‘Transitivity’

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The lay of the land

A catalog of ‘transitivities’

What implies what

The lay of the land

Transitivity

A binary relation R on a set S is transitive iff:
for every $x, y, z \in S$, if xRy and yRz then xRz.

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for every $x, y, z \in S$, if xRy and yRz then xRz.

Also:

If R is not a binary relation on a set,
it's not transitive.

A consequence relation is some kind of relation.

Many usual suspects are not binary relations on any set at all:
they relate collections of premises to single conclusions.

That is, they are relations between $\mathcal{C}$ and $\mathcal{F}$,
where $\mathcal{C}$ is the set of collections of formulas
and $\mathcal{F}$ is the set of formulas.

For transitivity to even be an option,
premises and conclusions must be treated the same.

Either we restrict them both to be single formulas,
or we allow (the same kind of) collections for both.

I'll consider only one option:
sets of premises and of conclusions.

Even here, transitivity is not usual.

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sets of premises and of conclusions.

Even here, transitivity is not usual.

In a wide variety of systems:

$A \vee B \vdash A, B$
$A, B \vdash A \wedge B$
$A \vee B \nvdash A \wedge B$

The lay of the land

'Transitivity'

Usual consequence relations, then, aren't transitive.

But it is usual to see consequence relations defined
as relations that are 'reflexive', 'monotonic', and 'transitive'.

What, then, does 'transitive' mean here?

Some common conditions for single-conclusion:

- If $\Gamma \vdash A$ and $A, \Gamma \vdash B$, then $\Gamma \vdash B$
- If $\Gamma \vdash A$ and $A, \Gamma' \vdash B$, then $\Gamma, \Gamma' \vdash B$
- If $\Gamma \vdash A$ for all $A \in \Theta$ and $\Theta, \Gamma' \vdash B$, then $\Gamma, \Gamma' \vdash B$

For sets of conclusions:

- If $\Gamma \vdash \Delta, A$ and $A, \Gamma \vdash \Delta$, then $\Gamma \vdash \Delta$
- If $\Gamma \vdash \Delta, A$ and $A, \Gamma' \vdash \Delta'$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$
- If $\Gamma, \Sigma^+ \vdash \Sigma^-, \Delta$ for every partition $\langle \Sigma^+, \Sigma^- \rangle$ of Σ, then $\Gamma \vdash \Delta$

The lay of the land

The setting

I assume an infinite set $\mathcal{F}$ of formulas.

Here, a consequence relation is a binary relation $\vdash$ on $\wp(\mathcal{F})$ that is

Monotonic:

If $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$

The use of sets imposes other structural properties automatically: contraction, expansion, exchange, associativity.

‘Reflexive’?

$A \vdash A$

‘Reflexivity’ turns out not to matter one way or the other.

Everything to follow holds either with it or without it.

Compact?

If $\Gamma \vdash \Delta$, then
there are finite $\Gamma_{\text{fin}} \subseteq \Gamma$ and $\Delta_{\text{fin}} \subseteq \Delta$ such that
$\Gamma_{\text{fin}} \vdash \Delta_{\text{fin}}$

Makes a difference;
I'll consider the situation both with and without
assuming compactness.

A catalog of ‘transitivities’

1024 properties

Name:	If	and	then
S	$C \vdash A$	$A \vdash D$	$C \vdash D$
KS	$\Gamma \vdash A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$
/F	$\Gamma \vdash A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
F/	$\Gamma \vdash \Delta, A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$
FG	$\Gamma \vdash \Delta, A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
/C	$\Gamma \vdash A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
C/	$\Gamma \vdash \Delta, \Sigma$	$A \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
/C$^+$	$\Gamma \vdash \Delta, A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
C$^+$/	$\Gamma \vdash \Delta, \Sigma$	$A, \Gamma \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
CG	$\Sigma^+, \Gamma \vdash \Delta, \Sigma^-$ for all partitions $\langle \Sigma^+, \Sigma^- \rangle$ of Σ		$\Gamma \vdash \Delta$

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C/	$\Gamma \vdash \Delta, \Sigma$	$A \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
$/\mathrm{C}^+$	$\Gamma \vdash \Delta, A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
$\mathrm{C}^+/$	$\Gamma \vdash \Delta, \Sigma$	$A, \Gamma \vdash \Delta$ for all $A \in \Sigma$	$\Gamma \vdash \Delta$
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That’s 10 properties.

I’ll consider arbitrary conjunctions of these.

This gives $2^{10} = 1024$ potential properties.

Fortunately, it’s not so many in the end!

A catalog of ‘transitivities’

Finite and complete

Compare:

	If	and	then
/F	$\Gamma \vdash A$	$A, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$
/C	$\Gamma \vdash A$ for all $A \in \Sigma$	$\Sigma, \Gamma \vdash \Delta$	$\Gamma \vdash \Delta$

/F implies all instances of /C with finite Σ.
This is why ‘finite’ and ‘complete’.

