Section 1.1

$1 \to 13$

1.1.1

1.1-1. Of a group of patients having injuries, 28% visit both a physical therapist and a chiropractor while 8% visit neither. Say that the probability of visiting a physical therapist exceeds the probability of visiting a chiropractor by 16%. What is the probability of a randomly selected person from this group visiting a physical therapist?

$P(T \cap C) = 0.28$

De Morgan's law

$P(T^c \cap C^c) = 0.08 = P((T \cup C)^c) \longrightarrow P(T \cup C) = 1 - 0.08 = 0.92$

$P(T) - P(C) = 0.16 \longrightarrow P(C) = P(T) - 0.16$

Inclusion- Exclusion principle:

$P(T \cup C) = P(T) + P(C) - P(T \cap C)$

$0.92 = P(T) + P(T) - 0.16 - 0.28 \quad \therefore 2P(T) = 1.36 \quad \therefore P(T) = 0.68$

1.1.3

1.1-3. Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns 1/52 to each of the 52 outcomes. Let

$A = \{x : x \text{ is a jack, queen, or king}\}$,

$B = \{x : x \text{ is a 9, 10, or jack and } x \text{ is red}\}$,

$C = \{x : x \text{ is a club}\}$,

$D = \{x : x \text{ is a diamond, a heart, or a spade}\}$.

Find **(a)** $P(A)$, **(b)** $P(A \cap B)$, **(c)** $P(A \cup B)$, **(d)** $P(C \cup D)$, and **(e)** $P(C \cap D)$.

(a) $P(A) = \frac{3.4}{52} = \frac{3}{13}$

(b) $P(A \cap B) = P(\text{red jack}) = \frac{2}{52} = \frac{1}{26}$

(c) $P(B) = \frac{3.2}{52} = \frac{3}{26} \longrightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{13} + \frac{3}{26} - \frac{1}{26} = \frac{4}{13}$

(d) $P(C \cup D) = P(C) + P(D) - \underbrace{P(C \cap D)}_{0} = \frac{13}{52} + \frac{3.13}{52} = 1$

(e) $P(C \cap D) = 0$

1.1.5

1.1-5. Consider the trial on which a 3 is first observed in successive rolls of a six-sided die. Let A be the event that 3 is observed on the first trial. Let B be the event that at least two trials are required to observe a 3. Assuming that each side has probability 1/6, find **(a)** $P(A)$, **(b)** $P(B)$, and **(c)** $P(A \cup B)$.

(a) $P(A) = \frac{1}{6}$

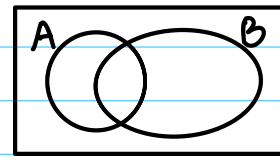
(b) $P(\infty) + \sum_{i=1}^{K} P(K^{th} \text{ roll}) = P(\infty) + \sum_{i=1}^{K} \left(\frac{5}{6}\right)^{K-1} \frac{1}{6} = P(\infty) + \frac{1/6}{1-5/6} = P(\infty) + 1 = 1$

$\therefore P(\infty) = 0 = P(0 \text{ trial})$

$\therefore P(B) = 1 - P(0 \text{ trial}) - P(1 \text{ trial}) = 1 - 0 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

1.1.7

1.1-7. Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.



$P(A \cup B^c) = 0.87 \longrightarrow P(B \setminus A) = 1 - 0.87 = 0.13$

$P(A) = P(A \cup B) - P(B \setminus A) = 0.76 - 0.13 = 0.63$

1.1.9

1.1-9. Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. It is given that $P(A_i) = 1/3$, $i = 1, 2, 3$; $P(A_i \cap A_j) = (1/3)^2$, $i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

(a) Use Theorem 1.1-6 to find $P(A_1 \cup A_2 \cup A_3)$.

(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

$P(A_1) = P(A_2) = P(A_3) = \frac{2}{6} = \frac{1}{3}$

$$P(A_1 \cup A_2 \cup A_3) = 1 - P((A_1 \cup A_2 \cup A_3)^c)$$
$$= 1 - P(A_1^c \cap A_2^c \cap A^c) = 1 - \frac{4 \times 4 \times 4}{6 \times 6 \times 6} = 1 - \left(\frac{2}{3}\right)^3$$
$$= 1 - \left(1 - \frac{1}{3}\right)^3$$

1.1.11

1.1-11. A typical roulette wheel used in a casino has 38 slots that are numbered 1, 2, 3, . . . , 36, 0, 00, respectively. The 0 and 00 slots are colored green. Half of the remaining slots are red and half are black. Also, half of the integers between 1 and 36 inclusive are odd, half are even, and 0 and 00 are defined to be neither odd nor even. A ball is rolled around the wheel and ends up in one of the slots; we assume that each slot has equal probability of 1/38, and we are interested in the number of the slot into which the ball falls.

(a) Define the sample space S.

(b) Let $A = \{0, 00\}$. Give the value of $P(A)$.

(c) Let $B = \{14, 15, 17, 18\}$. Give the value of $P(B)$.

(d) Let $D = \{x : x \text{ is odd}\}$. Give the value of $P(D)$.

(a) $S = \{00, 0, 1 \dots 36\}$

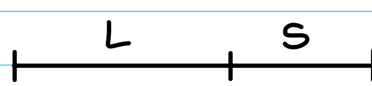
(b) $P(A) = \frac{2}{38} = \frac{1}{19}$

(c) $P(B) = \frac{4}{38} = \frac{2}{19}$

(d) $P(D) = \frac{18}{38} = \frac{9}{19}$

1.1.13

1.1-13. Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times longer than the shorter segment.



$P(L \geqslant 2S) = P(L + S \geqslant 3S) = P(3S \leq 1) = P(S \leq \frac{1}{3}) = \frac{2}{3}$

1.1.15

1.1-15. Let $S = A_1 \cup A_2 \cup \cdots \cup A_m$, where events $A_1, A_2, \dots, A_m$ are mutually exclusive and exhaustive.

(a) If $P(A_1) = P(A_2) = \cdots = P(A_m)$, show that $P(A_i) = 1/m$, $i = 1, 2, \dots, m$.

(b) If $A = A_1 \cup A_2 \cup \cdots \cup A_h$, where $h < m$, and (a) holds, prove that $P(A) = h/m$.

(a) $1 = P(S) = \sum_{i=1}^{m} P(A_i) = \sum_{i=1}^{m} P(A_1) = m.P(A_1) \rightarrow P(A_1) = P(A_i) = \frac{1}{m}$ Q.E.D!

(b) $P(A) = \sum_{i=1}^{h} P(A_i) = h.P(A_i) = h.\frac{1}{m} = \frac{h}{m}$ Q.E.D!

Section 1.2

1, 3, 5, 7, 9, 10, 11, 12

1.2.1

1.2-1. A combination lock was left at a fitness center. The correct combination is a three-digit number $d_1d_2d_3$, where d_i, $i = 1, 2, 3$, is selected from 0, 1, 2, 3, ..., 9. How many different lock combinations are possible with such a lock?

$10 \times 10 \times 10 = 1000$

1.2.3

1.2-3. How many different license plates are possible if a state uses

(a) Two letters followed by a four-digit integer (leading zeros are permissible and the letters and digits can be repeated)?

(b) Three letters followed by a three-digit integer? (In practice, it is possible that certain "spellings" are ruled out.)

26 letters in total

(a) $10^4 \times 26^2 = 6760000$ (b) $10^3 \times 26^3 = 17576000$

1.2.5

1.2-5. How many four-letter code words are possible using the letters in IOWA if

(a) The letters may not be repeated?

(b) The letters may be repeated?

(a) $4 \times 3 \times 2 \times 1 = 24$ (b) $4^4 = 256$

1.2.7

1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

(a) 6, 7, 8, 9.

(b) 6, 7, 8, 8.

(c) 7, 7, 8, 8.

(d) 7, 8, 8, 8.

(a) $\dfrac{4 \times 3 \times 2 \times 1}{10^4} = 0.0024$ (c) $\dfrac{\binom{4}{2}}{10^4} = 0.0006$

(b) $\dfrac{\binom{4}{2} \times 2 \times 1}{10^4} = 0.0012$ (d) $\dfrac{4}{10^4} = 0.0004$

1.2.9

1.2-9. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., *ANNAAA* means the American League team wins in six games) if the series goes

(a) Four games?

(b) Five games?

(c) Six games?

(d) Seven games?

(a) AAAA or NNNN → 2

(b) Lose any first 4 games → $2 \times 4 = 8$

(c) Lose 2 games in first 5 games

$\binom{5}{2} \times 2 = 20$

(d) Lose any 3 games in first 6 games

$\binom{6}{3} \times 2 = 40$

1.2.10

1.2-10. Pascal's triangle gives a method for calculating the binomial coefficients; it begins as follows:

