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Linear Algebra, Math 2101-003
Homework set #1

1. Explain why the set $H^{n\times n} = \{n \times n \text{ symmetric matrices }\}$ is a subspace of $\mathbb{R}^{n\times n}$.

2. **(a)** Let A be a matrix in $\mathbb{R}^{n\times n}$. Show that $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.
(b) Prove that any matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = H + S$, where H is symmetric, and S is skew-symmetric.
(c) Show that H, S, in part (b) are unique.

3. Show that the set $\{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is an affine space.

1 To show the set $H^{n\times n}$ to be a subspace of $\mathbb{R}^{n\times n}$, it must satisfy 3 properties

(i) The zero vector is in the set ($0 \in V$)

$(0_{ij})^T = 0_{ji} = 0_{ij}$

definition of zero matrices

definition of transposition

$\therefore$ $n\times n$ zero matrix (all entries 0) is symmetric $\therefore 0 \in V$

(ii) The set is closed under addition

Suppose A and B are $n\times n$ symmetric matrices, need to show that $(A+B)$ is also symmetric

$(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T + B^T)_{ij}$

definition of transposition

definition of transposition

definition of $(A+B)$

$\therefore (A+B)$ is symmetric

(iii) The set is closed under scalar multiplication

Suppose α is a scalar, A is an $n\times n$ symmetric matrix; need to show that (αA) is also symmetric definition of transposition

commutative law

definition of transposition

$$(\alpha A)^T_{ij} = (\alpha A)_{ji} = \alpha A_{ji} = \alpha (A^T)_{ij}$$

$\therefore$ (αA) is symmetric

Since the set $H^{n\times n}$ satisfies all 3 properties above, QED!

② (a) A is a matrix in $\mathbb{R}^{n\times n}$ (square matrix)

$$\begin{aligned}(A+A^T)^T_{ij} &= (A^T)_{ij} + (A^T)^T_{ij} && \text{by transpose property: } (A+B)^T = A^T + B^T \\ &= (A^T)_{ij} + A_{ij} && \text{by transpose property: } (A^T)^T = A \\ &= (A^T + A)_{ij} && \text{by definition of } ij \\ &= (A + A^T)_{ij} && \text{by commutative law}\end{aligned}$$

$\therefore$ $(A+A^T)$ is symmetric QED!

$$\begin{aligned}(A-A^T)^T_{ij} &= (A^T)_{ij} - (A^T)^T_{ij} && \text{by transpose property: } (A-B)^T = A^T - B^T \\ &= (A^T)_{ij} - A_{ij} && \text{by transpose property: } (A^T)^T = A \\ &= (A^T - A)_{ij} && \text{by definition of } ij \\ &= -(A - A^T)_{ij}\end{aligned}$$

$\therefore$ $(A-A^T)$ is skew-symmetric QED!

(b) By part (a), for any square matrix $A \in \mathbb{R}^{n\times n}$, $(A+A^T)$ is symmetric and $(A-A^T)$ is skew-symmetric.

Let $H = \frac{1}{2}(A+A^T) \longrightarrow H^T = \left[\frac{1}{2}(A+A^T)\right]^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A+A^T) = H$

transpose property proven in (a)

$\therefore$ H is symmetric

Let $S = \frac{1}{2}(A-A^T) \longrightarrow S^T = \left[\frac{1}{2}(A-A^T)\right]^T = \frac{1}{2}(A-A^T)^T = \frac{-1}{2}(A-A^T) = -S$

transpose property proven in (a)

$\therefore$ S is skew-symmetric

$\therefore$ $H+S = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = \frac{1}{2}(A+A^T+A-A^T) = \frac{1}{2}(2A) = A$

$\therefore$ $A = H+S$ where H is symmetric and S is skew-symmetric QED!

(c) Proof by contradiction: Suppose such combination of H and S in part (b) is not unique. Hence, we can construct X and Y that:

(i) $A = X + Y$

(ii) X is symmetric $\quad \therefore X = X^T$ $\quad\}$ by definitions

(iii) Y is skew-symmetric $\quad \therefore Y = -Y^T$ or $Y^T = -Y$ $\quad\}$

Thus,
$$\begin{aligned} A^T &= (X+Y)^T && \text{by assumption} \\ &= X^T + Y^T && \text{by transpose property} \\ &= X - Y && \text{by definitions above} \end{aligned}$$

$\therefore A + A^T = (X+Y) + (X-Y) = 2X \rightarrow X = \frac{A+A^T}{2} = H$ (from (b))

$A - A^T = (X+Y) - (X-Y) = X + Y - X + Y = 2Y \rightarrow Y = \frac{A-A^T}{2} = S$ (from (b))

This is a contradiction to such assumption that H and S are not unique

$\therefore$ Such assumption is invalid

$\therefore$ H and S are unique QED!

3. To show that the set $A = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is an affine space, need to show that $A = p + X = \{v \in V \mid v = p + x,\ x \in X,\ X \subseteq V = \mathbb{R}^3\}$

(i) Take point $p = (1,0,0)$ then $p_1 + p_2 + p_3 = 1 + 0 + 0 = 1$

$\therefore p \in V$ or $0 \notin V$ $\quad \therefore$ A does not contain vector 0

(ii) Define $X = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

Let $v, w \in X$, $\alpha \in \mathbb{R}$; wish to show that $(\alpha v + w) \in X$

$v \in X \rightarrow v_1 + v_2 + v_3 = 0$

$w \in X \rightarrow w_1 + w_2 + w_3 = 0$

Consider $\alpha v + w$:
$$\begin{aligned} &(\alpha v + w)_1 + (\alpha v + w)_2 + (\alpha v + w)_3 \\ &= \alpha v_1 + w_1 + \alpha v_2 + w_2 + \alpha v_3 + w_3 \\ &= \alpha(v_1 + v_2 + v_3) + (w_1 + w_2 + w_3) \\ &= \alpha . 0 + 0 = 0 \end{aligned}$$

$\therefore (\alpha v + w) \in X$ $\quad \therefore$ B is a subspace

Can express A as $A = p + X = \{(1,0,0) + \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

Need to verify:

(a) A is non-empty (proven in (i))

(b) A is closed under affine combinations

i.e: For any point $q \in A$ and any vector $v \in X$, the point $(q+v) \in A$

Let $q \in A \rightarrow q_1 + q_2 + q_3 = 1$

$v \in X \rightarrow v_1 + v_2 + v_3 = 0$

$\therefore (q+v) = (q_1 + v_1) + (q_2 + v_2) + (q_3 + v_3)$

$= (q_1 + q_2 + q_3) + (v_1 + v_2 + v_3) = 1 + 0 = 1 \in A$

$\therefore (q+v) \in A$

$\therefore$ A is non-empty and closed under affine combinations

$\therefore$ A is an affine space QED!