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REVISING UP: STRENGTHENING CLASSICAL LOGIC IN THE FACE OF PARADOX

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This paper provides a defense of the full strength of classical logic, in a certain form, against those who would appeal to semantic paradox or vagueness in an argument for a weaker logic. I will not argue that these paradoxes are based on mistaken principles; the approach I recommend will extend a familiar formulation of classical logic by including a fully transparent truth predicate and fully tolerant vague predicates. It has been claimed that these principles are not compatible with classical logic; I will argue, by both drawing on previous work[1] and presenting new work in the same vein, that this is not so. We can combine classical logic with these intuitive principles, so long as we allow the result to be nontransitive. In the end, I hope the paper will help us to handle familiar paradoxes within classical logic; along the way, I hope to shed some light on what classical logic might be for.

Here's the plan: §1 presents the supposed problems for classical logic, and §2 shows how to address them. §3 reflects on the situation and answers some objections, and §4 concludes.

1. A false dilemma

— *In the beginning, there was classical propositional logic.* —

Of course this is not so. Classical propositional logic (CPL) is a recent achievement; it was not handed down from on high. Further, it has never been uncontroversial; in its career it's been attacked from many angles. For example, the principle of explosion (from a contradiction, infer anything) is wildly counterintuitive;[2] the classical negation rules lack harmony;[3] important distinctions in content are collapsed;[4] and there are any number of other complaints one might have.

I won't address any of *these* complaints in this paper, or offer any positive argument at all for classical logic. My goal here is simply to

1. Primarily Cobreros et al. (2012); Ripley (2013).
2. Anderson and Belnap (1975); Routley et al. (1982).
3. Dummett (1991); Tennant (1987).
4. Barwise and Perry (1999); Routley and Routley (1975).

defend classical logic against one particular line of attack, the line that builds on paradoxes of truth and vagueness to attempt to undermine classical logic. This section lays out that line of attack, which proceeds by (what I'll claim is a false) dilemma. I'll save the formal details for §2 and beyond; this section will stay closer to intuitive motivations. For my purposes here, then, in the beginning there was CPL.

1.1 *Building up*

CPL, so I will claim, governs any propositions whatsoever. But of course there are valid and valuable inferences that CPL misses—those that hold just between certain particular propositions. Some of these inferences have given rise to familiar extensions of CPL.

For example, the inference from 'Everything is happy' to 'Alice is happy' is valid, and it would be nice to be able to capture inferences of this form. But from a purely CPL point of view, this is just an argument from p to q; the connections between the sentences are subpropositional, and so invisible to CPL. The solution is familiar: we allow ourselves to discern further structure within our propositions: names, variables, quantifiers, and maybe identity, and we take this further structure to obey appropriate rules. These rules pay attention to the shared predicate 'is happy' in the above argument, and so can yield the (correct) verdict that it is valid. The result is classical first-order logic (CL).

The strategy, we might hope, is a general one. When we find an area that supports valid inferences not captured by our logic, we can extend our logic with new rules that allow us to capture those inferences. Modal logics, higher-order logics, epistemic logics, imperative logics, and many others all follow this basic strategy.

1.2 *Forbidden fruit*

In some cases, however, the strategy seems to lead to trouble. For example, the argument from ' 'Snow is white' is true' to 'Snow is white' is valid. But even if we go past CPL to CL, this is just an argument from Pa to Qb; we cannot capture its validity. Two things are missing: the connection between the name ' 'Snow is white' ' and the sentence 'Snow is white', and recognition of the special role the truth predicate plays with regard to that connection.

One natural way to accommodate these is to add to CL a name-forming operator $\langle\ \rangle$, so that $\langle A\rangle$ is a name of the formula A, for any A. We can then add a predicate T that interacts with $\langle\ \rangle$ in an appropriate way. The most straightforward way is one that takes T to be *transparent*: that is, one that takes $T\langle A\rangle$ and A to be everywhere intersubstitutable, a way that takes these to behave identically, so far as our logic can yet see.[5]

It is sometimes claimed that this leads us into immediate trouble, at least when combined with classical logic. For example, Field offers: "Intersubstitutivity would lead to $T\langle A\rangle \equiv A$, which we know that no classical theorist can consistently accept".[6] One worry comes from the familiar liar paradox. Suppose we have a sentence λ that is $\neg T\langle\lambda\rangle$. Reasoning classically, as is well known, we can reach the conclusion $\lambda \wedge \neg\lambda$; and from this contradiction, as from any contradiction, anything at all follows. It seems like our straightforward strategy—just adding intuitively valid principles—has gotten us into trouble. Classical logic plus the intuitively valid rules for truth seems to allow us to prove too much.

Similar problems arise when we consider vagueness. The argument from 'Alice is happy' to 'Anyone very similar in happiness to Alice is happy' certainly seems valid,[7] but CL can only see it as an argument from Ha to $\forall x(xSa \supset Hx)$, and this is of course invalid. The problem is that CL does not respect the connection between 'happy' and 'similar in happiness'.

5. If we are later concerned with the logic of highly intensional notions like belief or proof, we might decide to relax this intersubstitution condition when A or $T\langle A\rangle$ occurs in the scope of such vocabulary.
6. Field (2008, p. 210, notation changed).
7. This is one manifestation of the so-called *tolerance* principle; see Wright (1975).

