§ 4.1 Prelimary Theory

4.1.1 Initial Value Problems

(Skip Boundary Value Prob.)

Book : nth order linear IVP.
see definition.

2nd order linear IVP

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

A solution curve to the IVP must pass through the (x_0, y_0) and have slope y_1 at that point.

eg The given family of fns is a general solution to the DE on the indicated interval. Find a member of the family that is a solution to the IVP.

$$y = c_1 e^{4x} + c_2 e^{-x}, \quad (-\infty, \infty)$$

$$\boxed{y'' - 3y' - 4y = 0,} \quad y(0) = 1, \quad y'(0) = 2.$$

$$y' = 4c_1 e^{4x} - c_2 e^{-x}$$

$y(0) = 1$ $1 = c_1 e^0 + c_2 e^0$

$y'(0) = 2$ $2 = 4c_1 - c_2$

$$\begin{cases} 1 = c_1 + c_2 \\ 2 = 4c_1 - c_2 \end{cases}$$

$$\begin{aligned} 1 &= c_1 + c_2 \\ 2 &= 4c_1 - c_2 \\ \hline 3 &= 5c_1 \\ c_1 &= \tfrac{3}{5} \end{aligned}$$

$$\begin{aligned} c_2 &= 1 - c_1 \\ &= 1 - \tfrac{3}{5} \\ &= \tfrac{2}{5} \end{aligned}$$

$$\boxed{PS: y = \tfrac{3}{5} e^{4x} + \tfrac{2}{5} e^{-x}}$$

Theorem 4.1.1

Existence of a Unique Solution

see book, p.119

Example 1:

$$3y''' + 5y'' - y' + 7y = 0,$$
$$y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

Note $y = 0$ is a solution to this IVP.

Because all the coefficients are constants and $3 \neq 0$, all conditions of T 4.1.1 are fulfilled. Hence, $y = 0$ is the only solution on any interval containing 1.

Example 2 :

$y = 3e^{2x} + e^{-2x} - 3x$ is a solution to

the IVP $\underbrace{y'' - 4y = 12x}_{\downarrow}$, $y(0) = 4$, $y'(0) = 1$.

(verify)

$x_0 = 0$

linear equation.

Coefficients $1, -4, \underset{g(x)}{12x}$ are continuous and $a_2(x) = 1 \neq 0$ on any interval, I, containing 0. So T 4.1.1 says that the given fn is the unique solution on I.

Example: $x^2 y'' - 2xy' + 2y = 6$, $y(0) = 2$

$y'(0) = 1$

$(-\infty, \infty)$

T. 4.1.1 does NOT apply since

$a_2(x) = x^2$ can equal 0 when $x = 0$

No uniqueness. There can be more than one solution on $(-\infty, \infty)$.