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Answers to HW #13 - Math 2101-02

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 43/25 - \lambda & -24/25 \\ 0 & -24/25 & 57/25 - \lambda \end{bmatrix} =$$

$$= (2-\lambda)\left[\left(\frac{43}{25} - \lambda\right)\left(\frac{57}{25} - \lambda\right) - \frac{24}{25} \cdot \frac{24}{25}\right] =$$

$$= (2-\lambda)\left[\frac{43 \cdot 57}{25 \cdot 25} + \lambda^2 - \lambda\left(\frac{43+57}{25}\right) - \frac{576}{625}\right]$$

$$= (2-\lambda)\left[\lambda^2 - \frac{100}{25}\lambda + \frac{2451 - 576}{625}\right] =$$

$$= (2-\lambda)\left[\lambda^2 - 4\lambda + 3\right]$$

$\lambda_1 = 2$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{16 - 12}}{2} = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$$

Eigenvalues of A, $\sigma(A) = \{1, 2, 3\}$

$$\lambda_1 = 2 \qquad \lambda_2 = 3 \qquad \lambda_3 = 2$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -7/25 & -24/25 \\ 0 & -24/25 & 7/25 \end{bmatrix}$$

$$N(A - 2I) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

check. $Av_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\, v_1$ ✓

$$A - 1I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18/25 & -24/25 \\ 0 & -24/25 & 32/25 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18/25 & -24/25 \\ 0 & 0 & 0 \end{bmatrix}$$

$$m = \frac{24}{18} = \frac{4}{3}$$

$$N(A - I) = \left\{ \alpha \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \right\} \qquad V_2 = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

$$\text{check} \quad A \cdot v_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{43 \cdot 4 - 24 \cdot 3}{25} \\ \frac{-24 \cdot 4 + 57 \cdot 3}{25} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

✓

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -32/25 & -24/25 \\ 0 & -24/25 & -18/25 \end{bmatrix}$$

$$m = \frac{24}{32} = \frac{3}{4} \qquad \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -32/25 & -24/25 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N(A - 3I) = \left\{ \alpha \begin{vmatrix} 0 \\ 3 \\ -4 \end{vmatrix} \right\}$$

$$\text{check } A \cdot v = \begin{vmatrix} 0 \\ (43 \cdot 3 + 24 \cdot 4)/25 \\ -(24 \cdot 3 + 57 \cdot 4)/25 \end{vmatrix} = \begin{vmatrix} 0 \\ 9 \\ -12 \end{vmatrix} = 3 \begin{vmatrix} 0 \\ 3 \\ -4 \end{vmatrix} \checkmark$$

We observe that $A^T = A$

and that $v_1^T v_2 = 0$
$v_1^T v_3 = 0$
$v_2^T v_3 = 0$

2).

let $A^T = A$ $\lambda_1 \neq \lambda_2$

$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2$$

Multiply first eq. by v_2^T and second by v_1^T

$$v_2^T A v_1 = \lambda_1 v_2^T v_1$$

$$v_1^T A v_2 = \lambda_2 v_1^T v_2$$

since $A^T = A$ the 2 left terms are identical

$$\left((v_1^T A v_2)^T = v_2^T A^T v_1 = v_2^T A v_1 \right)$$

Subtracting and noting that $v_1^T v_2 = v_2^T v_1$,

We have $(\lambda_1 - \lambda_2) v_2^T v_1 = 0$

The First factor is not zero since $\lambda_1 \neq \lambda_2$

Then we must have $v_2^T v_1 = 0$

q.e.d.