SURREAL ORDERED EXPONENTIAL FIELDS

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Abstract. In [15], the algebraico-tree-theoretic simplicity hierarchical structure of J. H. Conway's ordered field **No** of surreal numbers was brought to the fore and employed to provide necessary and sufficient conditions for an ordered field (ordered K-vector space) to be isomorphic to an initial subfield (K-subspace) of **No**, i.e. a subfield (K-subspace) of **No** that is an initial subtree of **No**. In this sequel to [15], piggybacking on the just-said results, analogous results are established for *ordered exponential fields*. It is further shown that a wide range of ordered exponential fields are isomorphic to initial exponential subfields of $(\mathbf{No}, \exp)$. These include all models of $T(\mathbb{R}_W, e^x)$, where $\mathbb{R}_W$ is the reals expanded by a *convergent Weierstrass system* W. Of these, those we call *trigonometric-exponential fields* are given particular attention. It is shown that the exponential functions on the initial trigonometric-exponential subfields of **No**, which includes **No** itself, extend to *canonical* exponential functions on their *surcomplex* counterparts. This uses the precursory result that trigonometric-exponential initial subfields of **No** and *trigonometric ordered initial subfields* of **No**, more generally, admit *canonical* sine and cosine functions. This is shown to apply to the members of a distinguished family of initial exponential subfields of **No**, to the image of the canonical map of the ordered exponential field $\mathbb{T}$ of *transseries* into **No**, which is shown to be initial, and to the ordered exponential fields $\mathbb{R}((\omega))^{EL}$ and $\mathbb{R}\langle\langle\omega\rangle\rangle$, which are likewise shown to be initial.

1. Introduction

In his monograph *On Numbers and Games* [5], J. H. Conway introduced a real closed field **No** of *surreal numbers* containing the reals and the ordinals as well as a great many less familiar numbers, including $-\omega$, $\omega/2$, $1/\omega$, and $\sqrt{\omega}$, to name only a few. Indeed, **No** is so remarkably inclusive that, subject to the proviso that numbers—construed here as members of ordered fields—be individually definable in terms of sets of NBG (von Neumann-Bernays-Gödel set theory with Global Choice), it may be said to contain "All Numbers Great and Small" [12, 13, 15, 18].

No also has a rich algebraico-tree-theoretic structure which was brought to the fore by Ehrlich [14, 15] and further developed and explored in [9, 16, 17, 18, 20, 22, 23]. This *simplicity hierarchical* (or *s-hierarchical*) *structure* depends upon **No**'s structure as a lexicographically ordered full binary tree and arises from the fact that the sums and products of any two members of the tree are the simplest possible elements of the tree consistent with **No**'s structure as an ordered group and an ordered field, respectively, it being understood that x is *simpler than* y just in case x is a predecessor of y in the tree.

Among the remarkable s-hierarchical features of **No** is that much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite recursion that provides an unfolding of the entire spectrum of numbers great and small (modulo the aforementioned provisos), the recursive process of defining **No**'s arithmetic in turn provides an unfolding of the entire spectrum of ordered fields (ordered K-vector spaces; ordered abelian groups) in such a way that an isomorphic copy of every such system either emerges as an initial substructure of **No**–a substructure of **No** that is an initial subtree of **No**–or is contained in a theoretically distinguished instance of such a system that does. More specifically, in [15] Ehrlich showed that:

Proposition 1.1. *Every ordered vector space over an Archimedean ordered field is isomorphic to an initial subspace of* **No***; in particular, every divisible ordered abelian group is isomorphic to an initial subgroup of* **No**.

Proposition 1.2. *Every real closed ordered field is isomorphic to an initial subfield of* **No**.

These results were obtained with the aid of the following more general results from [15] that provide necessary and sufficient conditions for an ordered K-vector space (ordered field) to be isomorphic to an initial K-subspace (subfield) of **No**.

Proposition 1.3. *An ordered K-vector space is isomorphic to an initial subspace of* **No** *if and only if K is isomorphic to an initial subfield of* **No**.

Proposition 1.4. *An ordered field K is isomorphic to an initial subfield of* **No** *if and only if K is isomorphic to a truncation closed, cross sectional subfield of a power series field $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$ where Γ is isomorphic to an initial subgroup of* **No**.

In [9], van den Dries and Ehrlich subsequently established:

Proposition 1.5. *The exponential field* $(\mathbf{No}, \exp)$ *of surreal numbers is an elementary extension of the exponential field* $(\mathbb{R}, e^x)$ *of real numbers,*

where exp is the recursively defined exponential function on **No** developed by Kruskal [5, page 38] and Gonshor [24, Ch. 10]. This result is obtained as a corollary of:

Proposition 1.6. *The field of surreal numbers equipped with restricted analytic functions and with* exp *is an elementary extension of the field of real numbers with restricted analytic functions and real exponentiation.*

Like a recent related work by the current authors [20], this is a sequel to [15]. Following some preliminary material in §2-§6, and piggybacking on Propositions 1.1-1.4, necessary and sufficient conditions are established for an *ordered exponential field* to be isomorphic to an initial exponential subfield of **No**. Using these conditions, it is further shown that a wide range of ordered exponential fields are isomorphic to initial exponential subfields of **No**. These include all models of the theory $T(\mathbb{R}_{\text{an}}, e^x)$ of real numbers with restricted analytic functions and exponentiation [10], a result previously established by Fornasiero [23], and, more generally, all models of the theory $T(\mathbb{R}_W, e^x)$ of real numbers with a *convergent Weierstrass system* W [7, 8] and exponentiation. Of these, those we call *trigonometric-exponential fields* are found to be of particular significance. More specifically, it is shown that the exponential functions on initial trigonometric-exponential subfields of **No**, which includes **No** itself, extend to *canonical* exponential functions on their *surcomplex* counterparts, thereby providing a positive answer to a questioned raised at the *mini-workshop on surreal numbers, surreal analysis, Hahn fields and derivations* held in Oberwolfach in 2016 [3, page 3315]. The proof of this uses the precursory result that trigonometric-exponential initial subfields of **No** and *trigonometric ordered initial subfields* of **No**, more generally, admit *canonical* sine and cosine functions. This is shown to apply to the members of a distinguished family of initial exponential subfields of **No** isolated by van den Dries and Ehrlich ([9], Corollary 5.5), to the image of the canonical map of the ordered exponential field $\mathbb{T}$ of *transseries* into **No** [2], which is shown to be initial, and to the ordered exponential fields $\mathbb{R}((\omega))^{EL}$ and $\mathbb{R}\langle\langle\omega\rangle\rangle$, due to Berarducci and Mantova ([4]; also see [26, 27, 28]), which are likewise shown to be initial.

Some of the methods employed in Sections 3.2, 6.2 and 8.2 are adaptations or expansions of methods developed by Ressayre [31], van den Dries, Macintyre and Marker [10, 11], and D'Aquino, Knight, Kuhlmann and Lange [6] in their treatments of truncation closed embeddings into Hahn fields of ordered exponential fields and ordered fields with additional structure more generally. However, as is evident from Proposition 1.4, even in the case of an ordered field K the existence of a truncation closed embedding into a Hahn field does not suffice to establish the existence of an initial embedding into **No**. This inadequacy is even more pronounced in the case of an initial embedding of an ordered exponential field K into **No**. Besides the properties required for an initial embedding of K, a number of conditions must be placed on K. While most of these conditions are familiar from the literature, one of them, due to the second author, appears to be new (Definition 6.3).

Throughout the paper the underlying set theory is NBG, which is a conservative extension of ZFC in which all proper classes are in bijective correspondence with the class of all ordinals (cf. [29]). By "set-model" ("class-model") we mean a model whose universe is a set (a proper class). The theories in the languages treated in §8 and §9 admit quantifier elimination, with the consequence that the results in those sections regarding class-models of such theories and elementary embeddings of models into class-models of such theories are provable in NBG. For this and details on formalizing the theory of surreal numbers in NBG more generally, see [13].

2. Preliminaries I: surreal numbers

A **tree** $(A, <_s)$ is a partially ordered class such that for each $x \in A$, the class $\{y \in A : y <_s x\}$ of **predecessors** of x, written '$\text{pr}_A(x)$', is a set well ordered by $<_s$. A maximal subclass of A well ordered by $<_s$ is called a **branch** of the tree. Two elements x and y of A are said to be **incomparable** if $x \neq y$, $x \not<_s y$ and $y \not<_s x$. An **initial subtree** of $(A, <_s)$ is a subclass A' of A with the induced order such that for each $x \in A'$, $\mathrm{pr}_{A'}(x) = \mathrm{pr}_A(x)$. The **tree-rank** of $x \in A$, written '$\rho_A(x)$', is the ordinal corresponding to the well-ordered set $(\mathrm{pr}_A(x), <_s)$; the αth **level** of A is $\{x \in A : \rho_A(x) = \alpha\}$; and a **root** of A is a member of the zeroth level. If $x, y \in A$, then y is said to be an **immediate successor** of x if $x <_s y$ and $\rho_A(y) = \rho_A(x) + 1$; and if $(x_\alpha)_{\alpha<\beta}$ is a chain in A (i.e., a subclass of A totally ordered by $<_s$), then y is said to be an **immediate successor of the chain** if $x_\alpha <_s y$ for all $\alpha < \beta$ and $\rho_A(y)$ is the least ordinal greater than the tree-ranks of the members of the chain. The **length** of a chain $(x_\alpha)_{\alpha<\beta}$ in A is the ordinal β.

