Preserving truth and preserving probability

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Section 1

Introduction

General background

- Logical consequence in the deductive tradition is all about truth preservation.
- But in the inductive tradition, it is about probability preservation
- The relation between both perspectives has not been explored systematically, but only partially, foremost in the work of Ernest Adams on conditionals, which characterizes classical logic probabilistically, and then offers a probabilistic definition of validity for conditional logic.

Our goal here: establish more bridges, with a specific focus on logics for vagueness.

The tolerance principle

(1) If x is tall and y is imperceptibly shorter than x, then y is tall.

The tolerance principle has all the hallmarks of a principle of inductive reasoning (Egré 2023, Ms), for several reasons:

- Our confidence in the principle is not perfect, otherwise sorites arguments would not lead to paradox.
- Our confidence in the principle can be increased or decreased depending on how similar consecutive items in a sorites sequence are taken to be, but also depending on their position in the sorites (see Egré, Ripley, Verheyen 2019, "The sorites in psychology").

A probabilistic take on tolerance

Various approaches exist that offer a probabilistic view of tolerance, including Borel (1907), Lassiter (2011), Egré (2011), Lassiter and Goodman (2017)

Here we are specifically interested in the approach of Lassiter and Goodman, who propose different probabilistic interpretations of the sorites main premise

Two readings of the tolerance conditional

(2) If a_{n+1} is tall, then a_n is tall.

LG (2017) distinguish two readings of the conditional:

(3)
a. Material : $Ta_{n+1} \supset Ta_n$.
b. Indicative (Adams) : $Ta_{n+1} \rightarrow Ta_n$.

Each can be assigned a probability value by calculating:

(4)
a. $Pr(\neg(Ta_{n+1} \wedge \neg Ta_n))$
b. $Pr(Ta_n | Ta_{n+1})$

The material reading

Basic idea: to be tall is to be taller than a threshold (viz. Kennedy 2007), but the threshold can be variable or uncertain.

Assume a probability distribution Pr on thresholds, then:

$$Pr(Ta_{n+1} \supset Ta_n) = 1 - Pr(Ta_{n+1} \wedge \neg Ta_n)$$

$$Pr(Ta_{n+1} \wedge \neg Ta_n) = \int_{h(a_n)}^{h(a_{n+1})} Pr(\theta) d\theta$$

This is the probability of the threshold falling between a_{n+1} and a_n

NB. The probability is typically also joint on heights, but we can factor out the uncertainty on heights if we consider that a sorites sequence is given with heights communicated to the speaker.

Step between consecutive items

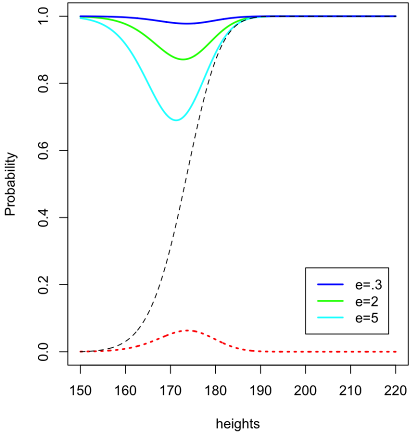
Let us understand: $y = a_n$ and $x = a_{n+1}$ to mean

$$h(x) = h(y) + e$$

So $Pr(Ta_{n+1} \supset Ta_n)$ is shorthand for:

$$Pr(Tx \supset Ty | h(x) = h(y) + e)$$

We can calculate this probability for different values of e (the tolerance step)



Red: $Pr(\theta)$; Black: $Pr(Tx)$; Other: $Pr(Tx \supset Ty | h(x) = h(y) + e)$

Observations

The approach vindicates the inductive take on the sorites paradox:

- The probability of material tolerance gets higher as the step between consecutive items reduces
- the probability of material tolerance decreases for instances closer to the center of the threshold distribution.
- the probability of material tolerance cannot be 1 on each instance

Three notions of consequence

Different instances of tolerance receive different probabilities. In this talk, we are interested in three different notions of probabilistic validity, which we call.