The same goes for:

- F/ and C/
- FG and /C$^+$
- FG and CG
- FG and C$^+$/

What implies what

Without compactness

One way for a property to imply another is for all instances of the second to be instances of the first.

KS implies S:

	If	and	then
S	$C \vdash A$	$A \vdash D$	$C \vdash D$
KS	$\Gamma \vdash A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$

In this way:

- /C implies /F
- C/ implies F/
- /C$^+$ implies FG
- C$^+$/ implies FG

Other implications depend on monotonicity.

F/ implies KS:

	If	and	then
KS	$\Gamma \vdash A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$
F/	$\Gamma \vdash \Delta, A$	$A \vdash \Delta$	$\Gamma \vdash \Delta$

In this way:

- /F implies KS
- FG implies /F
- FG implies F/
- /C$^+$ implies /C
- C$^+$/ implies C/
- CG implies /C$^+$
- CG implies C$^+$/

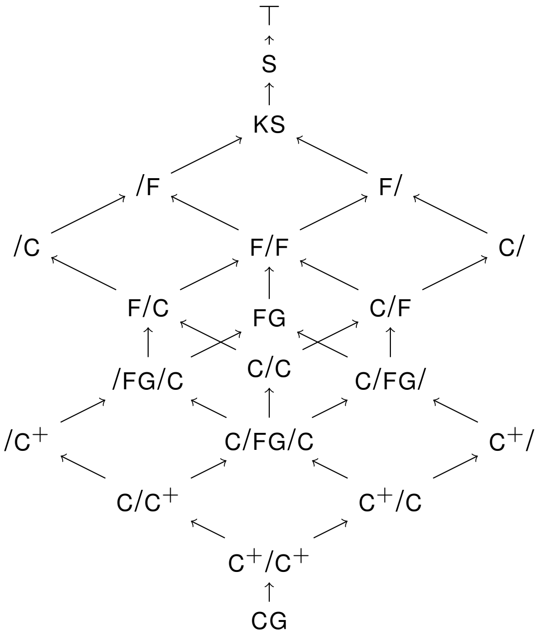
Other implications follow by semilattice properties of implication and conjunction.

- If P implies Q and Q implies R, then P implies R.
- If P implies both Q and R, it implies their conjunction.
- The conjunction of P and Q implies both P and Q.

These implications are enough to get our 1024 down to 21, and settle a number of implications among these.

In addition to the ten base properties,
the following eleven conjunctions remain:

Name:	Definition:	Name:	Definition:
$\top$	The empty conjunction.		
F/F	F/ and /F.	F/C	C/ and /F.
C/F	F/ and /C.	C/C	C/ and /C.
/FG/C	/C and FG.	C/FG/	C/ and FG.
C/FG/C	/C, C/, and FG.	C/C$^+$	/C$^+$ and C/.
C$^+$/C	C$^+$/ and /C.	C$^+$/C$^+$	/C$^+$ and C$^+$/.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F/F
C/
F/C
FG
C/F
/FG/C
C/C
C/FG/
/C+
C/FG/C
C+/
C/C+
C+/C
C+/C+
CG

What implies what

With compactness

If we assume compactness, we get a bunch more implications.

‘Finite’ properties, which already implied all finite instances of their ‘complete’ relatives, now fully imply them.

With compactness:

- /F implies /C
- F/ implies C/
- FG implies CG, and so everything

This brings 21 down to 7:

$$\begin{tikzcd}
 & \top & \\
 & \mathsf{S} \arrow[u] & \\
 & \mathsf{KS} \arrow[u] & \\
/\mathsf{F} = /\mathsf{C} \arrow[ur] & & \mathsf{F}/ = \mathsf{C}/ \arrow[ul] \\
 & \mathsf{F}/\mathsf{F} = \mathsf{C}/\mathsf{C} \arrow[ul] \arrow[ur] & \\
 & \mathsf{FG} = \mathsf{CG} \arrow[u] &
\end{tikzcd}$$

What implies what

Lattices

A folklore result is relevant.

In any lattice:

If $x \leq y$ and $x \wedge y \leq z$, then $x \leq z$.
(Think /F: if $\Gamma \vdash A$ and $\Gamma, A \vdash \Delta$, then $\Gamma \vdash \Delta$.)

If $x \leq y$ and $x \leq y \vee z$, then $x \leq z$.
(Think F/: if $A \vdash \Delta$ and $\Gamma \vdash A, \Delta$, then $\Gamma \vdash \Delta$.)

What about $\texttt{FG}$?

A lattice is such that
if $x \leq y \vee z$ and $x \wedge y \leq z$ then $x \leq z$
iff the lattice is distributive.

What implies what

Nonimplications

So there are at most 21 distinct properties,
or 7 with compactness required.

