On the supposed unity of soritical and semantic paradox

David Ripley

University of Connecticut
http://davewripley.rocks

(UConn logo, 1959)

The targets

Soritical

The targets

Semantic

The targets

Unity

Are these in some important sense the same phenomenon?

Priest (BTL, p. 183):

“Let us call this the Principle of Uniform Solution (PUS): same kind of paradox, same kind of solution.

The PUS puts a lot of weight on the notion of kind. To convince ourselves that two paradoxes are of the same kind we must convince ourselves:

- (a) that there is a structure that is common to the paradoxes, and
- (b) that this structure is responsible for the contradictions.”

Priest (BTL, p. 183):

“Let us call this the Principle of Uniform Solution (PUS): same kind of paradox, same kind of solution.

The PUS puts a lot of weight on the notion of kind. To convince ourselves that two paradoxes are of the same kind we must convince ourselves:

- (a) that there is a structure that is common to the paradoxes, and
- (b) that this structure is responsible for the contradictions.”

The question of unity is a question about explanations: why do the paradoxical conclusions follow?

Inclosure

The schema

Priest’s explanation for the paradoxical conclusions,
in both cases,
turns on the inclosure schema.

An inclosure paradox is one with two ingredients subject to two conditions.

The ingredients:

- Ω A set
- δ A partial function $\wp(\Omega) \nrightarrow \Omega$

The conditions:

- $\delta(\Omega)$ is defined
- $\delta(X) \notin X$

These conditions are contradictory.

$\delta(\Omega) \in \Omega$, since $\delta(\Omega)$ is defined and δ's codomain is Ω.

But $\delta(\Omega) \notin \Omega$, since $\delta(X) \notin X$.

Inclosure

Capturing the paradoxes

To cast the liar into the inclosure schema:

The ingredients:

- Ω A set — The set of true sentences in $\mathcal{L}$
- δ A p.f. $\psi \nrightarrow \Omega$ — $X \mapsto$ 'This sentence is not in X.'

($\delta(X)$ is defined when X is definable in $\mathcal{L}$.)

The conditions:

- $\delta(\Omega)$ is defined — 'The true sentences' defines Ω.
- $\delta(X) \notin X$ — If it was, it wouldn't be true.

$\delta(\Omega)$ is the liar sentence.

To cast the sorites into the inclosure schema,
fix a given sorites series for the predicate P.

The ingredients:

- Ω A set — The set of P things in the series
- δ A p.f. $\wp(\Omega) \nrightarrow \Omega$ — $X \mapsto$ the next thing in the series beyond X

($\delta(X)$ is total, by tolerance.)

The conditions:

- $\delta(\Omega)$ is defined — δ is total.
- $\delta(X) \notin X$ — By defn.

$\delta(\Omega)$ is the first non-P thing in the series.

Inclosure

Does this matter?

Inclosure paradoxes are crucially tied to negation.
They lead to a contradiction: $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$.

Paradoxes that don't involve negation do not fit the schema.

“If this sentence is true, then A”

Curry$_A$ leads to A for the same reasons the liar leads to contradiction.

$$\lambda := \neg T\lambda \qquad\qquad \kappa_A := T\kappa_A \supset A$$

$$\text{}^{\neg\mathrm{I}^1:}\dfrac{\text{}^{\neg\mathrm{E}:}\dfrac{\text{}^{\text{which is:}}\dfrac{\text{}^{T\mathrm{E}:}\dfrac{[T\lambda]^1}{\lambda}}{\neg T\lambda} \qquad [T\lambda]^1}{\bot}}{\neg T\lambda} \qquad\qquad \text{}^{\supset\mathrm{I}^1:}\dfrac{\text{}^{\supset\mathrm{E}:}\dfrac{\text{}^{\text{which is:}}\dfrac{\text{}^{T\mathrm{E}:}\dfrac{[T\kappa_A]^1}{\kappa_A}}{T\kappa_A \supset A} \qquad [T\kappa_A]^1}{A}}{T\kappa_A \supset A}$$

$$\text{}^{\neg\mathrm{E}:}\dfrac{\neg T\lambda \qquad \text{}^{T\mathrm{I}:}\dfrac{\text{}^{\text{which is:}}\dfrac{\neg T\lambda}{\lambda}}{T\lambda}}{\bot} \qquad\qquad \text{}^{\supset\mathrm{E}:}\dfrac{T\kappa_A \supset A \qquad \text{}^{T\mathrm{I}:}\dfrac{\text{}^{\text{which is:}}\dfrac{T\kappa_A \supset A}{\kappa_A}}{T\kappa_A}}{A}$$

