A generalization of Mertens' theorem

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In 1874 [Mer], Mertens' proved an interesting and useful theorem about the partial product of the Riemann zeta function at $s = 1$. Namely,

Theorem 1. *Let γ denote Euler's constant. Then,*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim e^{\gamma} \log(x).$$

There are various ways to define Euler's constant. The one we will use is $\gamma = -\int_0^\infty e^{-x} \log(x)\, dx \approx .577216$.

A nice proof of Mertens' theorem can be found in the classic text of Hardy and Wright [HW]. See Theorem 429 and its proof. In this paper we will consider possible generalizations of Mertens' theorem to other Dirichlet series beyond the Riemann zeta function. In Section 1 we will prove a version which is valid in any global function field. In Section 2 we prove a generalization to any algebraic number field. To give the flavor of these generalizations, we state that result here.

Theorem 2. *Let $K/\mathbb{Q}$ be a finite algebraic number field, $\zeta_K(s)$ the associated Dedekind zeta function, and α_K the residue of $\zeta_K(s)$ at $s = 1$. Then,*

$$\prod_{NP \leq x} \left(1 - \frac{1}{NP}\right)^{-1} = e^{\gamma} \alpha_K \log(x) + O_K(1),$$

where the product is over all prime ideals, P, in the ring of integers of K whose norm, NP, is less than or equal to x.

The notation $O_K(f(x))$ will be used in the statement of several theorems to draw attention to the fact that the implied constant depends on the field K under consideration. We will suppress this dependence on K in the course of the proofs.

It should be pointed out that Theorem 2 is stronger, even over Q, than the version of Mertens' theorem stated in Theorem 1. This stronger form is not hard to achieve and is used in [M-R-S].

In Section 3 we prove a result, Theorem 4, which is a speculative generalization of Merten's theorem to a broad class of Dirichlet series. Our work will depend on assuming a certain hypothesis which is like a prime number theorem for these series. Many of the series we have in mind are expected to satisfy a generalized Riemann hypothesis, so our assumption, which is much weaker, is not unreasonable. It has been suggested to the author that both of the conditions in Theorem 4 are unnecessarily restrictive. Moreover, by allowing the local factors to be power series, suitably restricted, rather than polynomials,

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still further generalizations would result. While these sugggestions undoubtedly lead in fruitful directions, we decided to leave matters at the present level of generality since the hypotheses as stated fit in well with the earlier sections of the paper and apply, provided that certain standard conjectures are correct, to most of the L-series arising from arithmetic geometry. We leave the formulation and proof of further generalizations to the interested reader.

As an illustration of Theorem 4, we prove that our general Mertens' theorem applies to Artin L-series.

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Section 1. Let F be a finite field with q elements, and K/F an algebraic function field in one variable with exact constant field F. Let $\zeta_K(s)$ be the zeta function of K. It is well known that this function is analytic in a neighborhood of 1 (actually in the whole complex plane) and has a simple pole at $s = 1$ of residue $\alpha = h_K / \big(q^g(1 - q^{-1}) \log(q)\big)$. Here, h_K is the number of divisor classes of degree zero and g is the genus of K.

The following result has been discovered independently by Bjorn Poonen, but without the precise determination of the constants involved.

Theorem 3. *With the above definitions and notation, we have*

$$\prod_{NP \le x} \left(1 - \frac{1}{NP}\right)^{-1} = \frac{e^{\gamma} h_K}{q^{g-1}(q-1)} \log_q(x) + O_K(1).$$

In this formula the product is over all prime divisors P whose norm, NP, is less than or equal to x.

Before proving this, we need the following Lemma.

Lemma 1.1. *There is a constant B_K such that*

$$\sum_{NP \le q^n} \frac{1}{NP} = \log(n) + B_K + O\left(\frac{1}{n}\right).$$

Proof. Let a_n be the number of prime divisors of degree n. It is well known that,

$$a_n = \frac{q^n}{n} + O\left(\frac{q^{n/2}}{n}\right).$$

This estimate uses the Riemann hypotheis for function fields. However, we could replace $n/2$ with $n\theta$ for some $\theta < 1$ which will suffice and is much more easily proven. In any case, define $r_n = a_n - q^n/n$. Then,

$$\sum_{NP \le q^n} \frac{1}{NP} = \sum_{k=1}^{n} \frac{a_k}{q^k} = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{r_k}{q^k}. \tag{1}$$

The first sum is known to equal $\log(n) + \gamma + O(1/n)$. The sum, $B'_K = \sum_{k=1}^{\infty} r_k/q^k$ is convergent by the above estimates. Thus the second sum on the right hand side of (1) is $B'_K + O\left(q^{-n/2}\right)$. The Lemma now follows with $B_K = \gamma + B'_K$.

