Derivation of the Influence Function for AUC for competing risk data and survival data

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Introduction

In this document we derive the influence function of the inverse probability of censoring weighted (IPCW) estimator of the time-dependent AUC in the cases with and without competing risks. We denote the uncensored time to event by T, the right censoring time by C, and by X a continuous marker. Let us first look at the case, where T is not censored (and there is only one event).

Influence function uncensored case

In the uncensored case, we observe (T, X) and denote Q for the joint probability measure Q, i.e., $(T, X) \sim Q$. The time-dependent AUC at time τ for marker X is defined as:

$$\mathrm{AUC}(\tau) = Q(X > X' | T \leqslant \tau, T' > \tau) = \frac{Q(T \leqslant \tau, T' > \tau, X > X')}{Q(T \leqslant \tau, T' > \tau)}$$

Let $\mathcal{Q}$ be the set of all probability measures of T, X and consider the functional $\nu_\tau : \mathcal{Q} \to [0, 1]$

$$\nu_\tau(Q) = \int\int 1_{\{t \leqslant \tau, t' > \tau, x > x'\}} dQ(t, x) dQ(t', x')$$

Note that the value of the functional is the numerator of the AUC:

$$\nu_\tau(Q) = Q(T \leqslant \tau, T' > \tau, X > X')$$

To derive the Gateaux derivative of the functional at Q in the direction of an observation T_i, X_i we introduce the corresponding path,

$$Q^i_\varepsilon = Q + \varepsilon \left\{ \delta_{\{T_i, X_i\}} - Q \right\}.$$

and the following short notation $\partial_\varepsilon = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0}$ to obtain the directional derivative:

$$\partial_\varepsilon Q^i_\varepsilon = \delta_{\{T_i, X_i\}} - Q.$$

The Gateaux derivative of the functional ν_τ is obtained using straight-forward calculus:

$$
\begin{aligned}
\partial_\varepsilon \nu_\tau(Q_\varepsilon) &= \int\int 1_{\{t \leqslant \tau, t' > \tau, x > x'\}} \partial_\varepsilon Q^i_\varepsilon(t,x) dQ(t',x') \\
&\quad + \int\int 1_{\{t \leqslant \tau, t' > \tau, x > x'\}} dQ(t,x) \partial_\varepsilon Q^i_\varepsilon(t',x') \\
&= \int\int 1_{\{t \leqslant \tau, t' > \tau, x > x'\}} d[\delta_{\{T_i, X_i\}} - Q](t,x) dQ(t',x') \\
&\quad + \int\int 1_{\{t \leqslant \tau, t' > \tau, x > x'\}} dQ(t,x) d[\delta_{\{T_i, X_i\}} - Q](t',x') \\
&= \int_{-\infty}^{\infty}\int_0^\tau Q(T > \tau, X < x) d\left[\delta_{\{T_i, X_i\}} - Q\right](t,x) \\
&\quad + \int_{-\infty}^{\infty}\int_\tau^\infty Q\left(T \leqslant \tau, X > x'\right) d\left[\delta_{\{T_i, X_i\}} - Q\right](t',x') \\
&= 1_{\{T_i \leqslant \tau\}} Q\left(T > \tau, X < X_i\right) + 1_{\{T_i > \tau\}} Q\left(T \leqslant \tau, X > X_i\right) - 2\nu_\tau(Q).
\end{aligned}
$$

Now we consider the functional $\mu_\tau : \mathcal{Q} \to [0,1]$ corresponding to the denominator of the AUC:

$$
\mu_\tau(Q) = \int\int 1_{\{t \leqslant \tau, t' > \tau\}} dQ(t,x) dQ(t',x') = Q(T \leqslant \tau, T' > \tau).
$$

An analogous calculation yields that

$$
\partial_\varepsilon \mu_\tau(Q_\varepsilon) = 1_{\{T_i \leqslant \tau\}} Q\left(T > \tau\right) + 1_{\{T_i > \tau\}} Q\left(T \leqslant \tau\right) - 2\mu_\tau(Q)
$$

