Core type theory I:
Implication and negation

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Introduction

Core logic (aka ‘intuitionistic relevant logic’)
is a system of logic devised and developed by Tennant
over the last 40 years.

In its usual presentation,
it's a first-order logic with vocabulary:
$\rightarrow, \neg, \wedge, \vee, \forall, \exists$

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it's a first-order logic with vocabulary:
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Here, I'll just consider the $\rightarrow, \neg$ propositional fragment.

Core logic has close connections to intuitionistic logic.
This will loom large here.

But its treatment of negation is distinctive.

Type theories are a family of formalisms with a wide range of uses.

Invented to block paradox, they have taken on a life of their own.

They have long been used in the semantics of natural languages and programming languages.

Their connection to proofs ties them to formalized, constructive, and proof-relevant mathematics as well.

The connection to proofs turns on the ability to see propositions as types, and vice versa.

Think of each type ϕ as the proposition 'ϕ is inhabited'. Each inhabitant of ϕ is a proof of it.

It is usual to suppose we can form function types:
given types ϕ and ψ there is a type $\phi \to \psi$ of functions from ϕ to ψ.

We may also suppose a empty type $\perp$:
a type that cannot be inhabited.

$$\Gamma, x : \phi \succ x : \phi$$

$$\frac{\Gamma, x : \phi \succ M : \psi}{\Gamma \succ (\lambda x.M) : \phi \to \psi} \qquad \frac{\Gamma \succ M : \phi \to \psi \qquad \Gamma \succ N : \phi}{\Gamma \succ MN : \psi}$$

$$\frac{\Gamma \succ M : \bot}{\Gamma \succ \texttt{explode}_{\phi}(M) : \phi}$$

Simultaneously a logic and a theory of functions.

(The logic is $\mathrm{Int}_{\to,\bot}$.)

The purpose of this talk is to develop core type theory:
the type theory that stands to core logic$_{\to,\neg}$
as a more usual type theory stands to Int$_{\to,\bot}$.

(Future work will extend this to include $\wedge, \vee$,
which present their own complications in core logic.)

Here’s the plan:

Core logic, its distinctive approach to negation, and ‘epistemic gain’

Core type theory and its properties

Tying the two together

One reason to pursue this project:

$$\frac{\Gamma \succ M : \bot}{\Gamma \succ \texttt{explode}_{\phi}(M) : \phi}$$

is pretty conspicuously ill-motivated.

Core logic

Formulas $\phi, \psi, \ldots$ are built from atoms with $\rightarrow, \neg$.

Sequents are $\Gamma \succ \mathfrak{C}$, where Γ is a set of formulas, and $\mathfrak{C}$ is either a formula or nothing.

Say $\mathfrak{D} \leq \mathfrak{C}$ iff either $\mathfrak{D}$ is empty or else $\mathfrak{D}$ is $\mathfrak{C}$.

Structural rule:

Id: $$\frac{}{\phi \succ \phi}$$

Important: no rules of weakening (either side) or cut!

$\to$ rules:

$$\to\text{L:}\quad \frac{\Gamma \succ \phi \qquad \Delta, \psi \succ \mathfrak{C}}{\Gamma, \Delta, \phi \to \psi \succ \mathfrak{C}} \qquad \to\text{R:}\quad \frac{\Gamma, \phi \succ \psi}{\Gamma \succ \phi \to \psi}$$

$$\to\text{R\_:}\quad \frac{\Gamma \succ \psi}{\Gamma \succ \phi \to \psi} \qquad \to\text{R!:}\quad \frac{\Gamma, \phi \succ}{\Gamma \succ \phi \to \psi}$$

Even without weakening, we can introduce the $\to$s that weakening would allow.

$\neg$ rules:

$$\neg\text{L:}\quad \frac{\Gamma \succ \phi}{\Gamma, \neg\phi \succ} \qquad\qquad \neg\text{R:}\quad \frac{\Gamma, \phi \succ}{\Gamma \succ \neg\phi}$$

These are almost usual (intuitionistic) negation rules.

But they don't absorb weakening like the $\to$ rules did.
So $\neg$ is where we can see core logic's nonmonotonicity.

