1 v_1^T + u_2 v_2^T$$

Tuesday 10/10

Recall:



$R(P) = W \qquad \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$P^2 = P \qquad v \in V \longrightarrow Pv = 0$

$P^T = P \qquad w \in W \longrightarrow Pw = w$

$v \perp W$

If $P^T = P$: $\langle x - Px, Px \rangle = 0 = [(I - P)x]^T Px = x^T (I - P)^T Px$

$= x^T \underbrace{(I - P) P}_{0} x = 0$ QED!

$V = \{\alpha u\} \longrightarrow V = \{\alpha_1 u_1 + \alpha_2 u_2 ; \alpha_1, \alpha_2 \in \mathbb{R}\}$

$\|u\| = 1 \qquad U = |u_1 u_2| \qquad P = I - uu^T \longrightarrow I - P = uu^T$

$\|u_1\| = \|u_2\| = 1 \qquad U^T U = I \qquad = I - UU^T$

$u_1^T u_2 = 0 \qquad P^2 = P$

$u \in V \longrightarrow Pu = 0$

$w \in R(P) \longrightarrow Pw = w$

$[A | b] \longrightarrow [E | c] \qquad M \ n \times n$ nonsingular

rank(MA) = rank(A)

If A $n \times n$: rank$(A) = n \longrightarrow$ rank$(MA) = n \quad ((MA)^{-1} = A^{-1} M^{-1})$

$\square$ A is $n \times k$, $\text{rank}(A) = k$ $\nearrow$ cols of A are linearly independent
$\searrow$ $Ax = 0 \longrightarrow x = 0$

$MAx = 0$

To show $\text{rank}(MA) = k \longrightarrow$ show $x = 0 \longrightarrow MA$ is nonsingular

Proof $\rightarrow$

$MAx = 0$

$M^{-1}MAx = M^{-1}0$

$Ax = 0 \longrightarrow x = 0$ QED!

If A has k basic cols then MA has k basic cols.

ELEMENTARY MATRIX

Def $\rightarrow$ $n \times n$ elementary matrix has the form $I - uv^T$ $(u, v \in \mathbb{R}^n ;\ u^Tv = v^Tu \neq 1)$

Proposition $\rightarrow$ Show that every elementary matrix is nonsingular & its inverse is also an elementary matrix

$$\boxed{(I - uv^T)^{-1} = I - \frac{uv^T}{u^Tv - 1} = I - \frac{1}{u^Tv - 1} uv^T}$$

a # $\rightarrow$ can factor $u\alpha v^T = \alpha(uv^T)$

Proof $\rightarrow$

$$(I - uv^T)\left(I - \frac{uv^T}{u^Tv - 1}\right) = I^2 - \frac{uv^T}{u^Tv - 1} - uv^T + \frac{uv^Tuv^T}{u^Tv - 1}$$

$$= I + \left(-1 - \frac{1}{u^Tv - 1} + \frac{v^Tu}{u^Tv - 1}\right) uv^T$$

$$= I + \frac{-(u^Tv - 1) - 1 + u^Tv}{u^Tv - 1} uv^T = I + \frac{0}{u^Tv - 1} uv^T = I \quad \text{QED!}$$

$E^T = (I - uv^T)^T = I - vu^T$

left multiply by E : row operation
right __________ col ________

unit lower triangular

$$E_{21} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad E_{31} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix}$$

e.g.:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} x = \begin{vmatrix} -1 \\ -2 \\ -1 \end{vmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 2 & 1 & 3 & -2 \\ 3 & 1 & 2 & -1 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 0 \\ 0 & -5 & -7 & 2 \end{array}\right]$$

$$E_{32} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{vmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{5}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] \qquad m_{32} = \frac{-a_{32}}{a_{22}} = \frac{-(-5)}{-3} = -\frac{5}{3}$$

$$E_{32}E_{31}E_{21}A = U \longrightarrow A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

$$A = \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{3} & 1 \end{vmatrix}}_{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{5}{3} & 1 \end{vmatrix}} U$$

order matters

LU factorization of A $\longrightarrow A = LU$

L

lower triangular

$$E_{32}E_{31}E_{21}[A \mid b] = [E \mid c]$$

rank maintained

Note

$\text{rank}(A^T) = \text{rank}(A)$

linearly independent rows = # of linearly independent cols

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$\text{rank}(A) = r \leq m$

$\downarrow$ full rank factorization

$A = B_{m \times r} \cdot C_{r \times n}$

both are full rank

(i) Basic col j in A $\longrightarrow$ c_{*j} has one 1, the rest 0

(ii) Non-basic col j $\longrightarrow$ $A_{*j} = \sum_{i \text{ basic}} \alpha_i A_{*j}$ $\qquad c_{*j} = \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_r \end{vmatrix}$

e.g.: $A = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 3 \end{vmatrix}\begin{vmatrix} 1 & 3 & 2 \end{vmatrix}$

rank 1 $\qquad B_{n \times 1} \quad C_{1 \times n}$

$$A = \begin{vmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix}\begin{vmatrix} 1 & 1/2 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}\begin{vmatrix} 2 & 1 \end{vmatrix}$$

Theorem Let $A \in \mathbb{R}^{m \times n}$

(i) $\text{Rank}(A^TA) = \text{Rank}(AA^T) = \text{Rank}(A)$

(ii) $AA^Tx = 0 \to A^Tx = 0$ $\qquad$ $A^TAx = 0 \to Ax = 0$

Proof $x^TAA^Tx = (A^Tx)^TA^Tx = \|A^Tx\|^2 = 0$ $\qquad$ $x^TA^TAx = (Ax)^TAx = \|Ax\|^2 = 0$

$\therefore A^Tx = 0$ $\qquad$ $\therefore Ax = 0$

Theorem $A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$ $\quad \text{rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$

Proof $\text{rank}(A) = r$

$PA = E_A$ (echlon form of A) $= \begin{bmatrix} C_{r\times n} \\ 0 \end{bmatrix}$

$\text{rank}(AB) = \text{rank}(\underbrace{P}_{\text{nonsingular}}AB) = \text{rank}(E_AB) = \text{rank}\begin{bmatrix} C_{r\times n}B \\ 0 \end{bmatrix} = \text{rank}\begin{bmatrix} (CB)_{r\times p} \\ 0 \end{bmatrix} \leq r$ QED!

$\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^TA^T) \leq \text{rank}(B^T) = \text{rank}(B)$ QED!

