UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Mechanical &
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MTE 203 - Advanced Calculus Fall 2021

MATLAB DESIGN PROJECT

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Summary of Analysis

Part I

Question a

BB-8 was plotted on MATLAB. The result can be seen in the following figure labelled Figure 1 for the first case and the second case, respectively. The MATLAB code is included in Appendix B and labelled as "MATLAB Code Question 1A".

To achieve this, the sphere for the body is created by using MATLAB 'sphere' to generate the coordinates. Then these coordinates are multiplied by the radius and generated three-dimensionally using surf.

The neck is created by creating a set of coordinates between the smaller circle of the neck and the larger circle of the neck then stretching this horizontally to give it a height. This is done by using 'linspace' and 'meshgrid' and then using 'mesh' to generate the three-dimensional shape.

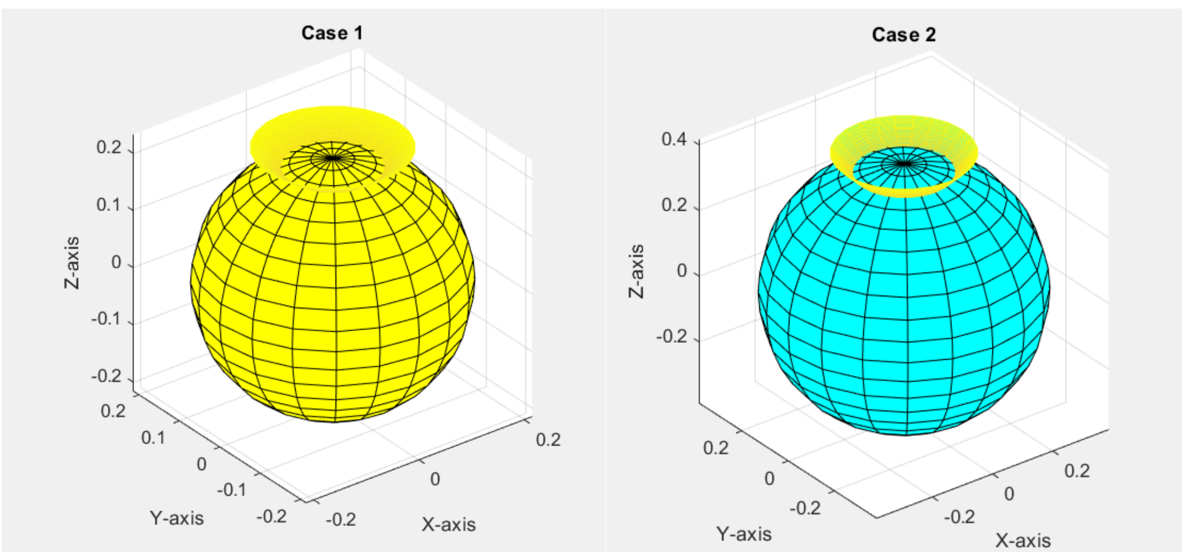


Figure 1: MATLAB Plot

Question b

The second case should provide more stability for the head. Even though the second case is larger overall, making the head in a higher spot than the first case, the second case would also have an increase in mass. Despite both the head and the body experiencing an increase in mass, the increase in mass in the body seems larger, which would bring the center of gravity down, therefore increasing the stability for the head.

Question c

The following are the triple integral equations for the volume of the socket, the head, the neck, and the body. 'r' is the radius of the body, 'd' is the depth of the socket, 'Φ' is the angle of the neck with the x-axis, 'h' is the radius of the head, and 't' is the thickness of the body wall. S_d

$$V_{socket} = \int_0^{2\pi} \int_0^{\sqrt{2rd-d^2}} \int_{r-d}^{\sqrt{r^2-R^2}} RdzdRd\theta$$

$$V_{neck} = \int_0^{2\pi} \int_0^{(h-\sqrt{2rd-d^2})\tan\emptyset} \int_0^{\sqrt{2rd-d^2}+z/tan\emptyset} RdzdRd\theta$$

$$V_{body} = \int_0^{2\pi} \int_0^{\pi} \int_{-\sqrt{h^2-R^2}}^{\sqrt{h^2-R^2}} \rho^2 sin\emptyset d\rho d\emptyset d\theta$$

$$V_{head} = \int_{\pi}^{2\pi} \int_0^{h} \int_{-\sqrt{h^2-R^2}}^{\sqrt{h^2-R^2}} RdzdRd\theta$$

$$V_{total} = V_{body} + V_{neck} + V_{head} - V_{socket}$$

To find the volume of the neck, an equation for the changing radius is needed. This is found using trigonometry. Refer to Figure 4 under Appendix A for the trigonometry used to find the equation.