Tennant defines core consequence $\vdash$ like so:
$\Gamma \vdash \mathfrak{C}$ iff there is some $\Gamma' \subseteq \Gamma$
such that $\Gamma' \succ \mathfrak{C}$ is derivable.

This builds in left weakening but not right,
and only at the end of a derivation.

There is no proposition $\perp$ in core logic (and so no E$\perp$Q rule).

Rather, there are proofs and refutations.

A proof of ϕ from Γ is a derivation of $\Gamma' \succ \phi$ for some $\Gamma' \subseteq \Gamma$.

A refutation of Γ is a derivation of $\Gamma' \succ$ for some $\Gamma' \subseteq \Gamma$.

Refuting Γ does not suffice for proving ϕ from Γ.

$\vdash$ is closely related to $\vdash_{Int}$:

Results (Tennant):

- $\Gamma \vdash$ iff $\Gamma \vdash_{\mathrm{Int}}$.
- If $\Gamma \nvdash$, then $\Gamma \vdash_{\mathrm{Int}} \phi$ iff $\Gamma \vdash \phi$.

But it is different: $\neg\phi, \phi \vdash_{\mathrm{Int}} \psi$,
but $\neg\phi, \phi \nvdash \psi$, although $\neg\phi, \phi \vdash$.

(These results hold for full first-order core logic too.)

Core logic does not admit cut.

$$\text{Cut:} \quad \frac{\Gamma \succ \phi \qquad \Delta, \phi \succ \mathfrak{C}}{\Gamma, \Delta \succ \mathfrak{C}}$$

For example, $\neg\phi \succ \phi \to \psi$ and $\phi \to \psi, \phi \succ \psi$ are both derivable, but $\neg\phi, \phi \succ \psi$ is not.

Instead, core logic has a property Tennant calls epistemic gain:

If $\Gamma \succ \phi$ and $\Delta, \phi \succ \mathfrak{C}$ are both derivable,
then there is a derivable sequent $\Sigma \succ \mathfrak{D}$ such that
$\Sigma \subseteq \Gamma \cup \Delta$ and $\mathfrak{D} \leq \mathfrak{C}$.

This is meant to supersede cut-admissibility;
not a partial replacement,
but a more precise statement of what really mattered all along.

In terms of consequence, this gives us:
if $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \mathfrak{C}$,
then $\Gamma, \Delta \vdash \mathfrak{D}$, for some $\mathfrak{D} \leq \mathfrak{C}$.

We can chain proofs together, so long as we're prepared
to maybe find a refutation of our combined premises instead.

One last fact about core logic:

adding cut to the system
gives precisely intuitionistic logic.

Seen this way, it is a distinctive proof system for good old Int,
with not all derivable sequents having cutfree derivations.

All derivable sequents do have cutfree derivable subsequents.
('Subsequent' here with $\subseteq$ on left and $\leq$ on right.)

Full disclosure:
I'm not yet 100% sure this is core logic.

Tennant's formulation involves more restrictive discharge policies, and his derivations are not closed under substitution.

My derivations are all substitution instances of core derivations.
I don't think this affects derivability,
even though it adds more derivations.

A term calculus

Types are our old formulas: built from atoms with $\rightarrow, \neg$.

$\phi \rightarrow \psi$ are function types: their canonical inhabitants give you a ψ if you give them a ϕ.

$\neg\phi$ are exception types: their canonical inhabitants cancel the current calculation when they encounter a ϕ.

Terms are either typed terms or refutation terms.

Handled Church-style; every term wears a hat:
either its (unique) type, or else $\perp$ if it's a refutation term.

(Hoping to extend this to Curry-style,
but the proofs were easier this way.)

Terms start from countably many variables $x^\phi, y^\phi, \ldots$ of each type ϕ.
There are no refutation variables.

There are six kinds of complex term:

$(M^{\phi\to\psi} N^\phi)^\psi$	$(\lambda x^\phi . M^\psi)^{\phi\to\psi}$
$(\lambda\_ . M^\psi)^{\phi\to\psi}$	$(\lambda ! x^\phi . M^\perp)^{\phi\to\psi}$
$(M^{\neg\phi} \text{⚯} N^\phi)^\perp$	$(\mathrm{R} x^\phi . M^\perp)^{\neg\phi}$

In the right column, x^ϕ must occur free in M, and becomes bound.
There is no vacuous binding.
α equivalents are identified.