$$\lambda := \neg T\lambda \qquad\qquad \kappa_A := T\kappa_A \supset A$$

$$\vee E^1: \dfrac{T\lambda \vee \neg T\lambda \qquad \text{which is: } \dfrac{TE: \dfrac{[T\lambda]^1}{\lambda}}{\neg T\lambda} \qquad [\neg T\lambda]^1}{\neg T\lambda}$$

$$\vee E^1: \dfrac{T\kappa_A \vee (T\kappa_A \supset A) \qquad \text{which is: } \dfrac{TE: \dfrac{[T\kappa_A]^1}{\kappa_A}}{T\kappa_A \supset A} \qquad [T\kappa_A \supset A]^1}{T\kappa_A \supset A}$$

$$\vee E^2: \dfrac{T\lambda \vee \neg T\lambda \qquad [T\lambda]^2 \qquad \text{which is: } \dfrac{[\neg T\lambda]^2}{TI: \dfrac{\lambda}{T\lambda}}}{T\lambda}$$

$$\vee E^2: \dfrac{T\kappa_A \vee (T\kappa_A \supset A) \qquad [T\kappa_A]^2 \qquad \text{which is: } \dfrac{[T\kappa_A \supset A]^2}{TI: \dfrac{\kappa_A}{T\kappa_A}}}{T\kappa_A}$$

$$\neg E: \dfrac{\begin{matrix} T\lambda \vee \neg T\lambda \\ \vdots \\ \neg T\lambda \end{matrix} \qquad \begin{matrix} T\lambda \vee \neg T\lambda \\ \vdots \\ T\lambda \end{matrix}}{\bot}$$

$$\supset E: \dfrac{\begin{matrix} T\kappa_A \vee (T\kappa_A \supset A) \\ \vdots \\ T\kappa_A \supset A \end{matrix} \qquad \begin{matrix} T\kappa_A \vee (T\kappa_A \supset A) \\ \vdots \\ T\kappa_A \end{matrix}}{A}$$

$$\lambda := \neg T\lambda \qquad\qquad \kappa_A := T\kappa_A \supset A$$

$$\text{TL: }\dfrac{\text{WL: }\dfrac{\text{which is: }\dfrac{\neg\text{L: }\dfrac{T\text{R: }\dfrac{\lambda \vdash \lambda}{\lambda \vdash T\lambda}}{\neg T\lambda, \lambda \vdash}}{\lambda, \lambda \vdash}}{\lambda \vdash}}{T\lambda \vdash}$$

$$\text{TL: }\dfrac{\text{WL: }\dfrac{\text{which is: }\dfrac{\supset\text{L: }\dfrac{T\text{R: }\dfrac{\kappa_A \vdash \kappa_A}{\kappa_A \vdash T\kappa_A} \qquad A \vdash A}{T\kappa_A \supset A, \kappa_A \vdash A}}{\kappa_A, \kappa_A \vdash A}}{\kappa_A \vdash A}}{T\kappa_A \vdash A}$$

$$\text{Cut: }\dfrac{T\text{R: }\dfrac{\text{which is: }\dfrac{\neg\text{R: }\dfrac{T\lambda \vdash}{\vdash \neg T\lambda}}{\vdash \lambda}}{\vdash T\lambda} \qquad T\lambda \vdash}{\vdash}$$

$$\text{Cut: }\dfrac{T\text{R: }\dfrac{\text{which is: }\dfrac{\supset\text{R: }\dfrac{T\kappa_A \vdash A}{\vdash T\kappa_A \supset A}}{\vdash \kappa_A}}{\vdash T\kappa_A} \qquad T\kappa_A \vdash A}{\vdash A}$$

$$\lambda := \neg T\lambda$$

λ must be either F or T.

If λ is F, then $T\lambda$ is F,
so $\neg T\lambda$ is T.
No good.

If λ is T, then $T\lambda$ is T,
so $\neg T\lambda$ is F.
No good.

$$\kappa_A := T\kappa_A \supset A$$

κ_A must be either F or T.

If κ_A is F, then $T\kappa_A$ is F,
so $T\kappa_A \supset A$ is T.
No good.

If κ_A is T, then $T\kappa_A$ is T,
so $T\kappa_A \supset A$ has the same value as A.
This must be T.

Any way to draw out a paradoxical conclusion from the liar
has a matching way to draw out A from curry$_A$,
and vice versa.

