§ 2.4 Exact Equations

Def: 2.4.1 Exact Equations

A differential expression

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an exact differential equation in a region R of the xy-plane if it corresponds to the differential of some function $F(x,y)$ defined in R, i.e. there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

If such a function exists, then by Chain Rule,

$$\frac{d}{dx}\left(F(x,y)\right) = \underset{\parallel}{\frac{\partial F}{\partial x}} \cdot \frac{dx}{dx} + \underset{\parallel}{\frac{\partial F}{\partial y}} \cdot \frac{dy}{dx}$$

$$= M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

ie. $dF = M(x,y)\,dx + N(x,y)\,dy = 0$.

Then $F(x,y)$ is a constant function.

ie. $F(x,y) = c$ is an implicit solution to the differential equation.

How do we know if an equation is exact?

We need $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

Then
$$\underset{\displaystyle \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}{\underset{\shortparallel}{\frac{\partial^2 f}{\partial y \partial x}}} = \frac{\partial M}{\partial y} \quad \text{and} \quad \underset{\displaystyle \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)}{\underset{\shortparallel}{\frac{\partial^2 f}{\partial x \partial y}}} = \frac{\partial N}{\partial x}.$$

From Calculus III, you know that for continuity, the mixed partials must be equal.

Requirement: $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

This is a necessary & sufficient conditions for a DE to be exact (in any rectangle.)

$M(x,y)\,dx + N(x,y)\,dy = 0$

eg $(x + \sin y) + (x\cos y - 2y)\,y' = 0$

i.e. $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$

$M(x,y) = x + \sin y \qquad N(x,y) = x\cos y - 2y$

Since $\dfrac{\partial M}{\partial y} = \cos y = \dfrac{\partial N}{\partial x}$, the DE is exact.

Now solve the exact DE:

(one way)

Step 1: Solve $\dfrac{\partial f}{\partial x}$ for f. Note: The constant of integration is $g(y)$.

$$\frac{\partial f}{\partial x} = \overset{M}{x + \sin y}$$

So $f(x,y) = \int (x + \sin y)\,dx$

$$= \frac{1}{2}x^2 + x\sin y + g(y).$$

Step 2: Find $\dfrac{\partial f}{\partial y}$ and set it equal to N and solve for $g'(y)$. $\dfrac{\partial f}{\partial y} = 0 + x\cos y + g'(y) \overset{set}{=} \overbrace{x\cos y - 2y}^{N}$.

So $\qquad g'(y) = -2y$

Step 3: Find $g(y)$ by integrating $g'(y)$.
write $F(x,y)$.

$$g'(y) = -2y \quad \Rightarrow \quad g(y) = -y^2.$$

Collect constant at end.

Hence, $F(x,y) = \frac{1}{2}x^2 + x\sin y - y^2$.

Step 4: $F(x,y) = c$ is an implicit

solution to the DE.

Solution to the DE: $\boxed{\frac{1}{2}x^2 + x\sin y - y^2 = c}$

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eg $$y' = \frac{2 + ye^{xy}}{3y^2 - xe^{xy}}$$

$$(3y^2 - xe^{xy})y' = 2 + ye^{xy}$$

$$(2 + ye^{xy}) + (-3y^2 + xe^{xy})y' = 0$$

$$\underbrace{(2 + ye^{xy})}_{M}dx + \underbrace{(-3y^2 + xe^{xy})}_{N}dy = 0$$

Put in correct form to find M and N.

The DE is exact since $\frac{\partial M}{\partial y} = y \cdot xe^{xy} + e^{xy} = \frac{\partial N}{\partial x}$.

Solve the exact DE:

$$\frac{\partial f}{\partial x} = M \quad \therefore \; f(x,y) = \int \overbrace{(2 + y e^{xy})}^{M} \, dx$$

$$= 2x + e^{xy} + g(y)$$

$$\frac{\partial f}{\partial y} = 0 + x e^{xy} + g'(y) \overset{\text{set}}{=} \underbrace{-3y^2 + x e^{xy}}_{N}$$

$$\Rightarrow \quad g'(y) = -3y^2$$

$$\Rightarrow \quad g(y) = -y^3 .$$

Hence, $f(x,y) = 2x + e^{xy} - y^3$.

Solution to the exact DE:

$$\boxed{2x + e^{xy} - y^3 = C.}$$

eg Solve the IVP

$$\underbrace{(2x-y)}_{M} + \underbrace{(2y-x)}_{N}\, y' = 0, \quad y(1) = 3$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} \quad \checkmark \quad \text{exact}$$

$$\frac{\partial f}{\partial x} = \underbrace{2x-y}_{M} \Rightarrow f(x,y) = \int (2x-y)\,dx$$

$$= x^2 - xy + g(y)$$

$$\frac{\partial f}{\partial y} = -x + g'(y) \overset{\text{set}}{=} \underbrace{2y-x}_{N}$$

$$\Rightarrow g'(y) = 2y \quad \text{and} \quad g(y) = \int 2y\,dy = y^2$$

Hence, $f(x,y) = x^2 - xy + y^2$.

Solution: ① general solution $\quad x^2 - xy + y^2 = C$

② Solve for C: $y(1) = 3$ $\quad 1^2 - 1(3) + 3^2 = C$

So $C = 7$.

③ Particular solution of IVP

$$\boxed{x^2 - xy + y^2 = 7}$$