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2021 Kandikaatopaikka

1 Hardy-Wrightin todistuksen perkaamista

G. H. Hardyn ja *E. M. Wrightin* kirjan *An Introduction to the theory of numbers* sivulla 469 olevan ϕ-funktion alarajan todistuksen läpikäyntiä.

1.1 Määrittely: mitä todistetaan

Aloitetaan määrittelemällä kuvaus

$$f(n) = \frac{\phi(n)e^{\gamma}\log\log n}{n},$$

missä γ on Eulerin vakio.

Halutaan todistaa, että $\liminf f(n) = 1$, mikä on yhtäpitävää sen kanssa, että ϕ-funktion alaraja on $\frac{n}{e^{\gamma}\log\log n}$.

1.2 Määrittely: miten todistetaan

Pitää kirjoittaa kokonaan uusiksi alkusepitykset nyt kun sigma joudutaankin ottamaan käyttöön

Riittää löytää funktiot $F_1(t)$ ja $F_2(t)$, joille pätee

1. $\lim_{t\to\infty} F_1(t) = 1$ ja $\lim_{t\to\infty} F_2(t) = 1$
2. $f(n) \geq F_1(\log n)$ kaikilla $n \geq 3$
3. $f(n_j) \leq \frac{1}{F_2(j)}$ äärettömällä kasvavalla jonolla $n_2, n_3, ...$

"Tämä tarkoittaa, että on löydetty funktio $F_1(\log n)$, jonka on sama limes infimum on yksi, mutta funktio on kaikkialla suurempi kuin $f(n)$. Tällöin funktion $f(n)$ limes infimum on enintään yksi. Vastaavasti alapuolen kanssa."

1.3 Todistus osa 1: $f(n) \geq F_1(\log n)$

Olkoot $p_1, p_2, ..., p_{r-\rho} \leq \log n$ ja $p_{r-\rho+1}, ..., p_r > \log n$ luvun n alkutekijöitä. Siis luvulla n on yhteensä r alkutekijää, joista $\log n$:ää suurempia on ρ kappaletta.

Nyt

$$(\log n)^{\rho} < p_{r-\rho+1} \cdot p_{r-\rho+2} \cdots p_r \leq n, \tag{1}$$

mistä seuraa

$$\rho < \frac{\log n}{\log\log n}. \tag{2}$$

Eli $\log n$:ää suurempia alkulukutekijöitä on alle $\frac{\log n}{\log \log n}$ kappaletta. Nyt tulokaavaa käyttäen ϕ-funktion suhde n:ään voidaan ilmaista seuraavasti

$$
\begin{aligned}
\frac{\phi(n)}{n} &= \prod_{i=1}^{r}(1-\frac{1}{p_i}) && (3)\\
&= \prod_{i=1}^{r-\rho}(1-\frac{1}{p_i}) \prod_{i=r-\rho+1}^{r}(1-\frac{1}{p_i}) && (4)\\
&\geq \left(\prod_{i=1}^{r-\rho}(1-\frac{1}{p_i})\right)(1-\frac{1}{\log n})^{\rho} && (5)\\
&> \left(\prod_{i=1}^{r-\rho}(1-\frac{1}{p_i})\right)(1-\frac{1}{\log n})^{\frac{\log n}{\log\log n}}. && (6)
\end{aligned}
$$

Näin ollen voidaan valita

$$F_1(t) = e^{\gamma} \log t \left(1-\frac{1}{t}\right)^{\frac{t}{\log t}} \prod_{p \leq t}\left(1-\frac{1}{p}\right),$$

jolloin

$$
\begin{aligned}
F_1(\log n) &= e^{\gamma} \log\log n \left(1-\frac{1}{\log n}\right)^{\frac{\log n}{\log\log n}} \prod_{p\leq \log n}\left(1-\frac{1}{p}\right)\\
&= e^{\gamma} \log\log n \left(1-\frac{1}{\log n}\right)^{\frac{\log n}{\log\log n}} \prod_{i=1}^{r-\rho}\left(1-\frac{1}{p}\right)\\
&\leq \frac{\phi(n)}{n} e^{\gamma}\log\log n = f(n).
\end{aligned}
$$