The quotient rule now yields that the Gateaux derivative for the functional $\mathrm{AUC}_\tau(Q) = \nu_\tau(Q)/\mu_\tau(Q)$ is given by

$$
\begin{aligned}
& IF_{\mathrm{AUC}}(T_i, X_i; \tau) = (\mu_\tau(Q))^{-2} \\
&\quad \left[1_{\{T_i \leqslant \tau\}} (Q\left(T > \tau, X < X_i\right) \mu_\tau(Q) - Q(T > \tau)\nu_\tau(Q)) \right. \\
&\quad \left. + 1_{\{T_i > \tau\}} (Q\left(T \leqslant \tau, X > X_i\right) \mu_\tau(Q) - \nu_\tau(Q) Q(T \leqslant \tau))\right].
\end{aligned}
$$

Influence function with censoring and multiple events

In this section, we derive the Influence function for the AUC with more than event and censoring (where the censoring is allowed to depend on the covariates). We now suppose that the censoring distribution depends on the baseline covariates but continue to assume that the censoring time is conditionally independent of the event time and event type given the covariates. To describe the situation with competing risks we introduce a new random variable $D \in \{1,2\}$ which indicates the cause (i.e., type of the event) observed at time T such that $D = 1$ means that the event of interest occurred, and $D = 2$ that a competing risk occurred. As in the survival setting we let Q denote the joint probability measure of the uncensored data, $(T, D, X) \sim Q$, and P the joint probability measure of the right censored data $Z = (\tilde{T}, \Delta, X) \sim P$ now with $\Delta = D 1_{\{T \leqslant C\}}$ taking values in the set $\{0, 1, 2\}$. We are interested in the following definition of the time-dependent discrimination measure for cause 1:

$$
\begin{aligned}
\mathrm{AUC}^1(\tau) &= Q(X_i > X_j \mid T_i \leqslant \tau, D_i = 1, (T_j > \tau \cup D_j = 2)) \\
&= \frac{Q(X_i > X_j, T_i \leqslant \tau, D_i = 1, (T_j > \tau \cup D_j = 2))}{Q(T_i \leqslant \tau, D_i = 1, (T_j > \tau \cup D_j = 2))}
\end{aligned}
$$

We define a functional $\nu_\tau^1(Q)$ to represent the numerator:

$$\begin{aligned}
\nu_\tau^1(Q) &= \iint 1_{\{x>x', t\leqslant\tau, d=1, (t'>\tau\cup d'=2)\}} dQ(t,d,x) dQ\left(t',d',x'\right) \\
&= \iint 1_{\{x>x', t\leqslant\tau, (t'>\tau\cup d'=2)\}} dQ(t,1,x) dQ\left(t',d',x'\right) \psi_\tau^1(P) \\
&= \iint 1_{\{x>x', t\leqslant\tau, t'>\tau\}} + 1_{\{x>x', t\leqslant\tau, t'\leqslant\tau, d'=2)\}} dQ(t,1,x) dQ\left(t',d',x'\right) \\
&= \iint 1_{\{x>x', t\leqslant\tau, t'>\tau\}} dQ(t,1,x) dQ\left(t',d',x'\right) \\
&\quad + \iint 1_{\{x>x', t\leqslant\tau, t'\leqslant\tau\}} dQ(t,1,x) dQ\left(t',2,x'\right) \\
&= \sum_{\delta'=0,1,2} \iint 1_{\{x>x', t\leqslant\tau, t'>\tau\}} \frac{dP(t,1,x)}{G(t-|x)} \frac{dP\left(t',\delta',x'\right)}{G\left(\tau|x'\right)} \\
&\quad + \iint 1_{\{x>x', t\leqslant\tau, t'\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} \frac{dP(t',2,x')}{G(t'-|x')}
\end{aligned}$$

