MVE155, Statistical Inference
Assignment 2

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Exercises

The code to the following solutions and data can be found in the source code files attached in the mail. Since the code is written in cells, you need to run through all the cells in order for the code to work.

a.

The source code for this subsection can be found in the attached file, by opening the script-file "ExerciseA".

(i) The histogram of the data of interarrival times can be seen in figure [1], which shows that the Gamma distribution is a plausible model. To plot the histogram, the built in function "histogram(X)", which creates a histogram plot of data set X and uses an automatic binning algorithm that returns bins with a uniform width.

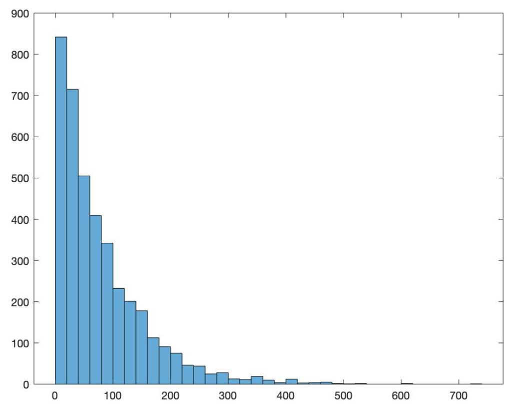

Figure 1: Histogram of the inter-arrival times

b.

The source code for this subsection can be found in the attached file, by opening the script-file "ExerciseB".

(i) The parameters α and λ of the gamma distribution was fitted by the method of moments. Here, the built in function "mean(A)" was used, which returns the mean of the elements of A as well as "std(A)", which returns the standard deviation of the elements of A.

The parameter estimations was given by:

$\tilde{\lambda} = \frac{\bar{x}}{\sigma^2}$ and $\tilde{\alpha} = \tilde{\lambda} \cdot \bar{x}$, where $\bar{x} = \frac{\sum x_i}{N}$ and $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{N}$.

The resulted values was $\tilde{\lambda} = 0.0127$ and $\tilde{\alpha} = 1.0121$.

(ii) The parameters α and λ of the gamma distribution was fitted by the maximum likelihood. The built in function "gamfit(data)", which returns the maximum likelihood estimates for the parameters of the gamma distribution given the data in data set. In the Matlab function, lambda in the gamma distribution is defined as 1/lambda, which is important to take into account.

The gamma density function is in the following form, $f(x) = \frac{1}{\Gamma(\alpha)}\lambda^\alpha x^{\alpha-1}e^{-\lambda x}, \quad x > 0$, with the gamma function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. The likelihood function then becomes

$L(\alpha, \lambda) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x_i^{\alpha-1} e^{-\lambda x_i} = \frac{\lambda^{n\alpha}}{\Gamma^n(\alpha)} (x_1 \cdots x_n)^{\alpha-1} e^{-\lambda(x_1+\ldots+x_n)} = \frac{\lambda^{n\alpha}}{\Gamma^n(\alpha)} t_2^{\alpha-1} e^{-\lambda t_1}$ where $(t_1, t_2) = (x_1 + \ldots + x_n, x_1 \cdots x_n)$.

The likelihood function is then maximized by taking the logarithm of the likelihood function and then taking the derivative of that function equal to zero, which gives the parameter estimates.

The resulted values was $\hat{\lambda} = 0.0128$ and $\hat{\alpha} = 1.0263$, which are both larger than the estimated parameters with the method of moments estimate.

(iii) From figure (2) one can see that both gamma densities with the estimated parameters from the method of moments and maximum likelihood, fit the data set well, and are similar to each other. The two gamma densities thereby looks reasonable. From figure (1) it is also clear that the data set has a gamma distribution and thereby the reasonable fit is expected. It is also possible to see that the parameter estimation by the maximum likelihood has a slightly better fit, since its follows the histogram a bit more accurate at the top. In this sub-exercise, the built in function "sort(A)" that sorts the elements of A in ascending order. Another built in function that was used was "gampdf(X,A,B)", which computes the gamma pdf at each of the values in X using the corresponding shape parameters in A and scale parameters in B.

