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1 Hardy-Wrightin todistuksen perkaamista

G. H. Hardy ja *E. M. Wrightin* kirjan *An Introduction to the theory of numbers* sivulla 469 olevan ϕ -funktion alarajan todistuksen läpikäyntiä.

1.1 Määrittely: mitä todistetaan

Aloitetaan määrittelemällä kuvaus

$$f(n) = \frac{\phi(n)e^\gamma \log \log n}{n},$$

missä γ on Eulerin vakio.

Halutaan todistaa, että $\liminf f(n) = 1$, mikä on yhtäpitävää sen kanssa, että ϕ -funktion alaraja on $\frac{n}{e^\gamma \log \log n}$.

1.2 Määrittely: miten todistetaan

Pitää kirjoittaa kokonaan uusiksi alkuseitykset nyt kun sigma joudutaankin ottamaan käyttöön

Riittää löytää funktiot $F_1(t)$ ja $F_2(t)$, joille pätee

1. $\lim_{t \rightarrow \infty} F_1(t) = 1$ ja $\lim_{t \rightarrow \infty} F_2(t) = 1$
2. $f(n) \geq F_1(\log n)$ kaikilla $n \geq 3$
3. $f(n_j) \leq \frac{1}{F_2(j)}$ äärettömällä kasvavalla jonolla n_2, n_3, \dots

”Tämä tarkoittaa, että on löydetty funktio $F_1(\log n)$, jonka on sama limes infimum on yksi, mutta funktio on kaikkialla suurempi kuin $f(n)$. Tällöin funktion $f(n)$ limes infimum on enintään yksi. Vastaavasti alapuolen kanssa.”

1.3 Todistus osa 1: $f(n) \geq F_1(\log n)$

Olko $p_1, p_2, \dots, p_{r-\rho} \leq \log n$ ja $p_{r-\rho+1}, \dots, p_r > \log n$ luvun n alkutekijöitä. Siis luvulla n on yhteensä r alkutekijää, joista $\log n$:ää suurempia on ρ kappaletta.

Nyt

$$(\log n)^\rho < p_{r-\rho+1} \cdot p_{r-\rho+2} \cdots p_r \leq n, \quad (1)$$

mistä seuraa

$$\rho < \frac{\log n}{\log \log n}. \quad (2)$$

Eli $\log n$:ää suurempia alkulukutekijöitä on alle $\frac{\log n}{\log \log n}$ kappaletta. Nyt tulokaavaa käyttäen ϕ -funktion suhde n :ään voidaan ilmaista seuraavasti

$$\frac{\phi(n)}{n} = \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \quad (3)$$

$$= \prod_{i=1}^{r-\rho} \left(1 - \frac{1}{p_i}\right) \prod_{i=r-\rho+1}^r \left(1 - \frac{1}{p_i}\right) \quad (4)$$

$$\geq \left(\prod_{i=1}^{r-\rho} \left(1 - \frac{1}{p_i}\right) \right) \left(1 - \frac{1}{\log n}\right)^\rho \quad (5)$$

$$> \left(\prod_{i=1}^{r-\rho} \left(1 - \frac{1}{p_i}\right) \right) \left(1 - \frac{1}{\log n}\right)^{\frac{\log n}{\log \log n}}. \quad (6)$$

Näin ollen voidaan valita

$$F_1(t) = e^\gamma \log t \left(1 - \frac{1}{t}\right)^{\frac{t}{\log t}} \prod_{p \leq t} \left(1 - \frac{1}{p}\right),$$

jolloin

$$\begin{aligned} F_1(\log n) &= e^\gamma \log \log n \left(1 - \frac{1}{\log n}\right)^{\frac{\log n}{\log \log n}} \prod_{p \leq \log n} \left(1 - \frac{1}{p}\right) \\ &= e^\gamma \log \log n \left(1 - \frac{1}{\log n}\right)^{\frac{\log n}{\log \log n}} \prod_{i=1}^{r-\rho} \left(1 - \frac{1}{p_i}\right) \\ &\leq \frac{\phi(n)}{n} e^\gamma \log \log n = f(n). \end{aligned}$$