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
  :   :   :   :   :   :
```

The nth row of this triangle gives the coefficients for $(a+b)^{n-1}$. To find an entry in the table other than a 1 on the boundary, add the two nearest numbers in the row directly above. The equation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},$$

called **Pascal's equation**, explains why Pascal's triangle works. Prove that this equation is correct.

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-1-r+1)!}$$

$$= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= \frac{(n-1)!}{(r-1)!(n-1-r)!}\left(\frac{1}{r} + \frac{1}{n-r}\right)$$

$$= \frac{(n-1)!}{(r-1)!(n-1-r)!}\left(\frac{n-r+r}{r(n-r)}\right) = \frac{(n-1)!}{(r-1)!(n-1-r)!} \times \frac{n}{r(n-r)}$$

$$= \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Q.E.D!

1.2.11

1.2-11. Three students (S) and six faculty members (F) are on a panel discussing a new college policy.

(a) In how many different ways can the nine participants be lined up at a table in the front of the auditorium?

(b) How many lineups are possible, considering only the labels S and F?

(c) For each of the nine participants, you are to decide whether the participant did a good job or a poor job stating his or her opinion of the new policy; that is, give each of the nine participants a grade of G or P. How many different "scorecards" are possible?

3S 6F

(a) $9! = 362880$ (b) $\binom{9}{3} = 84$ (c) $2^9 = 512$

1.2.12

1.2-12. Prove:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0 \quad \text{and} \quad \sum_{r=0}^{n}\binom{n}{r} = 2^n.$$

Hint: Consider $(1-1)^n$ and $(1+1)^n$, or use Pascal's equation and proof by induction.

(a) $$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \sum_{r=0}^{n}\binom{n}{r}1^{n-r}(-1)^r = (1-1)^n = 0^n = 0$$ Q.E.D!

(b) $$\sum_{r=0}^{n}\binom{n}{r} = \sum_{r=0}^{n}\binom{n}{r}1^{n-r}1^r = (1+1)^n = 2^n$$ Q.E.D!

Section 1.3

1, 3, 5, 7, 9, 11

1.3.1

1.3-1. A common screening test for HIV is called the ELISA (enzyme-linked immunosorbent assay) test. Among 1 million people who are given the ELISA test, we can expect results similar to those given in the following table:

	B_1: Has HIV Virus	B_2: Does Not Have HIV Virus	Totals
A_1: Test Positive	4,885	73,630	78,515
A_2: Test Negative	115	921,370	921,485
Totals	5,000	995,000	1,000,000

If one of these 1 million people is selected randomly, find the following probabilities: **(a)** $P(B_1)$, **(b)** $P(A_1)$, **(c)** $P(A_1 \mid B_2)$, and **(d)** $P(B_1 \mid A_1)$. **(e)** In words, what do parts (c) and (d) say?

(a) $P(B_1) = \dfrac{5000}{10^6} = 0.005$

(b) $P(A_1) = \dfrac{78515}{10^6} = 0.078515$

(c) $P(A_1 \mid B_2) = \dfrac{P(A_1 \cap B_2)}{P(B_2)} = \dfrac{73630}{995000} = 0.074$

(d) $P(B_1 \mid A_1) = \dfrac{P(B_1 \cap A_1)}{P(A_1)} = \dfrac{4885}{78515} = 0.0622$

(e) Part (c) is the prob. a person w/o HIV has a (+) test
Part (d) is the prob. a person w/ a (+) test has HIV

1.3.3

1.3-3. Let A_1 and A_2 be the events that a person is left-eye dominant or right-eye dominant, respectively. When a person folds his or her hands, let B_1 and B_2 be the events that the left thumb and right thumb, respectively, are on top. A survey in one statistics class yielded the following table:

	B_1	B_2	Totals
A_1	5	7	12
A_2	14	9	23
Totals	19	16	35

If a student is selected randomly, find the following probabilities: **(a)** $P(A_1 \cap B_1)$, **(b)** $P(A_1 \cup B_1)$, **(c)** $P(A_1 \mid B_1)$, and **(d)** $P(B_2 \mid A_2)$. **(e)** If the students had their hands folded and you hoped to select a right-eye-dominant student, would you select a "right thumb on top" or a "left thumb on top" student? Why?

(a) $P(A_1 \cap B_1) = \dfrac{5}{35} = \dfrac{1}{7}$

(b) $P(A_1 \cup B_1) = P(A_1) + P(B_1) - P(A_1 \cap B_1)$

$= \dfrac{12 + 19 - 5}{35} = \dfrac{26}{35}$

(c) $P(A_1 \mid B_1) = \dfrac{P(A_1 \cap B_1)}{P(B_1)} = \dfrac{5/35}{19/35} = \dfrac{5}{19}$

(d) $P(B_2 \mid A_2) = \dfrac{P(B_2 \cap A_2)}{P(A_2)} = \dfrac{9/35}{23/35} = \dfrac{9}{23}$

(e) $P(A_2 \mid B_1) = \dfrac{P(B_1 \cap A_2)}{P(B_1)} = \dfrac{14}{19}$ vs. $P(A_2 \mid B_2) = \dfrac{P(A_2 \cap B_2)}{P(B_2)} = \dfrac{9}{16}$

$\therefore P(A_2 \mid B_1) > P(A_2 \mid B_2) \longrightarrow$ left thumb is more likely

1.3.5

1.3-5. Suppose that the gene for eye color for a certain male fruit fly has alleles R (red) and W (white) and the gene for eye color for the mating female fruit fly also has alleles R and W. Their offspring receives one allele for eye color from each parent.

(a) Define the sample space of the alleles for eye color for the offspring.

(b) Assume that each of the four possible outcomes has equal probability. If an offspring ends up with either two red alleles or one red and one white allele, its eyes will look red. Given that an offspring's eyes look red, what is the conditional probability that it has two red alleles?

(a) $S = \{R, W\} \times \{R, W\}$

(b) $P(RR \mid red) = \dfrac{P(RR \cap red)}{P(red)} = \dfrac{P(RR)\,P(red \mid RR)}{P(red)}$

$= \dfrac{1/4 \times 1}{3/4} = \dfrac{1}{3}$

1.3.7

1.3-7. An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that both balls are orange?

20 26

$P(oo \mid \text{at least } 1\, o) = \dfrac{P(oo \cap \text{at least 1 orange})}{P(\text{at least 1 orange})} = \dfrac{P(oo)}{P(\text{at least 1 orange})}$

$\hookrightarrow P(ob) + P(oo)$

$$= \frac{\frac{\binom{2}{2}}{\binom{4}{2}}}{\frac{\binom{2}{1}\binom{2}{1}+\binom{2}{2}}{\binom{4}{2}}} = \frac{1}{2\times 2+1} = \frac{1}{5}$$

1.3-9. An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the mth ball selected. Let the event A_i denote a match on the ith draw, $i = 1, 2, 3, 4$.

(a) Show that $P(A_i) = \frac{3!}{4!}$ for each i.

(b) Show that $P(A_i \cap A_j) = \frac{2!}{4!}, i \neq j$.

(c) Show that $P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}, i \neq j, i \neq k, j \neq k$.

(d) Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}.$$

(e) Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!} = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

(f) What is the limit of this probability as n increases without bound?

$\boxed{1.3.9}$

fixed

(a) ○ ○ ⓘ ○ $\quad P(A_i) = \frac{3!}{4!}$ Q.E.D!

2 fixed

(b) ○ ○ ⓘ ⓙ $\quad P(A_i \cap A_j) = \frac{2!}{4!}$ Q.E.D!

3 fixed

(c) ○ ⓚ ⓘ ⓙ $\quad P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}$ Q.E.D!

(d) $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + \ldots + P(A_4) - P(A_1 \cap A_2) - \ldots + P(A_1 \cap A_2 \cap A_3) + \ldots - P(A_1 \cap A_2 \ldots \cap A_4)$

$$= 4 \times \frac{3!}{4!} - \binom{4}{2}\frac{2!}{4!} + \binom{4}{3}\frac{1!}{4!} - \frac{1}{4!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$$ Q.E.D!

(e) $P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \frac{1}{2!} + \ldots + (-1)^{n+1}\frac{1}{n!}$

$$= 1 - 1 + \frac{1}{1!} - \frac{1}{2!} + \ldots - \frac{(-1)^n}{n!} = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right)$$ Q.E.D!

(f) $1 - P(A_1 \cup A_2 \cup \ldots A_n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}$

$\therefore P(A_1 \cup A_2 \cup \ldots A_n) = 1 - \frac{1}{e}$

1.3-11. Consider the birthdays of the students in a class of size r. Assume that the year consists of 365 days.

(a) How many different ordered samples of birthdays are possible (r in sample), allowing repetitions (with replacement)?

(b) The same as part (a), except requiring that all the students have different birthdays (without replacement).

(c) If we can assume that each ordered outcome in part (a) has the same probability, what is the probability that at least two students have the same birthday?

(d) For what value of r is the probability in part (c) about equal to 1/2? Is this number surprisingly small? Hint: Use a calculator or computer to find r.

$\boxed{1.3.11}$

(a) 365^r $\qquad$ (b) $_{365}P_r$

(c) $P(\geq 2 \text{ same}) = 1 - P(\text{no same}) = 1 - \frac{_{365}P_r}{365^r}$

(d) $1 - \frac{_{365}P_r}{365^r} = \frac{1}{2} \rightarrow \frac{_{365}P_r}{365^r} = \frac{1}{2} \rightarrow r = 23$

Section 1.4 1, 3, 5, 7, 9, 11, 13

1.4.1

1.4-1. Let A and B be independent events with $P(A) = 0.7$ and $P(B) = 0.2$. Compute **(a)** $P(A \cap B)$, **(b)** $P(A \cup B)$, and **(c)** $P(A' \cup B')$.

(a) $P(A \cap B) = P(A).P(B) = 0.7 \times 0.2 = 0.14$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.7 + 0.2 - 0.14 = 0.76$

(c) $P(A^c \cup B^c) = 1 - P(A \cap B) = 1 - 0.14 = 0.86$

1.4.3

1.4-3. Let A and B be independent events with $P(A) = 1/4$ and $P(B) = 2/3$. Compute **(a)** $P(A \cap B)$, **(b)** $P(A \cap B')$, **(c)** $P(A' \cap B')$, **(d)** $P[(A \cup B)']$, and **(e)** $P(A' \cap B)$.

(a) $P(A \cap B) = P(A).P(B) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$

(b) $P(A \cap B^c) = P(A).\ P(B^c) = \frac{1}{4} \times (1 - \frac{2}{3}) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

(c) $P(A^c \cap B^c) = P(A^c).\ P(B^c) = (1 - \frac{1}{4}) \times (1 - \frac{2}{3}) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$

(d) $P((A \cup B)^c) = P(A^c \cap B^c) = \frac{1}{4}$

(e) $P(A^c \cap B) = P(A^c).\ P(B) = (1 - \frac{1}{4}) \times \frac{2}{3} = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

1.4.5

1.4-5. If $P(A) = 0.8$, $P(B) = 0.5$, and $P(A \cup B) = 0.9$, are A and B independent events? Why or why not?

$$\left.\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) = 0.8 + 0.5 - 0.9 = 0.4 \\ P(A).\ P(B) &= 0.8 \times 0.5 = 0.4 \end{aligned}\right\}$$

$\therefore P(A \cap B) = P(A).\ P(B)$ $\therefore$ A, B independent events

1.4.7

1.4-7. Each of three football players will attempt to kick a field goal from the 25-yard line. Let A_i denote the event that the field goal is made by player i, $i = 1, 2, 3$. Assume that A_1, A_2, A_3 are mutually independent and that $P(A_1) = 0.5$, $P(A_2) = 0.7$, and $P(A_3) = 0.6$.

(a) Compute the probability that exactly one player is successful.

(b) Compute the probability that exactly two players make a field goal (i.e., one misses).

(a) $P(\text{1 field goal}) = P(A_1 \cap A_2^c \cap A_3^c) + P(A_1^c \cap A_2 \cap A_3^c) + P(A_1^c \cap A_2^c \cap A_3)$

$= P(A_1).\ P(A_2^c).\ P(A_3^c) + P(A_1^c).\ P(A_2).\ P(A_3^c) + P(A_1^c).\ P(A_2^c).\ P(A_3)$

$= 0.5 \times 0.3 \times 0.4 + 0.5 \times 0.7 \times 0.4 + 0.5 \times 0.3 \times 0.6 = 0.29$

(b) $P(\text{2 field goals}) = P(A_1 \cap A_2 \cap A_3^c) + P(A_1 \cap A_2^c \cap A_3) + P(A_1^c \cap A_2 \cap A_3)$

$= 0.5 \times 0.7 \times 0.4 + 0.5 \times 0.3 \times 0.6 + 0.5 \times 0.7 \times 0.6 = 0.44$

1.4-9. Suppose that A, B, and C are mutually independent events and that $P(A) = 0.5$, $P(B) = 0.8$, and $P(C) = 0.9$. Find the probabilities that **(a)** all three events occur, **(b)** exactly two of the three events occur, and **(c)** none of the events occurs.

$\boxed{1.4.9}$

(a) $P(A \cap B \cap C) = P(A).\ P(B).\ P(C) = 0.5 \times 0.8 \times 0.9 = 0.36$

(b) $P(2 \text{ events}) = P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$

$= P(A).\ P(B).\ P(C^c) + P(A).\ P(B^c).\ P(C) + P(A^c).\ P(B).\ P(C)$

$= 0.5 \times 0.8 \times 0.1 + 0.5 \times 0.2 \times 0.9 + 0.5 \times 0.8 \times 0.9 = 0.49$

(c) $P(\text{no events}) = P(A^c \cap B^c \cap C^c) = P(A^c).\ P(B^c).\ P(C^c) = 0.5 \times 0.2 \times 0.1 = 0.01$

1.4-11. Let A and B be two events.

(a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain.

(b) If $A \subset B$, can A and B ever be independent events? Explain.

$\boxed{1.4.11}$

(a) $A \cap B = \emptyset$ hence not always independent

Unless $A = B = \emptyset$: $P(A \cap B) = P(\emptyset) = 0 = P(A).\ P(B) \longrightarrow$ then independent

(b) If $P(A) = 0$, $P(B) = 1$ that $P(A \cap B) = P(\emptyset) = 0 = P(A).\ P(B)$

↳ then independent

1.4-13. An urn contains two red balls and four white balls. Sample successively five times at random and with replacement, so that the trials are independent.

(a) Compute the probability that no two balls drawn consecutively in the sequence of five balls have the same color.

(b) How would your answer to part (a) change if the sampling is without replacement?

2r

4W

$\boxed{1.4.13}$

(a) r w r w r + w r w r w $= \left(\frac{2}{6}\right)^3\left(\frac{4}{6}\right)^2 + \left(\frac{2}{6}\right)^2\left(\frac{4}{6}\right)^3 = \frac{12}{243}$

(b) w r w r w $= \frac{4}{6} \times \frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{1}{15}$

Section 1.5

1, 3, 5, 7

1.5.1

1.5-1. Bowl B_1 contains two white chips, bowl B_2 contains two red chips, bowl B_3 contains two white and two red chips, and bowl B_4 contains three white chips and one red chip. The probabilities of selecting bowl B_1, B_2, B_3, or B_4 are 1/2, 1/4, 1/8, and 1/8, respectively. A bowl is selected using these probabilities and a chip is then drawn at random. Find

(a) $P(W)$, the probability of drawing a white chip.

(b) $P(B_1 \mid W)$, the conditional probability that bowl B_1 had been selected, given that a white chip was drawn.

1/2	1/4	1/8	1/8
B_1	B_2	B_3	B_4
2w	2r	2w	3w
		2r	1r

(a) $P(w) = P(B_1) \cdot P(w \mid B_1) + P(B_2) \cdot P(w \mid B_2) + P(B_3) \cdot P(w \mid B_3) + P(B_4) \cdot P(w \mid B_4)$

$$= \frac{1}{2} \times \frac{2}{2} + \frac{1}{4} \times \frac{0}{2} + \frac{1}{8} \times \frac{2}{4} + \frac{1}{8} \times \frac{3}{4} = 0.65625$$

(b) $$P(B_1 \mid w) = \frac{P(B_1 \cap w)}{P(w)} = \frac{P(B_1) \cdot P(w \mid B_1)}{P(w)} = \frac{1/2 \times 2/2}{0.65625} = 0.7619$$

1.5.3

1.5-3. A doctor is concerned about the relationship between blood pressure and irregular heartbeats. Among her patients, she classifies blood pressures as high, normal, or low and heartbeats as regular or irregular, finding that (a) 16% have high blood pressure; (b) 19% have low blood pressure; (c) 17% have an irregular heartbeat; (d) of those with an irregular heartbeat, 35% have high blood pressure; and (e) of those with normal blood pressure, 11% have an irregular heartbeat. What percentage of her patients have a regular heartbeat and low blood pressure?

	R	IR	
H	0.1005	$0.35 \times 0.17 = 0.0595$	0.16
N	0.5785	$0.11 \times 0.65 = 0.0715$	0.65 $\rightarrow$ 1 − 0.16 − 0.19
L	0.151	0.039	0.19
	0.83	0.17	

1.5.5

1.5-5. At a hospital's emergency room, patients are classified and 20% of them are critical, 30% are serious, and 50% are stable. Of the critical ones, 30% die; of the serious, 10% die; and of the stable, 1% die. Given that a patient dies, what is the conditional probability that the patient was classified as critical?

	C	SR	ST
D	$0.3 \times 0.2 = 0.06$	$0.1 \times 0.3 = 0.03$	$0.01 \times 0.5 = 0.005$
ND	0.14	0.27	0.495
	0.2	0.3	0.5

$$P(C \mid D) = \frac{P(C \cap D)}{P(D)}$$

$$= \frac{0.06}{0.06 + 0.03 + 0.005} = 0.6316$$

1.5.7

1.5-7. A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemist's output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

	+	−	
I	$0.9 \times 0.2 = 0.18$	0.02	0.2
NI	$0.05 \times 0.8 = 0.04$	0.76	0.8
	0.22	0.78	1

$$P(I \mid +) = \frac{P(I \cap +)}{P(+)} = \frac{0.18}{0.22} = 0.8182$$

Section 2.1

1, 3, 5, 7, 11

2.1.1

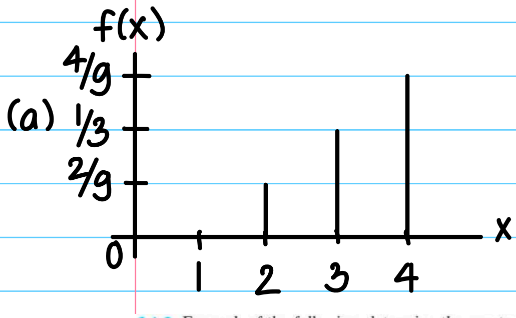
2.1-1. Let the pmf of X be defined by $f(x) = x/9$, $x = 2, 3, 4$.

(a) Draw a bar graph for this pmf.

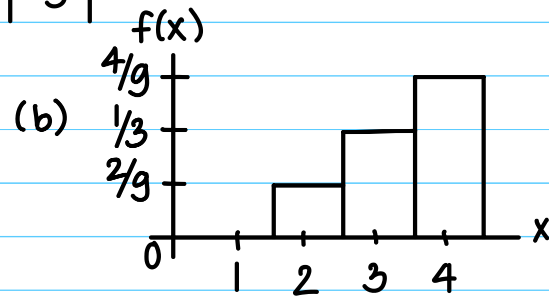
(b) Draw a probability histogram for this pmf.

X	2	3	4
f_X	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

(a)



(b)



2.1.3

2.1-3. For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a bar graph:

(a) $f(x) = x/c, \quad x = 1, 2, 3, 4.$

(b) $f(x) = cx, \quad x = 1, 2, 3, \ldots, 10.$

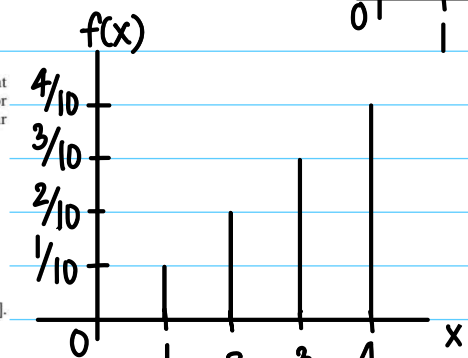
(c) $f(x) = c(1/4)^x, \quad x = 1, 2, 3, \ldots.$

(d) $f(x) = c(x+1)^2, \quad x = 0, 1, 2, 3.$

(e) $f(x) = x/c, \quad x = 1, 2, 3, \ldots, n.$

(f) $f(x) = \dfrac{c}{(x+1)(x+2)}, \quad x = 0, 1, 2, 3, \ldots.$

HINT: In part (f), write $f(x) = c[1/(x+1) - 1/(x+2)]$.



(a) $$\sum_{x=1}^{4} f(x) = \sum_{x=1}^{4} \frac{x}{c} = \frac{1}{c}(1+2+3+4) = \frac{10}{c} = 1 \longrightarrow c = 10$$

(b) $$\sum_{x=1}^{10} f(x) = \sum_{x=1}^{10} cx = c(1+2+\ldots+10) = 55c = 1 \longrightarrow c = \frac{1}{55}$$

(c) $$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} c\left(\frac{1}{4}\right)^x = c\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x = c\frac{1/4}{1-1/4} = \frac{1}{3}c = 1 \rightarrow c = 3$$

(d) $$\sum_{x=0}^{3} f(x) = \sum_{x=0}^{3} c(x+1)^2 = c(1+2^2+3^2+4^2) = 30c = 1 \rightarrow c = \frac{1}{30}$$

(e) $$\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} \frac{x}{c} = \frac{1}{c}\sum_{x=1}^{n} x = \frac{1}{c} \times \frac{n(n+1)}{2} = 1 \rightarrow c = \frac{n(n+1)}{2}$$

(f) $$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{c}{(x+1)(x+2)} = c\sum_{x=0}^{\infty}\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

$$= c\left(\frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \ldots\right) = c = 1$$

2.1.5

2.1-5. The pmf of X is $f(x) = (5 - x)/10,\ x = 1, 2, 3, 4.$

(a) Graph the pmf as a bar graph.

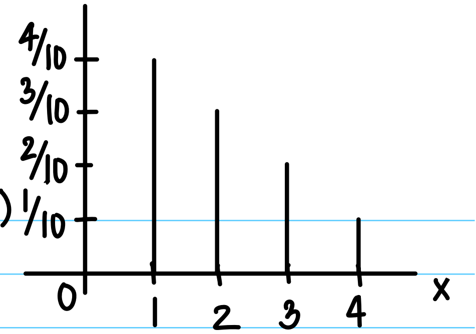
(b) Use the following independent observations of X, simulated on a computer, to construct a table like Table 2.1-1:

3	1	2	2	3	2	2	2	1	3	3	2	3	2	4	4	2	1	1	3
3	1	2	2	1	1	4	2	3	1	1	1	2	1	3	1	1	3	3	1
1	1	1	1	1	4	1	3	1	2	4	1	1	2	3	4	3	1	4	2
2	1	3	2	1	4	1	1	1	2	1	3	4	3	2	1	4	4	1	3
2	2	2	1	2	3	1	1	4	2	1	4	2	1	2	3	1	4	2	3

(c) Construct a probability histogram and a relative frequency histogram like Figure 2.1-2.

X	1	2	3	4
f_X	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

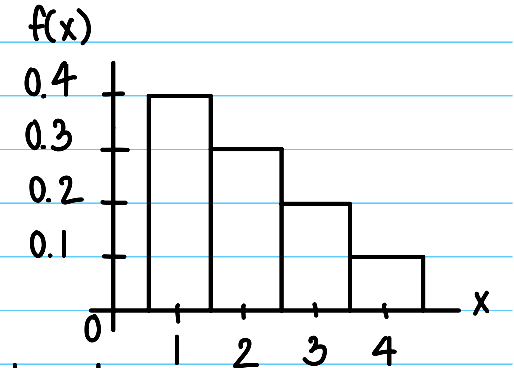
(a)



(b)

x	Frequency	Relative Frequency	f(x)
1	38	$38/100 = 0.38$	$4/10 = 0.4$
2	27	$27/100 = 0.27$	$3/10 = 0.3$
3	21	$21/100 = 0.21$	$2/10 = 0.2$
4	14	$14/100 = 0.14$	$1/10 = 0.1$

(c)



2.1.7

2.1-7. Let a random experiment be the casting of a pair of fair six-sided dice and let X equal the smaller of the outcomes if they are different and the common value if they are equal.

(a) With reasonable assumptions, find the pmf of X.

(b) Draw a probability histogram of the pmf of X.

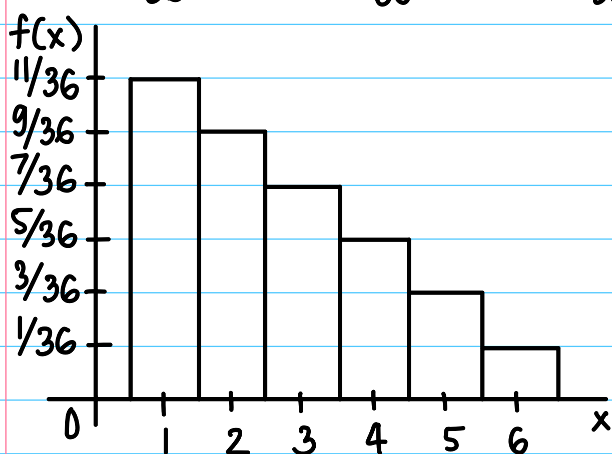
(c) Let Y equal the range of the two outcomes (i.e., the absolute value of the difference of the largest and the smallest outcomes). Determine the pmf $g(y)$ of Y for $y = 0, 1, 2, 3, 4, 5$.

(d) Draw a probability histogram for $g(y)$.

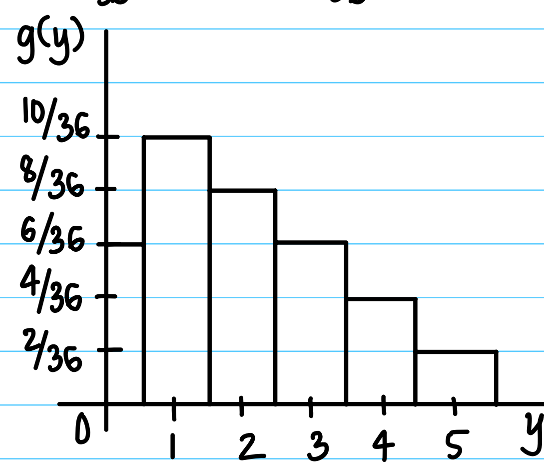
X	1	2	3	4	5	6
f_X	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(a) $f_X(1) = \frac{11}{36},\ f_X(2) = \frac{9}{36},\ f_X(3) = \frac{7}{36},\ f_X(4) = \frac{5}{36},\ f_X(5) = \frac{3}{36},\ f_X(6) = \frac{1}{36}$

(b)



(d)



(c) $g_Y(0) = \frac{6}{36} = \frac{1}{6},\ g_Y(1) = \frac{10}{36} = \frac{5}{18},\ g_Y(2) = \frac{8}{36} = \frac{2}{9},$

$g_Y(3) = \frac{6}{36} = \frac{1}{6},\ g_Y(4) = \frac{4}{36} = \frac{1}{9},\ g_Y(5) = \frac{2}{36} = \frac{1}{18}$

2.1.11

2.1-11. Let X be the number of accidents per week in a factory. Let the pmf of X be

$$f(x) = \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}, \qquad x = 0, 1, 2, \ldots$$

Find the conditional probability of $X \geq 4$, given that $X \geq 1$.

$$P(X \geq 4 \mid X \geq 1) = \frac{P(X \geq 4 \cap X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 4)}{P(X \geq 1)}$$

$$= \frac{\frac{1}{5} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} + \ldots + \cancel{\frac{1}{x+1}} - \cancel{\frac{1}{x+2}} + \ldots}{\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \ldots + \cancel{\frac{1}{x+1}} - \cancel{\frac{1}{x+2}} + \ldots} = \frac{\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}$$

Section 2.2 1, 3, 5, 7, 9

2.2.1

2.2-1. Find $E(X)$ for each of the distributions given in Exercise 2.1-3.

(a) $f(x)=x/c$, $x=1,2,3,4$.
(b) $f(x)=cx$, $x=1,2,3,\ldots,10$.
(c) $f(x)=c(1/4)^x$, $x=1,2,3,\ldots$.
(d) $f(x)=c(x+1)^2$, $x=0,1,2,3$.
(e) $f(x)=x/c$, $x=1,2,3,\ldots,n$.
(f) $f(x)=\dfrac{c}{(x+1)(x+2)}$, $x=0,1,2,3,\ldots$.
HINT: In part (f), write $f(x)=c[1/(x+1)-1/(x+2)]$.

(a) $f(x)=\dfrac{x}{10}$, $x=1,2,3,4$

$$E[X]=\sum_{x=1}^{4} x f_X(x)=\sum_{x=1}^{4}\frac{x^2}{10}=\frac{1}{10}(1^2+2^2+3^2+4^2)=\frac{30}{10}=3$$

(b) $f(x)=\dfrac{x}{55}$, $x=1\to 10$

$$E[X]=\sum_{x=1}^{10} x f_X(x)=\sum_{x=1}^{10}\frac{x^2}{55}=\frac{1}{55}\sum_{x=1}^{10}x^2=\frac{385}{55}=7$$

(c) $f(x)=\dfrac{3}{4^x}$, $x=1,2,3\ldots$

$$E[X]=\sum_{x=1}^{\infty} x f_X(x)=\sum_{x=1}^{\infty}\frac{3x}{4^x}=3\Big[\overbrace{\left(\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}\right)^2+3\left(\tfrac{1}{4}\right)^3+\ldots}^{S}\Big]$$

$\to$ use $Geom(p)=Geom\left(\frac{3}{4}\right)\to \mu=\frac{1}{p}=\frac{4}{3}$

$$S=\left(\tfrac{1}{4}\right)+2\left(\tfrac{1}{4}\right)^2+3\left(\tfrac{1}{4}\right)^3+\ldots$$
$$4S=1+2\left(\tfrac{1}{4}\right)+3\left(\tfrac{1}{4}\right)^2+\ldots$$
$$3S=\left(\tfrac{1}{4}\right)^0+\tfrac{1}{4}+\left(\tfrac{1}{4}\right)^2=\frac{1}{1-1/4}=\frac{4}{3}$$

(d) $f(x)=\dfrac{(x+1)^2}{30}$, $x=0,1,2,3$

$$E[X]=\sum_{x=0}^{3} x f_X(x)=\sum_{x=0}^{3}\frac{x(x+1)^2}{30}=\frac{1}{30}(0+1\times2^2+2\times3^2+3\times4^2)=\frac{7}{3}$$

(e) $f(x)=\dfrac{x}{c}$, $x=1\to n$

$$E[X]=\sum_{x=1}^{n} x f_X(x)=\sum_{x=1}^{n}\frac{x^2}{c}=\frac{1}{c}\sum_{x=1}^{n}x^2=\frac{2}{n\cancel{(n+1)}}\times\frac{n\cancel{(n+1)}(2n+1)}{6}=\frac{2n+1}{3}$$

(f) $f(x)=\dfrac{1}{(x+1)(x+2)}$, $x=0,1,2,3\ldots$

$$E[X]=\sum_{x=0}^{\infty} x f_X(x)=\sum_{x=0}^{\infty}\frac{x}{(x+1)(x+2)}=\frac{1}{2\times 3}+\frac{2}{3\times 4}+\frac{3}{4\times 5}+\dots$$

$$=\frac{1}{2}-\frac{1}{3}+2\left(\frac{1}{3}-\frac{1}{4}\right)+3\left(\frac{1}{4}-\frac{1}{5}\right)+\dots=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots=\sum_{x=2}^{\infty}\frac{1}{x}=\infty$$

2.2.3

2.2-3. Let X be a discrete random variable with the Benford distribution introduced in Exercise 2.1-4, which has pmf $f(x)=\log_{10}(\frac{x+1}{x})$, $x=1,2,\dots,9$. Show that $E(X)=9-\log_{10}(9!)$.

$$E[X]=\sum_{x=1}^{9} x f_X(x)=\sum_{x=1}^{9} x\log_{10}\left(\frac{x+1}{x}\right)=\sum_{x=1}^{9} x\left(\log_{10}(x+1)-\log_{10}x\right)$$

$$=\log_{10}2-\underbrace{\log_{10}1}_{0}+2(\log_{10}3-\log_{10}2)+3(\log_{10}4-\log_{10}3)+\dots+9(\log_{10}10-\log_{10}9)$$

$$=9\log_{10}10-\log_{10}2-\log_{10}3-\dots-\log_{10}9$$

Q.E.D!

$$=9-(\log_{10}2+\log_{10}3+\dots+\log_{10}9)=9-\log_{10}(2\times 3\times\dots\times 9)=9-\log_{10}9!$$

2.2.5

2.2-5. Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the pmf

$$f(x)=\frac{5-x}{10},\qquad x=1,2,3,4.$$

If the patient is to receive \$200 from an insurance company for each of the first two days in the hospital and \$100 for each day after the first two days, what is the expected payment for the hospitalization?

X	1	2	3	4
f_X	0.4	0.3	0.2	0.1
Y	200	400	500	600

$$E[Y]=\sum_{y=1}^{4} y.f_Y(y)=\sum_{y=1}^{4} y.f_X(x)=200\times 0.4+400\times 0.3+500\times 0.2+600\times 0.1=360$$

2.2.7

2.2-7. In Example 2.2-1 let $Z=u(X)=X^3$.

(a) Find the pmf of Z, say $h(z)$.

(b) Find $E(Z)$.

(c) How much, on average, can the young man expect to win on each play if he charges \$10 per play?

$$f(x)=\frac{4-x}{6},\quad x=1,2,3\longrightarrow z=1,8,27$$

(a) $Z=X^3\longrightarrow X=\sqrt[3]{Z}\longrightarrow h(z)=\dfrac{4-\sqrt[3]{z}}{6}$

(b) $$E[Z]=\sum z.h(z)=1\times\frac{4-1}{6}+8\times\frac{4-\sqrt[3]{8}}{6}+27\times\frac{4-\sqrt[3]{27}}{6}$$

$$=\frac{1}{2}+8\times\frac{1}{3}+27\times\frac{1}{6}=\frac{1}{2}+\frac{8}{3}+\frac{9}{2}=\frac{23}{3}$$

(c) $10-\dfrac{23}{3}=\dfrac{7}{3}$

2.2.9