	Consistent	Inconsistent system
$Ax = b$	at least 1 sln, $b \in R(A)$	no sln, $b \notin R(A)$
$[E \mid c]$	has no row of $[0 \dots 0 \mid \alpha]$ $\alpha \neq 0$ $\text{rank}(A) = \text{rank}([A \mid b])$	has 1 row of $[0 \dots 0 \mid \alpha]$ $\alpha \neq 0$ $\text{rank}(A) < \text{rank}([A \mid b])$

e.g.: $\begin{bmatrix} \boxed{1} & 2 & \boxed{3} & 4 \\ \boxed{2} & 4 & \boxed{8} & 10 \\ \boxed{3} & 6 & \boxed{11} & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ no sln

basic cols

= 7 instead

If asked for modification to get consistent system, make this = 0

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$Ax_p = b$

$A(x_p + \underbrace{x_h}_{\text{solution to } Ax=0}) = Ax_p + Ax_h = b + 0 = b$

e.g.: $\left[\begin{array}{cccc|c} \boxed{1} & 2 & \boxed{3} & 4 & 1 \\ \boxed{2} & 4 & \boxed{8} & 10 & 6 \\ \boxed{3} & 6 & \boxed{11} & 14 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\text{Rank}(A) = 2$

$\text{Rank}[A \mid b] = 2$

Consistent system:

(1) $\exists$ at least 1 sln

(2) $b = \sum_{i \text{ basic}} a_{i*}$

(3) $\nexists$ row $[E_A | c]$ of the form $[0 \ldots 0 | \alpha]$ $\alpha \neq 0$

(4) $\text{rank}(A) = \text{rank}([A|b])$

For particular sln, set free variables to 0, solve for the others

$$2x_3 + 2x_4 = 4 \xrightarrow{x_4 = 0} 2x_3 = 4 \rightarrow x_3 = 2$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \xrightarrow[x_4 = 0]{x_2 = 0} x_1 + 3(2) = 1 \rightarrow x_1 = -5 \rightarrow x_p = \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Verify: $-5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$

Write basic variables in terms of free variables

$2x_3 + 2x_4 = 0 \longrightarrow x_3 = -x_4$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0 \longrightarrow x_1 = -2x_2 - 3x_3 - 4x_4 = -2x_2 + 3x_4 - 4x_4$
$= -2x_2 - x_4$

$$\therefore x_h = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = \underset{\alpha}{x_2}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \underset{\beta}{x_4}\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

General sln of $Ax=b$ is $X = \begin{bmatrix} -5 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ $\forall \alpha, \beta \in \mathbb{R}$

e.g.: $\begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{bmatrix} = U$

$m_{21} = -1$ $\quad m_{32} = -4$

$m_{31} = -1$

$$E_{21} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \rightarrow A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{bmatrix}}_{U}$$

$$b = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix} \to \text{Solve} \overset{L}{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{vmatrix}} \overset{y}{\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix}} = \overset{b}{\begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}} \to \begin{array}{l} y_1 = 3 \\ y_1 + y_2 = -2 \to y_2 = -5 \\ y_1 + 4y_2 + y_3 = 1 \to y_3 = 18 \end{array}$$

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$Ax = b \to LUx = b \to$ useful when solving for multiple bs.

(1) $Ly = b$ ◺ $\underset{y}{|}\; = \;\underset{b}{|} \to y_1 = b_1$ (2) $Ux = y$ ◹ $\underset{x}{|}\; = \;\underset{y}{|} \to y_3 = b_3$

(cont.) e.g.: $y = \begin{vmatrix} 3 \\ -5 \\ 18 \end{vmatrix} \to$ Solve $Ux = y$ $\begin{vmatrix} 2 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 3 \\ -5 \\ 18 \end{vmatrix} \to \begin{array}{l} x_3 = 2 \\ x_2 = -1 \\ x_1 = 1 \end{array} \to x = \begin{vmatrix} 1 \\ -1 \\ 2 \end{vmatrix}$

Verify: $Ax = 1 \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix} - 1 \begin{vmatrix} 3 \\ 2 \\ -1 \end{vmatrix} + 2 \begin{vmatrix} 2 \\ -1 \\ -1 \end{vmatrix} = \begin{vmatrix} 3 \\ -2 \\ 1 \end{vmatrix}$

For fix order of the rows, LU factorization is unique. (A nxn nonsingular)

L: lower unit triangular U: upper unit triangular

Proof

$A = L_1U_1 = L_2U_2$

$L_1U_1 = L_2U_2$

$U_1 = L_1^{-1}L_2U_2$

$U_1U_2^{-1} = L_1^{-1}L_2$

Since L_1^{-1} & L_2 have ones in the diagonal then product is also lower unit triangular.

$\therefore L_1^{-1}L_2 = I$

$\therefore L_1 = L_2$ & $U_1 = U_2$ $\therefore$ L, U are unique QED!

◹ = ◺ → diagonal matrix

SECTION 9 – DETERMINANT

Let $A \ni \det A \neq 0$ = Let A be nonsingular

$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ If A singular, $\det A = 0$ $\det I = 1$

nonsingular, $\det A \neq 0$

↓

linear map for each col → multilinear

e.g.: $A = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ $B = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}$ $\det B = 2 \det A$ $\boxed{\det(\alpha A) = \alpha^n \det A}$

(1) a $1\times 1 \rightarrow \det a = a$ (4) $\det(A^{-1}) = \dfrac{1}{\det A}$

(2) $\det A^T = \det A$

(3) $\det(AB) = \det A . \det B$ $AA^{-1} = I \rightarrow \det A . \det A^{-1} = \det I = 1$

(5) $\det(\text{diagonal matrix}) = \prod_i a_{ii}$ (product of diagonal entries)

(6) det(◥) = det(◣) = $\prod_i a_{ii}$ (7) $\det P = -1$

↳ col (or row) exchange

How to find determinant? → Use $A = LU$ factorization

$\det A = \underbrace{\det L}_{=1} . \det U = \det U.$

e.g: $A = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix}$

$m_{21} = -2$

$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \longrightarrow \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{vmatrix}$ $m_{21} = \dfrac{-a_{21}}{a_{11}}$

$L = \begin{vmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{vmatrix}$ $\boxed{\det A = \det U = a_{11}a_{22} - a_{21}a_{12}}$

e.g: $\det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\det A = \sum_{\text{all permutations of } p \text{ of } \{1,2,3\ldots n\}} \sigma(p)\, a_{1p_1} a_{2p_2} a_{3p_3} \ldots a_{np_n}$ $\sigma(p) = (-1)^{\#\text{exchanges}}$

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$\det A = \begin{cases} 0 & A \text{ singular} \\ \neq 0 & A \text{ nonsingular} \end{cases}$ $\det A = a_{11}a_{22} - a_{12}a_{21}$ 2×2

$\det(a) = a \leftarrow \#$

$\det(AB) = \det A . \det B$

diagonal/triangular

$\det A = \det(LU) = \underbrace{\det L}_{1} . \det U = \prod u_{ii}$

$\mathring{A}_{ij}$ $(n-1)\times(n-1)$ matrix where row i, col j deleted

delete row 1 →

e.g: $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$ $\mathring{A}_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$ $\mathring{A}_{31} = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$ $\mathring{A}_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$ ...

the rest is

delete col 2 ↑

$$\boxed{\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det \mathring{A}_{ij}}$$

$$\det A = 0 . \det \mathring{A}_{31} - 1 . \det \mathring{A}_{32} + 2 \det \mathring{A}_{33}$$
$$= 0 - (1-3) + 2(-1-2) = -4$$

SECTION 4.1 & 4.2

$\text{span}\{v_1, v_2 \ldots v_n\} = \{ \sum_{i=1}^{k} \alpha_i v_i ,\ \alpha \in \mathbb{R} \} \rightarrow$ subspace

& linearly independent $\rightarrow$ a basis $\rightarrow \dim S = n$ $\Big\}$ <=> B is a minimal spanning set; <=> B is a maximal linearly independent set

e.g: $\text{span}\left\{ \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ 2 \\ 0 \end{vmatrix} \right\}$ is a basis $\rightarrow \dim S = 2$

Hyperplane in $\mathbb{R}^n = \{x \mid x^T v = 0,$ fixed $v\}$ has dimension $n-1$

Consider 2 subspaces of V
- $\rightarrow$ if $M \subseteq N$, $\dim M \leq \dim N$
- $\rightarrow$ if $\dim M = \dim N$, $M = N$