We also need the influence function for the censoring as part of our calculations:

$$\begin{aligned}
\kappa_{t,z}(P_\varepsilon^i) &= \exp(-\int_0^t \frac{dP_\varepsilon(s,0|z)}{P_\varepsilon(\tilde{T}>s|Z=z)}) \\
\partial_\varepsilon P_\varepsilon(\tilde{T}>s, Z=z) &= \mathbb{1}_{\{\tilde{T}_i>s, Z_i=z\}} - P(\tilde{T}>s, Z=z) \\
\partial_\varepsilon P_\varepsilon(\tilde{T}\leqslant s, \Delta=0, Z=z) &= \mathbb{1}_{\{\tilde{T}_i\leqslant s, \Delta_i=0, Z_i=z\}} - P(\tilde{T}\leqslant s, \Delta=0, Z=z)
\end{aligned}$$

We have when Z is discrete,

$$\begin{aligned}
&\partial_\varepsilon \left[\int_0^t \frac{dP_\varepsilon(s,0|z)}{P_\varepsilon(\tilde{T}>s|Z=z)}\right] \\
&= \int_0^t \frac{\partial_\varepsilon dP_\varepsilon(s,0,z) P(\tilde{T}_i>s, Z_i=z)}{P(\tilde{T}>s, Z_i=z)^2} - \frac{dP(s,0,z)\partial_\varepsilon P_\varepsilon(\tilde{T}>s, Z_i=z)}{P(\tilde{T}>s, Z_i=z)^2} \\
&= \int_0^t \frac{(d\mathbb{1}_{\{\tilde{T}_i\leqslant s, \Delta_i=0, Z_i=z\}} - dP(s,0,z))P(\tilde{T}>s, Z_i=z)}{P(\tilde{T}>s, Z=z)^2} \\
&\quad - \frac{dP(s,0,z)(\mathbb{1}_{\{\tilde{T}_i>s, Z_i=z\}} - P(\tilde{T}>s, Z=z))}{P(\tilde{T}>s, Z=z)^2} \\
&= \int_0^t \frac{d\mathbb{1}_{\{\tilde{T}_i\leqslant s, \Delta_i=0, Z_i=z\}} P(\tilde{T}>s, Z=z) - dP(s,0,z)\mathbb{1}_{\{\tilde{T}_i>s, Z_i=z\}}}{P(\tilde{T}>s, Z=z)^2} \\
&= \frac{\mathbb{1}_{\{\tilde{T}_i\leqslant t, \Delta_i=0, Z_i=z\}}}{P(\tilde{T}>\tilde{T}_i, Z=Z_i)} - \int_0^t \frac{\mathbb{1}_{\{\tilde{T}_i>s, Z_i=z\}} dP(s,0,z)}{P(\tilde{T}>s, Z=z)^2} \\
&= \frac{\mathbb{1}_{\{\tilde{T}_i\leqslant t, \Delta_i=0, Z_i=z\}}}{G(\tilde{T}_i|Z_i)S(\tilde{T}_i|Z_i)P(Z=Z_i)} - \int_0^{\tilde{T}_i\wedge t} \frac{\mathbb{1}_{\{Z_i=z\}} dP(s,0,z)}{G(s|Z_i)^2 S(s|Z_i)^2 P(Z=Z_i)^2} \\
&:= f_i(t,z)
\end{aligned}$$

When Z is continuous or mixed, the above has to be modified. Luckily, if we use the trick from before, we can do the calculation again which will mostly be the same, i.e.

$$f_i(t,z) = \frac{\mathbb{1}_{\{\tilde{T}_i\leqslant t, \Delta_i=0\}} \delta_{Z_i}(z)}{G(\tilde{T}_i|Z_i)S(\tilde{T}_i|Z_i)} - \int_0^{\tilde{T}_i\wedge t} \frac{\delta_{Z_i}(z) dP(s,0|z)}{G(s|Z_i)^2 S(s|Z_i)^2}$$

This just corresponds to setting $\delta_{Z_i}(z) = \frac{1_{\{Z_i=z\}}}{P(Z=Z_i)}$ in the previous calculation.