We can, however, add a way to respect this. For each predicate P, add a similarity relation $\sim_P$.[8] Then we can add rules that respect the connection between P and $\sim_P$. For example (as above), the argument from Pa to $\forall x(x \sim_P a \supset Px)$ should be valid. Also, the argument from Pa and $a \sim_P b$ to Pb should be valid, and the principle of tolerance itself—$\forall x \forall y((Px \land x \sim_P y) \supset Py)$—should be a theorem.

Again, though, there is immediate trouble. It is not hard to find a sorites sequence: a sequence of individuals such that the first is P for some vague predicate P, the last is not P, and each is similar P-wise to the next. But the existence of such a series seems incompatible with the just-cited principles governing P and $\sim_P$. Suppose there is such a sequence. Start from the claim that the first thing is P and the claim that it's P-similar to the second thing; from these, validly conclude that the second thing is P. Then, work from the claim that the second thing is P and the claim that it's P-similar to the third thing to validly conclude that the third thing is P too. Continue in this manner for long enough, and you'll reach the conclusion that the last thing is P. But we supposed that the last thing was not P—contradiction. Thus, there cannot be such a sequence.

Both transparent truth and tolerant vagueness, then, seem to leave us with a dilemma. Each of them, when added to classical logic, allows us to prove *too much*: either anything whatsoever, in the case of transparent truth, or the nonexistence of sorites sequences, in the case of tolerant vagueness. The line of attack I'm concerned with here claims that we must thus either: i) do without transparent truth and tolerant vagueness, or else ii) weaken our logic until it is safe for transparent truth and tolerant vagueness, until it no longer allows us to draw the unwanted conclusions. If we accept this dilemma, then any successful defense of transparent truth or tolerant vagueness will ipso facto be a successful argument against classical logic.

However, recent work on truth and vagueness has given us resources to see this to be a false dilemma. We can reject the dilemma and instead accept all of transparent truth, tolerant vagueness, and classical logic. §2 surveys the core of the strategy.

8. This strategy closely follows that of Cobreros et al. (2012).

2. How to address truth and vagueness

This section presents and briefly describes logical approaches that conservatively extend classical logic with both transparent truth and tolerant vagueness.[9]

2.1 Classical logic

Before we can see how to extend classical logic in a way that respects transparent truth and tolerant vagueness, it will help to pick a particular presentation of classical logic to work from. Here, I'll work with a multiple-conclusion sequent presentation of classical logic. One reason for this is to show, so to speak, that there's nothing up my sleeves. Multiple-conclusion presentations individuate logics more finely than single-conclusion presentations;[10] if I were to work with a single-conclusion presentation, you'd be justified in wondering whether I had hidden some secret nonclassicality "offstage", as supervaluationist logics do.[11] The multiple-conclusion setting will make plain that this is not the case.

First, the language itself. We take a first-order language without identity, containing a truth predicate T, a distinguished term $\langle A \rangle$ for each formula A, and a $2n$ary predicate $\sim_P$ for every $n$ary predicate P. Allow for formulas to contain distinguished terms for themselves.[12] This allows for paradoxical formulas galore, such as a liar sentence λ that is $\neg T\langle \lambda \rangle$. Throughout, classical logic will apply to the full vocabulary, including those bits of vocabulary that will later be used to handle

9. The approach to truth recommended here is a slight modification of that in Ripley (2013); the approach to vagueness, while related to that of Cobreros et al. (2012), is novel, as is the combined system treating both truth and vagueness.
10. See Humberstone (2012); Shoesmith and Smiley (1978) for details.
11. Hyde (1997).
12. This can be achieved as outlined in Ripley (2012).

truth and vagueness. Of course, classical logic on its own can't recognize the connections between $T\langle A \rangle$ and A, or between $\sim_P$ and P. We'll have to add those connections in later. For now, I'm just making sure the vocabulary is in the language, so we can put those connections in when they're required. (I work throughout without identity; it poses no special problems to add it.)

For our classical logic, I'll use the sequent calculus G1c,[13] given in Figure 1. Here, a sequent is of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite multisets of formulas.[14] I abbreviate freely: eg Γ, A, A is the multiset just like Γ but with two more occurrences of A, and $\Gamma, A/B$ is either Γ, A or Γ, B. In addition to its axioms, G1c uses the two structural rules of weakening (K) and contraction (W), and an operational rule to introduce each connective or quantifier on each side of the sequent turnstile. In the rules for $\forall$ and $\exists$, t can be any term; a is any nondistinguished term that does not occur free in $\Gamma \cup \Delta$.

This, then, is our starting point. It gives us first-order classical logic in a multiple conclusion setting. The sense in which the project of this paper is to *strengthen* classical logic can now be made fully precise: at no point will I take back, weaken, modify, restrict the application of, or otherwise fiddle with *any* axiom or rule of G1c. They remain in force throughout; they will only be *added* to.

In fact, this specific formulation of classical logic can be varied in any number of respects without affecting the points at issue; the choice of G1c is just for concreteness. One thing does matter a great deal, though: G1c includes no rule of *cut*:

$$\text{Cut:} \quad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

13. Taken from Troelstra and Schwichtenberg (2000, p. 61–62).

14. A multiset is like a set (and unlike a list) in that it pays no attention to order, but like a list (and unlike a set) in that it does pay attention to number of occurrences. For example, $\langle A, B, A \rangle$ is a different list from $\langle A, A, B \rangle$, but $[A, B, A]$ is the same multiset as $[A, A, B]$. On the other hand, $\{A, A, B\}$ is the same set as $\{A, B\}$, but $[A, A, B]$ is a different multiset from $[A, B]$.

