Paradoxes and the structure of reasoning

David Ripley

Monash University
http://davewripley.rocks

Paradoxes

Think about testing a hypothesis.

A simplified picture:

1. Suppose the hypothesis is true.
2. Figure out what else would follow.
3. Check whether those other things are really true.
4. If not, the hypothesis was wrong.

2. Figure out what else would follow.

Here, some basic assumptions are helpful, like:

- things either are a certain way or they're not,
- if things are one way, they're not also any incompatible way,
- for something to be true is for things to be as it says they are,
- we can think about collections of things that are a certain way,

and so on.

If those basic assumptions aren't trustworthy,
the whole project falls apart.

“This sentence is not true.”

“This sentence is not true.”

If it’s true, then it’s not true.
So if it’s true, it’s both true and not true.
But that’s a contradiction!
So it’s not true after all.

"This sentence is not true."

If it's true, then it's not true.
So if it's true, it's both true and not true.
But that's a contradiction!
So it's not true after all.

But that's what it says! So it is true.
It's both true and not true.
We have a contradiction.

"If this sentence is true, then 2 + 2 = 5."

"If this sentence is true, then 2 + 2 = 5."

If it's true, then if it's true, then 2 + 2 = 5.
So if it's true, 2 + 2 does = 5.

"If this sentence is true, then $2 + 2 = 5$."

If it's true, then if it's true, then $2 + 2 = 5$.
So if it's true, 2 + 2 does = 5.

But that's what it says! So it is true.
So $2 + 2 = 5$.

Some collections don't contain themselves.
Others do.

Think of the collection of all limes,
and the collection of everything else.

Now, think of the collection of all collections that don't contain themselves.

(It contains, among other things, the collection of all limes.)

But does it contain itself?
If it does, it doesn't.
If it doesn't, it does.

It's a lot like the liar sentence;
it leads to contradiction in the same way.

The basic assumptions we use to investigate anything seem to be broken.

If it follows from the mere existence of a Curry sentence that $2 + 2 = 5$, what right do we have to say the Earth isn't flat?

We have a choice:

- Give up. Our reasoning really is broken. Maybe we can find another way to learn. Maybe not.

OR

- Push on. Find some trustworthy reasoning, even if it's not what we're used to.

We have a choice:

- Give up. Our reasoning really is broken. Maybe we can find another way to learn. Maybe not.

OR

- Push on. Find some trustworthy reasoning, even if it’s not what we’re used to.

$\neg$	Negation, not
$\&$	Conjunction, and
T	True
λ	the liar sentence
$\vdash$	Entailment, follows from

The liar sentence λ is $\neg T\lambda$.

We can derive a contradiction from $T\lambda$:

$$\dfrac{T\lambda \qquad \dfrac{\dfrac{T\lambda}{\lambda}}{\neg T\lambda}}{T\lambda \,\&\, \neg T\lambda}$$

So by reductio, we can conclude $\neg T\lambda$.

We can go on to derive a contradiction from $\neg T\lambda$ (which we have now proved!):

$$\dfrac{\dfrac{\dfrac{\neg T\lambda}{\lambda}}{T\lambda} \qquad \neg T\lambda}{T\lambda \,\&\, \neg T\lambda}$$

So by explosion, everything follows.

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \,\&\, B}$$

$$\begin{array}{c}[A] \\ \vdots \\ \dfrac{B \,\&\, \neg B}{\neg A}\end{array}$$

$$\frac{A \,\&\, \neg A}{B}$$

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \,\&\, B}$$

$$\begin{array}{c}[A]\\ \vdots \\ \dfrac{B \,\&\, \neg B}{\neg A}\end{array}$$

$$\frac{A \,\&\, \neg A}{B}$$

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \,\&\, B}$$

$$\frac{\begin{matrix}[A] \\ \vdots \\ B \,\&\, \neg B\end{matrix}}{\neg A}$$

$$\frac{A \,\&\, \neg A}{B}$$

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \,\&\, B}$$

$$\begin{array}{c}[A]\\ \vdots \\ \dfrac{B \,\&\, \neg B}{\neg A}\end{array}$$

$$\frac{A \,\&\, \neg A}{B}$$

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \mathbin{\&} B}$$

$$\begin{array}{c}[A] \\ \vdots \\ \dfrac{B \mathbin{\&} \neg B}{\neg A}\end{array}$$

$$\frac{A \mathbin{\&} \neg A}{B}$$

Here are the steps we've used:

$$\frac{\neg T\lambda}{\lambda} \qquad \frac{\lambda}{\neg T\lambda}$$

$$\frac{A}{TA} \qquad \frac{TA}{A}$$

$$\frac{A \qquad B}{A \,\&\, B}$$

$$\begin{array}{c}[A] \\ \vdots \\ \dfrac{B \,\&\, \neg B}{\neg A}\end{array}$$

$$\frac{A \,\&\, \neg A}{B}$$

Vocabulary?

One way to undermine this argument is to focus on negation.

Two main flavours:

- Maybe reductio is the problem?
- Maybe explosion is the problem?

That is, maybe it’s wrong to think that something leading to a contradiction must be false.

Or maybe it’s wrong to think that things can’t be two incompatible ways.