We have

$$\partial_\varepsilon \left[\frac{dP_\varepsilon\,(t,\delta,x)}{\kappa(P_\varepsilon)\,(s)} \right] = \frac{d(\delta_{\{\tilde{T}_i,\Delta_i,X_i\}})(t,\delta,x) + dP(t,\delta,x)\,[f_i(s,x)-1]}{G(s|x)}$$

So calculations for $\nu_\tau^1(Q)$ can now be done using the above. This yields that the influence function for the numerator can be written as:

$$\begin{aligned}
\mathrm{IF}_\nu(Z_i;\tau) = {} & \frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{G(\tilde{T}_i-|X_i)} \int 1_{\{X_i>x',t'>\tau\}} \frac{dP(t',x')}{G(\tau|x')} && (1)\\
& + \int 1_{\{t\leqslant\tau\}} \int 1_{\{x>x',t'>\tau\}} \frac{dP(t',x')}{G(\tau|x')} \frac{[f_i(t-,x)-1]\,dP(t,1,x)}{G(t-|x)} && (2)\\
& + \iint 1_{\{x'<x,t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} \frac{1_{\{t'>\tau\}}\,[f_i(\tau,x')-1]\,dP(t',x')}{G(\tau|x')} && (3)\\
& + \left(\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=2\}}}{G(\tilde{T}_i-|X_i)} + \frac{1_{\{\tilde{T}_i>\tau\}}}{G(\tau|X_i)} \right) \int 1_{\{X_i<x,t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} && (4)\\
& + \iint 1_{\{x'<x,t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} 1_{\{t'\leqslant\tau\}} (f_i(t'-,x')-1) \frac{dP(t',2,x')}{G(t'-|x')} && (5)\\
& + \frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{G(\tilde{T}_i-|X_i)} \int 1_{\{X_i>x',t'\leqslant\tau\}} \frac{dP\,(t',2,x')}{G(t'-|x')} && (6)\\
& + \iint 1_{\{x>x',t'\leqslant\tau\}} \frac{dP\,(t',2,x')}{G(t'-|x')} 1_{\{t\leqslant\tau\}} \frac{[f_i(t-,x)-1]\,dP(t,1,x)}{G(t-|x)} && (7)
\end{aligned}$$

Next, the denominator is

$$\begin{aligned}
\mathrm{IF}_\mu(Z_i;\tau) = {} & \frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{G(\tilde{T}_i-|X_i)} \int 1_{\{t'>\tau\}} \frac{dP(t',x')}{G(\tau|x')} && (8)\\
& + \int 1_{\{t'>\tau\}} \frac{dP(t',x')}{G(\tau|x')} \int 1_{\{t\leqslant\tau\}} \frac{[f_i(t-,x)-1]\,dP(t,1,x)}{G(t-|x)} && (9)\\
& + \int 1_{\{t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} \int \frac{1_{\{t'>\tau\}}\,[f_i(\tau,x')-1]\,dP(t',x')}{G(\tau|x')} && (10)\\
& + \left(\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=2\}}}{G(\tilde{T}_i-|X_i)} + \frac{1_{\{\tilde{T}_i>\tau\}}}{G(\tau|X_i)} \right) \int 1_{\{t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} && (11)\\
& + \int 1_{\{t\leqslant\tau\}} \frac{dP(t,1,x)}{G(t-|x)} \int 1_{\{t'\leqslant\tau\}} (f_i(t'-,x')-1) \frac{dP(t',2,x')}{G(t'-|x')} && (12)\\
& + \frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{G(\tilde{T}_i-|X_i)} \int 1_{\{t'\leqslant\tau\}} \frac{dP\,(t',2,x')}{G(t'-|x')} && (13)\\
& + \int 1_{\{t\leqslant\tau\}} (f_i(t-,x)-1) \frac{dP(t,1,x)}{G(t-|x)} \int 1_{\{t'\leqslant\tau\}} \frac{dP(t',2,x')}{G(t'-|x')} && (14)
\end{aligned}$$

