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31 October 2023

Due Tuesday 7 November 2023, 11 AM

Linear Algebra, Math 2101-003
Homework set #7

1. (2 points).
Show that the following matrix does not have real eigenvalues
$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

2. (3 points). Compute the eigenvalues and eigenvectors of the following three matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}.$$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

3. (2 points).
(a). For the symmetric case of Exercise 2, check that the eigenvectors are orthogonal.
(b). Let u_1, u_2 be the two eigenvectors from part (a). Normalize them (so that they have norm 1). Let us call them v_1, v_2, and let $V = [v_1, v_2]$ be a matrix that has these two vectors. (Check that $V^T V = I$). Let λ_1, λ_2 be the corresponding eigenvalues. Let $\Lambda = diag(\lambda_1, \lambda_2)$ be a diagonal matrix with the two eigenvalues in the diagonal. Check that $AV = V\Lambda$ and therefore $A = V\Lambda V^T$.

4. (3 points).
Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{bmatrix}$. Compute all eigenvalues and eigenvectors of A. Check that the eigenvalues λ and the eigenvectors v satisfy $Av = \lambda v$, and in this case, for any two different eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, we have $v_1^T v_2 = 0$.

Name: Elle Nguyen HW #7

1. $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2 - (-4) = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 1 + 4 = \lambda^2 - 2\lambda + 5 = 0$

$\therefore \lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = \dfrac{2 \pm \sqrt{-16}}{2}$ are not real eigenvalues QED!

2. (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (2-\lambda)^2 - 1 = (2-\lambda-1)(2-\lambda+1) = (1-\lambda)(3-\lambda) = 0$

$\therefore \lambda = 1, 3$

(i) $\lambda = 1 \rightarrow A - I = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 + x_2 = 0 \therefore x_1 = -x_2$

$\rightarrow u = \begin{vmatrix} -1 \\ 1 \end{vmatrix} \rightarrow N(A - I) = \left\{ \alpha \begin{vmatrix} -1 \\ 1 \end{vmatrix} \right\}, \alpha \in \mathbb{R}$

(ii) $\lambda = 3 \rightarrow A - 3I = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -x_1 + x_2 = 0 \therefore x_1 = x_2$

$\rightarrow u = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \rightarrow N(A - 3I) = \left\{ \alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix} \right\}, \alpha \in \mathbb{R}$

(b) $A = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)^2 - 0 = (3-\lambda)^2 = 0 \rightarrow 3 - \lambda = 0 \rightarrow \lambda = 3$

$A - 3I = \begin{bmatrix} 3-3 & 0 \\ 1 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 0 \rightarrow u = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$\rightarrow N(A - 3I) = \left\{ \alpha \begin{vmatrix} 0 \\ 1 \end{vmatrix} \right\}, \alpha \in \mathbb{R}$

(c) $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (3-\lambda)(2-\lambda) - 0 = (3-\lambda)(2-\lambda) = 0 \therefore \lambda = 2, 3$

(i) $\lambda = 2 \rightarrow A - 2I = \begin{bmatrix} 3-2 & 0 \\ 1 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 0$

$\rightarrow u = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \rightarrow N(A-2I) = \left\{ \alpha \begin{vmatrix} 0 \\ 1 \end{vmatrix} \right\}, \ \alpha \in \mathbb{R}$

(ii) $\lambda = 3 \rightarrow A - \ I = \begin{bmatrix} 3-3 & 0 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 - x_2 = 0 \rightarrow x_1 = x_2$

$\rightarrow u = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \rightarrow N(A-3I) = \left\{ \alpha \begin{vmatrix} 1 \\ 1 \end{vmatrix} \right\}, \ \alpha \in \mathbb{R}$

3 (a) Symmetric matrix is $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with 2 eigenvectors $u_1 = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$ & $u_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

Verify: $u_1^T u_2 = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha_1 \alpha_2 (-1 \times 1 + 1 \times 1) = \alpha_1 \alpha_2 (-1+1) = 0$

$\therefore u_1^T u_2 = 0$ $\therefore$ Eigenvectors are orthogonal QED!

(b) $u_1 = \begin{vmatrix} -1 \\ 1 \end{vmatrix} \rightarrow v_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} -1 \\ 1 \end{vmatrix}$ & $u_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \rightarrow v_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Verify: $V^T V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \ \therefore V^T V = I$

QED!

$\alpha_1 = 1, \alpha_2 = 3 \rightarrow \Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$\left.\begin{aligned} &\text{Verify: } AV = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix} \\ &V\Lambda = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix} \end{aligned}\right\} \therefore AV = V\Lambda \quad \text{QED!}$$

$V \Lambda V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = A$ QED!