The proof of the Lemma was adapted from [Kn].

The key to the proof of Theorem 3 is the following lemma. Our proof is a "discrete" version of the proof of Merten's theorem due to G.H.Hardy and exposited in Hardy and Wright.

Lemma 1.2. *Recall that $\alpha_K = \mathrm{Res}_{s=1}\zeta_K(s)$. Let B_K be the constant in the statement of Lemma 1.1. Then,*

$$\sum_P \left[\frac{1}{NP} + \log\left(1 - \frac{1}{NP}\right)\right] = -\gamma - \log\log(q) - \log(\alpha) + B_K.$$

Proof. Define, for $s > 1$, $g(s) = \sum_P \frac{1}{NP^s}$ and consider the sum

$$h(s) = g(s) - \log\zeta_K(s) = \sum_P \left[\frac{1}{NP^s} + \log\left(1 - \frac{1}{NP^s}\right)\right].$$

A standard estimation shows the sum on the right is bounded term by term by the convergent series $\sum_P \frac{1}{NP(NP-1)}$. It follows that the above infinite series for $h(s)$ is uniformly convergent for all $s \geq 1$ and so $h(s) \to h(1)$ as $s \to 1$ from above. Of course, $h(1)$ is the sum given in the statement of the Lemma.

Let $f(n) = \sum_{\deg P \leq n} \frac{1}{NP}$. For convenience, set $f(0) = 0$. Then, for $\delta > 0$, we have

$$\sum_{NP \leq q^n} \frac{1}{NP^{1+\delta}} = \sum_{k=1}^{n}(f(k) - f(k-1))q^{-k\delta} = \sum_{k=1}^{n-1} f(k)(q^{-k\delta} - q^{-(k+1)\delta}) \ + \frac{f(n)}{q^{n\delta}}.$$

The right hand side easily simplifies to

$$\left(1 - \frac{1}{q^\delta}\right)\sum_{k=1}^{n-1}\frac{f(k)}{q^{k\delta}} \ + \ \frac{f(n)}{q^{n\delta}}.$$

Now, using Lemma 1.1 and letting $n \to \infty$ we find

$$g(1+\delta) = \left(1 - \frac{1}{q^\delta}\right)\sum_{k=1}^{\infty}\frac{f(k)}{q^{k\delta}}. \qquad (2)$$

We want to see what happens in this formula as $\delta \to 0$. To this end we invoke Lemma 1.1 once more in the form, $f(k) = \log(k) + B_K + O(1/k)$. We will also repeatedly use the elementary fact that $(1 - q^{-\delta}) = \delta\log(q) + O(\delta^2)$. To see what happens, we investigate the sums obtained by replacing $f(k)$ in equation (2) first by B_K, then by $\log(k)$, and finally by $1/k$.

Replacing $f(k)$ by B_K we find

$$\left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{B_K}{q^{k\delta}} \ = \ B_K q^{-\delta} = \ B_K + o(1). \tag{3}$$

Here, and in what follows, $o(1)$ refers to a quantity that tends to 0 as δ tends to 0.

Replacing $f(k)$ with $\log(k)$ we find

$$\left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{\log(k)}{q^{k\delta}} = \left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{\log(k\delta)}{q^{k\delta}} - \log(\delta)q^{-\delta}. \tag{4}$$

Since $\log(\delta)q^{-\delta} = -\log(\delta) + O(\delta)$, we concentrate on the other sum which is almost an approximating sum for a definite integral. In fact,

$$\sum_{k=1}^{\infty}\frac{\log(k\delta)}{q^{k\delta}}\delta \to \int_0^{\infty}(\log(x))q^{-x}\ dx = \frac{-\gamma-\log\log(q)}{\log(q)}.$$

We have used elementary calculus and the formula $\gamma = -\int_0^{\infty}\log(u)e^{-u}\ du$.

Putting all this together, we find

$$\left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{\log(k)}{q^{k\delta}} = -\gamma - \log\log(q) - \log(\delta) + o(1). \tag{5}$$

Finally, replace $f(k)$ by $1/k$ in equation (2). We find

$$\left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{1}{kq^{k\delta}} = -\left(1-\frac{1}{q^{\delta}}\right)\log\left(1-\frac{1}{q^{\delta}}\right) = o(1). \tag{6}$$

The final equality follows from the fact that $u\log(u) \to 0$ as $u \to 0$.