A tree $(A, <_s)$ is said to be **binary** if each member of A has at most two immediate successors and every chain in A of limit length has at most one immediate successor. If every member of A has two immediate successors and every chain in A of limit length (including the empty chain) has an immediate successor, then the binary tree is said to be **full**. Since a full binary tree has a level for each ordinal, the universe of a full binary tree is a proper class.

Following [15, Definition 1], a binary tree $(A, <_s)$ together with a total ordering $<$ defined on A will be said to be **lexicographically ordered** if for all $x, y \in A$, x is incomparable with y if and only if x and y have a common predecessor lying between them (i.e. there is a $z \in A$ such that $z <_s x$, $z <_s y$ and either $x < z < y$ or $y < z < x$). The appellation "lexicographically ordered" is motivated by the fact that: $(A, <, <_s)$ is a lexicographically ordered binary tree if and only if $(A, <, <_s)$ is isomorphic to an initial ordered subtree of the **lexicographically ordered canonical full binary tree** $(B, <_{lex(B)}, <_B)$, where B is the class of all sequences of $-$'s and $+$'s indexed over some ordinal, $x <_B y$ signifies that x is a proper initial subsequence of y, and $(x_\alpha)_{\alpha<\mu} <_{lex(B)} (y_\alpha)_{\alpha<\sigma}$ if and only if $x_\beta = y_\beta$ for all $\beta <$ some δ, but $x_\delta < y_\delta$, it being understood that $- <$ *undefined* $< +$ [15, Theorem 1].

Let $(A, <, <_s)$ be a lexicographically ordered binary tree. If (L, R) is a pair of subclasses of A for which every member of L precedes every member of R, then we will write '$L < R$'. Also, if x and y are members of A, then '$x <_s y$' will be read "x **is simpler than** y"; and if there is an $x \in I = \{y \in A : L < \{y\} < R\}$ such that $x <_s y$ for all $y \in I \setminus \{x\}$, then we will denote this **simplest member of** A **lying between the members of** L **and the members of** R by '$\{L \mid R\}$'. Following Conway's game-theoretic terminology, the members of L and R are called the **options** of x. For all $x \in A$, by '$L_{s(x)}$' we mean $\{a \in A : a <_s x \text{ and } a < x\}$ and by '$R_{s(x)}$' we mean $\{a \in A : a <_s x \text{ and } x < a\}$.

The following proposition collects together a number of properties of, or results about, lexicographically ordered binary trees that will be appealed to in subsequent portions of the paper.

Proposition 2.1. [15, Theorem 2] *Let $(A, <, <_s)$ be a lexicographically ordered binary tree.*

(i) For all $x \in A$, $x = \{L_{s(x)} \mid R_{s(x)}\}$;

(ii) for all $x, y \in A$, $x <_s y$ if and only if $L_{s(x)} < \{y\} < R_{s(x)}$ and $y \neq x$;

(iii) for all $x \in A$ and all $L, R \subseteq A$, $x = \{L \mid R\}$ if and only if L is cofinal with $L_{s(x)}$ and R is coinitial with $R_{s(x)}$ if and only if $L < \{x\} < R$ and $\{y \in A : L < \{y\} < R\} \subseteq \{y \in A : L_{s(x)} < \{y\} < R_{s(x)}\}$.

Let $(\mathbf{No}, <, <_s)$ be the **lexicographically ordered binary tree of surreal numbers** constructed in any of the manners found in the literature (cf. [14, 15, 16, 18]), including simply letting $(\mathbf{No}, <, <_s) = (B, <_{lex(B)}, <_B)$. Central to the development of the s-hierarchical theory of surreal numbers is the following result where a lexicographically ordered binary tree $(A, <, <_s)$ is said to be **complete** [15, Definition 6] if whenever L and R are sub*sets* of A for which $L < R$, there is an $x \in A$ such that $x = \{L \mid R\}$.

Proposition 2.2. [15, Theorem 4 and Proposition 2] *A lexicographically ordered binary tree is complete if and only if it is full if and only if it is isomorphic to $(\mathbf{No}, <, <_s)$.*

No's canonical class **On** of ordinals consists of the members of the "rightmost" branch of $(\mathbf{No}, <, <_s)$, i.e. the unique branch of $(\mathbf{No}, <, <_s)$ whose members satisfy the condition: $x < y$ if and only if $x <_s y$.

By a **cut** in an ordered class $(A, <)$ we mean a pair (X, Y) of subclasses of A where $X < Y$ and $X \cup Y = A$. If $A \subsetneq A'$, (X, Y) is a cut in A, and $X < \{z\} < Y$ where $z \in A' \setminus A$, we say z **realizes the cut** (X, Y). If z realizes a cut in A, on occasion we denote the cut by $(A^{<z}, A^{>z})$ where $A^{<z} = \{a \in A : a < z\}$ and $A^{>z} = \{a \in A : a > z\}$. For $A \subsetneq \mathbf{No}$ and for $x \in \mathbf{No}$, we say that x **is the simplest element realizing a cut in** A if $x \notin A$ and if $x = \{A^{<x} \mid A^{>x}\}$. Note that if A is an initial subclass of **No** and if x is the simplest element realizing a cut in A, then $A \cup \{x\}$ is also initial.

2.1. **Conway names.** Let $\mathbb{D}$ be the set of all surreal numbers having finite tree-rank, and

$$\mathbb{R} \;=\; \mathbb{D} \cup \big\{\{L \,|\, R\} : (L,R) \text{ is a Dedekind gap in } \mathbb{D}\big\}.$$

The following result regarding the structure of $\mathbb{R}$ is essentially due to Conway [5, pages 12, 23-25].

Proposition 2.3. $\mathbb{R}$ *(with* $+, -, \cdot$ *and* $<$ *defined à la* **No***) is isomorphic to the ordered field of real numbers defined in any of the more familiar ways,* $\mathbb{D}$ *being* **No***'s ring of dyadic rationals (i.e., rationals of the form* $m/2^n$ *where* m *and* n *are integers);*

$$n \;=\; \{0,\dots,n-1 \,|\, \varnothing\} \quad \textit{and} \quad -n \;=\; \{\varnothing \,|\, -(n-1),\dots,0\}$$

for each positive integer n*,* $0 = \{\varnothing \,|\, \varnothing\}$*, and the remainder of the dyadics are the arithmetic means of their left and right predecessors of greatest tree-rank; e.g.,* $1/2 = \{0 \,|\, 1\}$*.*

$\mathbb{R}$ is the unique Dedekind complete initial subfield of **No**. Henceforth, all references to the reals are understood to be references to $\mathbb{R}$.

A striking s-hierarchical feature of **No** is that every surreal number can be assigned a canonical "proper name" that is a reflection of its characteristic s-hierarchical properties. These **Conway names** or **normal forms** are expressed as formal sums of the form

$$\sum_{\alpha<\beta} r_\alpha \omega^{y_\alpha}$$

where β is an ordinal, $(y_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence of surreals, and $(r_\alpha)_{\alpha<\beta}$ is a sequence of nonzero real numbers, the Conway name of an ordinal being just its Cantor normal form, it being understood that 0 is the empty sum indexed over $\alpha < \beta = 0$ [5, pages 31-33] and [15, §3.1 and §5].

Every nonzero surreal x is the sum of three components, each of which can be succinctly characterized in terms of the Conway name of x: the **purely infinite** component of x, whose terms solely have positive exponents; the **real** component of x, whose sole term (if it is not the empty sum) has exponent 0; and the **infinitesimal** component of x, whose terms solely have negative exponents. Notice that 0, being the empty sum, may be regarded as purely infinite.

The surreal numbers having Conway names of the form ω^y are called **leaders** since they denote the simplest positive members of the various Archimedean classes of **No**. More formally, they may be inductively defined by formula

$$\omega^y \;=\; \left\{0, n\omega^{y^L} \;\middle|\; \frac{1}{2^n}\omega^{y^R}\right\},$$

where n ranges over the positive integers, and y^L and y^R range over the elements of $L_{s(y)}$ and $R_{s(y)}$, respectively.

2.2. **Distinguished ordered binary subtrees of No.** Henceforth, the classes of **No**'s leaders and purely infinite numbers will be denoted '$Lead_{\mathbf{No}}$' and '$\mathbf{No}_{\mathrm{PI}}$', respectively. **Oz** is the canonical integer part of **No** consisting of the surreals whose Conway names have no negative exponents and whose coefficient for any term whose exponent is 0 is an integer [5, p. 45]. $Lead_{\mathbf{No}}$, $\mathbf{No}_{\mathrm{PI}}$ and **Oz** all have ordered tree structures inherited from $(\mathbf{No}, <, <_s)$. In the subsequent discussion we will appeal to the following results about these substructures of $(\mathbf{No}, <, <_s)$, the first and second of which are known from the literature ([17, page 3: Note 2] and [15, page 1245: Theorem 11]) and the third of which appears to be new.