Kuitenkin funktiolle F_1 pätee Mertenin kolmannen lauseen nojalla

$$
\begin{aligned}
\lim_{t\to\infty} F_1(t) &= \lim_{t\to\infty} e^{\gamma} \log t \left(1-\frac{1}{t}\right)^{\frac{t}{\log t}} \prod_{p\leq t}\left(1-\frac{1}{p}\right)\\
&= \lim_{t\to\infty} e^{\gamma} \left(1-\frac{1}{t}\right)^{\frac{t}{\log t}} \left(\log t \prod_{p\leq t}\left(1-\frac{1}{p}\right)\right)\\
&= \lim_{t\to\infty} e^{\gamma} \left(1-\frac{1}{t}\right)^{\frac{t}{\log t}} e^{-\gamma}\\
&= \lim_{t\to\infty} \left(1-\frac{1}{t}\right)^{\frac{t}{\log t}}\\
&= 1
\end{aligned}
$$

Täten funktion f limes infimum on korkeintaan 1.

1.4 Todistus osa 2: $f(n_j) \le \frac{1}{F_2(j)}$

Next, to prove the part (**??**), let's define

$$g(n) = \frac{\sigma(n)}{n\, e^{\gamma} \log\log n}$$

and show that $g(n_j) \ge F_2(j)$ for an infinite increasing sequence. By theorem 2.2 the desired result will follow.

Let

$$n_j = \prod_{p \le e^j} p^j, \text{ where } j \ge 2\,.$$

By the lemma **??**

$$\log n_j = \log \prod_{p \le e^j} p^j = j \log \prod_{p \le e^j} p = j\vartheta(e^j) \le Aje^j\,.$$

Hence

$$\log\log n_j = \log Aje^j = \log A + \log j + \log e^j = \log A + \log j + j\,. \tag{7}$$

Since n_j is the product of the primes smaller than e^j to the power of j, by the lemma 2.1.1 we have

$$\sigma(n_j) = \prod_{p \le e^j} \frac{p^{j+1} - 1}{p - 1}$$

and

$$\frac{\sigma(n_j)}{n_j} = \prod_{p \le e^j} \frac{p^{j+1} - 1}{(p-1)p^j} = \prod_{p \le e^j} \frac{p^{j+1}\left(1 - \frac{1}{p^{j+1}}\right)}{p^{j+1}\left(1 - \frac{1}{p}\right)} = \prod_{p \le e^j} \frac{1 - \frac{1}{p^{j+1}}}{1 - \frac{1}{p}}\,. \tag{8}$$

Also, by the lemma 5.2

$$\prod_{p \le e^j} \left(1 - \frac{1}{p^{j+1}}\right) > \prod \left(1 - \frac{1}{p^{j+1}}\right) = \frac{1}{\zeta(j+1)}\,. \tag{9}$$

Now we can define

$$F_2(t) = \frac{1}{e^{\gamma}\, \zeta(t+1)(A + t + \log t)} \prod_{p \le e^t} \left(\frac{1}{1 - \frac{1}{p}}\right)$$

because by combining the results (7), (8) and (9)

$$
\begin{aligned}
F_2(j) &= \frac{1}{e^{\gamma}\,\zeta(j+1)(A+j+\log j)} \prod_{p \le e^j} \left(\frac{1}{1-\frac{1}{p}} \right) \\
&\le \frac{1}{e^{\gamma}\,\log\log n_j} \prod_{p \le e^j} \frac{1-\frac{1}{p^{j+1}}}{1-\frac{1}{p}} \\
&= \frac{\sigma(n_j)}{n_j\, e^{\gamma}\,\log\log n_j} = g(n_j)\,.
\end{aligned}
$$

By the Merten's third theorem (theorem **??**)

$$
\begin{aligned}
\lim_{t\to\infty} F_2(t) &= \lim_{t\to\infty} \frac{1}{e^{\gamma}\,\zeta(t+1)(A+t+\log t)} \prod_{p \le e^t} \left(\frac{1}{1-\frac{1}{p}} \right) \\
&= \lim_{t\to\infty} \frac{e^{\gamma}\,\log e^t}{e^{\gamma}\,\zeta(t+1)(A+t+\log t)} \\
&= \lim_{t\to\infty} \frac{t}{\zeta(t+1)(A+t+\log t)} \\
&= \lim_{t\to\infty} \frac{t}{A+t+\log t} \\
&= 1\,.
\end{aligned}
$$

By the theorem 2.2

$$
f(n)\,g(n) = \frac{\phi(n)\,e^{\gamma}\log\log n}{n} \cdot \frac{\sigma(n)}{n\,e^{\gamma}\log\log n} = \frac{\phi(n)\sigma(n)}{n^2} < 1
$$

and since $g(n_j) \ge F_2(j)$

$$
f(n_j) \le \frac{1}{F_2(j)}\,.
$$

Viel semmonen johtopäätös

2 Okei, sigma-funktio tarvitaan

Definition 2.1. *The σ-function*

$$
\sigma(n) = \sum_{d|n} d\,,
$$

meaning $\sigma(n)$ is the sum of the divisors of n.