Axioms:

$$\text{Id:} \quad \frac{}{A \vdash A} \qquad \bot\text{L:} \quad \frac{}{\bot \vdash}$$

Structural rules:

$$\text{KL:} \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \text{KR:} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}$$

$$\text{WL:} \quad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \text{WR:} \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$$

Operational rules:

$$\neg\text{L:} \quad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \neg\text{R:} \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}$$

$$\wedge\text{L:} \quad \frac{\Gamma, A/B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\text{R:} \quad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

$$\vee\text{L:} \quad \frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \qquad \vee\text{R:} \quad \frac{\Gamma \vdash A/B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

$$\supset\text{L:} \quad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \supset B \vdash \Delta} \qquad \supset\text{R:} \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta}$$

$$\forall\text{L:} \quad \frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x A(x) \vdash \Delta} \qquad \forall\text{R:} \quad \frac{\Gamma \vdash A(a), \Delta}{\Gamma \vdash \forall x A(x), \Delta}$$

$$\exists\text{L:} \quad \frac{\Gamma, A(a) \vdash \Delta}{\Gamma, \exists x A(x) \vdash \Delta} \qquad \exists\text{R:} \quad \frac{\Gamma \vdash A(t), \Delta}{\Gamma \vdash \exists x A(x), \Delta}$$

Figure 1: The calculus G1c

Cut encodes a slight generalization of *transitivity*, one that holds for many logics, including classical logic. Given the other rules of G1c, however, there is no need to have a separate rule of cut in the system; G1c already determines classical logic.[15]

As far as G1c is concerned, then, cut makes little difference. But presently I will add rules to G1c, and those rules will create systems in which cut makes a big difference. These systems will avoid bad consequences precisely by allowing for counterexamples to cut; I will recommend *nontransitive* extensions of classical logic. (Any formulation of classical logic that includes a rule of cut, then, would not work for my purposes here.)

2.2 *Transparent truth*

First, to add transparent truth. Start from G1c, and add the following *truth rules* (the double line indicates that these rules may be used in either direction, deriving the bottom sequent from the top one or the top from the bottom):[16]

$$TL: \quad \frac{\Gamma, A \vdash \Delta}{\Gamma, T\langle A\rangle \vdash \Delta} \qquad TR: \quad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash T\langle A\rangle, \Delta}$$

15. That is, cut is *admissible* in G1c: adding it to G1c would not change the stock of derivable sequents. It would only allow for new derivations of things that are already derivable. See Troelstra and Schwichtenberg (2000, p. 67).

16. In Ripley (2013), the approach to truth was formalized with the single-line versions of these rules, which can only be used from top to bottom. So long as these are the only rules beyond the usual logical rules, this makes no difference to what can be derived (as is proved there).

But here, I will add rules for vague predicates as well, and this will make the difference matter. The simplest example is this: a transparent, tolerant consequence relation ought to validate the sequent $A, \langle A\rangle \sim_T \langle B\rangle \vdash B$. With only the single-line truth rules, though, this sequent and others like it are not derivable. This is because, in the presence of tolerant vagueness, the single-line rules do not suffice for transparency. The double-line rules ensure transparency in many more settings, including the settings to be developed here.

These rules govern the interaction between the truth predicate T and the distinguished terms. The resulting logic, which I'll call G1cT, has some attractive features.[17] First, it is a *conservative extension* of classical logic, in the sense that, whenever there is no T in $\Gamma \cup \Delta$, the sequent $\Gamma \vdash \Delta$ is derivable in G1cT iff it is derivable in G1c.[18] Thus, the addition of the truth rules has no effect on which arguments in the T-free language are valid.

Second, G1cT is an *inference-preserving* extension of classical logic. That is, if $\Gamma \vdash \Delta$ is derivable in G1c, then $\Gamma^\star \vdash \Delta^\star$ is derivable in G1cT, where $\star$ is any uniform substitution on the full language. (Since G1c is closed under uniform substitution, this amounts to the claim that when $\Gamma \vdash \Delta$ is derivable in G1c, it remains derivable in G1cT.) This tells us that all our familiar classically-valid arguments—excluded middle, explosion, modus ponens, contraction, everything—remain valid *even* when Ts are around.[19]

Third, G1cT features a fully transparent truth predicate, in the sense outlined in §1.2. Swapping wffs for their T-wffs or vice versa, even when they appear as subformulas, does not affect the G1cT-derivability of any sequent, so it does not affect the validity of any argument.

Transparent truth plus classical logic was supposed to cause serious trouble. In particular, it was supposed to leave us with a logic in which we could derive anything whatsoever. However, the conservative extension result guarantees that this hasn't happened here. Adding trans-

17. For proofs of these claims, see Ripley (2012, 2013).

18. The usual sense of 'conservative extension' requires expanding the language under consideration. In this paper, though, I work with a single language, which includes from the start all the vocabulary wanted for eventual treatments of truth and vagueness. (This is to make it as clear as possible that all the rules of G1c apply to the whole language.) Thus, I'll use a slightly extended sense of 'conservative extension': adding rules to a system conservatively extends the system in my extended sense iff the new rules don't affect the derivability of any sequent not including the vocabulary governed by the new rules. (The resulting system, then, is a conservative extension in the usual sense of the system that results by simply throwing the affected vocabulary out of the language entirely.)