Lemma 2.1. $(\mathbf{Oz}, <\restriction_{\mathbf{Oz}}, <_s\restriction_{\mathbf{Oz}})$ *is an initial subtree of* $(\mathbf{No}, <, <_s)$*.*

Lemma 2.2. $(Lead_{\mathbf{No}}, <\restriction_{Lead_{\mathbf{No}}}, <_s\restriction_{Lead_{\mathbf{No}}})$ *is a lexicographically ordered full binary tree.*

Lemma 2.3. $(\mathbf{No}_{\mathrm{PI}}, <\restriction_{\mathbf{No}_{\mathrm{PI}}}, <_s\restriction_{\mathbf{No}_{\mathrm{PI}}})$ *is a lexicographically ordered full binary tree.*

Proof. First note that $\mathbf{No}_{\mathrm{PI}}$ is a subclass of **No**'s ring **Oz** of omnific integers, the latter of which is an initial subtree of **No**. Moreover, the members of **Oz** consist of those surreals having sign-expansions which neither contain a plus immediately followed by a minus nor a minus immediately followed by a plus [24, p. 111: Theorem 8.1]. $\mathbf{No}_{\mathrm{PI}}$ is the subclass of **Oz** whose members have sign-expansions of limit length. Accordingly, to complete the proof, it suffices to note that if x is a purely infinite surreal, then its purely infinite immediate successor $> x$ (resp. $< x$) is the surreal number whose sign-expansion consists of the sign-expansion of x followed by ω pluses (resp. ω minuses). And, if $(x_\alpha)_{\alpha<\beta}$ is chain of limit length of purely infinite surreals ordered by $<_s$, then the purely infinite immediate successor of the chain is the surreal having the shortest sign-expansion s for which the sign-expansions of the x_α's are initial subsequences of s (i.e. the immediate successor of the chain in $\mathbf{No}$). $\square$

Corollary 2.1. *The following are are lexicographically ordered full binary trees:*

(i) $(Lead^{>1}_{\mathbf{No}}, <\restriction_{Lead^{>1}_{\mathbf{No}}}, <_s\restriction_{Lead^{>1}_{\mathbf{No}}})$,

(ii) $(\mathbf{No}^{>0}_{\mathrm{PI}}, <\restriction_{\mathbf{No}^{>0}_{\mathrm{PI}}}, <_s\restriction_{\mathbf{No}^{>0}_{\mathrm{PI}}})$,

(iii) $(Lead^{<1}_{\mathbf{No}}, <\restriction_{Lead^{<1}_{\mathbf{No}}}, <_s\restriction_{Lead^{<1}_{\mathbf{No}}})$, *and*

(iv) $(\mathbf{No}^{<0}_{\mathrm{PI}}, <\restriction_{\mathbf{No}^{<0}_{\mathrm{PI}}}, <_s\restriction_{\mathbf{No}^{<0}_{\mathrm{PI}}})$.

Proof. This follows from Lemmas 2.2 and 2.3 and the simple fact that deleting the root of a lexicographically ordered full binary tree, in this case 1 and 0 respectively, results in two such trees. $\square$

3. Preliminaries II: ordered abelian groups

Let Γ be an ordered abelian group. For $x, y \in \Gamma$, we set

$$\begin{aligned} x \preccurlyeq y \ &:\Longleftrightarrow\ |x| < n|y| \text{ for some } n \in \mathbb{N} \\ x \prec y \ &:\Longleftrightarrow\ n|x| < |y| \text{ for all } n \in \mathbb{N} \\ x \asymp y \ &:\Longleftrightarrow\ x \preccurlyeq y \text{ and } y \preccurlyeq x \text{ (equivalently, if } x \preccurlyeq y \text{ and } x \not\prec y). \end{aligned}$$

Then $\asymp$ is an equivalence relation on $\Gamma \setminus \{0\}$ and the equivalence classes corresponding to $\asymp$ are called the **Archimedean classes of** Γ. We say that Γ is **Archimedean** if $\Gamma \setminus \{0\}$ consists of exactly one Archimedean class.

3.1. **Hahn groups.** Let $\mathbb{R}((t^S))_{\mathbf{On}}$ be the ordered group of power series (defined á la Hahn [25]) consisting of *all* formal power series of the form $\sum_{\alpha<\beta} r_\alpha t^{s_\alpha}$ where $(s_\alpha)_{\alpha<\beta\in\mathbf{On}}$ is a possibly empty descending sequence of elements of an ordered class S and $r_\alpha \in \mathbb{R}^\times$ for each $\alpha < \beta$. When S is a set, then $\mathbb{R}((t^S))_{\mathbf{On}}$ is a set as well, and it is often simply written $\mathbb{R}((t^S))$. When S is a proper class, then $\mathbb{R}((t^S))_{\mathbf{On}}$ is also a proper class. We call S the **value class of** $\mathbb{R}((t^S))_{\mathbf{On}}$. In the literature, the appellation "Hahn group" is usually reserved for those structures $\mathbb{R}((t^S))_{\mathbf{On}} = \mathbb{R}((t^S))$, where S is a set. However, we refer to $\mathbb{R}((t^S))_{\mathbf{On}}$ as a Hahn group whether S is a set or a proper class.

An element $x \in \mathbb{R}((t^S))_{\mathbf{On}}$ is said to be a **truncation** of $\sum_{\alpha<\beta} r_\alpha t^{s_\alpha} \in \mathbb{R}((t^S))_{\mathbf{On}}$ if $x = \sum_{\alpha<\sigma} r_\alpha t^{s_\alpha}$ for some $\sigma \leqslant \beta$. A subgroup Γ of $\mathbb{R}((t^S))_{\mathbf{On}}$ is said to be **truncation closed** if every truncation of every member of Γ is itself a member of Γ. A subgroup Γ of $\mathbb{R}((t^S))_{\mathbf{On}}$ is said to be **cross sectional** if $\{t^s : s \in S\} \subseteq \Gamma$.

3.2. **Developments.** Let Γ be a divisible ordered abelian group, let Δ be a divisible ordered abelian subgroup of Γ, and let S be a set. Suppose that we have a truncation closed, cross sectional embedding $\imath : \Delta \to \mathbb{R}((t^S))$.

Definition 3.1. For $y \in \Gamma$, we say that an element $\sum_{\alpha<\beta} r_\alpha t^{s_\alpha} \in \mathbb{R}((t^S))$ is a **partial development of** y **over** Δ if for all $\sigma < \beta$:

(i) $\sum_{\alpha\leqslant\sigma} r_\alpha t^{s_\alpha}$ is in $\imath(\Delta)$ and

(ii) $\imath^{-1}\Big(\sum_{\alpha\leqslant\sigma} r_\alpha t^{s_\alpha}\Big) - y \prec \imath^{-1}(t^{s_\sigma})$.

There is a unique maximal partial development of y over Δ which we refer to as the **development of** y **over** Δ (with respect to $\imath$) and which we denote by $D_\Delta(y)$.

Lemma 3.1. *The development of y over Δ only depends on the cut that y realizes over Δ.*

Proof. Suppose that $x, y \in \Gamma$ realize the same cut over Δ and that $D_\Delta(x) \neq D_\Delta(y)$. Let $\sum_{\alpha<\beta} r_\alpha t^{s_\alpha} \in \mathbb{R}((t^S))$ be the greatest common partial development of x and y over Δ. Without loss of generality, we may assume that z is *not* the development of x over Δ, so there is $r_\beta \in \mathbb{R}$ and $s_\beta \in S$ such that $\sum_{\alpha\leqslant\beta} r_\alpha t^{s_\alpha}$ is a partial development of x but not of y. Set $z := \imath^{-1}\big(\sum_{\alpha\leqslant\beta} r_\alpha t^{s_\alpha}\big) \in \Delta$. We have

$$z - x \ \prec\ \imath^{-1}(t^{s_\beta}), \qquad z - y \ \succcurlyeq\ \imath^{-1}(t^{s_\beta}).$$

If $z-x$ and $z-y$ have opposite signs, then z is between x and y. Suppose that $z-x$ and $z-y$ are both positive. Take $n>0$ such that $z-y>\frac{1}{n}\imath^{-1}(t^{s_\beta})$. Then

$$z-x \;<\; \frac{\imath^{-1}(t^{s_\beta})}{n} \;<\; z-y$$

and so $x>z-\frac{1}{n}\imath^{-1}(t^{s_\beta})>y$. Since $z-\frac{1}{n}\imath^{-1}(t^{s_\beta})$ is in Δ, this shows that x and y do not realize the same cut over Δ. The case that $z-x$ and $z-y$ are both negative is similar. □

Definition 3.2. Let γ be an element of $\Gamma\setminus\Delta$. We say that γ **realizes a v-cut over** Δ if $D_\Delta(\gamma)$ lies in $\imath(\Delta)$ and if $\gamma-\imath^{-1}\big(D_\Gamma(\gamma)\big)\not\asymp\delta$ for all $\delta\in\Delta$.

The terminology "v-cut" indicates that this definition should only depend on the cut realized by γ. The following result indicates that this is indeed the case:

Lemma 3.2. *Suppose that γ realizes a v-cut over Δ and suppose that $\gamma^*\in\Gamma\setminus\Delta$ realizes the same cut as γ over Δ. Then γ^* also realizes a v-cut over Δ.*

Proof. By Lemma 3.1, we know that $D_\Delta(\gamma^*)=D_\Delta(\gamma)$. This shows that $D_\Delta(\gamma^*)$ lies in Δ and that $\gamma^*-\imath^{-1}\big(D_\Delta(\gamma^*)\big)$ realizes the same cut in Γ as $\gamma-\imath^{-1}\big(D_\Delta(\gamma)\big)$. Suppose that $\gamma^*-\imath^{-1}\big(D_\Delta(\gamma^*)\big)$ is positive and assume towards contradiction that there is $\delta\in\Delta^{>0}$ with $\delta\asymp\gamma^*-\imath^{-1}\big(D_\Gamma(\gamma^*)\big)$. Then there is $n>0$ with

$$\frac{1}{n}\delta \;<\; \gamma^*-\imath^{-1}\big(D_\Gamma(\gamma^*)\big) \;<\; n\delta.$$

Therefore, we have $\frac{1}{n}\delta<\gamma-\imath^{-1}\big(D_\Gamma(\gamma)\big)<n\delta$ as well, but then $\gamma-\imath^{-1}\big(D_\Gamma(\gamma)\big)\asymp\delta$, a contradiction. □

If Γ is itself a divisible subgroup of a Hahn group $\mathbb{R}((\omega^S))$ and if Δ is truncation closed and cross sectional, then each $y\in\Gamma$ has a development $D_\Delta(y)$ with respect to the identity map on Δ. This development is always a truncation of y.