In fact, these counts are exact.

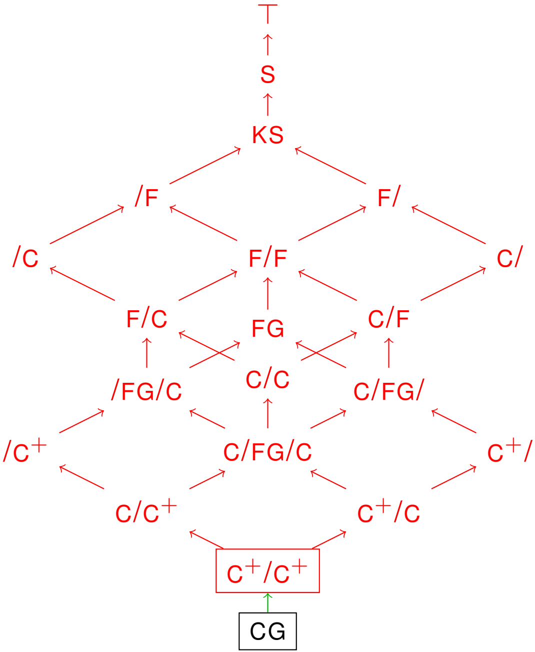
To be sure no implications are missing,
it takes ten consequence relations.

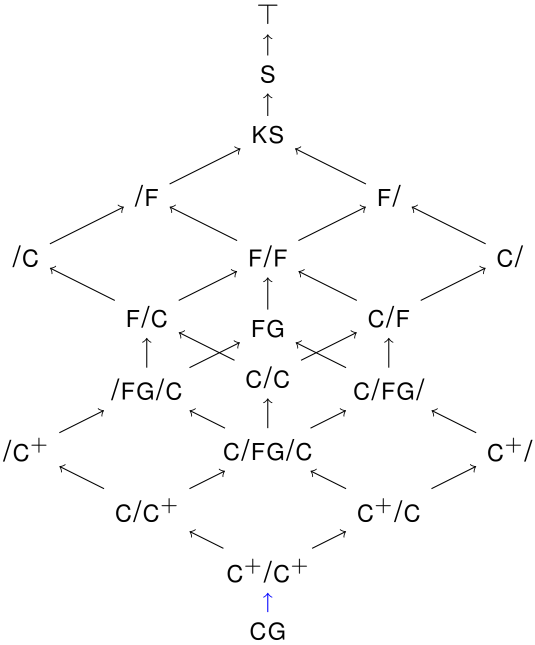
$$
\begin{tikzcd}
 & & \top & & \\
 & & \mathrm{S} \arrow[u] & & \\
 & & \mathrm{KS} \arrow[u] & & \\
 & /\mathrm{F} \arrow[ur] & & \mathrm{F}/ \arrow[ul] & \\
/\mathrm{C} \arrow[ur] & & \mathrm{F}/\mathrm{F} \arrow[ul] \arrow[ur] & & \mathrm{C}/ \arrow[ul] \\
 & \mathrm{F}/\mathrm{C} \arrow[ul] \arrow[ur] & \mathrm{FG} \arrow[u] & \mathrm{C}/\mathrm{F} \arrow[ul] \arrow[ur] & \\
 & /\mathrm{FG}/\mathrm{C} \arrow[u] \arrow[ur] & \mathrm{C}/\mathrm{C} \arrow[ul] \arrow[ur] & \mathrm{C}/\mathrm{FG}/ \arrow[u] \arrow[ul] & \\
/\mathrm{C}^+ \arrow[ur] & & \mathrm{C}/\mathrm{FG}/\mathrm{C} \arrow[ul] \arrow[u] \arrow[ur] & & \mathrm{C}^+/ \arrow[ul] \\
 & \mathrm{C}/\mathrm{C}^+ \arrow[ul] \arrow[ur] & & \mathrm{C}^+/\mathrm{C} \arrow[ul] \arrow[ur] & \\
 & & \mathrm{C}^+/\mathrm{C}^+ \arrow[ul] \arrow[ur] & & \\
 & & \mathrm{CG} \arrow[u] & &
\end{tikzcd}
$$

$\Gamma \vdash \Delta$ iff:
$(\Gamma \cap \Delta \neq \emptyset$ or)

$\Gamma \cup \Delta$ is infinite.

Not compact.
Has C$^+$/C$^+$.
Lacks CG.





