§ 7.3.2 Translation on the t-axis

gives

Additional properties of the Laplace transform (LT) that are useful for solving IVP with discontinuous or impulsive forcing functions.

Definition 7.3.1 Unit Step Function $\mathcal{U}(t-a)$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

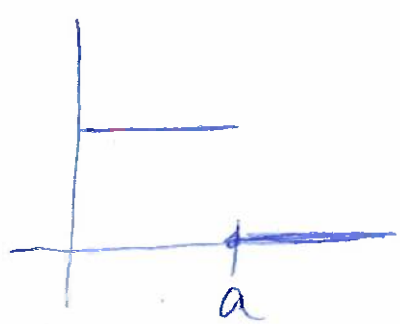
(For LT, only care about $a \ge 0$)

eg $f(t) = 1 - \mathcal{U}(t-a)$

$$f(t) = \begin{cases} 1, & 0 \le t < a \\ 0, & t \ge a \end{cases}$$

$1-0$

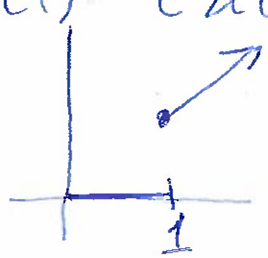
$1-1$



eg $f(t) = t\,\mathcal{U}(t-1) = \begin{cases} 0, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$

$t \cdot 0$

$t \cdot 1$



eg $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$

$0 + g(t)$

$0 + g(t) - g(t) + h(t)$

Then $f(t) = g(t) - g(t)\,u(t-a) + h(t)\,u(t-a)$.

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eg $f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$

$f(t) = g(t)\left[u(t-a) - u(t-b)\right]$

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eg $f(t) = 5 - \underbrace{5\,u(t-8)}$

$5\,u(t-8) = \begin{cases} \overset{0\times 5}{0}, & 0 \le t < 8 \\ \underset{1\times 5}{5}, & t \ge 8 \end{cases}$

$f(t) = \begin{cases} \overset{5-0}{5}, & 0 \le t < 8 \\ \underset{5-5}{0}, & t \ge 8 \end{cases}$

Now, we write a piecewise function in terms of the unit step function.

(This is what we will need to do to use LP if $f(t)$ is given as a piecewise fn).

$$f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \ge 9 \end{cases}$$

+3

-6

+2

$f_1(t) = 2$

$$f_2(t) = 2 + 3\,\mathcal{U}(t-4) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & t \ge 4 \end{cases}$$

$$f_3(t) = 2 + 3\,\mathcal{U}(t-4) \underbrace{- 6\,\mathcal{U}(t-7)}_{\text{negative jump of } 6} = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & t \ge 7 \end{cases}$$

$$f(t) = 2 + 3\,\mathcal{U}(t-4) - 6\,\mathcal{U}(t-7) + \underbrace{2\,\mathcal{U}(t-9)}_{\begin{cases} 0, & 0 \le t < 9 \\ 2, & t \ge 9 \end{cases}}$$

$$\left(= \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & t \ge 9 \end{cases} \right)$$

T.7.3.2 Second Translation Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

Pf: see book

Corollary: Letting $f(t) = 1$, we get

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

eg $f(t) = 2 - 3\,\mathcal{U}(t-2) + \mathcal{U}(t-3)$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{\mathcal{U}(t-2)\} + \mathcal{L}\{\mathcal{U}(t-3)\}$$

$$= \frac{2}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}.$$

Inverse Laplace:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}$$ see Example 7

$$= f(t-a)\mathcal{U}(t-a)$$

Awkward

Last formula is awkward.

