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Area of Archimedean spiral

The graph of $r(\theta) = \theta a$, where a is a constant, is called Archimedean spiral. Let's consider the area of such spiral when it has made one whole revolution. In that case we have the angle $\theta = 2\pi$ and radius $r(2\pi) = 2\pi a$.

The length of spiral curve given an angle θ gets closer to the length of the corresponding arc of a circle, as θ gets smaller. This said, the differential $d\theta$ draws a curve with length $d\theta\, r(\theta)$ and sweeps the area $\frac{1}{2} d\theta\, r(\theta) r(\theta) = \frac{1}{2}(r(\theta))^2\, d\theta$.

Now, by integrating from 0 to 2π we get the area of one revolution

$$\int_0^{2\pi} \frac{1}{2}(r(\theta))^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(\theta a)^2\, d\theta = \left[\frac{\theta^3 a^2}{6}\right]_0^{2\pi} = \frac{(2\pi)^3 a^2}{6} = \frac{4\pi^3 a^2}{3}\,.$$

When comparing this result to the area of a circle with radius $r(2\pi) = 2\pi a$, we see that it equals three times the area of the Archimedean spiral after one revolution:

$$\pi(2\pi a)^2 = 4\pi^3 a^2 = 3 \cdot \frac{4\pi^3 a^2}{3}\,.$$