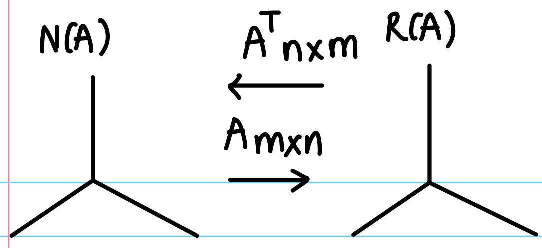
e.g: $M = \left\{ x \in \mathbb{R}^3 \mid x^T v = 0 ,\ v = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \right\}$
- $\rightarrow$ hyperplane in $\mathbb{R}^3$
- $\rightarrow \dim M = 2$

$N = \left\{ x \in \mathbb{R}^3 \mid x = \alpha_1 \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} + \alpha_2 \begin{vmatrix} 1 \\ 2 \\ 0 \end{vmatrix} \right\}$
- $\rightarrow$ basis
- $\rightarrow \dim N = 2$

Given $\{v_1, v_2 \ldots v_k\}$ basis of a subspace S, any vector $v \in S$ can be written in a unique manner as a linear combination of the vectors of the basis

$$\boxed{v = \sum_{i}^{k} \alpha_i v_i = \sum \beta_i v_i} \rightarrow 0 = \sum (\alpha_i - \beta_i) v_i \rightarrow \alpha_i = \beta_i$$

coordinates of v in their basis



$\dim N(A) =$ # free variables $= n-r$
$\dim R(A) = \text{rank}(A) = r$

$\boxed{\dim N(A) + \dim R(A) = n}$ $\boxed{\dim N(A^T) + \dim R(A^T) = m}$

e.g: $A = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ $\quad L = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

$m_{21} = -2,\ m_{31} = -1$
basic cols $A_1, A_2 \longrightarrow$ range spanned by 2 basic cols $R(A) = \left\{ \alpha_1 \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} + \alpha_2 \begin{vmatrix} 2 \\ 2 \\ 2 \end{vmatrix} \right\}$

$N(A) = \{x \mid Ax = 0\} = \{x \mid Ux = 0\}$

Solve $Ux = 0 \begin{cases} \text{free variables } x_3, x_4 \\ -2x_2 - 2x_3 - 3x_4 = 0 \rightarrow x_2 = -x_3 - \frac{3}{2}x_4 \\ x_1 + 2x_2 + x_3 + 2x_4 = 0 \rightarrow x_1 - 2x_3 - 3x_4 + x_3 + 2x_4 = 0 \end{cases}$
$\rightarrow x_1 - x_3 - x_4 = 0 \rightarrow x_1 = x_3 + x_4$

$x = \begin{vmatrix} x_3 + x_4 \\ -x_3 - 3/2\, x_4 \\ x_3 \\ x_4 \end{vmatrix} = \underset{\alpha_1}{x_3} \begin{vmatrix} 1 \\ -1 \\ 1 \\ 0 \end{vmatrix} + \underset{\alpha_2}{x_4} \begin{vmatrix} 1 \\ -3/2 \\ 0 \\ 1 \end{vmatrix} \rightarrow N(A)$

Let X, Y be subspaces of V
$\rightarrow X + Y = \{v \in V \mid v = x + y,\ x \in X,\ y \in Y\}$ is a subspace
$\searrow \dim(X+Y) = \dim X + \dim Y - \dim(X \cap Y)$

if $X \cap Y = \{0\}$ basis of $X = \{v_1, v_2 \dots v_k\}$
basis of $Y = \{w_1, w_2 \dots w_k\}$
$\hookrightarrow$ basis of $(X+Y)$ = basis of $X \cup$ basis of Y
$= \{v_1, v_2 \dots v_k\} \cup \{w_1, w_2 \dots w_k\}$
$= \{v_1, v_2 \dots v_k, w_1, w_2 \dots w_k\}$

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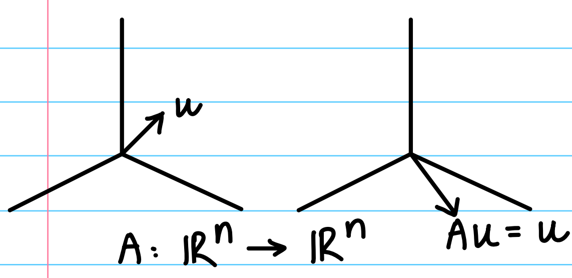
SECTION 3.1



$Au = \lambda u \longrightarrow$ multiple of u
eigenvector } eigenpair
eigenvalues } privileged directions

e.g.: $A = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}$ $\quad u = \begin{vmatrix} 2 \\ -1 \end{vmatrix} \xrightarrow{\text{normalize}} v = \frac{1}{\sqrt{5}} \begin{vmatrix} 2 \\ -1 \end{vmatrix}$

$Au = \begin{vmatrix} -2 \\ 1 \end{vmatrix} = \overset{\lambda}{(-1)} \begin{vmatrix} 2 \\ -1 \end{vmatrix}$

$A(\alpha u) = \alpha A u = \alpha \lambda u = \alpha(\lambda u)$

$\therefore$ Every multiple of an eigenvalue is an eigenvalue

If A singular → 0 is an eigenvalue
→ eigenvector in $N(A)$: $Ax = 0 = 0.x$

If u, v exist $\exists$ $Av = \lambda v$ $(v \neq 0)$ then $Av - \lambda v = 0$

$\iff (A - \lambda I) v = 0 \iff \det(A - \lambda I) = 0$

$\therefore$ $N(A - \lambda I)$ is the set of eigenvectors corresponding to λ
& also eigenspace

$A - \lambda I = \begin{vmatrix} -1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} \rightarrow \det(A - \lambda I) = -(1+\lambda)(1-\lambda) = \lambda^2 - 1 = \underbrace{p(\lambda)}_{\text{characteristic polynomial}}$

$\therefore$ eigenvalues $\lambda = \pm 1$

(i) $\lambda = 1 \rightarrow A - \lambda I = \begin{vmatrix} -2 & 0 \\ 1 & 0 \end{vmatrix} \rightarrow u = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$ $\quad -2x_1 = 0 \rightarrow x_1 = 0$

$N(A - I) = \left\{ \alpha \begin{vmatrix} 0 \\ 1 \end{vmatrix} \right\}, \alpha \in \mathbb{R}$

(ii) $\lambda = -1 \rightarrow A - \lambda I = \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} \rightarrow u = \begin{vmatrix} -2 \\ 1 \end{vmatrix}$ $\quad x_1 + 2x_2 = 0 \rightarrow x_1 = -2x_2$

$N(A + I) = \left\{ \alpha \begin{vmatrix} -2 \\ 1 \end{vmatrix} \right\} = \left\{ \alpha \begin{vmatrix} 2 \\ -1 \end{vmatrix} \right\}, \alpha \in \mathbb{R}$

$\sigma(A)$ spectrum of $A = \{-1, 1\}$
↳ set of eigenvalues of A

→ $\dim(N(A - \lambda I)) = 1$
→ $\dim(N(A - \lambda I)) \geq 1$ if ≥ 1 eigenvector

e.g.: $A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(1-\lambda)^2(1+\lambda) = p_A(\lambda)$

(i) $\lambda = 1 \rightarrow$ double root
$\rightarrow$ multiplicity $= 2$

$$N(A-\lambda I) = N\left(\begin{vmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{vmatrix}\right) = \left\{\alpha_1 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + \alpha_2 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}\right\} \text{ dim } 2$$

