1. A. We will label "canned" as 0 and "bagged" as 1. "No" is 0 and "Yes" is 1. Then our table looks like

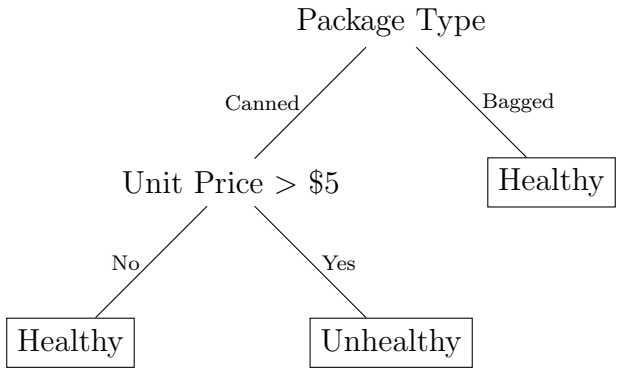
No.	Package Type	Unit Price > \$5	Contains > 5 grams of fat	Healthy?
1	0	1	1	0
2	1	1	0	1
3	1	0	1	1
4	0	0	0	1

We split based on the largest decrease in entropy:

$$L(S) = -|S|(p_S \log p_S + (1 - p_S) \log(1 - p_S)).$$

We start with S includes the entire training set. $p_S = 3/4$ and we get $L(S) \approx 2.249$. If we split by package type = 0, then we get $L(S_{\text{left}}) + L(S_{\text{right}}) \approx 1.386$. If we split by Unit Price > \$5 = 0, then we get $L_{\text{left}} + L(S_{\text{right}}) \approx 1.386$. Similarly, splitting by contains > 5 grams of fat = 0 yields the same split reduction. We choose the first split by default.

By the same argument, we can choose between splitting along unit price or grams of fat as either give the same loss reduction. By default, we choose the former, and we get the final table.



B. No. For example in Figure 1 we clearly see that for this linearly separable data, the perceptron is much simpler than the tree, and both have zero training error.

C. i. The initial loss according to the gini index is ($p_S = 0.5$)

$$L(S) = 4\left(1 - \frac{1}{4} - \frac{1}{4}\right) = 2$$

A single split along the vertical axis ($x_2 = 0$) yields

$$L(S) = 1 + 1 = 2,$$

so there's no reduction in impurity. The same results in a split along $x_1 = 0$. So the classification error is 0.5.

ii. The tree with no training error is shown below in Figure 2.

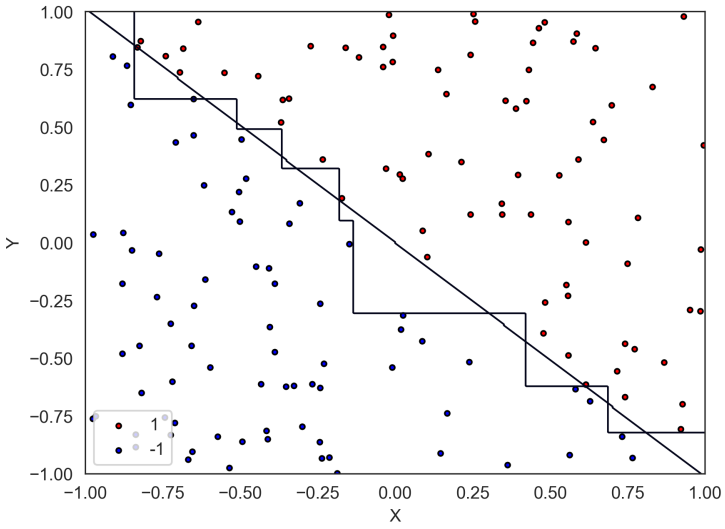


Figure 1: Training linearly separable data with perceptron versus tree classifier.

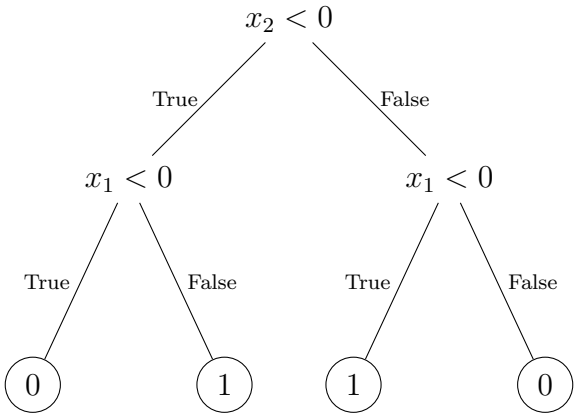


Figure 2: Tree for 4 data points in Problem C.

