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Linear Algebra, Math 2101-002
Exam #2

Write your name on your response sheet. This is a closed book and notes exam. Explain your answers in as much detail as possible. Check your examples and results. You need to explain every step of your logic, development and calculation.

1. (10 pts.)
Consider the set $S = \{x \in \mathbb{R}^3 | 2x_1 + 3x_2 + 2x_3 = 0\}$.
(a) Show that S is a subspace of $\mathbb{R}^3$.
(b) Show that S is not all of $\mathbb{R}^3$.

2. (10 pts.)
Let $S \subset \mathbb{R}^n$ be a subspace. Let $S^\perp = \{x \in \mathbb{R}^n | x^T y = 0 \text{ for all } y \in S\}$. Prove that $S^\perp$ is also a subspace.

3. (20 pts.)
For each of the following statements indicate if the statement is TRUE or FALSE.
If it is true, prove the statement. If it is false, give a counter-example.

(a) If A is nonsingular, then, $A + A$ is nonsingular.

(b) For any pair of matrices $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{k\times m}$, then, $(AB)^T = B^T A^T$.

(c) If $A = LU$ is an LU factorization of A. If $Ux = 0$, then $Ax = 0$.

(d) If $A = LU$ is an LU factorization of A. If $Ax = 0$, then $Ux = 0$ (the converse of (c))

4. (20 pts.)
Let $A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & -10 \end{vmatrix}$.
(a) Compute the LU factorization of A.
(b) Find a set of spanning vectors for $\mathcal{R}(A)$ (the range or column space of A).
(c) Find a set of spanning vectors for $\mathcal{N}(A)$ (the null space of A).
(d) For which values of α does the vector $c = \begin{vmatrix} 1 \\ -1 \\ \alpha \end{vmatrix}$ belong to $\mathcal{R}(A)$?

SEE REVERSE FOR MORE QUESTIONS

5. (20 pts.)

Let $A = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}$.

(a) Compute the rank of A, using its LU Factorization.

(b) Give **three** different reasons why A is a nonsingular matrix, and explain why each characterization you give applies to this particular matrix.

(c) Using the LU factorization, solve $Ax = b$, where $b = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$.

6. (5 pts.)

Let

$$B = \begin{vmatrix} 1 & 3/2 \\ a & 1/2 \end{vmatrix}.$$

For which values of a is B singular? Explain your answer.

7. (15 pts.) (a) Let A be $n \times n$. Prove that

$$S = \frac{A + A^T}{2} \quad \text{and} \quad K = \frac{A - A^T}{2}$$

are respectively symmetric and skew-symmetric matrices. Confirm that $A = S + K$.

(b) Let A be $n \times n$. Prove that if $A = S_1 + K_1$, with S_1 symmetric and K_1 skew-symmetric, then $S_1 = S$ and $K_1 = K$.

END OF REGULAR PART OF EXAM (100 pts.)
THE REST, ONLY IF TIME ALLOWS.

8. (10 pts. extra credit)

(a) Let A be a symmetric $n \times n$ matrix of rank 1. Prove that $A = uu^T$ for some vector $u \in \mathbb{R}^n$.

(b) Prove that for such matrices, each diagonal entry is nonnegrative, i.e., $a_{ii} \geq 0$, $i = 1, \ldots, n$, and for at least one i, $a_{ii} > 0$ (at least one diagonal entry is positive).

9. (10 pts. extra credit) Prove that there does not exist a skew-symmetric matrix of rank 1.

10. (5 pts. extra extra credit)

Let P be such that $P^2 = P$. Prove that if $P \neq I$, then P is singular.