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These six ways of compounding terms correspond directly to the six connective-introducing rules.

$$\to\mathrm{L}:\quad \frac{\Gamma \succ M^{\phi} \qquad \Delta, x^{\psi} \succ N^{\mathfrak{C}}}{\Gamma, \Delta, y^{\phi\to\psi} \succ (N^{\mathfrak{C}}[x^{\psi} \mapsto (yM)^{\psi}])^{\mathfrak{C}}} \qquad \to\mathrm{R}:\quad \frac{\Gamma, x^{\phi} \succ M^{\psi}}{\Gamma \succ (\lambda x^{\phi}.M^{\psi})^{\phi\to\psi}}$$

$$\to\mathrm{R\_}:\quad \frac{\Gamma \succ M^{\psi}}{\Gamma \succ (\lambda\_.M^{\psi})^{\phi\to\psi}} \qquad \to\mathrm{R!}:\quad \frac{\Gamma, x^{\phi} \succ M^{\perp}}{\Gamma \succ (\lambda ! x^{\phi}.M^{\perp})^{\phi\to\psi}}$$

$$\neg\mathrm{L}:\quad \frac{\Gamma \succ M^{\phi}}{\Gamma, y^{\neg\phi} \succ (y^{\neg\phi} \circ\!\!\!\circ M^{\phi})^{\perp}} \qquad \neg\mathrm{R}:\quad \frac{\Gamma, x^{\phi} \succ M^{\perp}}{\Gamma \succ (\mathrm{R}x^{\phi}.M^{\perp})^{\neg\phi}}$$

A redex is an occurrence of one of these four forms.

$$((\lambda x^{\phi}.M^{\psi})^{\phi\to\psi}\ N^{\phi})^{\psi}$$

$$((\lambda\_.M^{\psi})^{\phi\to\psi}\ N^{\phi})^{\psi}$$

$$((\mathsf{R}x^{\phi}.M^{\perp})^{\neg\phi} \multimapboth N^{\phi})^{\perp}$$

$$((\lambda!x^{\phi}.M^{\perp})^{\phi\to\psi}\ N^{\phi})^{\psi}$$

The last is an explosive redex; the rest are nonexplosive.

All redexes are applications or contradictions
with a complex left component.

So no redex is created in any core derivation:
all applications and contradictions have variables on the left.

Redexes are instead created by cuts;
the term calculus corresponds to core logic plus cut.

Recall that this is intuitionistic logic.

The core term calculus is a core-logic lens on intuitionistic logic.

Here's how cut works with terms:

$$\text{Cut:} \quad \frac{\Gamma \succ M^{\phi} \qquad \Delta, x^{\phi} \succ N^{\mathfrak{C}}}{\Gamma, \Delta \succ (N^{\mathfrak{C}}[M^{\phi} \mapsto x^{\phi}])^{\mathfrak{C}}}$$

Since cuts can put complex terms where variables used to be, they can create redexes.

$$\text{Cut: } \dfrac{\rightarrow\text{R\_: } \dfrac{\Gamma \succ M^{\psi}}{\Gamma \succ (\lambda\_.M)^{\phi\rightarrow\psi}} \qquad \rightarrow\text{L: } \dfrac{\Delta \succ N^{\phi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Delta, \Sigma, z^{\phi\rightarrow\psi} \succ O^{\mathfrak{C}}[y \mapsto zN]}}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda\_.M]}$$

$$\text{which is: } \dfrac{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda\_.M]}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto (\lambda\_.M)N]}$$

$$\text{Cut: } \dfrac{\text{→R\_: } \dfrac{\Gamma \succ M^{\psi}}{\Gamma \succ (\lambda\_.M)^{\phi\to\psi}} \qquad \text{→L: } \dfrac{\Delta \succ N^{\phi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Delta, \Sigma, z^{\phi\to\psi} \succ O^{\mathfrak{C}}[y \mapsto zN]}}{\text{which is: } \dfrac{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda\_.M]}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto (\lambda\_.M)N]}}$$