See Figure 5 and Figure 6 in Appendix A for the derivation of the socket and neck volume equations.

Question d

The following Table 1 shows the volumes of both cases. The values were solved in MATLAB by layering three 'int' functions to address the triple integral. The code is attached in Appendix B as "MATLAB Code Question 1D".

Table 1: Volume Values

Volume [m^3]	Case 1	Case 2
Body	0.01057	0.03535
Head	0.00409	0.01553
Neck	0.00111	0.0051
Socket	0. 00015	0.00074
Total	0.01562	0.05524

Problems Encountered

In the creation of the MATLAB code to create the 3D shape for the body and the neck, some problems were encountered while creating the neck. Due to the neck being a cone that is cut off and at a specific angle with respect to the x-y plane, some troubles were initially encountered to make its height, upper radius and lower radius correct. Eventually the problem was solved by creating two separate meshgrids for the upper and lower radius, and the height of the neck. This solution can be seen in the MATLAB code attached in Appendix B as “MATLAB Code Question 1A”.

Part II

Question a

Refer to Appendix A for Figure 7, Figure 8, and Figure 9 that illustrate the hand calculations used to find the expression for BB-8’s center of mass.

The y values are determined to be 0 without calculations because BB-8 is symmetrical along the y-axis.

The x and z center of mass values are calculated using the below formulas.

$$X_{Center\ of\ Mass} = \frac{Mass_{yz}}{Mass}$$

$$Z_{Center\ of\ Mass} = \frac{Mass_{xy}}{Mass}$$

Therefore, the $Mass_{yz}$ and $Mass_{xy}$ values had to be calculated for the head and the body.

When calculating mass, the triple integrals for the volume of the particular part is taken and multiplied by its respective density to obtain the mass.

In the case of the head, the original volume integral had to be changed slightly because the shape it carved out was in the wrong orientation and therefore, could not be used in conjunction with the density formula.

The Z center of mass for the head also had to be adjusted by adding the radius of the body, the height of the neck, and subtracting the socket height because the head is not located at the origin.

Question b

The equations found in Part II Question a are entered onto MATAB using layered ‘int’ and simple algebra functions to find the center of mass of each body part. Then the total center of mass is averaged using the following formula:

$$Center\ of\ Mass = \frac{CoM_{Head}M_{Head} + CoM_{Body}M_{Body} + CoM_{Weight}M_{Weight}}{M_{Head} + M_{Body} + M_{Weight}}$$

Refer to Table 2 below for the center of mass values with respect to the origin. This was calculated in MATLAB and the code is appended in Appendix B titled “MATLAB Code Question 2”.

Table 2: Center of Mass Values

Location from origin (meters)	Case 1	Case 2
X	-0.1817629	-0.2156015
Y	0	0
Z	0.2793535	0.4900664

Question c

The torque equations are shown in Figure 10 (under Appendix A).

The idea is to take the force in the x, y, z direction and multiplying by the orthogonal component of the center of the mass. In this case, there is only the friction and the normal force at play here. The friction force is multiplied by the Z-center of mass and the normal force is multiplied by the X-center of mass.

The resulting torque for case 1 is 63.4623606 N*m and case 2 is 63.5242443 N*m.

Question d

Horsepower = RPM * torque / 5252. Since the minimum RPM is 1100, then the minimum horsepower would be 9.8 for the first case and 10.8 for the second case. For these requirements, a successful motor is found and the linked in Appendix C under "Motor Link".

The horsepower of this motor is 20 HP and the RPM is 1800. Since horsepower = rpm * torque / 5252, the torque of this motor ends up being 58.4 ft*lbs, which translates to 79.18 N*m. Since the largest dimension of this motor is 642mm, it would not be able to fit inside case 1 however it will fit inside case 2 very well. The torque required for case 2 is 70 N*m, which this motor can supply.

Problems Encountered

When calculating the center of mass, double integrals were initially used instead of triple integrals. The line of thinking was that since the y-component is 0 for all masses, therefore, calculations in 2D is enough. However, since the density formula provided is in m^3, which is 3D, triple integrals had to be used to find the correct values.