3.3. **Initial subgroups and subspaces of No.** Let Γ be an initial divisible ordered abelian subgroup of **No**. The **value class of** Γ is the class

$$S \;:=\; \{s\in\mathbf{No}:\omega^s\in\Gamma\}.$$

Then S is an initial subclass of **No** and Γ is a truncation closed, cross sectional subgroup of $\mathbb{R}((\omega^S))_{\mathbf{On}}$. Thus, each $\gamma\in\mathbf{No}$ has a development $D_\Gamma(\gamma)$ over Γ. This development is a truncation of γ.

Let $\boldsymbol{k}$ be an Archimedean ordered field. Then there is a unique initial ordered field embedding $\imath:\boldsymbol{k}\to\mathbf{No}$, and we identify $\boldsymbol{k}$ with its image under this embedding. By Proposition 1.3, we get that every ordered $\boldsymbol{k}$-vector space V admits an initial $\boldsymbol{k}$-linear embedding into **No**. For future use, we record the induction step in the proof of this result [15, pages 1241-1242]:

Lemma 3.3. *Let V be an initial ordered $\boldsymbol{k}$-vector subspace of* **No** *and let $x\in\mathbf{No}\setminus V$. Suppose that x is the simplest element realizing a cut in V. Then the $\boldsymbol{k}$-vector subspace $V+\boldsymbol{k}x\subseteq\mathbf{No}$ is initial. In particular, if S is the value class of V and if $s\in\mathbf{No}\setminus S$ is the simplest element realizing a cut in S, then $V+\boldsymbol{k}\omega^s$ is initial.*

4. Preliminaries III: ordered fields

Let K be an ordered field. The Archimedean classes of K as well as the relations $\preccurlyeq$, $\prec$, and $\asymp$ are defined with respect to the underlying ordered additive group of K. We say that an element $x\in K$ is **infinite** if $x\succ 1$. A **cross section** for K is an ordered subgroup $\mathfrak{M}\subseteq K^{>0}$ such that for each $x\in K^\times$ there is exactly one $\mathfrak{m}\in\mathfrak{M}$ with $x\asymp\mathfrak{m}$.

Let Γ be an ordered abelian group. Then the Hahn group $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ is in fact a *Hahn field*. We call Γ the **value group of** $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$. Let K be a truncation closed, cross sectional subfield of $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$. The set $\mathbb{R}_K=\{r\in\mathbb{R}:rt^0\in K\}$ is an Archimedean ordered field, which we will call the **coefficient field** of K. Note that $\mathbb{R}_K$ is isomorphic to the residue class field of K with respect to the Archimedean valuation.[1] The

[1]The traditional definition of the residue class field does not work in NBG if K is a proper class. For a suitable modification applicable to sets and proper classes, see [15, page 1253].

multiplicative subgroup $t^{\Gamma} \subseteq K^{>0}$ is a canonical cross section for K. We also single out the (nonunital) ring of **purely infinite elements of** K:

$$K_{\mathrm{PI}} \;:=\; \Big\{ \sum_{\alpha<\beta} r_\alpha t^{\gamma_\alpha} \in K : \text{ all } \gamma_\alpha > 0 \Big\} \;=\; K \cap \mathbb{R}((t^{\Gamma^{>0}}))_{\mathbf{On}}.$$

The following result, which is employed in the proof of Proposition 1.4, is critical in the proof of the main theorem.

Proposition 4.1 ([15], Theorem 18; [18], Theorem 14)**.** *If K is a truncation closed, cross sectional subfield of a Hahn field $\mathbb{R}((t^{\Gamma}))$ and $\imath : \Gamma \to \mathbf{No}$ is an initial group embedding, then the mapping that sends*

$$\sum_{\alpha<\beta} r_\alpha t^{\gamma_\alpha} \in K$$

to the surreal number having Conway name

$$\sum_{\alpha<\beta} r_\alpha \omega^{\imath(\gamma_\alpha)}$$

is an initial embedding of K into $\mathbf{No}$.

If K is an initial ordered subfield of $\mathbf{No}$, then the class

$$\Gamma \;:= \{\gamma \in \mathbf{No} : \omega^{\gamma} \in K\}$$

is an initial subgroup of $\mathbf{No}$, which we call the **value group of** K. We have that K is a truncation closed, cross sectional subgroup of $\mathbb{R}((\omega^{\Gamma}))_{\mathbf{On}}$. We set $K_{\mathrm{PI}} := K \cap \mathbf{No}_{\mathrm{PI}}$.

5. Surreal exponentiation

An **exponential ordered field** is an ordered field A together with an **exponential** map $\exp_A$ which is an order-preserving isomorphism from the ordered additive group of A onto the ordered multiplicative group $A^{>0}$ of positive elements of A. The Kruskal-Gonshor surreal exponential function exp is defined by recursion as follows:

$$\exp(x) \;=\; \Big\{ 0, (\exp x^L)[x - x^L]_n, (\exp x^R)[x - x^R]_{2n+1} \;\Big|\; \frac{\exp x^L}{[x^L - x]_{2n+1}}, \frac{\exp x^R}{[x^R - x]_n} \Big\},$$

where x^L and x^R range over the predecessors of x in $(\mathbf{No}, <_s, <)$ that are less than x and greater than x respectively, and where $[y]_n$ denotes $1 + y + \frac{y^2}{2!} + ... + \frac{y^n}{n!}$ for all surreal y. Recall that, being an elementary extension of $(\mathbb{R}, e^x)$ (Proposition 1.5), $(\mathbf{No}, \exp)$ satisfies the condition: $\exp(x) > x^n$ for each positive infinite surreal number x and each natural number n.

While the definition of exp is quite complicated for the general surreal case, the following result of Gonshor [24, pages 149-157] shows it reduces to more revealing and manageable forms for the three theoretically significant cases.

Proposition 5.1. *Let* exp *be the Kruskal-Gonshor exponential on* $\mathbf{No}$.

(i) $\exp(x) = e^x$ *for all* $x \in \mathbb{R}$*;*
(ii) $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ *for all infinitesimal* x*;*
(iii) if x is purely infinite, then

$$\exp(x) \;=\; \Big\{ 0, (\exp x^L)(x - x^L)^n \;\Big|\; \frac{\exp x^R}{(x^R - x)^n} \Big\},$$

where x^L (resp. x^R) now range over all purely infinite predecessors of x less than (resp. greater than) x.

The significance of cases (i)–(iii) accrues from the fact that for an arbitrary surreal number x,

$$\exp(x) \;=\; \exp(x_P) \cdot \exp(x_R) \cdot \exp(x_I)$$

where x_P, x_R and x_I are the purely infinite, real and infinitesimal components of x, respectively.

From an algebraic point of view, it is already clear from Proposition 5.1 (i)–(ii) what $\exp(x)$ is for real and infinitesimal values of x. To shed further algebraic light on $\exp(x)$ when x is purely infinite we turn to the following additional result of Gonshor [24, Theorem 10.9]

Proposition 5.2. *The restriction of* exp *to the class of purely infinite surreal numbers is an isomorphism of ordered groups onto* $Lead_{\mathbf{No}}$.

The following lemma, which provides even more information on the exponentials of purely infinite numbers, will come into play later:

Lemma 5.1. *Let K be an initial ordered subfield of* **No** *whose universe is a set and let Γ be the value group of K. Let $x \in K_{\mathrm{PI}}$ and suppose that $\exp x \notin K$ and that $\exp y \in K$ for each $y \in K_{\mathrm{PI}}$ with $y <_s x$. Then $\exp x = \omega^\gamma$ for some $\gamma \in \mathbf{No}$ which is the simplest element realizing a cut in Γ.*

Proof. By Proposition 5.2, we know that $\exp x = \omega^\gamma$ for some $\gamma \in \mathbf{No}$. By Proposition 5.1, we have

$$\omega^\gamma \;=\; \left\{0, (\exp x^L)(x - x^L)^n \;\middle|\; \frac{\exp x^R}{(x^R - x)^n}\right\}$$

where x^L and x^R range over the left and right purely infinite predecessors of x. We claim that for each purely infinite left predecessor x^L of x and every n, there is a $\delta \in \Gamma$ such that $(\exp x^L)(x - x^L)^n \asymp \omega^\delta$. By our assumption that x is a simplest purely infinite element without an exponential in K, we know that there is a $\delta_1 \in \Gamma$ with $\exp x^L = \omega^{\delta_1}$. Since $x - x^L \in K$, there is a $\delta_2 \in \Gamma$ such that $x - x^L$ is comparable to ω^{δ_2}. Then $(x - x^L)^n$ is comparable to $\omega^{n\delta_2}$ and so $(\exp x^L)(x - x^L)^n$ is comparable to $\omega^{\delta_1 + n\delta_2}$. Since

$$(\exp x^L)(x - x^L)^{n+1} \;\succ\; (\exp x^L)(x - x^L)^n$$

for each n, we see that the set $\{\omega^\delta : \delta \in \Gamma^{<\gamma}\}$ is cofinal in the left options of ω^γ. Applying the same argument on the right, we see that the set $\{\omega^\delta : \delta \in \Gamma^{>\gamma}\}$ is coinitial in the right options of ω^γ. Thus, $\gamma = \{\Gamma^{<\gamma} \mid \Gamma^{>\gamma}\}$. □

In addition to its inductively defined exponential function exp, Norton and Kruskal independently provided inductive definitions of the inverse function log, but thus far only an inductive definition of log for surreals of the form ω^y has appeared in print. Nevertheless, since each positive surreal x, written in normal form, has a unique decomposition of the form

$$x \;=\; \omega^y r(1 + \varepsilon),$$

where ω^y is a leader, r is a positive member of $\mathbb{R}$ and ε is an infinitesimal, $\log(x)$ may be obtained for an arbitrary positive surreal x from the equation

$$\log(x) \;=\; \log(\omega^y) + \ln(r) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon^k}{k},$$

where $\log(\omega^y)$ is inductively defined by

$$\left\{ \log\left(\omega^{y^L}\right) + n, \log\left(\omega^{y^R}\right) - \omega^{\frac{y^R - y}{n}} \;\middle|\; \log\left(\omega^{y^R}\right) - n, \log\left(\omega^{y^L}\right) + \omega^{\frac{y - y^L}{n}} \right\}.$$

Moreover, since log is analytic, an inductive definition of $\log(1 + z)$ for infinitesimal values of the variable can be provided in the manner discussed in [21].

