# § 4.2 Reduction of Order (only on Quiz 4)

Homogeneous linear $2^{nd}$ order DE:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

We saw in § 4.1.2 that the general solution to this DE is a linear combination

$$y = c_1 y_1 + c_2 y_2 \, ,$$

where $y_1$ and $y_2$ are linearly independent, i.e. they form a fundamental set of solutions to the DE.

---

Problem: Given $y_1$, find $y_2$.

Step 1: Put into standard form by dividing by $a_2(x)$.

$$P(x) = \frac{a_1(x)}{a_2(x)} \qquad Q(x) = \frac{a_0(x)}{a_2(x)}$$

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

where $P(x)$, $Q(x)$ are continuous on some interval.

$y_1(x) \neq 0$
$\forall x$ in $I$

Step 2: Given $y_1$, to find $y_2(x)$:

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^{2}(x)}\,dx$$

---

Example 2:

$y_1(x) = x^{-1}$ is a solution to the HLDE

$2x^2y'' + 3xy' - y = 0$, $x > 0$. Find a fundamental set of solutions to the DE and the general solution.

Step 1: Standard form:

$$y'' + \frac{3}{2x}y' - \frac{1}{2x^2}y = 0$$

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

so $P(x) = \dfrac{3}{2x}$.

Step 2: Find $y_2(x)$. See next page.

$$y_2(x) = x^{-1}\int \frac{e^{-\int \frac{3}{2x}\,dx}}{(x^{-1})^2}\,dx \quad , x>0$$

$$= x^{-1}\int \frac{e^{-\frac{3}{2}\ln x}}{x^{-2}}\,dx \quad , x>0$$

$$= x^{-1}\int \frac{e^{\ln x^{-3/2}}}{x^{-2}}\,dx$$

$$= x^{-1}\int \frac{x^{-3/2}}{x^{-2}}\,dx$$

$$= x^{-1}\int x^{\frac{1}{2}}\,dx$$

$$= x^{-1}\cdot \frac{2}{3}x^{3/2} + \underset{\text{Assume } C=0}{\underset{\shortparallel}{C}}$$

$$= \frac{2}{3}x^{1/2}$$

Fundamental Set $\{x^{-1},\ x^{1/2}\}$

General Solution :

$$y = c_1 x^{-1} + c_2 x^{1/2}$$