Single Value Decomposition of a Mattrix

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1 Compute the $\mathbf{A^T}$ and $\mathbf{A^TA}$ for the given matrix

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$\mathbf{A^T} = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

2 Find the eigenvalues of $\mathbf{A^TA}$, sort them in descending order

$$\mathbf{A^TA} = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} = (25-\lambda)^2 - 15^2 = (25-\lambda-15)(25-\lambda+15)$$

$$= (10-\lambda)(40-\lambda) = 0$$
$$\lambda_1 = 40, \lambda_2 = 10$$

3 Construct Σ as a diagonal matrix of the singular values = square roots of the eigenvalues of $\mathbf{A^TA}$ and find Σ^{-1}

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

Since Σ is diagonal, its inverse is simply the reciprocal of the diagonal elements:

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sqrt{40}} & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

4 Constrruct V with columns the unit eigenvectors of $\mathbf{A^TA}$

$$\boldsymbol{\lambda = 40}:$$

$$(\mathbf{A} - 40\mathbf{I})\vec{X} = \begin{bmatrix} (25-40) & -15 \\ -15 & (25-40) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

Solving the system of equations:

$$\begin{cases} -15x_1 - 15x_2 & = 0 \\ -15x_1 - 15x_2 & = 0 \end{cases} \tag{1}$$

which results in $x_2 = -x_1$. Thus, for any real number x_1, a vector of the form $\vec{X} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ is an eigenvector of $\mathbf{A^T A}$ corresponding to $\lambda = 40$. To make it an unit length divide by its magnitude

$$\|X\| = \sqrt{x_1^2 + x_2^2} = \sqrt{2x_1^2} = \sqrt{2}x_1$$

Hence, $\hat{X} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is the unit eigenvector coresponding to $\lambda = 40$

$$\boldsymbol{\lambda = 40}:$$

$$(\mathbf{A} - 10\mathbf{I})\vec{Y} = \begin{bmatrix} (25-10) & -15 \\ -15 & (25-10) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{0}$$

Solving the system of equations:

$$\begin{cases} 15y_1 - 15y_2 &= 0 \\ -15y_1 + \quad 15y_2 &= 0 \end{cases} \tag{2}$$

which results in $y_2 = -y_1$. Thus, for any real number y_1, a vector of the form $\vec{X} = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix}$ is an eigenvector of $\mathbf{A^T A}$ corresponding to $\lambda = 10$. To make it an unit length divide by its magnitude

$$\|Y\| = \sqrt{y_1^2 + y_2^2} = \sqrt{2y_1^2} = \sqrt{2}y_1$$

Hence, $\hat{Y} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ is the unit eigenvector coresponding to $\lambda = 10$

Thus the matrix $\mathbf{V} = \left[\hat{X}, \hat{Y}\right]$ is given by:

$$\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and its transpose is:

$$\mathbf{V^T} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

5 Construct $\mathbf{U} = \mathbf{AV\Sigma^{-1}}$

$$\mathbf{U} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{10} \\ -\frac{\sqrt{5}}{20} & \frac{\sqrt{5}}{10} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix} \approx \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}$$

6 Check if we get the matrix A from the matrix multiplication

$$\mathbf{U} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 6.3246 & 0 \\ 0 & 3.1623 \end{bmatrix} \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 4.4721 & -4.4721 \\ 2.2361 & 2.2361 \end{bmatrix}$$

$$= \begin{bmatrix} 3.9998 & 0.00004 \\ 2.9999 & -4.9998 \end{bmatrix} \approx \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \mathbf{A}$$