Part III

Question a

The r̄ equation is found by taking into account the position of the asteroid and the spacecraft.

$$\bar{r} = (-x - 2500)\hat{\imath} + (-y - 1000)\hat{\jmath}$$

In order to find $\hat{r}$, the magnitude is also needed.

$$|\bar{r}| = \sqrt{(-x - 2500)^2 + (-y - 1000)^2}$$

Since $\hat{r}$ is r̄/|r̄|, the $\hat{r}$ components would be following:

$$\hat{\mathrm{r}}_x = \frac{-x - 2500}{\sqrt{(-x - 2500)^2 + (-y - 1000)^2}}\hat{\imath}$$

$$\hat{\mathrm{r}}_y = \frac{-y - 1500}{\sqrt{(-x - 2500)^2 + (-y - 1000)^2}}\hat{\jmath}$$

The force magnitude is found by using the (– $G\mathrm{m}/\mathrm{r}^\wedge 2$) equation. The negative sign of the equation is negated so that the vector field would point towards the asteroid and not away from it.

$$|F| = \frac{6.67e - 11 * 1.3e20}{(-x - 2500)^2 + (-y - 1000)^2}\hat{\jmath}$$

The formulas for $\hat{r}$ and the force magnitude are shown below in Table 3 as anonymous MATLAB functions.

Table 3: Part III formulas

$\hat{r}$ x-component	`rx = @(x, y) (-x-2500)./(sqrt((-x-2500).^2+(-y-1000).^2));`
$\hat{r}$ y-component	`ry = @(x, y) (-y-1000)./(sqrt((-x-2500).^2+(-y-1000).^2));`
Force magnitude (– $G\mathrm{m}/\mathrm{r}^\wedge 2$)	`f_mag = @(x, y) (g.*m)./((-x-2500).^2+(-y-1000).^2);`

Question b

Below is Figure 2, the plot of the gravitational field and the original path plotted in MATLAB. The MATLAB code is attached to Appendix B as “MATLAB Code Question 3”.

