Section 5.6 Orthogonal Matrices

We saw already Q orthogonal matrix if $Q^TQ = I$ that is its columns are orthonormal

$$\langle q_i, q_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Recall $\|Qx\| = \|x\| \Rightarrow \|Q\| = 1$

$$\langle Qx, Qy \rangle = \langle x, y \rangle$$

i.e. Orthogonal matrices maintain length and angles.

$$Q^TQ = I \Leftrightarrow Q^{-1} = Q^T$$

$$\Rightarrow QQ^T = I$$

Examples. Rotations by θ

$$Q_\theta = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$Q_\theta^T Q_\theta = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \overset{1}{\cos^2+\sin^2} & \overset{0}{\cos\sin - \sin\cos} \\ 0 & 1 \end{vmatrix}$$

Geometrically. Rotation maintains length and angles.

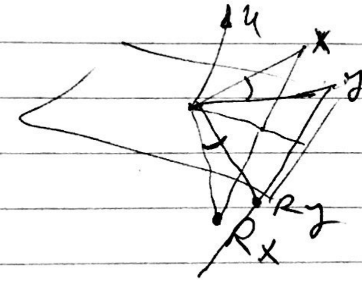
Rotation angle θ in i, j plane in $\mathbb{R}^n$

$$\begin{array}{c|c} \begin{array}{c} \\ i \to \\ \\ j \text{ —} \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & c & & -s & & \\ & & & 1 & & & \\ & & s & & c & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \\ & \begin{array}{ccccccc} & & i & & j & & \end{array} \end{array} \qquad c^2+s^2=1$$

Reflections across a Hyperplane (subspace of dimension $n-1$)

Let $H = \{ x \,/\, x^T u = 0 \} =$

$$= \{ x \,/\, x^T \frac{u}{\|u\|} = 0 \}$$



Maintains length and angles

Formula:

$$R = I - \frac{2uu^T}{u^Tu}$$

or if $u^Tu = 1$, $\|u\| = 1$

$$R = I - uu^T$$

We can show that $R^T = R$, $RR = I$

$Ru = -u$ and

if $u^Tw = 0$, then $Rw = w$ i.e. $w \in H$

Indeed for $R = I - 2uu^T$, $\|u\| = 1$

i.e. $u^Tu = 1$

$$R^T = I^T - 2(uu^T)^T = I - 2uu^T = R$$

$$R^TR = RR = (I - 2uu^T)(I - 2uu^T) =$$

$$= I - 2uu^T - 2uu^T + 4\,u\underbrace{u^Tu}_{=1}u^T = I$$

$$Ru = (I - 2uu^T)u = u - 2u\underbrace{u^Tu}_{=1} = u - 2u = -u$$

If $u^Tw = 0$

$$(I - 2uu^T)w = w - 2u\underbrace{u^Tw}_{=0} = w$$

Example $u = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ $\quad u^T u = 3$

$H = \{ x / x^T u = u^T x = 0 \} =$

$= \{ x \in \mathbb{R}^3 / x_1 + x_2 + x_3 = 0 \}$

$R = I - \frac{2}{3} u u^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} =$

$= \begin{vmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{vmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

clearly $R^T = R$

$R^T R = R \cdot R = \frac{1}{3} \cdot \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$

$R u = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = - \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = -u$

for $w \ni u^T w = 0$ example $w = \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix}$

$R w = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \\ 1 \end{vmatrix} = w$