$$\text{Cut: } \dfrac{\Gamma \succ M^{\psi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Gamma, \Sigma \succ O^{\mathfrak{C}}[y \mapsto M]}$$

$$\text{Cut: } \dfrac{\to\text{R\_: } \dfrac{\Gamma \succ M^{\psi}}{\Gamma \succ (\lambda\_.M)^{\phi\to\psi}} \qquad \to\text{L: } \dfrac{\Delta \succ N^{\phi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Delta, \Sigma, z^{\phi\to\psi} \succ O^{\mathfrak{C}}[y \mapsto zN]}}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda\_.M]}$$

which is: $$\dfrac{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda\_.M]}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto (\lambda\_.M)N]}$$

$$\text{Cut: } \dfrac{\Gamma \succ M^{\psi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Gamma, \Sigma \succ O^{\mathfrak{C}}[y \mapsto M]}$$

Each redex has a reduct, as follows:

$$((\lambda x^{\phi}.M^{\psi})^{\phi\to\psi} N^{\phi})^{\psi} \qquad (M[x \mapsto N])^{\psi}$$

$$((\lambda\_.M^{\psi})^{\phi\to\psi} N^{\phi})^{\psi} \qquad M^{\psi}$$

$$((\mathsf{R}x^{\phi}.M^{\perp})^{\neg\phi} \multimap N^{\phi})^{\perp} \qquad (M[x \mapsto N])^{\perp}$$

$$((\lambda! x^{\phi}.M^{\perp})^{\phi\to\psi} N^{\phi})^{\psi} \qquad (M[x \mapsto N])^{\perp}$$

A step of gentle reduction takes a term $O[R]$
with indicated nonexplosive redex R,
and yields $O[R']$, with R' the reduct of R.

A step of reduction is either a step of gentle reduction,
or else takes a term $O[R]$ with indicated explosive redex R,
and yields R', with R' the reduct of R.

Reducing an explosive redex discards its context,
and produces a refutation term.

$$
\text{Cut: } \dfrac{\rightarrow\text{R!: } \dfrac{\Gamma, x^{\phi} \succ M^{\perp}}{\Gamma \succ (\lambda ! x.M)^{\phi\rightarrow\psi}} \qquad \rightarrow\text{L: } \dfrac{\Delta \succ N^{\phi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Delta, \Sigma, z^{\phi\rightarrow\psi} \succ O^{\mathfrak{C}}[y \mapsto zN]}}{\text{which is: } \dfrac{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda ! x.M]}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto (\lambda ! x.M)N]}}
$$

$$\text{Cut: } \dfrac{\text{→R!: } \dfrac{\Gamma, x^{\phi} \succ M^{\perp}}{\Gamma \succ (\lambda! x.M)^{\phi\to\psi}} \qquad \text{→L: } \dfrac{\Delta \succ N^{\phi} \qquad \Sigma, y^{\psi} \succ O^{\mathfrak{C}}}{\Delta, \Sigma, z^{\phi\to\psi} \succ O^{\mathfrak{C}}[y \mapsto zN]}}{\text{which is: } \dfrac{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto zN][z \mapsto \lambda! x.M]}{\Gamma, \Delta, \Sigma \succ O^{\mathfrak{C}}[y \mapsto (\lambda! x.M)N]}}$$

$$\text{Cut: } \dfrac{\Delta \succ N^{\phi} \qquad \Gamma, x^{\phi} \succ M^{\perp}}{\Delta, \Gamma \succ M^{\perp}[x \mapsto N]}$$

A typed term N^{ϕ} is a plan to calculate a ϕ.

But plans can run into exceptions,
and terminate before they produce their output.

$(\lambda!x^{\phi}.M^{\perp})^{\phi\to\psi}$ promises a ψ if given a ϕ,
but what it will do is terminate.

So it doesn't matter what we were going to do
with the ψ we were planning to produce;
the context can be discarded.

We do not in general have that reduction preserves type, because of explosive reduction. But:

Lemmas:

If $M^{\perp}$ reduces to N, then N is a refutation term.
If M^{ϕ} reduces to N^{ψ}, then the reduction is gentle.
If M^{ϕ} gently reduces to N, then N has type ϕ.

