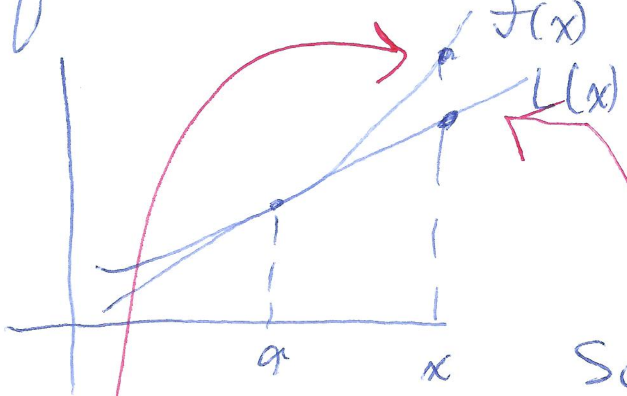
§ 2.6 Numerical Methods – Euler's Method

Recall Linearization & Linear Approximation from Calculus I.



$(a, f(a))$, $m = f'(a)$

$y - f(a) = f'(a)(x-a)$
(point - slope)

So, $y = f(a) + f'(a)(x-a)$.

Equation of tangent line to f at $x = a$.

Now Replace y by $L(x)$

$$\underbrace{L(x) = f(a) + f'(a)(x-a)}_{\text{Linearization of } f \text{ at } a.}$$

Use $L(x)$ to approximate $f(x)$.

$f(x) \approx L(x)$

$$\underbrace{\text{So } f(x) \approx f(a) + f'(a)(x-a)}_{\text{Linear Approximation}}$$

Now suppose we have a 1st order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0$$

and we want to approximate the value for the solution $y(x)$ at some specific value of x (not necessarily near x_0).

We can use Euler's method - an extension of linear approximation to do this (without solving for $y(x)$).

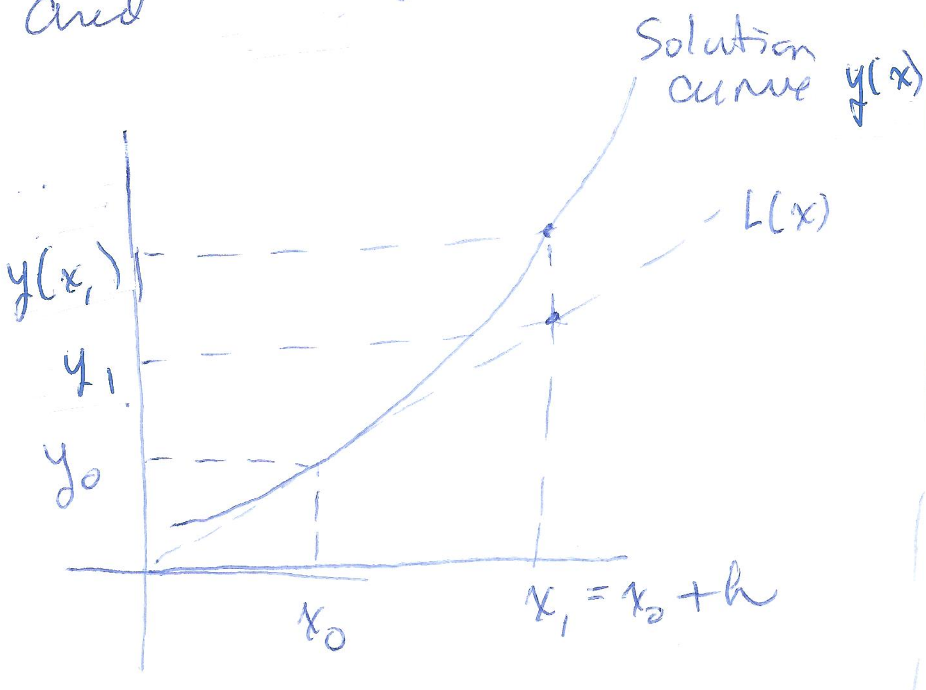
$L(x) = f(a) + f'(a)(x-a)$
$f(x) \approx f(a) + f'(a)(x-a)$

Using the above notation,

$$L(x) = y_0 + f(x_0, y_0)(x - x_0)$$

$$y(x_1) \approx L(x_1) = y_1$$

and



Letting $x_1 = x_0 + h$,

$$L(x_1) = y_0 + h\, f(x_0, y_0)$$

Now we are at (x_1, y_1).

Reevaluate the derivative at this point.

$$y'(x_1, y_1) = F(x_1, y_1).$$

So $y(x_2) = y(x_0 + 2h) = y(x_1 + h)$

and $y(x_2) \approx y_2 = y_1 + h \cdot F(x_1, y_1)$

Continuing in this fashion, we get

$$y_{n+1} = y_n + h\,F(x_n, y_n)$$

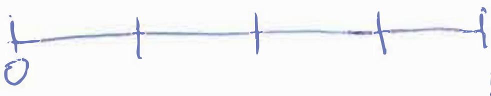
where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$

Euler's Method.

equivalently, $y_n = y_{n-1} + h\,F(x_{n-1}, y_{n-1})$

and so, $\boxed{y_n = y_{n-1} + h \cdot y'_{n-1}}$

eg Estimate $y(1)$ for $y' = y - x$; $y(0) = 2$ using Euler's method with $h = 0.25$



So 4 steps.

n	h	x_n	$y_n = y_{n-1} + h y'_{n-1}$	$y_n' = y_n - x_n$
0	–	$x_0 = 0$	$y_0 = 2$	$y_0' = 2 - 0 = 2$
1	.25	$x_1 = .25$	$y_1 = 2 + .25(2) = 2.5$	$y_1' = 2.5 - .25 = 2.25$
2	.25	$x_2 = .5$	$y_2 = 2.5 + .25(2.25) = 3.0625$	$y_2' = 3.0625 - .5 = 2.5625$
3	.25	$x_3 = .75$	$y_3 = 3.0625 + .25(2.5625) = 3.703125$	$y_3' = 3.703125 - .75 = 2.953125$
4	.25	$x_4 = 1$	$y_4 = 3.703125 + .25(2.953125) = 4.4414...$	

So $y(1) \approx 4.4414$

Note : $y' = y - x, \; y(0) = 2$

$y' - y = -x, \; y(0) = 2$ 1st order linear.

$\frac{d}{dx}\left[e^{-x}y\right] = -xe^{-x}$

$\mu(x) = e^{-\int 1 dx} = e^{-x}$

$e^{x}y = xe^{-x} + e^{-x} + C$

$y = x + 1 + Ce^{x}$

u		dv
$-x$	+	e^{-x}
-1	−	$-e^{-x}$
0		e^{-x}

$(0,2) \Rightarrow 2 = 0 + 1 + Ce^{0}$

So $C = 1$.

$y(1) = 2 + e \approx 4.718$

Approximation was 4.4414. (not bad).

HW #4 Hint. Absolute Error = |actual value − approximation|

$$\text{Relative \% Error} = \frac{\text{Absolute Error}}{|\text{actual value}|} \times 100$$

See Example 2 in the book.

Illustration of Example.
see Demonstration of ...
in Modules

Note:

$$L_1(x) = 2 + 2(x-0)$$

$$L_2(x) = 2.5 + 2.25(x-.25)$$

$$L_3(x) = 3.0625 + 2.5625(x-.5)$$

$$L_4(x) = 3.703125 + 2.953125(x-.75)$$

Use the above to explain the illustration of this example.