(ii) $\lambda = -1 \rightarrow$ multiplicity 1

$$N(A-\lambda I) = N\left(\begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}\right) = \left\{\alpha \begin{vmatrix} 0 \\ -2 \\ 1 \end{vmatrix}\right\}$$

$2x_1 = 0$
$x_2 + 2x_3 = 0 \rightarrow x_2 = -2x_3$

$p_A(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\dots(a_{nn}-\lambda)\dots$
$= (-1)^n \lambda^n + \dots$

n odd $\rightarrow \geq 1$ real eigenvalues $\forall A$ $n\times n \rightarrow \leq n$ eigenvalues

A is diagonal/triangular $\rightarrow$ $\det(A-\lambda I) = \prod_{i=1}^{n} (a_{ii}-\lambda) = 0$
$\rightarrow$ $\lambda = a_{ii}$ are the roots
$\rightarrow$ $\sigma(A) = \{a_{11}, a_{22} \dots a_{nn}\}$

Algebraic multiplicity of λ = multiplicity of λ as a root of $p_A(\lambda)$
Geometric multiplicity = $\dim N(A-\lambda I) \leq$ Algebraic multiplicity

$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i$, $\det(A) = \prod_{i=1}^{n} \lambda_i$ If $<$: λ is defective
A is defective

e.g: $A = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \rightarrow \det(A-\lambda I) = \det\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$
$\therefore \lambda = 2$, multiplicity 2

$$N(A-2I) = N\left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\right) = \left\{\alpha \begin{vmatrix} 0 \\ 1 \end{vmatrix}\right\}$$
$\dim N(A-2I) = 1 < 2$

$\sigma(A^T) = \sigma(A)$ $\rightarrow$ same eigenvalues
$\rightarrow$ possibly diff eigenvectors

e.g: $A^T = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} \rightarrow \lambda = 1 \rightarrow A^T - I = \begin{vmatrix} -2 & 1 \\ 0 & 0 \end{vmatrix}$ & $\lambda = -1 \rightarrow A^T + I = \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}$

$x = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$ $x = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$

left eigenvector

$A^Tu = \lambda u \rightarrow (A^Tu)^T = (\lambda u)^T \rightarrow u^TA = \lambda u^T \rightarrow u^TAv = \lambda u^Tv$

If $A^T = A \rightarrow \sigma(A) \subset \mathbb{R}$
$\searrow$ eigenvectors are orthogonal if eigenvalues distinct

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \end{aligned}\right\} \lambda_1 \neq \lambda_2 \rightarrow v_1^Tv_2 = 0$$

e.g: $A = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \rightarrow \det\left(\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}\right) = (3-\lambda)^2 - 1 = 0 \therefore \lambda = 4, 2$

(i) $\lambda = 4 \rightarrow A - 4I = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \rightarrow N(A-4I) = \left\{\alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix}\right\} = \left\{\alpha \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}\right\}$

(ii) $\lambda = 2 \rightarrow A - 2I = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \rightarrow N(A-2I) = \left\{\alpha \begin{vmatrix} 1 \\ -1 \end{vmatrix}\right\} = \left\{\alpha \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \end{vmatrix}\right\}$

$Av_1 = \lambda_1 v_1$ $\quad V = [v_1 \ \ v_2]$ $\quad AV = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}$
$Av_2 = \lambda_2 v_2$

$\therefore AV = V\Lambda$ $\quad \Lambda = \mathrm{diag}(\lambda_i)$
$\therefore A = V\Lambda V^{-1} = V\Lambda V^T \rightarrow$ diagonalization of A

$\rho(A)$ spectral radius $= \max|\lambda_i|$ $(\lambda_i \in \sigma(A))$

$Ax = \lambda x \rightarrow \|Ax\| = \|\lambda x\| = |\lambda|\|x\| \rightarrow |\lambda| \leq \frac{\|Ax\|}{\|x\|} \leq \|A\|$

$\therefore \rho(A) \leq \|A\|$

$N(A - \lambda I) = \mathrm{span}\{v_1, v_2 \dots v_n\} \rightarrow A\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \lambda\left(\sum_{i=1}^{k} \alpha_i v_i\right)$

Proof $\rightarrow$ $A\left(\sum \alpha_i v_i\right) = \sum \alpha_i A v_i = \sum \alpha_i \lambda v_i = \lambda\left(\sum \alpha_i v_i\right)$ QED!

$\therefore$ For a multiple eigenvalue λ, any linear combination of vectors in $N(A - \lambda I)$ is an eigenvector of λ

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e.g: $A = \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{2}}{2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$ $\quad A^TA = I$

rotation $45°$

$$\det(A-\lambda I) = \det\begin{vmatrix} \frac{\sqrt{2}}{2}-\lambda & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}-\lambda \end{vmatrix} = \left(\frac{\sqrt{2}}{2}-\lambda\right)^2 + \frac{1}{2} = \frac{1}{2} - \sqrt{2}\lambda + \lambda^2 + \frac{1}{2}$$

$$= \lambda^2 - \sqrt{2}\lambda + 1 = 0$$

$$\therefore \lambda_{12} = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm \sqrt{-2}}{2}$$

e.g: $A = \begin{vmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{vmatrix} \rightarrow A - \lambda I = \begin{vmatrix} -3-\lambda & 1 & -3 \\ 20 & 3-\lambda & 10 \\ 2 & -2 & 4-\lambda \end{vmatrix}$

$$\det(A-\lambda I) = (-3-\lambda)[(3-\lambda)(4-\lambda)+20] - 1[20(4-\lambda)-20] - 3[-40-2(3-\lambda)]$$

$$= -(\lambda-3)^2(\lambda+2) = 0 \rightarrow \begin{cases} \lambda = 3 & \text{algebraic multiplicity} = 2 \\ \lambda = -2 & 1 \end{cases}$$

(i) $\lambda = -2 \rightarrow A + 2I = \begin{vmatrix} -1 & 1 & -3 \\ 20 & 5 & 10 \\ 2 & -2 & 6 \end{vmatrix} \rightarrow N(A+2I) = \left\{ \alpha \begin{vmatrix} -1 \\ 2 \\ 1 \end{vmatrix} \right\}$

(ii) $\lambda = 3 \rightarrow A - 3I = \begin{vmatrix} -6 & 1 & -3 \\ 20 & 0 & 10 \\ 2 & -2 & 1 \end{vmatrix} \rightarrow \begin{vmatrix} -6 & 1 & -3 \\ 0 & 10/3 & 0 \\ 0 & -5/3 & 0 \end{vmatrix} \rightarrow \begin{vmatrix} -6 & 1 & -3 \\ 0 & 10/3 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

$$m_{21} = \frac{-a_{21}}{a_{11}} = \frac{-20}{-6} = \frac{10}{3} \qquad m_{32} = \frac{-a_{32}}{a_{22}} = \frac{5/3}{10/3} = \frac{1}{2} \qquad \begin{cases} x_2 = 0 \\ 2x_1 = -x_3 \end{cases}$$

$$m_{31} = \frac{-a_{31}}{a_{11}} = \frac{-2}{-6} = \frac{1}{3}$$

$$\therefore x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}x_3 \\ 0 \\ x_3 \end{vmatrix} = x_3 \begin{vmatrix} -1/2 \\ 0 \\ 1 \end{vmatrix} = \alpha \begin{vmatrix} -1 \\ 0 \\ 2 \end{vmatrix} \rightarrow N(A-3I) = \left\{ \alpha \begin{vmatrix} -1 \\ 0 \\ 2 \end{vmatrix} \right\}$$

How to find determinant of 3 x 3 matrix ?