- Preservation consequence: when the premises are above a threshold, the conclusions are above the same threshold.
- Symmetric consequence: when the premises are above a threshold, the conclusions are above a dual threshold
- Material consequence: the material conditional resulting from conjoining premises and disjoining conclusions is above a threshold

Looking for ST

Symmetric consequence is "ST-like", in particular:

- it uses different thresholds for premises and conclusions, and its fuzzy counterpart (using truth-functional degrees of truth, instead of non-truth-functional probabilities) is provably ST under specific assumptions (Cobreros, Egré, Ripley, van Rooij 2023, "Tolerance and Degrees of Truth")

However, as we will see, it won't coincide with ST in this case, despite sharing structural similarities in some cases.

Caveat

In this talk, we stick fo a perfectly classical language, without the indicative conditional.

While we plan to extend the results to the indicative conditional, there is enough work to do with the material conditional to begin with!

However, the main observations we made about the material conditional can be made with the Adams conditional, except that the probability of tolerance instances cannot be bounded below across the spectrum

Section 2

Preservation consequence

Thresholds

Let a thresho<u>ld</u> α be any upset $\{1\} \subseteq \alpha \subseteq (0, 1]$.
That is, whenever $x \in \alpha$ and $x \leq y$, then $y \in \alpha$.

Once we have a 'good' status for a sentence to have —in this case, having probability in some threshold— it's natural to ask about preserving that status.

Preservation consequence

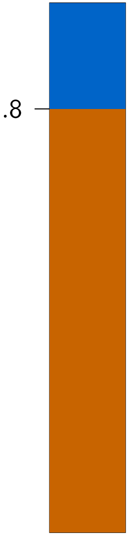
For a given threshold α, we define:

α-preservation consequence

A probability distribution Pr is an α-preservation counterexample to $\Gamma \succ \Delta$ iff:
$Pr[\Gamma] \subseteq \alpha$ and $Pr[\Delta] \subseteq [0,1] \setminus \alpha$.

And $\Gamma \succ \Delta$ is α-preservation valid (written $\models^{p}_{\alpha} \Gamma \succ \Delta$ or $\Gamma \models^{p}_{\alpha} \Delta$) iff:
no probability distribution is an α-preservation counterexample to it.

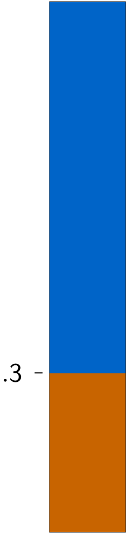
Adjunction



For example, $p, q \not\models^{p}_{[.8,1]} p \wedge q$, since there is a probability distribution Pr with $Pr(p), Pr(q) \geq .8$ and $Pr(p \wedge q) < .8$.

This happens for every threshold except $\{1\}$.

Adjunction



For example, $p, q \not\models^{p}_{[.8,1]} p \wedge q$, since there is a probability distribution Pr with $Pr(p), Pr(q) \geq .8$ and $Pr(p \wedge q) < .8$.

This happens for every threshold except $\{1\}$.

Thresholds containing .5 yield "weakly paraconsistent" logics: eg $p, \neg p \not\models^{p}_{(.3,1]} q$, although $p \wedge \neg p \models^{p}_{(.3,1]} q$.

Conditionals

Modus ponens failures

Take any failure of adjunction $A, B \not\models^p_\alpha A \wedge B$

This gives rise to a failure of modus ponens $A, A \supset (A \wedge B) \not\models^p_\alpha A \wedge B$, since always $Pr(A \supset (A \wedge B)) \geq Pr(B)$.

So modus ponens fails for $\models^p_\alpha$ for all $\alpha \neq \{1\}$.

Transitivity failures

We also have $A \supset B, B \supset C \not\models^p_\alpha A \supset C$ for $\alpha \neq \{1\}$

Super- and Sub-valuationism

Extreme cases: preserving probability 1, probability $\neq 0$

The extreme cases of preservation consequence have known non-probabilistic descriptions:

$\vDash^{p}_{\{1\}}$ is supervaluationist logic, and $\vDash^{p}_{(0,1]}$ is subvaluationist logic.

Among other things, that means $\vDash^{p}_{\{1\}}$ is classical for arguments $\Gamma \succ A$, and $\vDash^{p}_{(0,1]}$ is classical for arguments $A \succ \Delta$.

