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Fibonacci sequence and golden ratio

The famous Fibonacci sequence starts with 1, 1, 2, 3, 5, 8, 13... and its members have the relationship $F_{n+1} - F_n = F_{n-1}$. Based on this, let's prove that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi \text{, where } \Phi \text{ is the golden ratio.}$$

By the relationship above

$$F_{n+1} - F_n = F_{n-1} \quad \Leftrightarrow \quad \frac{F_{n+1}}{F_n} - 1 = \frac{F_{n-1}}{F_n} \quad \Leftrightarrow \quad \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

and using this result recursively, we get

$$\begin{aligned} \frac{F_{n+1}}{F_n} &= 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{1}{1 + \frac{F_{n-1}}{F_{n-2}}} \\ &= 1 + \frac{1}{1 + \frac{1}{\frac{F_{n-2}}{F_{n-1}}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{F_{n-2}}{F_{n-3}}}} = 1 + \frac{1}{1 + \frac{1}{1 + \dots}}, \end{aligned}$$

which is a known form for golden ratio Φ.