Or maybe it’s wrong to think that not being a certain way is incompatible with being that way.

Problem 1: What, then, makes negation negation?

Problem 2: Paradoxes that have nothing to do with negation.

Another way to undermine the argument is to focus on truth.

Two main flavours:

- Maybe $T\lambda$ doesn't really entail λ?
- Maybe λ doesn't really entail $T\lambda$?

That is, maybe being true is something different from telling it like it is.

Problem 1: We can just rebuild the paradoxes
with ‘tells it like it is’ instead of ‘is true’.

The thought must be that ‘telling it like it is’ is incoherent.

But this undermines inquiry even more directly than the paradoxes originally did!

Problem 2: Paradoxes that have nothing to do with truth.

Pseudo-Scotus:

God exists

Therefore, this argument is invalid.

Suppose God exists, and suppose the argument is valid.
Then the argument must be invalid.
So it is both valid and invalid; contradiction.

Thus, if God exists the argument must be invalid.
But this is to prove its conclusion from its premise,
so it really is valid!

Since the argument is valid, if God exists, then it is invalid.
But this would be a contradiction.
So God does not exist.

Solutions that focus on particular vocabulary are limited to paradoxes where that vocabulary plays some role.

As long as we consider only the liar, solutions focusing on negation or truth can seem plausible.

But many very similar paradoxes
have no vocabulary in common at all!

Curry:	If this sentence is true, then $2 + 2 = 5$.
Russell:	The collection of all collections that do not contain themselves.
Pseudo-Scotus:	God exists. Therefore this argument is invalid.

No approach that focuses on particular vocabulary can get at the general phenomenon.

Structure

If it’s not particular vocabulary, then what is it?

There are two main families of response.

Both focus not on the steps in our proof,
but on its structure.

Return to the derivation of a contradiction from $T\lambda$:

$$\dfrac{T\lambda \qquad \dfrac{\dfrac{T\lambda}{\lambda}}{\neg T\lambda}}{T\lambda \mathbin{\&} \neg T\lambda}$$

Note that this uses $T\lambda$ twice.

$T\lambda$ on its own does not lead to contradiction.

So our reductio should conclude
not $\neg T\lambda$ outright,

but just that if $T\lambda$ holds, then $\neg T\lambda$ holds.
(And we already knew this!)

Keeping track of number in this way
means rejecting contraction:

$$\frac{A, A \vdash C}{A \vdash C}$$

Two As might suffice for C where one does not.

Blocking contraction is enough
to prevent the paradoxes from causing trouble.

The liar and Russell no longer lead to contradiction,
Curry no longer leads to $2 + 2 = 5$,
and so on.

Valid reasoning now not only uses its premises,
but uses them up.

A different approach focuses on the very last step.

At this step, we have proved $T\lambda \,\&\, \neg T\lambda$,
and then explosion gives us any B at all.

That is, we chain together our argument to $T\lambda \,\&\, \neg T\lambda$
with the argument from $T\lambda \,\&\, \neg T\lambda$ to B.

The move is a case of transitivity:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

It's surprising (and it takes a lot of work to show!)
but transitivity is completely dispensable most of the time.
It's a shortcut only; we can do the same things without it.

One instance where this is not the case, though,
is exactly the crucial link in the paradoxical argument.

A valid argument from A to C rules out asserting A while denying C.

If we've ruled out asserting A while denying B,	$A \vdash B$
and ruled out asserting B while denying C,	$B \vdash C$
have we ruled out asserting A while denying C?	$A \overset{?}{\vdash} C$

A valid argument from A to C rules out asserting A while denying C.

If we've ruled out asserting A while denying B,	$A \vdash B$
and ruled out asserting B while denying C,	$B \vdash C$
have we ruled out asserting A while denying C?	$A \overset{?}{\vdash} C$

No!

Asserting A while denying C can still be fine,

so long as we remain silent about B.

Blocking transitivity is a different way
to prevent the paradoxes from causing trouble.

We still get a contradiction: $\vdash T\lambda \,\&\, \neg T\lambda$.
And we can have $A \,\&\, \neg A \vdash B$.

But without transitivity, this is ok!

Consider the following procedure
for reasoning from some stock of premises:

Cumulative reasoning:

1. Start from some stock of premises.
2. Draw conclusions that follow from your stock.
3. Add those conclusions to your original stock,
   resulting in an expanded stock.
4. Go back to step 2 and repeat.

$$\frac{X \vdash A \qquad X, A \vdash C}{X \vdash C}$$

$$\frac{X \vdash A \qquad X, A \vdash C}{X \vdash C}$$

Noncontractive and nontransitive approaches agree here:
this is not ok!

For the noncontractivist, this uses X twice to get to C;
its conclusion needs to be $X, X \vdash C$.

For the nontransitivist, this chains things together on A;
it cannot be repaired, but much of the time is dispensable.

- Paradoxes seem to show that something is seriously wrong in our usual practices of inquiry.
- Solutions that focus on particular vocabulary like negation or truth miss how widespread paradoxes are.
- Solutions that focus on the structure of reasoning do better.
- They have the upshot that cumulative reasoning is not always trustworthy.