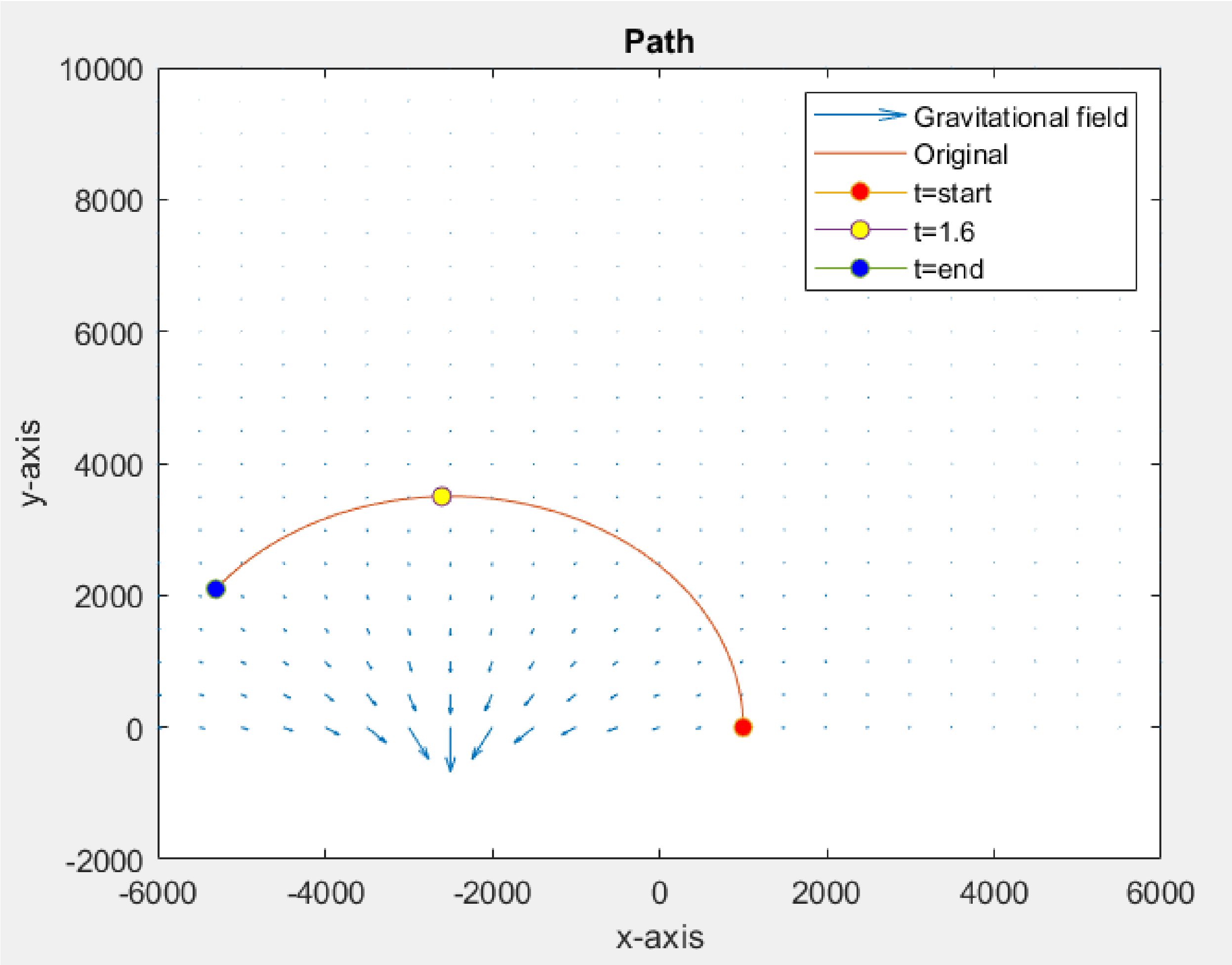


Figure 2: Vector Field and Original Path

This is achieved in MATLAB by using 'quiver' to plot the gravitational vector field. It took in the X and Y within the domain between -6000 to 10000 and the direction of the arrow determined by the force magnitude and the unit directional vector from the point to the asteroid. The path is entered as a parametric function and plotted using 'fplot'.

Question c

Work can be calculated using the following formula:

$$W = \int_{a}^{b} \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$$

As the path is given and the force can be found by using the anonymous functions in question a, work can be calculated once the starting and ending points are plugged into the integral.

In MATLAB, this formula is plugged in with the values to determine the work. The code can be found in Appendix B under "MATLAB Code Question 3".

In the interval between t = 0 to t = 2.5, the work done by the field is -305737.59771 J. In the interval between t = 1.6 to t = 2.5 the work done by the field is 149338.38938 J. Over the first interval, the Falcon's displacement is in the opposite direction of the force, which means the spaceship is losing energy. In the second interval, the Falcon's displacement is in the same direction as the force, which means the Falcon is gaining energy.

Question d

Below is Figure 3, the plot of the gravitational field with the original and new path plotted in MATLAB. The MATLAB code is attached to Appendix B as "MATLAB Code Question 3".