19. It thus goes beyond the logic sketched in Beall (2009, p. 16), which gives a conservative extension of classical logic that is not inference-preserving.

parent truth simply hasn't caused the problems that were supposed to result. (Of course *something* odd is going on: cut is no longer admissible. I'll come back to that in §2.5.)

2.3 *Tolerant vagueness*

I'll take a similar approach to tolerant vagueness, defining a system G1cV by adding to G1c. The additions will govern the similarity predicates $\sim_P$ and their relations to predicates P.

I'll write $\overrightarrow{a_n}$, $\overrightarrow{b_n}$, etc, for $n$tuples of terms. The intended reading of $\overrightarrow{a_n} \sim_P \overrightarrow{b_n}$ is that the as are P-similar to the bs. For example, imagine a sorites series for the two-place predicate 'near'. It will consist of a series of pairs of items, such that the difference between any two consecutive pairs in terms of within-pair nearness is very slight. In the present terminology, each pair is 'near'-similar to each of its neighboring pairs.

Although it is not strictly necessary for tolerant vagueness, it seems plausible enough that P-similarity is reflexive and symmetric for all P. We can build this into our sequent system by adding an axiom for reflexivity and two rules for symmetry, as in Figure 2. The rule $\sim$ref-drop is perhaps a bit unfamiliar, but it is quite sensible. It tells us: if an argument would be valid with the extra premise that the as are P-similar to themselves, then the argument is already valid, even without that premise. This is one (strong) way to assume that the as are P-similar to themselves.[20]

The real action, though, is in guaranteeing tolerance. This is handled by the *tolerance rule*:

$$\text{Tol:}\quad \frac{\Gamma \vdash \overrightarrow{a_n} \sim_P \overrightarrow{b_n}, \Delta}{\Gamma, P\overrightarrow{a_n} \vdash P\overrightarrow{b_n}, \Delta}$$

20. This precise formulation plays no role in the main point of the paper, but is convenient for an incidental reason, which I'll come back to in footnote 34. For further discussion of rules with this shape, see Negri and von Plato (1998).

$$\textbf{Axiom:}\quad \sim\text{ref-drop:}\quad \frac{\Gamma, \overrightarrow{a_n} \sim_P \overrightarrow{a_n} \vdash \Delta}{\Gamma \vdash \Delta}$$

$$\textbf{Rules:}\quad \sim\text{symL:}\quad \frac{\Gamma, \overrightarrow{a_n} \sim_P \overrightarrow{b_n} \vdash \Delta}{\Gamma, \overrightarrow{b_n} \sim_P \overrightarrow{a_n} \vdash \Delta} \qquad \sim\text{symR:}\quad \frac{\Gamma \vdash \overrightarrow{a_n} \sim_P \overrightarrow{b_n}, \Delta}{\Gamma \vdash \overrightarrow{b_n} \sim_P \overrightarrow{a_n}, \Delta}$$

Figure 2: Reflexivity and symmetry of $\sim_P$

The system G1cV is G1c, plus the rules in Figure 2, plus the tolerance rule.

Like G1cT, G1cV is a conservative extension of classical logic: if there are no similarity predicates in $\Gamma \cup \Delta$, then $\Gamma \vdash \Delta$ is derivable in G1c iff it's derivable in G1cV. Also like G1cT, G1cV is an inference-preserving extension: if $\Gamma \vdash \Delta$ is derivable in G1c, then $\Gamma^\star \vdash \Delta^\star$ is derivable in G1cV, for any uniform substitution $\star$. As before, this guarantees that all classically valid arguments—modus ponens, universal instantiation, whatever—remain valid when we move to G1cV.

G1cV, however, supports tolerant vagueness in all its forms. For example, we can derive $Pa, a \sim_P b \vdash Pb$ as follows:

$$\begin{array}{ll} \text{Id:} & \dfrac{}{a \sim_P b \vdash a \sim_P b} \\ \text{Tol:} & \dfrac{}{Pa, a \sim_P b \vdash Pb} \end{array}$$

In words, the argument from 'Alice is happy' and 'Gertrude is similar in happiness to Alice' to 'Gertrude is happy' can now be recognized as valid. We can get to other ways of stating tolerance as well. Building on the above derivation, for example, we can proceed in either of the following directions:

$$\forall\text{R:}\ \dfrac{\supset\text{R:}\ \dfrac{Pa, a \sim_P b \vdash Pb}{Pa \vdash a \sim_P b \supset Pb}}{Pa \vdash \forall x(a \sim_P x \supset Px)}$$

$$\forall\text{R:}\ \dfrac{\forall\text{R:}\ \dfrac{\supset\text{R:}\ \dfrac{\text{WL:}\ \dfrac{\wedge\text{L:}\ \dfrac{\wedge\text{L:}\ \dfrac{Pa, a \sim_P b \vdash Pb}{Pa, Pa \wedge a \sim_P b \vdash Pb}}{Pa \wedge a \sim_P b, Pa \wedge a \sim_P b \vdash Pb}}{Pa \wedge a \sim_P b \vdash Pb}}{\vdash (Pa \wedge a \sim_P b) \supset Pb)}}{\vdash \forall y((Pa \wedge a \sim_P y) \supset Py)}}{\vdash \forall x \forall y((Px \wedge x \sim_P y) \supset Py)}$$

The first of these derivations gets us to the validity of the argument from 'Alice is happy' to 'Everyone similar in happiness to Alice is happy', and the second gets us that 'Given any two things, if one of them is happy and they are similar in happiness to each other, then the other one is happy too' is a theorem (a valid consequence of no premises).