We have shown,

$$g(1+\delta) = \left(1-\frac{1}{q^{\delta}}\right)\sum_{k=1}^{\infty}\frac{f(k)}{q^{k\delta}} = -\gamma - \log\log(q) - \log(\delta) + B_K + o(1). \tag{7}$$

Note that

$$\alpha_K = \lim_{s\to 1}(s-1)\zeta_K(s) = \lim_{\delta\to 0}\delta\zeta_K(1+\delta).$$

Thus, $\log(\delta) = -\log\zeta_K(1+\delta) + \log(\alpha_K) + o(1)$. Substituting this expression for $\log(\delta)$ into equation (7) yields

$$h(1+\delta) = g(1+\delta) - \log\zeta_K(1+\delta) = -\gamma - \log\log(q) - \log(\alpha_K) + B_K + o(1). \tag{8}$$

The proof of Lemma 1.2 is concluded by letting δ tend to 0 in equation (8).

Proof of Theorem 3. We have shown

$$-\gamma - \log\log(q) - \log(\alpha_K) + B_K = \sum_P \left[\frac{1}{NP} + \log\left(1 - \frac{1}{NP}\right)\right]. \tag{9}$$

The right hand side can be written as

$$\sum_{NP \le x} \left[\frac{1}{NP} + \log\left(1 - \frac{1}{NP}\right)\right] + \epsilon_1(x),$$

where $\epsilon_1(x) \to 0$ as $x \to \infty$. Using Lemma 1.1 once again, we see this can be rewritten as

$$\log(\log_q(x)) + B_K + \sum_{NP \le x} \log\left(1 - \frac{1}{NP}\right) + \epsilon_2(x), \tag{10}$$

where $\epsilon_2(x) \to 0$ as $x \to \infty$.

Combining equations (9) and (10) and exponentiating we find

$$e^{-\gamma}(\log(q))^{-1}\alpha_K^{-1} \sim \log_q(x) \prod_{NP \le x} \left(1 - \frac{1}{NP}\right).$$

Since $\log_q(x) = \log(x)/\log(q)$, we have

$$\prod_{NP \le x} \left(1 - \frac{1}{NP}\right)^{-1} \sim e^{\gamma}\alpha_K \log(x). \tag{11}$$

Recall the formula, $\alpha_K = h_K/\big(q^{g-1}(q-1)\log(q)\big)$, and substitute this into the above asymptotic relation. This gives the asymptotic form of our analogue of Mertens' theorem. To get the stronger form given in the statement of Theorem 3, we have to analyze the functions $\epsilon_1(x)$ and $\epsilon_2(x)$. We will do this in the proof of Theorem 2 given in Section 2. A similar analysis applies here and gives the full result.

Section 2. In this section K will denote an algebraic number field and $\zeta_K(s)$ the Dedekind zeta function of K. Let $\mathcal{O}_K$ denote the ring of integers of K. The letter A will be used to denote a non-zero ideal in $\mathcal{O}_K$ and P will be used to denote a non-zero prime ideal in $\mathcal{O}_K$. Thus,

$$\zeta_K(s) = \sum_A \frac{1}{NA^s} = \prod_P \left(1 - \frac{1}{NP^s}\right)^{-1}.$$

It is known that the sum and product converge for $\mathrm{Re}(s) > 1$, that $\zeta_K(s)$ can be analytically continued to a neighborhood of 1 (in fact, to the whole complex plane), and that at $s = 1$ it has a simple pole. Let α_K denote the residue of that pole at $s = 1$. One knows further that

$$\alpha_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}}.$$

The notations are standard.

The goal of this section is to prove Theorem 2, which was stated in the introduction. The proof is along the lines of that given for the case $K = Q$ in [H-W], but with a number of modifications. The following lemma is essentially Theorem 421 in [H-W] and the reader is referred there for the proof.

Lemma 2.1. *Suppose $c_1, c_2, \ldots$ is a sequence of real numbers. Set $C(t) = \sum_{n \le t} c_n$. Suppose $f(t)$ is a function of t with a continuous derivative for $t \ge 1$. Then,*

$$\sum_{n \le x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t)\ dt. \tag{12}$$

If $c_n = 0$ for $n < n_1$ then the lower limit on the integral in the above equation can be taken to be n_1.

The second tool which we shall use is the following theorem which is due to E. Landau (see Satz 190 of [L]). It is a result which is a small step away from the prime ideal theorem for number fields. Over the rational numbers, Hardy and Wright are able to use a much more elementary starting place. Perhaps this can also be accomplished in the case of more general number fields. However, our proof does proceed quite naturally from this result and it forms the basis for the proposed generalizations of Section 3. So, we shall make use of it.

