Influence function calculation for Brier score for event time data

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IF Calculation

To describe the situation with competing risks (and also survival) we introduce a random variable $D \in \{1, 2\}$ which indicates the cause (i.e., type of the event) observed at time T such that $D = 1$ means that the event of interest occurred, and $D = 2$ that a competing risk occurred. As in the survival setting we let Q denote the joint probability measure of the uncensored data, $(T, D, X) \sim Q$, and P the joint probability measure of the right censored data $Z = (\tilde{T}, \Delta, X) \sim P$ now with $\Delta = D1_{\{T \leqslant C\}}$ taking values in the set $\{0, 1, 2\}$. We are interested in the following definition of the time-dependent discrimination measure for cause 1. We can easily calculate the influence function for the Brier score, which can be written as:

$$
\begin{aligned}
&\int \left\{1_{\{t \leqslant \tau\}} - R(\tau \mid x)\right\}^2 Q(dt, dx) \\
&= \int 1_{\{t \leqslant \tau\}} - 21_{\{t \leqslant \tau\}} R(\tau \mid x) + R(\tau \mid x)^2 Q(dt, dx) \\
&= \int 1_{\{t \leqslant \tau\}} (1 - 2R(\tau \mid x)) Q(dt, dx) + \int R(\tau \mid x)^2 Q(dx) \\
&= \int \frac{1_{\{t \leqslant \tau\}}}{G(t- | x)} (1 - 2R(\tau \mid x)) P(dt, 1, dx) + \int R\left(\tau | x\right)^2 P(dx)
\end{aligned}
$$

We find

$$
\begin{aligned}
IC_{\text{Brier}}(\tilde{T}_i, \Delta_i, X_i; \tau) &= \partial_\varepsilon \int \frac{1_{\{t \leqslant \tau\}}}{G(t- | x)} (1 - 2R(\tau \mid x)) P_\varepsilon(dt, 1, dx) + \partial_\varepsilon \int R\left(\tau | x\right)^2 P_\varepsilon(dx) \\
&= \int 1_{\{t \leqslant \tau\}} (1 - 2R(\tau \mid x)) \frac{d(\delta_{\{\tilde{T}_i, \Delta_i, X_i\}})(t, 1, x) + dP(t, 1, x)\left[f_i(t-, x) - 1\right]}{G(t- | x)} \\
&+ \int R\left(\tau | x\right)^2 (\delta_{\{\tilde{T}_i, \Delta_i, X_i\}} - P) \\
&= 1_{\{\tilde{T}_i \leqslant \tau, \Delta_i = 1\}} (1 - 2R(\tau \mid X_i)) \frac{1}{G(\tilde{T}_i - | X_i)} \\
&+ \int 1_{\{t \leqslant \tau\}} (1 - 2R(\tau \mid x)) f_i(t-, x) \frac{dP(t, 1, x)}{G(t- | x)} + R\left(\tau | X_i\right)^2 \\
&- \int R\left(\tau | x\right)^2 dP(x) - \int 1_{\{t \leqslant \tau\}} (1 - 2R(\tau \mid x)) \frac{dP(t, 1, x)}{G(t- | x)}
\end{aligned}
$$

The last term that is subtracted is the Brier score. Then using that

$$
f_i(t, x) = \frac{\mathbb{1}_{\{\tilde{T}_i \leqslant t, \Delta_i = 0\}} \delta_{X_i}(x)}{G(\tilde{T}_i | X_i) S(\tilde{T}_i | X_i)} - \int_0^{\tilde{T}_i \wedge t} \frac{\delta_{X_i}(x) dP(s, 0 | x)}{G(s | X_i)^2 S(s | X_i)^2}
$$

we see that (plugging in $f(t,x)$ instead of $f(t-,x)$!)

$$\int_X \int_0^\tau (1-2R(\tau|x)) f_i(t-,x) \frac{P(dt,1,x)}{G(t-|x)} = \int_X \int_0^\tau (1-2R(\tau|x)) \left[\frac{\mathbb{1}_{\{\tilde{T}_i \leqslant t, \Delta_i = 0\}} \delta_{X_i}(x)}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} - \int_0^{\tilde{T}_i \wedge t} \frac{\delta_{X_i}(x) P(ds,0|x)}{G(s|X_i)^2 S(s|X_i)^2} \right] \frac{P(dt,1,x)}{G(t-|x)}$$

$$= \int_0^\tau (1-2R(\tau|X_i)) \left[\frac{\mathbb{1}_{\{\tilde{T}_i \leqslant t, \Delta_i = 0\}}}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} - \int_0^{\tilde{T}_i \wedge t} \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \right] \frac{P(dt,1|X_i)}{G(t-|X_i)}$$

$$= (i) - (ii)$$

where

$$\begin{aligned} (i) &= \int_0^\tau (1-2R(\tau|X_i)) \frac{\mathbb{1}_{\{\tilde{T}_i \leqslant t, \Delta_i = 0\}}}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} \frac{P(dt,1|X_i)}{G(t-|X_i)} \\ &= (1-2R(\tau|X_i)) \frac{\mathbb{1}_{\{\tilde{T}_i \leqslant \tau, \Delta_i = 0\}}}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} \int_{\tilde{T}_i}^\tau \frac{P(dt,1|X_i)}{G(t-|X_i)} \\ &= (1-2R(\tau|X_i)) \frac{\mathbb{1}_{\{\tilde{T}_i \leqslant \tau, \Delta_i = 0\}}}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} (F_1(\tau|X_i) - F_1(\tilde{T}_i|X_i)) \end{aligned}$$