5.1. **Gonshor's map h.** In studying surreal exponentiation, Gonshor [24, page 172] defined the map $h : \mathbf{No} \to \mathbf{No}^{>0}$ as follows:

$$h(s) \;=\; \left\{0, h(s^L) \;\middle|\; h(s^R), \frac{1}{k}\omega^s\right\}$$

where s^L ranges over all left predecessors of s, where s^R ranges over all right predecessors of s, and where k ranges over the positive integers. The importance of this map comes from the following result of Gonshor [24]:

Proposition 5.3. *Let $\gamma = \sum_{\alpha<\beta} r_\beta \omega^{s_\beta} \in \mathbf{No}$. Then*

$$\log \omega^\gamma \;=\; \sum_{\alpha<\beta} r_\beta \omega^{h(s_\beta)}.$$

In particular, we have $\log \omega^{\omega^s} = \omega^{h(s)}$ for each $s \in \mathbf{No}$.

Let K be an initial exponential subfield of $\mathbf{No}$ whose universe is a set, let Γ be the value group of K, and let S be the value set of Γ. Note that Γ is an initial ordered $\mathbb{R}_K$-vector subspace of $\mathbf{No}$ since

$$\omega^{r\gamma} \;=\; \exp(r\log\omega^{\gamma}) \in K$$

for each $r \in \mathbb{R}_K$ and each $\gamma \in \Gamma$. In this subsection, we establish two results about K involving this map h.

Lemma 5.2. *For each $\gamma \in \mathbf{No}$, we have $\log\omega^{D_\Gamma(\gamma)} = D_K(\log\omega^{\gamma})$.*

Proof. We write $\gamma = \sum_{\alpha<\beta} r_\alpha\omega^{s_\alpha}$, so

$$\log\omega^{\gamma} \;=\; \sum_{\alpha<\beta} r_\alpha\omega^{h(s_\alpha)}.$$

Since h is strictly increasing, we have for each $\beta_0 \leqslant \beta$ that

$$\log\omega^{\sum_{\alpha<\beta_0} r_\alpha\omega^{s_\alpha}} \;=\; \sum_{\alpha<\beta_0} r_\alpha\omega^{h(s_\alpha)}$$

is a partial development of $\log\omega^{\gamma}$. Thus, $\sum_{\alpha<\beta_0} r_\alpha\omega^{s_\alpha}$ is a maximal partial development of γ if and only if $\log\omega^{\sum_{\alpha<\beta_0} r_\alpha\omega^{s_\alpha}}$ is a maximal partial development of $\log\omega^{\gamma}$. $\square$

Lemma 5.3. *Let $\gamma_0,\ldots,\gamma_n$ be elements of $\mathbf{No}\setminus\Gamma$ and let $s_0 > s_1 > \ldots > s_{n-1}$ be elements of $\mathbf{No}\setminus S$. Set $\Gamma_0 := \Gamma$ and for $m \leqslant n$, set $\Gamma_{m+1} := \Gamma_m + \mathbb{R}_K\gamma_m$. Fix $n \geqslant 0$ and suppose that the following conditions are met:*

- *(i) γ_m realizes a v-cut over Γ for each $m \leqslant n$;*
- *(ii) γ_m is the simplest element realizing a cut over Γ_m for each $m \leqslant n$;*
- *(iii) $\gamma_m = D_\Gamma(\gamma_m) \pm \omega^{s_m}$ and $\gamma_{m+1} = h(s_m)$ for each $m < n$.*

Then there is an $s \in \mathbf{No}\setminus S$ with $s < s_{n-1}$ such that $\gamma_n = D_\Gamma(\gamma_n) \pm \omega^{s}$ and such that $h(s)$ is the simplest element realizing a cut over Γ_{n+1}.

Proof. By assumption (ii) and Lemma 3.3, we know that Γ_m is an initial $\mathbb{R}_K$-subspace of $\mathbf{No}$ for each $m \leqslant n+1$. For each $m \leqslant n$, let S_m be the value set of Γ_m. Then $S_0 = S$ and each S_m is an initial subset of $\mathbf{No}$. By condition (iii), we have that $S_m = S \cup \{s_0,\ldots,s_{m-1}\}$ for $m > 0$.

Since γ_n realizes a v-cut over Γ, we have that $\gamma_n = D_\Gamma(\gamma_n) + r\omega^{s} + z$ for some $s \in \mathbf{No}\setminus S$, some $r \in \mathbb{R}^\times$, and some $z \in \mathbf{No}$ with $z \prec \omega^{s}$. We have $s < s_{n-1}$ since

$$r\omega^{s} \;=\; \gamma_n - D_\Gamma(\gamma_n) \;\preccurlyeq\; \gamma_n \;=\; h(s_{n-1}) \;\prec\; \omega^{s_{n-1}}.$$

Let $\varepsilon \in \{\pm 1\}$ be the sign of r. We claim that γ_n and $D_\Gamma(\gamma_n)+\varepsilon\omega^{s}$ realize the same cut over Γ_n. Suppose not, and take $\delta \in \Gamma_n$ lying between γ_n and $D_\Gamma(\gamma_n)+\varepsilon\omega^{s}$. Then $\delta - D_\Gamma(\gamma_n) \asymp \omega^{s}$, and so $s \in S_n$, a contradiction. Since $D_\Gamma(\gamma_n) + \varepsilon\omega^{s} \leqslant_s \gamma_n$ and since γ_n is the simplest element realizing a cut over Γ_n, we conclude that $\gamma_n = D_\Gamma(\gamma_n) + \varepsilon\omega^{s}$.

We now show that $h(s)$ is the simplest element realizing a cut over Γ_{n+1}. Since Γ_{n+1} is initial, we know that its value set $S_n \cup \{s\}$ is initial as well. Therefore, all predecessors of s lie in S_n. Since $h(S_n) = h(S) \cup \{\gamma_1,\ldots,\gamma_n\}$ and since $h(S) \subseteq \Gamma$, we know that $h(S_n) \subseteq \Gamma_{n+1}$. Additionally, we have

$$\omega^{s} \;=\; \left|\gamma_n - D_\Gamma(\gamma_n)\right| \in \Gamma_{n+1}$$

and so $\frac{1}{k}\omega^{s} \in \Gamma_{n+1}$ for each k. Since all left and right options in the definition of $h(s)$ lie in Γ_{n+1}, we have that $h(s)$ is the simplest element realizing a cut in Γ_{n+1}. $\square$

6. Subfields of ordered exponential fields

In this section, let K be an ordered exponential field whose universe is a set and suppose that

- (i) K is a truncation closed, cross sectional subfield of a Hahn field $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$;
- (ii) $\exp_K x = e^x$ for all $x \in \mathbb{R}_K$;
- (iii) $\exp_K x = \sum_{n=0}^{\infty} x^n/n!$ for all infinitesimal $x \in K$;
- (iv) $\exp_K x > x^n$ for each positive infinite $x \in K$ and each n;
- (v) $\log_K(t^{\Gamma}) = K_{\mathrm{PI}}$.

We let $v : K^\times \to \Gamma$ be the corresponding valuation: $v(x) = \gamma :\Longleftrightarrow x \asymp t^{\gamma}$ for each $x \in K^\times$.

6.1. **Subfields parametrized by subspaces of Γ.** Let Δ be an ordered $\mathbb{R}_K$-subspace of Γ. We set

$$K_\Delta \;:=\; \Big\{ \sum_{\alpha<\beta} r_\alpha t^{\gamma_\alpha} \in K : \beta \in \mathbf{On} \text{ and all } \gamma_\alpha \text{ are in } \Delta \Big\}.$$

Note that $D_{K_\Delta}(y) \in K_\Delta$ for each $y \in K$.

Definition 6.1. We say that Δ is a log**-subspace of** Γ if $\log_K(t^\Delta) \subseteq (K_\Delta)_{\mathrm{PI}}$. We say that Δ is a log-exp**-subspace of** Γ if $\log_K(t^\Delta) = (K_\Delta)_{\mathrm{PI}}$.

Note that Γ is a log-exp-subspace of itself. Any $\mathbb{R}_K$-vector space embedding $\imath : \Delta \to \mathbf{No}$ induces an ordered field embedding $\tilde{\imath} : K_\Delta \to \mathbf{No}$ given by

$$\tilde{\imath}\Big(\sum_{\alpha<\beta} r_\alpha t^{\gamma_\alpha} \Big) \;=\; \sum_{\alpha<\beta} r_\alpha \omega^{\imath(\gamma_\alpha)}.$$

If Δ is a log-subspace of Γ and if $\imath : \Delta \to \mathbf{No}$ is an $\mathbb{R}_K$-vector space embedding, then we say that $\imath$ is a log**-embedding** if

$$\tilde{\imath}(\log_K t^\delta) \;=\; \log \omega^{\imath(\delta)}$$

for each $\delta \in \Delta$.

