Section 5.4 Orthogonality

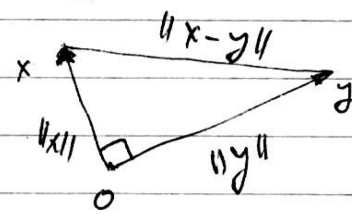
We say that $x, y \in V$ are <u>orthogonal</u> (or perpendicular) if $(x, y) = 0$.

In the standard inner product in $\mathbb{R}^n$

This is $(x,y) = x^T y = \sum_{i=1}^{n} x_i y_i = 0$

and it corresponds to 90°

<u>Pythagoras theorem</u> (in $\mathbb{R}^n$, or in any V).



If $x^T y = 0$ $\quad (x, y) = 0$

then $\|x\|^2 + \|y\|^2 = \|x - y\|^2$

<u>Proof</u>: $\|x\|^2 + \|y\|^2 - \|x - y\|^2 =$

$(x,x) + (y,y) - (x-y, x-y) =$

$(x,x) + (y,y) - (x,x) - (y,y) + 2(x,y) = 0$

q.e.d.

Examples

$$x = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad y = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \quad x^T y = 0$$

$$x = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \quad y = \begin{vmatrix} 1 \\ -1 \end{vmatrix} \quad x^T y = 0$$

$$x = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix} \quad y = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} \quad x^T y = 0$$

$$x = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix} \quad y = \begin{vmatrix} 4 \\ 0 \\ 0 \\ -1 \end{vmatrix} \quad x^T y = 0$$

or $x = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix} \quad y = \begin{vmatrix} -1 \\ 1 \\ 1 \\ -1 \end{vmatrix} \quad x^T y = 0$

or $p(x) = 1 \quad q(x) = x - \frac{1}{2}$

$$(p, q) = \int_0^1 p(x)\, q(x)\, dx = \int_0^1 \left(x - \frac{1}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2}x \Big|_0^1 = \frac{1}{2} - \frac{1}{2} - 0 - 0 = 0$$

A set $\{u_1, u_2 \dots u_n\}$ is
<u>orthogonal</u> if $(u_i, u_j) = 0 \quad i \neq j$

It is <u>orthonormal</u>, if in addition $\|u_i\| = 1$

i.e., $(u_i, u_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Example $u_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad u_2 = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad u_3 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

the Euclidean basis

Example $u_1 = \frac{1}{\sqrt{3}} \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \quad u_3 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}$

<u>Lemma.</u> An <u>orthonormal</u> (or just orthogonal) set is <u>linearly independent</u>

Proof. Let $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$

Wish to show $\alpha_j = 0 \quad \forall j = 1, \dots n$.
take u_j and take the inner product with the linear combination

$(u_j, \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) =$

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$$\alpha_1 (u_1, u_j) + \alpha_2 (u_2, u_j) + \dots + \alpha_j (u_j, u_j) + \dots + \alpha_n (u_j, u_n) =$$

$$= \alpha_j \underbrace{(u_j, u_j)}_{>0} \qquad \text{(because } (u_i, u_j) = 0 \text{ for } i \neq j\text{)}$$

$$\Rightarrow \alpha_j = 0 \qquad \text{q.e.d}$$

Recall CBS inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Define Angle between vectors θ such that

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

Clearly $|\cos\theta| \leq 1$

$\cos\theta = 0$ when $\langle x, y \rangle = 0$, $90° = \pi$

Thus if $\dim V = n$

n orthonormal vectors form a basis.

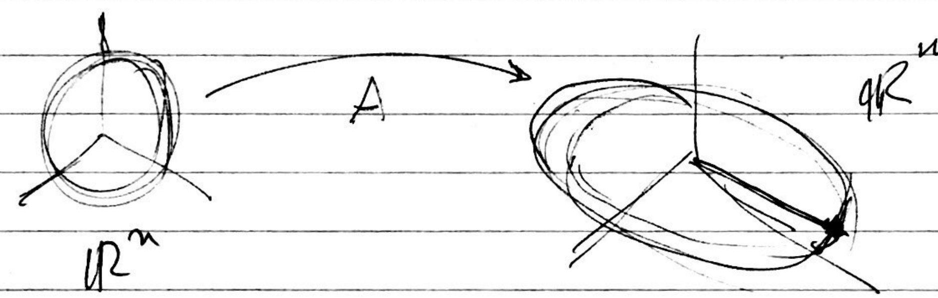
Section 5.2

Matrix Norm. $\mathbb{R}^{m\times n}$ a vector space.
~~same~~ definition of norm for any vector space.

We saw $\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$

We also ask that $\|AB\| \le \|A\|\,\|B\|$.

Matrix norm as operator from $\mathbb{R}^n$ to $\mathbb{R}^m$



$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

Largest length of vector in the image of the unit sphere.

Special case, let U be a matrix whose columns are orthonormal.

This implies that $U^T U = I$

Such matrices are coll unitary matrices, or orthogonal matrices and $\|U\|=1$, that is the image of the unit sphere is a unit sphere.

More later section 5.6

Such U maintains lengths and angles.

$$\|Ux\|^2 = (Ux, Ux) = (Ux)^T Ux$$

$$= x^T U^T U x = x^T I x = x^T x = \|x\|^2$$

$$(Ux)^T (Uy) = x^T U^T U y = x^T I y = x^T y$$

so that $\cos\theta = \dfrac{(Ux, Uy)}{\|Ux\| \, \|Uy\|} = \dfrac{(x, y)}{\|x\| \, \|y\|}$.

Note if $V^T V = I$ V square $(n \times n)$

then V nonsingular and $V^{-1} = V^T$.

So for $Ax = b$ if $A^T A = I$

$$x = A^T b \quad (A^{-1} b).$$