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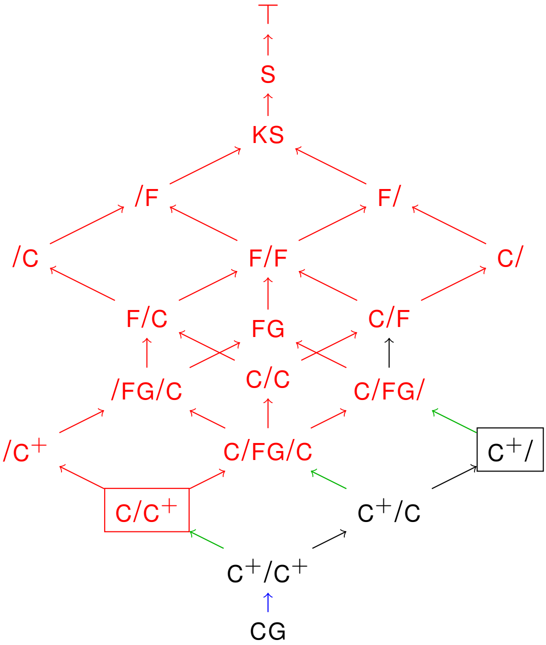
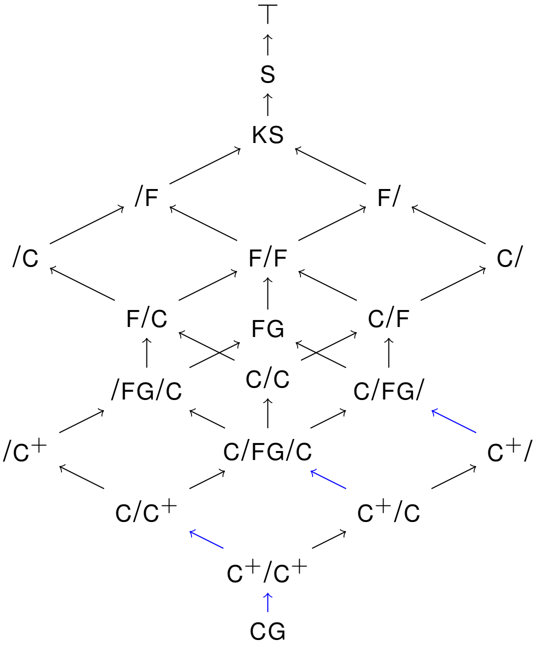
C/C$^+$
C$^+$/C
C$^+$/C$^+$
CG

$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

Δ is infinite or
$|\Gamma| \geq 2$.

Not compact.
Has c/c^+.
Lacks $c^+/$.

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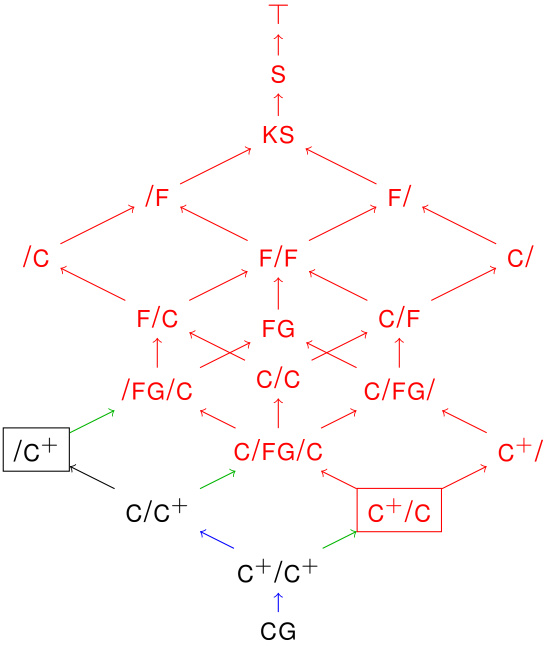
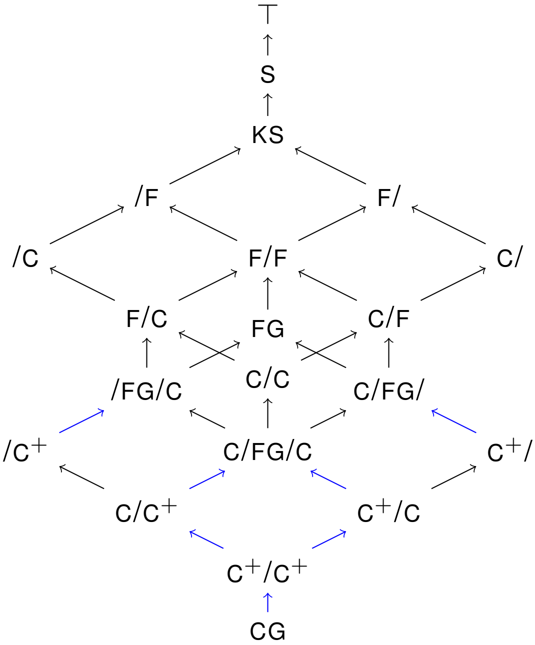
C+/C
C+/C+
CG

$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

Γ is infinite or
$|\Delta| \geq 2$.

Not compact.
Has c^+/c.
Lacks $/c^+$.