Theorem 2.2. *Let K be an algebraic number field. Then there is a constant c (depending on K) such that*

$$\theta_K(x) \stackrel{\text{def}}{=} \sum_{NP \le x} \log(NP) = x + O_K\left(xe^{-c\sqrt{\log(x)}}\right).$$

Lemma 2.3. *Let K be an algebraic number field. Then*

$$\sum_{NP \le x} \frac{\log(NP)}{NP} = \log(x) + O_K(1).$$

Proof. Let $b_n = \#\{P \mid NP = n\}$. Then, $\theta_K(x) = \sum_{n \le x} b_n \log(n)$. Apply Lemma 2.1 with $c_n = b_n \log(n)$, $f(t) = 1/t$, and $n_1 = 2$. We find

$$\sum_{NP \le x} \frac{\log(NP)}{NP} = \sum_{n \le x} b_n \log(n) \frac{1}{n} = \frac{\theta_K(x)}{x} + \int_2^x \frac{\theta_K(t)}{t^2}\ dt.$$

The result now follows directly from Theorem 2.2 and the fact that the integral $\int_2^\infty t^{-1} e^{-c\sqrt{\log(t)}}\ dt$ is convergent.

We are now in a position to prove the analogue of Lemma 1.1.

Lemma 2.4. *Let K be an algebraic number field. There is a constant B_K such that*

$$\sum_{NP\le x}\frac{1}{NP}=\log\log(x)+B_K+O_K\left(\frac{1}{\log(x)}\right).$$

Proof. Lemma 2.3 can be rewritten as follows

$$\sum_{NP\le x}\frac{\log(NP)}{NP}=\sum_{n\le x}\frac{b_n}{n}\log(n)=\log(x)+\tau(x). \tag{14}$$

Here, $\tau(x)$ is a bounded function of x. We now apply Lemma 2.1 once again with $c_n = b_n\log(n)/n$, $f(t)=1/\log(t)$, and $n_1=2$. We find

$$\sum_{NP\le x}\frac{1}{NP}=\sum_{n\le x}\frac{b_n}{n}=\frac{C(x)}{\log(x)}+\int_2^x\frac{C(t)}{t(\log(t))^2}\,dt. \tag{15}$$

Here, $C(x)$ is the sum in equation (14). The proof is concluded by substituting $C(x) = \log(x)+\tau(x)$ into equation (15) and a short calculation.

The following lemma, which generalizes Theorem 428 in [H-W] is the main step in the proof of Theorem 2.

Lemma 2.5. *Let B_K be the constant in the statement of Lemma 2.4. Then,*

$$\sum_P\left[\frac{1}{NP}+\log\left(1-\frac{1}{NP}\right)\right]=-\gamma-\log(\alpha_K)+B_K.$$

Proof. We begin as in the proof of Lemma 2.2 by defining for $s>1$, $g(s)=\sum_P\frac{1}{NP^s}$ and

$$h(s)=g(s)-\log\zeta_K(s)=\sum_P\left[\frac{1}{NP^s}+\log\left(1-\frac{1}{NP^s}\right)\right].$$

As before, the sum on the right is majorized term by term by the convergent series $\sum_P\frac{1}{NP(NP-1)}$. It follows that the series defining $h(s)$ is uniformly convergent for $s\ge 1$ and so $h(s)\to h(1)$ as $s\to 1$ from above. The sum we are trying is evaluate is, clearly, $h(1)$. Set $s=1+\delta$. We are going to use Lemma 2.1 yet again. This time $c_n=b_n/n$, $f(t)=t^{-\delta}$, and $n_1=2$ (there are no prime ideals of norm 1). In this case,

$$C(x)=\sum_{n\le x}c_n=\sum_{NP\le x}\frac{1}{NP}=\log\log(x)+B_K+E(x), \tag{16}$$

by Lemma 2.4. Here, $E(x)=O(1/\log(x))$. We deduce,

$$\sum_{NP\le x}\frac{1}{NP^{1+\delta}}=\sum_{n\le x}\frac{b_n}{n}\frac{1}{n^\delta}=\frac{C(x)}{x^\delta}+\delta\int_2^x\frac{C(t)}{t^{1+\delta}}\,dt. \tag{17}$$

Using equation (16) and letting $x \to \infty$ yields,

$$g(1+\delta) = \delta \int_2^\infty \frac{C(t)}{t^{1+\delta}}\, dt. \tag{18}$$

Since $C(t) = \log\log(t) + B_K + E(t)$ we will replace $C(t)$ first by B_K, then by $\log\log(t)$, and then by $E(t)$ in equation (18) and see what happens as $\delta \to 0$.