Alternative:

$$\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}$$

+a

add a

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}\Big|_{t \to t-a} \cdot \mathcal{U}(t-a)$$

−a

Subtract a

eg $f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}), & t \ge \frac{\pi}{4} \end{cases}$

$$f(t) = \sin t + \cos(t - \tfrac{\pi}{4})\,\mathcal{U}(t - \tfrac{\pi}{4})$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\cos(t - \tfrac{\pi}{4})\,\mathcal{U}(t - \tfrac{\pi}{4})\}$$

$$= \frac{1}{s^2+1} + e^{-\pi/4\, s}\,\mathcal{L}\{\underbrace{\cos(t - \tfrac{\pi}{4} + \tfrac{\pi}{4})}_{\cos t}\}$$

$$= \frac{1}{s^2+1} + e^{-\pi/4\, s}\,\frac{s}{s^2+1}$$

Example 8 :

$$\mathcal{L}\{\cos t\, u(t-\pi)\}$$

add π

$$= e^{-\pi s}\,\mathcal{L}\{\cos(t+\pi)\} \rightarrow \frac{s+\pi}{(s+\pi)^2+1}e^{-\pi s}$$

Identity

$$= e^{-\pi s}\,\mathcal{L}\{-\cos t\}$$

$$= -\frac{s}{s^2+1}e^{-\pi s}$$

Now we will find $\mathcal{L}^{-1}\{F(s)\}$

eg $\mathcal{L}^{-1}\left\{e^{-3s}\frac{s}{s^2+4}\right\} = \cos(2(t-3))\,u(t-3)$

Subtract 3

NOTE: $\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t\right)$

eg $F(s) = \frac{8}{s+3}e^{-s}$ $\qquad \left(\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}\right)$

subtract 1

$$\mathcal{L}^{-1}\{F(s)\} = 8e^{-3(t-1)}\,u(t-1)$$

eg $F(s) = \frac{8}{s^2-2s+5} e^{-2s} = 4 \cdot \frac{2}{(s-1)^2+4} e^{-2s}$

$a = 2$

$$\mathcal{L}^{-1}\{F(s)\} = 4 \cdot \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}\Bigg|_{t \to t-2} \cdot u(t-2)$$

subtract $2 = a$

$$= 4\left(\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\Bigg|_{s \to s-1}\right\}\right)\Bigg|_{t \to t-2} \cdot u(t-2)$$

$$= 4\left(e^{t}\sin(2t)\right)\Big|_{t \to t-2} \cdot u(t-2)$$

$$= 4e^{t-2}\sin(2(t-2)) \cdot u(t-2)$$

Example 9 : IVP

Review

$\Longrightarrow$

Another IVP

eg $y'' + 4y = \sin t + \underbrace{u(t-\pi)\sin(t-\pi)}_{+\pi}$,

$y(0) = 0, \quad y'(0) = 0$

$$s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 4Y(s) = \frac{1}{s^2+1} + e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$Y(s)(s^2+4) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{e^{-\pi s}}{(s^2+1)(s^2+4)}$$

PFD (works for both terms)

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

s^3: $0 = A + C \implies C = -A$

s^2: $0 = B + D \implies D = -B$

s: $0 = 4A + C \implies 0 = 4A - A$

$0 = 3A$

$\implies A = 0$ and $C = 0$

$s = 0$ Constant: $1 = 4B + D$

$1 = 4B - B$

$1 = 3B \implies B = 1/3$ and $D = -1/3$

So

$$Y(s) = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + e^{-\pi s}\left(\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right)$$

$-\frac{1}{3} = -\frac{1}{6}\cdot 2$

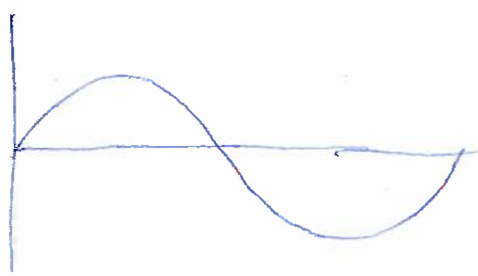
$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

$-\pi = -a$

$$+\, u(t-\pi)\left(\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin(2(t-\pi))\right)$$

Identities used

$$= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}u(t-\pi)(2\sin t + \sin 2t)$$



$\sin(t-\pi) = -\sin t$

$\sin(2t-2\pi) = \sin 2t$

periodicity

Know periodicity

$\sin(\theta \pm 2\pi n) = \sin\theta$

$\cos(\theta \pm 2\pi n) = \cos\theta$