2.2-9. In the gambling game chuck-a-luck, for a \$1 bet it is possible to win \$1, \$2, or \$3 with respective probabilities 75/216, 15/216, and 1/216. One dollar is lost with probability 125/216. Let X equal the payoff for this game and find $E(X)$. Note that when a bet is won, the \$1 that was bet, in addition to the \$1, \$2, or \$3 that is won, is returned to the bettor.

$$E[X]=\sum_{x} x f_X(x)$$

$$=(-1)\times\frac{125}{216}+1\times\frac{75}{216}+2\times\frac{15}{216}+3\times\frac{1}{216}\approx -0.0787$$

Section 2.3 1, 3, 5, 7, 9, 11, 13, 15, 18, 19

2.3.1

2.3-1. Find the mean, variance, and index of skewness for the following discrete distributions:

(a) $f(x) = 1, \quad x = 5.$

(b) $f(x) = 1/5, \quad x = 1, 2, 3, 4, 5.$

(c) $f(x) = 1/5, \quad x = 3, 5, 7, 9, 11.$

(d) $f(x) = x/6, \quad x = 1, 2, 3.$

(e) $f(x) = (1 + |x|)/5, \quad x = -1, 0, 1.$

(f) $f(x) = (2 - |x|)/4, \quad x = -1, 0, 1.$

(a) $E[X] = \sum x f_X(x) = 5 \times 1 = 5$

$Var(X) = E[X^2] - E[X]^2 = 5^2 \times 1 - 5^2 = 0 \longrightarrow \sigma = \sqrt{Var(X)} = 0$

$E[(X-\mu)^3] = E[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3] = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$
$= 5^3 \times 1 - 3 \times 5 \times 25 + 3 \times 5^2 \times 5 - 5^3 = 0$

$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{0}{0} = \text{undefined}$

(b) $E[X] = \sum_X x f_X(x) = \frac{1}{5}(1+2+3+4+5) = 3$

$E[X^2] = \sum_{X^2} x^2 f_X(x) = \frac{1}{5}(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 11$

$\therefore Var(X) = E[X^2] - E[X]^2 = 11 - 3^2 = 2 \longrightarrow \sigma = \sqrt{2}$

$E[(X-\mu)^3] = E[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3] = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$
$= E[X^3] - 3\mu E[X^2] + 3\mu^3 - \mu^3 = E[X^3] - 3\mu E[X^2] + 2\mu^3$

$= \frac{1}{5}(1^3 + 2^3 + 3^3 + 4^3 + 5^3) - 3 \times 3 \times 11 + 2 \times 3^3 = 0$

$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{0}{\sqrt{2}} = 0$

(c) $E[X] = \sum_X x f_X(x) = \frac{1}{5}(3+5+7+9+11) = 7$

$E[X^2] = \sum_{X^2} x^2 f_X(x) = \frac{1}{5}(3^2 + 5^2 + 7^2 + 9^2 + 11^2) = 57$

$\therefore Var(X) = E[X^2] - E[X]^2 = 57 - 7^2 = 8 \longrightarrow \sigma = 2\sqrt{2}$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= \frac{1}{5}(3^3+5^3+7^3+9^3+11^3) - 3\times 7\times 57 + 2\times 7^3 = 0$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{0}{2\sqrt{2}} = 0$$

(d) $E[X] = \sum_X x f_X(x) = 1\times\frac{1}{6} + 2\times\frac{2}{6} + 3\times\frac{3}{6} = \frac{7}{3}$

$E[X^2] = \sum_{X^2} x^2 f_X(x) = 1^2\times\frac{1}{6} + 2^2\times\frac{2}{6} + 3^2\times\frac{3}{6} = 6$

$\therefore Var(X) = E[X^2] - E[X]^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9} \rightarrow \sigma = \frac{\sqrt{5}}{3}$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= 1^3\times\frac{1}{6} + 2^3\times\frac{2}{6} + 3^3\times\frac{3}{6} - 3\times\frac{7}{3}\times 6 + 2\times\left(\frac{7}{3}\right)^3 = \frac{-7}{27}$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{-7}{27}\times\left(\frac{3}{\sqrt{5}}\right)^3 = \frac{-7}{5\sqrt{5}} = \frac{-7\sqrt{5}}{25}$$

(e) $E[X] = \sum_X x f_X(x) = -1\times\frac{1+1}{5} + 0 + 1\times\frac{1+1}{5} = 0$

$E[X^2] = \sum_{X^2} x^2 f_X(x) = (-1)^2\times\frac{1+1}{5} + 0 + 1^2\times\frac{1+1}{5} = \frac{4}{5}$

$\therefore Var(X) = E[X^2] - E[X]^2 = \frac{4}{5} - 0^2 = \frac{4}{5} \rightarrow \sigma = \frac{2}{\sqrt{5}}$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= (-1)^3\times\frac{1+1}{5} + 0 + 1^3\times\frac{1+1}{5} - 3\times 0\times\frac{4}{5} + 2\times 0^3 = 0$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = 0\times\left(\frac{\sqrt{5}}{2}\right)^3 = 0$$

(f) $E[X] = \sum_X x f_X(x) = -1\times\frac{2-1}{4} + 0 + 1\times\frac{2-1}{4} = 0$

$$E[X^2] = \sum_{x^2} x^2 f_X(x) = (-1)^2 \times \frac{2-1}{4} + 0 + 1 \times \frac{2-1}{4} = \frac{1}{2}$$

$$\therefore Var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - 0^2 = \frac{1}{2} \rightarrow \sigma = \frac{1}{\sqrt{2}}$$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= (-1)^3 \times \frac{2-1}{4} + 0 + 1^3 \times \frac{2-1}{4} - 3 \times 0 \times \frac{1}{2} + 2 \times 0^3 = 0$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = 0 \times (\sqrt{2})^3 = 0$$

2.3.3

2.3-3. If the pmf of X is given by $f(x)$, **(i)** depict the pmf as a probability histogram and find the values of **(ii)** the mean, **(iii)** the standard deviation, and **(iv)** the index of skewness.

(a)

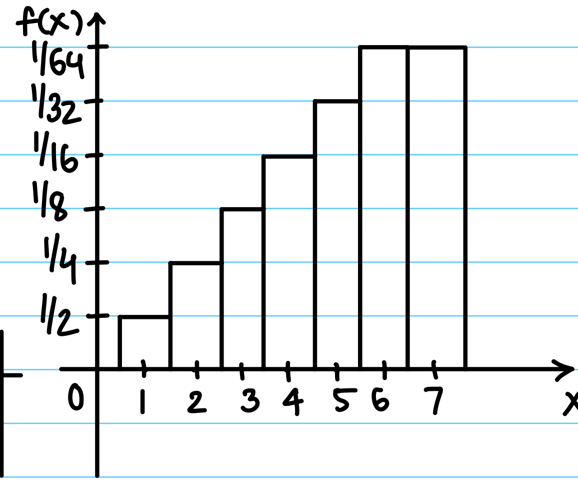
$$f(x) = \begin{cases} \frac{2^{6-x}}{64}, & x = 1, 2, 3, 4, 5, 6, \\ \frac{1}{64}, & x = 7. \end{cases}$$

(b)

$$f(x) = \begin{cases} \frac{1}{64}, & x = 1, \\ \frac{2^{x-2}}{64}, & x = 2, 3, 4, 5, 6, 7. \end{cases}$$

(a) pmf:

X	1	2	3	4	5	6	7
f_X	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$



$$E[X] = \sum_{x} x f_X(x) = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{13}{64} = \frac{127}{64}$$

$$E[X^2] = \sum_{x^2} x^2 f_X(x) = \frac{1}{2} + 1 + \frac{9}{8} + 1 + \frac{25}{32} + \frac{36+49}{64} = \frac{367}{64}$$

$$\therefore Var(X) = E[X^2] - E[X]^2 = \frac{367}{64} - \left(\frac{127}{64}\right)^2 = \frac{7359}{4096} \rightarrow \sigma = \frac{\sqrt{7359}}{64}$$

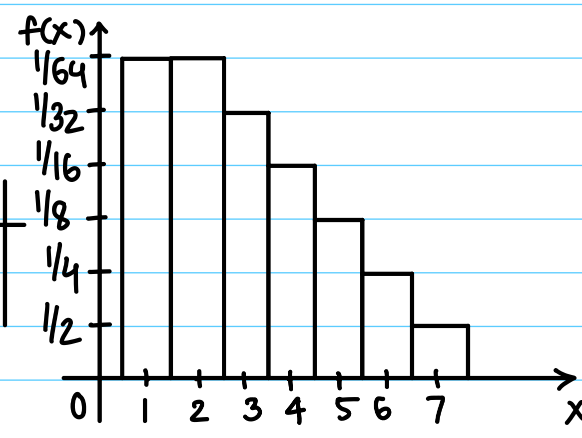
$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= \frac{1}{2} + 2 + \frac{27}{8} + 4 + \frac{125}{32} + \frac{6^3 + 7^3}{64} - 3 \times \frac{127}{64} \times \frac{367}{64} + 2 \times \left(\frac{127}{64}\right)^3 \approx 4.0060959$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = 1.6635$$

(b) pmf:

X	1	2	3	4	5	6	7
f_X	$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$



$$E[X]=\sum_{X} x f_X(x) = \frac{3}{64}+\frac{3}{32}+\frac{1}{4}+\frac{5}{8}+\frac{3}{2}+\frac{7}{2} = \frac{385}{64}$$

$$E[X^2]=\sum_{X^2} x^2 f_X(x) = \frac{5}{64}+\frac{9}{32}+1+\frac{25}{8}+9+\frac{49}{2} = \frac{2431}{64}$$

$$\therefore Var(X)=E[X^2]-E[X]^2 = \frac{2431}{64}-\left(\frac{385}{64}\right)^2 = \frac{7359}{4096} \longrightarrow \sigma = \frac{\sqrt{7359}}{64}$$

$$E[(X-\mu)^3]=E[X^3]-3\mu E[X^2]+2\mu^3$$

$$=\frac{9}{64}+\frac{27}{32}+4+\frac{125}{8}+\frac{6^3}{4}+\frac{7^3}{2}-3\times\frac{385}{64}\times\frac{2431}{64}+2\times\left(\frac{385}{64}\right)^3 \approx -4.0061$$

$$\therefore \frac{E[(X-\mu)^3]}{\sigma^3} = -1.6635$$

2.3.5

2.3-5. Consider an experiment that consists of selecting a card at random from an ordinary deck of cards. Let the random variable X equal the value of the selected card, where Ace = 1, Jack = 11, Queen = 12, and King = 13. Thus, the space of X is $S = \{1, 2, 3, \ldots, 13\}$. If the experiment is performed in an unbiased manner, assign probabilities to these 13 outcomes and compute the mean μ of this probability distribution.

$$f(x)=\frac{1}{13},\quad x=1,2\ldots 13$$

$$\mu=\sum_{X} x f_X(x) = \frac{1}{13}\times(1+2+\ldots+13) = 7$$

2.3.7

2.3-7. Let X equal an integer selected at random from the first m positive integers, $\{1, 2, \ldots, m\}$. Find the value of m for which $E(X) = \mathrm{Var}(X)$. (See Zerger in the references.)

$$f(x)=\frac{1}{m},\quad x=1,2\ldots m$$

$$\mu=\sum_{X} x f_X(x) = \frac{1}{m}(1+2+\ldots+m) = \frac{1}{m}\times\frac{m(m+1)}{2} = \frac{m+1}{2}$$

$$E[X^2]=\sum_{X^2} x^2 f_X(x) = \frac{1}{m}(1^2+2^2+\ldots+m^2) = \frac{1}{m}\times\frac{m(m+1)(2m+1)}{6}$$

$$Var(X)=E[X^2]-E[X]^2 = E[X]$$

$$\therefore \frac{(m+1)(2m+1)}{6}-\left(\frac{m+1}{2}\right)^2-\frac{m+1}{2} = \frac{2m^2+3m+1}{6}-\frac{m^2+2m+1}{4}-\frac{m+1}{2}$$

$$=2(2m^2+3m+1)-3(m^2+2m+1)-6(m+1)$$

$$=4m^2+6m+2-3m^2-6m-3-6m-6$$

$$=m^2-6m-7=\underbrace{(m+1)}_{>0}(m-7)=0 \longrightarrow m=7$$

2.3.9

2.3-9. A warranty is written on a product worth \$10,000 so that the buyer is given \$8000 if it fails in the first year, \$6000 if it fails in the second, \$4000 if it fails in the third, \$2000 if it fails in the fourth, and zero after that. The probability of the product's failing in a year is 0.1; failures are independent of those of other years. What is the expected value of the warranty?

$P(\text{fail}) = 0.1 \rightarrow P(\text{good}) = 0.9$

$P(X = 8000) = P(1^{st}\text{ fail}) = 0.1$

$P(X = 6000) = P(1^{st}\text{ good}, 2^{nd}\text{ fail}) = 0.9 \times 0.1 = 0.09$

$P(X = 4000) = P(1^{st}\text{ good}, 2^{nd}\text{ good}, 3^{rd}\text{ fail}) = 0.9 \times 0.9 \times 0.1 = 0.081$

$P(X = 2000) = P(1^{st}\text{ good}, 2^{nd}\text{ good}, 3^{rd}\text{ good}, 4^{th}\text{ fail}) = 0.9 \times 0.9 \times 0.9 \times 0.1$

$= 0.0729$

$E[X] = \sum_x x f_X(x) = 8000 \times 0.1 + 6000 \times 0.09 + 4000 \times 0.081 + 2000 \times 0.0729$

$= \$1809.80$

2.3.11

2.3-11. If the moment-generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t},$$

find the mean, variance, and pmf of X.

pmf:

X	1	2	3
f_X	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

$E[X] = \sum_x x f_X(x) = \frac{2}{5} + \frac{2}{5} + \frac{6}{5} = 2$

$\sigma^2 = E[X^2] - E[X]^2 = \frac{2}{5} + \frac{4}{5} + \frac{18}{5} - 2^2 = \frac{4}{5}$

2.3.13

2.3-13. For each question on a multiple-choice test, there are five possible answers, of which exactly one is correct. If a student selects answers at random, give the probability that the first question answered correctly is question 4.

$P(T) = \frac{1}{5} \rightarrow P(F) = \frac{4}{5}$

$P(FFFT) = \left(\frac{4}{5}\right)^3 \times \frac{1}{5} = 0.1024$

2.3.15

2.3-15. Apples are packaged automatically in 3-pound bags. Suppose that 4% of the time the bag of apples weighs less than 3 pounds. If you select bags randomly and weigh them in order to discover one underweight bag of apples, find the probability that the number of bags that must be selected is

(a) At least 20.

(b) At most 20.

(c) Exactly 20.

$P(\text{under}) = 0.04 \rightarrow P(\text{over}) = 0.96$

(a) $P(X \geq 20) = P(X > 19) = P(\text{over 19 times}) = 0.96^{19} = 0.4604$

(b) $P(X \leq 20) = 1 - P(X > 20) = 1 - P(\text{over 20 times}) = 1 - 0.96^{20} = 0.558$

(c) $P(X = 20) = P(\text{over 19 times} \cap \text{under } 20^{th}\text{ time})$

$= P(\text{over 19 times}) \cdot P(\text{under } 20^{th}\text{ time}) = 0.96^{19} \times 0.04 = 0.0184$

2.3.18

2.3-18. Let X have a geometric distribution. Show that

$$P(X > k + j \mid X > k) = P(X > j),$$

where k and j are nonnegative integers. Note: We sometimes say that in this situation there has been loss of memory.

Geometric distribution: $f(x) = q^{k-1}p, \ k = 1, 2 \ldots$

$P(X > k+j \mid X > k) = \frac{P(X > k+j \cap X > k)}{P(X > k)} = \frac{P(X > k+j)}{P(X > k)} = \frac{q^{k+j}}{q^k} = q^j = P(X > j)$ Q.E.D!

2.3.19

2.3-19. Given a random permutation of the integers in the set $\{1, 2, 3, 4, 5\}$, let X equal the number of integers that are in their natural position. The moment-generating function of X is

$$M(t) = \frac{44}{120} + \frac{45}{120}e^{t} + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t} + \frac{1}{120}e^{5t}.$$

(a) Find the mean and variance of X.

(b) Find the probability that at least one integer is in its natural position.

(c) Draw a graph of the probability histogram of the pmf of X.

pmf:

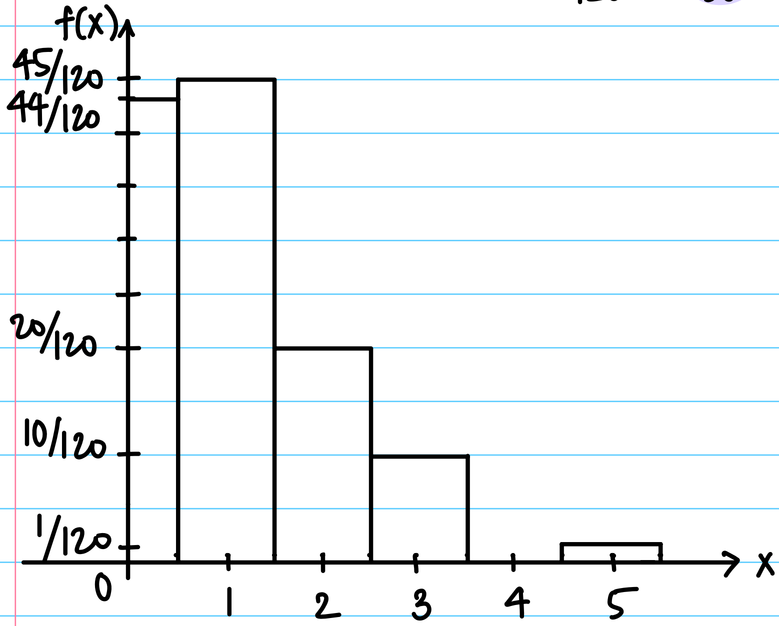
X	0	1	2	3	5
f_X	$\frac{44}{120}$	$\frac{45}{120}$	$\frac{20}{120}$	$\frac{10}{120}$	$\frac{1}{120}$

(a) $$E[X] = \sum_{X} x f_X(x) = \frac{45+40+30+5}{120} = 1$$

$$\sigma^2 = E[X^2] - E[X]^2 = \frac{45+80+90+25}{120} - 1^2 = 1$$

(b) $$P(X \geq 1) = 1 - P(X=0) = 1 - \frac{44}{120} = \frac{19}{30}$$

(c)



Section 2.4 1, 3, 5, 7, 9, 11, 13, 19, 20

2.4.1

2.4-1. An urn contains seven red and 11 white balls. Draw one ball at random from the urn. Let $X = 1$ if a red ball is drawn, and let $X = 0$ if a white ball is drawn. Give the pmf, mean, and variance of X.

pmf: $f(x) = \left(\frac{7}{18}\right)^x \left(\frac{11}{18}\right)^{1-x}$, $x = 0, 1$

$\mu = p = \frac{7}{18}$, $Var(x) = pq = \frac{7 \times 11}{18^2} = \frac{77}{324}$

2.4.3

2.4-3. On a six-question multiple-choice test there are five possible answers for each question, of which one is correct (C) and four are incorrect (I). If a student guesses randomly and independently, find the probability of

(a) Being correct only on questions 1 and 4 (i.e., scoring C, I, I, C, I, I).

(b) Being correct on two questions.

(a) $\left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4 = 0.016384$

(b) $\binom{6}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4 = 0.24576$

2.4.5

2.4-5. In a lab experiment involving inorganic syntheses of molecular precursors to organometallic ceramics, the final step of a five-step reaction involves the formation of a metal–metal bond. The probability of such a bond forming is $p = 0.20$. Let X equal the number of successful reactions out of $n = 25$ such experiments.

(a) Find the probability that X is at most 4.

(b) Find the probability that X is at least 5.

(c) Find the probability that X is equal to 6.

(d) Give the mean, variance, and standard deviation of X.

(a) $P(X \le 4) = \sum_{k=0}^{4} \binom{25}{k} 0.20^k \, 0.80^{25-k} = 0.4207$

(b) $P(X \geq 5) = 1 - P(X \le 4) = 1 - 0.4207 = 0.5793$

(c) $P(X = 6) = \binom{25}{6} 0.20^6 \, 0.8^{25-6} = 0.1633$

(d) $X \sim Bin(25, 0.20) \rightarrow \mu = np = 25 \times 0.20 = 5$

$\sigma^2 = npq = 25 \times 0.20 \times 0.80 = 4 \rightarrow \sigma = 2$

2.4.7

2.4-7. Suppose that 2000 points are selected independently and at random from the unit square $\{(x, y) : 0 \le x < 1, 0 \le y < 1\}$. Let W equal the number of points that fall into $A = \{(x, y) : x^2 + y^2 < 1\}$.

(a) How is W distributed?

(b) Give the mean, variance, and standard deviation of W.

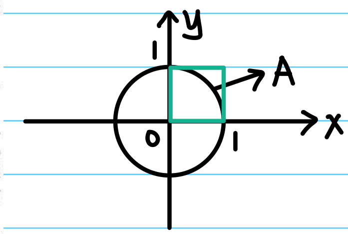
(c) What is the expected value of $W/500$?

(d) Use the computer to select 2000 pairs of random numbers. Determine the value of W and use that value to find an estimate for π. (Of course, we know the real value of π, and more will be said about estimation later on in this text.)

(e) How could you extend part (d) to estimate the volume $V = (4/3)\pi$ of a ball of radius 1 in 3-space?

(f) How could you extend these techniques to estimate the "volume" of a ball of radius 1 in n-space? NOTE: The answer given in terms of the Γ function (see Equation 3.2-2) is

$$V_n = \pi^{n/2} / \Gamma(n/2 + 1).$$



(a) $W \sim Bin(2000, \pi/4)$

(b) $\mu = np = 2000 \times \pi/4 = 500\pi$

$\sigma^2 = npq = 2000 \times \frac{\pi}{4} \times \left(1 - \frac{\pi}{4}\right) = 337.096$

$\therefore \sigma = 18.3602$

(c) $E\left[\frac{W}{500}\right] = \frac{E[W]}{500} = \frac{\mu_W}{500} = \frac{500\pi}{500} = \pi$

(d) $V = \frac{4/3\pi}{2^3} = \frac{\pi}{6} \rightarrow Bin(n, \frac{\pi}{6})$

each dimension have positive + negative sides

2.4.9

2.4-9. Suppose that the percentage of American drivers who multitask (e.g., talk on cell phones, eat a snack, or text at the same time they are driving) is approximately 80%. In a random sample of $n = 20$ drivers, let X equal the number of multitaskers.

(a) How is X distributed?

(b) Give the values of the mean, variance, and standard deviation of X.

(c) Find **(i)** $P(X = 15)$, **(ii)** $P(X > 15)$, and **(iii)** $P(X \le 15)$.

(a) $X \sim Bin(20, 0.8)$

(b) $\mu_X = np = 20(0.8) = 16 \qquad \sigma^2 = npq = 20(0.8)(0.2) = 3.2 \longrightarrow \sigma = 1.789$

(c) $P(X=15) = \binom{20}{15} 0.8^{15} 0.2^5 = 0.1746$

$$P(X > 15) = \sum_{k=16}^{20} \binom{20}{k} 0.8^k 0.2^{20-k} = 0.6296$$

$$P(X \le 15) = 1 - P(X > 15) = 1 - 0.6296 = 0.3704$$

2.4.11

2.4-11. Find the index of skewness for the $b(n, p)$ distribution, and verify that it is negative if $p < 0.5$, zero if $p = 0.5$, and positive if $p > 0.5$.

$$X \sim Bin(n,p) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \longrightarrow \sigma^2 = npq \longrightarrow \sigma = \sqrt{npq}$$

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = (q + pe^t)^n$$

Let $g(t) = \ln M_X(t) = n \ln(q + pe^t)$

$$g'(t) = n \times \frac{pe^t}{q + pe^t} = npe^t (q + pe^t)^{-1}$$

$$g''(t) = npe^t (q+pe^t)^{-1} + npe^t(-1)(pe^t)(q+pe^t)^{-2}$$
$$= npe^t(q+pe^t)^{-1} - np^2e^{2t}(q+pe^t)^{-2}$$

$$g'''(t) = npe^t(q+pe^t)^{-1} - np^2e^{2t}(q+pe^t)^{-2} - 2np^2e^{2t}(q+pe^t)^{-2}$$
$$+ np^2e^{2t}(-2)(pe^t)(q+pe^t)^{-3}$$

$$g'''(0) = np - np^2 - 2np^2 - 2np^3$$
$$= np - 3np^2 - 2np^3 = np(1 - 3p - 2p^2) = np(1-2p)(1-p)$$
$$= npq(1-2p)$$

$$\therefore \gamma = \frac{g'''(0)}{\sigma^3} = \frac{npq(1-2p)}{npq\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}}$$

$1-2p < 0 \rightarrow p > 0.5 \rightarrow \gamma < 0$

$1-2p = 0 \rightarrow p = 0.5 \rightarrow \gamma = 0$

$1-2p > 0 \rightarrow p < 0.5 \rightarrow \gamma > 0$

2.4.13

2.4-13. It is claimed that for a particular lottery, 1/10 of the 50 million tickets will win a prize. What is the probability of winning at least one prize if you purchase **(a)** ten tickets or **(b)** 15 tickets?

$X \sim Geom(0.1)$

(a) $P(X \geq 1) = 1 - P(X=0) = 1 - 0.9^{10} = 0.6513$

(b) $P(X \geq 1) = 1 - P(X=0) = 1 - 0.9^{15} = 0.7941$

2.4.19

2.4-19. Define the pmf and give the values of μ, σ^2, and σ when the moment-generating function of X is defined by

(a) $M(t) = 1/3 + (2/3)e^t$.

(b) $M(t) = (0.25 + 0.75e^t)^{12}$.

(a) $X \sim Ber(\frac{2}{3}) \rightarrow$ pmf:

X	0	1
f(x)	1/3	2/3

$\mu_X = 2/3$

$\sigma^2 = 2/3 - (2/3)^2 = 2/9$

$\hookrightarrow \sigma = \sqrt{2}/3$

(b) $X \sim Bin(12, 0.75) \rightarrow$ pmf: $f(x) = \sum_{k=0}^{12} \binom{12}{k} 0.75^k \, 0.25^{12-k}$

$\mu_X = np = 12 \times 0.75 = 9$ $\quad \sigma^2 = npq = 12(0.75)(0.25) = 2.25 \rightarrow \sigma = 1.5$

2.4.20

2.4-20. **(i)** Give the name of the distribution of X (if it has a name), **(ii)** find the values of μ and σ^2, and **(iii)** calculate $P(1 \leq X \leq 2)$ when the moment-generating function of X is given by

(a) $M(t) = (0.3 + 0.7e^t)^5$.

(b) $M(t) = \dfrac{0.3e^t}{1 - 0.7e^t}$, $\quad t < -\ln(0.7)$.

(c) $M(t) = 0.45 + 0.55e^t$.

(d) $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$.

(e) $M(t) = \sum_{x=1}^{10} (0.1)e^{tx}$.

(a) $X \sim Bin(5, 0.7) \rightarrow \mu_X = np = 5(0.7) = 3.5$, $\sigma^2 = npq = 5(0.7)(0.3) = 1.05$

$P(1 \leq X \leq 2) = \sum_{k=1}^{2} \binom{5}{k} 0.7^k \, 0.3^{5-k} = 0.1607$

(b) $X \sim Geom(0.3) \rightarrow \mu_X = \frac{1}{p} = \frac{1}{0.3} = \frac{10}{3}$, $\sigma^2 = \frac{p}{q^2} = \frac{0.3}{0.7^2} = 0.6122$

$P(1 \leq X \leq 2) = P(X=1) + P(X=2) = 0.3 + 0.7 \times 0.3 = 0.51$

(c) $X \sim Ber(0.55) \rightarrow \mu_X = p = 0.55$, $\sigma^2 = pq = 0.55(0.45) = 0.2475$

$P(1 \leq X \leq 2) = P(X=1) = p = 0.55$

(d) pmf:

X	1	2	3	4
f(x)	0.3	0.4	0.2	0.1

$\mu_X = 0.3 + 0.8 + 0.6 + 0.4 = 2.1$

$\sigma^2 = 0.3 + 1.6 + 1.8 + 1.6 - 2.1^2 = 0.89$

$P(1 \leq X \leq 2) = P(X=1) + P(X=2) = 0.3 + 0.4 = 0.7$

(e) $f(x) = 0.1 \rightarrow \mu_X = \sum_{x=1}^{10} 0.1x = 5.5$, $\sigma^2 = \sum_{x=1}^{10} 0.1x^2 - 5.5^2 = 8.25$

$P(1 \leq X \leq 2) = P(X=1) + P(X=2) = 0.1 + 0.1 = 0.2$

Section 2.5

1, 3, 5, 7, 9

2.5.1

2.5-1. In a lot (collection) of 100 light bulbs, there are five bad bulbs. An inspector inspects ten bulbs selected at random. Find the probability of finding at least one defective bulb. HINT: First compute the probability of finding no defectives in the sample.

$X \sim HG(5, 95, 10)$

$$P(X \geq 1) = 1 - P(X=0) = 1 - \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}} = 0.416$$

2.5.3

2.5-3. A professor gave her students six essay questions from which she will select three for a test. A student has time to study for only three of these questions. What is the probability that, of the questions studied,

(a) at least one is selected for the test?

(b) all three are selected?

(c) exactly two are selected?

$X \sim HG(3, 3, 3)$

(a) $$P(X \geq 1) = 1 - P(X=0) = 1 - \frac{\binom{3}{0}\binom{3}{3}}{\binom{6}{3}} = 1 - 0.05 = 0.95$$

(b) $$P(X=3) = \frac{\binom{3}{3}\binom{3}{0}}{\binom{6}{3}} = 0.05$$

(c) $$P(X=2) = \frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}} = 0.45$$

2.5.5

2.5-5. Five cards are selected at random without replacement from a standard, thoroughly shuffled 52-card deck of playing cards. Let X equal the number of face cards (kings, queens, jacks) in the hand. Forty observations of X yielded the following data:

2 1 2 1 0 0 1 0 1 1 0 2 0 2 3 0 1 1 0 3
1 2 0 2 0 2 0 1 0 1 1 2 1 0 1 1 2 1 1 0

(a) Argue that the pmf of X is

$$f(x) = \frac{\binom{12}{x}\binom{40}{5-x}}{\binom{52}{5}}, \qquad x = 0, 1, 2, 3, 4, 5,$$

and thus, that $f(0) = 2109/8330$, $f(1) = 703/1666$, $f(2) = 209/833$, $f(3) = 55/833$, $f(4) = 165/21{,}658$, and $f(5) = 33/108{,}290$.

(b) Draw a probability histogram for this distribution.

(c) Determine the relative frequencies of 0, 1, 2, 3, and superimpose the relative frequency histogram on your probability histogram.

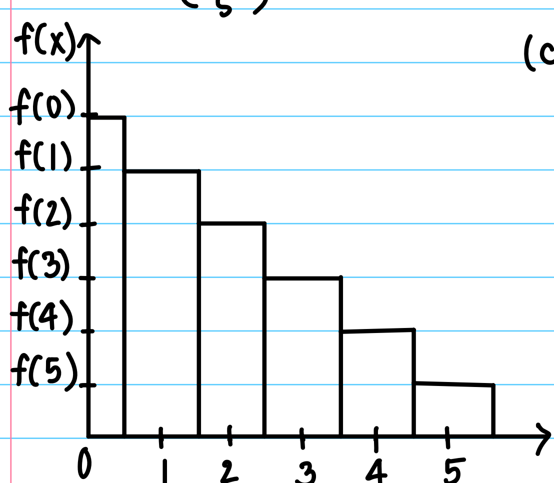
(a) $X \sim HG(\underbrace{12}_{\text{total face cards}}, \underbrace{40}_{\text{remaining cards}}, \underbrace{5}_{\text{sample size}})$

$$f(0) = \frac{\binom{12}{0}\binom{40}{5}}{\binom{52}{5}} = \frac{2109}{8330}, \quad f(1) = \frac{\binom{12}{1}\binom{40}{4}}{\binom{52}{5}} = \frac{703}{1666}$$

$$f(2) = \frac{\binom{12}{2}\binom{40}{3}}{\binom{52}{5}} = \frac{209}{833}, \quad f(3) = \frac{\binom{12}{3}\binom{40}{2}}{\binom{52}{5}} = \frac{55}{833}$$

$$f(4) = \frac{\binom{12}{4}\binom{40}{1}}{\binom{52}{5}} = \frac{165}{21658}, \quad f(5) = \frac{\binom{12}{5}\binom{40}{0}}{\binom{52}{5}} = \frac{33}{108290}$$

(b)



(c)

x	Frequency	Relative Frequency	f(x)
0	13	13/40 = 0.325	0.2532
1	16	16/40 = 0.4	0.4220
2	9	9/40 = 0.225	0.2509
3	2	2/40 = 0.05	0.0660
4	0	0	0.0076
5	0	0	0.0003

2.5-7. In the Michigan lottery game, LOTTO 47, the state selects six balls randomly out of 47 numbered balls. The player selects six different numbers out of the first 47 positive integers. Prizes are given for matching all six numbers (the jackpot), five numbers (\$2500), four numbers (\$100), and three numbers (\$5.00). A ticket costs \$1.00 and this dollar is not returned to the player. Find the probabilities of matching **(a)** six numbers, **(b)** five numbers, **(c)** four numbers, and **(d)** three numbers. **(e)** What is the expected value of the game to the player if the jackpot equals \$1,000,000? **(f)** What is the expected value of the game to the player if the jackpot equals \$2,000,000?

2.5.7

(a) $P(X=6)=\dfrac{\binom{6}{6}\binom{41}{0}}{\binom{47}{6}}=\dfrac{1}{10737573}$

(c) $P(X=4)=\dfrac{\binom{6}{4}\binom{41}{2}}{\binom{47}{6}}=\dfrac{4100}{3579191}$

(b) $P(X=5)=\dfrac{\binom{6}{5}\binom{41}{1}}{\binom{47}{6}}=\dfrac{82}{3579191}$

(d) $P(X=3)=\dfrac{\binom{6}{3}\binom{41}{3}}{\binom{47}{6}}=\dfrac{213200}{10737573}$

(e) $E[X]=\sum_{x=3}^{6} \underbrace{p}_{\text{prizes}} \times f(x)-1 = 5\times P(X=3)+100\times P(X=4)+2500\times P(X=5)+10^6\times P(X=6)-1$

$= -0.636$

(f) $E[X]=\sum_{x=3}^{6} \underbrace{p}_{\text{prizes}} \times f(x)-1 = 5\times P(X=3)+100\times P(X=4)+2500\times P(X=5)+2\times10^6\times P(X=6)-1$

$= -0.543$

2.5-9. Suppose there are three defective items in a lot (collection) of 50 items. A sample of size ten is taken at random and without replacement. Let X denote the number of defective items in the sample. Find the probability that the sample contains

(a) Exactly one defective item.

(b) At most one defective item.

2.5.9

$X\sim HG(3,47,10)$

(a) $P(X=1)=\dfrac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}}=\dfrac{39}{98}$

(b) $P(X\le 1)=P(X=0)+P(X=1)=\dfrac{\binom{3}{0}\binom{47}{10}}{\binom{50}{10}}+\dfrac{39}{98}=\dfrac{221}{245}$