Lemma 2.1.1. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of n, where $p_1, p_2, ..., p_r$ are distinct primes. Then

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{k_i+1} - 1}{p_i - 1} \,,$$

also denoted by

$$\sigma(n) = \prod_{p|n} \frac{p^{k+1} - 1}{p - 1} \,.$$

Proof. Theorem 275 in *Hardy & Wright: Introduction to the Theory of Numbers.* □

Theorem 2.2.

$$\frac{\phi(n)\,\sigma(n)}{n^2} < 1$$

Proof. By the Euler's product formula and lemma 2.1.1 we get

$$\begin{aligned}
\phi(n)\,\sigma(n) &= n \prod_{p|n} \left(1 - \frac{1}{p}\right) \cdot \prod_{p|n} \frac{p^{k+1} - 1}{p - 1} \\
&= n \prod_{p|n} \left(1 - \frac{1}{p}\right) \cdot \prod_{p|n} p^k \cdot \prod_{p|n} \frac{p - \frac{1}{p^k}}{p - 1} \\
&= n \prod_{p|n} \left(1 - \frac{1}{p}\right) \cdot n \prod_{p|n} \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} \\
&= n^2 \prod_{p|n} \left(\frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} - \frac{1 - \frac{1}{p^{k+1}}}{p - 1} \right) \\
&= n^2 \prod_{p|n} \frac{p - 1 - \frac{1}{p^k} + \frac{1}{p^{k+1}}}{p - 1} \\
&= n^2 \prod_{p|n} \frac{p - 1 - (p - 1)\frac{1}{p^{k+1}})}{p - 1} \\
&= n^2 \prod_{p|n} \left(1 - \frac{1}{p^{k+1}}\right) < n^2 \,.
\end{aligned}$$

Equivivalently

$$\frac{\phi(n)\,\sigma(n)}{n^2} < 1 \,.$$

□

3 Multiplikatiivisuustodistus

Tarttetaan alkuun modulomääritelmät ja muut (jippii lisää määriteltävää ja todistettavaa)

Definition 3.1. *Congruence*

Let $m \neq 0$. We say that a is congruent to b modulo m if $m|(a-b)$.
It it denoted by

$$a \equiv b \pmod{m}.$$

Lemma 3.1.1. Joku lemma on varmaan tarpeen

Theorem 3.2. Euler's totient function is multiplicative:

$$gcd(m,n) = 1 \quad \Rightarrow \quad \phi(mn) = \phi(m)\phi(n).$$

Proof. (Theorem 59+60 in *Hardy & Wright: Introduction to the Theory of Numbers.*)

Assume $gcd(m,n) = 1$ and $a \in \{1, 2, \ldots, m\}$ and $b \in \{1, 2, \ldots, n\}$.

Let C be a set containing all the numbers of the form $bm + an$. Since m and n are co-prime and a and b run through a complete set of residues (mod m) and (mod n) respectively, there is exactly mn numbers in the set C.

Let $b_1 m + a_1 n \in C$ and $b_2 m + a_2 n \in C$ be congruent to each other modulo mn. Now

$$b_1 m + a_1 n \equiv b_2 m + a_2 n \pmod{mn}$$

then

$$b_1 m \equiv b_2 m \pmod{m} \quad \text{and} \quad a_1 n \equiv a_2 n \pmod{n}$$

and furthermore

$$b_1 \equiv b_2 \pmod{m} \quad \text{and} \quad a_1 \equiv a_2 \pmod{n}.$$

This yields $a_1 = a_2$ and $b_1 = b_2$, since a and b En osaa sanoo tätä että a ja b sisältää menee vaan yhden kerran kaikki jäännökset läpi. Thus all of the mn numbers in C are incongruent to each other and therefore C forms a complete residue system modulo mn.