Lemma 6.1. *If Δ is a* log*-subspace of Γ then K_Δ is closed under $\log_K$ and*

$$\tilde{\imath}(\log_K x) \;=\; \log \tilde{\imath}(x)$$

for each log*-embedding $\imath : \Delta \to \mathbf{No}$ and each $x \in K_\Delta$. If Δ is in fact a* log-exp*-subspace of Γ, then K_Δ is also closed under $\exp_K$.*

Proof. Let Δ be a log-subspace of Γ and let $\imath$ be a log-embedding. Fix $\delta \in \Delta$, $r \in \mathbb{R}^{>0}$, and $\varepsilon \in K_\Delta$ with $\varepsilon \prec 1$. We need to show that $\log_K \big(rt^\delta(1+\varepsilon)\big) \in K_\Delta$ and that

$$\tilde{\imath}\big(\log_K \big(rt^\delta(1+\varepsilon)\big)\big) \;=\; \log \big(\tilde{\imath}\big(rt^\delta(1+\varepsilon)\big)\big) \;=\; \log \big(r\omega^{\imath(\delta)}\big(1+\tilde{\imath}(\varepsilon)\big)\big).$$

To see this, note that

(i) $\log_K t^\delta \in K_\Delta$ and $\tilde{\imath}(\log_K t^\delta) = \log \omega^{\imath(\delta)}$ by assumption;
(ii) $\log_K r = \ln r \in \mathbb{R}_K \subseteq K_\Delta$ and $\tilde{\imath}(\ln r) = \ln r = \log \tilde{\imath}(r)$; and
(iii) $\log_K(1+\varepsilon) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon^k}{k} \in K_\Delta$ (since the sum is in K) and ε is in K_Δ and
$$\tilde{\imath}\Big(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon^k}{k} \Big) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\tilde{\imath}(\varepsilon)^k}{k} \;=\; \log \tilde{\imath}(\varepsilon).$$

The claim follows, since $\log_K \big(rt^\delta(1+\varepsilon)\big) = \log_K(t^\delta) + \log_K(r) + \log_K(1+\varepsilon)$.

If Δ is a log-exp-subspace of Γ, then it is enough to show that $\exp_K x \in K_\Delta$ in the case that $x \in K_\Delta$ is purely infinite, the case that x is real, and the case that x is infinitesimal. The first case holds by assumption, the second follows since $\exp_K(\mathbb{R}_K) \subseteq \mathbb{R}_K$, and the third holds because $\exp_K x = \sum_{n=0}^{\infty} x^n/n! \in K_\Delta$ whenever $x \in K_\Delta$ is infinitesimal. □

Lemma 6.2. *Suppose that Δ is a proper* log-exp*-subspace of Γ and that $\imath : \Delta \to \mathbf{No}$ is an initial* log*-embedding. Fix $\gamma \in \Gamma \setminus \Delta$ and let $\gamma^* \in \mathbf{No}$ realize the same cut over $\imath(\Delta)$ that γ realizes over Δ. Then γ^* realizes a v-cut over $\imath(\Delta)$.*

Proof. We first note that ω^{γ^*} realizes the same cut over $\tilde{\imath}(K_\Delta)$ that t^γ realizes over K_Δ. We claim that $\log \omega^{\gamma^*}$ realizes the same cut over $\tilde{\imath}(K_\Delta)$ that $\log_K t^\gamma$ realizes over K_Δ. Indeed, for $x \in K_\Delta$, we have that

$$x < \log_K t^\gamma \iff \exp_K x < t^\gamma \iff \tilde{\imath}(\exp_K x) < \omega^{\gamma^*} \iff \exp \tilde{\imath}(x) < \omega^{\gamma^*} \iff \tilde{\imath}(x) < \log \omega^{\gamma^*}.$$

In light of Lemma 3.1, this shows that

$$D_{\tilde{\imath}(K_\Delta)}(\log \omega^{\gamma^*}) \;=\; \tilde{\imath}\big(D_{K_\Delta}(\log_K t^\gamma)\big).$$

Since $D_{K_\Delta}(\log_K t^\gamma)$ is in K_Δ, we have that $D_{\tilde{\imath}(K_\Delta)}(\log \omega^{\gamma^*})$ is in $\tilde{\imath}(K_\Delta)$. By Lemma 5.2, we have that $\log \omega^{D_{\imath(\Delta)}(\gamma^*)}$ is in $\tilde{\imath}(K_\Delta)$ and so $D_{\imath(\Delta)}(\gamma^*)$ is in Δ.

Since $D_{\imath(\Delta)}(\gamma^*) \in \Delta$, we may reduce to the situation that $D_{\imath(\Delta)}(\gamma^*) = 0$: we replace γ^* with $\gamma^* - D_{\imath(\Delta)}(\gamma^*)$ and accordingly, we replace γ with $\gamma - \imath^{-1}\big(D_{\imath(\Delta)}(\gamma^*)\big)$. It remains to show that $\gamma^* \not\asymp \imath(\delta)$ for all $\delta \in \Delta$. Let $S \subseteq \mathbf{No}$ be the value set of $\imath(\Delta)$. Suppose towards contradiction that $\gamma^* \asymp \omega^s$ for some $s \in S$. Then there is an $r \in \mathbb{R}$ with $\gamma^* - r\omega^s \prec \omega^s$. If $r \in \mathbb{R}_K$, then $r\omega^s$ is a partial development of γ^* over $\imath(\Delta)$, contradicting our assumption that $D_{\imath(\Delta)}(\gamma^*) = 0$, so we may assume that $r \notin \mathbb{R}_K$. Then we have

$$\{p\omega^s : p \in \mathbb{R}_K^{<r}\} \;<\; \gamma^* \;<\; \{q\omega^s : q \in \mathbb{R}_K^{>r}\}.$$

Set $\delta := \imath^{-1}(\omega^s)$, so we have

$$\{p\delta : p \in \mathbb{R}_K^{<r}\} \;<\; \gamma \;<\; \{q\delta : q \in \mathbb{R}_K^{>r}\}.$$

Assume that $\log_K(t^\delta) > 0$. Then we have

$$\left\{\frac{\log_K t^{p\delta}}{\log_K t^\delta} : p \in \mathbb{R}_K^{<r}\right\} \;<\; \frac{\log_K t^\gamma}{\log_K t^\delta} \;<\; \left\{\frac{\log_K t^{q\delta}}{\log_K t^\delta} : q \in \mathbb{R}_K^{>r}\right\}.$$

Set $x := \frac{\log_K t^\gamma}{\log_K t^\delta}$. Since $\frac{\log_K t^{p\delta}}{\log_K t^\delta} = p$ for all $p \in \mathbb{R}_K$, this shows that

$$\mathbb{R}_K^{<r} \;<\; x \;<\; \mathbb{R}_K^{>r}.$$

Then the residue of x is r, a contradiction. $\square$

Lemma 6.3. *If Δ is a* log*-subspace of Γ then there is a smallest* log-exp*-subspace of Γ containing Δ, which we denote by Δ^E.*

Proof. We define an increasing family (Δ_α) of $\mathbb{R}_K$-subspaces of Γ by setting

$$\Delta_0 \;:=\; \Delta, \qquad \Delta_{\alpha+1} \;:=\; \big\{\gamma \in \Gamma : \log_K(t^\gamma) \in K_{\Delta_\alpha}\big\}, \qquad \Delta_\alpha \;:=\; \bigcup_{\beta<\alpha} \Delta_\beta \text{ when } \alpha \text{ is a limit ordinal.}$$

Since Γ is a set, this family is eventually no longer strictly increasing, i.e. there is α_0 such that $\Delta_\alpha = \Delta_{\alpha_0}$ for each $\alpha \geqslant \alpha_0$. Then Δ_{α_0} is a log-exp-subspace of Γ: if $\gamma \in \Gamma$ and $\log_K(t^\gamma) \in K_{\Delta_{\alpha_0}}$, then $\gamma \in \Delta_{\alpha_0+1} = \Delta_{\alpha_0}$. Set $\Delta^E := \Delta_{\alpha_0}$. To see that Δ^E is minimal, note that any log-exp-subspace of Γ containing Δ_β for each $\beta < \alpha$ must contain Δ_α. By induction, we see that any log-exp-subspace of Γ containing $\Delta_0 = \Delta$ must contain Δ^E. $\square$

Lemma 6.4. *Suppose that Δ is a* log*-subspace of Γ and that $\imath$ is a* log*-embedding. Then $\imath$ extends uniquely to a* log*-embedding $\jmath : \Delta^E \to \mathbf{No}$. Moreover, if $\imath(\Delta)$ is initial then $\jmath(\Delta^E)$ is as well.*

Proof. We first show that any such embedding is unique. Let $\jmath_1$ and $\jmath_2$ be any two such embeddings and let (Δ_α) be as in the proof of Lemma 6.3. We know that $\jmath_1$ and $\jmath_2$ agree with each other (and with $\imath$) on $\Delta_0 = \Delta$. If $\jmath_1$ and $\jmath_2$ agree on Δ_β for all β less than a limit ordinal α, then they agree on Δ_α as well. We will show that if they agree on Δ_α, then they must agree on $\Delta_{\alpha+1}$. Let $\gamma \in \Delta_{\alpha+1}$. Then

$$\log \omega^{\jmath_1(\gamma)} \;=\; \tilde{\jmath}_1(\log_K t^\gamma) \;=\; \tilde{\jmath}_2(\log_K t^\gamma) \;=\; \log \omega^{\jmath_2(\gamma)},$$

where the middle equality follows from the fact that $\log_K(t^\gamma) \in K_{\Delta_\alpha}$. Injectivity of the logarithm and Conway's ω-map yields $\jmath_1(\gamma) = \jmath_2(\gamma)$. Thus, $\jmath_1$ and $\jmath_2$ agree on Δ^E.

