Linear Algebra Math 2101

# Exam #3

1. a) Determine if $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

Since these are 3 vectors and $\mathbb{R}^3$ has dimension 3.
All we need is to see if they are linearly independent

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

3 pivots. rank 3. linearly independent columns.

So, yes, it is a basis.

b) Find a basis of $\mathbb{R}^3$ which includes $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

All we need are 3 linearly independent vectors one of which is $v_1$

For example $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$ rank 3

(c) orthonormal basis including $w_1 = \frac{1}{5}\begin{vmatrix} 0 \\ 3 \\ 4 \end{vmatrix}$

For example $w_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ $\quad w_3 = \frac{1}{5}\begin{vmatrix} 0 \\ -4 \\ 3 \end{vmatrix}$

$\|w_i\| = 1 \quad w_i^T w_j = 0 \quad i \neq j$

(d) $v_1 = \begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix}$ expand with two other linear independent vectors,

for example $\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$ $\begin{vmatrix} 1 \\ 2 \\ 0 \end{vmatrix}$, and

do Gram-Schmidt orthogonalization (or equivalently QR factorization)

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2. $v = \begin{vmatrix} 3 \\ 4 \end{vmatrix}$ (a) $\|v\| = \sqrt{3^2 + 4^2} = 5$

$v_1 = \frac{v}{\|v\|} = \frac{1}{5}\begin{vmatrix} 3 \\ 4 \end{vmatrix}$

(b) you could 'guess' $w = \begin{vmatrix} -4 \\ 3 \end{vmatrix}$, or

you can just pick any vector so that $\{v, w\}$ are linearly independent and orthogonalize:

take $z = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$ for example

then $w = z - \langle z, v_1 \rangle v_1$

$$= \begin{vmatrix} 1 \\ 0 \end{vmatrix} - \frac{3}{5} \cdot \frac{1}{5} \begin{vmatrix} 3 \\ 4 \end{vmatrix} = \begin{vmatrix} 1 - 9/25 \\ 0 - 12/25 \end{vmatrix} = \frac{1}{25} \begin{vmatrix} 16 \\ -12 \end{vmatrix}$$

or $w = \frac{4}{25} \begin{vmatrix} 4 \\ -3 \end{vmatrix}$

$$\langle z, v_1 \rangle = 1 \cdot \frac{3}{5} + 0 \cdot \frac{4}{5} = \frac{3}{5}$$

and clearly $w^T v = 0 = w^T v_1$

$$\|w\| = \frac{4}{25} \cdot \sqrt{16 + 9} = \frac{4}{5}$$

c) $w_1 = \frac{w}{\|w\|} = \frac{5}{4} \cdot \frac{4}{25} \cdot \begin{vmatrix} 4 \\ -3 \end{vmatrix} = \frac{1}{5} \begin{vmatrix} 4 \\ -3 \end{vmatrix}$

d). $v_1$ $w_1$ two orthogonal vectors

thus, they are linearly independent

Since they are two, they are a basis of $\mathbb{R}^2$

3. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ $S = \{x \,/\, x^T v = 0\}$

(a) $S$ is an hyperplane in $\mathbb{R}^5$, dimension is 4.

A different way: $L = \{x \,/\, x = \alpha v, \alpha \in \mathbb{R}\}$

a line in $\mathbb{R}^5$, so $S = L^\perp$

$L$ has dimension 1

$\dim L + \dim S = 5$

$\Rightarrow \dim S = 5 - 1 = 4$

b) Let $w = \frac{v}{\|v\|}$. $\|v\| = \sqrt{1+4+9+1+1} = \sqrt{16} = 4$

$w = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$ $P = ww^T = \frac{1}{16}\begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 6 & 2 & 2 \\ 3 & 6 & 9 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix}$

$P^2 = w\underbrace{w^T w}_{=1}w^T = ww^T = P$

if $x \in L$ $x = \beta w = \frac{\alpha}{\|v\|} \cdot v$

$Px = ww^T(\beta w) = \beta w w^T w = \beta w = x$

if $z \perp L$, $z \perp w$, $w^T z = 0$

$Pz = ww^T z = 0$

(c) $Q = I - P = I - ww^T$

check $Q^2 = Q$ : $(I-P)(I-P) =$

$= (I - ww^T)(I - ww^T) =$

$I - 2ww^T + w\underbrace{w^T w}_{1}w^T = I - ww^T = Q$

$Qw = (I - ww^T)w = w - w\underbrace{w^T w}_{=1} = 0$

and if $z \in S$ $\quad w^T z = 0$

$Qz = z - w\underbrace{w^T z}_{=0} = z.$

4. $\quad U = \begin{vmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 2 \end{vmatrix} \qquad V = \begin{vmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{vmatrix}$

Rank $U = 2$ Rank $V = 2$ (each has two linearly independent colums)

$A = UV^T$ is $3 \times 3$ with at most rank 2
since $R(A) \subseteq R(U)$
so Rank $A = \dim R(A) \leq \dim R(U) =$ Rank $U$

Clearly $A \neq 0$ so rank $A > 0$

So rank $A$ is either 1 or 2.

But a rank-one matrix has all its columns a multiple of one another, and here clearly this is not the case: each column of $A$ is a different linear combination of the columns of $V$ just picking the first column and adding 1, 2, or 3 times the second.

Thus Rank $A = 2$

5. We have the SVD of A

$$A = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\Sigma = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \qquad \text{rank } A = \text{rank } \Sigma = 2$$

$$A = 2 \cdot \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$B = \frac{2}{2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\|A - B\| = \sigma_2 = 1$$