Replacing $C(t)$ by B_K we find,

$$\delta \int_2^\infty \frac{B_K}{t^{1+\delta}}\, dt = B_K - \delta \int_1^2 \frac{B_K}{t^{1+\delta}}\, dt = B_K + O(\delta). \tag{19}$$

Next, replace $C(t)$ by $\log\log(t)$. We find,

$$\delta \int_2^\infty \frac{\log\log(t)}{t^{1+\delta}}\, dt = \delta \int_1^\infty \frac{\log\log(t)}{t^{1+\delta}}\, dt - \delta \int_1^2 \frac{\log\log(t)}{t^{1+\delta}}\, dt =$$

$$-\gamma - \log(\delta) + O(\delta). \tag{20}$$

The integral from 1 to ∞ is evaluated by means of the substitution $t = e^{u/\delta}$. The error term is the result of the observation that $\int_1^2 t^{-1} |\log\log(t)|\, dt$ is a convergent integral.

Finally, we substitute $1/\log(t)$ for $C(t)$ and deduce

$$\delta \int_2^\infty \frac{dt}{\log(t)\ t^{1+\delta}} = \delta \int_{\log(2)}^\infty \frac{du}{u\ e^{\delta u}} = \delta \int_{\delta \log(2)}^\infty \frac{dw}{w\ e^{w}} = o(1). \tag{21}$$

The last equality follows from an application of L'Hôpital's rule.

Putting together equations (19), (20), and (21), we find

$$g(1+\delta) = -\gamma - \log(\delta) + B_K + o(1). \tag{22}$$

From $\lim_{\delta\to 0} \delta\zeta_K(1+\delta) = \alpha_K$ we see $\log(\delta) + \log\zeta_K(1+\delta) \to \log(\alpha_K)$ as $\delta \to 0$. Substituting this in equation (22) gives us the result

$$h(1+\delta) = g(1+\delta) - \log\zeta_K(1+\delta) = -\gamma - \log(\alpha_K) + B_K + o(1). \tag{23}$$

The proof of Lemma 2.5 is concluded by letting $\delta \to 0$ in equation (23).

Proof of Theorem 2. The proof is a slightly sharper version of the last part of the proof of Theorem 3 in the last section. We rewrite the assertion of Lemma 2.5 as

$$-\gamma - \log(\alpha_K) + B_K = \sum_{NP \le x} \left[\frac{1}{NP} + \log\left(1 - \frac{1}{NP}\right)\right] + \epsilon_1(x). \tag{24}$$

Let $d = [K : \mathbf{Q}]$. Then,

$$|\epsilon_1(x)| = -\sum_{NP > x} \left[\frac{1}{NP} + \log\left(1 - \frac{1}{NP}\right)\right] < \sum_{NP > x} \frac{1}{NP(NP-1)}$$

$$< \sum_{p > \sqrt[d]{x}} \frac{d}{p(p-1)} < \sum_{n > \sqrt[d]{x}} \frac{d}{n(n-1)} = \frac{d}{[\sqrt[d]{x}]}.$$

It follows that $\epsilon_1(x) = 0(x^{-\frac{1}{d}})$

Using this fact about $\epsilon_1(x)$ and Lemma 2.4, one can rewrite the right hand side of equation (24) as

$$\log\log(x) + B_K + \sum_{NP \leq x} \log\left(1 - \frac{1}{NP}\right) + \epsilon_2(x),$$

where $\epsilon_2(x) = O(\frac{1}{\log(x)}) + O(x^{-\frac{1}{d}}) = O(\frac{1}{\log(x)})$. Note that

$$e^{O\left(\frac{1}{\log(x)}\right)} = 1 + O\left(\frac{1}{\log(x)}\right).$$

Exponentiating yields

$$e^{-\gamma}\alpha_K^{-1} = \log(x) \prod_{NP \leq x} \left(1 - \frac{1}{NP}\right)\left(1 + O\left(\frac{1}{\log(x)}\right)\right).$$

Theorem 2 follows by simply rearranging the terms.

Section 3. We preserve the notations of Section 2. In particular, K will denote a finite algebraic number field. We will consider a general class of Dirichlet series based on K of the form

$$D(s) = \sum_A \frac{a(A)}{NA^s} = \prod_P L_P(NP^{-s})^{-1}. \tag{25}$$

In this definition,

$$L_P(T) = \prod_{i=1}^{m} (1 - \rho(P,i)T) \tag{26}$$

denotes a polynomial of fixed degree m independent of P.