Reduction preserves refutationhood.
Explosive reduction always produces refutation terms.
Gentle reduction preserves type.

Theorem:
Reduction (and so gentle reduction) is strongly normalising.

That is, from any term, every sequence of reductions is finite.

A term that can't be reduced is a normal form.

A term that can't be gently reduced is a gentle normal form.

Every normal form is a gentle normal form, but not vice versa.

SN gives: every reduction sequence reaches a normal form,
and every gentle reduction sequence reaches a gentle normal form.

A relation $\triangleright$ is confluent iff
whenever $M \triangleright N$ and $M \triangleright O$, then there is a P with $N \triangleright P$ and $O \triangleright P$.

It is weakly confluent iff
whenever $M \triangleright N$ and $M \triangleright O$, then there is a P with $N \triangleright^{\star} P$ and $O \triangleright^{\star} P$.

(* is reflexive transitive closure.)

Uh-oh:

Reduction isn't confluent.

One-step reduction isn't weakly confluent.

$$\left((\lambda\_.w^{\delta})^{\psi\to\delta}\ \left((\lambda!x^{\phi}.(y^{\neg\phi}⅋x^{\phi})^{\perp})^{\phi\to\psi}z^{\phi}\right)^{\psi}\right)^{\delta}$$

reduces in one step via its outer (left) redex to w^{δ},

and in one step via its inner (right) redex to $(y^{\neg\phi}⅋z^{\phi})^{\perp}$,

both of which are in normal form.

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$$\Big((\lambda\_.w^{\delta})^{\psi\to\delta}\ \big((\lambda !x^{\phi}.(y^{\neg\phi} \parr x^{\phi})^{\perp})^{\phi\to\psi} z^{\phi}\big)^{\psi}\Big)^{\delta}$$

reduces in one step via its outer (left) redex to w^{δ},

and in one step via its inner (right) redex to $(y^{\neg\phi} \parr z^{\phi})^{\perp}$,

both of which are in normal form.

$$\Bigg(\Big((\lambda !x^{\phi}.(y^{\neg\phi} \parr x^{\phi})^{\perp})^{\phi\to\psi\to\theta} v^{\phi}\Big)^{\psi\to\theta} \ \Big((\lambda !s^{\delta}.(r^{\neg\delta} \parr s^{\delta})^{\perp})^{\delta\to\psi} q^{\delta}\Big)^{\psi}\Bigg)^{\theta}$$

reduces in one step via its left redex to $(y^{\neg\phi} \parr v^{\phi})^{\perp}$,

and in one step via its right redex to $(r^{\neg\delta} \parr q^{\delta})^{\perp}$,

both of which are in normal form.

$$\Big(\big((\lambda!x^{\phi}.(y^{\neg\phi} ☍ x^{\phi})^{\perp})^{\phi\to\psi\to\theta} v^{\phi}\big)^{\psi\to\theta} \; \big((\lambda!s^{\delta}.(r^{\neg\delta} ☍ s^{\delta})^{\perp})^{\delta\to\psi} q^{\delta}\big)^{\psi}\Big)^{\theta}$$

reduces in one step via its left redex to $(y^{\neg\phi} ☍ v^{\phi})^{\perp}$,

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reduces in one step via its left redex to $(y^{\neg\phi} \parr v^{\phi})^{\perp}$,

and in one step via its right redex to $(r^{\neg\delta} \parr q^{\delta})^{\perp}$,

both of which are in normal form.

Theorem:

Gentle reduction is confluent.

Nonconfluence comes from explosive redexes.

A term is forking iff it contains either:
a $\lambda\_$ redex with an explosive redex in its argument, or
two nonoverlapping explosive redexes.

Theorem:

One-step reduction is weakly confluent on nonforking terms.

A term is hereditarily nonforking iff
it does not reduce to any forking term.

Corollary:

Reduction is confluent on hereditarily nonforking terms.