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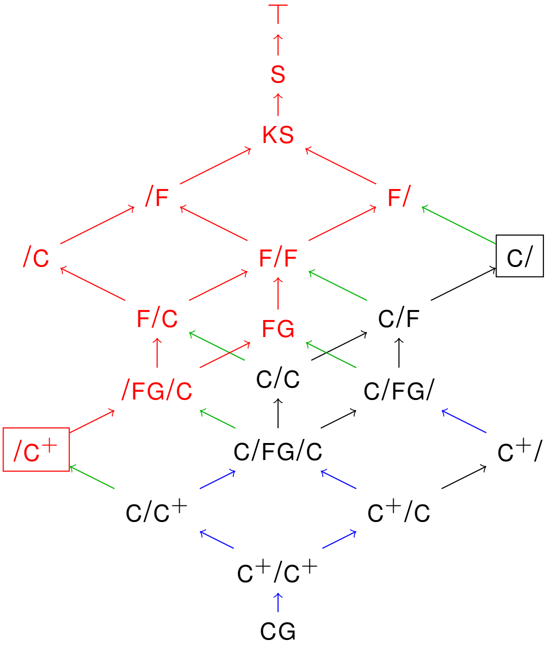
C/C+
C+/C
C+/C+
CG

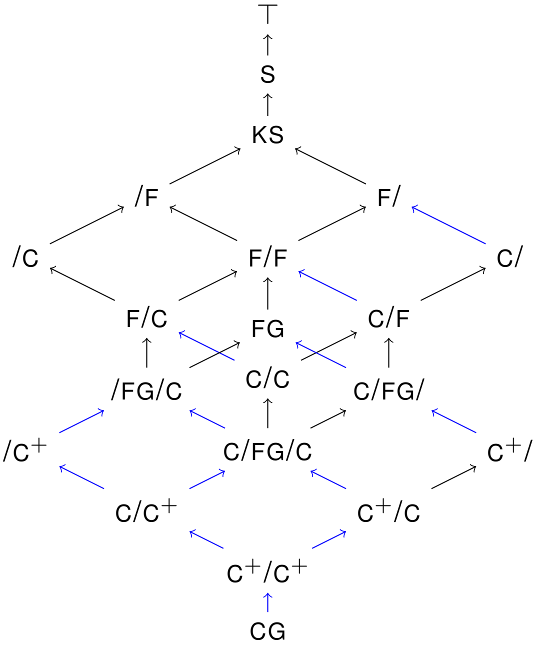
Choose some infinite Θ.

$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

$\Gamma \cap \Theta \neq \emptyset$ or
Δ is infinite.

Not compact.
Has $/\mathrm{C}^+$.
Lacks C/.

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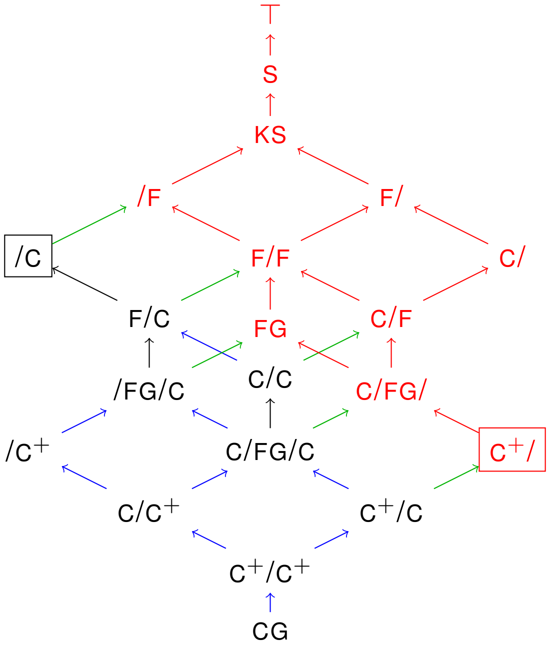

