Linear Algebra cheat sheet

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May 23, 2018

1 Chapter 1

$\boxed{\text{Thm}}$ Given $\mathbf{A}\vec{x} = \vec{b}$, $\mathbf{A}$ is $m \times n$, the following statements are logically equivalent:

a) For each $\vec{b}$ in $\mathbb{R}^m$, $\mathbf{A}\vec{x} = \vec{b}$ has a solution.
b) Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of columns of $\mathbf{A}$.
c) The columns of $\mathbf{A}$ span $\mathbb{R}^m$.
d) $\mathbf{A}$ has a pivot position in every row.

1.1 Transformations

$\boxed{\text{Def}}$ A transformation T is linear if:

a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \forall \vec{u}, \vec{v} \in T$ check!!
b) $T(c\vec{u}) = c\,T(\vec{u}), \forall c \in \mathbb{R}, \vec{u} \in T$

$\boxed{\text{Def}}$ A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$. This is assuming the mapping is Ax=b.

– pivots in every row iff columns span $\mathbb{R}^m$.

$\boxed{\text{Def}}$ A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$. This is assuming the mapping is Ax=b.

– pivots in every column iff columns are linearly independent.

2 Chapter 2

$\boxed{\text{Thm}}$ Matrix transpose, inverse identities:

a) $(\mathbf{A}^\intercal)^\intercal = \mathbf{A}$
b) $(\mathbf{A} + \mathbf{B})^\intercal = \mathbf{A}^\intercal + \mathbf{B}^\intercal$
c) $(r\,\mathbf{A})^\intercal = r\mathbf{A}^\intercal, \forall r \in \mathbb{R}$
d) $(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal$
e) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
f) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
g) $(\mathbf{A}^\intercal)^{-1} = (\mathbf{A}^{-1})^\intercal$

$\boxed{\text{Thm}}$ Inverse of a 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3 Chapter 3

$\boxed{\text{Thm}}$ Determinant properties

a) Multiple of one row of $\mathbf{A}$ added to another to form $\mathbf{B}$, $\det \mathbf{B} = \det \mathbf{A}$.

b) Two rows of $\mathbf{A}$ interchanged to produce $\mathbf{B}$, $\det \mathbf{B} = -\det \mathbf{A}$.

c) One row of $\mathbf{A}$ multiplied by k to produce $\mathbf{B}$, $\det \mathbf{B} = k \det \mathbf{A}$.

$\boxed{\text{Thm}}$ If $\mathbf{A}_{n\times n}$, then $\det \mathbf{A}^\intercal = \det \mathbf{A}$.

$\boxed{\text{Thm}}$ If $\mathbf{A}_{n\times n}, \mathbf{B}_{n\times n}$, then $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$.

3.1 Cramer's Rule

$\boxed{\text{Thm}}$ $\mathbf{A}_{n\times n}$ and invertible. $\mathbf{A}_i(\vec{b})$ is made by replacing the i^{th} column of $\mathbf{A}$ with $\vec{b}$. x_i is the i^{th} element of $\vec{x}$.

For any $\vec{b} \in \mathbb{R}^n$, $x_i = \frac{\det \mathbf{A}_i(\vec{b})}{\det \mathbf{A}}, i = 1, 2, ..., n$.

4 Chapter 4

$\boxed{\text{Def}}$ A **vector space** is a nonempty set of vectors for which the following ten axioms hold: note that u, v, w in V

1) $\vec{u}+\vec{v} \in \mathcal{V}$
2) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
4) $\vec{u}+\vec{0}=\vec{u}$ (a $\vec{0}$ exists in $\mathcal{V}$)
5) $\forall \vec{u} \in \mathcal{V}, \vec{u}+(-\vec{u})=\vec{0}$
6) $c\vec{u} \in \mathcal{V}$
7) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
8) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
9) $c(d\vec{u})=(cd)\vec{u}$
10) $1\vec{u}=\vec{u}$

$\boxed{\text{Def}}$ A **subspace** of $\mathcal{V}$ is a subset $\mathcal{H}$ of $\mathcal{V}$ with the following three properties:

a) The $\vec{0}$ of $\mathcal{V}$ is in $\mathcal{H}$
b) $\vec{u}+\vec{v} \in \mathcal{H}, \forall \vec{u}, \vec{v} \in \mathcal{H}$
c) $c\vec{u} \in \mathcal{H}, \forall c \in \mathbb{R}, \forall \vec{u} \in \mathcal{H}$

$\boxed{\text{Thm}}$ If $\vec{v_1}, ..., \vec{v_p} \in \mathcal{V}$, then Span$\{\vec{v_1}, ..., \vec{v_p}\}$ is a subspace of $\mathcal{V}$.