The quotient rule now yields that the Gateaux derivative for the functional $\mathrm{AUC}_\tau(P) = \nu_\tau(Q)/\mu_\tau(Q)$ is given by

$$\mathrm{IF}_{\mathrm{AUC}}(Z_i,\tau) = \frac{\mathrm{IF}_\nu(Z_i;\tau)\mu_\tau(Q) - \nu_\tau(Q)\mathrm{IF}_\mu(Z_i;\tau)}{\mu_\tau(Q)^2}$$

Some simpler cases

- Whenever we have survival data instead of competing risk data, the terms corresponding to $\Delta = 2$ can be set to zero, i.e. term 5-7 and 12-14. Also, the first part of 4 and 11 is set to zero.
- Whenever we have censoring that does not depend on the covariates, terns 2-3 simplify a bit and are instead written as:

$$\frac{f_i(\tau)-2}{G(\tau)}\int 1_{\{t\leqslant\tau\}}P(X<x,\tilde{T}>\tau)\frac{1}{G(t-)}dP(t,1,x)$$
$$+\frac{1}{G(\tau)}\int 1_{\{t\leqslant\tau\}}P(X<x,\tilde{T}>\tau)\frac{f_i(t-)}{G(t-)}dP(t,1,x)$$

Similar corrections hold for the denominator. In the code, terms (1-7) are now termed (8-14) and (8-14) are now termed (15-21). Note that $P(X<X_i,\tilde{T}>\tau)=\int 1_{\{t>\tau,x<X_i\}}dP(t,x)$.

Estimation

We estimate as follows:

$$\mathrm{IF}_\nu(Z_i;\tau)=\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{\hat{G}(\tilde{T}_i-|X_i)}\int 1_{\{X_i>x',t'>\tau\}}\frac{d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')} \tag{15}$$
$$+\int 1_{\{t\leqslant\tau\}}\int 1_{\{x>x',t'>\tau\}}\frac{d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')}\frac{\left[\hat{f}_i(t-,x)-1\right]d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)} \tag{16}$$
$$+\iint 1_{\{x'<x,t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)}\frac{1_{\{t'>\tau\}}\left[\hat{f}_i(\tau,x')-1\right]d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')} \tag{17}$$
$$+\left(\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=2\}}}{\hat{G}(\tilde{T}_i-|X_i)}+\frac{1_{\{\tilde{T}_i>\tau\}}}{\hat{G}(\tau|X_i)}\right)\int 1_{\{X_i<x,t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)} \tag{18}$$
$$+\iint 1_{\{x'<x,t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)}1_{\{t'\leqslant\tau\}}(f_i(t'-,x')-1)\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')} \tag{19}$$
$$+\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{\hat{G}(\tilde{T}_i-|X_i)}\int 1_{\{X_i>x',t'\leqslant\tau\}}\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')} \tag{20}$$
$$+\iint 1_{\{x>x',t'\leqslant\tau\}}\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')}1_{\{t\leqslant\tau\}}\frac{\left[\hat{f}_i(t-,x)-1\right]d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)} \tag{21}$$

Next, the denominator is

$$\mathrm{IF}_\mu(Z_i;\tau)=\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{\hat{G}(\tilde{T}_i-|X_i)}\int 1_{\{t'>\tau\}}\frac{d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')} \tag{22}$$
$$+\int 1_{\{t'>\tau\}}\frac{d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')}\int 1_{\{t\leqslant\tau\}}\frac{\left[\hat{f}_i(t-,x)-1\right]d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)} \tag{23}$$
$$+\int 1_{\{t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)}\int\frac{1_{\{t'>\tau\}}\left[\hat{f}_i(\tau,x')-1\right]d\mathbb{P}_n(t',x')}{\hat{G}(\tau|x')} \tag{24}$$
$$+\left(\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=2\}}}{\hat{G}(\tilde{T}_i-|X_i)}+\frac{1_{\{\tilde{T}_i>\tau\}}}{\hat{G}(\tau|X_i)}\right)\int 1_{\{t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)} \tag{25}$$
$$+\int 1_{\{t\leqslant\tau\}}\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)}\int 1_{\{t'\leqslant\tau\}}(\hat{f}_i(t'-,x')-1)\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')} \tag{26}$$
$$+\frac{1_{\{\tilde{T}_i\leqslant\tau,\Delta_i=1\}}}{\hat{G}(\tilde{T}_i-|X_i)}\int 1_{\{t'\leqslant\tau\}}\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')} \tag{27}$$
$$+\int 1_{\{t\leqslant\tau\}}(\hat{f}_i(t-,x)-1)\frac{d\mathbb{P}_n(t,1,x)}{\hat{G}(t-|x)}\int 1_{\{t'\leqslant\tau\}}\frac{d\mathbb{P}_n(t',2,x')}{\hat{G}(t'-|x')} \tag{28}$$