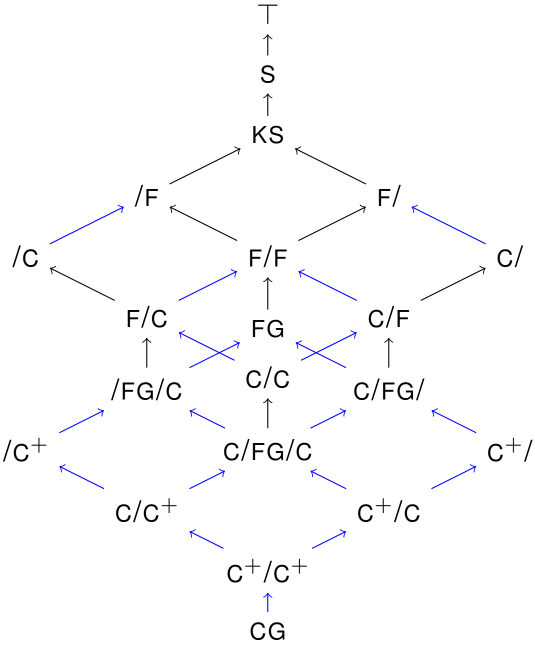

⊤
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$/C^+$
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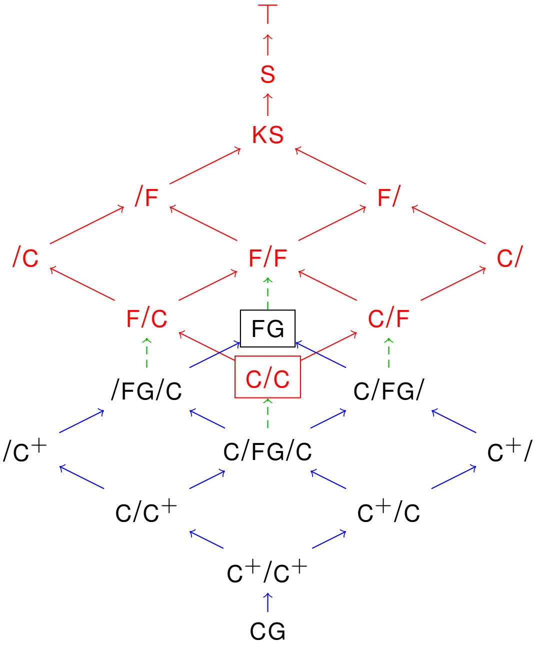
$C^+/$
C/C^+
C^+/C
C^+/C^+
CG

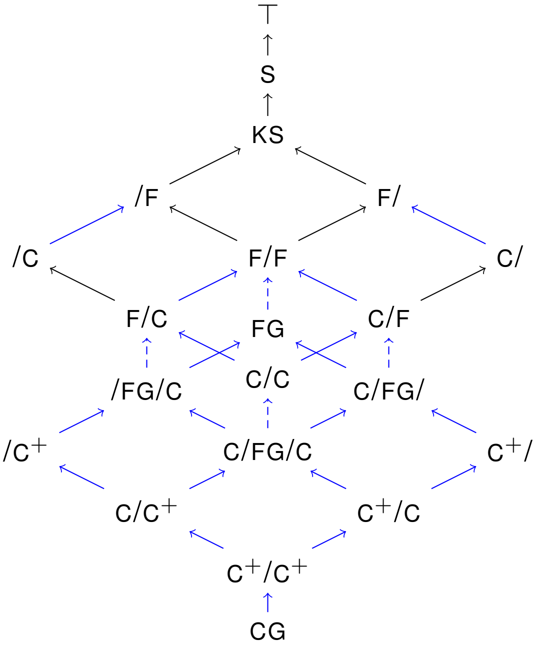
Choose distinct B, C, D.

$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

$B, C : D \sqsubseteq \Gamma : \Delta$ or
$C : D, B \sqsubseteq \Gamma : \Delta$.

Compact.
Has C/C.
Lacks FG.

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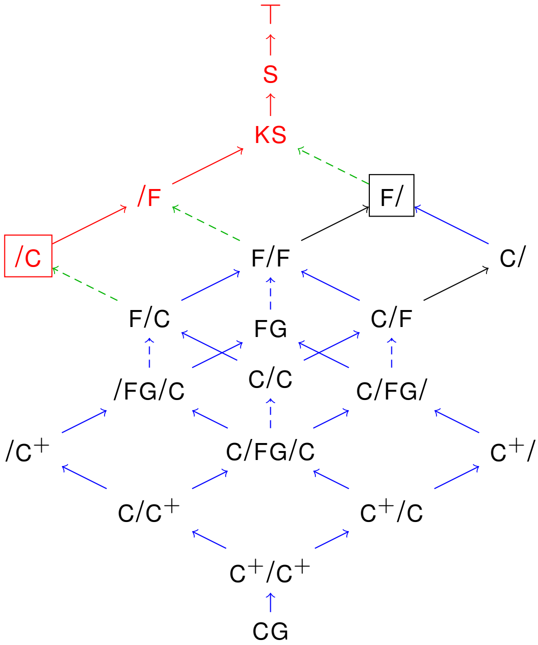
CG

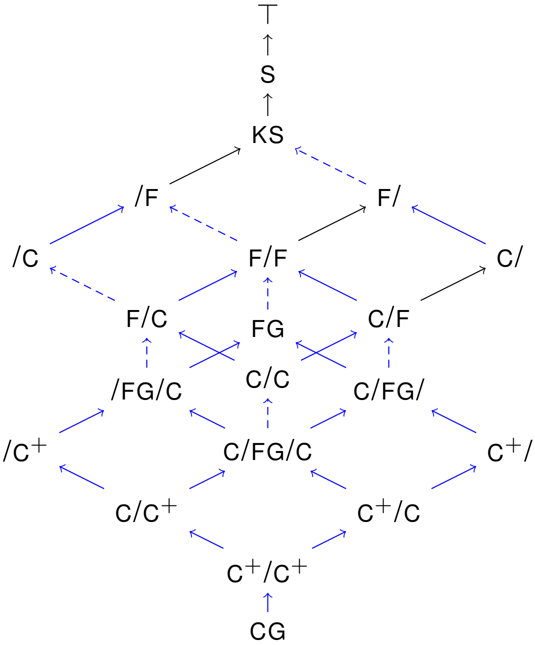
Choose distinct B–E.