As an example of the type of thing we have in mind, let V/K be an absolutely irreducible, projective, smooth variety defined over K. For an appropriate open subscheme Y of $\mathrm{Spec}(\mathcal{O}_K)$ one can construct a projective and smooth morphism $f : X \to Y$ of schemes, where X is irreducible of finite type over $\mathbf{Z}$ and with generic fiber the given morphism $V \to \mathrm{Spec}(K)$. Suppose V has dimension d. Then the zeta function of X can be factored as follows

$$\zeta(X,s) = \frac{D_0(s)D_2(s)\ldots D_{2d}(s)}{D_1(s)\ldots D_{2d-1}(s)},$$

where for each $0 \leq j \leq 2d$,

$$D_j(s) = \prod_P L_{P,j}(NP^{-s})^{-1}.$$

The $L_{P,j}(T)$ are polynomials of fixed degree independent of P with reciprocal roots of absolute value $NP^{j/2}$ (Deligne's theorem). It follows that the series $D_j(s)$ converges absolutely for $\mathrm{Re}(s) > 1 + \frac{j}{2}$. It is conjectured (A. Weil) that these series can be continued meromorphically to the whole complex plane. There are conjectures of J. Tate which state that the order of the pole of $D_{2j}(s)$ at $1+j$ is equal to the rank of the group of algebraic cycles defined over K on V of codimension j modulo homological equivalence. For details on all of this see Tate's famous article [Ta].

Other examples of the type of Dirichlet series we have in mind come from compatible systems of l-adic representations over K, automorphic representations, etc.

In all these examples, the inverse roots $\rho(P,i)$ of $L_P(T)$, have a fixed size, at least conjecturally, usually a power of NP. We will assume that this fixed size is 1. This is not a serious restriction because it can always be obtained in these examples by means of an appropriate translation $s \to s+m$.

Assuming $|\rho(P,i)| = 1$ for all P and i, we see that the series defining $D(s)$ converges for $\mathrm{Re}(s) > 1$. Assume it can be analytically continued to a neighborhood of $s=1$ and that it has a pole at $s=1$ of order k. Define $\alpha = \lim_{s\to 1}(s-1)^k D(s)$. Here is a conjectural generalization of Merten's theorem.

$$\prod_{NP \le x} L_P(NP^{-1})^{-1} = e^{k\gamma}\alpha \log^k(x) + O\left(\log^{k-1}(x)\right). \tag{28}$$

To show that this is plausible, suppose the left hand side is equal to $c\log(x)^k + O\left(\log(x)^{k-1}\right)$ for some constant c. Then consider the quotient

$$\frac{\prod_{NP\le x} L_P(NP^{-s})^{-1}}{\prod_{NP\le x}(1-NP^{-s})^{-k}}.$$

First let $x \to \infty$ and then $s \to 1$. The result is α/α_K^k. Now let $s \to 1$ first and then let $x \to \infty$. Using Theorem 2 and the above supposition, the result is $c/e^{k\gamma}\alpha_K^k$. Equating these results we find $c = e^{k\gamma}\alpha$.

Of course, this argument is not rigorous since the supposition and the interchange of the limits has not been justified. We will show how to establish the formula in equation (28) on the basis of a rather plausible assumption about the series $D(s)$. Assuming $D(s)$ has a pole of order k at $s=1$ consider the logarithmic derivative of $D(s)$. One finds

$$-\frac{D'(s)}{D(s)} = \sum_P \sum_j \left(\sum_i \rho(P,i)^j\right) \log(NP) NP^{-js}.$$

When $j=1$ one has $\sum_i \rho(P,i) = a(P)$. A standard calculation shows that the sum of the terms where $j > 1$ converges absolutely when $\mathrm{Re}(s) > \frac{1}{2}$. Finally, notice that $-D'(s)/D(s)$ has a simple pole with residue k at $s=1$. It follows that

$$\sum_P \frac{a(P)\log(NP)}{NP^s}$$

has a simple pole with residue k at $s = 1$. Thus , assumption b) in the following theorem can be taken to be a "prime number theorem" for our series $D(s)$.