Section 2.6 — 1, 3, 5, 7, 9

2.6.1

2.6-1. An excellent free-throw shooter attempts several free throws until she misses.

(a) If $p = 0.9$ is her probability of making a free throw, what is the probability of having the first miss on the 13th attempt or later?

(b) If she continues shooting until she misses three, what is the probability that the third miss occurs on the 30th attempt?

(a) $X \sim Geom(0.9) \longrightarrow P(X \geq 13) = P(X > 12) = 0.9^{12} = 0.2824$

(b) $X \sim Negbin(3, 0.1) \longrightarrow P(X = 30) = \binom{30-1}{3-1} 0.1^3 \, 0.9^{30-3} = 0.0236$

2.6.3

2.6-3. Suppose that a basketball player different from the ones in Example 2.6-2 and in Exercise 2.6-1 can make a free throw 60% of the time. Let X equal the minimum number of free throws that this player must attempt to make a total of ten shots.

(a) Give the mean, variance, and standard deviation of X.

(b) Find $P(X = 16)$.

$P(\text{free throw}) = 0.6 \qquad X \sim Negbin(10, 0.6)$

(a) $\mu_X = \frac{r}{p} = \frac{10}{0.6} = \frac{50}{3}, \quad \sigma^2 = \frac{rq}{p^2} = \frac{10(0.4)}{0.6^2} = \frac{100}{9} \longrightarrow \sigma = \frac{10}{3}$

(b) $P(X = 16) = \binom{16-1}{10-1} 0.6^{10} \, 0.4^{16-10} = 0.124$

2.6.5

2.6-5. Let the moment-generating function $M(t)$ of X exist for $-h < t < h$. Consider the function $R(t) = \ln M(t)$. The first two derivatives of $R(t)$ are, respectively,

$$R'(t) = \frac{M'(t)}{M(t)} \quad \text{and} \quad R''(t) = \frac{M(t)M''(t) - [M'(t)]^2}{[M(t)]^2}.$$

Setting $t = 0$, show that

(a) $\mu = R'(0)$.

(b) $\sigma^2 = R''(0)$.

(a) $R'(0) = \frac{M'(0)}{M(0)} = \frac{\mu_X}{1} = \mu_X$ Q.E.D!

(b) $R''(0) = \frac{M(0)M''(0) - M'(0)^2}{M(0)^2} = \frac{1 \times M''(0) - \mu_X^2}{1^2} = \sigma_X^2 - \mu_X^2 = E[X^2] - 2\mu_X^2$

$= E[X^2] - 2E[X]\mu_X + \mu_X^2 - \mu_X^2$

$= E[X^2 - 2X\mu_X + \mu_X^2] - \mu_X^2 = E[(X-\mu_X)^2] - \mu_X^2 = \sigma_X^2$

2.6.7

2.6-7. If $E(X^r) = 5^r$, $r = 1, 2, 3, \ldots$, find the moment-generating function $M(t)$ of X and the pmf of X.

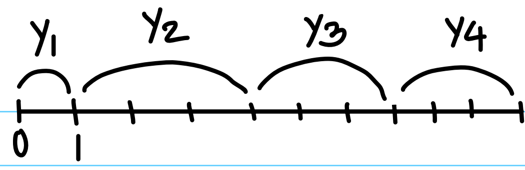
$M_X(t) = E[e^{tX}] = E\left[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right] = \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{5^k t^k}{k!} = e^{5t}$

$\therefore$ pmf:

X	5
f_X	1

$\boxed{2.6.9}$

2.6-9. One of four different prizes was randomly put into each box of a cereal. If a family decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?



Times $T = Y_1 + Y_2 + Y_3 + Y_4$

$Y_1 \to 1$, $Y_2 \to \text{Geom}\left(\frac{3}{4}\right)$, $Y_3 \to \text{Geom}\left(\frac{1}{2}\right)$, $Y_4 \to \text{Geom}\left(\frac{1}{4}\right)$

$$E[T] = E[Y_1] + E[Y_2] + E[Y_3] + E[Y_4] = 1 + \frac{1}{3/4} + \frac{1}{1/2} + \frac{1}{1/4}$$

$$= \frac{4}{4} + \frac{4}{3} + \frac{4}{2} + \frac{4}{1} = 4\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{3}$$

General n $\rightarrow$ $n\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) \approx n \ln n$

Section 2.7 1, 3, 5, 7, 9

2.7.1

2.7-1. Let X have a Poisson distribution with a mean of 4. Find
(a) $P(3 \le X \le 5)$.
(b) $P(X \ge 3)$.
(c) $P(X \le 3)$.

$\mu_X = \lambda = 4 \rightarrow X \sim \text{Poisson}(4)$

(a) $P(3 \le X \le 5) = \sum_{k=3}^{5} \frac{4^k}{k!} e^{-4} = 0.547$

(b) $P(X \geq 3) = 1 - P(X \le 2) = 1 - \sum_{k=0}^{2} \frac{4^k}{k!} e^{-4} = 0.762$

(c) $P(X \le 3) = \sum_{k=0}^{3} \frac{4^k}{k!} e^{-4} = 0.433$

2.7.3

2.7-3. Customers arrive at a travel agency at a mean rate of 11 per hour. Assuming that the number of arrivals per hour has a Poisson distribution, give the probability that more than ten customers arrive in a given hour.

$\mu_X = 11 = \lambda \rightarrow X \sim \text{Poisson}(11)$ $N_{[0,1]} \sim \text{Poisson}(\lambda \underset{=1}{t})$
$= \text{Poisson}(11)$

$P(X > 10) = 1 - P(X \le 10) = 1 - \sum_{k=0}^{10} \frac{11^k}{k!} e^{-11} = 0.540$ $\hookrightarrow P(N_{[0,1]} > 10)$

2.7.5

2.7-5. Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume a Poisson distribution, find the probability of at most one flaw appearing in 225 square feet.

$\mu_X = \lambda = \frac{1}{150} \times 225 = 1.5 \rightarrow X \sim \text{Poisson}(1.5)$

$P(X \le 1) = \sum_{k=0}^{1} \frac{1.5^k}{k!} e^{-1.5} = 0.558$

2.7.7

2.7-7. With probability 0.001, a prize of \$499 is won in the Michigan Daily Lottery when a \$1 straight bet is placed. Let Y equal the number of \$499 prizes won by a gambler after placing n straight bets. Note that Y is $b(n, 0.001)$. After placing $n = 2000$ \$1 bets, the gambler is behind or even if $\{Y \le 4\}$. Use the Poisson distribution to approximate $P(Y \le 4)$ when $n = 2000$.

$Y \sim \text{Bin}(2000, 0.001)$
$\rightarrow \mu_Y = \lambda = np = 2000 \times 0.001 = 2$
$\therefore Y \sim \text{Poisson}(2)$

$P(Y \le 4) = \sum_{k=0}^{4} \frac{2^k}{k!} e^{-2} = 0.947$

2.7.9

2.7-9. A store selling newspapers orders only $n = 4$ of a certain newspaper because the manager does not get many calls for that publication. If the number of requests per day follows a Poisson distribution with mean 3,
(a) What is the expected value of the number sold?
(b) How many should the manager order so that the chance of having more requests than available newspapers is less than 0.05?

$X = \#\text{requests/day} \rightarrow X \sim \text{Poisson}(3)$
$Y = \#\text{sold}$

(a) $\mu_Y = \sum_{k=0}^{4} k \cdot f_X(k) = \sum_{k=0}^{3} k \cdot \frac{3^k}{k!} e^{-3} + 4P(X \geq 4)$

$= 1.2696 + 4(1 - P(X \le 3)) = 1.2696 + 4(1 - 0.6472) = 2.681$

(b) $P(X > n) < 0.05$

$$P(X>n) = 1 - P(X \le n) = 1 - \sum_{k=0}^{n} \frac{3^k}{k!} e^{-3} < 0.05 \longrightarrow \sum_{k=0}^{n} \frac{3^k}{k!} e^{-3} > 0.95$$

Test: $P(X \le 4) = \sum_{k=0}^{4} \frac{3^k}{k!} e^{-3} = 0.815$

$$P(X \le 5) = \sum_{k=0}^{5} \frac{3^k}{k!} e^{-3} = 0.916$$

$$P(X \le 6) = \sum_{k=0}^{6} \frac{3^k}{k!} e^{-3} = 0.966 > 0.95 \longrightarrow n = 6$$

correction

$X \sim$ Poisson(3)

$Y = \min(X, 4) = X \wedge 4$

$$(X \wedge 4) = \begin{cases} k \text{ if } X = k, & k \le 4 \\ 4 \text{ if } X = k, & k \ge 5 \end{cases}$$

(a) $E[X \wedge 4] = \sum_{k=0}^{4} k f_X(x) + \sum_{k=5}^{\infty} 4 f_X(x)$

$$= 0 \cdot \frac{3^0}{0!} e^{-3} + 1 \cdot \frac{3^1}{1!} e^{-3} + 2 \cdot \frac{3^2}{2!} e^{-3} + 3 \cdot \frac{3^3}{3!} e^{-3} + 4P(X \ge 5)$$

$P(X \ge 5) = 1 - P(X \le 4) = 1 - 0.815 = 0.185$

table

(b) $P(X > n) = 0.05 \longrightarrow P(X \le n) = 0.95 \longrightarrow$ table

$\boxed{\text{Section 3.1}}$ 1, 3, 5, 7, 9, 11, 13, 21

$\boxed{3.1.1}$ **3.1-1.** Show that the mean, variance, and mgf of the uniform distribution are as given in this section. Also verify that the index of skewness is equal to zero.

$$X \sim \text{Unif}[a,b] = \begin{cases} 0 & , x \notin [a,b] \\ \frac{1}{b-a} & , x \in [a,b] \end{cases}$$

$$\mu_X = E[X] = \int_a^b x.f_X(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{2(b-a)} x^2 \Big|_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2.f_X(x)\,dx = \int_a^b \frac{x^2}{b-a}\,dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$\hookrightarrow \sigma_X^2 = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

$$M_X(t) = E[e^{tX}] = \int_a^b e^{tx}.f_X(x)\,dx = \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{e^{tx}}{t(b-a)}\Big|_a^b = \frac{e^{tb}-e^{ta}}{t(b-a)}$$

$(t \neq 0)$

$$E[X^3] = \int_a^b x^3.f_X(x)\,dx = \int_a^b \frac{x^3}{b-a}\,dx = \frac{x^4}{4(b-a)}\Big|_a^b = \frac{b^4-a^4}{4(b-a)}$$

$$= \frac{(b-a)(b+a)(b^2+a^2)}{4(b-a)} = \frac{(b+a)(b^2+a^2)}{4}$$

$$\hookrightarrow E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= \frac{(b+a)(b^2+a^2)}{4} - 3 \times \frac{a+b}{2} \times \frac{b^2+ab+a^2}{3} + 2 \times \left(\frac{a+b}{2}\right)^3$$

$$= \frac{a+b}{2}\left(\frac{a^2+b^2}{2} - b^2 - ab - a^2 + \frac{a^2+2ab+b^2}{2}\right)$$

$$= \frac{a+b}{2}\left(\frac{2a^2+2ab+2b^2}{2} - b^2 - ab - a^2\right) = \frac{a+b}{2} \times 0 = 0 \longrightarrow \gamma = 0$$

3.1-3. Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular ten-minute period, let X equal the time within the ten minutes that the customer arrived. If X is $U(0, 10)$, find

(a) The pdf of X.

(b) $P(X \geq 8)$.

(c) $P(2 \leq X < 8)$.

(d) $E(X)$.

(e) $\text{Var}(X)$.

3.1.3

(a) $X \sim U(0,10) \longrightarrow$ pdf: $f(x) = \frac{1}{10}, \; 0 < x < 10$

(b) $P(X \geq 8) = \int_8^{10} \frac{1}{10}\,dx = \frac{1}{10}x \Big|_8^{10} = \frac{10-8}{10} = \frac{2}{10} = \frac{1}{5}$

(c) $P(2 \leq X < 8) = 1 - P(X \geq 8) = 1 - \frac{1}{5} = \frac{4}{5}$

(d) $\mu_X = \frac{a+b}{2} = \frac{10+0}{2} = 5$ (e) $\sigma^2 = \frac{(a-b)^2}{12} = \frac{10^2}{12} = \frac{25}{3}$

3.1-5. Let Y have a uniform distribution $U(0, 1)$, and let

$$W = a + (b-a)Y, \qquad a < b.$$

(a) Find the cdf of W.

HINT: Find $P[a + (b-a)Y \leq w]$.

(b) How is W distributed?

3.1.5

$Y \sim U(0,1) \longrightarrow$ pdf: $f_Y(y) = 1, \; 0 < y < 1$

(a) $F_W(w) = P(W \leq w) = P(a + (b-a)Y \leq w) = P\left(Y \leq \frac{w-a}{b-a}\right)$

$\therefore$ cdf: $G(w) = \frac{w-a}{b-a}, \; a \leq w \leq b$

(b) pdf: $g(w) = \frac{1}{b-a}, \; w \in (a,b) \longrightarrow W \sim U(a,b)$

3.1-7. For each of the following functions, **(i)** find the constant c so that $f(x)$ is a pdf of a random variable X; **(ii)** find the cdf, $F(x) = P(X \leq x)$; **(iii)** sketch graphs of the pdf $f(x)$ and the cdf $F(x)$; and **(iv)** find μ, σ^2, and the index of skewness, γ:

(a) $f(x) = 4x^c, \quad 0 \leq x \leq 1$.

(b) $f(x) = c\sqrt{x}, \quad 0 \leq x \leq 4$.

(c) $f(x) = c/x^{3/4}, \quad 0 < x < 1$.

3.1.7

(a) $\int_0^1 f(x)\,dx = \int_0^1 4x^c\,dx = \frac{4x^c}{c+1}\Big|_0^1 = \frac{4}{c+1}(1-0) = \frac{4}{c+1} = 1 \longrightarrow c = 3$

$\therefore$ pdf: $f(x) = 4x^3 \longrightarrow$ cdf: $F(x) = \int_0^x f(x)\,dx = \int_0^x 4x^3\,dx = x^4, \; 0 \leq x \leq 1$

$\mu_X = \int_0^1 x f(x)\,dx = \int_0^1 x(4x^3)\,dx = \int_0^1 4x^4\,dx = \frac{4x^5}{5}\Big|_0^1 = \frac{4}{5}$

$E[X^2] = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2(4x^3)\,dx = \int_0^1 4x^5\,dx = \frac{4x^6}{6}\Big|_0^1 = \frac{2}{3}$

$\hookrightarrow \sigma_X^2 = E[X^2] - \mu_X^2 = \frac{2}{3} - \frac{16}{25} = \frac{2}{75}$

$$E[X^3] = \int_0^1 x^3 f(x)dx = \int_0^1 x^3(4x^3)dx = \int_0^1 4x^6 dx = \frac{4x^7}{7}\Big|_0^1 = \frac{4}{7}$$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3 = \frac{4}{7} - 3\left(\frac{4}{5}\right)\left(\frac{2}{3}\right) + 2\left(\frac{4}{5}\right)^3 = \frac{-4}{875}$$

$$\therefore \gamma = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{-4}{875} \times \left(\frac{2}{75}\right)^{-3/2} = \frac{-3\sqrt{6}}{7}$$

(b) $$\int_0^4 f(x)dx = \int_0^4 c\sqrt{x}\,dx = \int_0^4 cx^{1/2}dx = \frac{2}{3}cx^{3/2}\Big|_0^4 = \frac{2}{3}c(8) = \frac{16c}{3} = 1 \rightarrow c = \frac{3}{16}$$

$$\therefore \text{pdf}: f(x) = \frac{3\sqrt{x}}{16} \rightarrow F(x) = \int_0^x f(x)dx = \frac{3}{16}\left(\frac{2}{3}\right)x^{3/2}\Big|_0^x = \frac{x^{3/2}}{8},\ 0 \le x \le 4$$

$$\mu_X = \int_0^4 xf(x)dx = \int_0^4 \frac{3}{16}x\sqrt{x}\,dx = \int_0^4 \frac{3}{16}x^{3/2}dx = \frac{3}{16}\left(\frac{2}{5}\right)x^{5/2}\Big|_0^4 = \frac{12}{5}$$

$$E[X^2] = \int_0^4 x^2 f(x)dx = \int_0^4 \frac{3}{16}x^{5/2}dx = \frac{3}{16}\left(\frac{2}{7}\right)x^{7/2}\Big|_0^4 = \frac{48}{7}$$

$$\hookrightarrow \sigma_X^2 = E[X^2] - \mu_X^2 = \frac{48}{7} - \left(\frac{12}{5}\right)^2 = \frac{192}{175}$$

$$E[X^3] = \int_0^4 x^3 f(x)dx = \int_0^4 \frac{3}{16}x^{7/2}dx = \frac{3}{16}\left(\frac{2}{9}\right)x^{9/2}\Big|_0^4 = \frac{64}{3}$$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3 = \frac{64}{3} - 3\left(\frac{12}{5}\right)\left(\frac{192}{175}\right) + 2\left(\frac{12}{5}\right)^3 = \frac{21568}{525}$$

$$\therefore \gamma = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{5696}{175} \times \left(\frac{192}{175}\right)^{-3/2} = \frac{-2\sqrt{21}}{27}$$

(c) $$\int_0^1 f(x)dx = \int_0^1 \frac{c}{x^{3/4}}dx = \int_0^1 cx^{-3/4}dx = c(4)x^{1/4}\Big|_0^1 = 4c = 1 \rightarrow c = \frac{1}{4}$$

$$\therefore \text{pdf}: f(x) = \frac{1}{4}x^{-3/4} \rightarrow \text{cdf}: F(x) = \int_0^x f(x)dx = \int_0^x \frac{1}{4}x^{-3/4}dx = \sqrt[4]{x},\ 0 \le x \le 1$$

$$\mu_X = \int_0^1 x f(x)\,dx = \int_0^1 \frac{1}{4} x^{1/4}\,dx = \frac{1}{4}\left(\frac{4}{5}\right) x^{5/4}\Big|_0^1 = \frac{1}{5}$$

$$E[X^2] = \int_0^1 x^2 f(x)\,dx = \int_0^1 \frac{1}{4} x^{5/4}\,dx = \frac{1}{4}\left(\frac{4}{9}\right) x^{9/4}\Big|_0^1 = \frac{1}{9}$$

$$\hookrightarrow \sigma_X^2 = E[X^2] - \mu_X^2 = \frac{1}{9} - \left(\frac{1}{5}\right)^2 = \frac{16}{225}$$

$$E[X^3] = \int_0^1 x^3 f(x)\,dx = \int_0^1 \frac{1}{4} x^{9/4}\,dx = \frac{1}{4}\left(\frac{4}{13}\right) x^{13/4}\Big|_0^1 = \frac{1}{13}$$

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3 = \frac{1}{13} - 3\left(\frac{1}{5}\right)\left(\frac{1}{9}\right) + 2\left(\frac{1}{5}\right)^3 = \frac{128}{4875}$$

$$\therefore \gamma = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{128}{4875} \times \left(\frac{4}{15}\right)^{-3} = \frac{18}{13}$$

$\boxed{3.1.9}$

3.1-9. Let the random variable X have the pdf $f(x) = 2(1-x)$, $0 \le x \le 1$, zero elsewhere.

(a) Sketch the graph of this pdf.

(b) Determine and sketch the graph of the cdf of X.

(c) Find **(i)** $P(0 \le X \le 1/2)$, **(ii)** $P(1/4 \le X \le 3/4)$, **(iii)** $P(X = 3/4)$, and **(iv)** $P(X \ge 3/4)$.

(b) $$F(x) = \int_0^x f(x)\,dx = \int_0^x 2(1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^x = \begin{cases} 0 & , x < 0 \\ 2x - x^2, & 0 \le x < 1 \\ 1 & , x \geq 1 \end{cases}$$

(c) $$P\left(0 \le X \le \frac{1}{2}\right) = \int_0^{1/2} 2(1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^{1/2} = 2\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}$$

$$P\left(\frac{1}{4} \le X \le \frac{3}{4}\right) = \int_{1/4}^{3/4} 2(1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_{1/4}^{3/4} = 2\left(\frac{3}{4} - \frac{3}{32} - \frac{1}{4} + \frac{1}{32}\right) = \frac{1}{2}$$

$$P\left(X = \frac{3}{4}\right) = \int_{3/4}^{3/4} 2(1-x)\,dx = 0 \qquad P\left(X \geq \frac{3}{4}\right) = \int_{3/4}^{1} 2(1-x)\,dx = \frac{1}{16}$$

$\boxed{3.1.11}$

3.1-11. The pdf of Y is $g(y) = c/y^3$, $1 < y < \infty$.

(a) Calculate the value of c so that $g(y)$ is a pdf.

(b) Find $E(Y)$.

(c) Show that $\mathrm{Var}(Y)$ is not finite.

(a) $$\int_1^\infty \frac{c}{y^3}\,dy = \lim_{t\to\infty} \int_1^t c y^{-3}\,dy = \lim_{t\to\infty} \frac{-c}{2} y^{-2}\Big|_1^t = \lim_{t\to\infty} \frac{-c}{2}\left(t^{-2} \xrightarrow{} 0 \; - 1\right) = \frac{c}{2} = 1 \rightarrow c = 2$$

(b) $\mu_Y = \int_1^\infty y \frac{2}{y^3} dy = \lim_{t\to\infty} \int_1^t 2y^{-2} dy = \lim_{t\to\infty} -2y^{-1}\Big|_1^t = \lim_{t\to\infty} -2(\cancel{\tfrac{1}{t}}^{\,0} - 1) = 2$

(c) $E[X^2] = \int_1^\infty y^2 \frac{2}{y^3} dy = \lim_{t\to\infty} \int_1^t \frac{2}{y} dy = \lim_{t\to\infty} 2\ln|y|\Big|_1^t = 2\ln t \to \infty \therefore \sigma^2 = \infty$

$\boxed{3.1.13}$

3.1-13. The logistic distribution is associated with the cdf $F(x) = (1 + e^{-x})^{-1}$, $-\infty < x < \infty$. Find the pdf of the logistic distribution and show that its graph is symmetric about $x = 0$.

$$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}(1+e^{-x})^{-1} = -(1+e^{-x})'(1+e^{-x})^{-2} = -(-1)e^{-x}(1+e^{-x})^{-2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} \times \frac{e^{2x}}{e^{2x}} = \frac{e^x}{(e^x+1)^2} = f(-x) \to \text{symmetric}$$

$\boxed{3.1.21}$

3.1-21. Let $X_1, X_2, \ldots, X_k$ be random variables of the continuous type, and let $f_1(x), f_2(x), \ldots, f_k(x)$ be their corresponding pdfs, each with sample space $S = (-\infty, \infty)$. Also, let $c_1, c_2, \ldots, c_k$ be nonnegative constants such that $\sum_{i=1}^k c_i = 1$.

(a) Show that $\sum_{i=1}^k c_i f_i(x)$ is a pdf of a continuous-type random variable on S.

(b) If X is a continuous-type random variable with pdf $\sum_{i=1}^k c_i f_i(x)$ on S, $E(X_i) = \mu_i$, and $\mathrm{Var}(X_i) = \sigma_i^2$ for $i = 1, \ldots, k$, find the mean and the variance of X.

(a) $$\int_{-\infty}^{\infty} \sum_{i=1}^{K} c_i f_i(x)dx = \underbrace{\sum_{i=1}^{K} c_i}_{1} \underbrace{\int_{-\infty}^{\infty} f_i(x)dx}_{\text{pdfs}} = 1 \times 1 = 1$$ Q.E.D!

(b) $$\mu_X = \int_{-\infty}^{\infty} \sum_{i=1}^{K} x_i c_i f_i(x)dx = \sum_{i=1}^{K} c_i \int_{-\infty}^{\infty} x_i f_i(x)dx = \sum_{i=1}^{K} c_i \mu_i$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \int_{-\infty}^{\infty} \sum_{i=1}^{K} x_i^2 c_i f_i(x)dx - \mu_X^2 = \sum_{i=1}^{K} c_i \int_{-\infty}^{\infty} x_i^2 f_i(x)dx - \mu_X^2$$
$$= \sum_{i=1}^{K} c_i(\sigma_i^2 + \mu_i^2) - \mu_X^2$$

correction

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \sum_{i=1}^{K} c_i E[X_i^2] - \left(\sum_{i=1}^{K} c_i \mu_{X_i}\right)^2$$

$$\int_{-\infty}^{\infty} x^2 \sum_{i=1}^{K} c_i f_i(x)dx = \sum_{i=1}^{K} c_i E[X_i^2]$$

Section 3.2   1, 3, 5, 7, 9, 11, 13, 15, 17

3.2.1

3.2-1. What are the pdf, the mean, and the variance of X if the moment-generating function of X is given by the following?

(a) $M(t) = \frac{1}{1-3t}$, $t < 1/3$.

(b) $M(t) = \frac{3}{3-t}$, $t < 3$.

(a) $M_X(t) = \frac{1}{1-3t} = \frac{1/3}{1/3 - t}, \ t < \frac{1}{3} \rightarrow X \sim \text{Exp}(\frac{1}{3})$

pdf: $f_X(x) = \frac{1}{3} e^{-x/3}, \ x \geq 0 \rightarrow \mu_X = \frac{1}{\lambda} = \frac{1}{1/3} = 3, \ \sigma_X^2 = \frac{1}{\lambda^2} = \frac{1}{(1/3)^2} = 9$

(b) $M_X(t) = \frac{3}{3-t}, \ t < 3 \rightarrow X \sim \text{Exp}(3)$

pdf: $f_X(x) = 3e^{-3x}, \ x \geq 0 \rightarrow \mu_X = \frac{1}{\lambda} = \frac{1}{3}, \ \sigma_X^2 = \frac{1}{\lambda^2} = \frac{1}{9}$

3.2.15

3.2-15. Let the distribution of X be $\chi^2(r)$.

(a) Find the point at which the pdf of X attains its maximum when $r \geq 2$. This is the mode of a $\chi^2(r)$ distribution.

(b) Find the points of inflection for the pdf of X when $r \geq 4$.

(c) Use the results of parts (a) and (b) to sketch the pdf of X when $r = 4$ and when $r = 10$.

(a) $f(x) = \frac{x^{r/2-1}}{\Gamma(\frac{r}{2}) 2^{r/2}} e^{-x/2}, \ x > 0$

$f'(x) = \frac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} \left[\left(\frac{r}{2} - 1\right) x^{r/2-2} e^{-x/2} + x^{r/2-1}\left(-\frac{1}{2} e^{-x/2}\right) \right] = 0$

$\therefore x^{r/2-2}\left(\frac{r}{2} - 1 - \frac{x}{2}\right) e^{-x/2} = 0 \quad \therefore \frac{r}{2} - 1 - \frac{x}{2} = 0 \quad \therefore x = r - 2$

(b) $f''(x) = \left(\frac{r}{2} - 2\right) x^{r/2-3} \left(\frac{r}{2} - 1 - \frac{x}{2}\right) e^{-x/2} + x^{r/2-2}\left(\frac{-1}{2}\right) e^{-x/2}$

$+ x^{r/2-2}\left(\frac{r}{2} - 1 - \frac{x}{2}\right) e^{-x/2}\left(-\frac{1}{2}\right) = 0$

$\therefore x^{r/2-3}\left[\left(\frac{r}{2} - 2\right)\left(\frac{r}{2} - 1 - \frac{x}{2}\right) + \frac{-x}{2} + x\left(\frac{r}{2} - 1 - \frac{x}{2}\right)\left(\frac{-1}{2}\right)\right] = 0$

$\therefore (r-4)(r-2-x) - 2x - x(r-2-x) = 0$

$(r-4)(r-2) - (r-4)x - 2x - (r-2)x + x^2 = 0$

$(r-4)(r-2) - x(r-4+2+r-2) + x^2 = 0$

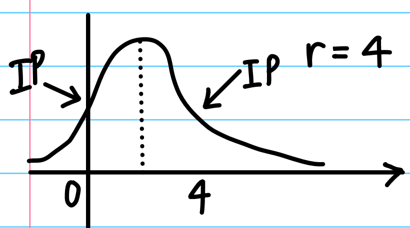
$$x^2-(2r-4)x+(r-4)(r-2)=0 \longrightarrow x^2-2(r-2)x+(r-4)(r-2)=0$$

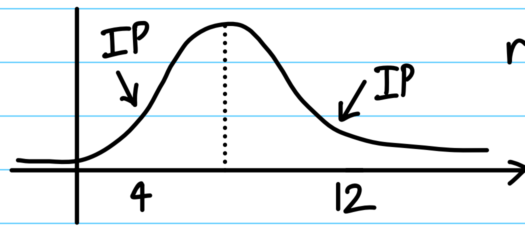
$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2(r-2)\pm\sqrt{4(r-2)^2-4(r-4)(r-2)}}{2}$$

$$=r-2\pm\sqrt{(r-2)(r-2-r+4)}=r-2\pm\sqrt{2(r-2)}$$

(c) $r=4 \longrightarrow \text{IP}: \ x=4-2\pm\sqrt{2(4-2)}=2\pm2=0,4$

$r=10 \longrightarrow \text{IP}: \ x=10-2\pm\sqrt{2(10-2)}=8\pm\sqrt{2(8)}=8\pm4=4,12$





3.2.7

3.2-7. Find the moment-generating function for the gamma distribution with parameters α and θ.

HINT: In the integral representing $E(e^{tX})$, change variables by letting $y=(1-\theta t)x/\theta$, where $1-\theta t>0$.

$X \sim \text{Gamma}(\alpha,\theta)$

$$M_X(t)=E[e^{tX}]=\int_0^\infty e^{tx}\frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha}e^{-x/\theta}dx=\left(\frac{\theta'}{\theta}\right)^\alpha\int_0^\infty\frac{x^{\alpha-1}}{\Gamma(\alpha)(\theta')^\alpha}e^{-\left(\frac{1}{\theta}-t\right)}dx$$

$$=\left(\frac{\theta'}{\theta}\right)^\alpha\cdot 1=\left(\frac{1}{\left(\frac{1}{\theta}-t\right)\theta}\right)^\alpha=\frac{1}{(1-\theta t)^\alpha},\ t<\frac{1}{\theta}$$

3.2.5

3.2-5. There are times when a shifted exponential model is appropriate. That is, let the pdf of X be

$$f(x)=\frac{1}{\theta}e^{-(x-\delta)/\theta},\qquad \delta<x<\infty.$$

(a) Define the cdf of X.

(b) Calculate the mean and variance of X.

(a) $$F(x)=\int_{-\infty}^x f(u)\,du=\begin{cases}\int_\delta^x \frac{1}{\theta}e^{-\frac{u-\delta}{\theta}}du, & x>\delta\\ 0, & x\le\delta\end{cases}$$

$w=\frac{u-\delta}{\theta}\qquad dw=\frac{du}{\theta}$

$$=\int_0^{\frac{x-\delta}{\theta}}e^{-w}dw=-e^{-w}\Big|_0^{\frac{x-\delta}{\theta}}=\begin{cases}1-e^{-\frac{x-\delta}{\theta}}, & x>\delta\\ 0, & x\le\delta\end{cases}$$

exp dist.

(b) $\frac{X-\delta}{\theta}\sim \text{Exp}(1)=Y$

$$E\left[\frac{X-\delta}{\theta}\right]=E[Y]=1\longrightarrow\frac{E[X]-\delta}{\theta}=1\longrightarrow E[X]=\theta+\delta$$

OR $\frac{X-\delta}{\theta} = Y \sim Exp(1)$

$\therefore X = \delta + \theta Y \longrightarrow \mu_X = \delta + \theta\mu_Y = \delta + \theta$

$\sigma_X^2 = \theta^2 \sigma_Y^2 = \theta^2$

3.2.11

3.2-11. Let X have a gamma distribution with parameters α and θ, where $\alpha > 1$. Find $E(1/X)$.

$X \sim Gamma(\alpha, \theta)$

$$E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} dx = \int_0^\infty \frac{x^{\alpha-2}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} dx$$

$$= \frac{\Gamma(\alpha-1)\theta^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha} \underbrace{\int_0^\infty \frac{x^{\alpha-2}e^{-x/\theta}}{\Gamma(\alpha-1)\theta^{\alpha-1}}}_{Gamma(\alpha-1,\theta)} dx = \frac{\Gamma(\alpha-1)\theta^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)\theta^\alpha} = \frac{1}{(\alpha-1)\theta}, \alpha > 1$$

3.2.3

3.2-3. Let X have an exponential distribution with mean $\theta > 0$. Show that

$$P(X > x + y \mid X > x) = P(X > y).$$

$X \sim Exp(\theta) \rightarrow f(x) = \frac{1}{\theta}e^{-x/\theta}, x \geq 0$

$$P(X > x) = 1 - P(X \leq x) = 1 - \int_0^x \frac{1}{\theta}e^{-x/\theta} dx = 1 + e^{-x/\theta}\Big|_0^x = 1 + e^{-x/\theta} - 1 = e^{-x/\theta}$$

$$P(X > x+y \mid X > x) = \frac{P(X > x+y \cap X > x)}{P(X > x)} = \frac{P(X > x+y)}{P(X > x)} = \frac{e^{-\frac{x+y}{\theta}}}{e^{-x/\theta}} = e^{-y/\theta} = P(X > y)$$

3.2.9

3.2-9. If the moment-generating function of a random variable W is

$$M(t) = (1 - 7t)^{-20}, \qquad t < 1/7,$$

find the pdf, mean, and variance of W.

$$M_X(t) = \frac{1}{(1-7t)^{20}} = \frac{1}{(1-\theta t)^\alpha} \longrightarrow X \sim Gamma(20, 7) \qquad \alpha = 20, \theta = 7$$

$$\text{pdf: } f(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} = \frac{x^{20-1}e^{-x/7}}{\Gamma(20)7^{20}} = \frac{x^{19}e^{-x/7}}{7^{20} \cdot 19!}$$

$\mu_X = \alpha\theta = 20(7) = 140 \qquad \sigma_X^2 = \alpha\theta^2 = 20(7^2) = 980$

3.2-13. If X is $\chi^2(23)$, find the following:

(a) $P(14.85 < X < 32.01)$.

(b) Constants a and b such that $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$.

(c) The mean and variance of X.

(d) $\chi^2_{0.05}(23)$ and $\chi^2_{0.95}(23)$.

3.2.13

$X \sim \chi^2(23) \longrightarrow r = 23$

(a) $P(14.85 < X < 32.01) = 0.9 - 0.1 = 0.8$

(b) $P(a < X < b) = 0.95 \longrightarrow \chi^2_{0.975}(23) = 38.08 = b \quad \chi^2_{0.025}(23) = 11.69 = a$

(c) $\mu_X = r = 23 \quad \sigma_X^2 = 2r = 46$ (d) $\chi^2_{0.95}(23) = 35.17 \quad \chi^2_{0.05}(23) = 13.09$

