Section 3.4

$\boxed{\text{Composition of two linear functions is linear}}$

Let

$$f: D \longrightarrow R$$

$$g: P \longrightarrow D$$

$$f \circ g(x) = f(g(x))$$

$$f \circ g: P \longrightarrow R$$

Example $g\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{bmatrix}$ $\quad g: \mathbb{R}^2 \to \mathbb{R}^3$

$f\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 + y_3 \\ 3y_2 - y_3 \\ y_1 - 2y_2 \end{bmatrix}$ $\quad f: \mathbb{R}^3 \to \mathbb{R}^3$

$$(f \circ g)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2) + 3x_2 \\ 3(x_1 - x_2) - 3x_2 \\ (x_1 + x_2) - 2(x_1 - x_2) \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 3x_1 - 4x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$

$$f(g(x)) = \begin{bmatrix} x_1 + 4x_2 \\ 3x_1 - 4x_2 \\ -x_1 + 3x_2 \end{bmatrix}$$

$$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

<u>Proof.</u> wish to show that
$f \circ g : P \to R$ is a linear map.

$$(f \circ g)(\alpha x + y) = f\left(g(\alpha x + y)\right) =$$

def. of composition
of Maps

$$= f\left[\alpha g(x) + g(y)\right] = \alpha f(g(x)) + f(g(y)) =$$

g linear f linear

$$= \alpha (f \circ g)(x) + (f \circ g)(y)$$ Q.E.D.

def of composition

Note this is <u>not</u> commutative

<u>it</u> may not even be defined.
(composition)

$\mathbb{R}^2 \to \mathbb{R}^2$ Rotation followed by expansion (or shrinkage) - or Reflection followed by Rotation etc.

Section 3.5 Matrix Multiplication

Recall Matrix times vector

$$A \cdot X = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \\ a_{21}X_1 + a_{21}X_2 + a_{23}X_3 \end{bmatrix}$$

Now think of two column vectors

$$A \cdot [X \mid y] =$$

$$\left[\begin{array}{c|c} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 & a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \end{array}\right]$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \qquad B = [X \mid y] = \begin{vmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -3 \end{vmatrix}$$

$$A \cdot B = \left[\begin{array}{c|c} 1+2+3 & 2+0+3\cdot(-3) \\ -1+0+1 & -2+0+1\cdot(-3) \end{array}\right] = \begin{bmatrix} 6 & -7 \\ 0 & -5 \end{bmatrix}$$

Note A 2×3 $\qquad B$ 3×2 $\qquad A\cdot B = 2\cdot 2$

In general

A is $n_1 \times n_2$ B is $n_2 \times n_3$

$A \cdot B$ is $n_1 \times n_3$

General Formula.

$\underset{n_1 \times n_2}{A}$ a_{ij} $\underset{n_2 \times n_3}{B}$ b_{ij}.

$\underset{n_1 \times n_3}{C = A \cdot B}$ c_{ij}

$$c_{ij} = \sum_{k=}^{n_2} a_{ik} \cdot b_{kj}$$

Special case A is $1 \times n$ B is $n \times 1$

$x^T = [x_1 \cdot x_2 — x_n]$ $y = \begin{vmatrix} y_1 \\ \vdots \\ y_n \end{vmatrix}$

$$x^T \cdot y = \sum_{i=1}^{n} x_i y_i$$

$$[3 \ 0 \ 1] \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = 3 \cdot 1 + 0 \cdot (-2) + 1 \cdot 4 = 7$$

So. each entry of $A \cdot B$

is the product of a row of A

with a colum of B

$$C_{ij} = A_{i*} \cdot B_{*j}$$

Another example $B \cdot A =$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 1-2 & 2+0 & 3+1 \\ 1+0 & 2+0 & 3+0 \\ 1+3 & 2+0 & 3-3 \end{bmatrix}}_{3\times 3}$$

$$= \begin{bmatrix} -1 & 2 & 4 \\ 1 & 2 & 3 \\ 4 & 2 & 0 \end{bmatrix}$$

Note $A \cdot B \neq B \cdot A$

$$\boxed{\text{Matrix product is not commutative}}$$

Consider the Matrix representing g from section 3.4 p. 68

$$g : \mathbb{R}^2 \to \mathbb{R}^3$$

Matrix B is thus 3×2

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}$$

let A represent $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix}$$

the representation of $f \circ g$ is $\underset{3\times 3}{A} \cdot \underset{3\times 2}{B} = C$, 3×2

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 1+0+3 \\ 0+3+0 & 0-3+0 \\ 1-2+0 & 1+2+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & -3 \\ -1 & 3 \end{bmatrix}$$

Matrix Multiplication is such that if B represents g and A represents f

A·B represents $f \circ g$.

Another view

A has columns a_1 a_2 -- a_n

A_{*1} A_{*2} -- A_{*n}

$$\underset{m\times n}{A} = \left[\, a_1 \,\middle|\, a_2 \,\middle|\, \cdots \,\middle|\, a_n \,\right] \qquad v \text{ vector } v = \begin{bmatrix} v_1 \\ v_2 \\ \downarrow \\ v_n \end{bmatrix}$$

$$A \cdot v = \sum_{i=1}^{n} v_i a_i = v_1 a_1 + v_2 a_2 \cdots + v_n a_n$$

$$v_1 A_{*1} + v_2 A_{*2} + \cdots + v_n A_{*n}$$

Linear combination of the columns of A.
Elements of the vector are the coefficients of the linear combination

Same for two vector $\underset{m\times n}{A} \underset{n\times 2}{\left[x \middle| y \right]} = \underset{m\times 2}{\left[Ax \middle| Ay \right]}$

back to example

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}$$

$$A \cdot B = \left[\; 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \;\middle|\; 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} \; \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ 1 \end{bmatrix} \;\right]$$

$$= \left[\begin{array}{c|c} 6 & -7 \\ 0 & -5 \end{array}\right].$$