Ordering, Classicality

SET-FMLA and FMLA-SET orderings:

When $\alpha \subseteq \beta$, then: $\Gamma \vDash^{p}_{\beta} A$ implies $\Gamma \vDash^{p}_{\alpha} A$ and $A \vDash^{p}_{\alpha} \Delta$ implies $A \vDash^{p}_{\beta} \Delta$.

(Paris 2004, "Deriving information from inconsistent knowledge bases")

FMLA-FMLA classicality:

For any α, we have $A \vDash^{p}_{\alpha} B$ iff $A \succ B$ is classically valid

Ordering

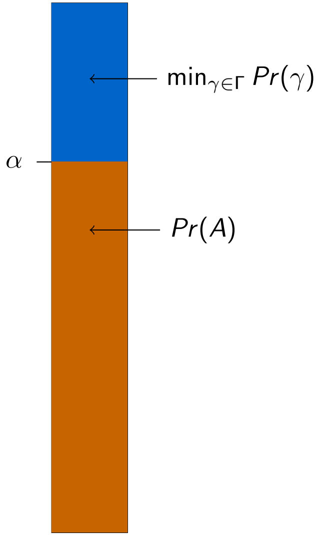
When $\alpha \subseteq \beta$, how to prove $\Gamma \vDash^{p}_{\beta} A$ implies $\Gamma \vDash^{p}_{\alpha} A$

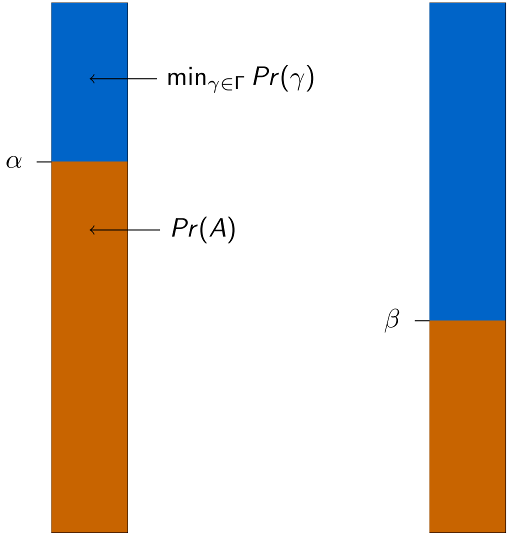
Suppose $\Gamma \nvDash^{p}_{\alpha} A$. Then there is a Pr with $Pr[\Gamma] \subseteq \alpha$ and $Pr(A) \notin \alpha$.

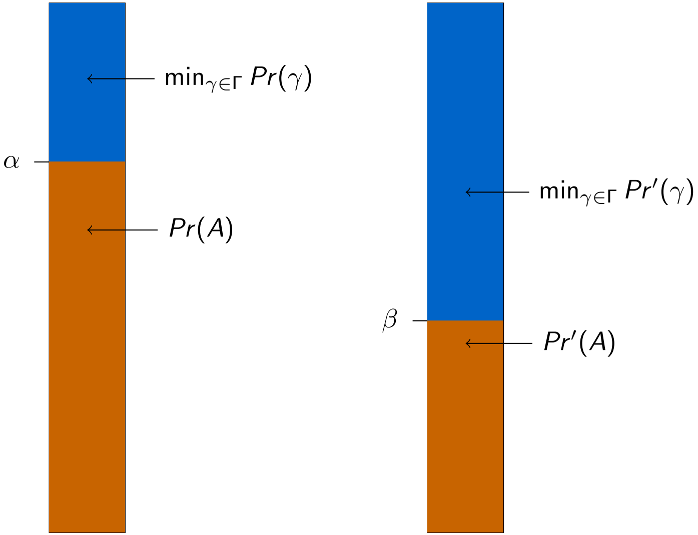
We generate a new Pr' with $Pr'[\Gamma] \subseteq \beta$ and $Pr'(A) \notin \beta$.

The key is Jeffrey conditioning:

- Given Pr, A, x with $Pr(A) \neq 0$, generates Pr' such that:
- $Pr'(A) = x$, and
- if $Pr'(A) > Pr(A)$, then $Pr'(A) - Pr(A) \geq Pr'(B) - Pr(B)$, and
- if $Pr(A) > Pr'(A)$, then $Pr(A) - Pr'(A) \geq Pr(B) - Pr'(B)$.


