Question 5 - Homework 6 - Number Systems

Show that if p and q are odd integers, then the equation, $x^2 + px + q = 0$ has no rational solutions.

Let us assume, for a contradiction, that it *has* rational solutions:

$$\begin{aligned}
&\implies x^2 + px + q = 0 \\
&\implies (x + \frac{a}{b})(x + \frac{c}{d}) = 0 \\
&\implies x = -\frac{a}{b} \; and \; x = -\frac{c}{d}
\end{aligned}$$

We must now note the behaviour of the quadratic under the following two cases:

$$\begin{aligned}
&(i) \; x = -\frac{a}{b} \\
&(ii) \; x = -\frac{c}{d}
\end{aligned}$$

For case (i)*:*

$$\begin{aligned}
&\implies x^2 + px + q = 0 \\
&\implies (-\frac{a}{b})^2 + p(-\frac{a}{b}) + q = 0 \\
&\implies \frac{a^2}{b^2} - \frac{pa}{b} + q = 0 \\
&\implies a^2 - pab + qb^2 = 0 \qquad (multiply \; by \; b^2)
\end{aligned}$$

We note that since $\frac{a}{b}$ is in its lowest terms (where both a and b aren't even), we have three different cases for the possible values of $\frac{a}{b}$:

1. a can be odd and b can be even (with p and q always odd)
2. a can be even and b can be odd (with p and q always odd)
3. a and b can both be odd (with p and q always odd)

For case (1):

Since a is odd then a^2 will also be odd. With a, p and q belong to odd integers and b being even, apb and qb^2 will be even. Therefore, $x^2 + px + q$ (odd - even +even) will be odd.

For case (2):

With a belonging to the even integers and b, p, q belonging to the odd integers, we can conclude that a^2 and apb will be even and qb^2 will be odd. Therefore, $x^2 + px + q$ (even - even + odd) will be odd.

For case (3):

With a, b, p, q all belonging to the odd integers, we can conclude that a^2, apb and qb^2 will be odd. Therefore, $x^2 + px + q$ (odd - odd +odd) will be odd.

For all of the above, possible values for $\frac{a}{b}$ we see that that quadratic when substituted with $x = \frac{a}{b}$ is always odd. 0 does not belong to the family of odd integers, therefore, we have a contradiction.

Overall, we can conclude for case (i), because of our proof by contradiction, $x^2 + px + q = 0$ has no rational solutions for $x = \frac{a}{b}$.

Similarly, we must look at case (ii):

For case (ii):

$$\begin{aligned}
&\implies x^2 + px + q = 0 \\
&\implies (-\frac{c}{d})^2 + p(-\frac{c}{d}) + q = 0 \\
&\implies \frac{c^2}{d^2} - \frac{pc}{d} + q = 0 \\
&\implies c^2 - pcd + qd^2 = 0 \qquad (\textit{multiply by } d^2)
\end{aligned}$$

Like earlier, we note that since $\frac{c}{d}$ is in its lowest terms (where both c and d aren't even), we have three different cases for the possible values of $\frac{c}{d}$:

1. c can be odd and d can be even (with p and q always odd)
2. c can be even and d can be odd (with p and q always odd)
3. c and d can both be odd (with p and q always odd)

For case (1):

Since c is odd then c^2 will also be odd. With d belonging to even integers and p and q belonging to odd integers, apb and qb^2 will be even. Therefore, $x^2 + px + q$ (odd - even + even) will be odd.

For case (2):

With c belonging to the even integers and d, p, q belonging to the odd integers, we can conclude that c^2 and cpd will be even and qd^2 will be odd. Therefore, $x^2 + px + q$ (even - even + odd) will be odd.

For case (3):

With c, d, p, q all belonging to the odd integers, we can conclude that c^2, cpd and qd^2 will be odd. Therefore, $x^2 + px + q$ (odd - odd +odd) will be odd.

For all of the above, possible values for $\frac{c}{d}$ we see that that quadratic when substituted with $x = \frac{c}{d}$ is always odd. 0 does not belong to the family of odd integers, therefore, we have a contradiction.

Overall, we can conclude for case (i), because of our proof by contradiction, $x^2 + px + q = 0$ has no rational solutions for $x = \frac{c}{d}$.

Through proof by exhaustion and proof by contradiction, we can conclude that $x^2 + px + q = 0$ has no rational solutions.