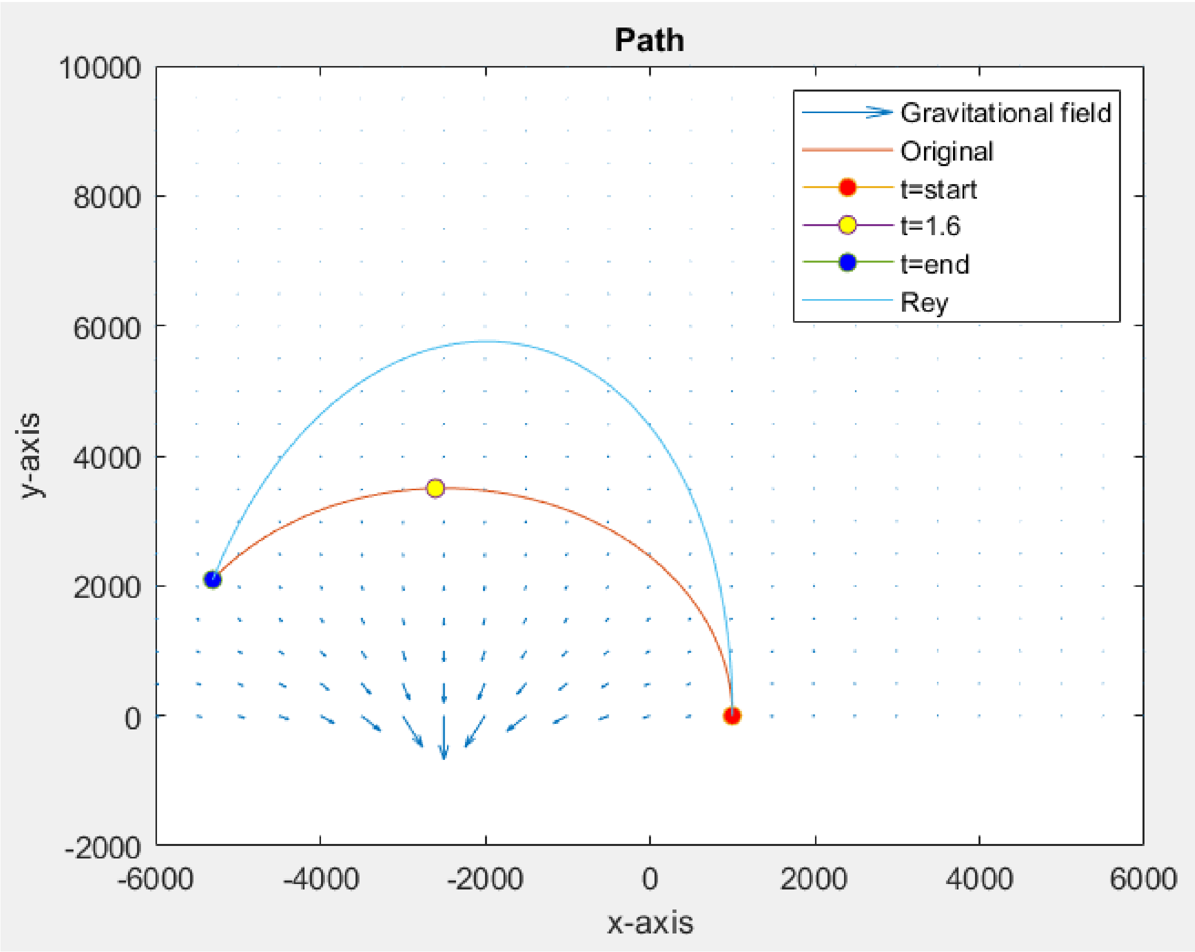


Figure 3: Vector Field with New and Old Path

The second path is plotted in the same way the first plot is. This is done by using 'fplot' on the parametric equation of the second path.

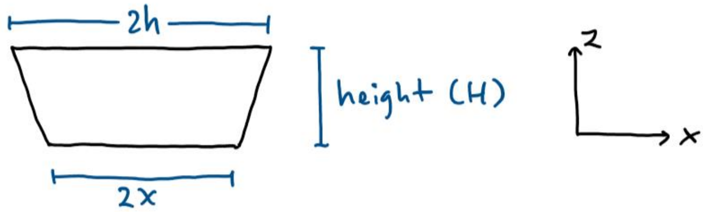
The work done in the new path is -305738.0 J. Which should be the same as the previous path (red path from t = 0 to t = 2.5). This is because the displacements of the two paths are the same, and since work is related to displacement, the work would also be the same. The very small difference between the two work values can be attributed to rounding errors.

Summary of Results

	Case 1	Case 2
R(m)	0.215	0.385
h(m)	0.125	0.195
Φ(°)	36	45
d(m)	0.015	0.025
Volume of the body [m^3]	0.01057	0.03535
Volume of the head [m^3]	0.00409	0.01553
Volume of the neck [m^3]	0.00111	0.0051
Volume of the socket [m^3]	0.00015	0.00074
Total Volume [m^3]	0.01562	0.05524
$\bar{x}$ body [m]	0	0
$\bar{y}$ body [m]	0	0
$\bar{z}$ body [m]	0	0
$\bar{x}$ head [m]	0	0
$\bar{y}$ head [m]	0	0
$\bar{z}$ head [m]	0.2793535	0.4900664
$\bar{x}$ total [m]	-0.1817629	-0.2156015
$\bar{y}$ total [m]	0	0
$\bar{z}$ total [m]	-0.0764474	-0.0526546
Body torque output [Nm]	63.5242443	69.9424265
Required HP at 1100 RPM [HP]	9.8	10.8
Work done from t = 0 to t = 2.5 (first path)	-305737.59771 J	
Work done from t = 1.6 to t = 2.5 (first path)	149338.38938 J	
Work done from t = 0 to t = 2.5 (second path)	-305738.0 J	

Appendix A

Part 1 : Neck's Changing radius

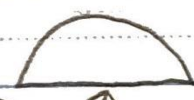


$$Z = H/(h-x)$$
$$X = z/(H/(h-x)) = z(h-x)/H$$
$$\text{Changing radius} = x + z(h-x)/H$$