3.2-17. If 15 observations are taken independently from a chi-square distribution with four degrees of freedom, find the probability that at most three of the 15 observations exceed 7.779.

3.2.17

$X \sim \chi^2(4) \rightarrow P(X > 7.779) = 1 - P(X \leq 7.779) = 1 - 0.9 = 0.1$

S_n = # trials exceeding 7.779

$$P(S_n \leq 3) = \sum_{k=0}^{3} \binom{15}{k} p^k q^{15-k} = \sum_{k=0}^{3} \binom{15}{k} 0.1^k \, 0.9^{15-k} = 0.9444$$

Section 3.3

1, 3, 5, 7, 9, 11, 13, 15, 17

3.3.1

3.3-1. If Z is $N(0, 1)$, find

(a) $P(0.47 < Z \le 2.13)$. **(b)** $P(-0.97 \le Z < 1.27)$.
(c) $P(Z > -1.56)$. **(d)** $P(Z > 2.78)$.
(e) $P(|Z| < 1.96)$. **(f)** $P(|Z| < 1)$.
(g) $P(|Z| < 2)$. **(h)** $P(|Z| < 3)$.

(a) $P(0.47 < Z \le 2.13) = P(Z \le 2.13) - P(Z \le 0.47) = 0.9834 - 0.6808 = 0.3026$

(b) $P(-0.97 \le Z < 1.27) = P(Z < 1.27) - P(Z \le -0.97)$
$= P(Z < 1.27) - P(Z \geq 0.97) = 0.898 - 0.166 = 0.732$

(c) $P(Z > -1.56) = P(Z < 1.56) = 0.9406$ (d) $P(Z > 2.78) = 0.0027$

(e) $P(|Z| < 1.96) = P(-1.96 < Z < 1.96) = P(Z < 1.96) - P(Z \le -1.96)$
$= P(Z < 1.96) - P(Z \geq 1.96)$
$= 0.975 - 0.025 = 0.95$

(f) $P(|Z| < 1) = P(-1 < Z < 1) = P(Z < 1) - P(Z \le -1)$
$= P(Z < 1) - P(Z \geq 1) = 0.8413 - 0.1587 = 0.6826$

(g) $P(|Z| < 2) = P(-2 < Z < 2) = P(Z < 2) - P(Z \le -2)$
$= P(Z < 2) - P(Z \geq 2) = 0.9772 - 0.0228 = 0.9544$

(h) $P(|Z| < 3) = P(-3 < Z < 3) = P(Z < 3) - P(Z \le -3)$
$= P(Z < 3) - P(Z \geq 3) = 0.9987 - 0.0013 = 0.9974$

3.3.3

3.3-3. If Z is $N(0, 1)$, find values of c such that

(a) $P(Z \ge c) = 0.025$. **(b)** $P(|Z| \le c) = 0.95$.
(c) $P(Z > c) = 0.05$. **(d)** $P(|Z| \le c) = 0.90$.

(a) $P(Z \geq c) = 0.025 \rightarrow c = 1.96$ (c) $P(Z > c) = 0.05 \rightarrow c = 1.645$

(b) $P(|Z| \le c) = P(-c \le Z \le c) = P(Z \le c) - P(Z < -c) = P(Z \le c) - P(Z > c)$
$= P(Z \le c) - 1 + P(Z \le c) = 2P(Z \le c) - 1 = 0.95$
$\therefore P(Z \le c) = 0.975 \rightarrow c = 1.96$

(d) $P(|Z| \le c) = 2P(Z \le c) - 1 = 0.9 \rightarrow P(Z \le c) = 0.95 \rightarrow c = 1.645$

3.3.5

3.3-5. If X is normally distributed with a mean of 6 and a variance of 25, find

(a) $P(6 \le X \le 14)$. **(b)** $P(4 \le X \le 14)$.
(c) $P(-4 < X \le 0)$. **(d)** $P(X > 15)$.
(e) $P(|X-6| < 5)$. **(f)** $P(|X-6| < 10)$.
(g) $P(|X-6| < 15)$. **(h)** $P(|X-6| < 12.4)$.

$X \sim N(6,25) \longrightarrow Z = \frac{X-\mu}{\sigma} = \frac{X-6}{5} \sim N(0,1)$

(a) $P(6 \le X \le 14) = P\left(\frac{6-6}{5} \le Z \le \frac{14-6}{5}\right) = P(0 \le Z \le 1.6)$
$= \Phi(1.6) - \Phi(0) = 0.9452 - 0.5 = 0.4452$

(b) $P(4 \le X \le 14) = P\left(\frac{4-6}{5} \le Z \le \frac{14-6}{5}\right) = P(-0.4 \le Z \le 1.6)$

$= P(Z \le 1.6) - P(Z < -0.4) = P(Z \le 1.6) - P(Z > 0.4)$
$= 0.9452 - 0.3446 = 0.6006$

(c) $P(-4 < X \le 0) = P\left(\frac{-4-6}{5} \le Z \le \frac{0-6}{5}\right) = P(-2 \le Z \le -1.2)$

$= P(Z \le -1.2) - P(Z < -2) = P(Z \geq 1.2) - P(Z > 2) = 0.1151 - 0.0228 = 0.0923$

(d) $P(X > 15) = P\left(Z > \frac{15-6}{5}\right) = P(Z > 1.8) = 0.0359$

(e) $P(|X-6| < 5) = P(-5 < X-6 < 5) = P(-1 < Z < 1) = P(Z < 1) - P(Z < -1)$
$= P(Z < 1) - P(Z > 1) = 0.8413 - 0.1587 = 0.6826$

(g) $P(|X-6| < 15) = P(-15 < X-6 < 15) = P(-3 < Z < 3) = P(Z < 3) - P(Z < -3)$
$= P(Z < 3) - P(Z > 3) = 0.9987 - 0.0013 = 0.9974$

(h) $P(|X-6| < 12.4) = P(-12.4 < X-6 < 12.4) = P(-2.48 < Z < 2.48)$
$= P(Z < 2.48) - P(Z > 2.48) = 0.9934 - 0.0066 = 0.9868$

3.3.7

3.3-7. If X is $N(650, 400)$, find
(a) $P(600 \le X < 660)$.
(b) A constant $c > 0$ such that $P(|X-650| \le c) = 0.95$.

$X \sim N(650,400) \longrightarrow Z = \frac{X-\mu}{\sigma} = \frac{X-650}{20} \sim N(0,1)$

(a) $P(600 \le X < 660) = P\left(\frac{600-650}{20} \le Z < \frac{660-650}{20}\right) = P(-2.5 \le Z < 0.5)$

$= P(Z < 0.5) - P(Z < -2.5) = P(Z < 0.5) - P(Z > 2.5) = 0.6915 - 0.0062 = 0.6853$

(b) $P(|X-650| \le c) = P(-c \le X-650 \le c) = P\left(\frac{-c}{20} \le Z \le \frac{c}{20}\right)$

$= 2P\left(Z \le \frac{c}{20}\right) - 1 = 0.95 \rightarrow P\left(Z \le \frac{c}{20}\right) = 0.975 \rightarrow c = 20(1.96) = 39.2$

3.3.9

3.3-9. Find the distribution of $W = X^2$ when
(a) X is $N(0, 4)$,
(b) X is $N(0, \sigma^2)$.

(a) $X \sim N(0,4) \rightarrow Z = \frac{X-\mu}{\sigma} = \frac{X-0}{2} = \frac{X}{2} \sim N(0,1) \rightarrow X = 2Z \rightarrow X^2 = 4Z^2 = W$

$Z^2 \sim \chi^2(1) \sim Gamma(\frac{1}{2}, 2)$

$M_W(t) = E[e^{tW}] = E[e^{t4Z^2}] = M_{Z^2}(4t) = \frac{1}{(1-2.4t)^{1/2}} = \frac{1}{(1-8t)^{1/2}}$

$\therefore W \sim Gamma(\alpha = \frac{1}{2}, \theta = 8)$

(b) $X \sim N(0,\sigma^2) \rightarrow Z = \frac{X-\mu}{\sigma} = \frac{X-0}{\sigma} = \frac{X}{\sigma} \sim N(0,1) \rightarrow X^2 = \sigma^2 Z^2 = W$

$Z^2 \sim \chi^2(1) \sim Gamma(\frac{1}{2}, 2)$

$M_W(t) = E[e^{tW}] = E[e^{t\sigma^2 Z^2}] = M_{Z^2}(t\sigma^2) = \frac{1}{(1-2t\sigma^2)^{1/2}} = \frac{1}{(1-2\sigma^2 t)^{1/2}}$

$\therefore W \sim Gamma(\alpha = \frac{1}{2}, \theta = 2\sigma^2)$

3.3.11

3.3-11. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $N(21.37, 0.16)$.
(a) Let X denote the weight of a single mint selected at random from the production line. Find $P(X > 22.07)$.
(b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $P(Y \le 2)$.

$X \sim N(21.37, 0.16)$

(a) $P(X > 22.07) = P(\frac{X-\mu}{\sigma} > \frac{22.07-21.37}{0.4}) = P(Z > 1.75) = 0.0401$

(b) $p = P(X < 20.857) = P(\frac{X-\mu}{\sigma} < \frac{20.857-21.37}{0.4}) = P(Z < -1.2825)$

$= P(Z > 1.2825) = P(Z > 1.29) = 0.0985 \rightarrow q = 0.9015$

$P(Y \le 2) = \sum_{K=0}^{2} \binom{15}{K} p^K q^{15-K} = \sum_{K=0}^{2} \binom{15}{K} 0.0985^K \, 0.9015^{15-K} = 0.8217$

3.3.13

3.3-13. The serum zinc level X in micrograms per deciliter for males between ages 15 and 17 has a distribution that is approximately normal with $\mu = 90$ and $\sigma = 15$. Compute the conditional probability $P(X > 114 \mid X > 99)$.

$P(X > 114 \mid X > 99) = \frac{P(X > 114)}{P(X > 99)} = \frac{P(Z > \frac{114-90}{15})}{P(Z > \frac{99-90}{15})} = \frac{P(Z > 1.6)}{P(Z > 0.6)} = \frac{0.0548}{0.2743} = 0.1998$

3.3.15

3.3-15. The "fill" problem is important in many industries, such as those making cereal, toothpaste, beer, and so on. If an industry claims that it is selling 12 ounces of its product in a container, it must have a mean greater than 12 ounces, or else the FDA will crack down, although the FDA will allow a very small percentage of the containers to have less than 12 ounces.
(a) If the content X of a container has a $N(12.1, \sigma^2)$ distribution, find σ so that $P(X < 12) = 0.01$.
(b) If $\sigma = 0.05$, find μ so that $P(X < 12) = 0.01$.

(a) $P(X<12) = P\left(\frac{X-\mu}{\sigma} < \frac{12-\mu}{\sigma}\right) = P\left(Z < \frac{12-12.1}{\sigma}\right) = P\left(Z < \frac{-0.1}{\sigma}\right) = P\left(Z > \frac{0.1}{\sigma}\right) = 0.01$

$\therefore \frac{0.1}{\sigma} = 2.33 \longrightarrow \sigma = 0.043$

(b) $P(X<12) = P\left(Z < \frac{12-\mu}{0.05}\right) = 0.01 = P\left(Z > \frac{\mu-12}{0.05}\right) \longrightarrow \frac{\mu-12}{0.05} = 2.33$

$\therefore \mu = 12.117$

$\boxed{3.3.17}$

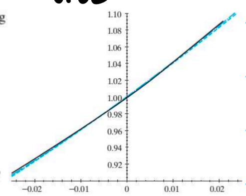
3.3-17. Figure 3.3-3(b) shows the graphs of the following three moment-generating functions:

$$g_1(t) = \frac{1}{1-4t}, \qquad t < 1/4,$$

$$g_2(t) = \frac{1}{(1-2t)^2}, \qquad t < 1/2,$$

$$g_3(t) = e^{4t+t^2/2}, \qquad -\infty < t < \infty.$$

Why do these three graphs look so similar around $t = 0$?



$g_1(t) = \frac{1}{1-4t} \longrightarrow \mathrm{Exp}\left(\lambda = \frac{1}{4}\right) \quad \mu = \frac{1}{\lambda} = 4$

$g_2(t) = \frac{1}{(1-2t)^2} \longrightarrow \mathrm{Gamma}(\alpha = 2, \theta = 2) \quad \mu = \alpha\theta = 4$

$g_3(t) = e^{4t + t^2/2} \longrightarrow N(4, 1) \quad \mu = 4$

same slope at $t = 0$

Section 4.1 1, 3, 5, 7, 9, 11

4.1.1

4.1-1. For each of the following functions, determine the constant c so that $f(x, y)$ satisfies the conditions of being a joint pmf for two discrete random variables X and Y:

(a) $f(x, y) = c(x + 2y), \quad x = 1, 2, \quad y = 1, 2, 3.$

(b) $f(x, y) = c(x + y), \quad x = 1, 2, 3, \quad y = 1, \ldots, x.$

(c) $f(x, y) = c, \quad x$ and y are integers such that $6 \le x + y \le 8$, $0 \le y \le 5$.

(d) $f(x, y) = c\left(\frac{1}{4}\right)^x\left(\frac{1}{3}\right)^y, \quad x = 1, 2, \ldots, \quad y = 1, 2, \ldots.$

(a) $\displaystyle\sum_{x=1}^{2}\sum_{y=1}^{3} f(x,y) = c\sum_{x=1}^{2}(x+2+x+4+x+6) = c\sum_{x=1}^{2}(3x+12)$

$= c(3+12+6+12) = c(33) = 1 \longrightarrow c = 1/33$

(b) $f(1,1) = c(1+1) = 2c \qquad f(2,2) = c(2+2) = 4c \qquad f(3,2) = c(3+2) = 5c$

$f(2,1) = c(2+1) = 3c \qquad f(3,1) = c(3+1) = 4c \qquad f(3,3) = c(3+3) = 6c$

$\therefore 24c = 1 \longrightarrow c = 1/24$

(c) $\left.\begin{array}{l} 6 \le x+y \le 8 \\ 0 \le y \le 5 \end{array}\right\} 1 \le x \le 8$ $\quad (1,5), (2,4), (2,5), (3,3), (3,4), (3,5), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (6,0), (6,1), (6,2), (7,0), (7,1), (8,0)$

$\therefore 18c = 1 \longrightarrow c = 1/18$

(d) $\displaystyle\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} f(x,y) = c\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x\sum_{y=1}^{\infty}\left(\frac{1}{3}\right)^y = c\left(\frac{1/4}{1/4-1}\right)\left(\frac{1/3}{1/3-1}\right) = \frac{1}{6}c \rightarrow c = 6$

4.1.3

4.1-3. Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(a) Find $f_X(x)$, the marginal pmf of X.

(b) Find $f_Y(y)$, the marginal pmf of Y.

(c) Find $P(X > Y)$.

(d) Find $P(Y = 2X)$.

(e) Find $P(X + Y = 3)$.

(f) Find $P(X \le 3 - Y)$.

(g) Are X and Y independent or dependent? Why or w not?

(h) Find the means and the variances of X and Y.

(a) $\displaystyle f_X(x) = \sum_y \frac{x+y}{32} = \frac{1}{32}(x+1+x+2+x+3+x+4) = \frac{4x+10}{32} = \frac{2x+5}{16} \quad x = 1, 2$

(b) $\displaystyle f_Y(y) = \sum_x \frac{x+y}{32} = \frac{1}{32}(y+1+y+2) = \frac{2y+3}{32}, \quad y = 1, 2, 3, 4$

(c) $P(X > Y) = P(X=2, Y=1) = \frac{2+1}{32} = \frac{3}{32}$

(d) $P(Y = 2X) = P(X=1, Y=2) + P(X=2, Y=4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$

(e) $P(X + Y = 3) = P(X=1, Y=2) + P(X=2, Y=1) = \frac{6}{32} = \frac{3}{16}$

(f) $P(X \le 3 - Y) = P(X+Y \le 3) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1)$

$$= \frac{8}{32} = \frac{1}{4}$$

(g) $f_X(x) \cdot f_Y(y) = \frac{2x+5}{16} \cdot \frac{2y+3}{32} \neq \frac{x+y}{32} \longrightarrow X, Y$ dependent

(h)

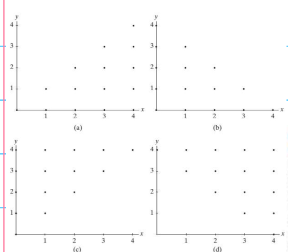
	$X=1$	$X=2$	
$Y=1$	2/32	3/32	5/32
$Y=2$	3/32	4/32	7/32
$Y=3$	4/32	5/32	9/32
$Y=4$	5/32	6/32	11/32
	14/32	18/32	

$$\mu_X = 1 \cdot \frac{14}{32} + 2 \cdot \frac{18}{32} = \frac{50}{32} = \frac{25}{16}$$

$$\mu_Y = 1 \cdot \frac{5}{32} + 2 \cdot \frac{7}{32} + 3 \cdot \frac{9}{32} + 4 \cdot \frac{11}{32} = \frac{90}{32} = \frac{45}{16}$$

$$\sigma_X^2 = 1^2 \cdot \frac{14}{32} + 2^2 \cdot \frac{18}{32} - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$$

$$\sigma_Y^2 = 1^2 \cdot \frac{5}{32} + 2^2 \cdot \frac{7}{32} + 3^2 \cdot \frac{9}{32} + 4^2 \cdot \frac{11}{32} - \left(\frac{45}{16}\right)^2 = \frac{295}{256}$$



4.1.5

4.1-5. Each part of Figure 4.1-5 depicts the sample space S of the joint pmf of discrete random variables X and Y. Suppose that, for each part, the joint pmf is constant over S. Give S in terms of inequalities for each of the parts **(a)**, **(b)**, **(c)**, and **(d)**, and determine the marginal pmfs of X and Y.

(a) $x = 0, 1, 2, 3, 4 \quad y = 0 \ldots x$

$f_X(x) = \frac{x+1}{15}$, $f_Y(y) = \frac{5-y}{15}$

(b) $x = 0, 1, 2, 3, 4 \quad y = 0 \ldots 4 - x$

$f_X(x) = \frac{5-x}{15}$, $f_Y(y) = \frac{5-y}{15}$

(c) $x = 0, 1, 2, 3, 4 \quad y = x \ldots 4$

$f_X(x) = \frac{5-x}{15}$, $f_Y(y) = \frac{y+1}{15}$

(d) $x = 0, 1, 2, 3, 4 \quad y = 4 - x \ldots 4$

$f_X(x) = \frac{x+1}{15}$, $f_Y(y) = \frac{y+1}{15}$

4.1.7

4.1-7. Roll a pair of four-sided dice, one red and one black. Let X equal the outcome of the red die and let Y equal the sum of the two dice.

(a) On graph paper, describe the space of X and Y.

(b) Define the joint pmf on the space (similar to Figure 4.1-1).

(c) Give the marginal pmf of X in the margin.

(d) Give the marginal pmf of Y in the margin.

(e) Are X and Y dependent or independent? Why or why not?

(b) $x = 1, 2, 3, 4 \quad y = x+1, x+2, x+3, x+4$

$$f(x,y) = \frac{1}{16}$$

(c) $f_X(x) = \sum_{x=1}^{4} f(x,y) = \frac{4}{16} = \frac{1}{4}$

(e) Dependent $\rightarrow$ not rectangular space

(d) $P(Y=2) = P(1,1) = 1/16$

$P(Y=3) = P(1,2) + P(2,1) = 1/8$

$P(Y=4) = P(1,3) + P(2,2) + P(3,1) = 3/16$

$P(Y=5) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = 1/4$

$P(Y=6) = P(2,4) + P(3,3) + P(4,2) = 3/16$

$P(Y=7) = P(3,4) + P(4,3) = 1/8$

$P(Y=8) = P(4,4) = 1/16$

$$f_Y(y) = \frac{4 - |y-5|}{16}$$

4.1.9

4.1-9. A particle starts at (0, 0) and moves in one-unit independent steps with equal probabilities of 1/4 in each of the four directions: north, south, east, and west. Let S equal the east–west position and T the north–south position after n steps.

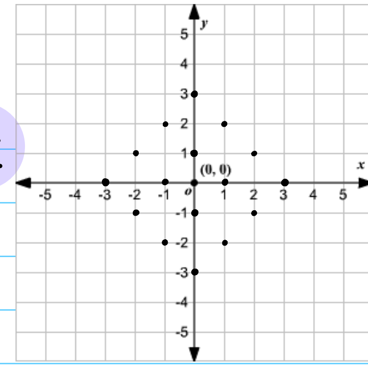
(a) Define the joint pmf of S and T with $n = 3$. On a two-dimensional graph, give the probabilities of the joint pmf and the marginal pmfs (similar to Figure 4.1-1).

(b) Let $X = S + 3$ and let $Y = T + 3$. Give the marginal pmfs of X and Y.

(a) $P((\pm 3, 0), (0, \pm 3)) = \frac{1}{64}$

$P((\pm 2, \pm 1)) = \frac{3}{64}$

$P((\pm 1, 0), (0, \pm 1)) = \frac{9}{64}$



(b) $p = \frac{1}{4} \times 2 = \frac{1}{2}$

$X, Y \sim Bin(6, \frac{1}{2})$

4.1.11

4.1-11. A manufactured item is classified as good, a "second," or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

(a) Give the joint pmf of X and Y, $f(x, y)$.

(b) Sketch the set of points for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?

(c) Find $P(X = 10, Y = 4)$.

(d) Give the marginal pmf of X.

(e) Find $P(X \le 11)$.

(a) $f(x, y) = \frac{15!}{x!\,y!\,(15-x-y)!} \left(\frac{6}{10}\right)^x \left(\frac{3}{10}\right)^y \left(\frac{1}{10}\right)^{15-x-y}, \quad 0 \le x+y \le 15$

(b) Dependent, not rectangular shape

(c) $P(X = 10, Y = 4) = \frac{15!}{10!\,4!\,(15-10-4)!} \left(\frac{6}{10}\right)^{10} \left(\frac{3}{10}\right)^{4} \left(\frac{1}{10}\right)^{1} = 0.0735$

(d) $X \sim Bin(15, \frac{6}{10})$

(e) $P(X \le 11) = \sum_{k=0}^{11} \binom{15}{k} \left(\frac{6}{10}\right)^k \left(\frac{4}{10}\right)^{15-k} = 0.9095$

Section 4.2

1, 3, 5, 7, 9, 12

4.2.1

4.2-1. Let the random variables X and Y have the joint pmf

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

Find the means μ_X and μ_Y, the variances σ_X^2 and σ_Y^2, the covariance $\mathrm{Cov}(X, Y)$, and the correlation coefficient ρ. Are X and Y independent or dependent?

$$f_X(x) = \frac{2x+5}{16}, \quad f_Y(y) = \frac{2y+3}{32} \longrightarrow f_X(x) \cdot f_Y(y) \neq f(x,y)$$

X, Y dependent

$$\mu_X = \sum_X x f_X(x) = \sum_X x \sum_y f(x,y) = \sum_X x\left(\frac{x+1+x+2+x+3+x+4}{32}\right)$$

$$= \sum_X x\,\frac{4x+10}{32} = \sum_X x\,\frac{2x+5}{16} = \frac{1(2+5)+2(4+5)}{16} = \frac{25}{16}$$

$$\mu_Y = \sum_Y y f_Y(y) = \sum_Y y \sum_X f(x,y) = \sum_Y y\left(\frac{1+y+2+y}{32}\right) = \sum_Y y\left(\frac{2y+3}{32}\right)$$

$$= \frac{1(2+3)+2(4+3)+3(6+3)+4(8+3)}{32} = \frac{90}{32} = \frac{45}{16}$$

$$\sigma_X^2 = \sum_X x^2 f_X(x) - \mu_X^2 = \sum_X x^2\,\frac{2x+5}{16} - \left(\frac{25}{16}\right)^2 = \frac{2+5+4(4+5)}{16} - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$$

$$\sigma_Y^2 = \sum_Y y^2 f_Y(y) - \mu_Y^2 = \sum_Y y^2\,\frac{2y+3}{32} - \left(\frac{45}{16}\right)^2$$

$$= \frac{2+3+4(4+3)+9(6+3)+16(8+3)}{32} - \left(\frac{45}{16}\right)^2 = \frac{295}{256}$$

$$\mathrm{Cov}(X,Y) = \sum_{X,Y} xy f(x,y) - \mu_X \mu_Y = \sum_{X=1}^{2}\sum_{y=1}^{4} xy\,\frac{x+y}{32} - \frac{25}{16}\left(\frac{45}{16}\right)$$

$$= \sum_{X=1}^{2} \frac{x(x+1)+2x(x+2)+3x(x+3)+4x(x+4)}{32} - \frac{1125}{256}$$

$$= \sum_{X=1}^{2} \frac{10x^2+30x}{32} - \frac{1125}{256} = \frac{35}{8} - \frac{1125}{256} = \frac{-5}{256}$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{-5}{256}}{\sqrt{\frac{63}{256} \times \frac{295}{256}}} = \frac{-5}{3\sqrt{7} \times \sqrt{295}} = \frac{-5}{3\sqrt{2065}}$$

4.2-3. Roll a fair four-sided die twice. Let X equal the outcome of the first roll, and let Y equal the sum of the two rolls.

(a) Determine μ_X, μ_Y, σ_X^2, σ_Y^2, $\text{Cov}(X, Y)$, and ρ.

(b) Find the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

4.2.3

$$f(x,y)=\frac{1}{16},\quad x=1,2,3,4 \quad y=x+1,x+2,x+3,x+4$$

(a) $$f_X(x)=\sum_y f(x,y)=\frac{4}{16}=\frac{1}{4}$$

$P(Y=2)=P(1,1)=1/16$

$P(Y=3)=P(1,2)+P(2,1)=1/8$

$P(Y=4)=P(1,3)+P(2,2)+P(3,1)=3/16$

$P(Y=5)=P(1,4)+P(2,3)+P(3,2)+P(4,1)=1/4$

$P(Y=6)=P(2,4)+P(3,3)+P(4,2)=3/16$

$P(Y=7)=P(3,4)+P(4,3)=1/8$

$P(Y=8)=P(4,4)=1/16$

$$f_Y(y)=\frac{4-|y-5|}{16}$$

$$\mu_X=\sum_x x f_X(x)=\sum_{x=1}^{4}\frac{x}{4}=\frac{1+2+3+4}{4}=\frac{5}{2}$$

$$\mu_Y=\sum_y y f_Y(y)=\sum_y y\,\frac{4-|y-5|}{16}=\frac{1}{8}+\frac{3}{8}+\frac{3}{4}+\frac{5}{4}+\frac{9}{8}+\frac{7}{8}+\frac{1}{2}=5$$

$$\sigma_X^2=\sum_x x^2 f_X(x)-\mu_X^2=\sum_{x=1}^{4}\frac{x^2}{4}-\left(\frac{5}{2}\right)^2=\frac{5}{4}$$

$$\sigma_Y^2=\sum_y y^2 f_Y(y)-\mu_Y^2=\sum_y y^2\,\frac{4-|y-5|}{16}-5^2$$

$$=\frac{1}{4}+\frac{9}{8}+3+\frac{25}{4}+\frac{27}{4}+\frac{49}{8}+4-25=\frac{5}{2}$$

$$\text{Cov}(X,Y)=\sum_{x,y} xy f(x,y)-\mu_X\mu_Y=\sum_{x=1}^{4}\sum_y\frac{xy}{16}-\left(\frac{5}{2}\right)5$$

$$=\frac{1(2)+1(3)+1(4)+1(5)+2(3)+2(4)+2(5)+2(6)+3(4)+3(5)+3(6)\ldots}{16}-\frac{25}{2}$$

$$=\frac{5}{4}\rightarrow \rho=\frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{5/4}{\sqrt{\frac{5}{4}\times\frac{5}{2}}}=\frac{\sqrt{2}}{2}$$

(b) $$y=\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)+\mu_Y=\frac{\sqrt{2}}{2}\times\frac{\sqrt{5/2}}{\sqrt{5/4}}\left(x-\frac{5}{2}\right)+5=x+\frac{5}{2}\rightarrow y=x+\frac{5}{2}$$

4.2-5. Let X and Y be random variables with respective means μ_X and μ_Y, respective variances σ_X^2 and σ_Y^2, and correlation coefficient ρ. Fit the line $y = a + bx$ by the method of least squares to the probability distribution by minimizing the expectation

$$K(a, b) = E[(Y - a - bX)^2]$$

with respect to a and b. Hint: Consider $\partial K/\partial a = 0$ and $\partial K/\partial b = 0$, and solve simultaneously.

4.2.5

$K(a,b)=E\left[(bX+a-Y)^2\right]$
minimize

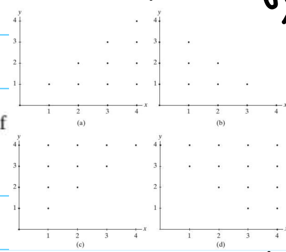
$$0=\frac{\partial K(a,b)}{\partial b}=E\left[2X(bX+a-Y)\right]=E\left[X(bX+a-Y)\right]$$

$$0=\frac{\partial K(a,b)}{\partial a}=E\left[2(bX+a-Y)\right]=E\left[bX+a-Y\right]$$

$$\therefore 0 = bE[X^2] + a\mu_X - E[XY]$$
$$0 = b\mu_X + a - \mu_Y \longrightarrow 0 = b\mu_X^2 + a\mu_X - \mu_X\mu_Y$$
$$0 = b\sigma_X^2 - Cov(X,Y) \longrightarrow b = \frac{Cov(X,Y)}{\sigma_X^2}$$
$$0 = \frac{Cov(X,Y)}{\sigma_X^2}\mu_X + a - \mu_Y \rightarrow a = \mu_Y - \frac{Cov(X,Y)}{\sigma_X^2}\mu_X = \mu_Y - \mu_X b$$

$\boxed{4.2.7}$

4.2-7. Determine the correlation coefficient ρ for each of the four distributions in Exercise 4.1-5.



(a) $X = 0,1,2,3,4 \quad y = 0 \ldots x \quad f_X(x) = \frac{x+1}{15}, \; f_Y(y) = \frac{5-y}{15} \quad f(x,y) = \frac{1}{15}$

$$\mu_X = \sum_x x f_X(x) = \sum_{x=0}^{4} \frac{x(x+1)}{15} = \frac{8}{3}$$
$$\mu_Y = \sum_y y f_Y(y) = \sum_y \frac{y(5-y)}{15} = \frac{4}{3}$$
$$\sigma_X^2 = \sum_x x^2 f_X(x) - \mu_X^2 = \sum_x \frac{x^2(x+1)}{15} - \left(\frac{8}{3}\right)^2 = \frac{14}{9}$$
$$\sigma_Y^2 = \sum_y y^2 f_Y(y) - \mu_Y^2 = \sum_y \frac{y^2(5-y)}{15} - \left(\frac{4}{3}\right)^2 = \frac{14}{9}$$

$$Cov(X,Y) = \sum_{x,y} xy f(x,y) - \mu_X\mu_Y = \sum_{x,y} \frac{xy}{15} - \left(\frac{8}{3}\right)\left(\frac{4}{3}\right)$$
$$= \frac{0(0)+1(0)+1(1)+2(0)+2(1)+2(2)+3(0)+3(1)+3(2)+3(3)+4(0)+\ldots}{15} - \frac{32}{9}$$
$$= \frac{7}{9} \rightarrow \rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{7/9}{\sqrt{\frac{14}{9} \times \frac{14}{9}}} = \frac{7/9}{14/9} = \frac{1}{2} = 0.5$$

(b) $X = 0,1,2,3,4 \quad y = 0 \ldots 4-x \quad f_X(x) = \frac{5-x}{15}, \; f_Y(y) = \frac{5-y}{15}$

$$\mu_X = \sum_x x f_X(x) = \sum_x \frac{x(5-x)}{15} = \frac{4}{3}$$
$$\mu_Y = \sum_y y f_Y(y) = \sum_y \frac{y(5-y)}{15} = \frac{4}{3}$$

$$\sigma_X^2 = \sum_X x^2 f_X(x) - \mu_X^2 = \sum_X \frac{x^2(5-x)}{15} - \left(\frac{4}{3}\right)^2 = \frac{14}{9}$$

$$\sigma_Y^2 = \sum_Y y^2 f_Y(y) - \mu_Y^2 = \sum_Y \frac{y^2(5-y)}{15} - \left(\frac{4}{3}\right)^2 = \frac{14}{9}$$

$$\mathrm{Cov}(X,Y) = \sum_{X,Y} xyf(x,y) - \mu_X\mu_Y = \sum_{X,Y} \frac{xy}{15} - \left(\frac{4}{3}\right)\left(\frac{4}{3}\right)$$

$$= \frac{0(0)+0(1)+0(2)+0(3)+0(4)+1(0)+1(1)+1(2)+1(3)+2(0)+2(1)+2(2)\ldots}{15} - \frac{16}{9}$$

$$= \frac{-7}{9} \rightarrow \rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-7/9}{\sqrt{\frac{14}{9} \times \frac{14}{9}}} = -0.5$$

(c) $x = 0,1,2,3,4 \quad y = x \ldots 4 \qquad f_X(x) = \frac{5-x}{15},\ f_Y(y) = \frac{y+1}{15}$

$$\mu_X = \sum_X x f_X(x) = \sum_X \frac{x(5-x)}{15} = \frac{4}{3}$$

$$\mu_Y = \sum_Y y f_Y(y) = \sum_Y \frac{y(y+1)}{15} = \frac{8}{3}$$

$$\sigma_X^2 = \sum_X x^2 f_X(x) - \mu_X^2 = \sum_X \frac{x^2(5-x)}{15} - \left(\frac{4}{3}\right)^2 = \frac{14}{9}$$

$$\sigma_Y^2 = \sum_Y y^2 f_Y(y) - \mu_Y^2 = \sum_Y \frac{y(y+1)}{15} - \left(\frac{8}{3}\right)^2 = \frac{14}{9}$$

$$\mathrm{Cov}(X,Y) = \sum_{X,Y} xyf(x,y) - \mu_X\mu_Y = \sum_{X,Y} \frac{xy}{15} - \left(\frac{4}{3}\right)\left(\frac{8}{3}\right)$$

$$= \frac{0(0)+0(1)+0(2)+0(3)+0(4)+1(1)+1(2)+1(3)+1(4)+2(2)+2(3)+\ldots}{15} - \frac{32}{9}$$

$$= \frac{7}{9} \longrightarrow \rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{7/9}{\sqrt{\frac{14}{9} \times \frac{14}{9}}} = 0.5$$

(d) $x = 0,1,2,3,4 \quad y = 4-x \ldots 4 \qquad f_X(x) = \frac{x+1}{15},\ f_Y(y) = \frac{y+1}{15}$

$$\mu_X = \sum_X x f_X(x) = \sum_x \frac{x(x+1)}{15} = \frac{8}{3}$$

$$\mu_Y = \sum_Y y f_Y(y) = \sum_Y \frac{y(y+1)}{15} = \frac{8}{3}$$

$$\sigma_X^2=\sum_X x^2 f_X(x)-\mu_X^2=\sum_X \frac{x^2(x+1)}{15}-\left(\frac{8}{3}\right)^2=\frac{14}{9}$$

$$\sigma_Y^2=\sum_Y y^2 f_Y(y)-\mu_Y^2=\sum_Y \frac{y(y+1)}{15}-\left(\frac{8}{3}\right)^2=\frac{14}{9}$$

$$\mathrm{Cov}(X,Y)=\sum_{X,Y} xyf(x,y)-\mu_X\mu_Y=\sum_{X,Y}\frac{xy}{15}-\left(\frac{8}{3}\right)\left(\frac{8}{3}\right)$$

$$=\frac{0(4)+1(3)+1(4)+2(2)+2(3)+2(4)+3(1)+3(2)+3(3)+3(4)+4(0)\ldots}{15}-\frac{64}{9}$$

$$=\frac{-7}{9}\rightarrow \rho=\frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{-7/9}{\sqrt{\frac{14}{9}\times\frac{14}{9}}}=-0.5$$

4.2.9

4.2-9. Let the joint pmf of X and Y be

$f(x,y)=1/4, \quad (x,y)\in S=\{(0,0),(1,1),(1,-1),(2,0)\}.$

(a) Are X and Y independent?

(b) Calculate $\mathrm{Cov}(X,Y)$ and ρ.

This exercise also illustrates the fact that dependent random variables can have a correlation coefficient of zero.

(a) Not rectangular $\rightarrow$ X, Y dependent $\rightarrow$ No

(b) $\mu_X=\frac{0+1+1+2}{4}=1 \qquad \mu_Y=\frac{0+1+(-1)+0}{4}=0$

$$\mathrm{Cov}(X,Y)=E[XY]-\mu_X\mu_Y=\frac{0(0)+1(1)+1(-1)+2(0)}{4}-1(0)=0$$

$$\rho=\frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{0}{\sigma_X\sigma_Y}=0$$

4.2.12

4.2-12. If the correlation coefficient ρ exists, show that $-1\le\rho\le 1$.

HINT: Consider the discriminant of the nonnegative quadratic function $h(v)=E\{[(X-\mu_X)+v(Y-\mu_Y)]^2\}$.

$$h(v)=E\{[(X-\mu_X)+v(Y-\mu_Y)]^2\}\geq 0$$

$$=E[(X-\mu_X)^2+2v(X-\mu_X)(Y-\mu_Y)+v^2(Y-\mu_Y)^2]$$

$$=E[(X-\mu_X)^2]+2E[v(X-\mu_X)(Y-\mu_Y)]+E[v^2(Y-\mu_Y)^2]$$