Despite its preserving classical logic and adding tolerance principles, however, G1cV is compatible with the existence of sorites sequences. In particular, we cannot derive $Pa, a \sim_P b, b \sim_P c \vdash Pc$; tolerance holds, but only one step of it, not two (and not more).[21] Thus, sorites series are not ruled out by G1cV. The trouble that was supposed to occur from combining tolerant vagueness with classical logic has simply failed to show up.

2.4 *Truth and vagueness together*

Moreover, by combining the T rules of G1cT with the $\sim$ rules of G1cV, we can reach a combined system, which I will call G1cTV. G1cTV, as you might expect, exhibits all the nice features of both G1cT and G1cV: it is an inference-preserving conservative extension of G1c, and it exhibits both fully transparent truth and fully tolerant vagueness. G1cTV thus confirms that if classical logic can handle transparent truth and tolerant vagueness separately, then there is no further obstacle to handling them together.

As I've pointed out, though, it's often been thought that classical logic *cannot* handle either of these phenomena. From that point of view, G1cT, G1cV, and G1cTV should all be seen as surprises. However, for the defender of classical logic who's felt uncomfortable about denying intuitive principles, or for the defender of intuitive principles who's felt uncomfortable about departures from classical logic, they should be welcome surprises.

2.5 *How it works*

This is all possible because G1cT, G1cV, and G1cTV are *nontransitive*. That is, there are instances in each system where $A \vdash B$ and $B \vdash C$ are both derivable, but $A \vdash C$ is not derivable. None admits the rule of cut (reproduced here) in full generality:

$$\text{Cut:}\ \dfrac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

For example, in G1cT, if we have a liar sentence λ that is $\neg T\langle\lambda\rangle$, we can derive both $\vdash \lambda$ and $\lambda \vdash$, as follows:

$$\text{WR:}\ \dfrac{\neg\text{R:}\ \dfrac{T\text{L:}\ \dfrac{\text{Id:}\ \dfrac{}{\lambda \vdash \lambda}}{T\langle\lambda\rangle \vdash \lambda}}{\vdash \lambda, \neg T\langle\lambda\rangle}}{\vdash \lambda} \qquad \text{WL:}\ \dfrac{\neg\text{L:}\ \dfrac{T\text{R:}\ \dfrac{\text{Id:}\ \dfrac{}{\lambda \vdash \lambda}}{\lambda \vdash T\langle\lambda\rangle}}{\lambda, \neg T\langle\lambda\rangle \vdash}}{\lambda \vdash}$$

Despite these two derivations, though, there is no derivation of the empty sequent in G1cT, as there would be if we could cut these derivations together. This is because there is no derivation of the empty sequent in G1c, no T occurs in the empty sequent (since nothing occurs

21. I'll show that this can't be derived in §2.5.

in it), and G1cT is a conservative extension of G1c.

Similarly, cut is not an admissible rule in G1cV. Although we can derive both $Pa, a \sim_P b \vdash Pb$ and $Pb, b \sim_P c \vdash Pc$, there is no derivation of $Pa, a \sim_P b, b \sim_P c \vdash Pc$. The derivations are quick:

$$\dfrac{\dfrac{}{a \sim_P b \vdash a \sim_P b}\text{Id}}{Pa, a \sim_P b \vdash Pb}\text{Tol} \qquad \dfrac{\dfrac{}{b \sim_P c \vdash b \sim_P c}\text{Id}}{Pb, b \sim_P c \vdash Pc}\text{Tol}$$

If we could cut these together, we could derive $Pa, a \sim_P b, b \sim_P c \vdash Pc$; but this sequent is not derivable, so again we see that cut is not an admissible rule.

To see that this sequent is not derivable, we can go through a model theory. Let a *model* for G1cV be a three-valued interpretation function I using values from $\{1, \frac{1}{2}, 0\}$ on the strong kleene scheme,[22] obeying the additional restrictions that 1) $I(\overrightarrow{a_n} \sim_P \overrightarrow{a_n}) = 1$, 2) $I(\overrightarrow{a_n} \sim_P \overrightarrow{b_n}) = I(\overrightarrow{b_n} \sim_P \overrightarrow{a_n})$, and 3) if $I(\overrightarrow{a_n} \sim_P \overrightarrow{b_n}) > 0$, then $|I(P\overrightarrow{a_n}) - I(P\overrightarrow{b_n})| < 1$. These constraints assure reflexivity of $\sim_P$, symmetry of $\sim_P$, and tolerance, respectively. Tolerance amounts to the constraint that if I assigns a positive value to the claim that two *n*tuples are P-similar, then I cannot simultaneously assign value 1 to the claim that one *n*tuple is P and assign value 0 to the claim that the other *n*tuple is P.