Figure 4: Formula for the Neck's Changing Radius

part 1 c)

SOCKET



radius of the socket (a)

$$= \sqrt{r^2-(r-d)^2} = \sqrt{2rd-d^2}$$

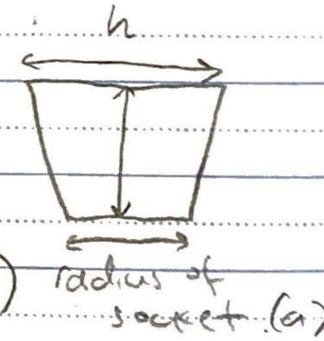
$$V_s = \int_0^{2\pi}\int_0^{\sqrt{2rd-d^2}}\int_{r-d}^{\sqrt{r^2-R^2}} R\,dz\,dR\,d\theta$$

let a be the radius of the socket

$$\sqrt{a^2-r^2} = \sqrt{2rd-d^2-r^2}$$

Figure 5: Socket Volume Derivation

NECK.



height of the cone

$$(h-\sqrt{rd2-d^2})\left(1/\tan(90-\phi)\right)$$

$$= (h-\sqrt{2rd-d^2})(\tan\phi)$$

$$a+\frac{z\cancel{(h-\sqrt{2rd-d^2})}}{\cancel{(h-\sqrt{2rd-d^2})}(\tan\phi)} = a+\frac{z}{\tan\phi}$$

$$\sqrt{2rd-d^2}+\frac{z}{\tan\phi}$$

$$V_N = \int_0^{2\pi}\int_0^{(h-\sqrt{2rd-d^2})(\tan\phi)}\int_0^{\sqrt{2rd-d^2}+\frac{z}{\tan\phi}} R\,dR\,dz\,d\theta$$

Figure 6: Neck Volume Derivation

Part 2

a) The y should be 0 because everything happens in the x-z plane and thus look symmetrical in the y-axis.

Head

$$M_h = \int_0^{\pi}\int_0^{h}\int_{-\sqrt{h^2-r^2}}^{\sqrt{h^2-r^2}} \rho h(r,\theta,y)\, r \, dy\, dr\, d\theta$$

$$M_{hyz} = \int_0^{\pi}\int_0^{h}\int_{-\sqrt{h^2-r^2}}^{\sqrt{h^2-r^2}} r^2 \cos\theta\, \rho h(r,\theta,y)\, dy\, dr\, d\theta$$

$$M_{hxy} = \int_0^{\pi}\int_0^{h}\int_{-\sqrt{h^2-r^2}}^{\sqrt{h^2-r^2}} r^2 \sin\theta\, \rho h(r,\theta,y)\, dy\, dr\, d\theta$$

$$\bar{X}h = \frac{M_{hyz}}{Mh} \qquad \bar{Z}h = \frac{M_{hxy}}{Mh}$$

Figure 7: Head Center of Mass

Body

$$M_b = \int_0^{2\pi}\int_0^{\pi}\int_{R-t}^{R} 300\rho^2 \sin(\varphi)\; d\rho\, d\varphi\, d\theta$$

$$M_{byz} = \int_0^{2\pi}\int_0^{\pi}\int_{R-t}^{R} 300\rho^2 \sin(\varphi)\, \rho \sin(\varphi) \cos(\theta)\, d\rho\, d\varphi\, d\theta$$

$$M_{bxy} = \int_0^{2\pi}\int_0^{\pi}\int_{R-t}^{R} 300\rho^2 \sin(\varphi)\, \rho \cos(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$\bar{X}_b = \frac{M_{byz}}{M_b} \qquad \bar{Z}_b = \frac{M_{hxy}}{M_b}$$

Figure 8: Body Center of Mass

Weight

$\bar{X}_w = -200mm \quad \bar{Z}_w = -90mm \quad M_w = 40kg$

Total

$$\frac{M_h \cdot \bar{X}_h + M_b \cdot \bar{X}_b + M_w \cdot \bar{X}_w}{M_h + M_b + M_w} = \bar{X}$$

$$\frac{M_h \cdot \bar{Z}_h + M_b \bar{Z}_b + M_w \bar{Z}_w}{M_h + M_b + M_w} = \bar{Z}$$

