$\phi, \psi, \theta, \ldots$	Formulas (atoms, $\to$, $\neg$)
$\Gamma, \Delta, \Sigma, \ldots$	Sets of formulas
$\mathfrak{C}, \mathfrak{D}, \ldots$	Either a formula or nothing
$\mathfrak{D} \leq \mathfrak{C}$	Either $\mathfrak{D}$ is nothing or $\mathfrak{D}$ is $\mathfrak{C}$

Conventions

Id: $\dfrac{}{\phi \succ \phi}$ $\quad$ $\to$L: $\dfrac{\Gamma \succ \phi \qquad \Delta, \psi \succ \mathfrak{C}}{\Gamma, \Delta, \phi \to \psi \succ \mathfrak{C}}$ $\quad$ $\neg$L: $\dfrac{\Gamma \succ \phi}{\Gamma, \neg\phi \succ}$

$\to$R: $\dfrac{\Gamma, \phi \succ \psi}{\Gamma \succ \phi \to \psi}$ $\quad$ $\to$R_: $\dfrac{\Gamma \succ \psi}{\Gamma \succ \phi \to \psi}$ $\quad$ $\to$R!: $\dfrac{\Gamma, \phi \succ}{\Gamma \succ \phi \to \psi}$ $\quad$ $\neg$R: $\dfrac{\Gamma, \phi \succ}{\Gamma \succ \neg\phi}$

Core logic (?)

Let $\Gamma \vdash \mathfrak{C}$ iff there is some derivable $\Gamma' \succ \mathfrak{C}$ with $\Gamma' \subseteq \Gamma$.

Then $\Gamma \vdash$ iff $\Gamma \vdash_{\text{Int}}$; and if $\Gamma \nvdash$, then $\Gamma \vdash \phi$ iff $\Gamma \vdash_{\text{Int}} \phi$.

Core logic is not closed under cut; $\neg\phi \vdash \phi \to \psi$ and $\phi \to \psi, \phi \vdash \psi$, but $\neg\phi, \phi \nvdash \psi$.

But if $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \mathfrak{C}$, then $\Gamma, \Delta \vdash \mathfrak{D}$, for some $\mathfrak{D} \leq \mathfrak{C}$.

Adding cut to the sequent system gives a system that derives all and only Int-valid sequents.

Some facts

Variables $x^\phi, y^\phi, \ldots$ for each formula ϕ.

$(M^{\phi\to\psi} N^\phi)^\psi$ $\qquad$ $(\lambda x^\phi . M^\psi)^{\phi\to\psi}$

$(\lambda\_.M^\psi)^{\phi\to\psi}$ $\qquad$ $(\lambda ! x^\phi . M^\perp)^{\phi\to\psi}$

$(M^{\neg\phi} \between N^\phi)^\perp$ $\qquad$ $(\mathrm{R} x^\phi . M^\perp)^{\neg\phi}$

Terms: formation

$((\lambda x^\phi . M^\psi)^{\phi\to\psi} N^\phi)^\psi$	$(M[x \mapsto N])^\psi$
$((\lambda\_.M^\psi)^{\phi\to\psi} N^\phi)^\psi$	M^ψ
$((\mathrm{R} x^\phi . M^\perp)^{\neg\phi} \between N^\phi)^\perp$	$(M[x \mapsto N])^\perp$
$((\lambda ! x^\phi . M^\perp)^{\phi\to\psi} N^\phi)^\psi$	$(M[x \mapsto N])^\perp$

This last kind is an *explosive redex*.

Terms: redexes and their reducts

A step of *gentle reduction* takes a term $O[R]$ with indicated nonexplosive redex R, and yields $O[R']$, with R' the reduct of R.

A step of *reduction* is either a step of gentle reduction, or else takes a term $O[R]$ with indicated explosive redex R, and yields R', with R' the reduct of R.

That's not a typo; reduction is not *compatible*, because explosive reduction discards context.

Terms: reduction

If $M^\perp$ reduces to N, then N is a refutation term.
If M^ϕ reduces to N^ψ, then the reduction is gentle.
If M^ϕ gently reduces to N, then N has type ϕ.

Handy lemmas

Reduction (and so gentle reduction) is *strongly normalising*:
every reduction sequence reaches a normal form,
and every gentle reduction sequence reaches a gentle normal form.

Reduction is *not* confluent; gentle reduction *is* confluent.

Theorems: strong normalisation and confluence

$$\Big((\lambda\_.w^{\delta})^{\psi\to\delta}\ \big((\lambda!x^{\phi}.(y^{\neg\phi}\mathbin{{}_{\circ}{}^{\circ}}x^{\phi})^{\perp})^{\phi\to\psi}\,z^{\phi}\big)^{\psi}\Big)^{\delta}$$

reduces to w^{δ} or $(y^{\neg\phi}\mathbin{{}_{\circ}{}^{\circ}}z^{\phi})^{\perp}$.

Nonconfluence type 1: _/!

$$\Big(\big((\lambda!x^{\phi}.(y^{\neg\phi}\mathbin{{}_{\circ}{}^{\circ}}x^{\phi})^{\perp})^{\phi\to\psi\to\theta}\,v^{\phi}\big)^{\psi\to\theta}\ \big((\lambda!s^{\delta}.(r^{\neg\delta}\mathbin{{}_{\circ}{}^{\circ}}s^{\delta})^{\perp})^{\delta\to\psi}\,q^{\delta}\big)^{\psi}\Big)^{\theta}$$

reduces to $(y^{\neg\phi}\mathbin{{}_{\circ}{}^{\circ}}v^{\phi})^{\perp}$ or $(r^{\neg\delta}\mathbin{{}_{\circ}{}^{\circ}}q^{\delta})^{\perp}$.

Nonconfluence type 2: !/!

A term is *forking* iff it contains either a $\lambda\_$ redex with an explosive redex in its argument,
or two nonoverlapping explosive redexes.

A term is *hereditarily nonforking* iff it does not reduce to a forking term.

Reduction is confluent on hereditarily nonforking terms.

Forking

Every term has a unique gentle normal form.
Not every term has a unique normal form, but hereditarily nonforking terms do.
Every term has at most one typed normal form.

Corollaries

Each derivation of $\Gamma \succ \mathfrak{C}$ determines a term of type $\mathfrak{C}$ with free variables of types Γ.
Each term of type $\mathfrak{C}$ with free variables of types Γ determines a derivation of $\Gamma \succ \mathfrak{C}$ with only principal cuts.

A term is in normal form iff its associated derivation is cutfree.
A derivation has no nonidentity cuts iff its associated term is in normal form.

Term $\mapsto$ derivation $\mapsto$ term is the identity.
Derivation $\mapsto$ term $\mapsto$ derivation removes identity cuts, and pushes other cuts up to their principal steps.

Correspondence