That's a sign that the paradoxes are driven by the same factors.
(This argument does not turn on identifying these factors!)

Inclosure misses the commonalities between liars and curries.

But these commonalities explain
how they lead to paradoxical conclusions.

So the inclosure schema fails to identify
the kind relevant for PUS.

- At most two of these sentences are true.
- At most two of these sentences are true.
- At most two of these sentences are true.

Cut elimination

The idea

Start from some logical system that does not require a rule of cut:

$$\text{Cut:} \quad \frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Id: $$\dfrac{}{A \vdash A}$$

D: $$\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

WL: $$\dfrac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}$$

WR: $$\dfrac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

¬L: $$\dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

¬R: $$\dfrac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

&L: $$\dfrac{A, B, \Gamma \vdash \Delta}{A \mathbin{\&} B, \Gamma \vdash \Delta}$$

&R: $$\dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \mathbin{\&} B}$$

∀L: $$\dfrac{A(t), \Gamma \vdash \Delta}{\forall x A(x), \Gamma \vdash \Delta}$$

∀R: $$\dfrac{\Gamma \vdash \Delta, A(a)}{\Gamma \vdash \Delta, \forall x A(x)}$$

(In ∀R, a must be an eigenvariable.)

This system admits cut.

Why?

Claim:

Gentzen's proof, as modified by Bimbó (henceforth, the G-B proof), is explanatory.

It does not merely establish that these systems admit cut;
it shows why they do.

The G-B proof has four key moves,
and an important overarching structure.

Key move 1: Cut on axioms and after D is idle.

$$\text{Cut: } \frac{A \vdash A \qquad A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \rightsquigarrow A, \Gamma \vdash \Delta$$

$$\text{Cut: } \frac{\text{D: } \dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta', A} \qquad A, \Gamma'' \vdash \Delta''}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''} \rightsquigarrow \text{D: } \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma', \Gamma'' \vdash \Delta, \Delta', \Delta''}$$

Key move 2: Cut on nonprincipals could have been done earlier.

$$\text{Cut: } \frac{\neg\text{R: } \dfrac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, B, \neg A} \qquad B, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', \neg A} \rightsquigarrow \neg\text{R: } \frac{\text{Cut: } \dfrac{A, \Gamma \vdash \Delta, B \qquad B, \Gamma' \vdash \Delta'}{A, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta', \neg A}$$

Key move 3: Contraction can be pushed below cuts.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,\text{WR:} \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut:}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, A, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', A}\,\text{Cut:} \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\,\text{Cut:}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Ws:}$$

Key move 4: Cut on principals can be moved to their components.

$$\dfrac{\neg R: \dfrac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad \neg L: \dfrac{\Gamma' \vdash \Delta', A}{\neg A, \Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut:} \quad \rightsquigarrow$$

$$\text{Cut:}\ \dfrac{\Gamma' \vdash \Delta', A \qquad A, \Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Overarching structure:

The following strategy terminates,
because branches of proofs are finite, and
because the component relation is well-founded:

- Take a cut with no cuts above it.
- Using KM2–KM4, push it up until you hit axioms or Ds.
- By KM1, these are idle.

Cut elimination

Capturing the paradoxes

Rules for paradoxical vocabulary break cut elimination.

$$T\text{L}: \quad \frac{A, \Gamma \vdash \Delta}{TA, \Gamma \vdash \Delta} \qquad T\text{R}: \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, TA}$$

$$\text{Tol}: \quad \frac{\Gamma \vdash \Delta, t \sim_P u}{Pt, \Gamma \vdash \Delta, Pu}$$

$$\lambda := \neg T\lambda \qquad\qquad \kappa_A := T\kappa_A \supset A$$

$$\text{Cut: }\dfrac{\text{WR: }\dfrac{\text{which is: }\dfrac{\neg\text{R: }\dfrac{T\text{L: }\dfrac{\lambda \vdash \lambda}{T\lambda \vdash \lambda}}{\vdash \lambda, \neg T\lambda}}{\vdash \lambda, \lambda}}{\vdash \lambda} \qquad \text{WR: }\dfrac{\text{which is: }\dfrac{\neg\text{L: }\dfrac{T\text{R: }\dfrac{\lambda \vdash \lambda}{\lambda \vdash T\lambda}}{\neg T\lambda, \lambda \vdash}}{\lambda, \lambda \vdash}}{\lambda \vdash}}{\vdash}$$