Let a model I be a *counterexample* to a sequent $\Gamma \vdash \Delta$ iff $I(\gamma) = 1$ for every $\gamma \in \Gamma$ and $I(\delta) = 0$ for every $\delta \in \Delta$. Let a sequent be *valid* iff no model I is a counterexample to it. Then every axiom of G1cV is valid, and all rules of G1cV preserve validity; G1cV is sound for these models.[23] But there is a counterexample to $Pa, a \sim_P b, b \sim_P c \vdash Pc$; there is no obstacle to assigning 1 to all the premises and 0 to the conclusion, so long as the value $\frac{1}{2}$ is assigned to Pb along the way. Since G1cV is sound for these models, it cannot derive any sequent that has a counterexample, so it cannot derive this sequent, so it does not admit the rule of cut.

Each of these counterexamples to cut applies in G1cTV as well; it's nontransitive for both truth-based reasons and vagueness-based reasons.

3. Do we need transitivity?

Thus, there is no problem maintaining full classical logic, as embodied in G1c, and conservatively extending it to accommodate transparent truth and tolerant vagueness. One simply needs to recognize that full classical logic does not, on its own, guarantee the transitivity of all its extensions. Of course classical logic itself remains transitive—nothing here should be construed as denying this well-known fact. But a transitive logic can have nontransitive extensions, and it's this possibility that's been exploited here by the target systems.

Full adherence here to a standard sequent-calculus presentation of classical logic has been, as promised, maintained throughout. The original dilemma—give up on both transparent truth and tolerant vagueness or else weaken classical logic—has been seen to be a false one. There is a middle path, a path that involves *strengthening* classical logic with principles that guarantee transparency and tolerance.

There is a natural worry, however: what if transitivity is such an important feature of a consequence relation that none of this matters? If transitivity is really nonnegotiable, then whatever other features the target systems do or don't have, they can't be correct theories of consequence. The purpose of this section is to explore and address some potential worries along these lines.

22. See Beall and van Fraassen (2003); Priest (2008) for details. Quantification can be handled substitutionally or objectually, to taste.

23. It is also complete, although that doesn't matter here. While we're at it, adding transparent T on the model of Kripke (1975) gives a model theory for G1cTV (sound and complete). See Cobreros et al. (2013); Ripley (2012, 2013) for the relations between Kripke-style models and logics like G1cT, and see Cobreros et al. (2012) for discussion of strong kleene models and a logic resembling G1cV. Combining vagueness and truth here poses no new problems.

3.1 Closure and constraint

There is at least prima facie reason to think that the target systems, because they are nontransitive, are not well-suited to play at least one of the roles consequence relations are typically taken to play. What we (implicitly) accept, or what we are committed to, is sometimes taken to be closed under consequence relations of various sorts, in the following sense: if a set of premises entails a conclusion, and someone is committed to all the premises, then they are committed to the conclusion as well.

But according to G1cT, λ follows from no premises. So if our commitments are closed under G1cT, we are committed to λ no matter what. Also according to G1cT, anything at all follows from λ. So if our commitments are closed under G1cT, if we are committed to λ then we are committed to absolutely everything. Putting this together, by the first step we are committed to λ, so by the second step we are committed to absolutely everything. This is a Very Bad result. Similar problems can be generated by assuming that our commitments are closed under the other target systems.

One possible response would be to note that the above line of reasoning requires chaining together two valid steps of reasoning about commitments. But if consequence itself is nontransitive, then we ought to be open to the possibility that our reasoning *about* consequence and its relation to our commitments should likewise allow for failures of transitivity. This might work to block the problem. Although I think it's worth pursuing this line of thought, I won't do it here. Instead, I'll assume that the target systems really cannot give a satisfactory theory of how our commitments are closed. There are other roles for a consequence relation to fill, however; closure is not the only game in town.

In other work, I and others explore these issues in more depth;[24] here I merely sketch the approach I favor. On this approach, the target systems play a *constraint* role rather than a closure role.[25] The idea is this: when a set Δ of conclusions is a consequence of a set Γ of premises, then we ought not accept all the members of Γ while rejecting all the members of Δ. (The 'ought' here has whatever force valid consequence brings with it; I won't take a stand here on what it might be.) This is the role the target systems should be understood as occupying.

This has the result that G1cT forbids us from rejecting λ (since it validates $\vdash \lambda$), and it forbids us from accepting it too (since it validates $\lambda \vdash$); the same holds for the other target systems, mutatis mutandis. Failures of transitivity occur just where neither acceptance nor rejection is appropriate; it is borderline statements that have this feature when vagueness is afoot.[26]

One upshot of this is that in patterns of acceptance and rejection, the adopter of the target systems ought to behave a lot like adopters of various nonclassical solutions. For example, they had better not reject $\lambda \wedge \neg\lambda$, since $\vdash \lambda \wedge \neg\lambda$ in G1cT and G1cTV. Similarly, they had better not accept $\lambda \vee \neg\lambda$, since $\lambda \vee \neg\lambda \vdash$. Indeed, one useful way to look at the approach I'm recommending is as blending some of the features of classical and nonclassical views of the paradoxes, as seeing how classical logic can still govern 'nonclassical' patterns of assertion and denial.[27]

Of course, any formal system can be interpreted in a wide variety of ways. My purpose in calling attention to the constraint role as I have

24. Especially Cobreros et al. (2012, 2013); Ripley (2013).

25. I take the idea from Restall; Restall (2005) is a clear articulation. This is roughly a multiple-conclusion version of the role dubbed Wo- in MacFarlane (2004). I owe the language of 'closure' vs 'constraint' to discussions with Jc Beall.

