The size of Euler's totient function when $n \to \infty$

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1 Summary

Viittaus [1] toinenkin [2]

2 Introduction

3 Notation and definitons

Notation 3.1. *Divisibility* $a|b$
Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be such that b is divisible by a. This is denoted by $a|b$.

Definition 3.2. *Greatest common divisor,* $gcd(a,b)$
Let $a \neq 0$ and $b \neq 0$. There is a unique $d \in \mathbb{N}$ with following properties:

1. $d|a$ and $d|b$
2. if $d'|a$ and $d'|b$, then $d'|d$

The number d is called the greatest common divisor of a and b. It's denoted by $gcd(a,b) = d$.

Definition 3.3. *Prime number*
Integer $p \in \mathbb{N}$ is a prime, if $p \geq 2$ and for every $k \in \mathbb{N}$ holds that if $k|p$ then $k \in 1, p$. The set of prime numbers is denoted by $\mathbb{P}$.
In other words, all integers greater than than 1, which are only divisible by themself and 1, are primes.

Definition 3.4. *Co-primes*
If $gcd(a,b) = 1$, a and b are called co-primes or relative primes.

Definition 3.5. Multiplicative number theoretic function
Function $f : \mathbb{N} \to \mathbb{R}$ is called number theoretic function. Function f is multiplicative if $f(ab) = f(a)f(b)$ when $gcd(a,b) = 1$.

4 Euler's totient function and its properties

Euler's totient function is a number theoretic function used in NÖNÖNÖNÖ

Definition 4.1. *Euler's totient function* $\phi : \mathbb{N} \to \mathbb{N}$
It's set that $\phi(1) = 1$. For all $n \geq 2$, $\phi(n)$ is the number of integers $a \in \{1, 2, ..., n\}$, for which $gcd(a,n) = 1$.
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That is, the value of the totient function in $n \in \mathbb{N}$ is the number of natural numbers smaller than n, which are its co-primes.

Theorem 4.2. Euler's product formula

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

where $\prod_{p|n}(1 - \frac{1}{p})$ means the product over *distinct* primes that divide n.

Proof. KIKKI

□

Theorem 4.3. Totient function and primes
For every $p \in \mathbb{P}$ holds $\phi(p) = p - 1$

Proof. KIKKI

□

5 The limits of Euler's totient function

As shown, there is an exact formula for the rather verbally defined totient function $\phi(n)$. Tough, using it requires factorization of n, which seems to cause the complicatedness to estimate its size as n gets bigger.

For example let $n = 2^p - 1 \in \mathbb{P}$ where also $p \in \mathbb{P}$ (meaning n is so called Mersenne prime). By theorem **??** we know $\phi(n) = n - 1$. On the other hand, from Euler's product formula follows that $\phi(n+1) = \phi(2^p) = 2^p(1 - \frac{1}{2}) = \frac{2^p}{2} = \frac{n+1}{2}$. Now we see that while n and $n + 1$ differ from each other only insignificantly, $\phi(n + 1)$ is half the value of $\phi(n)$.

As we see the size of Euler's totient function fluctuates as it grows, making its size quite difficult to define. After shortly proving the trivial upper limit of the totient function, we move on to the more complicated lower limit.

5.1 Upper limit of Euler's totient function TÄHÄN JÄIN

Theorem 5.2. *Eulerin ϕ-funktion yläraja*
Kaikilla luonnollisilla luvuilla $n \geq 2$ pätee $\phi(n) < n$.

Proof. Suoraan määritelmästä seuraa, että $\phi(n) \leq n$, koska joukossa $\{1, 2, ..., n\}$ on n alkiota ja siten niiden joukosta ei voi löytyä yli n kappaletta ehtoa täyttävää lukua. Lisäksi jokaisella n pätee $syt(n, n) = n$. Täten millään $n \geq 2$ ei voi olla $\phi(n) = n$.

Siis $\phi(n) < n$ jokaisella $n \geq 2$.

□

Theorem 5.3. *Alkuluvuilla $\phi(p) = p - 1$*
Jokaisella alkuluvulla $p \in \mathbb{P}$ pätee $\phi(p) = p - 1$.

Proof. Olkoon $p \in \mathbb{P}$. Tällöin jokaisella $k < p$, $k \in \mathbb{N}$ pätee $syt(k, p) = 1$, mistä seuraa suoraan $\phi(p) = p - 1$. PITÄISIKÖ TÄÄ TODISTAA PAREMMIN

□

Theorem 5.4. ϕ-funktion pienin yläraja
Jokaisella $n \in \mathbb{N}$ pätee $\phi(n) \leq n - 1$.

Proof. Tulos saadaan suoraan yhdistämällä lauseet ÄSKEINEN ja SITÄ EDELLINEN.

□

5.5 Eulerin ϕ-funktion alaraja

5.6 $\phi(n) < \sqrt(n)$?

Lähdetään tutkimaan ϕ-funktion alarajaa tarkastelemalla onko olemassa suuria luonnollisia lukuja, joilla $\phi(n) < \sqrt{n}$. Huomataan, että ainakin vielä luvulla $n = 6$ pätee $\phi(6) = 2 < \sqrt(6)$, mutta sen jälkeen arvot näyttäisivät järjestään ylittävän vastaavan neliöjuuren arvon.

Tarkastellaan tilannetta tarkemmin jos osataan ehehe

6 Eulerin ϕ-funktion keskiarvo

7 Asiaaa

8 Lähteet

[1] E. M. Wright G. H. Hardy. *An Introduction to the Theory of Numbers.* 2008.

[2] Eero Saksman. "Introduction to Number Theory". 2019.