Written Homework 8 (Chapter 9)

Name(s): Elliot Marshall

9.1.5 Let A and B be sets. Prove that

(a) If A is a finite set, then $A \cap B$ is a finite set.

$A \cap B$ contains all the elements of B that are also in A. Thus, $A \cap B$ cannot contain any elements that are not in A; therefore, $|A \cap B| \leq |A|$. Which means that since A is a finite set, $A \cap B$ must also be a finite set

(b) If $A \cup B$ is a finite set, then A and B are finite sets.

$A \cup B$ contains all the elements in either A or B. Since $A \cup B$ is finite, $|A \cup B|$ must also be finite. Since $|A \cup B| > |A|$, $|A \cup B| > |B|$, and $|A \cup B|$ is finite, we can conclude that A and B are both finite.

(c) If $A \cap B$ is an infinite set, then A is an infinite set.

Since $A \cap B$ is an infinite set, and it contains the elements in both A and B, we can conclude that A (and B) is an infinite set as it contains an infinite number of elemens.

(d) If A is an infinite set or B is an infinite set, then $A \cup B$ is an infinite set.

Since $A \cup B$ is made up of the elements in either A or B, if A is an infinite set, then it contributes an infinite number of elements to $A \cup B$. Likewise if B is an infinite set, it too would contribute an infinite number of elements to $A \cup B$. In either case (or a combination of both), $A \cup B$ is an infinite set if at least one of its subsets is infinite.

9.1.7 Prove the following proposition:

(b) If A, B, C, and D are sets with $A \approx B$ and $C \approx D$, and if A and C are disjoint and B and D are disjoint, then $A \cup C \approx B \cup D$.

Since A and C are disjoint, $A \cap C = \emptyset$. Likewise, since B and D are disjoint, $B \cap D = \emptyset$. Therefore since both $A \cap C$ and $B \cap D$ are equal to $\emptyset$, $A \cup C \approx B \cup D \approx \emptyset$

9.2.2. Prove that each of the following sets is countably infinite.

(a) The set F^+ of all natural numbers that are multiples of 5

Because the set F^+ is equivalent to multiples of 5, it can be rewritten as $F^+ = 5n$, such that $n \,\epsilon\, \mathbb{N}$. Therefore, since the natural numbers ($\mathbb{N}$) are countably infinite, F^+ must also be countable infinite.

(b) The set F of all integers that are multiples of 5

Because the set F^+ is equivalent to multiples of 5, it can be rewritten as $F^+ = 5n$, such that $n \,\epsilon\, \mathbb{Z}$. Therefore since the integers ($\mathbb{Z}$) are countably infinite, F^+ must also be countable infinite.

(c) $\left\{ \dfrac{1}{2^k} \,\middle|\, k \in \mathbb{N} \right\}$

Because the set is based on the set of natural numbers and the set of natural numbers is countably infinite, the set is also countably infinite.

(d) $\{n \in \mathbb{Z} \mid n \geq -10\}$

Similar to part C, since the set depends on the integers greater than or equal to -10 (also known as $\{\mathbb{N}\} \cup \{-10, -9, ..., -2, -1\}$) and the set of natural numbers plus the negetive numbers greater than -11 is countably infinite, the set listed above is also countably infinite.

($\star$) Optional Challenge: Replace one of the problems above with 9.2.9.

9.2.9. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as follows: For each $(m, n) \in \mathbb{N} \times \mathbb{N}$,

$$f(m, n) = 2^{m-1}(2n - 1).$$

1. Prove that f is an injection. (See the textbook for hints.)
2. Prove that f is a surjection. (See the textbook for hints.)
3. Prove that $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ and hence that $\text{card}(\mathbb{N} \times \mathbb{N}) = \aleph_0$.