We now show that such an embedding exists. Suppose that $\Delta \neq \Delta^E$. It is enough to show that $\imath$ can be extended to a log-embedding $\imath^* : \Delta^* \to \mathbf{No}$ where Δ^* is a log-subspace of Δ^E properly containing Δ. Fix $x \in (K_\Delta)_{\mathrm{PI}}$ with $\exp_K x \notin t^\Delta$. We have $\exp_K x = t^\gamma$ for some $\gamma \in \Delta^E \setminus \Delta$ and we have $\exp \tilde{\imath}(x) = \omega^{\gamma^*}$ for some $\gamma^* \in \mathbf{No} \setminus \imath(\Delta)$. We claim that γ^* realizes the same cut over $\imath(\Delta)$ that γ realizes over Δ. To see this, take $\delta \in \Delta$ and note that

$$\begin{aligned} \delta < \gamma \iff t^\delta < t^\gamma \iff \log_K t^\delta < x \iff \tilde{\imath}(\log_K t^\delta) < \tilde{\imath}(x) \\ \iff \log \omega^{\imath(\delta)} < \tilde{\imath}(x) \iff \omega^{\imath(\delta)} < \omega^{\gamma^*} \iff \imath(\delta) < \gamma^*. \end{aligned}$$

We set $\Delta^* := \Delta + \mathbb{R}_K\gamma$ and we extend $\imath$ to an embedding $\imath^* : \Delta^* \to \mathbf{No}$ by setting $\imath^*(\gamma) := \gamma^*$. By the argument above, $\imath^*$ is an ordered group embedding. To see that Δ^* is a log-subspace of Γ and that $\imath^*$ is log-preserving, consider an element $\delta + r\gamma$ where $\delta \in \Delta$ and where $r \in \mathbb{R}_K$. We have

$$\log_K t^{\delta + r\gamma} \;=\; \log_K t^\delta + rx \in (K_\Delta)_{\mathrm{PI}}$$

and we have

$$\tilde{\imath}^*(\log_K t^{\delta+r\gamma}) \;=\; \tilde{\imath}(\log_K t^{\delta}+rx) \;=\; \log \omega^{\imath(\delta)} + r\tilde{\imath}(x) \;=\; \log \omega^{\imath(\delta)} + \log \omega^{r\gamma^*} \;=\; \log \omega^{\imath^*(\delta+r\gamma)}.$$

Finally, suppose that $\imath(\Delta)$ is initial. Then $\tilde{\imath}(K_\Delta)$ is initial as well and we may choose x above which is simplest, in the sense that if $y \in (K_\Delta)_{\mathrm{PI}}$ and if $\tilde{\imath}(y) <_s \tilde{\imath}(x)$, then $\exp_K y \in \Delta$. By Lemma 5.1, we see that γ^* is the simplest element realizing a cut in $\imath(\Delta)$, so $\imath^*(\Delta^*) = \imath(\Delta) + \mathbb{R}_K\gamma^*$ is initial by Lemma 3.3. $\square$

6.2. **Δ-paths.** In this subsection, we fix a proper log-exp-subspace $\Delta \subsetneq \Gamma$ and a positive infinite $y \in K$ with $v(y) \notin \Delta$. We define a sequence (y_n) of elements of K by setting $y_0 := y$ and

$$y_{n+1} \;:=\; \big|\log_K y_n - D_{K_\Delta}(\log_K y_n)\big|.$$

Since K_Δ is closed under $\exp_K$, we have for each n that $y_n \notin K_\Delta$ and so $v(y_n) \notin \Delta$. We call (y_n) the **Δ-path of** y. The definition of a Δ-path is motivated by the Main Lemma in [31, page 286].

Lemma 6.5. *y_n is infinite for each n and $v(y_0), v(y_1), \ldots$ are $\mathbb{R}_K$-linearly independent over Δ.*

Proof. For each n, set $d_n := D_{K_\Delta}(\log_K y_n)$. By assumption, y_0 is infinite. Now suppose towards contradiction that there is an n with $y_{n+1} \preccurlyeq 1$. Then we have $\log_K y_n - d_n \preccurlyeq 1$ and so

$$\exp_K\big(\log_K y_n - d_n\big) \;=\; y_n/\exp_K d_n \;\asymp\; 1.$$

This gives $y_n \asymp \exp_K d_n$, a contradiction.

We now turn to linear independence. Suppose towards contradiction that there is some $n \geqslant 0$, some $\delta \in \Delta$, and some $r_0, \ldots, r_{n-1} \in \mathbb{R}_K$ with

$$v(y_n) \;=\; \delta + r_0 v(y_0) + \ldots + r_{n-1}v(y_{n-1}).$$

Then we have $y_n = ut^\delta y_0^{r_0}\cdots y_{n-1}^{r_{n-1}}$ for some $u \in K$ with $u \asymp 1$. We have

$$\log_K y_n \;=\; \log u + \log_K t^\delta + r_0 \log_K y_0 + \ldots + r_{n-1}\log_K y_{n-1}.$$

For each m, we have $\log_K y_m = d_m + \varepsilon_m y_{m+1}$ where $\varepsilon_m = \pm 1$. Set

$$a \;:=\; (\log_K t^\delta + r_0 d_0 + \ldots + r_{n-1}d_{n-1} - d_n) \in K_\Delta.$$

Since y_{n+1} is infinite and $\log u \preccurlyeq 1$, we have

$$\varepsilon_n y_{n+1} \;\asymp\; a + r_0\varepsilon_0 y_1 + \ldots + r_{n-1}\varepsilon_{n-1}y_n.$$

Since each y_m is infinite, we have $y_{m+1} \preccurlyeq \log_K y_m \prec y_m$. In particular, $v(a)$ and $v(y_1), \ldots, v(y_{n+1})$ are all distinct, a contradiction. $\square$

Definition 6.2. We say that y is **Δ-atomic** if $y_n = t^{v(y_n)}$ for each n. If y is Δ-atomic, then we set

$$\Delta\langle y\rangle := \Delta \oplus \bigoplus_{n=0}^{\infty}\mathbb{R}_K v(y_n).$$

Lemma 6.6. *If y is Δ-atomic then $\Delta\langle y\rangle$ is a* log-*subspace of* Γ.

Proof. Since $(K_{\Delta\langle y\rangle})_{\mathrm{PI}}$ is an $\mathbb{R}_K$-subspace of K, it is enough to show that $\log y_n \in (K_{\Delta\langle y\rangle})_{\mathrm{PI}}$ for each n. We have

$$\log_K y_n \;=\; D_{K_\Delta}(\log_K y_n) \pm y_{n+1} \;=\; D_{K_\Delta}(\log_K y_n) \pm t^{v(y_{n+1})}.$$

Since y_{n+1} is infinite, we know that both $D_{K_\Delta}(\log_K y_n)$ and $t^{v(y_{n+1})}$ are purely infinite. Thus, $\log_K y_n$ is purely infinite as well. $\square$

As the reader will recall, a family of structures $(A_\alpha)_{\alpha<\beta\leqslant\mathbf{On}}$ is said to be a **continuous chain** if $A_\sigma \subseteq A_\alpha$ for all $\sigma < \alpha < \beta$ and if $A_\alpha = \bigcup_{\sigma<\alpha} A_\sigma$ for each infinite limit ordinal $\alpha < \beta$. This notion is employed repeatedly in the remainder of the paper beginning with:

Definition 6.3. Let $\Delta \subseteq \Gamma$ be a log-subspace of Γ. We say that K is **molecular over** Δ if there is a continuous chain $(\Delta_\alpha)_{\alpha\leqslant\beta<\mathbf{On}}$ of log-exp-subspaces of Γ and a corresponding family of elements $(y_\alpha)_{\alpha<\beta}$ from K such that the following holds:

(i) $\Delta_0 = \Delta^E$ (see Lemma 6.3) and $\Delta_\beta = \Gamma$.
(ii) y_α is positive, infinite, and Δ_α-atomic and $\Delta_{\alpha+1} = \Delta_\alpha\langle y_\alpha\rangle^E$ for each $\alpha < \beta$.

We say that K is **molecular** if it is molecular over $\Delta = \{0\}$.