$\Gamma \vdash \Delta$ iff:
$(\Gamma \cap \Delta \neq \emptyset$ or)

$B : C, D \sqsubseteq \Gamma : \Delta$ or
$E : B, C, D \sqsubseteq \Gamma : \Delta$.

Compact.
Has /C.
Lacks F/.

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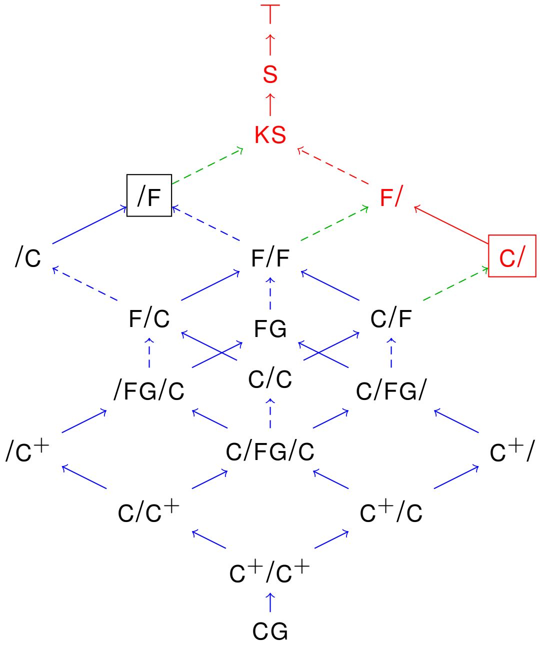
C⁺/C
C⁺/C⁺
CG

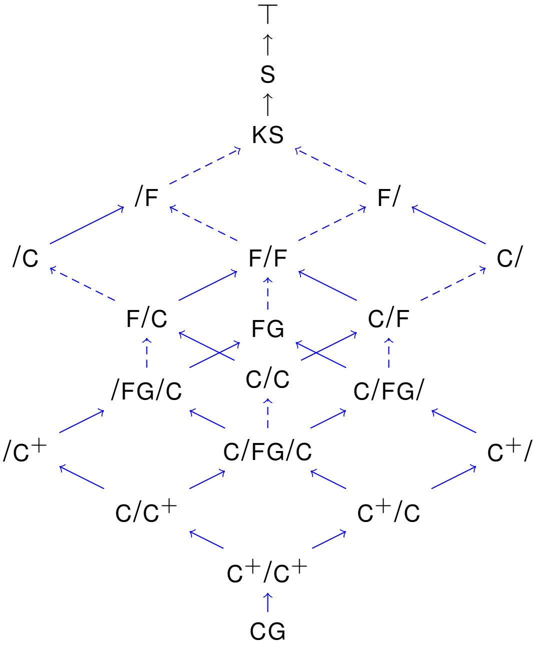
Choose distinct B–E.

$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

$C, D : B \sqsubseteq \Gamma : \Delta$ or
$B, C, D : E \sqsubseteq \Gamma : \Delta$.

Compact.
Has C/.
Lacks /F.





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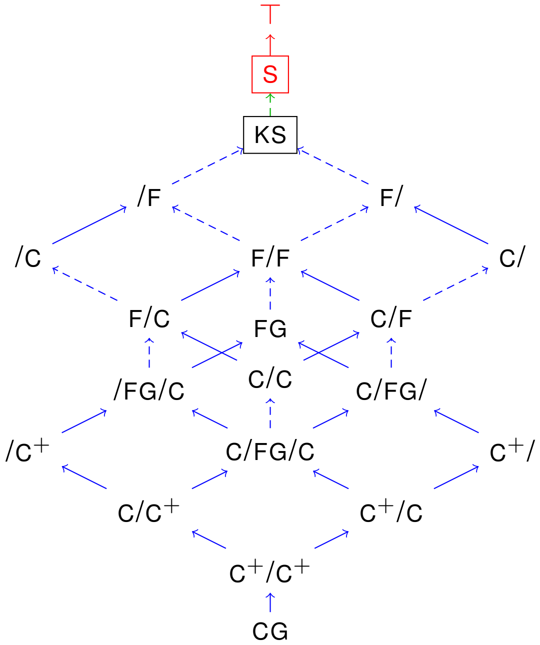
C+/C
C+/C+
CG

