Question 4 - Homework 1 - Sets, Series and Sequences

Let A $= \{\frac{n}{m} : n \in \mathbb{N}, m \in \mathbb{N}, m > n\}$.

(a) Prove that inf $A = 0$

Solution:

To show that 0 is a infimum of A, we must prove the following:
(i) That 0 is a lower bound in A such that $\forall x \in A, x \geqslant 0$.
(ii) That for all b, with $b > 0$, b is not a lower bound such that $\forall b, b > 0, \exists x \in A, b > x$.

So, for (i):

Take $n = 1$ and $m \to \infty$ since $m > n$, then, $\frac{n}{m} \to 0$ which is the lowest possible value for A.
Taking the maximum value of as $n \to \infty$, and $m = n + 1$, $\frac{m}{n} \to 1$.
So, A takes the values $(0, 1)$. Hence, 0 is always less than these values so condition (i) is satisfied and 0 is a lower bound.

For condition (ii):
Let $\epsilon > 0$ be given. By the Archimedean Property, we can find an $n \in \mathbb{N}$, such that $\frac{1}{n} < \epsilon$. As $m > n$, for all $n > 0$, we find that:

$$\frac{1}{m} < \frac{1}{n} < \epsilon \implies \frac{1}{m} < \epsilon$$

We know that when $n = 1$, $\frac{n}{m} = \frac{1}{m}$, so $\frac{1}{m} \in A$. So,

$$0 + \epsilon < x, \ \text{ for } \ x \in A$$

Thus, inf $A = 0$ as both condition (i) and (ii) are satisfied.

(b) Sup $A = 1$

Solution:

To show that 1 is a supremum of A, we must prove the following:
(i) That 1 is an upper bound in A such that $\forall x \in A, x \leqslant 1$.
(ii) That for all b, with $b < 1$, b is not an upper bound such that $\forall b, b < 1, \exists x \in A, b < x$.
So, for (i):
Take $n = 1$ and $m \to \infty$ since $m > n$, then, $\frac{n}{m} \to 0$ which is the lowest possible value for A.
Taking the maximum value of as $n \to \infty$, and $m = n + 1$, $\frac{m}{n} \to 1$
So, we can conclude that A takes the values $(0, 1)$. Hence, 1 is always greater than these values, so condition (i) is satisfied and 1 is an upper bound.

For condition (ii):

Let $\epsilon > 0$ be given. By the Archimedean Property, we can find an $n \in \mathbb{N}$, such that $\frac{1}{n} < \epsilon$. As $m > n$, for all $n > 0$, we find that:

$$\frac{1}{m} < \frac{1}{n} < \epsilon \implies -\epsilon < -\frac{1}{m}$$

Adding 1 to both sides, we get:

$$1 - \epsilon < 1 - \frac{1}{m}$$

Then, when $m = n + 1$ because $m > n$, we get $\frac{n}{m} = 1 - \frac{1}{m}$, so $1 - \frac{1}{m} \in A$. Then,

$$1 - \epsilon < x \text{ for } x \in A$$

Thus, $\sup A = 1$ as both condition (i) and (ii) are satisfied.