D Szyld Math 2101

Section 4.3 Linear Independence

Let $v_1\ v_2, \dots, v_r \in V$, we say that they are linearly <u>dependent</u> if one can be written as a linear combination of the others.

Example $v_1 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ $v_2 = \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix}$ $v_3 = \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix}$

$$v_3 = 2 \begin{vmatrix} 1 \\ 2 \\ 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$$

Note, if v_3 is dependent of $v_1\ v_2$, then, v_1 is dependent of $v_2\ v_3$, etc

e.g. since $v_3 = 2v_2 + (-1)\,v_1$

then $v_2 = \frac{1}{2} v_3 + \frac{1}{2} v_1$

Equivalently if $A = [v_1\ v_2 \dots v_r]$

then $v_1\ v_2 \dots v_r$ are linearly dependent if

$Ax = 0$ has a nontrivial solution

In this example $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

Then for $x = \begin{vmatrix} -1 \\ 2 \\ -1 \end{vmatrix}$ $\quad Ax = 0$

Another example

$\begin{vmatrix} 2 \\ 0 \\ 1 \end{vmatrix} \begin{vmatrix} 1 \\ -1 \\ 0 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}$ linearly dependent

$v_1 \quad v_2 \quad v_3$

$v_1 - v_2 - v_3 = 0 \qquad Ax = 0 \quad x = \begin{vmatrix} 1 \\ -1 \\ -1 \end{vmatrix}$

$v_1 = v_2 + v_3 \qquad A = [v_1\ v_2\ v_3]$

$v_1, v_2 \ldots v_r$ are linearly dependent vectors or the <u>set</u> $\{v_1, v_2, \ldots v_r\}$ is linearly dependent.

Similarly, the set $\{v_1, v_2, \ldots, v_r\}$ is $\boxed{\text{linearly independent}}$ if <u>none</u> of the vectors can be written as a linear combination of the others.

Equivalently

If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$$

$\{v_1 v_2 \ldots v_r\}$ linearly independent

if their linear combination $=0 \Rightarrow$ coeffs. are zero.

Equivalently

$A = [v_1 v_2 \ldots v_n] \qquad Ax = 0 \Rightarrow x = 0$

Examples $\begin{vmatrix}1\\0\\0\end{vmatrix} \begin{vmatrix}0\\1\\0\end{vmatrix} \begin{vmatrix}0\\0\\1\end{vmatrix}$

$\begin{vmatrix}1\\1\\1\end{vmatrix} \begin{vmatrix}1\\2\\2\end{vmatrix} \begin{vmatrix}1\\2\\3\end{vmatrix}$

$\begin{vmatrix}1\\1\end{vmatrix} \begin{vmatrix}1\\-1\end{vmatrix}$

<u>Exercise.</u> Determine if

$\begin{vmatrix}1\\2\\1\end{vmatrix}$ $\begin{vmatrix}1\\0\\2\end{vmatrix}$ $\begin{vmatrix}5\\6\\7\end{vmatrix}$ are linearly independent or dependent.

$$A = \begin{vmatrix}1&1&5\\2&0&6\\1&2&7\end{vmatrix} \to \begin{vmatrix}1&1&5\\0&-2&-4\\0&1&2\end{vmatrix} \to \begin{vmatrix}1&1&5\\0&-2&-4\\0&0&0\end{vmatrix}$$

rank $A = 2$, 1 free variable

thus $\exists\, x \neq 0 \quad Ax = 0$

$\Rightarrow$ vectors are linearly dependent

If we solve $Ax = 0$, we get coefficients so that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

In this exercise

$$-2x_2 - 4x_3 = 0$$
$$x_2 = -2x_3$$

$$x_1 + x_2 + 5x_3 = x_1 - 2x_3 + 5x_3$$
$$= x_1 + 3x_3$$
$$x_1 = -3x_3$$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{vmatrix} = x_3 \begin{vmatrix} -3 \\ -2 \\ 1 \end{vmatrix}$$

$$-3\begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} - 2\begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix} + \underline{1}\begin{vmatrix} 5 \\ 6 \\ 7 \end{vmatrix} = 0$$

or $$\begin{vmatrix} 5 \\ 6 \\ 7 \end{vmatrix} = 3\begin{vmatrix} 1 \\ 2 \\ 1 \end{vmatrix} + 2\begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix}$$

□ A $m \times n$ – the following are equivalent

- columns of A are linearly independent
- $Ax = 0 \Rightarrow x = 0$
- $N(A) = \{0\}$
- $\text{Rank}(A) = n$

and of course the complement.

- columns of A are linearly dependent
- $Ax = 0$ for ~~s~~ some $x \neq 0$
- $N(A) \supsetneq \{0\}$
- $\text{Rank } A < n$

---

rows of A are linearly independent
(columns of A^T)

$\Leftrightarrow$ $N(A^T) = 0$

$\Leftrightarrow$ $\text{Rank } A = m$

A $n \times n$ (square)

the following are equivalent

- columns of A are linearly independent
- A is nonsingular
- $N(A) = \{0\}$ $[Ax = 0 \Rightarrow x = 0]$
- rows of A are linearly independent
- $N(A^T) = \{0\}$ $[A^T x = 0 \Rightarrow x = 0]$
- rank $A = n$
- Range$(A) = R(A) = \mathbb{R}^n$
- $R(A^T) = \mathbb{R}^n$