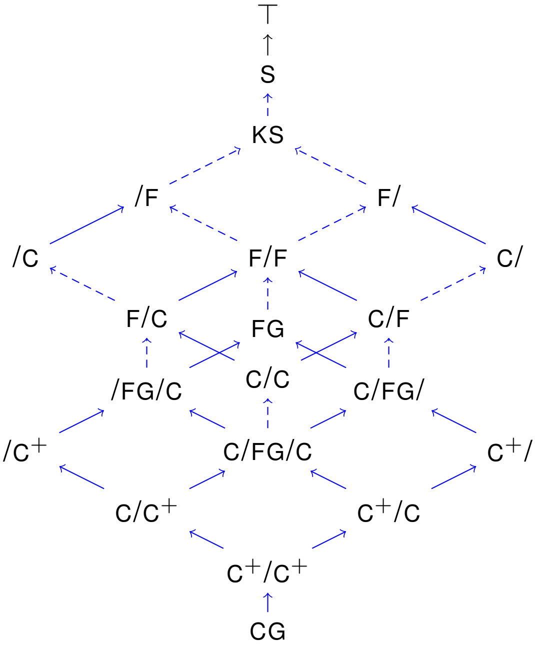
Choose some B.

$\Gamma \vdash \Delta$ iff:
$(\Gamma \cap \Delta \neq \emptyset$ or)

$\max(|\Gamma|, |\Delta|) \geq 2$ and
$B \in \Gamma \cup \Delta$.

Compact.
Has S.
Lacks KS.

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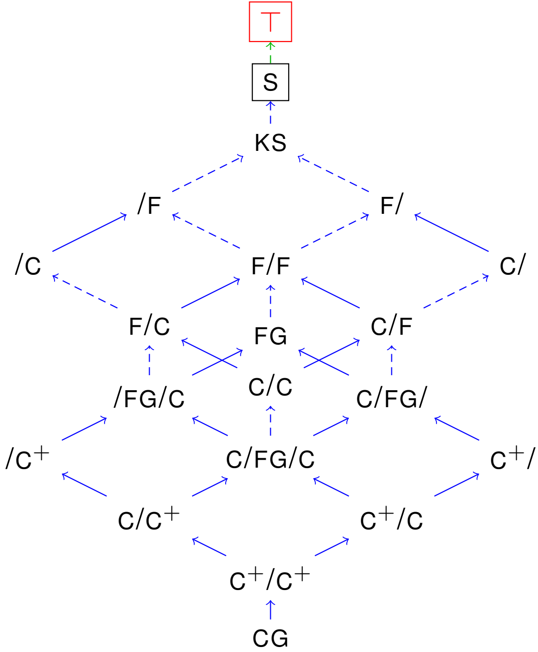
C+/C
C+/C+
CG

Choose distinct B, C, D.

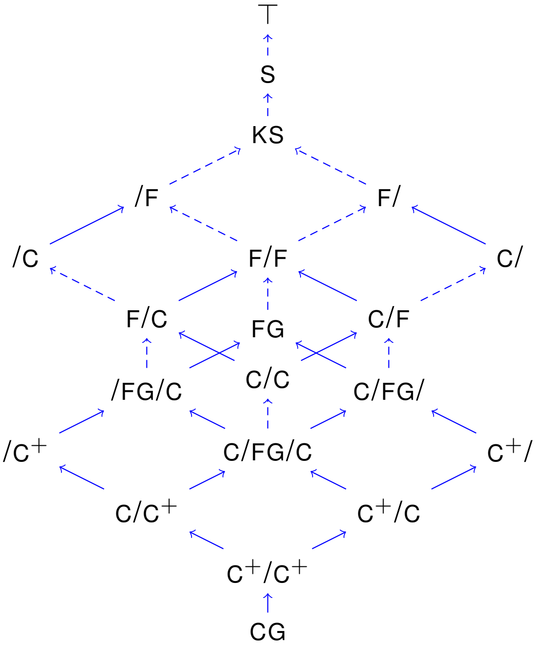
$\Gamma \vdash \Delta$ iff:
($\Gamma \cap \Delta \neq \emptyset$ or)

$B : C \sqsubseteq \Gamma : \Delta$ or
$C : D \sqsubseteq \Gamma : \Delta$.

Compact.
Has $\top$.
Lacks S.



There is a counterexample among these ten to every potential implication not already recorded.



So all 21 properties are distinct,
and the indicated implications are exhaustive.

Assuming compactness, all 7 properties are distinct,
and the indicated implications are exhaustive.

The moral:

Don't just say 'transitive'!
There's a ton of texture here.
(And none of it involves transitivity.)

And remember:

Things only get more complex
when we ditch other structural properties;
I've assumed just about everything.

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