We suggest for estimating this, that the inner part of the integrals are estimated first, because (1) some of the inner integrals appear more than once and (2) for efficiency reasons. Note that, when we have that the censoring is independent of the covariates, we might do some other optimizations as well: We assume that the survival times are sorted and possibly with ties such that $\tilde{T}_1 < \ldots < \tilde{T}_i = \ldots = \tilde{T}_{i+k} < \tilde{T}_{i+k+1} < \ldots < \tilde{T}_n$. We use the following algorithm to preserve memory and the number of iterations for say $\mu^{(i)} = \int 1_{\{t \leqslant \tau\}} \frac{f_i(t-)}{G(t-)} dP(t,1,x)$. The idea is to split the sum into two terms:

$$\frac{1}{n} \sum_{j=1}^{n} \frac{\hat{f}_i(\tilde{T}_j-) 1_{\{\tilde{T}_j \leqslant \tau, \Delta_j = 1\}}}{\hat{G}(\tilde{T}_j-)} =$$
$$\frac{1}{n} \left(\sum_{j=2}^{i+k} \frac{g(j) 1_{\{\tilde{T}_j \leqslant \tau, \Delta_j = 1\}}}{\hat{G}(\tilde{T}_j-)} + h(i) \sum_{j=i+k+1}^{n} \frac{1_{\{\tilde{T}_j \leqslant \tau, \Delta_j = 1\}}}{\hat{G}(\tilde{T}_j-)} \right)$$

since $\hat{f}_i(\tilde{T}_j-)$ only depends on i for $i+k > j$ and only depends on j for $i+k \leqslant j$, so these values are calculated a priori. Also the first term will always be zero, since we are looking at the value of the integral before any observed event (hence the sum starts at $j=2$). One can check in the estimation of the Influence Curve for the censoring, which does not depend on the covariates that we need to calculate $2n$ values (i.e. n values for $g(i)$ and n for $h(j)$). This is how we can avoid memory issues. The algorithm is:

```
t := 1
μ̂_2 := Σ_{j=1}^{n} 1{T̃_j ≤ τ, Δ_j = 1} / Ĝ(T̃_j−)
while T̃_1 = T̃_t and t ≤ n do
    if T̃_t ≤ τ and Δ_t = 1 then
        μ̂_2 = μ̂_2 − 1 / G(T̃_t−)
    end
    t = t + 1
end
tieEnd := t − 1
μ̂_1 := 0
for i = 1 to n do
    μ̂^(i) = (1/n) (μ̂_1 + h(i) μ̂_2)
    if tieEnd ≤ i then
        t = i + 1
        while T̃_1 = T̃_t and t ≤ n do
            if T̃_t ≤ τ and Δ_t = 1 then
                μ̂_2 = μ̂_2 − 1 / G(T̃_t−)
                μ̂_1 = μ̂_1 + g(t) 1{T̃_t ≤ τ, Δ_t = 1} / Ĝ(T̃_t−)
            end
            Let t = t + 1
        end
    end
    Let tieEnd = t − 1
end
return μ̂^(i) for each i = 1, ..., n
```

The idea is that but we keep on adding and subtracting the terms with tied values in the event times. Then we do not need to calculate a sum for each i.

On the other hand, efficient calculation of some of the estimates can be done for example by sorting the risks X in (1) for the integral there.