Theorem 4. *Let $D(s)$ and the polynomials $L_P(T)$ be defined as in equations (25) and (26). Assume $D(s)$ converges for $\mathrm{Re}(s) > 1$ and can be analytically continued to a neighborhood of $s = 1$ where it either has a pole of order $k \geq 1$, or is holomorphic and not zero at $s = 1$ in which case we set $k = 0$. Define $\alpha = \lim_{s\to 1}(s-1)^k D(s)$. Finally, assume*

$$a) \quad |\rho(P,i)| = 1 \quad \text{for all} \quad P, i.$$

$$b) \quad \sum_{NP\leq x} a(P)\log(NP) = kx + O\left(xe^{-c\sqrt{\log(x)}}\right).$$

In part b), c is some suitable constant greater than 0. With these definitions and assumptions we have

$$\prod_{NP\leq x} L_P(NP^{-1})^{-1} = e^{k\gamma}\alpha\log^k(x) + O\left(\log^{k-1}(x)\right).$$

Proof. The proof proceeds in the same manner as the proof of Theorem 2. Because of this we merely outline the steps.

Using Lemma 2.1 and assumption b) in the statement of the theorem, we deduce

$$\sum_{NP\leq x} \frac{a(P)\log(NP)}{NP} = k\log(x) + O(1).$$

Next, using this relation and Lemma 2.1 one shows there is a constant B such that

$$\sum_{NP\leq x} \frac{a(P)}{NP} = k\log\log(x) + B + O\left(\frac{1}{\log(x)}\right). \tag{29}$$

Finally, one proves the following relation which generalizes Lemma 2.5. The proof follows the same sequence of steps as the proof of that Lemma.

$$\sum_P \left[\frac{a(P)}{NP} + \log L_P(NP^{-1})\right] = -k\gamma - \log(\alpha) + B. \tag{30}$$

Theorem 4 follows as before by combining equations (29) and (30).

It should be remarked that the case $k = 0$ works perfectly well and results in the relation,

$$\prod_{NP\leq x} L_P(NP^{-1})^{-1} = D(1) + O\left(\frac{1}{\log(x)}\right).$$

Let K be an algebraic number field, and χ a non-trivial Hecke character of finite order. Landau has shown (see [L2], Satz LXXXIII) that hypothesis $b)$ of Theorem 4 hold for the L-series $L(s,\chi)$. Thus, one has

$$\prod_{NP\leq x}\left(1-\chi(P)NP^{-1}\right)^{-1}=L(1,\chi)+O_K\left(\frac{1}{\log(x)}\right).$$

It is of interest to point out that in his "Handbuch" on prime numbers [L3], Landau explicitly deals with this result when χ is a Dirichlet character on the field of rational numbers. In [W], K. Williams uses Landau's result to provide the asymptotics of

$$\prod_{p\leq x,\ p\equiv a \pmod{l}}(1-p^{-1})^{-1}\ ,$$

thus generalizing Merten's theorem in another direction. It would be of interest to extend William's work to number fields and function fields using the results of this paper.

It is useful to remark that hypothesis $b)$ of Theorem 4 can be reformulated in a somewhat more flexible way. Recall that $D(s)$ is the Dirichlet series under consideration. Let's define

$$\theta(x,D)=\sum_{NP\leq x}a(P)\log(NP),$$

and

$$\psi(x,D)=\sum_{NP^m\leq x}a(P^m)\log(NP).$$

To be very explicit, the sum in the definition of $\psi(x,D)$ is over all prime ideals P and integers m such that $NP^m\leq x$.

Using assumption $a)$ of Theorem 4, one easily shows (see [H-W], Theorem 413) that $\psi(x,D)=\theta(x,D)+O\left(\sqrt{x}\log(x)^2\right)$. It follows that hypothesis $b)$ holds for $\theta(x,D)$ if and only if it holds for $\psi(x,D)$. The reason this is useful is that $\psi(x,D)$ has a natural interpretation. Namely, write

$$\frac{D'(s)}{D(s)}=\sum_{n=1}^{\infty}\frac{d_n}{n^s}.$$

Then, a simple calculation shows that $\psi(x,D)=\sum_{n\leq x}d_n$.

Theorem 5. *Let L/K be an Galois extension of algebraic number fields with Galois group G. Let χ run over the irreducible complex characters of G with χ_o denoting the trivial character. Let $\rho=\sum n(\chi)\chi$ be a character of G, and $L(s,\rho)$ the corresponding Artin L-function. Denoting the local factors of $L(s,\rho)$ by $L_P(s,\rho)$, we have*

$$\prod_{NP\leq x}L_P(1,\rho)^{-1}=e^{k\gamma}\alpha\log^k(x)+O_{K,\rho}\left(\log^{k-1}(x)\right),$$

where $k = n(\chi_o)$ *and* $\alpha = \alpha_K^k \prod_{\chi \neq \chi_o} L(1,\chi)^{n(\chi)}$.

Proof. When $D(s) = L(s,\chi)$ we will use the notation $\psi(x,D) = \psi(x,\chi)$. It is almost immediate from the definition that $\psi(x,\chi_1+\chi_2) = \psi(x,\chi_1) + \psi(x,\chi_2)$.