$\boxed{\text{Def}}$ $\text{Nul}\,\mathbf{A} = \{\vec{x} : \vec{x} \in \mathbb{R}^n, \mathbf{A}\vec{x}=\vec{0}\}$

$\boxed{\text{Def}}$ $\text{Col}\,\mathbf{A} = \{\vec{b} : \vec{b}=\mathbf{A}\vec{x}, \text{for some } \vec{x} \in \mathbb{R}^n\}$

- pivot columns of $\mathbf{A}$ forms a basis for Col $\mathbf{A}$

$\boxed{\text{Def}}$ Suppose $\mathcal{H}$ is a subspace of $\mathcal{V}$. $\beta = \{\vec{b_1}, ..., \vec{b_p}\}$ in $\mathcal{V}$ is a **basis** for $\mathcal{H}$ if:

a) β is a linearly independent set
b) $\mathcal{H} = \text{Span}\{\vec{b_1}, ..., \vec{b_p}\}$

$\boxed{\text{Def}}$ $\text{rank}\,\mathbf{A} = \dim \text{Col}\,\mathbf{A} = \dim \text{Row}\,\mathbf{A} = \dim \text{Col}\,\mathbf{A}^\intercal$
$\text{rank}\,\mathbf{A} + \dim \text{Nul}\,\mathbf{A} = n$

5 Chapter 5

$\boxed{\text{Thm}}$ The eigenvalues of a triangular matrix are the entries on its main diagonal.

$\boxed{\text{Thm}}$ If $\vec{v}_1, ..., \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_r$, then the set $\{\vec{v}_1, ..., \vec{v}_r\}$ is linearly independent.

$\boxed{\text{Fact}}$ $\det \mathbf{A} = \begin{cases} (-1)^r \cdot (\text{product of pivots in echelon form}), & \mathbf{A} \text{ is invertible} \\ 0, & \mathbf{A} \text{ is not invertible} \end{cases}$

$\boxed{\text{Thm}}$ Determinant properties: $\mathbf{A}_{n\times n}, \mathbf{B}_{n\times n}$

a) $\mathbf{A}$ is invertible $\iff$ $\det \mathbf{A} \neq 0$

b) $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$

c) $\det A^{\intercal} = \det \mathbf{A}$

d) If $\mathbf{A}$ is triangular, then $\det \mathbf{A}$ = product of entries on main diagonal

$\boxed{\text{Fact}}$ λ is an eigenvalue of $A_{n\times n}$ $\iff$ $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$\boxed{\text{Def}}$ Suppose $\mathbf{A}_{n\times n}, \mathbf{B}_{n\times n}$. $\mathbf{A}$ is similar to $\mathbf{B}$, denoted $\mathbf{A} \sim \mathbf{B}$, if $\exists\, \mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$.

$\boxed{\text{Thm}}$ If $\mathbf{A} \sim \mathbf{B}$, then they have the same characteristic polynomial and the same eigenvalues with the same multiplicities.

$\boxed{\text{Thm}}$ $\mathbf{A}_{n\times n}$ is diagonalizable $\iff$ $\mathbf{A}$ has n linearly independent eigenvectors.

$\boxed{\text{Thm}}$ Suppose $\mathbf{A}_{n\times n}$ with distinct eigenvalues: $\lambda_1, ..., \lambda_p$

a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k.

b) $\mathbf{A}$ is diagonalizable $\iff$ sum of dimensions of the eigenspaces $= n$.

c) If $\mathbf{A}$ is diagonalizable and β_k is the basis for the eigensapce corresponding to λ_k for each k, then $\beta_1, ..., \beta_p$ forms an eigenvector basis for $\mathbb{R}^n$.

6 Chapter 6

later!

7 Chapter 7

$\boxed{\text{Thm}}$ Invertible Matrix Theorem: Given $\mathbf{A}$ is $n \times n$, the following statements are logically equivalent:

a) $\mathbf{A}$ is invertible.

b) $\mathbf{A}$ is row equivalent to $\mathbf{I}_{n\times n}$.

c) $\mathbf{A}$ has n pivot positions.

d) $\mathbf{A}\vec{x} = \vec{0}$ has only trivial solutions.

e) Columns of $\mathbf{A}$ form a linearly independent set.

f) $\mathbf{A}\vec{x} = \vec{b}$ has at least one solutions for each $\vec{b}$ in $\mathbb{R}^n$.

g) $\vec{x} \mapsto \mathbf{A}\vec{x}$ is one-to-one.

h) Columns of $\mathbf{A}$ span $\mathbb{R}^n$.

i) $\vec{x} \mapsto \mathbf{A}\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) $\exists\, \mathbf{C}_{n\times n}$ such that $\mathbf{CA} = \mathbf{I}$.

k) $\exists\, \mathbf{D}_{n\times n}$ such that $\mathbf{AD} = \mathbf{I}$.

l) $\mathbf{A}^\intercal$ is an invertible matrix.

m) Columns of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.

n) Col $\mathbf{A} = \mathbb{R}^n$.

o) dim Col $\mathbf{A} = n$.

p) rank $\mathbf{A} = n$.

q) Nul $\mathbf{A} = \vec{0}$ check this...

r) dim Nul $\mathbf{A} = 0$.

s) 0 is not an eigenvalue of $\mathbf{A}$.

t) The determinant of $\mathbf{A}$ is not 0.