HW10 Responses. Math 2101 - D. Szyld (1)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} = [a_1 \quad a_2 \quad a_3]$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{9}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$w_2 = a_2 - \langle a_2, q_1 \rangle q_1 \quad , \quad a_2^T q_1 = \frac{1}{3}(1+2+2) = 5/3$$

$$w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3} \cdot \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 5/9 \\ 1 - 10/9 \\ 1 - 10/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

check $w_2^T q_1 = 0$ ✓ $\quad \|w_2\| = \frac{1}{9}\sqrt{16+1+1} = \frac{\sqrt{18}}{9} \left( = \frac{\sqrt{2}}{\sqrt{9}} \right)$

$$q_2 = \frac{9}{\sqrt{18}} \cdot \frac{1}{9} \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$$w_3 = a_3 - \langle a_3, q_1 \rangle q_1 - \langle a_3, q_2 \rangle q_2$$

$$\langle a_3, q_1 \rangle = \frac{1}{3}(2+4+8) = 14/3$$

$$\langle a_3, q_2 \rangle = \frac{1}{\sqrt{18}} [8-2-4] = \frac{2}{\sqrt{18}}$$

$$w_3 = \begin{vmatrix} 2 \\ 2 \\ 4 \end{vmatrix} - \frac{14}{3} \cdot \frac{1}{3} \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} - \frac{2}{\sqrt{18}} \cdot \frac{1}{\sqrt{18}} \begin{vmatrix} 4 \\ -1 \\ -1 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 - \frac{14}{9} - \frac{2}{18} \cdot 4 \\ 2 - \frac{28}{9} + \frac{2}{18} \\ 4 - \frac{28}{9} + \frac{2}{18} \end{vmatrix} = \frac{1}{18} \begin{vmatrix} 36 - 28 - 8 \\ 36 - 56 + 2 \\ 72 - 56 + 2 \end{vmatrix}$$

$$= \frac{1}{18} \begin{vmatrix} 0 \\ -18 \\ 18 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix}$$

check $w_3^T q_1 = 0$ , $w_3^T q_2 = 0$ ✓

$\|w_3\| = \sqrt{2}$ $\qquad q_3 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix}$

$$Q = \begin{vmatrix} 1/3 & 4/\sqrt{18} & 0 \\ 2/3 & -1/\sqrt{18} & -1/\sqrt{2} \\ 2/3 & -1/\sqrt{18} & 1/\sqrt{2} \end{vmatrix}$$

We collect the entries in $R$:

$$R = \begin{vmatrix} 3 & 5/3 & 14/3 \\ 0 & \sqrt{18}/9 & 2/\sqrt{18} \\ 0 & 0 & \sqrt{2} \end{vmatrix}$$

check $Q.R = A$ ✓

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$$A = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix}$$

$m_{21} = 2$

$m_{31} = 2$

$m_{32} = 1$

$$L = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix}$$

check $L.U = A$ ✓

To solve $Ax = b$

$$QRx = b$$

$$Rx = Q^T b$$

$$Q^T = \begin{vmatrix} 1/3 & 2/3 & 2/3 \\ 4/\sqrt{18} & -1/\sqrt{18} & -1/\sqrt{18} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} \qquad b = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$$

$$Q^T . b = \begin{vmatrix} \frac{2}{3} + \frac{8}{3} + \frac{12}{3} \\ 8/\sqrt{18} - \frac{4}{\sqrt{18}} - \frac{6}{\sqrt{18}} \\ -4/\sqrt{2} + 6/\sqrt{2} \end{vmatrix} = \begin{vmatrix} 22/3 \\ -2/\sqrt{18} \\ 2/\sqrt{2} \end{vmatrix} = \begin{vmatrix} 22/3 \\ -2/\sqrt{18} \\ \sqrt{2} \end{vmatrix}$$

solve: $Rx = \begin{vmatrix} 22/3 \\ -2/\sqrt{18} \\ \sqrt{2} \end{vmatrix}$ $\qquad x_3 = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$$\frac{\sqrt{18}}{9} x_2 + 2/\sqrt{18} \cdot 1 = -2/\sqrt{18}$$

$$x_2 = \frac{-4}{\sqrt{18}} \Big/ \frac{\sqrt{18}}{9} = \frac{-36}{18} = -2$$

$$3x_1 + \frac{5}{3}(-2) + \frac{14}{3} \cdot 1 = \frac{22}{3}$$

$$3x_1 = \frac{22}{3} - \frac{4}{3} = \frac{18}{3} = 6$$

$$x_1 = \frac{6}{3} = 2$$

$$x = \begin{vmatrix} 2 \\ -2 \\ 1 \end{vmatrix}$$

Check $A \cdot x = b$ ✓

Solve $Ly = b = \begin{vmatrix} 2 \\ 4 \\ 6 \end{vmatrix}$

$$y_1 = 2$$

$$2 \cdot 2 + y_2 = 4 \qquad y_2 = 0$$

$$2 \cdot 2 + 1 \cdot 0 + y_3 = 6 \qquad y_3 = 2$$

Solve $Ux = y = \begin{vmatrix} 2 \\ 0 \\ 2 \end{vmatrix}$

$$x_3 = \frac{2}{2} = 1 \qquad -x_2 - 2 \cdot 1 = 0, \; x_2 = -2$$

$$x_1 + 1 \cdot (-2) + 2 \cdot 1 = 2 \qquad x_1 = 2$$

$$x = \begin{vmatrix} 2 \\ -2 \\ 1 \end{vmatrix} \checkmark$$

Exercise 5.6.5.

(a). $U$ orthogonal, i.e. $U^T U = I$
$V$ orthogonal, i.e. $V^T V = I$

What about $U.V$

$$(U.V)^T U V = V^T \underbrace{U^T U}_{=I} V = V^T V = I$$

$(AB)^T = B^T A^T$

thus $U.V$ orthogonal matrix

(b) What about $U+V$

$$(U+V)^T (U+V) = (U^T + V^T)(U+V)$$

$$= U^T U + U^T V + V^T U + V^T V$$
$$= I + U^T V + V^T U + I$$

unlikely to be $= I$

need an example.
simple example $U = V = I$

$$U + V = I + I = 2I$$

not orthogonal $2I \neq I$.

(c) Consider

$$A = \begin{bmatrix} U & O_{n \times m} \\ O_{m \times n} & V \end{bmatrix}$$

$U$ is $n \times n$ $V$ is $m \times m$

$$A^T A = \begin{bmatrix} U^T U & 0 \\ 0 & V^T V \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix} = I$$

$\Rightarrow$ $A$ orthogonal matrix