Now

$$\begin{aligned} gcd(bm+an, mn) = 1 \quad &\Leftrightarrow \quad gcd(bm+an, m) = 1 \ \text{ and } \ gcd(bm+an, n) = 1 \\ &\Leftrightarrow \quad gcd(an, m) = 1 \ \text{ and } \ gcd(bm, n) = 1 \\ &\Leftrightarrow \quad gcd(a, m) = 1 \ \text{ and } \ gcd(b, n) = 1, \end{aligned}$$

meaning

TÄÄ TODISTUS JOUTAA ROSKIIN

$\square$

PAREMPI TODISTUS (EHKÄ)

Proof. Assume $m > 1$, $n > 1$ and $gcd(m,n) = 1$. Consider the array which

$$\begin{array}{ccccc} 0 & 1 & \dots & m-2 & m-1 \\ m & m+1 & \dots & m+(m-2) & m+(m-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-2)m & (n-2)m+1 & \dots & (n-2)m+(m-2) & (n-2)m+(m-1) \\ (n-1)m & (n-1)m+1 & \dots & (n-1)m+(m-2) & (n-1)m+(m-1) \end{array}$$

consists of integers from 0 to $mn-1$, forming a complete residue system (mod mn).

Clearly, each row of the array forms a complete residue system (mod m) and all the elements of any column are congruent to each other (mod m). Now there are two types of columns: $\phi(n)$ columns containing only co-primes to m and the rest containing none of them.

Now consider the co-prime columns. Every column forms a complete residue system (mod n) (LeVeque: chapter 3.2, theorem 3.5, p. 52), meaning each includes $\phi(n)$ elements co-prime to n. Counting $\phi(n)$ elements from all the $\phi(m)$ columns we get in total $\phi(m)\phi(n)$ numbers that are relatively prime to both m and n.

On the other hand, since $gcd(m,n) = 1$, an integer k is co-prime to mn if and only if both $gcd(m,k) = 1$ and $gcd(n,k) = 1$. Hence there are $\phi(m)\phi(n)$ numbers relatively prime to mn. Thus by definition $\phi(mn) = \phi(m)\phi(n)$.

The case $m = 1$ or $n = 1$ is trivial, since $\phi(1) = 1$ and thus $\phi(mn) = \phi(m)\phi(n)$.

□

4 Tulokaavan todistus

Eulerin tulokaava arvon $\phi(n)$ laskemiseksi on hyvinkin tärkeä palanen eli todistetaan se nyt suoraan englanniksi niin ei tarvitse erikseen kääntää.

4.1 Eulers's product formula

Theorem 4.2. *Euler's product formula*

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where $\prod_{p|n}(1 - \frac{1}{p})$ means the product over *distinct* primes that divide n.

Proof. Assume first that $n = p^k$, where $p \in \mathbb{P}$. Now for every x, for which $gdc(p^k, x) > 1$, holds $x = mp^{k-1}$ for some $m \in \{1, 2, ..., p^{k-1}\}$.

Hence

$$\phi(n) = \phi(p^k) = p^k - p^{k-1} = p^k - \frac{p^k}{p} = \left(1 - \frac{1}{p}\right) p^k = \left(1 - \frac{1}{p}\right) n.$$

Then, in the general case, assume $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = \prod_{i=1}^r p_i^{k_i}$, where $p_1, p_2, ..., p_r$ are distinct primes that divide n and $k_1, k_2, ..., k_r$ their powers respectively.

Now, since ϕ is a multiplicative function

$$\begin{aligned}\phi(n) &= \phi(p_1^{k_1} p_1^{k_1} \cdots p_r^{k_r}) \\ &= \phi(p_1^{k_1})\, \phi(p_2^{k_2}) \cdots \phi(p_r^{k_r}) \\ &= \left(1 - \frac{1}{p_1}\right) p_1^{k_1} \left(1 - \frac{1}{p_2}\right) p_2^{k_2} \cdots \left(1 - \frac{1}{p_r}\right) p_r^{k_r} \\ &= \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) p_i^{k_i} \\ &= n \prod_{p|n} \left(1 - \frac{1}{p}\right).\end{aligned}$$

□

5 Funktioita ja muita

Definition 5.1. The zeta-function

$$\zeta(s) = \sum_{n=1]^\infty \frac{1}{n^s}}$$

The zeta-funtion converges, when $s > 1$.

Theorem 5.2. For all $s > 1$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Definition 5.3. Von Mangoldt function

Let $p \in \mathbb{P}$ and $k \geq 1$.