Reduction can produce forks.

$$\left(\left(\lambda y^{\phi}.((\lambda\_.w^{\psi})^{\phi\to\psi}y^{\phi})^{\psi}\right)^{\phi\to\psi}\ \left((\lambda!z^{\delta}.(v^{\neg\delta}\mathbin{\circ\!\!\!\circ}z^{\delta})^{\perp})^{\delta\to\phi}u^{\delta}\right)^{\phi}\right)^{\psi}$$

is nonforking.
Reduced at its outer (left) redex, it yields

$$\left((\lambda\_.w^{\psi})^{\phi\to\psi}\ \left((\lambda!z^{\delta}.(v^{\neg\delta}\mathbin{\circ\!\!\!\circ}z^{\delta})^{\perp})^{\delta\to\phi}u^{\delta}\right)^{\phi}\right)^{\psi}$$

which is forking.
Both terms reduce at their explosive redex directly to $(v^{\neg\delta}\mathbin{\circ\!\!\!\circ}u^{\delta})$.

Reduction can produce forks.

$$\Big(\big(\lambda y^{\phi}.((\lambda\_.w^{\psi})^{\phi\to\psi}y^{\phi})^{\psi}\big)^{\phi\to\psi}\ \big((\lambda !z^{\delta}.(v^{\neg\delta}\multimapboth z^{\delta})^{\perp})^{\delta\to\phi}u^{\delta}\big)^{\phi}\Big)^{\psi}$$

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Reduced at its outer (left) redex, it yields

$$\Big((\lambda\_.w^{\psi})^{\phi\to\psi}\ \big((\lambda !z^{\delta}.(v^{\neg\delta}\multimapboth z^{\delta})^{\perp})^{\delta\to\phi}u^{\delta}\big)^{\phi}\Big)^{\psi}$$

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Reduction can produce forks.

$$\Bigg(\Big(\lambda y^{\phi}.((\lambda\_.w^{\psi})^{\phi\to\psi} y^{\phi})^{\psi} \Big)^{\phi\to\psi} \ \Big((\lambda ! z^{\delta}.(v^{\neg\delta} \text{⚯}\, z^{\delta})^{\bot})^{\delta\to\phi} u^{\delta} \Big)^{\phi} \Bigg)^{\psi}$$

is nonforking.

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$$\Bigg((\lambda\_.w^{\psi})^{\phi\to\psi} \ \Big((\lambda ! z^{\delta}.(v^{\neg\delta} \text{⚯}\, z^{\delta})^{\bot})^{\delta\to\phi} u^{\delta} \Big)^{\phi} \Bigg)^{\psi}$$

which is forking.

Both terms reduce at their explosive redex directly to $(v^{\neg\delta} \text{⚯}\, u^{\delta})$.

Every term $M^{\mathfrak{C}}$ has a unique gentle normal form $M^{\mathfrak{C}}_{\mathrm{gnf}}$.

If $M^{\mathfrak{C}}_{\mathrm{gnf}}$ is a typed normal form, it is $M^{\mathfrak{C}}$'s unique typed normal form;
if it is not, $M^{\mathfrak{C}}$ does not have a typed normal form.

So no term has multiple distinct typed normal forms;
all or all but one of a term's normal forms must be refutation terms.

Hereditarily nonforking terms have unique normal forms;
the two examples of nonconfluence are the only kinds.

Every calculation produces at most one result:
its gentle normal form, if that's a normal form,
and no result if it isn't.

A calculation might be able to fail in multiple distinct ways,
but its result, if any, remains unique, and gentle reduction will find it.

Correspondence

So we have a sequent calculus on the one hand,
and a bunch of terms with reduction behaviour on the other.

Each derivation determines a term in a straightforward way.

If the derivation is of $\Gamma \succ \phi$,
the term has type ϕ and free variables of types in Γ.

If the derivation is of $\Gamma \succ$,
it's a refutation term with free variables of types in Γ.

The term is in normal form iff the derivation contains no proper cuts.

Each term determines a derivation in a straightforward way.

If the term has type ϕ and free variables of types Γ,
the derivation is of $\Gamma \succ \phi$.

If it's a refutation term with free variables of types Γ,
the derivation is of $\Gamma \succ$.

All cuts in these derivations are principal.

The term is in normal form iff the derivation contains no cuts.

The round trip is the identity on terms,
and pushes cuts up to principal cases on derivations.