iii. Yes, we could have the impurity measure

$$L(S) = |S|^2 \left(1 - p_S^2 - (1 - p_S)^2\right).$$

This would create the tree shown in Figure 2. The problem with this loss metric is that while it prioritizes too much on the size of the dataset size with the $|S|^2$ factor. As a result, a node prioritizes making the two children nodes have equal sizes over separating classes.

iv. In the worst case scenario, each partition or leaf only contains one point in the training set. Therefore, we need at least 100 leaf nodes. Starting at the root node corresponding to depth level $\ell = 0$, the nth depth level has 2^ℓ nodes. Note that $2^6 = 64$, so we need at least 7 levels, where the 7th level has $2 \times (100 - 64) = 72$ nodes (all of them leaf nodes). As a result, the 6th level has 36 internal nodes.

Summing from the root to the 5th level internal nodes yields

$$\sum_{k=0}^{5} 2^k = 63.$$

Therefore, the number of internal nodes in the worst case scenario of every leaf node containing one point gives $63 + 36 = 99$ internal nodes. Of course, most datapoints won't produce trees with 99 data points (the average value is about 36 nodes from averaging random datasets).

D. For each continuous feature, we must consider $N - 1$ splits. Therefore, we must consider $D(N-1)$ splits $\sim O(ND)$.

2. A. We see from Figure 3 that from the E_{out} curve, the optimal minimum leaf size for this dataset is 12.

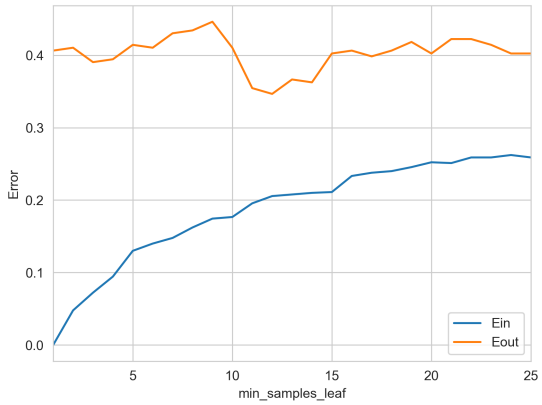


Figure 3: The optimal leaf size is 12.

B. We see from Figure 4 that from the E_{out} curve, the optimal depth for the tree is 2.

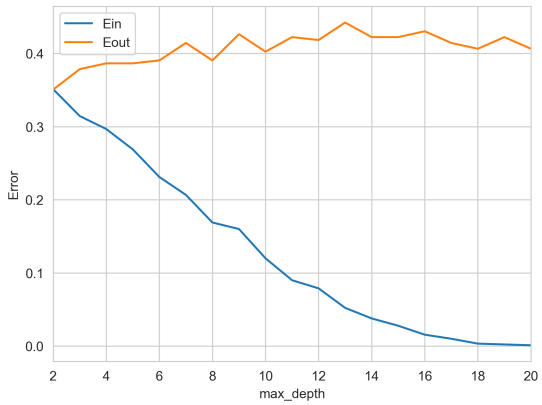


Figure 4: The optimal depth is 2.

C. We see that neither stopping criterion has a profound impact on the accuracy of the model. We see some overfitting in the depth.

D. We see from Figure 5 that from the E_{out} curve, the optimal minimum leaf sisze for the tree is 3.

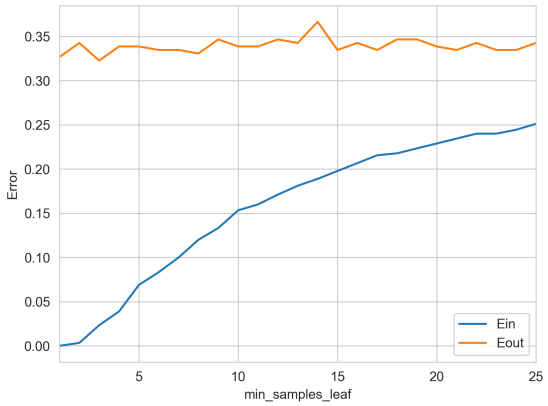


Figure 5: The optimal leaf size is 3.

E. We see from Figure 6 that from the E_{out} curve, the optimal depth for the tree is 12.

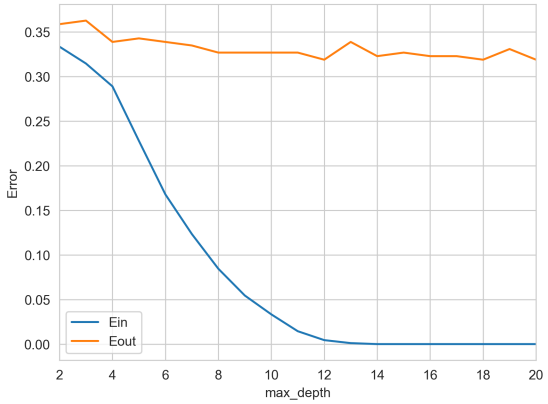


Figure 6: The optimal depth is 12.