$\min_{\gamma \in \Gamma} Pr(\gamma)$
α
$Pr(A)$


$\min_{\gamma\in\Gamma} Pr(\gamma)$
α
$Pr(A)$
β


$\min_{\gamma\in\Gamma} Pr(\gamma)$
α
$Pr(A)$
$\min_{\gamma\in\Gamma} Pr'(\gamma)$
β
$Pr'(A)$

Summary

Probability preservation matches truth-preservation frameworks, but in the SET-SET case we can see that:

- Probability 1 preservation matches preservation of supertruth
- Probability $\neq 0$ preservation matches preservation of subtruth
- Adams's characterization of classical logic in the Set-Fmla case falls out as a special case

NB. Probability is non-truth functional, just as are super- and subvaluationist truth.

Section 3

Symmetric consequence

Motivation

- No preservation logic, even at the extremes, gives SET-SET classical logic Our ordering does not approach a classical limit, but s'valuationist ones.
- Previous work on three-valued and fuzzy approaches to vagueness has matched full classical logic by using 'mixed consequence', where conclusions can be held to a different standard from premises (see Cobreros, Egré, Ripley, van Rooij 2023, "Tolerance and Degrees of truth").
- We adapt this to our probabilistic setting, and show that it works here as well.

Symmetric consequence

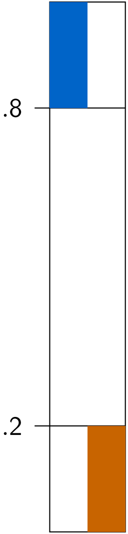
For a given threshold α, we define $\overline{\alpha} = \{x \in [0,1] \mid 1 - x \in \alpha\}$, and then:

α-symmetric consequence

A probability distribution Pr is an α-symmetric counterexample to $\Gamma \succ \Delta$ iff:
$Pr[\Gamma] \subseteq \alpha$ and $Pr[\Delta] \subseteq \overline{\alpha}$.

And $\Gamma \succ \Delta$ is α-symmetric valid (written $\models^{s}_{\alpha} \Gamma \succ \Delta$ or $\Gamma \models^{s}_{\alpha} \Delta$) iff:
no probability distribution is an α-symmetric counterexample to it.

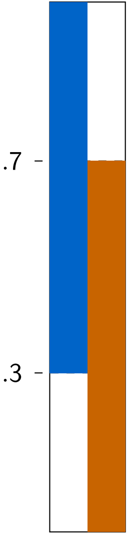
Adjunction



We have $p, q \vDash^{s}_{[.8,1]} p \wedge q$; when $Pr(p), Pr(q) \geq .8$, the lowest $Pr(p \wedge q)$ can be is .6.

Still, $p, q, r, s \nvDash^{s}_{[.8,1]} p \wedge q \wedge r \wedge s$.

Adjunction



We have $p, q \models^{s}_{[.8,1]} p \wedge q$; when $Pr(p), Pr(q) \geq .8$, the lowest $Pr(p \wedge q)$ can be is .6.

Still, $p, q, r, s \not\models^{s}_{[.8,1]} p \wedge q \wedge r \wedge s$.

Thresholds containing .5 still give weakly paraconsistent logics, now nonreflexive as well.

For example, let $Pr(p) = .5$.
Then $Pr(p) \in \alpha$ and $Pr(p) \in \overline{\alpha}$, so $p \not\models^{s}_{\alpha} p$

Classicality, Monotonicity

Extreme cases:

The extreme cases of symmetric consequence:

$\vDash^{\mathsf{S}}_{\{1\}}$ is exactly classical logic,
and $\Gamma \vDash^{\mathsf{S}}_{(0,1]} \Delta$ iff there is a contradiction in Γ or a tautology in Δ.

Ordering:

When $\alpha \subseteq \beta$, it's immediate that $\vDash^{\mathsf{S}}_{\beta} \subseteq \vDash^{\mathsf{S}}_{\alpha}$;
every α-symmetric counterexample just is a β-symmetric one as well.

Adjunction and modus ponens

Conjunctions:

Let P_n be the first n atomic sentences, and let $Conj_n$ be $P_n \succ \bigwedge P_n$.