26. In fact, I think the situation is a bit more complex than this, but this is enough to sketch the basics. See Ripley (2013) for details. Cobreros et al. (2012) discusses related issues.

27. An anonymous referee, on noting that $\vdash \lambda \wedge \neg\lambda$, points out that this violates "the spirit of classical logic", and I think they are probably right. At the very least, it is out of step—quite deliberately—with the spirit in which classical logic is often deployed. The interesting thing, to me at least, is that this is compatible with holding to the letter of classical logic in a certain way; the spirit may have been the whole problem.

is not to insist that it is the only role the target systems can be seen as playing. These systems exhibit interesting behavior—combining classical logic with transparent truth and tolerant vagueness—which may be of wider use. My purpose is simply to point out that there is at least this important and familiar role for a consequence relation to play that the target systems are well-suited for. This is especially important because of the potential difficulty around seeing them as filling the distinct important and familiar role of closure.

3.2 *Consequence and reasoning*

This all fits with the idea that the target systems are excellent theories of closure *in those cases* that don't provide counterexamples to transitivity.[28] The remainder of this section focuses on these cases, and looks at the potential for endorsing familiar classical reasoning, including use of the cut rule, from within the target systems.

The idea is to see whether adopting the target systems would force any revision of existing classical reasoning in nonparadoxical cases; I will argue that it wouldn't. As an anonymous referee points out, many nonclassical approaches to paradox also endorse full classical reasoning in nonparadoxical domains. This, then, is a way in which the present approach is no different than these other approaches. The differences are elsewhere.[29]

The relation between consequence and reasoning, of course, is a relatively loose one. Consequence is a binary relation between sets of sentences; it holds just when the argument from the first set to the second is valid. Reasoning, though, is an activity we engage in. Moreover, we engage in it for a wide variety of reasons, only some of which have anything at all to do with consequence.

So a theory of the extension of the consequence relation (which is all I'm offering here), whatever its content, can only tell us so much about reasoning. In particular, it is helpless (on its own) to say anything about when a particular course of reasoning is wise or foolish, useless or useful, difficult or easy, callous or compassionate. All these distinctions matter a great deal for a full evaluation of any reasoning practice, but I won't discuss them here.

The only aspect of a course of reasoning that a theory of the extension of the consequence relation *can* tell us about is its validity: has it led us to conclusions that validly follow from its premises or not? This is the only aspect of reasoning, then, that I'll be concerned with here. Every course of reasoning that we are used to endorsing as valid from a classical point of view remains endorsed as valid by the target systems.

The starting point for this argument is what I've been at pains to emphasize: the target systems are all overwhelmingly classical. No classically-valid argument fails—ever—in these systems. Thus, they validate, for example, all of classical mathematics; no revision in mathematical practice is necessary from the point of view recommended here.

Too, the target systems preserve the *operational rules* governing the classical connectives. Some authors have taken these rules to be a crucial part of fixing the classical meaning of the connectives.[30] Insofar as these patterns of reasoning are important to a theory of meaning, it's

28. Restall (2005) gives an argument in favor of cut that is useful here. While the argument is question-begging as support for cut (see Ripley (2013) for discussion), it does succeed in showing that filling the constraint role suffices for filling the closure role *if and only if* cut holds for the case in question.

29. One subtle difference lives here, actually, although I'm not sure how much it matters. Nonclassical approaches that preserve classical reasoning in limited domains tend to delimit those domains by paying attention to what *vocabulary* occurs (Is there a truth predicate? Is there a vague predicate? etc). The target systems, on the other hand, don't do this; what matters for uses of cut, as will emerge presently, is whether *transparency* or *tolerance* has been appealed to, not what vocabulary occurs.

30. For example, Restall (2005); Ripley (2013); Rumfitt (2000).

important that the target systems preserve them. They do; in fact, these operational rules are part of the very definition of the target systems.[31]

3.3 Saving cut where it counts

However, there might remain a worry that, by doing without cut, users of the target systems deprive themselves of an important tool indispensable for classical reasoning. The usual admissibility and elimination theorems for cut show that this can't be right for indispensability-in-principle: anything derivable with cut is already derivable without it Troelstra and Schwichtenberg (2000). But there are other forms of indispensability, including practical indispensability.

Indeed, Boolos has argued that we cannot practically do without cut.[32] His main point is that "the development and utilization within derivations of *subsidiary* conclusions" [p. 377, emphasis in original] is practically indispensable to ordinary reasoning. Since cut is the very feature of a sequent system that allows for such utilization, doing without cut seems to dispense with the practically indispensable.

The reason we can't do without such subsidiary conclusions is that they allow us to take *shortcuts* in our derivations. By allowing for these shortcuts, Boolos shows how to construct a derivation in 3175 symbols that would take over 10^{38} symbols without cut.[33] 3175 symbols is not too much to ask from a real live deducer; 10^{38} symbols, on the other hand, certainly is. Although both derivations are possible in principle, what is possible in principle may not be possible in practice, and this is the difference that Boolos's argument turns on.

The status of cut in the target systems is quite different, though. In these systems, cut is not just an innocent shortcut; it would allow for derivations of sequents that are not derivable without it, and that should not be derivable. We cannot, then, allow it in full generality within the target systems. But does this doom us, in situations like the one Boolos considers, to giving derivations that would require billions of hellaseconds to write out?