$$=\sigma_X^2+2v\sigma_X\sigma_Y\frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X\sigma_Y}+v^2\sigma_Y^2$$

$$=\sigma_X^2+2v\sigma_X\sigma_Y\rho+v^2\sigma_Y^2=\sigma_Y^2v^2+2v\sigma_X\sigma_Y\rho+\sigma_X^2$$

Discriminant $=b^2-4ac\leq 0$ (either 1 or no real roots)

$$=4\sigma_X^2\sigma_Y^2\rho^2-4\sigma_Y^2\sigma_X^2=4\sigma_X^2\sigma_Y^2(\rho^2-1)\leq 0\rightarrow -1\leq\rho\leq 1$$

Section 4.3

1, 3, 5, 7, 9, 11

4.3.1

4.3-1. Let X and Y have the joint pmf

$$f(x, y) = \frac{x+y}{32}, \qquad x = 1, 2, \qquad y = 1, 2, 3, 4.$$

(a) Display the joint pmf and the marginal pmfs on a graph like Figure 4.3-1(a).

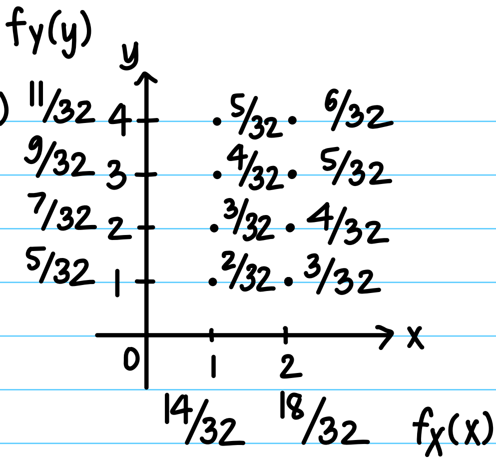
(b) Find $g(x \mid y)$ and draw a figure like Figure 4.3-1(b), depicting the conditional pmfs for $y = 1, 2, 3,$ and 4.

(c) Find $h(y \mid x)$ and draw a figure like Figure 4.3-1(c), depicting the conditional pmfs for $x = 1$ and 2.

(d) Find $P(1 \le Y \le 3 \mid X = 1)$, $P(Y \le 2 \mid X = 2)$, and $P(X = 2 \mid Y = 3)$.

(e) Find $E(Y \mid X = 1)$ and $\mathrm{Var}(Y \mid X = 1)$.

(a)



(b) $$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{x+y}{32}}{\sum_{x=1}^{2} \frac{x+y}{32}} = \frac{x+y}{2y+3}, \quad x = 1,2 \qquad y = 1,2,3,4$$

(c) $$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x+y}{32}}{\sum_{y=1}^{4} \frac{x+y}{32}} = \frac{x+y}{4x+10}, \quad x = 1,2 \qquad y = 1,2,3,4$$

(d) $$P(1 \le Y \le 3 \mid X = 1) = \frac{P(1 \le Y \le 3,\ X = 1)}{P(X = 1)} = \frac{\frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32}}{\frac{4(1)+10}{32}} = \frac{9}{14}$$

$$P(Y \le 2 \mid X = 2) = \frac{P(Y \le 2,\ X = 2)}{P(X = 2)} = \frac{\frac{2+1}{32} + \frac{2+2}{32}}{\frac{4(2)+10}{32}} = \frac{7}{18}$$

$$P(X = 2 \mid Y = 3) = \frac{P(X = 2,\ Y = 3)}{P(Y = 3)} = \frac{\frac{2+3}{32}}{\frac{2(3)+3}{32}} = \frac{5}{9}$$

(e) $$E[Y \mid X = 1] = \sum_{y=1}^{4} y\, h(y \mid x = 1) = \sum_{y=1}^{4} \frac{y(1+y)}{4(1)+10} = \frac{20}{7}$$

$$\mathrm{Var}(Y \mid X = 1) = E[Y^2 \mid X = 1] - (E[Y \mid X = 1])^2 = \sum_{y=1}^{4} \frac{y^2(1+y)}{4(1)+10} - \left(\frac{20}{7}\right)^2 = \frac{55}{49}$$

4.3.3

4.3-3. Let W equal the weight of laundry soap in a 1-kilogram box that is distributed in Southeast Asia. Suppose that $P(W < 1) = 0.02$ and $P(W > 1.072) = 0.08$. Call a box of soap light, good, or heavy, depending on whether $\{W < 1\}$, $\{1 \le W \le 1.072\}$, or $\{W > 1.072\}$, respectively. In $n = 50$ independent observations of these boxes, let X equal the number of light boxes and Y the number of good boxes.

(a) What is the joint pmf of X and Y?

(b) Give the name of the distribution of Y along with the values of the parameters of this distribution.

(c) Given that $X = 3$, how is Y distributed conditionally?

(d) Determine $E(Y \mid X = 3)$.

(e) Find ρ, the correlation coefficient of X and Y.

$$P(1 \le W \le 1.072) = 1 - 0.02 - 0.08 = 0.9$$

(a) $$f(x,y) = \binom{50}{x,\ y,\ 50-x-y} 0.02^{x}\, 0.9^{y}\, 0.08^{50-x-y}$$

(b) $Y \sim Bin(50, 0.9)$ (c) $Y \sim Bin(47, \frac{0.9}{0.98})$

(d) $$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\binom{50}{x, y, 50-x-y} 0.02^x 0.9^y 0.08^{50-x-y}}{\binom{50}{x} 0.02^x 0.98^{50-x}}$$

$$= \binom{50-x}{y} \frac{0.9^y 0.08^{50-x-y}}{0.98^{50-x}}$$ OR $h(y|x) = \binom{47}{y}\left(\frac{0.9}{0.98}\right)^k \left(\frac{0.08}{0.98}\right)^{47-k}$

$$E[Y|X=3] = \sum_y y h(y|x=3) = \sum_y y \binom{47}{y} \frac{0.9^y 0.08^{47-y}}{0.98^{47}} = \frac{2115}{49}$$

(e) $$\rho = -\sqrt{\frac{p_X p_Y}{(1-p_X)(1-p_Y)}} = -\sqrt{\frac{0.02 \times 0.9}{(1-0.02)(1-0.9)}} = \frac{-3}{7}$$

4.3.5

4.3-5. Let X and Y have a trinomial distribution with $n = 2$, $p_X = 1/4$, and $p_Y = 1/2$.

(a) Give $E(Y \mid x)$.

(b) Compare your answer in part (a) with the equation of the line of best fit in Example 4.2-2. Are they the same? Why or why not?

(a) $$E[Y|X] = (n-x)\frac{p_Y}{1-p_X} = (2-x)\frac{1/2}{1-1/4} = \frac{2(2-x)}{3}, \quad x = 0, 1, 2$$

(b) Same, $y = \frac{-2}{3}x + \frac{4}{3}$ because $E[Y|X]$ is linear

4.3.7

4.3-7. Using the joint pmf from Exercise 4.2-3, find the value of $E(Y \mid x)$ for $x = 1, 2, 3, 4$. Do the points $[x, E(Y \mid x)]$ lie on the best-fitting line?

$f(x,y) = \frac{1}{16}$, $x = 1, 2, 3, 4$ $y = x+1, x+2, x+3, x+4$

$$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/16}{4 \cdot 1/16} = \frac{1}{4}$$

$x = 1, 2, 3, 4$

$$E[Y|X] = \sum_y y h(y|x) = \sum_y \frac{y}{4} = \frac{1}{4}(x+1+x+2+x+3+x+4) = \frac{2x+5}{2}$$

↳ $(x, E[Y|X])$ on best-fitting line $y = x + \frac{5}{2}$

4.3.9

4.3-9. Let X and Y have a uniform distribution on the set of points with integer coordinates in $S = \{(x, y) : 0 \le x \le 7, x \le y \le x + 2\}$. That is, $f(x, y) = 1/24$, $(x, y) \in S$, and both x and y are integers. Find

(a) $f_X(x)$.

(b) $h(y \mid x)$.

(c) $E(Y \mid x)$.

(d) $\sigma^2_{Y|x}$.

(e) $f_Y(y)$.

(a) $f_X(x) = \sum_y f(x,y) = f(x,x) + f(x,x+1) + f(x,x+2) = \frac{3}{24} = \frac{1}{8}$, $x = 0, 1, 2 \ldots 7$

(b) $h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/24}{1/8} = \frac{1}{3}$, $y = x, x+1, x+2$
$x = 0, 1 \ldots 7$

(c) $E[Y|X=x] = \sum_y y\,h(y|x) = \sum_x^{x+2} \frac{1}{3} y = \frac{1}{3}(x + x+1 + x+2) = \frac{3x+3}{3} = x+1$
$x = 0, 1 \ldots 7$

(d) $\sigma^2_{Y|x} = E[Y^2|X=x] - E[Y|X=x]^2 = \sum_y y^2 h(y|x) - (x+1)^2$

$= \sum_x^{x+2} \frac{1}{3} y^2 - (x+1)^2 = \frac{x^2 + (x+1)^2 + (x+2)^2}{3} - (x+1)^2$

$= \frac{x^2 + x^2 + 2x + 1 + x^2 + 4x + 4}{3} - x^2 - 2x - 1 = \frac{3x^2 + 6x + 5 - 3x^2 - 6x - 3}{3} = \frac{2}{3}$

(e)

X/Y	0	1	2	3	4	5	6	7	8	9
0	p	p	p							
1		p	p	p						
2			p	p	p					
3				p	p	p				
4					p	p	p			
5						p	p	p		
6							p	p	p	
7								p	p	p
	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$

$$f_Y(y) = \begin{cases} 1/24 \,;\; y = 0, 9 \\ 2/24 \,;\; y = 1, 8 \\ 3/24 \,;\; y = 2, 3 \ldots 7 \end{cases}$$

4.3.11

4.3-11. Suppose that X has a geometric distribution with parameter p, and suppose the conditional distribution of Y, given $X = x$, is Poisson with mean x. Find $E(Y)$ and $\mathrm{Var}(Y)$.

$X \sim \mathrm{Geom}(p) \rightarrow \mu_X = 1/p$, $\sigma_X^2 = \frac{q}{p^2}$

$Y|_{X=x} \sim \mathrm{Poisson}(x)$

$\rightarrow \mu_{Y|X=x} = x \rightarrow \mu_{Y|X} = X$

$\rightarrow \sigma^2_{Y|X=x} = x \rightarrow \sigma^2_{Y|X} = X$

$E[Y] = E[E[Y|X]] = E[X] = \frac{1}{p}$

$$Var(Y) = E\left[Var(Y \mid X)\right] + Var(E[Y \mid X])$$

$$= E[X] + Var(X) = \frac{1}{p} + \frac{q}{p^2} = \frac{p+q}{p^2} = \frac{1}{p^2}$$

Section 4.4

1, 3, 5, 7, 11, 13, 15, 17, 19, 21

4.4.1

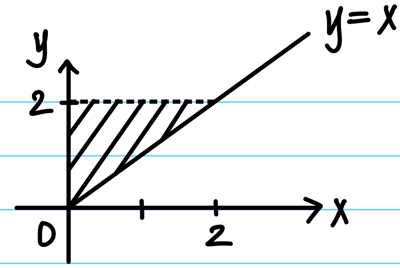
4.4-1. Let $f(x, y) = (3/16)xy^2$, $0 \le x \le 2$, $0 \le y \le 2$, be the joint pdf of X and Y.

(a) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions.

(b) Are the two random variables independent? Why or why not?

(c) Compute the means and variances of X and Y.

(d) Find $P(X \le Y)$.



(a) $f_X(x) = \int_y f(x,y)\,dy = \int_0^2 \frac{3}{16} xy^2\,dy = \frac{3x}{16}\left(\frac{1}{3}y^3\right)\Big|_0^2 = \frac{x(2^3-0)}{16} = \frac{x}{2},\ 0 \le x \le 2$

$f_Y(y) = \int_x f(x,y)\,dx = \int_0^2 \frac{3}{16} xy^2\,dx = \frac{3}{16}y^2\left(\frac{1}{2}\right)x^2\Big|_0^2 = \frac{3y^2}{32}(2^2-0) = \frac{3y^2}{8}$

(b) $f_X(x)\,f_Y(y) = \frac{x}{2}\left(\frac{3y^2}{8}\right) = \frac{3}{16}xy^2 = f(x,y) \rightarrow$ X, Y independent

(c) $\mu_X = \int_x x f_X(x)\,dx = \int_0^2 x\left(\frac{x}{2}\right)dx = \frac{1}{2}\left(\frac{1}{3}\right)x^3\Big|_0^2 = \frac{2^3}{6} = \frac{4}{3}$

$\sigma_X^2 = \int_x x^2 f_X(x)\,dx - \mu_X^2 = \int_0^2 x^2\left(\frac{x}{2}\right)dx - \left(\frac{4}{3}\right)^2 = \frac{x^4}{8}\Big|_0^2 - \frac{16}{9} = \frac{2^4}{8} - \frac{16}{9} = \frac{2}{9}$

$\mu_Y = \int_y y f_Y(y)\,dy = \int_0^2 y\left(\frac{3y^2}{8}\right)dy = \frac{3}{8}\left(\frac{1}{4}\right)y^4\Big|_0^2 = \frac{3(2^4)}{32} = \frac{3}{2}$

$\sigma_Y^2 = \int_y y^2 f_Y(y)\,dy - \mu_Y^2 = \int_0^2 y^2\left(\frac{3y^2}{8}\right)dy - \left(\frac{3}{2}\right)^2 = \frac{3}{8}\left(\frac{1}{5}\right)y^5\Big|_0^2 - \frac{9}{4} = \frac{3}{20}$

(d) $P(X \le Y) = \int_0^2\int_0^y f(x,y)\,dx\,dy = \int_0^2\int_0^y \frac{3}{16}xy^2\,dx\,dy = \int_0^2 \frac{3}{16}y^2\left(\frac{x^2}{2}\right)\Big|_0^y dy$

$= \int_0^2 \frac{3}{32}y^2(y^2)\,dy = \frac{3}{32}\left(\frac{1}{5}\right)y^5\Big|_0^2 = \frac{3(2^5)}{32(5)} = \frac{3}{5}$

4.4.3

4.4-3. Let $f(x, y) = 2e^{-x-y}$, $0 \le x \le y < \infty$, be the joint pdf of X and Y. Find $f_X(x)$ and $f_Y(y)$, the marginal pdfs of X and Y, respectively. Are X and Y independent?

$u = -x-y \rightarrow du = -dy$

$f_X(x) = \int_y f(x,y)\,dy = \int_x^\infty 2e^{-x-y}\,dy = \lim_{t\to\infty}\int_{-x-t}^{-2x} 2e^u\,du = \lim_{t\to\infty} 2e^u \Big|_{-x-t}^{-2x}$

$$= \lim_{t\to\infty} 2\left(e^{-2x} - \cancel{e^{-x-t}}^{\,0}\right) = 2e^{-2x}, \quad 0 \le x < \infty$$

$$u = -x-y \rightarrow du = -dx$$

$$f_Y(y) = \int_x f(x,y)\,dx = \int_0^y 2e^{-x-y}\,dx = \int_{-2y}^{-y} 2e^{u}\,du = 2e^{u}\Big|_{-2y}^{-y} = 2(e^{-y} - e^{-2y}), \quad 0 < y < \infty$$

$$f_X(x) f_Y(y) = 2e^{-2x} \cdot 2(e^{-y} - e^{-2y}) = 4e^{-2x-y} - 2e^{-2x-2y} \neq f(x,y)$$

$\therefore$ X, Y dependent

4.4-5. For each of the following functions, determine the value of c for which the function is a joint pdf of two continuous random variables X and Y.

(a) $f(x, y) = cxy, \quad 0 \le x \le 1, \quad x^2 \le y \le x.$

(b) $f(x, y) = c(1 + x^2 y), \quad 0 \le x \le y \le 1.$

(c) $f(x, y) = cye^{x}, \quad 0 \le x \le y^2, \quad 0 \le y \le 1.$

(d) $f(x, y) = c\sin(x + y), \quad 0 \le x \le \pi/2, \quad 0 \le y \le \pi/2.$

$\boxed{4.4.5}$

(a) $$\int_x\int_y f(x,y)\,dy\,dx = \int_0^1\int_{x^2}^{x} cxy\,dy\,dx = c\int_0^1 \frac{xy^2}{2}\Big|_{x^2}^{x} dx = \frac{c}{2}\int_0^1 x(x^2 - x^4)\,dx$$

$$= \frac{c}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{c}{2}\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{c}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{c}{24} = 1 \rightarrow c = 24$$

(b) $$\int_x\int_y f(x,y)\,dy\,dx = \int_0^1\int_x^1 c(1 + x^2 y)\,dy\,dx = c\int_0^1 \left(y + \frac{x^2y^2}{2}\right)\Big|_x^1 dx$$

$$= c\int_0^1 \left(1 + \frac{x^2}{2} - x - \frac{x^4}{2}\right) dx = c\left(x + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x^5}{10}\right)\Big|_0^1 = \frac{17}{30}c = 1 \rightarrow c = \frac{30}{17}$$

(c) $$\int_y\int_x f(x,y)\,dx\,dy = \int_0^1\int_0^{y^2} cye^{x}\,dx\,dy = c\int_0^1 ye^{x}\Big|_0^{y^2} dy = c\int_0^1 y(e^{y^2} - 1)\,dy$$

$$= c\int_0^1 (ye^{y^2} - y)\,dy = c\left(\int_1^e \frac{du}{2} - \frac{y^2}{2}\Big|_0^1\right) = c\left(\frac{u}{2}\Big|_1^e - \frac{1}{2}\right) = \frac{c}{2}(e - 1 - 1) = \frac{c(e-2)}{2} = 1$$

$$u = e^{y^2} \rightarrow du = 2ye^{y^2}dy \rightarrow \frac{du}{2} = ye^{y^2}dy \qquad \therefore c = \frac{2}{e-2}$$

(d) $$\int_x\int_y f(x,y)\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2} c\sin(x+y)\,dy\,dx = c\int_0^{\pi/2} -\cos(x+y)\Big|_0^{\pi/2} dx$$

$$= c\int_{\pi/2}^{0} \left(\cos\left(x + \frac{\pi}{2}\right) - \cos x\right) dx = c\left(\sin\left(x + \frac{\pi}{2}\right) - \sin x\right)\Big|_{\pi/2}^{0}$$

$$=c\left(\sin\frac{\pi}{2}-\sin 0-\sin\pi+\sin\frac{\pi}{2}\right)=c(1-0-0+1)=2c=1\rightarrow c=1/2$$

4.4-7. Let $f(x,y)=4/3,\ 0<x<1,\ x^3<y<1$, zero elsewhere.

4.4.7

(a) Sketch the region where $f(x,y)>0$.

(b) Find $P(X>Y)$.

$$\text{(b)}\quad P(X>Y)=\int_0^1\int_{x^3}^{x} f(x,y)\,dy\,dx=\int_0^1\int_{x^3}^{x}\frac{4}{3}\,dy\,dx=\int_0^1\frac{4}{3}y\Big|_{x^3}^{x}dx=\int_0^1\frac{4}{3}(x-x^3)\,dx$$

$$=\frac{4}{3}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1=\frac{4}{3}\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{4}{3}\left(\frac{1}{4}\right)=\frac{1}{3}$$

4.4-11. Let X and Y have the joint pdf $f(x,y)=cx(1-y)$, $0<y<1$, and $0<x<1-y$.

4.4.11

(a) Determine c.

(b) Compute $P(Y<X\mid X\le 1/4)$.

$$\text{(a)}\quad \int_y\int_x f(x,y)\,dx\,dy=\int_0^1\int_0^{1-y} cx(1-y)\,dx\,dy=c\int_0^1(1-y)\frac{x^2}{2}\Big|_0^{1-y}dy=\frac{c}{2}\int_0^1(1-y)^3dy$$

$$=\frac{c}{2}\int_0^1(1-3y+3y^2-y^3)\,dy=\frac{c}{2}\left(y-\frac{3y^2}{2}+y^3-\frac{y^4}{4}\right)\Big|_0^1=\frac{c}{2}\left(1-\frac{3}{2}+1-\frac{1}{4}\right)=1\rightarrow c=8$$

$$\text{(b)}\quad f_X(x)=\int_y f(x,y)\,dy=\int_0^1 8x(1-y)\,dy=\int_0^1(8x-8xy)\,dy=(8xy-4xy^2)\Big|_0^1=4x$$

$$h(y\mid x)=\frac{f(x,y)}{f_X(x)}=\frac{8x(1-y)}{4x}=2(1-y),\quad 0<y<1,\ 0<x<1-y$$

$$P\left(Y<X\mid X\le\frac{1}{4}\right)=\frac{P(Y<X\le 1/4)}{P(X\le 1/4)}=\frac{29/768}{1/8}=\frac{29}{96}$$

$$P\left(Y<X\le\frac{1}{4}\right)=\int_0^{1/4}\int_y^{1/4}8x(1-y)\,dx\,dy=\int_0^{1/4}4x^2(1-y)\Big|_y^{1/4}dy=\int_0^{1/4}4(1-y)\left(\frac{1}{16}-y^2\right)dy$$

$$=4\int_0^{1/4}\left(\frac{1}{16}-y^2-\frac{y}{16}+y^3\right)dy=4\left(\frac{y}{16}-\frac{y^3}{3}-\frac{y^2}{32}+\frac{y^4}{4}\right)\Big|_0^{1/4}=\frac{29}{768}$$

$$P\left(X\le\frac{1}{4}\right)=\int_0^{1/4}f_X(x)\,dx=\int_0^{1/4}4x\,dx=2x^2\Big|_0^{1/4}=\frac{1}{8}$$

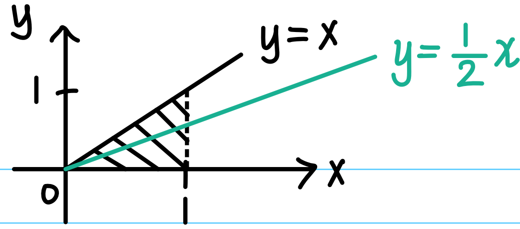
4.4.13

4.4-13. Let X and Y be random variables of the continuous type having the joint pdf

$$f(x, y) = 2, \qquad 0 \le y \le x \le 1.$$

Draw a graph that illustrates the domain of this pdf.
(a) Find the marginal pdfs of X and Y.
(b) Compute μ_X, μ_Y, σ_X^2, σ_Y^2, $\mathrm{Cov}(X, Y)$, and ρ.
(c) Determine the equation of the least squares regression line and draw it on your graph. Does the line make sense to you intuitively?

(a)



(b) $f_X(x) = \int_y f(x,y)\,dy = \int_0^x 2\,dy = 2y\Big|_0^x = 2x, \quad 0 \le x \le 1$

$f_Y(y) = \int_x f(x,y)\,dx = \int_y^1 2\,dx = 2x\Big|_y^1 = 2(1-y), \quad 0 \le y \le 1$

(c) $\mu_X = \int_x x f_X(x)\,dx = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

$\mu_Y = \int_y y f_Y(y)\,dy = \int_0^1 y2(1-y)\,dy = \int_0^1 (2y - 2y^2)\,dy = \left(y^2 - \frac{2y^3}{3}\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

$\sigma_X^2 = \int_x x^2 f_X(x)\,dx - \mu_X^2 = \int_0^1 2x^3\,dx - \frac{4}{9} = \frac{x^4}{2}\Big|_0^1 - \frac{4}{9} = \frac{1}{18}$

$\sigma_Y^2 = \int_y y^2 f_Y(y)\,dy - \mu_Y^2 = \int_0^1 2y^2(1-y)\,dy - \frac{1}{9} = \left(\frac{2y^3}{3} - \frac{y^4}{2}\right)\Big|_0^1 - \frac{1}{9} = \frac{1}{18}$

$\mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y = \int_0^1\int_0^x xy\,f(x,y)\,dy\,dx - \frac{2}{3} \times \frac{1}{3} = \int_0^1\int_0^x 2xy\,dy\,dx - \frac{2}{9}$

$= \int_0^1 xy^2\Big|_0^x\,dx - \frac{2}{9} = \int_0^1 x^3\,dx - \frac{2}{9} = \frac{x^4}{4}\Big|_0^1 - \frac{2}{9} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$

$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{1/36}{\sqrt{\frac{1}{18} \times \frac{1}{18}}} = \frac{1}{2}$

(c) $y = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = \frac{1}{3} + \frac{1}{2} \times \sqrt{\frac{1/18}{1/18}}\left(x - \frac{2}{3}\right) = \frac{1}{2}x$

4.4.15

4.4-15. An automobile repair shop makes an initial estimate X (in thousands of dollars) of the amount of money needed to fix a car after an accident. Say X has the pdf

$$f(x) = 2e^{-2(x-0.2)}, \qquad 0.2 < x < \infty.$$

Given that $X = x$, the final payment Y has a uniform distribution between $x - 0.1$ and $x + 0.1$. What is the expected value of Y?

$Y|_X \sim \mathrm{Unif}(x - 0.1,\ x + 0.1)$

$E[Y] = E[E[Y|X]] = E\left[\frac{x - 0.1 + x + 0.1}{2}\right] = \mu_X = \int_{0.2}^{\infty} x f(x)\,dx$

$$u = x \qquad dv = e^{-2x}dx$$
$$du = dx \qquad v = \frac{e^{-2x}}{-2}$$

$$= \lim_{t\to\infty} \int_{0.2}^{t} 2xe^{-2(x-0.2)}dx = \lim_{t\to\infty}\int_{0.2}^{t} 2xe^{-2x+0.4}dx = \lim_{t\to\infty}\int_{0.2}^{t} 2e^{0.4}xe^{-2x}dx$$

$$= \lim_{t\to\infty} 2e^{0.4}\left(\frac{-xe^{-2x}}{2}\Big|_{0.2}^{t} + \int_{0.2}^{t}\frac{e^{-2x}}{2}dx\right)$$

$$= \lim_{t\to\infty} e^{0.4}\left(xe^{-2x}\Big|_{t}^{0.2} + \int_{0.2}^{t}e^{-2x}dx\right)$$

$$= \lim_{t\to\infty} e^{0.4}\left(0.2e^{-0.4} - te^{-2t} + \frac{e^{-2x}}{2}\Big|_{t}^{0.2}\right)$$

$$= \lim_{t\to\infty} e^{0.4}\left(0.2e^{-0.4} - \cancelto{0}{te^{-2t}} + \frac{1}{2}e^{-0.4} - \frac{1}{2}\cancelto{0}{e^{-2t}}\right) = 0.2 + \frac{1}{2} = 0.7 \rightarrow \$700$$

4.4.17

4.4-17. Let $f(x, y) = 1/40, 0 \le x \le 10, 10-x \le y \le 14-x$, be the joint pdf of X and Y.

(a) Sketch the region for which $f(x, y) > 0$.

(b) Find $f_X(x)$, the marginal pdf of X.

(c) Determine $h(y \mid x)$, the conditional pdf of Y, given that $X = x$.

(d) Calculate $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

(b) $$f_X(x) = \int_y f(x,y)dy = \int_{10-x}^{14-x}\frac{1}{40}dy = \frac{y}{40}\Big|_{10-x}^{14-x} = \frac{14-x-10+x}{40} = \frac{4}{40} = \frac{1}{10} \qquad 0 \le x \le 10$$

(c) $$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/40}{1/10} = \frac{1}{4}, \quad 0 \le x \le 10, \ 10-x \le y \le 14-x$$

(d) $$E[Y|x] = \int_y y\,h(y|x)dy = \int_{10-x}^{14-x}\frac{y}{4}dy = \frac{y^2}{8}\Big|_{10-x}^{14-x} = \frac{1}{8}((14-x)^2 - (10-x)^2)$$
$$= \frac{1}{8}(14-x-10+x)(14-x+10-x) = \frac{4(24-2x)}{8} = 12-x, \quad 0 \le x \le 10$$

4.4.19

4.4-19. Let X have a uniform distribution $U(0, 2)$, and let the conditional distribution of Y, given that $X = x$, be $U(0, x^2)$.

(a) Determine $f(x, y)$, the joint pdf of X and Y.

(b) Calculate $f_Y(y)$, the marginal pdf of Y.

(c) Compute $E(X \mid y)$, the conditional mean of X, given that $Y = y$.

(d) Find $E(Y \mid x)$, the conditional mean of Y, given that $X = x$.

$$X \sim Unif(0,2) \qquad Y|x \sim Unif(0,x^2)$$

(a) $$f_X(x) = \frac{1}{b-a} = \frac{1}{2-0} = \frac{1}{2}, \quad h(y|x) = \frac{1}{b-a} = \frac{1}{x^2-0} = \frac{1}{x^2}$$

$$f(x,y) = f_X(x).\,h(y|x) = \frac{1}{2x^2}, \quad 0<x<2, \ 0<y<x^2 \rightarrow 0<\sqrt{y}<x<2$$

(b) $$f_Y(y) = \int_x f(x,y)dx = \int_{\sqrt{y}}^{2}\frac{1}{2x^2}dx = \frac{1}{2}\int_{\sqrt{y}}^{2}x^{-2}dx = \frac{-1}{2x}\Big|_{\sqrt{y}}^{2} = \frac{-1}{4} + \frac{1}{2\sqrt{y}} = \frac{2-\sqrt{y}}{4\sqrt{y}}$$

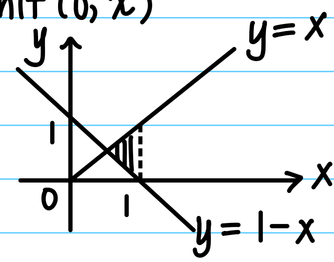
(c) $$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2x^2} \times \frac{4\sqrt{y}}{2-\sqrt{y}} = \frac{2\sqrt{y}}{x^2(2-\sqrt{y})}$$

$$E[X|Y=y] = \int_x x\,g(x|y)\,dx = \int_{\sqrt{y}}^{2} \frac{2x\sqrt{y}}{x^2(2-\sqrt{y})}\,dx = \int_{\sqrt{y}}^{2} \frac{2\sqrt{y}}{x(2-\sqrt{y})}\,dx = \frac{2\sqrt{y}\ln x}{2-\sqrt{y}}\Bigg|_{\sqrt{y}}^{2}$$
$$= \frac{2\sqrt{y}}{2-\sqrt{y}} \ln\left(\frac{2}{\sqrt{y}}\right)$$

(d) $E[Y|x] = \dfrac{a+b}{2} = \dfrac{0+x^2}{2} = \dfrac{x^2}{2}$

$\boxed{4.4.21}$ **4.4-21.** Let X have the uniform distribution $U(0, 1)$, and let the conditional distribution of Y, given $X = x$, be $U(0, x)$. Find $P(X + Y \geq 1)$.

$X \sim Unif(0,1) \qquad Y|_x \sim Unif(0,x)$



$$f_X(x) = \frac{1}{b-a} = \frac{1}{1-0} = 1, \quad 0<x<1$$

$$h(y|x) = \frac{1}{b-a} = \frac{1}{x-0} = \frac{1}{x}, \quad 0<y<x,\ 0<x<1$$

$$f(x,y) = f_X(x).\,h(y|x) = 1 \times \frac{1}{x} = \frac{1}{x}, \quad 0<y<x,\ 0<x<1$$

$$P(X+Y \geq 1) = P(Y \geq 1-X) = \iint_{x\,y} f(x,y)\,dy\,dx = \int_{1/2}^{1}\int_{1-x}^{x} \frac{1}{x}\,dy\,dx = \int_{1/2}^{1} \frac{y}{x}\Bigg|_{1-x}^{x} dx$$
$$= \int_{1/2}^{1} \frac{x-1+x}{x}\,dx = \int_{1/2}^{1} \left(2-\frac{1}{x}\right)dx = (2x - \ln x)\Big|_{1/2}^{1}$$
$$= 2 - \ln 1 - 1 + \ln\frac{1}{2} = 1 - \ln 2$$

Alternative

$$P(A) = \int_x P(A \mid X=x)\, f_X(x)\,dx$$

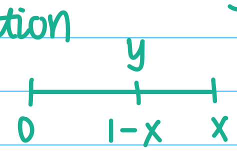
$$P(X+Y \geq 1) = \int P(X+Y \geq 1 \mid X=x)\, f_X(x)\,dx = \int P(x+Y \geq 1 \mid X=x)\, f_X(x)\,dx$$
$$= \int P(Y \geq 1-x \mid X=x)\, f_X(x)\,dx = \int_{1/2}^{1} \frac{x-(1-x)}{x} . 1\,dx = 1 - \ln 2$$

$\rightarrow \dfrac{\text{distance}}{\text{total length}}$

Y is uniform distribution

$\hookrightarrow 0 < y < x$

$\hookrightarrow y \geq 1-x \leftrightarrow x > 1/2$



$1-x < x \rightarrow x > 1/2$

Section 4.5 1, 3, 5, 7, 9, 11, 13

4.5.1

4.5-1. Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3$, $\mu_Y = 10$, $\sigma_X^2 = 25$, $\sigma_Y^2 = 9$, and $\rho = 3/5$. Compute
(a) $P(-5 < X < 5)$.
(b) $P(-5 < X < 5 \mid Y = 13)$.
(c) $P(7 < Y < 16)$.
(d) $P(7 < Y < 16 \mid X = 2)$.

(a) $P(-5<X<5) = P\left(\frac{-5-\mu_X}{\sigma_X} < \frac{X-\mu_X}{\sigma_X} < \frac{5-\mu_X}{\sigma_X}\right) = P\left(\frac{-5-(-3)}{5} < Z < \frac{5-(-3)}{5}\right)$

$= P\left(\frac{-2}{5} < Z < \frac{8}{5}\right) = \Phi(1.6) - \Phi(-0.4) = \Phi(1.6) - (1-\Phi(0.4))$

$= 0.9452 - 1 + 0.6554 = 0.6006$

(b) $\mu_{X|Y=13} = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y) = -3 + \frac{3}{5}\left(\frac{5}{3}\right)(13-10) = 0$

$\sigma_{X|Y=13} = \sqrt{\sigma_X^2(1-\rho^2)} = \sqrt{25(1-(3/5)^2)} = 4$

$P(-5<X<5 \mid Y=13) = P\left(\frac{-5-\mu_{X|y}}{\sigma_{X|y}} < \frac{X-\mu_{X|y}}{\sigma_{X|y}} < \frac{5-\mu_{X|y}}{\sigma_{X|y}}\right)$

$= P\left(\frac{-5-0}{4} < Z < \frac{5-0}{4}\right) = \Phi(1.25) - \Phi(-1.25) = 2\Phi(1.25) - 1$

$= 2(0.8944) - 1 = 0.7888$

(c) $P(7<Y<16) = P\left(\frac{7-\mu_Y}{\sigma_Y} < \frac{Y-\mu_Y}{\sigma_Y} < \frac{16-\mu_Y}{\sigma_Y}\right) = P\left(\frac{7-10}{3} < Z < \frac{16-10}{3}\right)$

$= P(-1<Z<2) = \Phi(2) - \Phi(-1) = \Phi(2) - 1 + \Phi(1) = 0.9772 - 1 + 0.8413 = 0.8185$

(d) $\mu_{Y|X=2} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X) = 10 + \frac{3}{5}\left(\frac{3}{5}\right)(2-(-3)) = 11.8$

$\sigma_{Y|X=2} = \sqrt{\sigma_Y^2(1-\rho^2)} = \sqrt{9(1-(3/5)^2)} = 2.4$

$P(7<Y<16 \mid X=2) = P\left(\frac{7-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{16-\mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= P\left(\frac{7-11.8}{2.4} < Z < \frac{16-11.8}{2.4}\right) = \Phi(1.75) - \Phi(-2) = \Phi(1.75) - 1 + \Phi(2)$

$= 0.9599 - 1 + 0.9772 = 0.9371$

4.5.3

4.5-3. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 2.8$, $\mu_Y = 110$, $\sigma_X^2 = 0.16$, $\sigma_Y^2 = 100$, and $\rho = 0.6$. Compute
(a) $P(108 < Y < 126)$.
(b) $P(108 < Y < 126 \mid X = 3.2)$.

(a) $P(108<Y<126) = P\left(\frac{108-\mu_Y}{\sigma_Y} < \frac{Y-\mu_Y}{\sigma_Y} < \frac{126-\mu_Y}{\sigma_Y}\right) = P\left(\frac{108-110}{10} < Z < \frac{126-110}{10}\right)$

$= P(-0.2 < Z < 1.6) = \Phi(1.6) - \Phi(-0.2) = \Phi(1.6) - 1 + \Phi(0.2)$

$= 0.9452 - 1 + 0.5793 = 0.5245$

(b) $\mu_{Y|X=3.2} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 110 + 0.6\left(\frac{10}{0.4}\right)(3.2 - 2.8) = 116$

$\sigma_{Y|X=3.2} = \sqrt{\sigma_Y^2(1-\rho^2)} = \sqrt{100(1-0.6^2)} = 8$

$P(108 < Y < 126 \mid X = 3.2) = P\left(\frac{108 - \mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y - \mu_{Y|X}}{\sigma_{Y|X}} < \frac{126 - \mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= P\left(\frac{108 - 116}{8} < Z < \frac{126 - 116}{8}\right) = \Phi(1.25) - \Phi(-1) = 0.8944 - 1 + 0.8413 = 0.7357$

4.5-5. Let X denote the height in centimeters and Y the weight in kilograms of male college students. Assume that X and Y have a bivariate normal distribution with parameters $\mu_X = 185$, $\sigma_X^2 = 100$, $\mu_Y = 84$, $\sigma_Y^2 = 64$, and $\rho = 3/5$.

4.5.5

(a) Determine the conditional distribution of Y, given that $X = 190$.

(b) Find $P(86.4 < Y < 95.36 \mid X = 190)$.

(a) $\mu_{Y|X=190} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 84 + \frac{3}{5}\left(\frac{8}{10}\right)(190 - 185) = 86.4$

$\sigma_{Y|X=190} = \sqrt{\sigma_Y^2(1-\rho^2)} = \sqrt{64(1-(3/5)^2)} = 6.4$

$h(y \mid X = 190) \sim N(\mu_{Y|X}, \sigma_{Y|X}^2) = N(86.4, 40.96)$

(b) $P(86.4 < Y < 95.36 \mid X = 190) = P\left(\frac{86.4 - \mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y - \mu_{Y|X}}{\sigma_{Y|X}} < \frac{95.36 - \mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= P\left(\frac{86.4 - 86.4}{6.4} < Z < \frac{95.36 - 86.4}{6.4}\right) = P(0 < Z < 1.4) = \Phi(1.4) - \Phi(0)$