$\bar{y} = 0$

Figure 9: Overall Center of Mass

Part 2 c)

assume gravity is 9.81 m/s^2

$$|\bar{x}| N + |\bar{y}| N - (r - |\bar{z}|) \mu_k N = \text{Torque}$$
$$= |\bar{x}| \, 9.81(m) + 0 - (r - |\bar{z}|) \mu_k (9.81)(m)$$

N = normal force

$m = m_h + m_b + m_w$

Figure 10: Torque Calculations

Appendix B

MATLAB Code Question 1A

```
clc, clear, close all

%% data
%data = [0.215, 0.02, 0.125, 36, 0.015];
%data2 = ["yellow", "Case 1"];
data = [0.385, 0.02, 0.195, 45, 0.025];
data2 = ["cyan", "Case 2"];

body_radius = data(1);
t = data(2);
h = data(3); %h
neck_angle = 90 - data(4);
d = data(5);
colour = data2(1);

%% body
[body_x, body_y, body_z] = sphere;
body = surf(body_x*body_radius, body_y*body_radius, body_z*body_radius, "FaceColor",
colour);
hold on;

%% neck
% the trig used to derive the equation for the changing radius of the neck.
top_r = h;
bottom_r = sqrt(body_radius^2-(body_radius-d)^2);
neck_ht = (top_r-bottom_r).*(1/tand(neck_angle)); %neck height
z_offset = body_radius-d; % the distance from the origin to the bottom of the neck.

r=linspace(top_r,bottom_r,25); % row vector of 25 evenly spaced points between the
upper and lower radius.
theta = linspace(0, 2*pi, 25); % row vector of 25 points to create a circle
[r,theta] = meshgrid(r,theta); % meshgrid for the upper and lower circles
neck_x = r.*cos(theta); % x in polar
neck_y = r.*sin(theta); % y in polar
[neck_z, temp] = meshgrid(linspace(neck_ht, 0, 25), linspace(0, 2*pi, 25)); %
meshgrid for the height of the neck
neck_z = neck_z + z_offset;
neck = mesh(neck_x, neck_y, neck_z, "FaceColor", colour); % final mesh to generate
the shape

% %% head
% head_radius = h;
% [head_x, head_y, head_z] = sphere;
% head_z(head_z<0) = 0;
% head_offset = body_radius-d+neck_ht;
% head = surf(head_x*head_radius, head_y*head_radius, head_z*head_radius+head_offset,
"FaceColor", colour);
```

```
xlabel("X-axis");
ylabel("Y-axis");
zlabel("Z-axis");

title (data2(2));

axis equal;
```

MATLAB Code Question 1D

```
clc, clear, close all

syms R z theta
syms p phi

data = [0.215, 0.02, 0.125, 36, 0.015];
%data = [0.385, 0.02, 0.195, 45, 0.025];

r = data(1);
t = data(2);
h = data(3);
angle = data(4);
d = data(5);

%% triple integrals to find volumes
v_head = round(int(int(int(R, z, -sqrt(h^2-R^2), sqrt(h^2-R^2)), R, 0, h), theta, pi,
2*pi), 5)
v_body = round(int(int(int(p^2*sin(phi), p, r - t, r), phi, 0, pi), theta, 0, 2*pi),
5)
v_socket = round(int(int(int(R, z, r-d, sqrt(r^2-R^2)), R, 0, sqrt(2*r*d-d^2)),
theta, 0, 2*pi), 5)
v_neck = round(int(int(int(R, R, 0, sqrt(2*r*d-d^2)+z/tand(angle)), z, 0, (h-
sqrt(2*r*d-d^2))*tand(angle)), theta, 0, 2*pi), 5)
v_total = round(v_head+v_body+v_neck-v_socket, 5)
```