Kuitenkin funktiolle F_1 pätee Mertenin kolmannen lauseen nojalla

$$\begin{aligned} \lim_{t \rightarrow \infty} F_1(t) &= \lim_{t \rightarrow \infty} e^\gamma \log t \left(1 - \frac{1}{t}\right)^{\frac{t}{\log t}} \prod_{p \leq t} \left(1 - \frac{1}{p}\right) \\ &= \lim_{t \rightarrow \infty} e^\gamma \left(1 - \frac{1}{t}\right)^{\frac{t}{\log t}} \left(\log t \prod_{p \leq t} \left(1 - \frac{1}{p}\right) \right) \\ &= \lim_{t \rightarrow \infty} e^\gamma \left(1 - \frac{1}{t}\right)^{\frac{t}{\log t}} e^{-\gamma} \\ &= \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t}\right)^{\frac{t}{\log t}} \\ &= 1 \end{aligned}$$

Täten funktion f limes infimum on korkeintaan 1.

1.4 Todistus osa 2: $f(n_j) \leq \frac{1}{F_2(j)}$

Next, to prove the part (??), let's define

$$g(n) = \frac{\sigma(n)}{n e^\gamma \log \log n}$$

and show that $g(n_j) \geq F_2(j)$ for an infinite increasing sequence. By theorem 2.2 the desired result will follow.

Let

$$n_j = \prod_{p \leq e^j} p^j, \text{ where } j \geq 2.$$

By the lemma ??

$$\log n_j = \log \prod_{p \leq e^j} p^j = j \log \prod_{p \leq e^j} p = j \vartheta(e^j) \leq A j e^j.$$

Hence

$$\log \log n_j = \log A j e^j = \log A + \log j + \log e^j = \log A + \log j + j. \quad (7)$$

Since n_j is the product of the primes smaller than e^j to the power of j , by the lemma 2.1.1 we have

$$\sigma(n_j) = \prod_{p \leq e^j} \frac{p^{j+1} - 1}{p - 1}$$

and

$$\frac{\sigma(n_j)}{n_j} = \prod_{p \leq e^j} \frac{p^{j+1} - 1}{(p - 1)p^j} = \prod_{p \leq e^j} \frac{p^{j+1} \left(1 - \frac{1}{p^{j+1}}\right)}{p^{j+1} \left(1 - \frac{1}{p}\right)} = \prod_{p \leq e^j} \frac{1 - \frac{1}{p^{j+1}}}{1 - \frac{1}{p}}. \quad (8)$$

Also, by the lemma 4.2

$$\prod_{p \leq e^j} \left(1 - \frac{1}{p^{j+1}}\right) > \prod_{p \leq e^j} \left(1 - \frac{1}{p^{j+1}}\right) = \frac{1}{\zeta(j+1)}. \quad (9)$$

Now we can define

$$F_2(t) = \frac{1}{e^\gamma \zeta(t+1)(A + t + \log t)} \prod_{p \leq e^t} \left(1 - \frac{1}{p}\right)$$

because by combining the results (7), (8) and (9)

$$\begin{aligned}
F_2(j) &= \frac{1}{e^\gamma \zeta(j+1)(A+j+\log j)} \prod_{p \leq e^j} \left(\frac{1}{1 - \frac{1}{p}} \right) \\
&\leq \frac{1}{e^\gamma \log \log n_j} \prod_{p \leq e^j} \frac{1 - \frac{1}{p^{j+1}}}{1 - \frac{1}{p}} \\
&= \frac{\sigma(n_j)}{n_j e^\gamma \log \log n_j} = g(n_j).
\end{aligned}$$

By the Merten's third theorem (theorem ??)

$$\begin{aligned}
\lim_{t \rightarrow \infty} F_2(t) &= \lim_{t \rightarrow \infty} \frac{1}{e^\gamma \zeta(t+1)(A+t+\log t)} \prod_{p \leq e^t} \left(\frac{1}{1 - \frac{1}{p}} \right) \\
&= \lim_{t \rightarrow \infty} \frac{e^\gamma \log e^t}{e^\gamma \zeta(t+1)(A+t+\log t)} \\
&= \lim_{t \rightarrow \infty} \frac{t}{\zeta(t+1)(A+t+\log t)} \\
&= \lim_{t \rightarrow \infty} \frac{t}{A+t+\log t} \\
&= 1.
\end{aligned}$$

By the theorem 2.2

$$f(n)g(n) = \frac{\phi(n) e^\gamma \log \log n}{n} \cdot \frac{\sigma(n)}{n e^\gamma \log \log n} = \frac{\phi(n)\sigma(n)}{n^2} < 1$$

and since $g(n_j) \geq F_2(j)$

$$f(n_j) \leq \frac{1}{F_2(j)}.$$

Viel semmonen johtopäätös

2 Okei, sigma-funktio tarvitaan

Definition 2.1. *The σ -function*

$$\sigma(n) = \sum_{d|n} d,$$

meaning $\sigma(n)$ is the sum of the divisors of n .