$= 0.9192 - 0.5 = 0.4192$

4.5-7. For a pair of gallinules, let X equal the weight in grams of the male and Y the weight in grams of the female. Assume that X and Y have a bivariate normal distribution with $\mu_X = 415$, $\sigma_X^2 = 611$, $\mu_Y = 347$, $\sigma_Y^2 = 689$, and $\rho = -0.25$. Find

4.5.7

(a) $P(309.2 < Y < 380.6)$.

(b) $E(Y \mid x)$.

(c) $\mathrm{Var}(Y \mid x)$.

(d) $P(309.2 < Y < 380.6 \mid X = 385.1)$.

(a) $P(309.2 < Y < 380.6) = P\left(\frac{309.2 - \mu_Y}{\sigma_Y} < \frac{Y - \mu_Y}{\sigma_Y} < \frac{380.6 - \mu_Y}{\sigma_Y}\right)$

$= P\left(\frac{309.2 - 347}{\sqrt{689}} < Z < \frac{380.6 - 347}{\sqrt{689}}\right) = \Phi(1.28) - \Phi(-1.44)$

$= 0.8997 - 0.0749 = 0.8248$

(b) $\mu_{Y|X=x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 347 + (-0.25)\sqrt{\frac{689}{611}}(x - 415)$

$= -0.2655x + 457.1735$

(c) $\mathrm{Var}(Y \mid X=x) = \sigma_Y^2(1-\rho^2) = 689(1-(-0.25)^2) = 645.9375$

(d) $\mu_{Y|X=385.1} = -0.2655(385.1) + 457.1735 = 354.92945$

$P(309.2 < Y < 380.6 \mid X = 385.1) = P\left(\frac{309.2-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{380.6-\mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= \Phi(1.01) - \Phi(-1.79) = 0.8438 - 0.0367 = 0.8071$

4.5-9. Let X and Y have a bivariate normal distribution. Find two different lines, $a(x)$ and $b(x)$, parallel to and equidistant from $E(Y \mid x)$, such that

$$P[a(x) < Y < b(x) \mid X = x] = 0.9544$$

for all real x. Plot $a(x)$, $b(x)$, and $E(Y \mid x)$ when $\mu_X = 2$, $\mu_Y = -1$, $\sigma_X = 3$, $\sigma_Y = 5$, and $\rho = 3/5$.

4.5.9

$\mu_{Y|X=x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X) = -1 + \frac{3}{5}\left(\frac{5}{3}\right)(x-2) = x-3$

$\sigma_{Y|X=x} = \sqrt{\sigma_Y^2(1-\rho^2)} = \sqrt{5^2(1-(3/5)^2)} = 4$

$P(A < Y < B \mid X = x) = P\left(\frac{A-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{B-\mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= P\left(\frac{A-x+3}{4} < Z < \frac{B-x+3}{4}\right) = \Phi\left(\frac{B-x+3}{4}\right) - \Phi\left(\frac{A-x+3}{4}\right) = 0.9544$

$\therefore \Phi\left(\frac{B-x+3}{4}\right) = 0.9772 \longrightarrow \frac{B-x+3}{4} = 2 \longrightarrow B = x+5$

$\Phi\left(\frac{A-x+3}{4}\right) = 0.0228 \longrightarrow \frac{-A+x-3}{4} = 2 \longrightarrow A = x-11$

4.5-11. For a female freshman in a health fitness program, let X equal her percentage of body fat at the beginning of the program and Y equal the change in her percentage of body fat measured at the end of the program. Assume that X and Y have a bivariate normal distribution with $\mu_X = 24.5$, $\sigma_X^2 = 4.8^2 = 23.04$, $\mu_Y = -0.2$, $\sigma_Y^2 = 3.0^2 = 9.0$, and $\rho = -0.32$. Find

4.5.11

(a) $P(1.3 \le Y \le 5.8)$.

(b) $\mu_{Y|x}$, the conditional mean of Y, given that $X = x$.

(c) $\sigma^2_{Y|x}$, the conditional variance of Y, given that $X = x$.

(d) $P(1.3 \le Y \le 5.8 \mid X = 18)$.

(a) $P(1.3 \le Y \le 5.8) = P\left(\frac{1.3-\mu_Y}{\sigma_Y} \le \frac{Y-\mu_Y}{\sigma_Y} \le \frac{5.8-\mu_Y}{\sigma_Y}\right) = P\left(\frac{1.3-(-0.2)}{3} \le Z \le \frac{5.8-(-0.2)}{3}\right)$

$= \Phi(2) - \Phi(0.5) = 0.9772 - 0.6915 = 0.2857$

(b) $\mu_{Y|X=x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X) = -0.2 + (-0.32)\frac{3}{4.8}(x-24.5) = -0.2x + 4.7$

(c) $\sigma^2_{Y|X=x} = \sigma_Y^2(1-\rho^2) = 9(1-(-0.32)^2) = 8.0784$

(d) $P(1.3 \le Y \le 5.8 \mid X = 18) = P\left(\frac{1.3-\mu_{Y|X}}{\sigma_{Y|X}} \le \frac{Y-\mu_{Y|X}}{\sigma_{Y|X}} \le \frac{5.8-\mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= P\left(\frac{1.3-1.1}{2.84} \le Z \le \frac{5.8-1.1}{2.84}\right) = \Phi(1.65) - \Phi(0.07) = 0.9505 - 0.5279 = 0.4226$

4.5.13

4.5-13. An obstetrician does ultrasound examinations on her patients between their 16th and 25th weeks of pregnancy to check the growth of each fetus. Let X equal the widest diameter of the fetal head, and let Y equal the length of the femur, both measurements in mm. Assume that X and Y have a bivariate normal distribution with $\mu_X = 60.6$, $\sigma_X = 11.2$, $\mu_Y = 46.8$, $\sigma_Y = 8.4$, and $\rho = 0.94$.

(a) Find $P(40.5 < Y < 48.9)$.

(b) Find $P(40.5 < Y < 48.9 \mid X = 68.6)$.

(a) $P(40.5 < Y < 48.9) = P\left(\frac{40.5-\mu_Y}{\sigma_Y} < \frac{Y-\mu_Y}{\sigma_Y} < \frac{48.9-\mu_Y}{\sigma_Y}\right)$

$= P\left(\frac{40.5-46.8}{8.4} < Z < \frac{48.9-46.8}{8.4}\right) = \Phi(0.25) - \Phi(-0.75)$

$= 0.5987 - 0.2266 = 0.3721$

(b) $\mu_{Y|X=68.6} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X) = 46.8 + 0.94\left(\frac{8.4}{11.2}\right)(68.6-60.6) = 52.44$

$\sigma_{Y|X=68.6} = \sqrt{\sigma_Y^2(1-\rho^2)} = \sqrt{8.4^2(1-0.94^2)} = \sqrt{8.213184}$

$P(40.5 < Y < 48.9 \mid X = 68.6) = P\left(\frac{40.5-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{Y-\mu_{Y|X}}{\sigma_{Y|X}} < \frac{48.9-\mu_{Y|X}}{\sigma_{Y|X}}\right)$

$= \Phi(-1.24) - \Phi(-4.17) = 0.1075 - 0 = 0.1075$

Section 5.1

1, 3, 5, 7, 9, $\overset{\times}{11}$, 13, 15, 17

$f_Y(y) = P(Y=y) = P(Y=K^2) = P(X=K)$

5.1.1

5.1-1. Let X have a geometric distribution with parameter p. Find the pmf of:
(a) $Y = 2X$.
(b) $Y = X^2$.

$X \sim Geom(p) \rightarrow f(x) = q^{x-1}p, \quad x = 1,2,3 \dots n$

(a) $g(y) = q^{y/2 - 1}p, \quad y = 2,4,6 \dots 2n$

(b) $g(y) = q^{\sqrt{y}-1}p, \quad y = 1,4,9 \dots n^2$

5.1.3

5.1-3. Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$. Find the pdf of $Y = X^2$.

$y = g(x) = x^2 \longrightarrow g'(x) = 2x, \; g^{-1}(y) = \sqrt{y}$

$$f_Y(y) = \sum_y f_X(x)\frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = \frac{4x^3}{2x}\Big|_{x=\sqrt{y}} = 2x^2\Big|_{x=\sqrt{y}} = 2y, \; 0<y<1$$

5.1.5

5.1-5. Let X have a gamma distribution with $\alpha = 3$ and $\theta = 2$. Determine the pdf of $Y = \sqrt{X}$.

$X \sim Gamma(3,2)$

$y = g(x) = \sqrt{x} \rightarrow x = g^{-1}(y) = y^2, \; g'(x) = \frac{1}{2\sqrt{x}}$

$$f_Y(y) = \sum_y f_X(x)\frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = 2\sqrt{x}\, f_X(x)\Big|_{x=y^2}$$

$$= 2\sqrt{x}\,\frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha}e^{-x/\theta}\Big|_{x=y^2} = 2\sqrt{y^2}\,\frac{(y^2)^{3-1}}{\Gamma(3)\,2^3}e^{-y^2/2}$$

$$= \frac{2y(y^4)}{(3-1)!\,8}e^{-y^2/2} = \frac{1}{8}y^5e^{-y^2/2}, \; 0<y<\infty$$

5.1.7

5.1-7. The pdf of X is $f(x) = \theta x^{\theta-1}$, $0<x<1$, $0<\theta<\infty$. Let $Y = -2\theta \ln X$. How is Y distributed?

$y = g(x) = -2\theta \ln x \longrightarrow x = g^{-1}(y) = e^{\frac{-y}{2\theta}}, \; g'(x) = \frac{-2\theta}{x}$

$$f_Y(y) = \sum_y f_X(x)\frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = \frac{\theta x^{\theta-1}}{2\theta/x}\Big|_{x=e^{\frac{-y}{2\theta}}} = \frac{1}{2}x^\theta\Big|_{x=e^{-y/2\theta}}$$

$$= \frac{1}{2}(e^{-y/2\theta})^\theta = \frac{1}{2}e^{-y/2} = \frac{1}{\theta}e^{-y/\theta} \longrightarrow Y \sim Exp(2)$$

5.1.9

5.1-9. A sum of \$50,000 is invested at a rate R, selected from a uniform distribution on the interval (0.03, 0.07). Once R is selected, the sum is compounded instantaneously for a year, so that $X = 50000e^R$ dollars is the amount at the end of that year.

(a) Find the cdf and pdf of X.

(b) Verify that $X = 50000e^R$ is defined correctly if the compounding is done instantaneously. Hint: Divide the year into n equal parts, calculate the value of the amount at the end of each part, and then take the limit as $n \to \infty$.

$R \sim Unif(0.03, 0.07) \longrightarrow$ pdf: $f(R) = \frac{1}{0.04} = 25$

$X = 50000e^R$

(a) cdf: $F_X(x) = P(X \le x) = P(50000e^R \le x) = P(R \le \ln(\frac{X}{50000})) = F_R(\ln\frac{X}{50000})$

pdf: $f_X(x) = \frac{d}{dx}F_X(x) = f_R(\ln\frac{X}{50000}) \cdot \frac{1}{X} = \frac{25}{X}, \; 50000e^{0.03} < x < 50000e^{0.07}$

$$\text{cdf}: F_X(x) = \int_{50000e^{0.03}}^{x} 25/x\,dx = 25\ln x \Big|_{50000e^{0.03}}^{x} = \boxed{25\ln x - 25\ln 50000 - 25(0.03) \quad (50000e^{0.03} < x < 50000e^{0.07})}$$

(b) $X = 50000\left(1+\frac{R}{n}\right)^n \xrightarrow{n\to\infty} \boxed{50000e^{R}}$

5.1.13

5.1-13. Let X have a Cauchy distribution. Find
(a) $P(X > 1)$.
(b) $P(X > 5)$.
(c) $P(X > 10)$.

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

(a) $P(X>1) = \int_1^\infty \frac{1}{\pi(1+x^2)}dx = \frac{\arctan x}{\pi}\Big|_1^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \boxed{0.25}$

(b) $P(X>5) = \int_5^\infty \frac{1}{\pi(1+x^2)}dx = \frac{\arctan x}{\pi}\Big|_5^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan 5\right) = \boxed{0.0628}$

(c) $P(X>10) = \int_{10}^\infty \frac{1}{\pi(1+x^2)}dx = \frac{\arctan x}{\pi}\Big|_{10}^\infty = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan 10\right) = 0.0317$

5.1.15

5.1-15. If X is $N(\mu, \sigma^2)$, then $M(t) = E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$, $-\infty < t < \infty$. We then say that $Y = e^X$ has a **lognormal distribution** because $X = \ln Y$.
(a) Show that the pdf of Y is

$$g(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp[-(\ln y - \mu)^2/2\sigma^2], \qquad 0 < y < \infty.$$

(b) Using $M(t)$, find **(i)** $E(Y) = E(e^X) = M(1)$, **(ii)** $E(Y^2) = E(e^{2X}) = M(2)$, and **(iii)** $\text{Var}(Y)$.

(a) $y = e^x \longrightarrow x = \ln y = g^{-1}(y)$, $g'(x) = e^x$

$$f_Y(y) = \sum_y f_X(x) \cdot \frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = \frac{f_X(x)}{e^x}\Big|_{x=\ln y}$$

$$= \frac{1}{e^{\ln y}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-1}{2\sigma^2}(\ln y - \mu)^2} = \frac{1}{y\sqrt{2\pi}\,\sigma} e^{\frac{-(\ln y-\mu)^2}{2\sigma^2}}, \quad 0<y<\infty$$

Q.E.D!

(b) $E[Y] = E[e^X] = M_X(1) = \boxed{e^{\mu + \frac{\sigma^2}{2}}}$ $\qquad \text{Var}(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = E[e^{2X}] = M_X(2) = \boxed{e^{2\mu + 2\sigma^2}}$ $\qquad = \boxed{e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}}$

5.1.17

5.1-17. Let $Y = X^2$.
(a) Find the pdf of Y when the distribution of X is $N(0, 1)$.
(b) Find the pdf of Y when the pdf of X is $f(x) = (3/2)x^2$, $-1 < x < 1$.

$y = x^2 \rightarrow x = \pm\sqrt{y} = g^{-1}(y)$, $g'(x) = 2x$

(a) $f_Y(y) = \sum_y f_X(x) \cdot \frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = \frac{f_X(x)}{|2x|}\Big|_{x=\pm\sqrt{y}}$

$= \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(-\sqrt{y})^2}{2}} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(\sqrt{y})^2}{2}} = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2}, & 0<y<\infty \\ 0, & y \leq 0 \end{cases}$

(b) $f_Y(y) = \sum_y f_X(x) \cdot \frac{1}{|g'(x)|}\Big|_{x=g^{-1}(y)} = \frac{3/2\, x^2}{|2x|}\Big|_{x=\pm\sqrt{y}}$

$= \frac{3}{4}\left(\frac{(-\sqrt{y})^2}{|-\sqrt{y}|} + \frac{(\sqrt{y})^2}{|\sqrt{y}|}\right) = \frac{3y}{2\sqrt{y}} = \begin{cases} \frac{3}{2}\sqrt{y}, & 0<y<1 \\ 0, & \text{else} \end{cases}$

Section 5.2 1, 3, 5, 7, 9, 11

5.2.1

5.2-1. Let X_1, X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 + X_1$. Note that the support of Y_1, Y_2 is $0 < y_1 < y_2 < \infty$. Also, find the marginal pdf of each of Y_1 and Y_2. Are Y_1 and Y_2 independent?

$X_1 \sim \chi^2(2) \sim$ Gamma$(1,2)$ $Y_1 = X_1$

$X_2 \sim \chi^2(2)$ $Y_2 = X_1 + X_2 \rightarrow X_2 = Y_2 - Y_1$

$$f_Y(y_1, y_2) = f_X(x_1, x_2)\left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| = f_X(x_1).f_X(x_2)\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = f_X(x_1).f_X(x_2)$$

$$= \frac{x_1^{\alpha-1} e^{-x_1/\theta}}{\Gamma(\alpha)\theta^\alpha} \cdot \frac{x_2^{\alpha-1} e^{-x_2/\theta}}{\Gamma(\alpha)\theta^\alpha} = \frac{x_1^{1-1} e^{-x_1/2}}{\Gamma(1)2^1} \cdot \frac{x_2^{1-1} e^{-x_2/2}}{\Gamma(1)2^1}$$

$$= \frac{e^{-(x_1+x_2)/2}}{2(2)} = \frac{1}{4}e^{-y_2/2}, \quad 0 < y_1 < y_2 < \infty$$

$$f_Y(y_1) = \int_{y_2} f_Y(y_1, y_2)\,dy_2 = \int_{y_1}^{\infty} \frac{1}{4}e^{-y_2/2}\,dy_2 = \frac{-1}{2}e^{-y_2/2}\Big|_{y_1}^{\infty} = \frac{1}{2}e^{-y_2/2}\Big|_{\infty}^{y_1}$$

$$= \frac{1}{2}\left(e^{-y_1/2} - \cancel{e^{-y_2/2}}_{\to 0}\right) = \boxed{\frac{1}{2}e^{-y_1/2}}, \quad 0 < y_1 < \infty$$

$$f_Y(y_2) = \int_{y_1} f_Y(y_1, y_2)\,dy_1 = \int_0^{y_2} \frac{1}{4}e^{-y_2/2}\,dy_1 = \frac{1}{4}y_1 e^{-y_2/2}\Big|_0^{y_2} = \boxed{\frac{1}{4}y_2 e^{-y_2/2}}$$

$0 < y_2 < \infty$

$f_Y(y_1)\,f_Y(y_2) \neq f_Y(y_1, y_2) \rightarrow$ Y_1, Y_2 not independent

5.2.3

5.2-3. Find the mean and the variance of an F random variable with r_1 and r_2 degrees of freedom by first finding $E(U)$, $E(1/V)$, $E(U^2)$, and $E(1/V^2)$.

$U \sim \chi^2(r_1)$ $F = \dfrac{U/r_1}{V/r_2}$ U, V independent

$V \sim \chi^2(r_2)$

$$E[U] = r_1 \rightarrow E[U^2] = Var(U) + E[U]^2 = 2r_1 + r_1^2$$

$$E\left[\frac{1}{V}\right] = \int_0^\infty \frac{1}{v}\,\frac{v^{r_2/2-1} e^{-v/2}}{\Gamma(r_2/2)2^{r_2/2}}\,dv = \frac{\Gamma(\frac{r_2}{2}-1)}{2\Gamma(r_2/2)}\int_0^\infty \underbrace{\frac{v^{r_2/2-2} e^{-v/2}}{2^{r_2/2-1}\Gamma(\frac{r_2}{2}-1)}}_{\text{Gamma}(\frac{r_2}{2}-1,\,2)}\,dv$$

$$= \frac{\Gamma(\frac{r_2}{2}-1)}{2(\frac{r_2}{2}-1)\Gamma(\frac{r_2}{2}-1)} = \frac{1}{r_2-2}$$

$$E\left[\frac{1}{V^2}\right] = \int_0^\infty \frac{1}{v^2}\,\frac{v^{r_2/2-1} e^{-v/2}}{\Gamma(r_2/2)2^{r_2/2}}\,dv = \frac{\Gamma(\frac{r_2}{2}-2)}{4\Gamma(\frac{r_2}{2})}\int_0^\infty \underbrace{\frac{v^{r_2/2-3} e^{-v/2}}{2^{r_2/2-2}\Gamma(\frac{r_2}{2}-2)}}_{\text{Gamma}(\frac{r_2}{2}-2,\,2)}\,dv$$

$$= \frac{\Gamma(\frac{r_2}{2}-2)}{4(\frac{r_2}{2}-1)(\frac{r_2}{2}-2)\Gamma(\frac{r_2}{2}-2)} = \frac{1}{(r_2-2)(r_2-4)}$$

$$\mu = E[F] = E\left[\frac{U/r_1}{V/r_2}\right] = \frac{r_2}{r_1} E[U]\, E\left[\frac{1}{V}\right] = \frac{r_2}{r_1}(r_1)\frac{1}{r_2-2} = \frac{r_2}{r_2-2}, \quad r_2 > 2$$

$$\sigma^2 = V(F) = E[F^2] - \mu^2 = E\left[\frac{U^2/r_1^2}{V^2/r_2^2}\right] = \frac{r_2^2}{r_1^2} E[U^2]\, E\left[\frac{1}{V^2}\right] - \left(\frac{r_2}{r_2-2}\right)^2$$

$$= \frac{r_2^2}{r_1^2}\left(2r_1 + r_1^2\right)\frac{1}{(r_2-2)(r_2-4)} - \frac{r_2^2}{(r_2-2)^2}$$

$$= \frac{r_2^2(2+r_1)(r_2-2) - r_2^2 r_1(r_2-4)}{r_1(r_2-2)^2(r_2-4)}$$

$$= \frac{r_2^2(2r_2 - 4 + \cancel{r_1 r_2} - 2r_1 - \cancel{r_1 r_2} + 4r_1)}{r_1(r_2-2)^2(r_2-4)} = \frac{2r_2^2(r_1+r_2-2)}{r_1(r_2-2)^2(r_2-4)}$$

$\boxed{5.2.5}$

5.2-5. Let the distribution of W be $F(8,4)$. Find the following:
(a) $F_{0.01}(8,4)$.
(b) $F_{0.99}(8,4)$.
(c) $P(0.198 \le W \le 8.98)$.

(a) $F_{0.01}(8,4) = 14.80$

(b) $F_{0.99}(8,4) = \dfrac{1}{F_{0.01}(4,8)} = \dfrac{1}{7.01} = 0.1427$

(c) $P(0.198 \le W \le 8.98) = 0.95$

$\boxed{5.2.7}$

5.2-7. Let X_1 and X_2 be independent chi-square random variables with r_1 and r_2 degrees of freedom, respectively. Show that
(a) $U = X_1/(X_1+X_2)$ has a beta distribution with $\alpha = r_1/2$ and $\beta = r_2/2$.
(b) $V = X_2/(X_1+X_2)$ has a beta distribution with $\alpha = r_2/2$ and $\beta = r_1/2$.

$X_1 \sim \chi^2(r_1) = \text{Gamma}\left(\frac{r_1}{2}, 2\right)$
$X_2 \sim \chi^2(r_2)$

(a) $U = \dfrac{X_1}{X_1+X_2} \rightarrow X_1 = UV$

$V = X_1 + X_2 \rightarrow X_2 = V - X_1 = V - UV = V(1-U)$

$$f(u,v) = f_X(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right| = f_X(x_1).f_X(x_2)\begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= \frac{x_1^{\alpha-1}e^{-x_1/\theta}}{\Gamma(\alpha)\theta^\alpha}\cdot\frac{x_2^{\alpha-1}e^{-x_2/\theta}}{\Gamma(\alpha)\theta^\alpha}\,|v - \cancel{uv} + \cancel{uv}|$$

$$= \frac{x_1^{\frac{r_1}{2}-1}e^{-x_1/2}}{\Gamma\left(\frac{r_1}{2}\right)2^{r_1/2}}\cdot\frac{x_2^{\frac{r_2}{2}-1}e^{-x_2/2}}{\Gamma\left(\frac{r_2}{2}\right)2^{r_2/2}}(v) = \frac{x_1^{\frac{r_1}{2}-1}x_2^{\frac{r_2}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)2^{r_1/2}2^{r_2/2}}\, v e^{-v/2}$$

$$= \frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \cdot \frac{(uv)^{\frac{r_1}{2}-1}(v(1-u))^{\frac{r_2}{2}-1}}{2^{r_1/2+r_2/2}\Gamma(\frac{r_1}{2}+\frac{r_2}{2})} ve^{-v/2}$$

$$= \frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \cdot u^{\frac{r_1}{2}-1}(1-u)^{\frac{r_2}{2}-1} \underbrace{\frac{v^{\frac{r_1}{2}+\frac{r_2}{2}-1}e^{-v/2}}{2^{r_1/2+r_2/2}\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}}_{\text{Gamma}(\frac{r_1}{2}+\frac{r_2}{2},\,2)}$$

$$f_U(u) = \int_v f(u,v)\,dv = \frac{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \cdot u^{\frac{r_1}{2}-1}(1-u)^{\frac{r_2}{2}-1}$$

$\therefore U \sim \text{Beta}(\frac{r_1}{2}, \frac{r_2}{2})$ Q.E.D!

(b) $V = \frac{X_2}{X_1+X_2} \longrightarrow X_2 = UV$

$U = X_1 + X_2 \longrightarrow X_1 = U - X_2 = U - UV = U(1-V)$

$$f(u,v) = f_X(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right| = f_X(x_1).f_X(x_2)\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= \frac{x_1^{\frac{r_1}{2}-1}e^{-x_1/2}}{\Gamma(\frac{r_1}{2})2^{r_1/2}} \cdot \frac{x_2^{\frac{r_2}{2}-1}e^{-x_2/2}}{\Gamma(\frac{r_2}{2})2^{r_2/2}} |u - \cancel{uv} + \cancel{uv}|$$

$$= \frac{(u(1-v))^{\frac{r_1}{2}-1}(uv)^{\frac{r_2}{2}-1}e^{-u/2}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{r_1/2+r_2/2}}(u)$$

$$= \frac{v^{\frac{r_2}{2}-1}(1-v)^{\frac{r_1}{2}-1}\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \cdot \underbrace{\frac{u^{\frac{r_1}{2}+\frac{r_2}{2}-1}e^{-u/2}}{\Gamma(\frac{r_1}{2}+\frac{r_2}{2})2^{r_1/2+r_2/2}}}_{\text{Gamma}(\frac{r_1}{2}+\frac{r_2}{2},\,2)}$$

$$f_V(v) = \int_u f(u,v)\,du = \frac{v^{\frac{r_2}{2}-1}(1-v)^{\frac{r_1}{2}-1}\Gamma(\frac{r_1}{2}+\frac{r_2}{2})}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})}$$

$\therefore V \sim \text{Beta}(\frac{r_2}{2}, \frac{r_1}{2})$ Q.E.D!

$\boxed{5.2.9}$

5.2-9. Determine the constant c such that $f(x) = cx^3(1-x)^6$, $0 < x < 1$, is a pdf.

$$\int_0^1 f(x)dx = c\int_0^1 x^3(1-x)^6 dx = c\frac{\Gamma(4)\Gamma(7)}{\Gamma(4+7)}\underbrace{\int_0^1 \frac{\Gamma(4+7)}{\Gamma(4)\Gamma(7)} x^{4-1}(1-x)^{7-1}dx}_{Beta(4,7)}$$

$$= c\,\frac{3!6!}{10!} = 1 \rightarrow c = 840$$

$\boxed{5.2.11}$

5.2-11. Evaluate

$$\int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2\, dy$$

(a) Using integration.

(b) Using the result of Exercise 5.2-10.

(a) $$\int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2dy = \int_0^{0.4} \frac{6!}{3!\,2!} y^3(1-y)^2 dy = \int_0^{0.4} 60y^3(1-2y+y^2)dy$$

$$= \int_0^{0.4} 60(y^3-2y^4+y^5)dy = 60\left(\frac{y^4}{4} - \frac{2y^5}{5} + \frac{y^6}{6}\right)\Big|_0^{0.4} = 60\left(\frac{0.4^4}{4} - \frac{2(0.4)^5}{5} + \frac{0.4^6}{6}\right) = 0.1792$$

(b) $$\int_0^p \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}\, dy = \sum_{y=\alpha}^{n} \binom{n}{y} p^y(1-p)^{n-y},$$ $n = \alpha + \beta - 1 = 4 + 3 - 1 = 6$

$$\int_0^{0.4} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} y^3(1-y)^2dy = \sum_{y=4}^{6} \binom{6}{y} 0.4^y (1-0.4)^{6-y} = 0.1792$$

Section 5.3   1, 3, 5, 7, 9

5.3.1

5.3-1. Let X_1 and X_2 be independent Poisson random variables with respective means $\lambda_1 = 2$ and $\lambda_2 = 3$. Find
(a) $P(X_1 = 3, X_2 = 5)$.
(b) $P(X_1 + X_2 = 1)$.
HINT. Note that this event can occur if and only if $\{X_1 = 1, X_2 = 0\}$ or $\{X_1 = 0, X_2 = 1\}$.

$X_1 \sim \text{Poisson}(2)$
$X_2 \sim \text{Poisson}(3)$

(a) $P(X_1 = 3, X_2 = 5) = P(X_1 = 3)P(X_2 = 5) = \frac{2^3}{3!}e^{-2} \times \frac{3^5}{5!}e^{-3} = 0.0182$

(b) $P(X_1 + X_2 = 1) = P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0)$

$$= \frac{2^0}{0!}e^{-2} \times \frac{3^1}{1!}e^{-3} + \frac{2^1}{1!}e^{-2} \times \frac{3^0}{0!}e^{-3} = 0.0337$$

5.3.3

5.3-3. Let X_1 and X_2 be independent random variables with probability density functions $f_1(x_1) = 2x_1$, $0 < x_1 < 1$, and $f_2(x_2) = 4x_2^3$, $0 < x_2 < 1$, respectively. Compute
(a) $P(0.5 < X_1 < 1 \text{ and } 0.4 < X_2 < 0.8)$.
(b) $E(X_1^2 X_2^3)$.

$X_1 \sim f_1(x_1) = 2x_1$ $X_2 \sim f(x_2) = 4x_2^3$

(a) $P(0.5 < X_1 < 1 \cap 0.4 < X_2 < 0.8) = P(0.5 < X_1 < 1) \cdot P(0.4 < X_2 < 0.8)$

$$= \int_{0.5}^{1} f(x_1)\,dx_1 \times \int_{0.4}^{0.8} f(x_2)\,dx_2 = \int_{0.5}^{1} 2x_1\,dx_1 \times \int_{0.4}^{0.8} 4x_2^3\,dx_2 = x_1^2\Big|_{0.5}^{1} \times x_2^4\Big|_{0.4}^{0.8} = \frac{36}{125}$$

(b) $E[X_1^2 X_2^3] = \int_0^1\int_0^1 x_1^2 x_2^3 f(x_1, x_2)\,dx_1 dx_2 = \int_0^1\int_0^1 x_1^2 x_2^3 f(x_1) \cdot f(x_2)\,dx_1 dx_2$

$$= \int_0^1\int_0^1 x_1^2 x_2^3 (2x_1)(4x_2^3)\,dx_1 dx_2 = \int_0^1 2x_1^3\,dx_1 \int_0^1 4x_2^6\,dx_2 = \frac{x_1^4}{2}\Big|_0^1 \times \frac{4x_2^7}{7}\Big|_0^1 = \frac{2}{7}$$

5.3.5

5.3-5. Let X_1 and X_2 be observations of a random sample of size $n = 2$ from a distribution with pmf $f(x) = x/6$, $x = 1, 2, 3$. Then find the pmf of $Y = X_1 + X_2$. Determine the mean and the variance of the sum in two ways.

$f(x_1, x_2) = f(x_1) \cdot f(x_2)$

1st way

$P(Y = 2) = P(X_1 = 1, X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 1) = 1/6 \times 1/6 = 1/36$

$P(Y = 3) = P(X_1 = 1, X_2 = 2) + P(X_1 = 2, X_2 = 1) = 2(1/6 \times 2/6) = 4/36$

$P(Y = 4) = P(X_1 = 2, X_2 = 2) + P(X_1 = 1, X_2 = 3) + P(X_1 = 3, X_2 = 1)$

$= 2/6 \times 2/6 + 1/6 \times 3/6 + 3/6 \times 1/6 = 10/36$

$P(Y = 5) = P(X_1 = 2, X_2 = 3) + P(X_1 = 3, X_2 = 2) = 2/6 \times 3/6 + 3/6 \times 2/6 = 12/36$

$P(Y = 6) = P(X_1 = 3, X_2 = 3) = 3/6 \times 3/6 = 9/36$

y	2	3	4	5	6
f(y)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

$\mu_Y = 2 \times \frac{1}{36} + 3 \times \frac{4}{36} + 4 \times \frac{10}{36} + 5 \times \frac{12}{36} + 6 \times \frac{9}{36} = \frac{14}{3}$

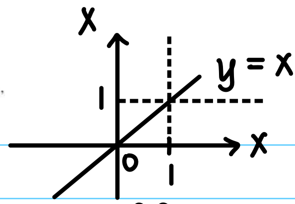
$\sigma_Y^2 = \frac{2^2 + 3^2 \times 4 + 4^2 \times 10 + 5^2 \times 12 + 6^2 \times 9}{36} - \left(\frac{14}{3}\right)^2 = \frac{10}{9}$

5.3.7

5.3-7. The distributions of incomes in two cities follow the two Pareto-type pdfs

$$f(x) = \frac{2}{x^3}, \quad 1 < x < \infty, \quad \text{and} \quad g(y) = \frac{3}{y^4}, \quad 1 < y < \infty,$$

respectively. Here one unit represents \$20,000. One person with income is selected at random from each city. Let X and Y be their respective incomes. Compute $P(X < Y)$.



$$P(X<Y) = \iint_{x\,y} f(x,y)\,dx\,dy = \iint_{x\,y} f(x)g(y)\,dx\,dy = \int_1^\infty\int_x^\infty \frac{2}{x^3}\cdot\frac{3}{y^4}\,dy\,dx = \int_1^\infty\int_x^\infty \frac{6}{x^3}y^{-4}\,dy\,dx$$

$$= \int_1^\infty \frac{6}{x^3}\left(\frac{1}{-3}\right)y^{-3}\Big|_x^\infty dx = \int_1^\infty \frac{2}{x^3}y^{-3}\Big|_\infty^x dx = \int_1^\infty \frac{2}{x^3}(x^{-3})\,dx = \int_1^\infty 2x^{-6}dx = \frac{2}{5}x^{-5}\Big|_\infty^1 = \frac{2}{5}$$

5.3.9

5.3-9. Let $X_1, X_2, \ldots, X_n$ be a random sample (of size n) from a distribution that has mean μ and variance σ^2. Define $Y_1 = X_1$, $Y_2 = X_1 - X_2$, $Y_3 = X_1 - X_2 + X_3, \ldots$, $Y_n = \sum_{i=1}^n (-1)^{i-1}X_i$. Find $E(Y_k)$ and $\mathrm{Var}(Y_k)$ for $k = 1, \ldots, n$.

$$E[Y_k] = E\left[\sum_{i=1}^k (-1)^{i-1}x_i\right] = \sum_{i=1}^k (-1)^{i-1}E[X_i] = \mu\sum_{i=1}^k(-1)^{i-1} = \begin{cases} 0, & k \text{ even} \\ \mu, & k \text{ odd}\end{cases}$$

$$\mathrm{Var}(Y_k) = \mathrm{Var}\left(\sum_{i=1}^k (-1)^{i-1}x_i\right) = \sum_{i=1}^k (-1)^{2(i-1)}\mathrm{Var}(X_i) = \sum_{i=1}^k 1^{i-1}\sigma^2 = k\sigma^2$$

2nd way

$$E[Y] = E[X_1 + X_2] = E[X_1] + E[X_2] = 2\left(1\times\frac{1}{6} + 2\times\frac{2}{6} + 3\times\frac{3}{6}\right) = \frac{14}{3}$$

inde

$$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 = 2\left(1^2\times\frac{1}{6} + 2^2\times\frac{2}{6} + 3^2\times\frac{3}{6} - \left(\frac{14}{3}\right)^2\right) = \frac{10}{9}$$

Section 5.4

1, 3, 5, 7, 9, 11, 21

5.4.1

5.4-1. Let X_1, X_2, X_3 be a random sample of size 3 from the distribution with pmf $f(x) = 1/4$, $x = 1, 2, 3, 4$. For example, observe three independent rolls of a fair four-sided die.