F. The early stopping on leaf size has little effect on the model accuracy. However, it does seem that depth has a stronger effect than leaf size.

G. The depth curve for the tree increases while it decreases for the random forest. The reason could be that the variance for the tree is greater than that for the random forest, so the E_{out} curve for this particular train-test split has this characterization. The random forest has a smaller variance, and so it produces an E_{out} curve more similar to what we would expect.

3. A. We look at the term

$$E = \frac{1}{N}\sum_{i=1}^{N} \exp\left(-y_i f(x_i)\right).$$

Now because y_i and $f(x_i)$ have the same or opposite signs, it follows that if y_i and $f(x_i)$ have the same size, then the term in the exponent is negative, so

$$0 < \exp\left(-y_i f(x_i) \leq 1\right), \qquad y_i f(x_i) \geq 0$$

and similarly if they have opposite signs

$$1 \leq \exp\left(-y_i f(x_i)\right), \qquad y_i f(x_i) \leq 0.$$

In either case, we see that

$$\exp\left(-y_i f(x_i)\right) \geq \mathbb{1}\left(H(x_i) \neq y_i\right),$$

so the desired result follows:

$$\frac{1}{N}\sum_{i=1}^{N} \exp\left(-y_i f(x_i)\right) \geq \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}\left(H(x_i) \neq y_i\right).$$

B. We have

$$D_{t+1}(i) = \frac{1}{Z_t} D_t(i) \exp\left(-\alpha_t y_i h_t(x_i)\right).$$

We substitute recursively and note that $D_1(i) = 1/N$, so we have

$$D_{T+1}(i) = \frac{1}{N}\prod_{t=1}^{T} \frac{1}{Z_t} \exp\left(-\alpha_t y_i h_t(x_i)\right).$$

C. Just plug in $f(x_i)$ as the weighted sum of weak classifiers.

$$\begin{aligned} E &= \frac{1}{N}\sum_{i=1}^{N} \exp\left(-y_i f(x_i)\right) \\ &= \frac{1}{N}\sum_{i=1}^{N} \exp\left(-y_i \sum_{t=1}^{T} \alpha_t h_t(x_i)\right). \end{aligned}$$

D. Expanding the result in C and using the result from B, we have

$$\begin{aligned} E &= \frac{1}{N}\sum_{i=1}^{N} \exp\left(-\sum_{t=1}^{T} \alpha_t y_i h_t(x_i)\right) \\ &= \frac{1}{N}\sum_{i=1}^{N} \left[\prod_{t=1}^{T} \exp\left(-\alpha_t y_i h_t(x_i)\right)\right] \\ &= \frac{1}{N}\sum_{i=1}^{N} \left[D_{T+1}(i)\prod_{t=1}^{T} Z_t\right] \\ &= \prod_{t=1}^{T} Z_t. \end{aligned}$$

In the last step, we used the fact that the D_t's over i must sum to 1 and Z_t is independent of i and can be pulled out of the summation.

E. We have

$$Z_t = \sum_{i=1}^{N} D_t(i) \exp\left(-\alpha_t y_i h_t(x_i)\right).$$

We can rewrite the exponential factor as

$$\exp\left(-\alpha_t y_i h_t(x_i)\right) = e^{-\alpha_t} + \left(e^{\alpha_t} - e^{-\alpha_t}\right) \mathbb{1}(h_t(x_i) \neq y_i).$$

Therefore,

$$\begin{aligned} Z_t &= \sum_{i=1}^{N} \left[D_t(i)e^{-\alpha_t} + D_t(i)\left(e^{\alpha_t} - e^{-\alpha_t}\right)\mathbb{1}(h_t(x_i) \neq y_i)\right] \\ &= e^{-\alpha_t} + \epsilon_t\left(e^{\alpha_t} - e^{-\alpha_t}\right) \\ &= \epsilon_t e^{\alpha_t} + (1-\epsilon_t)e^{-\alpha_t} \end{aligned}$$

F. Differentiating Z_t with respect to α_t and setting the result to zero yields

$$\begin{aligned} \epsilon_t e^{\alpha_t} + (\epsilon_t - 1)e^{-\alpha_t} &= 0 \\ \epsilon_t e^{2\alpha_t} &= 1 - \epsilon_t \\ \alpha_t &= \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right). \end{aligned}$$