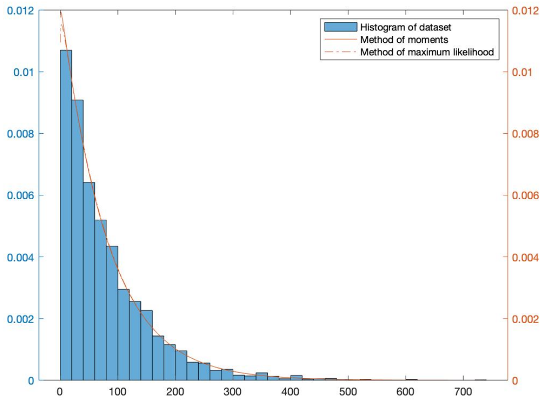

Figure 2: The two fitted gamma densities on top of the normalized histogram

c.

The source code for this subsection can be found in the attached file, by opening the script-file "ExerciseCandD".

(i) The standard errors of the parameters was fitted by method of moments by using bootstrap. The built in function "gamrnd(A,B)" was used, which generates random numbers from the gamma distribution with shape parameters in A and scale parameters in B. The following equations was used to estimate the standard errors of the parameters,

$s_{\tilde{\alpha}} = \sqrt{s_{\tilde{\alpha}}^2} = \sqrt{\frac{1}{B-1} \sum_{j=1}^{B} (\tilde{\alpha}_j - \bar{\alpha})^2}$ where $\bar{\alpha} = \frac{1}{B} \sum_{j=1}^{B} \tilde{\alpha}_j$, and $s_{\tilde{\lambda}} = \sqrt{s_{\tilde{\lambda}}^2} = \sqrt{\frac{1}{B-1} \sum_{j=1}^{B} \left(\tilde{\lambda}_j - \bar{\lambda}\right)^2}$ where $\bar{\lambda} = \frac{1}{B} \sum_{j=1}^{B} \tilde{\lambda}_j$.

The standard error for $s_{\tilde{\alpha}} = 0.0327$ and for $s_{\tilde{\lambda}} = 4.683 \cdot 10^{-4}$

(ii) The standard errors of the parameters was fitted by maximum likelihood by using bootstrap.

The same equations was used to estimate the standard errors of the parameters as in sub-exercise ci) but with parameter estimates $\hat{\alpha}$ and $\hat{\lambda}$

The standard error for $s_{\hat{\alpha}} = 0.0322$ and for $s_{\hat{\lambda}} = 4.467 \cdot 10^{-4}$

We can see that the result from the maximum likelihood gives a smaller standard error than for the method of moments. This result follows the theory since the Cramer-Rao inequality states that the maximum likelihood estimators are have minimal variance, i.e minimal standard error.

d.

The source code for this subsection can be found in the attached file, by opening the script-file "ExerciseCandD".

(i) Bootstrap was used to form a 95% approximate confidence intervals for the parameter estimates that was obtained by the method of moments. The confidence intervals for $\hat{\alpha}$ is $(2\tilde{\alpha} - c_2, 2\tilde{\alpha} - c_1)$, where c_1, c_2 are calculated by the built in Matlab function "prctile(Y,p)", which returns percentiles of the elements in an array Y for the percentages p in the interval [0,100]. The confidence interval for $\hat{\lambda}$ was calculated in the same way.

The resulting confidence interval is $\tilde{\alpha} = [0.9492, 1.0723]$ and $\tilde{\lambda} = [0.0118, 0.0136]$.

(ii) Bootstrap was used to form a 95% approximate confidence intervals for the parameter estimates that was obtained by maximum likelihood. The equation for calculating the confidence interval is the same as for the method of moments, i.e as described in sub-exercise di).

The resulting confidence interval is $\hat{\alpha} = [0.9660, 1.0889]$ and $\hat{\lambda} = [0.0120, 0.0137]$.

From the result one can see that the confidence interval for the maximum likelihood is more narrow then the confidence interval for the method of moments, which implies a more accurate approximation for the maximum likelihood. This result is inline with previous result that states that maximum likelihood provides a more accurate parameter estimation, i.e a lower standard error and a more narrow confidence interval.