§ 7.3 Intro

Recall : Completing the Square

$$s^2 + bs + c$$

$$\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$$

$$= \left(s^2 + bs + \left(\frac{b}{2}\right)^2\right) + c - \frac{b^2}{4}$$

$$= \left(s + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

eg $s^2 + 2s + 10$

$$\left(\frac{2}{2}\right)^2 = 1$$

$$= (s^2 + 2s + 1) + 10 - 1$$

$$= (s+1)^2 + 9$$

(Note : $(s-a)^2 + b^2$ we will need this form.

$$(s+1)^2 + 9 = (s-(-1))^2 + 3^2$$

So $a = -1$, $b = 3$.)

§ 7.3.1 Translation on s-Axis

T.7.3.1 If $\mathcal{L}\{f(t)\} = F(s)$ and $a \in \mathbb{R}$,

then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ $= \mathcal{L}\{f(t)\}\Big|_{s \to s-a}$

Pf: $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a) \quad \square.$$

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eg $\mathcal{L}\{e^{5t} t^3\} = \mathcal{L}\{t^3\}\Big|_{s \to s-5}$

$$= \frac{3!}{s^4}\Bigg|_{s \to s-5} = \frac{6}{(s-5)^4}$$

eg $\mathcal{L}\{e^{-2t} \cos 4t\} = \mathcal{L}\{\cos 4t\}\Big|_{s \to s-(-2)}$

$$= \frac{s}{s^2+16}\Bigg|_{s \to s+2} = \frac{s+2}{(s+2)^2+16}$$

Inverse Form of T.7.3.1

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\big|_{s \to s-a}\}$$

$$= e^{at} f(t)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

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eg $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$

PFD

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$2s+5 = A(s-3) + B$$

$s=3$: $\quad 11 = B$

s: $\quad 2 = A$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{(s-3)^2}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s \to s-3}\right\}$$

$$= 2e^{3t} + 11e^{3t}t$$

$$= 2e^{3t} + 11te^{3t}$$

eg $\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{2}s+\frac{5}{3}}{s^2+4s+6}\right\}$ Already PFD

$(s^2+4s+4)+6-4$

$=(s+2)^2+2$

$$=\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}s+\frac{5}{3}}{(s+2)^2+2}\right\}$$

$\frac{1}{2}s+\frac{5}{3}$

$=\frac{1}{2}(s+2)+\frac{5}{3}-1$

$=\frac{1}{2}(s+2)+\frac{2}{3}$

$$=\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}(s+2)}{(s+2)^2+2}\right\}+\mathcal{L}^{-1}\left\{\frac{\frac{2}{3}}{(s+2)^2+2}\right\}$$

$\frac{2}{3}=x\cdot\sqrt{2}$

$x=\frac{2}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$

$=\frac{2\sqrt{2}}{6}$

$=\frac{\sqrt{2}}{3}$

$$=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\}+\frac{\sqrt{2}}{3}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$=\frac{1}{2}\mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\to s+2}\right\}+\frac{\sqrt{2}}{3}\mathcal{L}^{-1}\left\{\left.\frac{\sqrt{2}}{s^2+2}\right|_{s\to s+2}\right\}$$

$$=\frac{1}{2}e^{-2t}\cos(\sqrt{2}\,t)+\frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}\,t)$$

eg $y'' - 2y' + 2y = e^{-t}, \; y(0) = 2, \; y'(0) = 1$

$$\big(s^2 Y(s) - \underbrace{s\,\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{1}}_{-s+1}\big) - 2\big(sY(s) - \underbrace{y(0)}_{2}\big) + 2Y(s) = \frac{1}{s+1}$$

$+4$

$$s^2 Y(s) - 2s - 1 - 2s\,Y(s) + 4 + 2Y(s) = \frac{1}{s+1}$$

$$Y(s)\big(\underbrace{s^2 - 2s + 2}_{\text{AE in } s}\big) = \frac{1}{s+1} + 2s - 3$$

$$Y(s) = \frac{1}{(s+1)(s^2-2s+2)} + \frac{2s-3}{s^2-2s+2}$$

already PFD

DO NOT COMBINE, BUT DO NOT FORGET

$\Rightarrow$

$$\frac{1}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

$$1 = A(s^2-2s+2) + (Bs+C)(s+1)$$

$s=-1$: $\quad 1 = A(1+2+2) \quad \Rightarrow A = \frac{1}{5}$

s^2 : $\quad 0 = A + B \quad \Rightarrow B = -\frac{1}{5}$

$s=0$: $\quad 1 = 2A + C \quad \Rightarrow C = 1 - 2\left(\frac{1}{5}\right)$

constant term

$C = \frac{3}{5}$

$\left(\frac{2}{2}\right)^2 = 1$

Since $s^2 - 2s + 2 = (s^2 - 2s + 1) + 2 - 1$
$= (s-1)^2 + 1$,

$$Y(s) = \frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s + \frac{3}{5}}{(s-1)^2+1} + \frac{2s-3}{(s-1)^2+1}$$

$$= \frac{\frac{1}{5}}{s+1} + \frac{\frac{9}{5}s - \frac{12}{5}}{(s-1)^2+1}$$

$$\frac{9}{5}s - \frac{12}{5} = \frac{9}{5}(s-1) - \frac{12}{5} + \frac{9}{5}$$
$$= \frac{9}{5}(s-1) - \frac{3}{5}$$

$$Y(s) = \frac{\frac{1}{5}}{s+1} + \frac{9}{5} \cdot \frac{s-1}{(s-1)^2+1} - \frac{3}{5} \cdot \frac{1}{(s-1)^2+1}$$

$$y(t) = \frac{1}{5}e^{-t} + \frac{9}{5}e^{t}\cos t - \frac{3}{5}e^{t}\sin t$$