(1) Forward elimination $\rightarrow A = L.U \rightarrow \det A = \underbrace{\det L}_{1} . \det U = \det U$

(2) Det formula

e.g: $A = \begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ & 1 & 1 & 1 & \\ & 2 & 2 & 1 & \\ 6 & 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 & 6 \\ 2 & 2 & 2 & 1 & 4 \end{array}$

$$\det A = (1+6+4) - (6+2+2) = 1 \checkmark$$

$$\overset{\circ}{A}_{11} \qquad \overset{\circ}{A}_{12} \qquad \overset{\circ}{A}_{13}$$

(3) Cofactor expansion $\det A = 1(1-2) - 1(2-6) + 2(2-3)$

$$= -1 + 4 - 2 = 1 \checkmark$$

SECTION 3.2

A & B are similar ($A \sim B$) if $\exists$ P nonsingular $\ni$ $B = P^{-1}AP$

$B \sim A$ $\qquad$ $A = P^{-1}BP$

(1) If $\lambda \in \sigma(A)$, $\exists v \neq 0 \rightarrow Av = \lambda v \rightarrow P^{-1}BPv = \lambda v$

$$\rightarrow B(Pv) = P(\lambda v) = \lambda(Pv)$$

$$\therefore \lambda \in \sigma(B)$$

(2) $p_A(\lambda) = \det(A - \lambda I) = \det(P^{-1}BP - \lambda I)$

$$= \det(P^{-1}BP - \lambda P^{-1}P)$$

$$= \det[P^{-1}(B - \lambda I)P]$$

det of product is product of dets

$$= \det P^{-1} . \det(B - \lambda I) . \det P \quad \left(\det P^{-1} = \frac{1}{\det P}\right)$$

$$= \det(B - \lambda I) = p_B(\lambda)$$

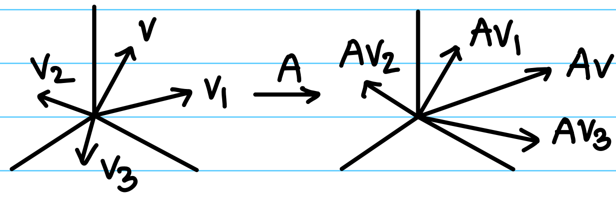
A $n \times n$ has a complete set of eigenvectors if $v_1, v_2 \dots v_n$ w/ $Av_i = \lambda_i v_i$ are linearly independent

$\rightarrow A \sim \Lambda \rightarrow$ A diagonalizable $\rightarrow$ basis of the space (dim S = n = n eigenvalues)

$V = [v_1 \ v_2 \dots v_n] \qquad \Lambda = \operatorname{diag}(\lambda_1 \ \lambda_2 \dots \lambda_n)$

$\therefore AV = V\Lambda \quad \therefore V^{-1}AV = \Lambda \quad \therefore A = V\Lambda V^{-1}$ $\rightarrow$ spectral decomposition

$\forall v \in \mathbb{R}^n, \exists! \alpha_i \ni v_i = \sum_{i=1}^{n} \alpha_i v_i$



$$Av = A\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i A v_i = \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

$|\lambda_{max}| = \max_i |\lambda_i| = \|A\| \qquad |\lambda_{min}| = \frac{1}{\|A^{-1}\|}$

$$\|Av\| \leq \sum_{i=1}^{n} |\alpha_i \lambda_i| \|v_i\| \leq \left(\sum_{i=1}^{n} |\alpha_i|\right) |\lambda_{max}|$$

$a = \begin{vmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \dots \\ \alpha_n \end{vmatrix}$ $\quad Va = \sum \alpha_i v_i = v \rightarrow a = V^{-1}v$

$\Lambda a = \Lambda V^{-1} v$

$AV = V\Lambda V^{-1} v = V\Lambda a = \sum \alpha_i \lambda_i v_i$

sum of product between entries of Λ & a

each col in V

e.g: $A = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}$ $\Lambda = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$ $V = \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} \rightarrow V^{-1} = \begin{vmatrix} 1/2 & 0 \\ 1/2 & 1 \end{vmatrix}$

$A = V \Lambda V^{-1} \rightarrow \Lambda = V^{-1} A V$

$$B = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}}^{V} \overbrace{\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}}^{\Lambda} \overbrace{\frac{1}{\sqrt{2}} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}}^{V^T}$$

SECTION 3.5

Tuesday 11/7

A rank 1 $\rightarrow A = uv^T$

A rank 2 $\rightarrow A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

LU factorization $\rightarrow A = LU \rightarrow$ give ranks, solve linear systems

spectral decomposition $\rightarrow A = V \Lambda V^{-1}$

today $\rightarrow$ singular value decomposition

$A = U \Sigma V^T$ ($m \times n$)
- U $m \times n$, V^T $n \times n$ orthogonal $\rightarrow$ orthonormal cols
- Σ diagonal $(\sigma_1, \sigma_2 \dots \sigma_n)$ $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ $\rightarrow$ always start w/ max

If $\left.\begin{array}{l} \sigma_n = 0 \\ \sigma_r \neq 0 \\ \sigma_{r+1} = 0 \end{array}\right\}$ rank$(A) = r$

$R(A) = \{ y \mid y = Ax \text{ for some } x \} = R(U)$
$= U \underbrace{\Sigma V^T x}_{\hat{x}}$

$\boxed{R(A) = R(U)}$

$N(A) = \{ x \mid Ax = 0 \} = N(V^T)$
$U \Sigma V^T x = 0 \quad \therefore V^T x = 0$

$\boxed{N(A) = N(V^T)}$

$\boxed{\begin{array}{l} R(A^T) = R(V) \\ N(A^T) = N(U^T) \end{array}}$

$A = U \Sigma V^T \rightarrow A^T = V \Sigma^T U^T = V \Sigma U^T$

diagonal matrix so transpose is itself

$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T \left\{ \begin{array}{l} A v_1 = \sigma_1 u_1 \\ A v_2 = \sigma_2 u_2 \\ A v_3 = \sigma_3 u_3 \end{array} \right.$

$$\boxed{\|A\|_2 = \max_{\|x\|=1} \|Ax\| = \sigma_1 = \sigma_{max}} \qquad \boxed{\|A^{-1}\|_2 = \frac{1}{\sigma_{min}} = \frac{1}{\sigma_n}}$$

$$Q^TQ = I \xrightarrow{SVD} Q = \overset{U\Sigma V^T}{QII}$$

Find B of rank $k < r \ni \|B-A\| = \inf_{\text{rank}B=k} \|B-A\| \rightarrow B = \sum_{i=1}^{k} \sigma_i u_i v_i^T$

$$\|B-A\| = \left\| \sum_{i=k+1}^{r} \sigma_i u_i v_i^T \right\| = \sigma_{k+1}$$ $\leftarrow$ norm is $\sigma_{max} = \sigma_{k+1}$ next largest σ of $B-A$

e.g.: $$A = \frac{1}{2}\begin{vmatrix} 5 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 5 & 0 \end{vmatrix} = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{vmatrix}\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}\frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix}$$

Find B of rank 2 $\ni \|B-A\|$ smallest of all B of rank 2

$$B = \sum_{i=1}^{2} \sigma_i u_i v_i^T = \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{vmatrix}\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{vmatrix}$$