It does not. The key is in what's already been noted: cut is admissible in G1c. This guarantees that it can continue to provide us with safe shortcuts, so long as they are shortcuts to places reachable in G1c. In other words, we already know that if $\Gamma \vdash \Delta$ is derivable in G1c, then it's derivable in our target systems; we also know that if $\Gamma \vdash A, \Delta$ and $\Gamma', A \vdash \Delta'$ are derivable in G1c, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is derivable in G1c. It follows directly that if $\Gamma \vdash A, \Delta$ and $\Gamma', A \vdash \Delta'$ are derivable *in G1c*, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is derivable *in the target systems*. When the premises of a cut are classically valid, the conclusion of the cut is—always, regardless of what vocabulary is involved—valid in our target systems.

This suggests adding a restricted rule of cut to the target systems. Such a rule could be used in a derivation only *above* truth or tolerance rules, never below them. That is, the premises of the cut would have to be derived without use of the truth or tolerance rules, although any rules at all could be freely applied to the conclusion of the cut. Since cut is admissible for G1c, it follows that this restricted cut is admissible for the target systems.[34] But this restricted cut can continue to provide

31. It's worth pointing out that preserving these rules goes beyond simply being an inference-preserving conservative extension. To see this, start from the set of G1c-derivable sequents, and add to our language a 0-place connective $\divideontimes$. Let a sequent $\Gamma(\divideontimes) \vdash \Delta(\divideontimes)$ containing one or more $\divideontimes$s be derivable iff:

- $\Gamma(q) \vdash \Delta(q)$, or
- $\Gamma = [\divideontimes, \divideontimes]$, or
- $\Delta = [\divideontimes, \divideontimes]$.

Call the resulting logic G1c$\divideontimes$. G1c$\divideontimes$ is a conservative extension of G1c, and it is also an inference-preserving extension (thanks to the first bullet point). But, as a bit of play will show, *none* of the rules of G1c preserves G1c$\divideontimes$-derivability. It would be difficult to defend the claim that G1c$\divideontimes$ contains, say, a classical negation. The target systems face no such difficulties.

32. Boolos (1984).

33. He considers different systems than the ones in this paper; the precise counts for the systems we're presently concerned with almost certainly differ slightly. But the overall point is not in doubt.

34. It almost follows, anyway. I've allowed for $\sim$ref-drop and the $\sim$syms, which are not part of G1c, to be used above the restricted cut. It's quick to show that cut is admissible for G1c plus these additions (this is the reason for the slightly

us with all the shortcuts cut ever provided for users of classical logic; those users never appealed to truth or tolerance rules, since it has been thought that these will not fit with classical logic, and so all their uses of cut are known to be safe.

With this restricted rule in place, we can give Boolos's shorter derivation (or any other classical derivation involving cut) without trouble. Cut is fine to use as a shortcut *when it is indeed just a shortcut*. The target systems thus do not in any way interfere with classical reasoning, even when such reasoning relies on cut.

3.4 Classical models

The situation is different in the model theory. As I pointed out in §2.5, we can give model theories for systems like these in relatively familiar ways, but they do not look much like classical two-valued model theory. Rather, they are built on a three-valued base. Moreover, consequence cannot be understood as *preservation* of any particular status on these models, whether that status is supposed to represent truth, assertibility, verifiability, or any other notion. Preservation, after all, is transitive; if consequence amounted to preservation, it too would be transitive.

Someone who identifies classical logic with its familiar two-valued model theory, then, will not accept that the target systems preserve classical logic.[35] I think such an identification, though, would be a mistake. We recognize two-valued model theory as classical because it gets the consequence relation right, not vice versa.

There is not space to argue for this claim here, so I leave it as a premise.[36] My claim that the target systems preserve classical logic, then, depends on identifying classical logic by features of its consequence relation rather than features of its model theory.

Interestingly, some well-known theorists whose *only reasons* for adopting a nonclassical logic come from transparency and/or tolerance in the end downplay the philosophical importance of model theory anyway, to avoid certain revenge problems.[37] But for someone willing to downplay the importance of model theory, the target systems are already classical; there is no need to turn away from classical logic at all. These theorists, then, occupy particularly unstable ground, given the existence of the target systems. I leave a fuller exploration of this issue for future work.

4. Conclusion

This paper has defended classical logic from an objection due to paradoxes of truth and vagueness. The familiar dilemma—do without transparent truth and tolerant vagueness, or else use a logic weaker than classical—is a false one. The defender of classical logic can acknowledge transparent truth, tolerant vagueness, or both, simply by moving to a nontransitive extension of classical logic like the extensions described in this paper. Classical logic is not itself directly threatened by these principles, as it has been thought to be.

This defense depends on identifying classical logic with its familiar consequence relation rather than its familiar model theory, and it depends on allowing for failures of transitivity. Defenders of the philosophical import of that familiar model theory, or of the claim that consequence must be transitive, may well face a difficult dilemma; I leave it to such defenders to make of that what they will.[38]

unorthodox 'drop'-formulation of $\sim$ref-drop); then the claim in the text follows. It's just the truth rules and the tolerance rule that cause trouble for cut.

35. For specialists: the nonclassicality of the model theory for the target systems extends to the algebraic. The target systems can't be given a model theory using boolean algebras in the usual way. I doubt there is even an *unusual* way that will work with any usefulness. However, a model theory using kleene algebras is relatively straightforward.

36. It's independently motivated in Restall (2009); Ripley (2013).

37. I have in mind Beall (2009); Field (2008).

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