Then $\vDash^S_\alpha$ validates $Conj_n$ iff $\alpha \subseteq (\frac{n}{n+1}, 1]$.

Modus ponenseses

Let MP_n be $p_1, p_1 \supset p_2, \ldots, p_{n-1} \supset p_n \succ p_n$.

Then $\vDash^S_\alpha$ validates MP_n iff $\alpha \subseteq (\frac{n}{n+1}, 1]$.

Reduction to preservation

Reduction to $\models^p$

$$\Gamma \models^s_\alpha \Delta \quad \text{iff} \quad \Gamma, \neg\Delta \models^s_\alpha \bot \quad \text{iff} \quad \Gamma, \neg\Delta \models^p_\alpha \bot$$

Section 4

Material consequence

Material Consequence

For a given threshold α, we define:

α-material consequence

A probability distribution Pr is an α-material counterexample to $\Gamma \succ \Delta$ iff:
$Pr[\bigwedge \Gamma \supset \bigvee \Delta] \not\subseteq \alpha$.

And $\Gamma \succ \Delta$ is α-material valid (written $\vDash^m_\alpha \Gamma \succ \Delta$ or $\Gamma \vDash^m_\alpha \Delta$) iff:
no probability distribution is an α-material counterexample to it.

Classicality

All classical

For any threshold α, we have $\vDash^m_\alpha$ is $\textsc{Set-Set}$ classical logic.

$$
\begin{aligned}
\Gamma \vDash^m_\alpha \Delta \quad & \Leftrightarrow \quad \top \vDash^m_\alpha \bigwedge \Gamma \supset \bigvee \Delta && (1)\\
& \Leftrightarrow \quad \top \vDash^p_\alpha \bigwedge \Gamma \supset \bigvee \Delta && (2)\\
& \Leftrightarrow \quad \top \vDash_{CL} \bigwedge \Gamma \supset \bigvee \Delta && (3)\\
& \Leftrightarrow \quad \Gamma \vDash_{CL} \Delta && (4)
\end{aligned}
$$

(1) and (2) by definitions; (3) by $\textsc{Fmla-Fmla}$ classicality of $\vDash^p_\alpha$

Section 5

Conclusion: back to the sorites

Summary of consequence relations

Preservation consequence is s'valuational, not fully classical.

We can achieve SET-SET classicality probabilistically,
in the limit for symmetric consequence
and everywhere for material consequence.

And both symmetric and material consequence
can be obtained from fragments of preservation consequence.

Transitivity?

Since material consequence is just classical consequence, it's transitive.

Since preservation consequence is based on preservation, it too is transitive.

But symmetric consequence can fail to be transitive;
any $\vDash^{S}_{\alpha}$ that validates $Conj_n$ but not $Conj_{n+1}$ is not transitive.

eg:

Suppose $A, B \vdash A \wedge B$ and $\vdash$ is transitive.

Then $p, q \vdash p \wedge q$ and $p \wedge q, r \vdash p \wedge q \wedge r$,
so by transitivity $p, q, r \vdash p \wedge q \wedge r$

Sorites

These different consequence relations give us different ways to evaluate sorites arguments.

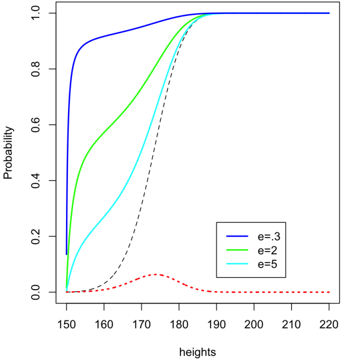
$$Pa_1, Pa_1 \supset Pa_2, \ldots, Pa_{n-1} \supset Pa_n \succ Pa_n$$

- Materially, all are valid.
- Preservationally, all are valid for $\{1\}$ and all invalid elsewhere.
- Symmetrically, which are valid depends on threshold:
  the narrower the threshold, the longer the valid chains.

Section 6

Appendix

Conditional probability of Ta_n given Ta_{n+1}



Red: $Pr(\theta)$; Black: $Pr(Tx)$; Other: $Pr(Ty|Tx, h(x) = h(y) + e)$

Containment relations