Similarly, we have

$$(ii) = (1-2R(\tau|X_i)) \int_0^\tau \int_0^{\tilde{T}_i \wedge t} \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \frac{P(dt,1|X_i)}{G(t-|X_i)}$$

If $\tilde{T}_i > \tau$, then this can be written as

$$\begin{aligned} (ii) &= (1-2R(\tau|X_i)) \int_0^\tau \int_s^\tau \frac{P(dt,1|X_i)}{G(t-|X_i)} \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \\ &= (1-2R(\tau|X_i)) \int_0^\tau (F_1(\tau|X_i) - F_1(s|X_i)) \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \end{aligned}$$

On the other hand, if $\tilde{T}_i \leqslant \tau$, then

$$\begin{aligned} (ii) &= (1-2R(\tau|X_i)) \int_0^{\tilde{T}_i} \int_s^\tau \frac{P(dt,1|X_i)}{G(t-|X_i)} \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \\ &= (1-2R(\tau|X_i)) \int_0^{\tilde{T}_i} (F_1(\tau|X_i) - F_1(s|X_i)) \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2} \end{aligned}$$

Thus

$$(ii) = (1-2R(\tau|X_i)) \int_0^{\tilde{T}_i \wedge \tau} (F_1(\tau|X_i) - F_1(s|X_i)) \frac{P(ds,0|X_i)}{G(s|X_i)^2 S(s|X_i)^2}$$

Hence,

$$(i)-(ii) = (1-2R(\tau|X_i)) \left(\frac{I(\tilde{T}_i \leqslant \tau, \Delta_i = 0)}{G(\tilde{T}_i|X_i) S(\tilde{T}_i|X_i)} (F_1(\tau|X_i) - F_1(\tilde{T}_i|X_i)) - \int_0^{\tilde{T}_i \wedge \tau} \frac{(F_1(\tau|X_i) - F_1(s|X_i))}{G(s|X_i)^2 S(s|X_i)^2} P(ds,0|X_i) \right)$$

Not expanding the paranthesis

We can also derive the Brier score as:

$$\sum_{\delta=0,1} \int \left\{ 1_{\{t \leqslant \tau\}} - R(\tau \mid x) \right\}^2 W_\tau(z; G) P(dt, \delta, dx)$$

with $W_\tau(z;G) = \frac{\delta I(t\leqslant\tau)}{G(t-|x)} + \frac{I(t>\tau)}{G(\tau|x)}$. Taking the Gateaux derivative yields,

$$\begin{aligned}
&I(\tilde{T}_i \leqslant \tau, \Delta_i = 1)\frac{(1-R(\tau \mid X_i))^2}{G(\tilde{T}_i - |X_i)} + I(\tilde{T}_i > \tau)\frac{R(\tau \mid X_i)^2}{G(\tau|X_i)} \\
&- Brier \\
&+ \int I(t \leqslant \tau)(1-R(\tau|x))^2 \frac{f_i(t-,x)}{G(t-|x)} P(dt,1,dx) \\
&+ \int I(t > \tau) R(\tau|x)^2 \frac{f_i(\tau,x)}{G(\tau|x)} P(dt,dx)
\end{aligned}$$

Whenever the censoring does not depend on covariates, the two last integrals are easy to estimate. However, when the censoring does depend on the covariates, we can use the tricks mentioned in IFAUC.pdf to get that,

$$\begin{aligned}
&\int I(t>\tau)R(\tau|x)^2\frac{f_i(\tau,x)}{G(\tau|x)}P(dt,dx) + \int I(t\leqslant\tau)(1-R(\tau|x))^2\frac{f_i(t-,x)}{G(t-|x)}P(dt,1,dx) \\
&= R(\tau|X_i)^2 S(\tau|X_i)\left(\frac{I(\tilde{T}_i\leqslant\tau,\Delta_i=0)}{G(\tilde{T}_i|X_i)S(\tilde{T}_i|X_i)} - \int_0^{\tilde{T}_i\wedge\tau}\frac{1}{G(s|X_i)^2S(s|X_i)^2}P(ds,0|X_i)\right) \\
&+ (1-R(\tau|X_i))^2\left(\frac{I(\tilde{T}_i\leqslant\tau,\Delta_i=0)}{G(\tilde{T}_i|X_i)S(\tilde{T}_i|X_i)}(F_1(\tau|X_i)-F_1(\tilde{T}_i|X_i)) - \int_0^{\tilde{T}_i\wedge\tau}\frac{(F_1(\tau|X_i)-F_1(s|X_i))}{G(s|X_i)^2S(s|X_i)^2}P(ds,0|X_i)\right) \\
&= R(\tau|X_i)^2 S(\tau|X_i)\left(\frac{I(\tilde{T}_i\leqslant\tau,\Delta_i=0)}{G(\tilde{T}_i|X_i)S(\tilde{T}_i|X_i)} - \int_0^{\tilde{T}_i\wedge\tau}\frac{1}{G(s|X_i)S(s|X_i)}\Lambda_C(ds|X_i)\right) \\
&+ (1-R(\tau|X_i))^2\left(\frac{I(\tilde{T}_i\leqslant\tau,\Delta_i=0)}{G(\tilde{T}_i|X_i)S(\tilde{T}_i|X_i)}(F_1(\tau|X_i)-F_1(\tilde{T}_i|X_i)) - \int_0^{\tilde{T}_i\wedge\tau}\frac{(F_1(\tau|X_i)-F_1(s|X_i))}{G(s|X_i)S(s|X_i)}\Lambda_C(ds|X_i)\right)
\end{aligned}$$