$$\text{Cut: }\dfrac{\text{WR: }\dfrac{\text{which is: }\dfrac{\supset\text{R: }\dfrac{\text{D: }\dfrac{T\text{L: }\dfrac{\kappa_A \vdash \kappa_A}{T\kappa_A \vdash \kappa_A}}{T\kappa_A \vdash \kappa_A, A}}{\vdash \kappa_A, T\kappa_A \supset A}}{\vdash \kappa_A, \kappa_A}}{\vdash \kappa_A} \qquad \text{WR: }\dfrac{\text{which is: }\dfrac{\supset\text{L: }\dfrac{T\text{R: }\dfrac{\kappa_A \vdash \kappa_A}{\kappa_A \vdash T\kappa_A} \quad A \vdash A}{T\kappa_A \supset A, \kappa_A \vdash A}}{\kappa_A, \kappa_A \vdash A}}{\kappa_A \vdash A}}{\vdash A}$$

But there is no derivation of $\vdash$ without cut,
and there is no derivation of $\vdash A$ (in general) without cut.

$$\Sigma := \{a_1 \sim_P a_2, a_2 \sim_P a_3, \ldots, a_{n-1} \sim_P a_n\}$$

$$\dfrac{\dfrac{\dfrac{a_1 \sim_P a_2 \vdash a_1 \sim_P a_2}{a_1 \sim_P a_2, Pa_1 \vdash Pa_2}\,\text{Tol} \qquad \dfrac{a_2 \sim_P a_3 \vdash a_2 \sim_P a_3}{a_2 \sim_P a_3, Pa_2 \vdash Pa_3}\,\text{Tol}}{a_1 \sim_P a_2, a_2 \sim_P a_3, Pa_1 \vdash Pa_3}\,\text{Cut} \qquad \dfrac{a_3 \sim_P a_4 \vdash a_3 \sim_P a_4}{a_3 \sim_P a_4, Pa_3 \vdash Pa_4}\,\text{Tol}}{a_1 \sim_P a_2, a_2 \sim_P a_3, a_3 \sim_P a_4, Pa_1 \vdash Pa_4}\,\text{Cut}$$

$$\ddots$$

$$\Sigma, Pa_1 \vdash Pa_n$$

But there is no derivation of $\Sigma, Pa_1 \vdash Pa_n$ without cut.

In each case, the paradoxical conclusion follows only via cut.

(Two possible lessons here.)

Explaining why cut is needed thus goes some way towards explaining why the paradoxical conclusions follow.

If the G-B proof is really explanatory,
then we can explain why cut is needed
by seeing where the new rules break the G-B proof.

Explaining why cut is needed thus goes some way towards explaining why the paradoxical conclusions follow.

If the G-B proof is really explanatory, then we can explain why cut is needed by seeing where the new rules break the G-B proof.

And it is in completely different places.

Truth rules:

We can push cuts on principal *TA* back to cuts on *A*.

$$\dfrac{\dfrac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, TA}\,TR \qquad \dfrac{A, \Gamma' \vdash \Delta'}{TA, \Gamma' \vdash \Delta'}\,TL}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut} \quad \rightsquigarrow \quad \dfrac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{Cut}$$

But the component relation is no longer well-founded,
due to sentences like λ and κ_A.

The G-B procedure works, but does not terminate.
The overarching structure is broken.
(See Tennant 1982, Hallnäs & Schroeder-Heister 1991.)

Tolerance rule:

There is no way to push cuts on principal Pu back.

$$\text{Cut: } \dfrac{\text{Tol: } \dfrac{\Gamma \vdash \Delta, t \sim_P u}{Pt, \Gamma \vdash \Delta, Pu} \qquad \text{Tol: } \dfrac{\Gamma' \vdash \Delta', u \sim_P v}{Pu, \Gamma' \vdash \Delta', Pv}}{Pt, \Gamma, \Gamma' \vdash \Delta, \Delta', Pv} \quad \rightsquigarrow \quad ???$$

The G-B procedure grinds to a halt immediately.

Key Move 4 is broken.

Argument sketch:

- The G-B proof explains why cut is admissible, when it is.
- Thus, the failure of admissibility due to truth rules
  is for a different reason than the failure due to the tolerance rule.
- These failures of admissibility are key to the paradoxical conclusions.
- So the paradoxical conclusions follow for different reasons.

Summary:

- The question about the unity of soritical and semantic paradoxes is an explanatory question.
- Inclosure fails to explain what is going on with the liar, since whatever is going on with the liar, it's the same with curry.
- The G-B proof of cut elimination gives a different perspective.
- The liar and curry go together; the sorites is something different.