Lemma 2.1.1. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of n , where p_1, p_2, \dots, p_r are distinct primes. Then

$$\sigma(n) = \prod_{i=1}^r \frac{p_i^{k_i+1} - 1}{p_i - 1}.$$

Proof. Theorem 275 in *Hardy & Wright: Introduction to the Theory of Numbers*. \square

Theorem 2.2.

$$\frac{\phi(n) \sigma(n)}{n^2} < 1$$

Proof. Theorem 329 in *Hardy & Wright: Introduction to the Theory of Numbers*. \square

3 Tulokaavan todistus

Eulerin tulokaava arvon $\phi(n)$ laskemiseksi on hyvinkin tärkeä palanen eli todistetaan se nyt suoraan englanniksi niin ei tarvitse erikseen kääntää.

3.1 Eulers's product formula

Theorem 3.2. *Euler's product formula*

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where $\prod_{p|n} (1 - \frac{1}{p})$ means the product over *distinct* primes that divide n .

Proof. Assume first that $n = p^k$, where $p \in \mathbb{P}$. Now for every x , for which $\gcd(p^k, x) > 1$, holds $x = mp^{k-1}$ for some $m \in \{1, 2, \dots, p^{k-1}\}$.

Hence

$$\phi(n) = \phi(p^k) = p^k - p^{k-1} = p^k - \frac{p^k}{p} = \left(1 - \frac{1}{p}\right) p^k = \left(1 - \frac{1}{p}\right) n.$$

Then, in the general case, assume $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = \prod_{i=1}^r p_i^{k_i}$, where p_1, p_2, \dots, p_r are distinct primes that divide n and k_1, k_2, \dots, k_r their powers respectively.

Now, since ϕ is a multiplicative function

$$\begin{aligned}
\phi(n) &= \phi(p_1^{k_1} p_1^{k_1} \cdots p_r^{k_r}) \\
&= \phi(p_1^{k_1}) \phi(p_2^{k_2}) \cdots \phi(p_r^{k_r}) \\
&= \left(1 - \frac{1}{p_1}\right) p_1^{k_1} \left(1 - \frac{1}{p_2}\right) p_2^{k_2} \cdots \left(1 - \frac{1}{p_r}\right) p_r^{k_r} \\
&= \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) p_i^{k_i} \\
&= n \prod_{p|n} \left(1 - \frac{1}{p}\right).
\end{aligned}$$

□

4 The zeta-function

Definition 4.1. The zeta-function

$$\zeta(s) = \sum_{n=1]^\infty \frac{1}{n^s}$$

The zeta-funtion converges, when $s > 1$.

Theorem 4.2. For all $s > 1$

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

5 Merten's (third) theorem

Theorem 5.1. *Merten's (third) theorem*

$$\lim_{n \rightarrow \infty} \log n \prod_{p \leq n} \left(1 - \frac{1}{p}\right) = e^{-\gamma}$$

where γ is the Euler's constant.

Proof. Oh, this seems like a työmaa

□

6 Edellisestä versiosta poistettua paskaa

6.0.1 Are there such integers n that $\phi(n) < \sqrt{n}$?

Let's begin with \sqrt{n} . Is there such large number n that $\phi(n) < \sqrt{n}$? When checking the values of $\phi(n)$ for smaller n , we see that at least with $n = 6$ the statement is true, as $\phi(6) = 2 < \sqrt{6}$. After that, however, the values seem to be consistently above the corresponding squareroot value.

Reasonable guess would be to assume that \sqrt{n} is a lower limit for $\phi(n)$ when $n \rightarrow \infty$. With more precise examination, we see that is indeed the case.