(a) Find the pmf of $Y = X_1 + X_2 + X_3$.

(b) Sketch a bar graph of the pmf of Y.

$$X_i \sim f(x) = \frac{1}{4} \longrightarrow f_Y(y) = f(x_1) \cdot f(x_2) \cdot f(x_3)$$

(a)

Y	3	4	5	6	7	8	9	10	11	12
$f_Y(y)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{6}{64}$	$\frac{10}{64}$	$\frac{12}{64}$	$\frac{12}{64}$	$\frac{10}{64}$	$\frac{6}{64}$	$\frac{3}{64}$	$\frac{1}{64}$

5.4.3

5.4-3. Let X_1, X_2, X_3 be mutually independent random variables with Poisson distributions having means 2, 1, and 4, respectively.

(a) Find the mgf of the sum $Y = X_1 + X_2 + X_3$.

(b) How is Y distributed?

(c) Compute $P(3 \le Y \le 9)$.

$$X_1 \sim \text{Poisson}(2) \longrightarrow M_{X_1}(t) = e^{\lambda(e^t - 1)}$$

(a) $$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2+X_3)}] \overset{\text{inde.}}{=} E[e^{tX_1}]E[e^{tX_2}]E[e^{tX_3}]$$
$$= e^{2(e^t-1)} \cdot e^{1(e^t-1)} \cdot e^{4(e^t-1)} = e^{7(e^t-1)}$$

(b) $Y \sim \text{Poisson}(7)$

(c) $$P(3 \le Y \le 9) = \sum_{k=3}^{9} \frac{7^k}{k!} e^{-7} = 0.8009$$

5.4.5

5.4-5. Let $Z_1, Z_2, \ldots, Z_7$ be a random sample from the standard normal distribution $N(0, 1)$. Let $W = Z_1^2 + Z_2^2 + \cdots + Z_7^2$. Find $P(1.69 < W < 14.07)$.

$W \sim \chi^2(7)$

$$P(1.69 < W < 14.07) = P(W < 14.07) - P(W < 1.69) = 0.95 - 0.025 = 0.925$$

5.4.7

5.4-7. Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$.

(a) Find the mgf of $Y = X_1 + X_2 + X_3$.

(b) How is Y distributed?

(c) How is $\overline{X}$ distributed?

$$X_i \sim \text{Gamma}(7, 5) \longrightarrow M_{X_i}(t) = \frac{1}{(1-5t)^7}$$

(a) $$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2+X_3)}] \overset{\text{inde.}}{=} E[e^{tX_1}]E[e^{tX_2}]E[e^{tX_3}]$$
$$= \left(\frac{1}{(1-5t)^7}\right)^3 = \frac{1}{(1-5t)^{21}}$$

(b) $Y \sim \text{Gamma}(21, 5)$

(c) $$M_{\overline{X}}(t) = [M_X(\tfrac{t}{n})]^n = \left(\frac{1}{(1-5(\frac{t}{3}))^7}\right)^3 = \frac{1}{(1-\frac{5}{3}t)^{21}} \longrightarrow \overline{X} \sim \text{Gamma}(21, \tfrac{5}{3})$$

5.4.9

5.4-9. Let X and Y, with respective pmfs $f(x)$ and $g(y)$, be independent discrete random variables, each of whose support is a subset of the nonnegative integers $0, 1, 2, \ldots$. Show that the pmf of $W = X + Y$ is given by the **convolution formula**

$$h(w) = \sum_{x=0}^{w} f(x)g(w-x), \qquad w = 0, 1, 2, \ldots.$$

HINT: Argue that $h(w) = P(W = w)$ is the probability of the $w + 1$ mutually exclusive events $\{X = x, Y = w - x\}$, $x \in \{0, 1, \ldots, w\}$.

$$h(w) = P(W = w) = \sum_{x=0}^{w} P(X = x, Y = w - x)$$

$$\overset{\text{inde.}}{=} \sum_{x=0}^{w} P(X = x)P(Y = w - x) = \sum_{x=0}^{w} f(x)g(w-x)$$ Q.E.D!

5.4.11

5.4-11. Let X and Y equal the outcomes when two fair six-sided dice are rolled. Let $W = X + Y$. Assuming independence, find the pmf of W when

(a) The first die has three faces numbered 0 and three faces numbered 2, and the second die has its faces numbered 0, 1, 4, 5, 8, and 9.

(b) The faces on the first die are numbered 0, 1, 2, 3, 4, and 5, and the faces on the second die are numbered 0, 6, 12, 18, 24, and 30.

(a) $f_X(x) = \frac{1}{2}$, $x = 0, 2$ $f_Y(y) = \frac{1}{6}$, $y = 0, 1, 4, 5, 8, 9$ $\xrightarrow{\text{inde.}}$ $f_W(w) = f_X(x) f_Y(y)$

W	0	1	2	3	4	5	6	7	8	9	10	11
$f_W(w)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

$f_W(w) = \frac{1}{12}$, $w = 0, 1 \dots 11$

(b) $f_X(x) = \frac{1}{6}$, $x = 0, 1, 2, 3, 4, 5$ $f_Y(y) = \frac{1}{6}$, $y = 0, 6, 12, 18, 24, 30$

$f_W(w) = \frac{1}{36}$, $w = 0, 1 \dots 35$

$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

5.4.21

5.4-21. Let X and Y be independent with distributions $N(5, 16)$ and $N(6, 9)$, respectively. Evaluate $P(X > Y) = P(X - Y > 0)$.

Let $W = X - Y$

$M_W(t) = E[e^{tW}] = E[e^{t(X-Y)}] \overset{\text{inde.}}{=} E[e^{tX}]E[e^{-tY}]$

$= e^{5t + \frac{1}{2}(16t^2)} e^{-6t + \frac{1}{2}(9t^2)} = e^{-t + \frac{1}{2}(25t^2)} \rightarrow W \sim N(-1, 25)$

$P(X > Y) = P(X - Y > 0) = P(W > 0) = P\left(\frac{W - \mu_W}{\sigma_W} > \frac{-\mu_W}{\sigma_W}\right) = P(Z > \frac{1}{5})$

$= P(Z > 0.2) = 0.4207$

Section 5.5 1, 3, 5, 7, 13, 15

5.5.1

5.5-1. Let $X_1, X_2, \ldots, X_{16}$ be a random sample from a normal distribution $N(77, 25)$. Compute

(a) $P(77 < \overline{X} < 79.5)$. **(b)** $P(74.2 < \overline{X} < 78.4)$.

$$\overline{X} \sim N(77, \tfrac{25}{16})$$

(a) $$P(77 < \overline{X} < 79.5) = P\left(\frac{77-77}{5/4} < \frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} < \frac{79.5-77}{5/4}\right) = P(0 < Z < 2)$$

$$= \Phi(2) - \Phi(0) = 0.9772 - 0.5 = 0.4772$$

(b) $$P(74.2 < \overline{X} < 78.4) = P\left(\frac{74.2-77}{5/4} < \frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} < \frac{78.4-77}{5/4}\right) = P(-2.24 < Z < 1.12)$$

$$= \Phi(1.12) - (1 - \Phi(2.24)) = 0.8686 + 0.9875 - 1 = 0.8561$$

5.5.3

5.5-3. Let X equal the widest diameter (in millimeters) of the fetal head measured between the 16th and 25th weeks of pregnancy. Assume that the distribution of X is $N(46.58, 40.96)$. Let $\overline{X}$ be the sample mean of a random sample of $n = 16$ observations of X.

(a) Give the values of $E(\overline{X})$ and $\mathrm{Var}(\overline{X})$.

(b) Find $P(44.42 \le \overline{X} \le 48.98)$.

(a) $$\mu_{\overline{X}} = 46.58 \qquad \sigma^2_{\overline{X}} = \frac{\sigma^2_X}{n} = \frac{40.96}{16} = 2.56 \rightarrow \overline{X} \sim N(46.58, 2.56)$$

(b) $$P(44.42 \le \overline{X} \le 48.98) = P\left(\frac{44.42-46.58}{1.6} \le \frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \le \frac{48.98-46.58}{1.6}\right)$$

$$= P(-1.35 \le Z \le 1.5) = \Phi(1.5) - (1 - \Phi(1.35)) = 0.9332 + 0.9115 - 1 = 0.8447$$

5.5.5

5.5-5. Let X equal the weight (in grams) of a nail of the type that is used for making decks. Assume that the distribution of X is $N(8.78, 0.16)$. Let $\overline{X}$ be the mean of a random sample of the weights of $n = 9$ nails.

(a) Sketch, on the same set of axes, the graphs of the pdfs of X and of $\overline{X}$.

(b) Let S^2 be the sample variance of the nine weights. Find constants a and b so that $P(a \le S^2 \le b) = 0.90$.

HINT: $P(a \le S^2 \le b)$ is equivalent to $P(8a/0.16 \le 8S^2/0.16 \le 8b/0.16)$, and $8S^2/0.16$ is $\chi^2(8)$. Find $8a/0.16$ and $8b/0.16$ from Table IV in Appendix B.

$$\frac{(n-1)S^2}{\sigma_X^2} \sim \chi^2(n-1) \longrightarrow \frac{8S^2}{0.16} \sim \chi^2(8)$$

(b) $$P(a \le S^2 \le b) = P\left(\frac{8a}{0.16} \le \frac{8S^2}{0.16} \le \frac{8b}{0.16}\right) = P\left(50a \le \frac{8S^2}{0.16} \le 50b\right) = 0.90$$

$$\therefore 50a = \chi^2_{0.95}(8) = 2.733 \longrightarrow a = 2.733/50 = 0.05466$$

$$50b = \chi^2_{0.05}(8) = 15.51 \longrightarrow b = 15.51/50 = 0.3102$$

5.5.7

5.5-7. Suppose that the distribution of the weight of a prepackaged "1-pound bag" of carrots is $N(1.18, 0.07^2)$ and the distribution of the weight of a prepackaged "3-pound bag" of carrots is $N(3.22, 0.09^2)$. Selecting bags at random, find the probability that the sum of three 1-pound bags exceeds the weight of one 3-pound bag. HINT: First determine the distribution of Y, the sum of the three, and then compute $P(Y > W)$, where W is the weight of the 3-pound bag.

$$X \sim N(1.18, 0.07^2) \qquad W \sim N(3.22, 0.09^2)$$

$$Y = X_1 + X_2 + X_3 \sim N(3 \times 1.18, 3 \times 0.07^2) = N(3.54, 0.0147)$$

$$T = Y - W \sim N(3.54 - 3.22, 0.0147 + 0.09^2) = N(0.32, 0.0228)$$

$$P(Y > W) = P(Y - W > 0) = P(T > 0) = P\left(\frac{T-\mu_T}{\sigma_T} > \frac{-\mu_T}{\sigma_T}\right) = P\left(Z > \frac{-0.32}{\sqrt{0.0228}}\right)$$

$$= P(Z > -2.12) = P(Z < 2.12) = 0.9830$$

5.5-13. Let Z_1, Z_2, and Z_3 have independent standard normal distributions, $N(0, 1)$.

(a) Find the distribution of

$$W = \frac{Z_1}{\sqrt{(Z_2^2 + Z_3^2)/2}}.$$

(b) Show that

$$V = \frac{Z_1}{\sqrt{(Z_1^2 + Z_2^2)/2}}$$

has pdf $f(v) = 1/(\pi\sqrt{2 - v^2})$, $-\sqrt{2} < v < \sqrt{2}$.

(c) Find the mean of V.

(d) Find the standard deviation of V.

(e) Why are the distributions of W and V so different?

5.5.13

(a) $Z_2, Z_3 \sim N(0,1) \rightarrow Z_2^2, Z_3^2 \sim \chi^2(1) \rightarrow Z_2^2 + Z_3^2 \sim \chi^2(2) \rightarrow W \sim t(2)$

(c) $\mu_V = \int_{-\sqrt{2}}^{\sqrt{2}} v f(v)\,dv = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{v}{\pi\sqrt{2-v^2}}\,dv = \int_0^0 \frac{-1}{2\pi\sqrt{u}}\,du = 0$ $\qquad u = 2 - v^2$, $du = -2v\,dv$

(d) $\sigma_V^2 = \int_{-\sqrt{2}}^{\sqrt{2}} v^2 f(v)\,dv - \mu_V^2 = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{v^2}{\pi\sqrt{2-v^2}}\,dv - 0^2 = \frac{-1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2-v^2-2}{\sqrt{2-v^2}}\,dv$

$= \frac{-1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-v^2} - \frac{2}{\sqrt{2-v^2}}\right)dv$ $\qquad x = \sqrt{2-v^2}$, $dx = \frac{-v}{\sqrt{2-v^2}}$ $\qquad dy = dv$, $y = v$

$= \frac{-1}{\pi}\left(v\sqrt{2-v^2}\Big|_{-\sqrt{2}}^{\sqrt{2}} + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{v^2\,dv}{\sqrt{2-v^2}} - 2\arcsin\left(\frac{v}{\sqrt{2}}\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}\right)$

$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{-v^2\,dv}{\pi\sqrt{2-v^2}} + \frac{2}{\pi}\left(\arcsin(1) - \arcsin(-1)\right)$

$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{-v^2\,dv}{\pi\sqrt{2-v^2}} + \frac{2}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = -\sigma_V^2 + 2 \quad \therefore 2\sigma_V^2 = 2 \quad \therefore \sigma_V^2 = 1$

(e) Because the numerator and denominator in (b) are dependent.

5.5-15. Let the distribution of T be $t(17)$. Find

(a) $t_{0.01}(17)$.

(b) $t_{0.95}(17)$.

(c) $P(-1.740 \le T \le 1.740)$.

5.5.15

(a) $t_{0.01}(17) = 2.567$ $\qquad$ (b) $t_{0.95}(17) = -t_{0.05}(17) = -1.740$

(c) $P(-1.740 \le T \le 1.740) = P(T \le 1.740) - P(T \le -1.740)$

$= 0.95 - 0.05 = 0.9$

$\boxed{\text{Section 5.6}}$ 1, 3, 5, 7, 8, 11

$\boxed{5.6.1}$

5.6-1. Let $\overline{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval $(0, 1)$. Approximate $P(1/2 \le \overline{X} \le 2/3)$.

$$X_i \sim U(0,1) \rightarrow \mu = \frac{1}{2},\ \sigma^2 = \frac{1}{12} \rightarrow \overline{X} \sim N\left(\frac{1}{2}, \frac{1}{144}\right)$$

$$P\left(\frac{1}{2} \le \overline{X} \le \frac{2}{3}\right) = P\left(\frac{1/2 - 1/2}{1/12} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{12}} \le \frac{2/3 - 1/2}{1/12}\right) = P(0 \le Z \le 2)$$

$$= \Phi(2) - \Phi(0) = 0.9772 - 0.5 = 0.4772$$

$\boxed{5.6.3}$

5.6-3. Let $\overline{X}$ be the mean of a random sample of size 36 from an exponential distribution with mean 3. Approximate $P(2.5 \le \overline{X} \le 4)$.

$$X_i \sim \mathrm{Exp}(3) \rightarrow \mu = 3,\ \sigma^2 = 3^2 = 9$$

$$\mu_{\overline{X}} = 3,\ \sigma^2_{\overline{X}} = \frac{\sigma^2}{n} = \frac{9}{36} = \frac{1}{4} \rightarrow \overline{X} \sim N\left(3, \frac{1}{4}\right)$$

$$P(2.5 \le \overline{X} \le 4) = P\left(\frac{2.5-3}{1/2} \le \frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \le \frac{4-3}{1/2}\right) = P(-1 \le Z \le 2)$$

$$= \Phi(2) - \Phi(-1) = 0.9772 - 1 + 0.8413 = 0.8185$$

$\boxed{5.6.5}$

5.6-5. Let $X_1, X_2, \ldots, X_{18}$ be a random sample of size 18 from a chi-square distribution with $r = 1$. Recall that $\mu = 1$ and $\sigma^2 = 2$.

(a) How is $Y = \sum_{i=1}^{18} X_i$ distributed?

(b) Using the result of part (a), we see from Table IV in Appendix B that

$$P(Y \le 9.390) = 0.05 \quad \text{and} \quad P(Y \le 34.80) = 0.99.$$

Compare these two probabilities with the approximations found with the use of the central limit theorem.

$$X_i \sim \chi^2(1) \rightarrow \mu = 1,\ \sigma^2 = 2 \rightarrow \mu_{\overline{X}} = 1,\ \sigma^2_{\overline{X}} = \frac{\sigma^2}{n} = \frac{2}{18} = \frac{1}{9}$$

$$\overline{X} \sim N\left(1, \frac{1}{9}\right)$$

(a) $Y = \sum_{i=1}^{18} X_i \sim \chi^2(18)$ as $X_i \sim \chi^2(1)$

(b) $Y \sim N(n\mu, n\sigma^2) = N(18, 36)$

$$P(Y \le 9.390) = P\left(\frac{Y-\mu_Y}{\sigma_Y} \le \frac{9.39-18}{\sqrt{36}}\right) = P(Z \le -1.435) = P(Z \geq 1.44) = 0.0749$$

$$P(Y \le 34.8) = P\left(\frac{Y-\mu_Y}{\sigma_Y} \le \frac{34.8-18}{\sqrt{36}}\right) = P(Z \le 2.8) = 0.9974$$

$\boxed{5.6.7}$

5.6-7. Let X equal the maximal oxygen intake of a human on a treadmill, where the measurements are in milliliters of oxygen per minute per kilogram of weight. Assume that, for a particular population, the mean of X is $\mu = 54.030$ and the standard deviation is $\sigma = 5.8$. Let $\overline{X}$ be the sample mean of a random sample of size $n = 47$. Find $P(52.761 \le \overline{X} \le 54.453)$, approximately.

$$\overline{X} \sim N\left(54.03, \frac{5.8^2}{47}\right)$$

$$P(52.761 \le \overline{X} \le 54.453) = P\left(\frac{52.761 - 54.03}{5.8/\sqrt{47}} \le \frac{\overline{X}-\mu_{\overline{X}}}{\sigma/\sqrt{47}} \le \frac{54.453 - 54.03}{5.8/\sqrt{47}}\right)$$

$$= P(-1.499 \le Z \le 0.5) = \Phi(0.5) - 1 + \Phi(1.5) = 0.6915 - 1 + 0.9332 = 0.6247$$

$\boxed{5.6.8}$

5.6-8. Let X equal the weight in grams of a miniature candy bar. Assume that $\mu = E(X) = 24.43$ and $\sigma^2 = \mathrm{Var}(X) = 2.20$. Let $\overline{X}$ be the sample mean of a random sample of $n = 30$ candy bars. Find **(a)** $E(\overline{X})$. **(b)** $\mathrm{Var}(\overline{X})$. **(c)** $P(24.17 \le \overline{X} \le 24.82)$, approximately.

$$\overline{X} \sim N\left(24.43, \frac{2.20}{30}\right) \quad (a)\ \mu_{\overline{X}} = 24.43 \quad (b)\ \sigma^2_{\overline{X}} = \frac{11}{150}$$

(c) $$P(24.17 \le \overline{X} \le 24.82) = P\left(\frac{24.17 - 24.43}{\sqrt{11/150}} \le \frac{\overline{X}-\mu_{\overline{X}}}{\sigma_{\overline{X}}} \le \frac{24.82 - 24.43}{\sqrt{11/150}}\right)$$

$= P(-0.96 \le Z \le 1.44) = \Phi(1.44) - 1 + \Phi(0.96) = 0.9251 - 1 + 0.8315 = 0.7566$

5.6.11

5.6-11. The tensile strength X of paper, in pounds per square inch, has $\mu = 30$ and $\sigma = 3$. A random sample of size $n = 100$ is taken from the distribution of tensile strengths. Compute the probability that the sample mean $\overline{X}$ is greater than 29.5 pounds per square inch.

$\overline{X} \sim N\left(30, \frac{3^2}{100}\right) = N\left(30, \frac{9}{100}\right)$

$P(\overline{X} > 29.5) = P\left(\frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} > \frac{29.5 - 30}{3/10}\right) = P(Z > -1.67) = \Phi(1.67) = 0.9525$

Section 5.7   1, 3, 5, 7, 9

5.7.1

5.7-1. Let the distribution of Y be $b(25, 1/2)$. Find the given probabilities in two ways: exactly, using Table II in Appendix B; and approximately, using the central limit theorem. Compare the two results in each of the three cases.

(a) $P(10 < Y \le 12)$. **(b)** $P(12 \le Y < 15)$. **(c)** $P(Y = 12)$.

$Y \sim Bin(25, \frac{1}{2})$

(a) $P(10 < Y \le 12) = P(Y \le 12) - P(Y \le 10) = 0.5 - 0.2122 = 0.2878$

$$P(10 < Y \le 12) = P(10.5 \le Y \le 12.5) = P\left(\frac{10.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}} \le \frac{Y - np}{\sqrt{npq}} \le \frac{12.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}}\right)$$

$= P(-0.8 \le Z \le 0) = \Phi(0) - 1 + \Phi(0.8) = 0.5 - 1 + 0.7881 = 0.2881$

(b) $P(12 \le Y < 15) = P(Y \le 14) - P(Y \le 11) = 0.7878 - 0.345 = 0.4428$

$$P(12 \le Y < 15) = P(11.5 \le Y \le 14.5) = P\left(\frac{11.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}} \le \frac{Y - np}{\sqrt{npq}} \le \frac{14.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}}\right)$$

$= P(-0.4 \le Z \le 0.8) = \Phi(0.8) - 1 + \Phi(0.4) = 0.7881 - 1 + 0.6554 = 0.4435$

(c) $P(Y = 12) = P(11 < Y \le 12) = P(Y \le 12) - P(Y \le 11) = 0.5 - 0.3450 = 0.155$

$$P(Y = 12) = P(11.5 \le Y \le 12.5) = P\left(\frac{11.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}} \le \frac{Y - np}{\sqrt{npq}} \le \frac{12.5 - 25(1/2)}{\sqrt{25(\frac{1}{2})(\frac{1}{2})}}\right)$$

$= P(-0.4 \le Z \le 0) = \Phi(0) - 1 + \Phi(0.4) = 0.5 - 1 + 0.6554 = 0.1554$

5.7.3

5.7-3. A public opinion poll in Southern California was conducted to determine whether Southern Californians are prepared for the "big earthquake" that experts predict will devastate the region sometime in the next 50 years. It was learned that "60% have not secured objects in their homes that might fall and cause injury and damage during a temblor." In a random sample of $n = 864$ Southern Californians, let X equal the number who "have not secured objects in their homes." Find $P(496 \le X \le 548)$, approximately.

$X \sim Bin(864, 0.6)$

$$P(496 \le X \le 548) = P\left(\frac{496 - 864(0.6)}{\sqrt{864(0.6)(0.4)}} \le \frac{X - np}{\sqrt{npq}} \le \frac{548 - 864(0.6)}{\sqrt{864(0.6)(0.4)}}\right)$$

$= P(-1.56 \le Z \le 2.06) = \Phi(2.06) - 1 + \Phi(1.56) = 0.9803 - 1 + 0.9406 = 0.9209$

5.7.5

5.7-5. Let $X_1, X_2, \ldots, X_{48}$ be a random sample of size 48 from the distribution with pdf $f(x) = 1/x^2$, $1 < x < \infty$. Approximate the probability that at most ten of these random variables have values greater than 4. HINT: Let the ith trial be a success if $X_i > 4$, $i = 1, 2, \ldots, 48$, and let Y equal the number of successes.

$$p = P(X_i > 4) = \int_4^\infty \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_4^\infty = \frac{1}{4} \rightarrow Y \sim Bin(48, \tfrac{1}{4})$$

$$P(Y \le 10) = P(Y \le 10.5) = P\left(\frac{Y - 48(1/4)}{\sqrt{48(\frac{1}{4})(\frac{3}{4})}} \le \frac{10.5 - 48(1/4)}{\sqrt{48(\frac{1}{4})(\frac{3}{4})}}\right) = P(Z \le -0.5) = P(Z \geq 0.5)$$

$= 0.3085$

5.7.7

5.7-7. Let X equal the number of alpha particles emitted by barium-133 per second and counted by a Geiger counter. Assume that X has a Poisson distribution with $\lambda = 49$. Approximate $P(45 < X < 60)$.

$X \sim \text{Poisson}(49) \longrightarrow N(49, 49)$

$$P(45 < X < 60) = P\left(\frac{45-49}{7} < \frac{X-\mu_X}{\sigma_X} < \frac{60-49}{7}\right) = P(-0.57 < Z < 1.57)$$

$$= \Phi(1.57) - 1 + \Phi(0.57) = 0.9418 - 1 + 0.7157 = 0.6575$$

5.7.9

5.7-9. Let $X_1, X_2, \ldots, X_{30}$ be a random sample of size 30 from a Poisson distribution with a mean of 2/3. Approximate

(a) $P\left(15 < \sum_{i=1}^{30} X_i \le 22\right)$. **(b)** $P\left(21 \le \sum_{i=1}^{30} X_i < 27\right)$.

$X_i \sim \text{Poisson}\left(\frac{2}{3}\right) \longrightarrow Y \sim \text{Poisson}(20) \longrightarrow N(20, 20)$

(a) $$P\left(15 < \sum_{i=1}^{30} X_i \le 22\right) = P(15 < Y \le 22) = P\left(\frac{15.5-20}{\sqrt{20}} < \frac{Y-\mu_Y}{\sigma_Y} \le \frac{22.5-20}{\sqrt{20}}\right)$$

$$= P(-1 < Z \le 0.56) = \Phi(0.56) - 1 + \Phi(1) = 0.7123 - 1 + 0.8413 = 0.5536$$

(b) $$P\left(21 \le \sum_{i=1}^{30} X_i < 27\right) = P(21 \le Y < 27) = P\left(\frac{20.5-20}{\sqrt{20}} < \frac{Y-\mu_Y}{\sigma_Y} \le \frac{26.5-20}{\sqrt{20}}\right)$$

$$= P(0.11 < Z \le 1.45) = \Phi(1.45) - \Phi(0.11) = 0.9265 - 0.5438 = 0.3827$$

Section 7.1

1, 3, 5, 7, 9, 11

7.1.1

7.1-1. A random sample of size 16 from the normal distribution $N(\mu, 25)$ yielded $\overline{x} = 73.8$. Find a 95% confidence interval for μ.

$$CI = \left[\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right] = \left[73.8 - z_{\frac{0.05}{2}}\frac{5}{\sqrt{16}}, 73.8 + z_{\frac{0.05}{2}}\frac{5}{\sqrt{16}}\right]$$

$$= \left[73.8 - 1.96\left(\frac{5}{4}\right), 73.8 + 1.96\left(\frac{5}{4}\right)\right] = [71.35, 76.25]$$

7.1.3

7.1-3. To determine the effect of 100% nitrate on the growth of pea plants, several specimens were planted and then watered with 100% nitrate every day. At the end of two weeks, the plants were measured. Here are data on seven of them:

17.5	14.5	15.2	14.0	17.3	18.0	13.8

Assume that these data are a random sample from a normal distribution $N(\mu, \sigma^2)$.

(a) Find the value of a point estimate of μ.

(b) Find the value of a point estimate of σ.

(c) Give the endpoints for a 90% confidence interval for μ.

(a) $$\overline{x} = \frac{17.5 + 14.5 + 15.2 + 14 + 17.3 + 18 + 13.8}{7} = 15.757$$

(b) $$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = 3.2095 \rightarrow s = 1.792$$

(c) $$CI = \left[\overline{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}, \overline{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}\right]$$

$$= \left[15.757 - t_{\frac{0.1}{2}}(7-1)\frac{1.792}{\sqrt{7}}, 15.757 + t_{\frac{0.1}{2}}(7-1)\frac{1.792}{\sqrt{7}}\right]$$

$$= \left[15.757 - 1.943\frac{1.792}{\sqrt{7}}, 15.757 + 1.943\frac{1.792}{\sqrt{7}}\right] = [14.441, 17.073]$$

7.1.5

7.1-5. As a clue to the amount of organic waste in Lake Macatawa (see Example 7.1-4), a count was made of the number of bacteria colonies in 100 milliliters of water. The number of colonies, in hundreds, for $n = 30$ samples of water from the east basin yielded

93	140	8	120	3	120	33	70	91	61
7	100	19	98	110	23	14	94	57	9
66	53	28	76	58	9	73	49	37	92

Find an approximate 90% confidence interval for the mean number (say, μ_E) of colonies in 100 milliliters of water in the east basin.

$$\overline{x} = \frac{\sum x}{30} = 60.367 \rightarrow s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = 1569.895$$

$$\rightarrow s = 39.622$$

$$CI = \left[\overline{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}, \overline{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}\right]$$

$$= \left[60.367 - t_{0.05}(29)\frac{39.622}{\sqrt{30}}, 60.367 + t_{0.05}(29)\frac{39.622}{\sqrt{30}}\right]$$

$$= \left[60.367 - 1.699\frac{39.622}{\sqrt{30}}, 60.367 + 1.699\frac{39.622}{\sqrt{30}}\right] = [48.077, 72.657]$$

7.1.7

7.1-7. Thirteen tons of cheese, including "22-pound" wheels (label weight), is stored in some old gypsum mines. A random sample of $n = 9$ of these wheels yielded the following weights in pounds:

21.50	18.95	18.55	19.40	19.15
22.35	22.90	22.20	23.10	

Assuming that the distribution of the weights of the wheels of cheese is $N(\mu, \sigma^2)$, find a 95% confidence interval for μ.

$$\overline{x} = 20.9 \qquad s^2 = 3.45375 \rightarrow s = 1.86$$

$$CI = \left[\overline{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}, \overline{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}\right]$$

$$= \left[20.9 - t_{0.025}(8)\frac{1.86}{3}, 20.9 + t_{0.025}(8)\frac{1.86}{3}\right]$$

$$= \left[20.9 - 2.306\left(\frac{1.86}{3}\right), 20.9 + 2.306\left(\frac{1.86}{3}\right)\right] = [19.47, 23.33]$$

7.1.9

7.1-9. During the Friday night shift, $n = 28$ mints were selected at random from a production line and weighed. They had an average weight of $\bar{x} = 21.45$ grams and $s = 0.31$ grams. Give the lower endpoint of an approximate 90% one-sided confidence interval for μ, the mean weight of all the mints.

$$\bar{x} - z_{\alpha}\frac{s}{\sqrt{n}} = 21.45 - z_{0.1}\frac{0.31}{\sqrt{28}} = 21.45 - 1.28\left(\frac{0.31}{\sqrt{28}}\right) = 21.38 \rightarrow [21.38, \infty)$$

7.1.11

7.1-11. The heart rates (in beats per minute) of 41 randomly selected finishers of the Chicago Marathon, five minutes after they completed the race, had sample mean $\bar{x} = 132$ and sample variance $s^2 = 105$. Assuming that the heart rates of all finishers of the Chicago Marathon five minutes after completing the race are normally distributed, obtain a 95% confidence interval for their mean.

$$CI = \left[\bar{x} - t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}, \bar{x} + t_{\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}}\right]$$

$$= \left[132 - t_{\frac{0.05}{2}}(40)\sqrt{\frac{105}{41}}, 132 + t_{\frac{0.05}{2}}(40)\sqrt{\frac{105}{41}}\right]$$

$$= \left[132 - 2.021\sqrt{\frac{105}{41}}, 132 + 2.021\sqrt{\frac{105}{41}}\right] = [128.77, 135.23]$$