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{othewise}. \end{cases}$$

Theorem 5.4.

$$\sum_{d|n} \Lambda(d) = \log n\,.$$

Proof. Let us denote $n = \prod p^k$. Now, by definition, we have

$$\sum_{d|n} \Lambda(d) = \sum_{p^k|n} \log p\,.$$

We notice that as the sum runs through all combinations of primes p and positive integer powers k such that $p^k|n$, each $\log p$ occurs k times. Hence

$$\sum_{p^k|n} \log p = \sum a \log p = \sum \log p^a = \log \prod p^a = \log n\,.$$

$\square$

Lemma 5.4.1.

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1)\,.$$

Proof. First, we have a week form of so called Striling's formula

$$\begin{aligned}\sum_{n\le x} \log n &= \int_1^x \log t \, d[t] \\ &= [x]\log x - [x] - \log 1 + 1 \\ &= x \log x - \{x\} \log x - x + \{x\} + 1 \\ &= x \log x - x + O(\log x)\,.\end{aligned}$$

On the other hand, by theorem 5.4 we can deduce

$$\sum_{n\le x} \log n = \sum_{n\le x} \sum_{d|n} \Lambda(d) = \sum_{d \le x} \Lambda(d) \left[\frac{x}{d}\right] = x \cdot \sum_{d\le x} \frac{\Lambda(d)}{d} + O(\psi(x))\,,$$

where $\psi(x) = \sum_{d\le x} \Lambda(d)$ [**?**].

Now we have

$$x \log x - x + O(\log x) = x \cdot \sum_{d \le x} \frac{\Lambda(d)}{d} + O(\psi(x))$$

yielding [**?**] the desired result

$$\log x + O(1) = \sum_{d\le x} \frac{\Lambda(d)}{d}\,,$$

when divided by x.

$\square$

6 Merten's theorem

Lemma 6.0.1. If $c_1, c_2, \ldots$ is a sequence of real numbers such that $c_i = 0$ for $i < 2$ and

$$C(t) = \sum_{n \le t} c_n$$

and $f(t)$ has continuous derivative for $t \ge 2 \in \mathbb{R}$, then

$$\sum_{n \le x} c_n\, f(n) = C(x)\, f([x]) - \int_2^x C(t)\, f'(t)\, dt\,.$$

Proof. First we observe that $C(t) = C(n)$ and $f(n) = f([t])$, when $n \le t \le n{+}1$.

We have

$$\begin{aligned}
\sum_{n \le x} c_n\, f(n) &= c_1\, f(1) + c_2\, f(2) + \cdots + c_n\, f(n) \\
&= C(1)\, f(1) + (C(2) - C(1))\, f(2) + \cdots + (C(n) - C(n-1))\, f(n) \\
&= C(1)\, (f(1) - f(2)) + C(2)\, (f(2) - f(3)) + \ldots \\
&\quad + C(n-1)\, (f(n-1) - f(n)) + C(n)\, f(n) \\
&= \sum_{n \le x-1} C(n)\, (f(n) - f(n+1)) + \underbrace{C(n)\, f(n)}_{C(x)\, f([x])}\,.
\end{aligned}$$

On the other hand, since $f(t)$ is continuously differentiable when $t \ge 2$ and $C(t) = 0$ elsewhere, we have

$$C(n)\, (f(n) - f(n+1)) = \int_{n+1}^{n} C(t)\, f'(t)\, dt = -\int_n^{n+1} C(t)\, f'(t)\, dt\,.$$

Finally, by combining these we get

$$\begin{aligned}
\sum_{n \le x} c_n\, f(n) &= C(x)\, f([x]) + \sum_{n \le x-1} C(n)\, (f(n) - f(n+1)) \\
&= C(x)\, f([x]) - \int_2^x C(t)\, f'(t)\, dt\,.
\end{aligned}$$

$\square$

Theorem 6.1. *Merten's theorem*

$$\lim_{n \to \infty} \log n \prod_{p \le n} \left(1 - \frac{1}{p}\right) = e^{-\gamma}$$

where γ is the Euler's constant.

Proof. We start by showing that for some constant B

$$\sum_{p \leq n} \left(\frac{1}{p}\right) = \log\log n + B + O\left(\frac{1}{\log x}\right)$$

which is also know as Mertens' second theorem.