Another useful property of the ψ-function is that it is invariant under induction. Namely, let λ be a character of a subgroup $H \subset G$. If $\lambda^* = Ind_H^G(\lambda)$ is the charcter of G induced from λ, then $\psi(x,\lambda^*) = \psi(x,\lambda)$. To see this, recall the corresponding property of Artin L-functions; $L(s,\lambda^*) = L(s,\lambda)$. Take the logarithmic derivative of both sides, write the result in the form $\sum a_n n^{-s} = \sum b_n n^{-s}$ and equate coefficients (see the remarks preceding the statement of the theorem).

To prove the theorem we must verify that both assumptions $a)$ and $b)$ of Theorem 4 are verified in this case. Assumption $a)$ is trivially verified because the inverse roots of $L_P(T,\rho)$ are roots of unity in the case of Artin L-functions. As for assumption $b)$ notice that

$$\psi(x,\rho) = n(\chi_o)\psi(x,\chi_o) + \sum_{\chi \neq \chi_o} n(\chi)\psi(x,\chi).$$

Since $L(s,\chi_o) = \zeta_K(s)$ we know (Theorem 2.2) that $\psi(x,\chi_o) = x + O\left(xe^{-c\sqrt{\log(x)}}\right)$ for an appropriate positive contant c. Thus, the proof comes down to showing that if χ is a non-trivial, irreducible character of G then $\psi(x,\chi) = O\left(xe^{-c\sqrt{\log(x)}}\right)$ for some $c > 0$. If χ is a linear character, then by the Artin reciprocity theorem $L(s,\chi)$ can be identified with a Hecke L-series and the result follows from Landau's Theorem LXXXIII in [L2] referred to previously. We now reduce the general case to this special case using the Brauer induction theorem.

Let χ be a non-trivial, irreducible character of G. We will let λ vary over all linear characters of all subgroups of G. Denote the subgroup on which λ is a character by H_λ. By the Brauer induction theorem there exist integers $m(\lambda)$ such that

$$\chi = \sum_\lambda m(\lambda)\lambda^*. \tag{31}$$

Using both of the above mentioned properties of the ψ-function we deduce

$$\psi(x,\chi) = \sum_\lambda m(\lambda)\psi(x,\lambda).$$

Some of the characters occuring in this sum could be trivial. So, invoking Landau's results once again we find that there is a constant $c > 0$ such that

$$\psi(x,\chi) = \left(\sum_{\lambda=\lambda_o} m(\lambda)\right) x + O\left(xe^{-c\sqrt{\log(x)}}\right).$$

It remains to prove that the sum in front of the x is zero. This follows from equation (31) and the Frobenius reciprocity law as follows.

$$0 = (\chi, \chi_o)_G = \sum_{\lambda} m(\lambda)(\lambda, Res^G_{H_\lambda}(\chi_o))_{H_\lambda} = \sum_{\lambda=\lambda_o} m(\lambda).$$

The proof is complete.

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We conclude by presenting a possible application to arithmetic geometry. Let's return to the Dirichlet series attached to algebraic varieties defined over K as described at the beginning of this section. Suppose V/K is an absolutely irreducible, smooth, projective surface defined over K and consider the Dirichlet series

$$D_2(s) = \prod_P L_{P,2}(NP^{-s})^{-1}.$$

In this case the degree of the polynomials $L_{P,2}(T)$ has degree equal to the second Betti number of V and the inverse roots have absolute value NP. Thus, the product for $D_2(s)$ converges absolutely for $\text{Re}(s) > 2$. Assuming analytic continuation to a neighborhood of $s = 2$, Tate's conjecture in this case asserts that $D_2(s)$ has a pole at $s = 2$ equal to k, the rank of the Néron-Severi group, $NS_K(V)$. Let $\alpha = \lim_{s\to 2}(s-2)^k D_2(s)$.

Notice that $D_2^*(s) = D_2(s+1)$ satisfies assumption a) of Theorem 4. It is easy to see that it has a pole of order k at $s = 1$ and $\alpha = \lim_{s\to 1}(s-1)^k D_2^*(s)$. Assume that D_2^* satisfies assumption b) as well. It then follows easily that

$$\prod_{NP\le x} L_{P,2}(NP^{-2})^{-1} = e^{k\gamma}\alpha \log^k(x) + O\left(\log^{k-1}(x)\right).$$

We have dealt with the case dim $V = 2$ just for the sake of concreteness. The same type of analysis would, of course, apply more widely to these geometric examples.

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