④ $A=\begin{bmatrix}2&0&0\\0&43/25&-24/25\\0&-24/25&57/25\end{bmatrix}\longrightarrow A-\lambda I=\begin{bmatrix}2-\lambda&0&0\\0&43/25-\lambda&-24/25\\0&-24/25&57/25-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(2-\lambda)\det\mathring{A}_{11}-0.\det\mathring{A}_{12}+0.\det\mathring{A}_{13}$

$=(2-\lambda)\det\begin{bmatrix}43/25-\lambda&-24/25\\-24/25&57/25-\lambda\end{bmatrix}=(2-\lambda)\left[\left(\frac{43}{25}-\lambda\right)\left(\frac{57}{25}-\lambda\right)-\left(\frac{24}{25}\right)^2\right]$

$=(2-\lambda)\left(\lambda^2-\frac{43}{25}\lambda-\frac{57}{25}\lambda+\frac{43(57)}{25^2}-\frac{24^2}{25^2}\right)=(2-\lambda)\left(\lambda^2-4\lambda+\frac{2451}{625}-\frac{576}{625}\right)$

$=(2-\lambda)(\lambda^2-4\lambda+3)=(2-\lambda)(\lambda-1)(\lambda-3)=0\quad\therefore\lambda=1,2,3$

(i) $\lambda=1\longrightarrow A-I=\begin{bmatrix}2-1&0&0\\0&43/25-1&-24/25\\0&-24/25&57/25-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&18/25&-24/25\\0&-24/25&32/25\end{bmatrix}$

$\xrightarrow{R_3\leftarrow R_3+\frac{4}{3}R_2}\begin{bmatrix}1&0&0\\0&18/25&-24/25\\0&0&0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\longrightarrow \begin{array}{l}x_1=0\\ \frac{18}{25}x_2-\frac{24}{25}x_3=0\longrightarrow 18x_2=24x_3\\ \longrightarrow x_2=\frac{4}{3}x_3\end{array}$

$\longrightarrow V=\begin{vmatrix}0\\4/3\\1\end{vmatrix}\longrightarrow N(A-I)=\left\{\alpha\begin{vmatrix}0\\4/3\\1\end{vmatrix}\right\},\ \alpha\in\mathbb{R}$

Verify: $AV=\begin{bmatrix}2&0&0\\0&43/25&-24/25\\0&-24/25&57/25\end{bmatrix}\begin{vmatrix}0\\4/3\\1\end{vmatrix}=\begin{vmatrix}0\\4/3\\1\end{vmatrix}=1\begin{vmatrix}0\\4/3\\1\end{vmatrix}=\lambda V$ ✓

(ii) $\lambda=2\longrightarrow A-2I=\begin{bmatrix}2-2&0&0\\0&43/25-2&-24/25\\0&-24/25&57/25-2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&-7/25&-24/25\\0&-24/25&7/25\end{bmatrix}$

$\xrightarrow{R_3\leftarrow R_3-\frac{24}{7}R_2}\begin{bmatrix}0&0&0\\0&-7/25&-24/25\\0&0&25/7\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\longrightarrow \begin{array}{l}\frac{25}{7}x_3=0\longrightarrow x_3=0\\ \frac{-7}{25}x_2-\frac{24}{25}x_3=0\longrightarrow x_2=0\end{array}\longrightarrow V=\begin{vmatrix}1\\0\\0\end{vmatrix}$

$$m_{32} = \frac{-a_{32}}{a_{22}} = \frac{24/25}{-7/25} = \frac{-24}{7} \longrightarrow N(A-2I) = \left\{ \alpha \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \right\},\ \alpha \in \mathbb{R}$$

$$\text{Verify: } Av = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{bmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix} = 2 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \lambda v \ \checkmark$$

(iii) $\lambda = 3 \longrightarrow A - 3I = \begin{bmatrix} 2-3 & 0 & 0 \\ 0 & 43/25 - 3 & -24/25 \\ 0 & -24/25 & 57/25 - 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -32/25 & -24/25 \\ 0 & -24/25 & -18/25 \end{bmatrix}$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{3}{4}R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -32/25 & -24/25 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \frac{-32}{25}x_2 - \frac{24}{25}x_3 = 0 \longrightarrow -32x_2 = 24x_3 \longrightarrow x_2 = \frac{-3}{4}x_3$$

$$m_{32} = \frac{-a_{32}}{a_{22}} = \frac{24/25}{-32/25} = \frac{-3}{4} \rightarrow v = \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} \longrightarrow N(A-3I) = \left\{ \alpha \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} \right\},\ \alpha \in \mathbb{R}$$

$$\text{Verify: } Av = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 43/25 & -24/25 \\ 0 & -24/25 & 57/25 \end{bmatrix} \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -9/4 \\ 3 \end{vmatrix} = 3 \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} = \lambda v \ \checkmark$$

$$\lambda_1 = 1,\ v_1 = \begin{vmatrix} 0 \\ 4/3 \\ 1 \end{vmatrix} \ \& \ \lambda_2 = 2,\ v_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \longrightarrow v_1^T v_2 = \begin{vmatrix} 0 & \frac{4}{3} & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = 0 \ \checkmark$$

$$\lambda_1 = 1,\ v_1 = \begin{vmatrix} 0 \\ 4/3 \\ 1 \end{vmatrix} \ \& \ \lambda_3 = 3,\ v_3 = \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} \longrightarrow v_1^T v_3 = \begin{vmatrix} 0 & \frac{4}{3} & 1 \end{vmatrix} \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} = 0 \ \checkmark$$

$$\lambda_2 = 2,\ v_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \ \& \ \lambda_3 = 3,\ v_3 = \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} \longrightarrow v_2^T v_3 = \begin{vmatrix} 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ -3/4 \\ 1 \end{vmatrix} = 0 \ \checkmark$$