FFR110, Computational biology 1
Problem set 3

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Problem 1

(a) The probability that there aren't any SNPs in a sample of size n can be calculated as,
$\mathrm{P}(\mathrm{S}_n = 0) =< e^{-\mu T_c} >=< e^{-\sum_{i=2}^{n} iT_i} >=< \prod_{i=2}^{n} e^{-\mu i T_i} >= \prod_{i=2}^{n} \int_0^\infty dT_i \lambda_i e^{-\lambda_i T_i} e^{-\mu i T_i}$
$= \prod_{i=2}^{n} \int_0^\infty dT_i \frac{\binom{i}{2}}{N} e^{-T_i(\frac{\binom{i}{2}}{N} + \mu_i)} = \prod_{i=2}^{n} \int_0^\infty d\tilde{T}_i \binom{i}{2} e^{-\tilde{T}_i(\binom{i}{2} + N\mu_i)} = \prod_{i}^{n} \int_0^\infty d\tilde{T}_i \binom{i}{2} e^{-\tilde{T}(\binom{i}{2} + \frac{\theta_i}{2})}$
Here $\tilde{T} = \frac{T_i}{N}$ and $\theta = 2\mu N$. When inserting the integral in Mathematica, one gets the following expression:

$$\prod_{i=2}^{n} \frac{i-1}{-1+\theta+i}$$

, which can be simplified to

$$\frac{(n-1)!}{(1+\theta)(2+\theta)\cdots(n-1+\theta)}$$

(b) The distribution of the SNPs in a sample of size n=2 can be calculated as,

$$P(S_2 = j) =< \frac{(\mu 2 T_2)^j}{j!} e^{-\mu 2 T_2} >= \int_0^\infty dT_2 \frac{e^{-\frac{T_2}{N}}}{N} \frac{(\mu 2 T_2)^j}{j!} e^{-2T_2\mu} =$$

$$\int_0^\infty d\tilde{T}_2 e^{-\tilde{T}_i(1+\theta)} \frac{(\tilde{T_i\theta})^j}{j!} = \frac{\theta^j}{j!} j!(1+\theta)^{-(j+1)}$$

, resulting in the following expression,

$$\frac{1}{1+\theta}(\frac{\theta}{1+})^j.$$

Problem 2

(a) When neglecting diffusion the following equations are obtained,

$$\frac{du}{dt} = a - (b+1)u + u^2 v$$

and

$$\frac{dv}{dt} = bu - u^2 v$$

with the fixed points: $(a, \frac{b}{a})$ and stability matrix $J(u^*, v^*) = \begin{pmatrix} -1+b & a^2 \\ -b & -a^2. \end{pmatrix}$

The stability of the fixed points are analysed by the eigenvalues of the stability matrix: $\lambda_{1,2} = (-1 - a^2 \pm \sqrt{-4a^2 + (1+a^2-b)^2} + b)$. One can thereby see that the fixed point is stable when $b < 1 + a^2$ (since $\tau(J) < 0$ then) and unstable else.

(b) By inserting the values for the constants, given in the problem, we get the following system,

$$\begin{cases} \frac{\partial u}{\partial t} = 3 - 9u + u^2 v + \nabla^2 u \\ \frac{\partial v}{\partial t} = 8u - 64v + D_v \nabla^2 v. \end{cases}$$

Here $d = \frac{D_v}{D_u}$, where $D_u = 1$ and $D_v = d > 1$.

The conditions for Turing instability are given by,

$$dJ_{11} + J_{22} > 0,$$

and

$$\frac{(dJ_{11} + J_{22})^2}{4d} > \det \mathbb{J}.$$

Which gives the following stability matrix,

$$J(u, v) = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}, J(u^*, v^*) = \begin{bmatrix} 7 & 9 \\ -8 & -9 \end{bmatrix}.$$

By using the conditions for Turing instability, we get, $D_v > \frac{7}{9}$ and $\frac{(D_v 7 - 9)^2}{4D_v} > 9$.

From these equations one can solve for D_v, where the resulting equation becomes $D_v = \frac{162}{98} \pm \sqrt{(\frac{162}{98})^2 - \frac{81}{49}}$. The resulting roots are $D_{v+} = 2,69$ and $D_{v-} = 0,614$ and since $D_v > 1$, the only allowed solution is $D_v = 2,69$.

Mathematica code

Exercise 1

```
Integrate[
 Binomial[i, 2]*Exp[-(Binomial[i, 2] + theta*i/2)*t], {t, 0, inf}]
```

Exercise 2

```
diffusionInstabilityModel = {u'[t] == a - (b + 1)*u[t] + u[t]^2*v[t],
   v'[t] == b*u[t] - u[t]^2*v[t]};
diffusionInstabilityModelEqualZero =
  diffusionInstabilityModel /. {u'[t] -> 0, v'[t] -> 0};
solution = Solve[diffusionInstabilityModelEqualZero, {u[t], v[t]}]
matrixDiffusionInstabilityModel = \
{diffusionInstabilityModel[[1]][[2]],
   diffusionInstabilityModel[[2]][[2]]};
stabilityMatrix = {{D[diffusionInstabilityModel[[1]][[2]], u[t]],
    D[diffusionInstabilityModel[[2]][[2]], u[t]]}, {D[
     diffusionInstabilityModel[[1]][[2]], v[t]],
    D[diffusionInstabilityModel[[2]][[2]], v[t]]}};
eigenValues = Simplify[Eigenvalues[stabilityMatrix] /. solution[[2]]]

u = a
v = b/a
{1/2 (-1 - b - u^2 + 2 u* v -
    Sqrt[-4 u^2 + (1 + b + u^2 - 2 u* v)^2]),
 1/2 (-1 - b - u^2 + 2 u* v +
    Sqrt[-4 u^2 + (1 + b + u^2 - 2 u* v)^2])}
```