OR $$B = 3\begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}\begin{vmatrix} 1 & -1 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{vmatrix}$$

$$B-A = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \longrightarrow \|B-A\| = \sigma_3 = 1$$

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$$\left.\begin{array}{l} A = U\Sigma V^T \\ A^T = V\Sigma^T U^T \end{array}\right\} \quad AA^T = U\Sigma \underbrace{V^TV}_{I}\Sigma^T U^T = U\underbrace{\Sigma\Sigma^T}_{\Sigma\Sigma=\Sigma^2} U^T = U\Lambda U^T$$

$$A^TA = V\Sigma^T \underbrace{U^TU}_{I}\Sigma V^T = V\Sigma^2 V^T = V\Lambda V^T$$

$$\boxed{\sigma_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)}} \longrightarrow \text{if } A \text{ symmetric } A^T = A \qquad \boxed{\sigma_i = |\lambda_i|}$$

$$\therefore A^2 = V\Sigma^2 V^T$$

$$\therefore \sigma_i = \sqrt{\lambda_i(A^TA)} = \sqrt{\lambda_i(A^2)} = \sqrt{(\lambda_i(A))^2} = |\lambda_i|$$

Let λ, v be eigenpair: $Av = \lambda v$

$$A^2 v = A.\lambda v = A\lambda v = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$$

$\boxed{A^K v = \lambda^K v}$

$$p(z) = a_0 + a_1 z + \ldots + a_K z^K$$
$$p(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_K A^K$$
$$\therefore\ p(A)v = p(\lambda)v$$

e.g.: $A = \frac{1}{5}\begin{vmatrix} 11 & 2 \\ 4 & 3 \\ 5 & 10 \end{vmatrix}$

$$\rightarrow AA^T = \frac{1}{25}\begin{vmatrix} 11 & 2 \\ 4 & 3 \\ 5 & 10 \end{vmatrix}\begin{vmatrix} 11 & 4 & 5 \\ 2 & 3 & 10 \end{vmatrix} = \frac{1}{25}\begin{vmatrix} 125 & 50 & 75 \\ 50 & 25 & 50 \\ 75 & 50 & 125 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix}$$

$$A^TA = \frac{1}{25}\begin{vmatrix} 11 & 4 & 5 \\ 2 & 3 & 10 \end{vmatrix}\begin{vmatrix} 11 & 2 \\ 4 & 3 \\ 5 & 10 \end{vmatrix} = \frac{1}{25}\begin{vmatrix} 162 & 84 \\ 84 & 113 \end{vmatrix}$$

$$\det(A^TA - \lambda I) = \frac{1}{25}\left[(162-\lambda)(113-\lambda) - 84^2\right]$$

$$= \frac{1}{25}(18306 - 6875\lambda + 625\lambda^2 - 7056) = \frac{1}{25}(625\lambda^2 - 6875\lambda + 11250)$$

$$= 25\lambda^2 - 275\lambda + 450 = 25(\lambda^2 - 11\lambda + 18) = (\lambda-2)(\lambda-9) = 0$$

normalize

(i) $\lambda_1 = 2 \rightarrow \sigma_1 = \sqrt{2} \rightarrow N(A^TA - 2I) = \left\{\alpha\begin{vmatrix} 4 \\ 3 \end{vmatrix}\right\} \rightarrow u_1 = \frac{1}{5}\begin{vmatrix} 4 \\ 3 \end{vmatrix}$

(ii) $\lambda_2 = 9 \rightarrow \sigma_2 = 3 \rightarrow N(A^TA - 9I) = \left\{\alpha\begin{vmatrix} 3 \\ -4 \end{vmatrix}\right\} \rightarrow u_2 = \frac{1}{5}\begin{vmatrix} 3 \\ -4 \end{vmatrix}$

$$\det(AA^T - \lambda I) = \det\begin{vmatrix} 5-\lambda & 2 & 3 \\ 2 & 1-\lambda & 2 \\ 3 & 2 & 5-\lambda \end{vmatrix} = -\lambda^3 + 11\lambda^2 - 18\lambda = -\lambda(\lambda-2)(\lambda-9) = 0$$

(i) $\lambda = 9 \rightarrow u_1 = \frac{1}{3}\begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix}$ (ii) $\lambda = 2 \rightarrow u_2 = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}$

$$\therefore A = \begin{array}{ccc} U & \Sigma & V^T \\ \begin{bmatrix} 2/3 & 1/\sqrt{2} \\ 1/3 & 0 \\ 2/3 & -1/\sqrt{2} \end{bmatrix} & \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{2} \end{bmatrix} & \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix} \end{array} \longleftarrow \text{reduced SVD w/o considering } \lambda = 0$$

OR

$$= \begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 & 0 \\ 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow \text{full SVD}$$

SECTION 4.7

X, Y subspaces of $V = \mathbb{R}^n \longrightarrow X + Y = \{ v = x + y, x \in X, y \in Y \}$

$$\boxed{\left.\begin{array}{l} \text{If } X + Y = V = \mathbb{R}^n \\ X \cap Y = \{0\} \end{array}\right\} \; X, Y \text{ are complementary subspaces}}$$

$\exists!\; x \in X, y \in Y \ni v = x + y$

Proof $\rightarrow$ $v \in V$

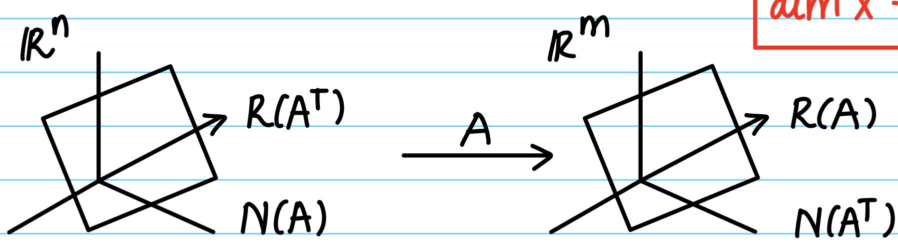
$v = x_1 + y_1 = x_2 + y_2 \quad (x \in X, y \in Y)$

$\therefore x_1 - x_2 + y_1 - y_2 = 0 \quad \therefore \underbrace{x_1 - x_2}_{\in X} = \underbrace{y_2 - y_1}_{\in Y} \in X \cap Y = \{\emptyset\}$

$\therefore \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \quad \therefore$ decomposition is unique QED!