MATLAB Code Question 2

```
clc, clear, close all

syms theta r y
syms p phi

data = [0.215, 0.02, 0.125, 36, 0.015];
weight = [-0.2, -0.09];
%data = [0.385, 0.02, 0.195, 45, 0.025];
%weight = [-0.29, -0.11];

R = data(1);
t = data(2);
```

```
h = data(3);
angle = data(4) * pi/180;
d = data(5);

%% head
density = 225 + 15*(cos(theta + pi/2)-sin(theta));
Mh = int(int(int(density * r, y, -sqrt(h^2-r^2), sqrt(h^2-r^2)), r, 0, h), theta, 0,
pi);
Mhyz = int(int(int(r^2*cos(theta)*density, y, -sqrt(h^2-r^2), sqrt(h^2-r^2)), r, 0,
h), theta, 0, pi);
Mhxy = int(int(int(r*sin(theta)*r*density, y, -sqrt(h^2-r^2), sqrt(h^2-r^2)), r, 0,
h), theta, 0, pi);
Xh = Mhyz/Mh;
bottom_r = sqrt(R^2-(R-d)^2);
neck = (h-bottom_r).*(tan(angle)); %neck height
Zh = Mhxy/Mh + R - d + neck; % the z-center of mass and plus the distance from the
origin

%% body
density = 300;
Mb = int(int(int(300*p^2*sin(phi), p, R - t, R), phi, 0, pi), theta, 0, 2*pi);
Mbyz = int(int(int(300*p^2*sin(phi)*p*sin(phi)*cos(theta), p, R - t, R), phi, 0, pi),
theta, 0, 2*pi);
Mbxy = int(int(int(300*p^2*sin(phi)*p*cos(phi), p, R - t, R), phi, 0, pi), theta, 0,
2*pi);
Xb = Mbyz/Mb;
Zb = Mbxy/Mb;

%% weight
Xw = weight(1);
Zw = weight(2);
Mw = 40;

%% total

x = (Mh*Xh+Mb*Xb+Mw*Xw)/(Mh+Mb+Mw);
y = 0;
z = (Mh*Zh+Mb*Zb+Mw*Zw)/(Mh+Mb+Mw);

% display data in a friendly way
round(x, 7)
round(z, 7)

%% torque
gravity = 9.81;
mass = Mb + Mh + Mw;

torque = round(abs(x)*gravity*mass - (R-abs(z))*0.25*9.81*mass, 7)
```

MATLAB Code Question 3

```
clc, clear, close all
```

```
%% ques a
% constants
g = 6.67e-11;
m = 1.3e20;

rx = @(x, y) (-x-2500)./(sqrt((-x-2500).^2+(-y-1000).^2)); % anonymous func for x-
component
ry = @(x, y) (-y-1000)./(sqrt((-x-2500).^2+(-y-1000).^2)); % anonymous func for y-
component

f_mag = @(x, y) (g.*m)./((-x-2500).^2+(-y-1000).^2); % anonymous func for the force
magnitude

%% ques b
% parametric equation of the Falcon's path
syms t
falx = @(a) 3500.*cos(a)-2500;
faly = @(a) 3500.*sin(a);

% plotting the gravitational field
[X,Y] = meshgrid(-6000:500:6000,0:500:10000); % between -6000 and 6000
quiver(X, Y, f_mag(X, Y).*rx(X, Y), f_mag(X, Y).*ry(X, Y), 1);
hold on

% plotting the Falcon's path and some key points
fplot(falx(t), faly(t), [0, 2.5]);
plot(falx(0), faly(0), 'o-', 'MarkerFaceColor','red');
plot(falx(1.6), faly(1.6), 'o-', 'MarkerFaceColor','yellow');
plot(falx(2.5), faly(2.5), 'o-', 'MarkerFaceColor','blue');


%% ques c
r = [falx(t), faly(t)]; % combining the parametric equation into one vector
r_d = diff(r); % finding the derivative with respect to t
f = [f_mag(falx(t), faly(t)).*rx(falx(t), faly(t)), f_mag(falx(t),
faly(t)).*ry(falx(t), faly(t))]; % force
%finding work
work1 = int(dot(f, r_d), t, 0, 2.5) % from 0 to 2.5
work2 = int(dot(f, r_d), t, 1.6, 2.5) % from 1.6 to 2.5


%% ques d
reyy = @(a) 3500.*sin(a)-1500.*a.^2+3750*a; % new path
fplot(falx(t), reyy(t), [0, 2.5]); % plotting the new path from 0 to 2.5
f2 = [f_mag(falx(t), reyy(t)).*rx(falx(t), reyy(t)), f_mag(falx(t),
reyy(t)).*ry(falx(t), reyy(t))]; % force for new path
work3 = vpaintegral(dot(f2, diff([falx(t), reyy(t)])), t, 0, 2.5) % calculating the
new work

%% labels
xlabel("x-axis");
ylabel("y-axis");
legend('Gravitational field', 'Original', 't=start', 't=1.6', 't=end', 'Rey');
title("Path");
```

Appendix C

Motor Link

https://www.emotorsdirect.ca/item/max-motion-ija160l-4-59-d