Let us use the lemma 6.0.1 with a sequence, where $c_p = \frac{\log p}{p}$ with prime indices and $c_n = 0$ otherwise. Now we have $C(x) = \sum_{p \leq x} \frac{\log p}{p}$.

We observe that

$$\begin{aligned}
\sum_{n \leq x} \frac{\Lambda(n)}{n} &= \sum_{k} \sum_{p^k \leq x} \frac{\log p}{p^k} \\
&= \sum_{p \leq x} \frac{\log p}{p} + \sum_{p \leq \sqrt{x}} \frac{\log p}{p^2} + \sum_{p \leq \sqrt[3]{x}} \frac{\log p}{p^3} + \dots \\
&< \sum_{p \leq x} \frac{\log p}{p} + \sum_{p} \left(\frac{1}{p^2} + \frac{1}{p^3} + \dots\right) \log p \\
&= \sum_{p \leq x} \frac{\log p}{p} + \sum_{p} \frac{\log p}{p(p-1)} \\
&< \sum_{p \leq x} \frac{\log p}{p} + \underbrace{\sum_{n \geq 2}^{\infty} \frac{\log n}{n(n-1)}}_{\text{constant } A} \\
&= C(x) + A \,.
\end{aligned}$$

Now, since A is a constant, we get

$$C(x) = \sum_{p \leq x} \frac{\log p}{p} = \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(1) = \log x + O(1)$$

by lemma 5.4.1.

Now, with $f(t) = \frac{1}{\log t}$, we get

$$\begin{aligned}
\sum_{p \le x} \frac{1}{p} &= \sum_{p \le x} \frac{c_p}{f(p)} \\
&= \frac{C(x)}{\log x} + \int_2^x \frac{C(t)}{t \log^2 t}\, dt \\
&= \frac{\log x + O(1)}{\log x} + \int_2^x \frac{\log t + O(1)}{t \log^2 t}\, dt \\
&= 1 + \frac{O(1)}{\log x} + \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{O(1)}{t \log^2 t}\, dt \\
&= 1 + O\left(\frac{1}{\log x}\right) + \log\log x - \log\log 2 + \int_2^\infty \frac{O(1)}{t \log^2 t}\, dt - \int_x^\infty \frac{O(1)}{t \log^2 t}\, dt \\
&= \log\log x + \underbrace{\left(\int_2^\infty \frac{O(1)}{t \log^2 t}\, dt + 1 - \log\log 2\right)}_{\text{constant } B} + O\left(\frac{1}{\log x}\right)
\end{aligned}$$

To reach the final form of Mertens' theorem, can be shown [?] that

$$B = \gamma + \sum_{p \le n} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right),$$

where γ is the Euler-Masheroni constant.

However, in its extent the proof would expand outside of the scope of this thesis. Let us take the value of B as given and deduce

$$\begin{aligned}
\sum_{p \le n} \frac{1}{p} &= \log\log x + \gamma + \sum_{p \le n} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) + O\left(\frac{1}{\log x}\right) \\
0 &= \log\log x + \gamma + \sum_{p \le n} \log\left(1 - \frac{1}{p}\right) + O\left(\frac{1}{\log x}\right) \\
0 &= \log\log x + \gamma + \log \prod_{p \le n} \left(1 - \frac{1}{p}\right) + O\left(\frac{1}{\log x}\right) \\
1 &= \log x + e^\gamma + \prod_{p \le n} \left(1 - \frac{1}{p}\right) + O(1)
\end{aligned}$$

HÄH miks tuolta ei nyt tuukaan ulos mertens heeeelp

KAIKKIALLA MERTENSIIN LIITTYVÄSSÄ ON NYT VIELÄ ÄKSÄT JA ÄNNÄT JA DEET SEKAISIN!!!!!

□

7 Edellisestä versiosta poistettua paskaa

7.0.1 Are there such integers n that $\phi(n) < \sqrt{n}$?

Let's begin with $\sqrt{n}$. Is there such large number n that $\phi(n) < \sqrt{n}$? When checking the values of $\phi(n)$ for smaller n, we see that at least with $n = 6$ the statement is true, as $\phi(6) = 2 < \sqrt{6}$. After that, however, the values seem to be consistently above the corresponding squareroot value.

Reasonable guess would be to assume that $\sqrt{n}$ is a lower limit for $\phi(n)$ when $n \to \infty$. With more precise examination, we see that is indeed the case.