If $B_X = \{x_1, x_2 \dots x_k\}$ is a basis of X
$B_Y = \{y_1, y_2 \dots y_l\}$ is a basis of Y $\Big\}$ complementary subspaces

$$\boxed{\dim X + \dim Y = \dim V}$$



$$\boxed{R(A) + N(A^T) = \mathbb{R}^m} \qquad \boxed{R(A^T) + N(A) = \mathbb{R}^n}$$

Tuesday 11/14

SECTION 4.3

X, Y complementary if $X + Y = \mathbb{R}^n$, $X \cap Y = \{0\}$

$v \in V \rightarrow v = x + y,\ x \in X,\ y \in Y$ $\qquad\|Q\| = \|I - P\| \geq 1$

$x = P_X v$ $\qquad \|P\| \geq 1$

$y = P_Y v = (I - P_X)v$ $\qquad \|P\| = 1$ iff $X \perp Y$

Let B_X basis of X

$U_{n \times k} = X$ matrix whose cols are in B_X

V (same order of U) matrix whose cols are a basis of $Y^{\perp}$

$\therefore P = U(V^TU)^{-1}V^T \rightarrow$ check if this is a projection $P^2 = P$

$P^2 = P.P = U\underbrace{(V^TU)^{-1}V^TU}_{I}(V^TU)^{-1}V^T = U(V^TU)^{-1}V^T = P \rightarrow$ projection ✓

(1) $R(P) = R(U) = X$

(2) $N(P) = Y$

$N(P) = \{x \mid Px = 0\} = \{x \mid V^Tx = 0\} = Y$

! Special case: $X \perp Y$ cols are orthonormal $\rightarrow U = V \rightarrow P = UU^T$

e.g: $X = \{x \mid x_1 + 2x_2 + x_3 = 0\} \subset \mathbb{R}^3 \rightarrow$ plane

$Y = \left\{ \alpha \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix},\ \alpha \in \mathbb{R} \right\} \rightarrow$ line

$U = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ basis for $Y^{\perp}$: $V = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow$ just 2 linearly inde. cols $\perp Y$

$V^TU = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -1 \\ 0 & -4/3 \end{bmatrix}$ $\quad L =$

$m = \frac{-2}{3}$

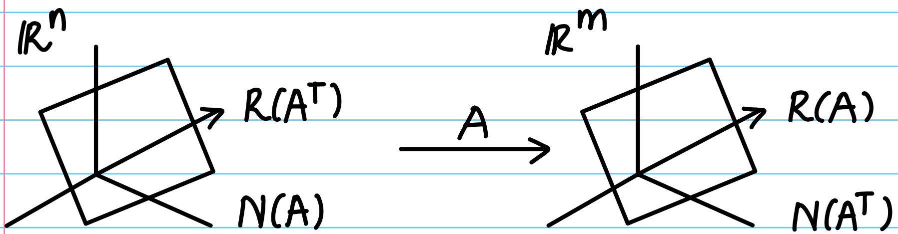
$U^{-1}_{upper} = \begin{vmatrix} -1/3 & 1/4 \\ 0 & -3/4 \end{vmatrix} \rightarrow (V^TU)^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{vmatrix} -1/3 & 1/4 \\ 0 & -3/4 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{vmatrix} = \begin{vmatrix} -1/2 & 1/4 \\ 1/2 & -3/4 \end{vmatrix}$

$\therefore P = U(V^TU)^{-1}V^T = \underset{3\times 2}{\begin{vmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix}}\underset{2\times 2}{\begin{vmatrix} -1/2 & 1/4 \\ 1/2 & -3/4 \end{vmatrix}}\underset{2\times 3}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}} = \begin{vmatrix} 3/4 & -1/2 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/2 & 3/4 \end{vmatrix}$

$R(P) = X \rightarrow \text{rank}(P) = \dim(X) = 3 - 1 = 2$

$$Q = I - P = \begin{vmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{vmatrix} \rightarrow \text{rank}(P) = 1 \text{ (bc cols are linearly dependent)}$$

$$\left.\begin{aligned} M^\perp + M &= \mathbb{R}^n \\ M \cap M^\perp &= \{0\} \end{aligned}\right\rangle M \oplus M^\perp$$



$\boxed{N(A)^\perp = R(A^T)}$ $\boxed{N(A^T)^\perp = R(A)}$ or $N(A^T) = R(A)^\perp$

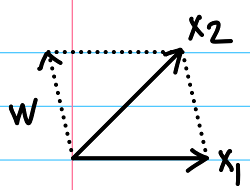
Proof $\rightarrow$ $\langle v, x \rangle = 0 \rightarrow v^T x = 0 = v^T A y = 0 \rightarrow (v^T A y)^T = 0 \rightarrow y^T A^T v = 0$

$x = Ay$ $\therefore A^T v = 0$ QED!

Thursday 11/30

$$x_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \quad x_2 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} \rightarrow \text{Find orthonormal basis of } S$$

(1) Normalize $q_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \rightarrow \langle x_2, q_1 \rangle = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = \sqrt{2}$



$$w = (I - q_1 q_1^T) x_2 = x_2 - \underbrace{\langle x_2, q_1 \rangle}_{= q_1^T x_2} q_1 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} - \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix}$$

$$q_2 = \frac{w}{\|w\|} = \frac{1}{2} \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \rightarrow Q = |q_1 \; q_2| = \begin{vmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{vmatrix}$$

General $\rightarrow$ Let $x_1, x_2 \ldots x_n$ a basis of an n-dimensional space V (an inner product space)

Find a basis of V $\{u_1, u_2 \ldots u_n\}$ orthonormal $\|u_i\| = 1$

$\langle u_i, u_j \rangle = 0 \quad i \neq j$

$\ni \text{span}\{u_1, u_2 \ldots u_k\} = \text{span}\{x_1, x_2 \ldots x_k\}$, $k = 1 \rightarrow n$

$$u_1 = \frac{x_1}{\|x_1\|}$$

$$k: 1 \to n-1: \quad w_{k+1} = x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, u_i \rangle u_i$$

$$= x_{k+1} - \langle x_{k+1}, u_1 \rangle u_1 - \langle x_{k+1}, u_2 \rangle u_2 - \dots - \langle x_{k+1}, u_k \rangle u_k$$

$$u_{k+1} = \frac{w_{k+1}}{\|w_{k+1}\|}$$

Proof $\to$ $\|u_i\| = 1$ for $i = 1, 2 \dots n \to \operatorname{span}\{u_1, u_2 \dots u_k\} = \operatorname{span}\{x_1, x_2 \dots x_k\}$

$w_2 = x_2 - \langle x_2, u_1 \rangle u_1$

$\therefore \langle w_2, u_1 \rangle = \langle x_2, u_1 \rangle - \langle x_2, u_1 \rangle \langle u_1, u_1 \rangle$

$\therefore$ General $j = k+1$

$$\therefore \langle w_{k+1}, u_i \rangle = \langle x_{k+1}, u_i \rangle - \sum_{j=1}^{k} \langle x_{k+1}, u_j \rangle \underbrace{\langle u_j, u_i \rangle}_{0}$$

$$= \langle x_{k+1}, u_i \rangle - \langle x_{k+1}, u_i \rangle = 0$$

$\therefore u_{k+1} \perp \operatorname{span}\{u_1, u_2 \dots u_k\}$ QED!

