Milestone 1: Eigenvalues and Eigenvectors of a Matrix

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1 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$$

2 Find the determinant of $\det(\mathbf{A} - \lambda\mathbf{I})$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{vmatrix}$$

$$= (8-\lambda)(7-\lambda) - 6$$

$$= \lambda^2 - 15\lambda + 50$$

3 Solve for the eigenvalues

$$\lambda^2 - 15\lambda + 50 = 0$$

$$\lambda = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(1)(50)}}{2(1)}$$

$$= \frac{15 \pm \sqrt{225 - 200}}{2}$$

$$= \frac{15 \pm \sqrt{25}}{2}$$

$$= \frac{15 \pm 5}{2}$$

$$= 10, 5$$

4 Solve for the eigenvectors

For $\boldsymbol{\lambda = 10}$:

$$(\mathbf{A} - 10\mathbf{I})\vec{x} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

Solving the system of equations:

$$\begin{cases} -2x_1 + 3x_2 &= 0 \\ 2x_1 - 3x_2 &= 0 \end{cases} \tag{1}$$

gives x_1 being any real number and $x_2 = \frac{2}{3}x_1$. Thus any vector of the form $\begin{bmatrix} x_1 \\ \frac{2}{3}x_1 \end{bmatrix}$ is an eigenvector for $\lambda = 10$. A specific example is $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

For $\boldsymbol{\lambda = 5}$:

$$(\mathbf{A} - 5\mathbf{I})\vec{y} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{0}$$

Solving the system of equations:

$$\begin{cases} 3y_1 + 3y_2 = 0 \\ 2y_1 + 2y_2 = 0 \end{cases}$$

gives y_1 being any real number and $y_2 = -y_1$. Thus, any vector of the form $\begin{bmatrix} y_1 \\ -y_1 \end{bmatrix}$ where y_1 is a any real number, is an eigenvector for $\lambda = 5$.

A specific example is

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$