3 vectors e.g. $x_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} \quad x_2 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} \quad x_3 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \to u_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}, \quad \langle x_2, u_1 \rangle = \sqrt{2}, \quad \langle x_3, u_1 \rangle = \frac{1}{\sqrt{2}}$

$$w_2 = x_2 - \langle x_2, u_1 \rangle u_1 = \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix} - \sqrt{2} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix} \to u_2 = \frac{1}{2} \begin{vmatrix} 0 \\ 0 \\ 2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$w_3 = x_3 - \langle x_3, u_1 \rangle u_1 - \langle x_3, u_2 \rangle u_2 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix} - 1 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1/2 \\ -1/2 \\ 0 \end{vmatrix}$$

$$\therefore u_3 = \frac{w_3}{\|w_3\|} = \sqrt{2} \begin{vmatrix} 1/2 \\ -1/2 \\ 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \to Q = \begin{vmatrix} u_1 & u_2 & u_3 \end{vmatrix}$$

$$\therefore \begin{cases} R_{11} = \sqrt{2} & R_{12} = \langle a_2, u_1 \rangle = \sqrt{2} \\ R_{22} = 2 & R_{13} = \langle a_3, u_1 \rangle = 1/\sqrt{2} \\ R_{33} = 1/\sqrt{2} & R_{23} = \langle a_3, u_2 \rangle = 1 \end{cases}$$

e.g: $A = \begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix} = |x_1 \;\; x_2|$ Find $Q \ni Q = |u_1 \;\; u_2|,\ q_1^T q_2 = 0$

$$x_1 = \begin{vmatrix}1\\2\end{vmatrix} \to u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt5}\begin{vmatrix}1\\2\end{vmatrix} \to \langle x_2, u_1\rangle = \frac{3}{\sqrt5}$$

$$W_2 = x_2 - \langle x_2, u_1\rangle u_1 = \begin{vmatrix}1\\1\end{vmatrix} - \frac{3}{\sqrt5}\,\frac{1}{\sqrt5}\begin{vmatrix}1\\2\end{vmatrix} = \begin{vmatrix}2/5\\-1/5\end{vmatrix} \to u_2 = \frac{1}{\sqrt5}\begin{vmatrix}2\\-1\end{vmatrix}$$

$$\therefore Q = |u_1 \;\; u_2| = \frac{1}{\sqrt5}\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}$$

$R_{ii} = \|W_i\|$

$R_{ij} = \langle a_j, u_i\rangle$ $\|x_1\|$

Q-R factorization $\to A = Q.R = \frac{1}{\sqrt5}\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\begin{vmatrix}\boxed{\sqrt5} & 3/\sqrt5\\ 0 & 1/\sqrt5\end{vmatrix}$

orthonormal — upper triangular — process above

$$\text{So } Q = \begin{vmatrix}1/\sqrt2 & 0 & 1/\sqrt2\\ 1/\sqrt2 & 0 & -1/\sqrt2\\ 0 & 1 & 0\end{vmatrix} = \frac{1}{\sqrt2}\begin{vmatrix}1 & 0 & 1\\ 1 & 0 & -1\\ 0 & \sqrt2 & 0\end{vmatrix} \;\; \& \;\; R = \begin{vmatrix}\sqrt2 & \sqrt2 & 1/\sqrt2\\ 0 & 2 & 1\\ 0 & 0 & 1/\sqrt2\end{vmatrix}$$

$$\text{Check: } Q.R = \begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 0 & 2 & 1\end{vmatrix} = A \;\checkmark$$

(1) $Ax = b \to QRx = b \to Rx = Q^Tb$

(2) $A = QR$ unique

Proof → $A = Q_1R_1 = Q_2R_2 \to R_1 = Q_1^TQ_2R_2 \to R_1R_2^{-1} = Q_1^TQ_2$ orthogonal

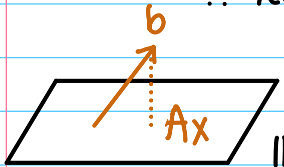
Tuesday 12/5

$A = Q.R \to R(A) = R(Q)$

$\searrow$ orthogonal projection onto the $R(A)$

$\therefore$ matrix P of rank $1 \ni P = QQ^T,\ P^2 = P,\ P^T = P$

$\therefore R(P) = R(A)$



Find best $x \ni \|b - Ax\|^2$ min

$$\|b - Ax\| = \|\underbrace{QQ^T}_{\text{maintain length}}b - QRx\| = \|Q(Q^Tb - Rx)\| = \|Q^Tb - Rx\|$$

$\therefore \min \|b - Ax\| = \|\tilde{Q}^Tb\|$ lower part only e.g.: $b = \begin{bmatrix}1\\2\\3\end{bmatrix}$ Check if $b \in R(A) \rightarrow$ if its projection is itself, $b \in R(A)$

$$\underset{A}{\begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 2\end{bmatrix}} = \underset{Q}{\begin{bmatrix}1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0\\ 0 & 1\end{bmatrix}} \underset{R}{\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 2\end{bmatrix}}$$

$$Q^Tb = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3/\sqrt{2}\\ 3\end{bmatrix} \ \& \ Q(Q^Tb) = \begin{bmatrix}1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3/\sqrt{2}\\ 3\end{bmatrix} = \begin{bmatrix}3/2\\ 3/2\\ 3\end{bmatrix} \neq b$$

$\therefore b \notin R(A)$

expanded version Q $\leftarrow$ $\tilde{Q} = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 0 & 1 & 0\end{bmatrix}$ $\quad \tilde{R} = \begin{bmatrix}R\\ 0\end{bmatrix} = \begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 2\\ 0 & 0\end{bmatrix}$

$\therefore \min \|b - Ax\| = \|\tilde{Q}^Tb - \tilde{R}x\|$

$$\tilde{Q}^Tb = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}3/\sqrt{2}\\ 3\\ -1/\sqrt{2}\end{bmatrix}$$

only 2 upper entries

Solve $Rx = \tilde{Q}^Tb \rightarrow \begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}3/\sqrt{2}\\ 3\end{bmatrix} \rightarrow x_2 = 3/2$

$x_1 + x_2 = 3/2 \rightarrow x_1 = 0$

$\therefore x = \begin{bmatrix}0\\ 3/2\end{bmatrix}$ $\quad \therefore Ax = \begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 3/2\end{bmatrix} = \begin{bmatrix}3/2\\ 3/2\\ 3\end{bmatrix} \rightarrow b - Ax = \begin{bmatrix}-1/2\\ -1/2\\ 0\end{bmatrix}$

$\therefore \|b - Ax\| = \frac{1}{\sqrt{2}} =$ same as norm of 3rd entry

Thursday 12/7

$Ax = b,\ b \notin R(A) \rightarrow$ Find $x \ni \min \|b - Ax\|$

$\|b - Ax\| = \|\tilde{Q}\tilde{Q}^Tb - \tilde{Q}\tilde{R}x\| = \|\tilde{Q}(\tilde{Q}^Tb - Rx)\| = \|\tilde{Q}^Tb - Rx\|$

vector maintaining length

e.g: $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Step 1
QR factor

$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \langle u_1, x_1 \rangle = 0 \rightarrow w_2 = x_2 - \langle u_2, x_1 \rangle x_2 = x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \therefore u_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

Step 2
expand Q $\longrightarrow$ choose 3^{rd} column as linearly inde. set

non-zero projections $\therefore$ 3^{rd} col works

$x_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \rightarrow \langle x_3, u_1 \rangle = \frac{1}{\sqrt{2}}, \ \langle x_3, u_2 \rangle = \frac{7}{\sqrt{6}}$

$w_3 = x_3 - \langle x_3, u_1 \rangle u_1 - \langle x_3, u_2 \rangle u_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore \tilde{Q} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \rightarrow \tilde{Q}^T b = \begin{bmatrix} \sqrt{2} \\ 8/\sqrt{6} \\ 2/\sqrt{3} \end{bmatrix}$

$\tilde{R} x = \tilde{Q}^T b \longrightarrow \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6} \end{bmatrix} x = \begin{bmatrix} \sqrt{2} \\ 8/\sqrt{6} \end{bmatrix}$ only 2 entries $\therefore x = \begin{bmatrix} 1 \\ 4/3 \end{bmatrix}$ Final ans.

Step 3
Verify

$b - Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4/3 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \| b - Ax \| = \frac{2}{\sqrt{3}}$ should match 3^{rd} entry in $\tilde{Q}^T b$