6.3. **Δ-paths of surreal numbers.** In this subsection, we fix a proper log-exp-subspace $\Delta \subseteq \Gamma$ and a positive infinite $y \in K$ with $v(y) \notin \Delta$. We also fix an initial log-embedding $\imath : \Delta \to \mathbf{No}$. Let $S \subseteq \mathbf{No}$ be the value set of $\imath(\Delta)$. Since $\imath(\Delta)$ is initial, S is also initial and $\omega^s \in \Delta$ for each $s \in S$. Let (y_n) be the Δ-path of y and for each n, set $d_n := D_{K_\Delta}(\log_K y_n)$. Let $y^* \in \mathbf{No}$ be an element realizing the same cut over $\tilde{\imath}(K_\Delta)$ that y realizes over K_Δ, and suppose that y^* is the simplest element in this cut. We define a sequence (y_n^*) in $\mathbf{No}$ by setting $y_0^* := y^*$ and by setting

$$y_{n+1}^* \ := \ \big|\log y_n^* - \tilde{\imath}(d_n)\big|$$

for each n. For each n, we set

$$\Delta_n \ := \ \Delta + \mathbb{R}_K v(y_0) + \ldots + \mathbb{R}_K v(y_{n-1}), \qquad \Delta_n^* \ := \ \imath(\Delta) + \mathbb{R}_K v(y_0^*) + \ldots + \mathbb{R}_K v(y_{n-1}^*).$$

Lemma 6.7. *y_n^* realizes the same cut over $\tilde{\imath}(K_\Delta)$ that y_n realizes over K_Δ for each n.*

Proof. This is clear for $n = 0$. Suppose that it holds for a given n. Fix $x \in K_\Delta$. We have

$$y_{n+1} < x \iff \big|\log_K y_n - d_n\big| < x.$$

Suppose that $\log y_n - d_n$ is positive (the case where it is negative is similar). We have

$$y_{n+1} < x \iff \log_K y_n - d_n < x \iff y_n < \exp_K(x + d_n).$$

Since $\imath$ is assumed to be a log-embedding, we have $\tilde{\imath}\big(\exp_K(x + d_n)\big) = \exp\big(\tilde{\imath}(x + d_n)\big)$. In light of our inductive assumption on y_n, this gives

$$y_{n+1} < x \iff y_n < \exp_K(x + d_n) \iff y_n^* < \exp\big(\tilde{\imath}(x + d_n)\big).$$

Working backwards, we see that $y_{n+1} < x \iff y_{n+1}^* < \tilde{\imath}(x)$. □

Lemma 6.8. *For each n, we have a unique isomorphism of ordered $\mathbb{R}_K$-vector spaces $\imath_n : \Delta_n \to \Delta_n^*$ which extends $\imath$ and which sends $v(y_m)$ to $v(y_m^*)$ for each $m < n$.*

Proof. We proceed by induction. This is clear for $n = 0$, so suppose that it holds for a fixed n. By Lemma 6.5, we know that $v(y_n) \notin \Delta_n$, so we need to show that $v(y_n^*)$ realizes the same cut over Δ_n^* that $v(y_n)$ realizes over Δ_n. An arbitrary $\delta \in \Delta_n$ has the form $\delta = \delta_0 + r_0 v(y_0) + \ldots + r_{n-1} v(y_{n-1})$ for some $\delta_0 \in \Delta$ and some $r_0, \ldots, r_{n-1} \in \mathbb{R}_K$. Then $t^\delta \asymp t^{\delta_0} y_0^{r_0} \ldots y_{n-1}^{r_{n-1}}$. Suppose that $y_n \succ t^{\delta_0} y_0^{r_0} \ldots y_{n-1}^{r_{n-1}}$. Since y_n and y_n^* are positive infinite, we may assume that $t^{\delta_0} y_0^{r_0} \ldots y_{n-1}^{r_{n-1}}$ is positive infinite as well. Then we have

$$\log y_n \ > \ \log t^{\delta_0} + r_0 \log y_0 + \ldots + r_{n-1} \log_K y_{n-1}.$$

As in the proof of Lemma 6.5, we write $\log_K y_m = d_m + \varepsilon_m y_{m+1}$ for each m, where $\varepsilon_m = \pm 1$, and we set

$$a \ := \ (\log_K t^{\delta_0} + r_0 d_0 + \ldots + r_{n-1} d_{n-1} - d_n).$$

We have

$$\varepsilon_n y_{n+1} \ > \ a + r_0 \varepsilon_0 y_1 + \ldots + r_{n-1} \varepsilon_{n-1} y_n.$$

Since $v(a)$ and $v(y_1), \ldots, v(y_{n+1})$ are all distinct, we see that this only depends on the signs of $r_0, \ldots, r_{n-1}$, on $\varepsilon_0, \ldots, \varepsilon_n$, and on whether $y_{n+1} > a$. This in turn only depends on the cut of y_{n+1} over K_Δ. By Lemma 6.7, this gives that

$$\varepsilon_n y_{n+1}^* \ > \ \tilde{\imath}(a) + r_0 \varepsilon_0 y_1^* + \ldots + r_{n-1} \varepsilon_{n-1} y_n^*.$$

We work backwards from this to deduce that $y_n^* \succ \omega^{\imath(\delta_0)} (y_0^*)^{r_0} \ldots (y_{n-1}^*)^{r_{n-1}}$. The case that $y_n \prec t^\delta y_0^{r_0} \ldots y_{n-1}^{r_{n-1}}$ is similar. □

Lemma 6.9. *For each n, we have $y_n^* = \omega^{v(y_n^*)}$ and $v(y_n^*)$ is the simplest element realizing a cut over Δ_n^*.*

Proof. By Lemma 6.7, we have that y_n^* realizes the same cut over $\tilde{\imath}(K_\Delta)$ that y_n realizes over K_Δ. This tells us that

$$\tilde{\imath}(d_n) \ = \ \tilde{\imath}\big(D_{K_\Delta}(\log_K y_n)\big) \ = \ D_{\tilde{\imath}(K_\Delta)}(\log y_n^*).$$

This also tells us that $v(y_n^*)$ realizes the same cut over Δ_0^* that $v(y_n)$ realizes over Δ_0, so $v(y_n^*)$ realizes a v-cut over Δ_0^* by Lemma 6.2. For each n, set $\gamma_n := v(y_n^*)$. We will show that the following holds for each $n \geqslant 0$:

(i) $y_m^* = \omega^{\gamma_m}$ for each $m \leqslant n$.
(ii) γ_m is the simplest element realizing a cut over Δ_m^* for each $m \leqslant n$.
(iii) There are $s_0 > s_1 > \ldots > s_{n-1} \in \mathbf{No}$ with $\gamma_m = D_{\Delta_0^*}(\gamma_m) \pm \omega^{s_m}$ and $\gamma_{m+1} = h(s_m)$ for $m < n$.

We begin with the case $n = 0$. Since $v(y_0^*) \notin \Delta_0^*$ we have that y_0^* and ω^{γ_0} realize the same cut over $\tilde{\imath}(K_\Delta)$. Since $\omega^{\gamma_0} \leqslant_s y_0^*$ and since y_0^* is assumed to be simplest, we have that $y_0^* = \omega^{\gamma_0}$. The assumption that y_0^* is simplest also gives that γ_0 is the simplest element realizing a cut over Δ_0^*. This takes care of (i) and (ii), and (iii) holds vacuously.

We now fix $n \geqslant 0$ and show that (i)–(iii) hold for $n+1$, assuming that they hold for n. Our assumptions ensure that the hypotheses of Lemma 5.3 are met with Δ_m^* in place of Γ_m. We conclude that $\gamma_n = D_{\Delta_0^*}(\gamma_n) \pm \omega^s$ where $s \in \mathbf{No} \setminus S$ is less than s_{n-1} and where $h(s)$ is the simplest element realizing a cut over Δ_{n+1}^*. We have

$$\begin{aligned} y_{n+1}^* &= \left|\log y_n^* - \tilde{\imath}(d_n)\right| = \left|\log y_n^* - D_{\tilde{\imath}(K_\Delta)}(\log y_n^*)\right| = \left|\log \omega^{\gamma_n} - D_{\tilde{\imath}(K_\Delta)}(\log \omega^{\gamma_n})\right| \\ &= \left|\log \omega^{D_{\imath(\Delta)}(\gamma_n)\pm\omega^s} - \log \omega^{D_{\imath(\Delta)}(\gamma_n)}\right| = \left|\log \omega^{\pm\omega^s}\right| = \log \omega^{\omega^s} = \omega^{h(s)}. \end{aligned}$$

From this, it follows that $h(s) = \gamma_{n+1}$, thereby establishing (i), (ii), and (iii) with $s_n := s$. □

Proposition 6.1. *Suppose that y is Δ-atomic. Then there is a unique* log*-embedding*

$$\jmath : \Delta\langle y\rangle^E \to \mathbf{No}$$

which extends $\imath$ and sends y to y^*. Moreover, $\jmath$ is initial.

Proof. We note that $\Delta\langle y\rangle = \bigcup_n \Delta_n$ and we set $\Delta^* := \bigcup_n \Delta_n^*$. By Lemma 6.8, we have a unique isomorphism of $\mathbb{R}_K$-vector spaces $\imath^* : \Delta\langle y\rangle \to \Delta^*$ which maps each $v(y_n)$ to $v(y_n^*)$. As y is Δ-atomic, we have that $y_n = t^{v(y_n)}$ for each n, so each y_n is in $K_{\Delta\langle y\rangle}$. By Lemma 6.9, we have $y_n^* = \omega^{v(y_n^*)}$ for each n. Thus, $\tilde{\imath}^*$ sends each y_n to y_n^*.

We claim that $\imath^*$ is a log-embedding. For this, it is enough to show that

$$\tilde{\imath}^*\big(\log_K t^{v(y_n)}\big) = \log \omega^{v(y_n^*)}$$

for each n. We may write $\log_K y_n = d_n \pm y_{n+1}$, so we have

$$\tilde{\imath}^*\big(\log_K t^{v(y_n)}\big) = \tilde{\imath}^*(\log_K y_n) = \tilde{\imath}^*(d_n \pm y_{n+1}) = \tilde{\imath}(d_n) \pm y_{n+1}^* = \log y_n^* = \log \omega^{v(y_n^*)}.$$

By Lemma 6.4, $\imath^*$ extends uniquely to a log-embedding $\jmath : \Delta\langle y\rangle^E \to \mathbf{No}$. By Lemma 6.9, we have that $\imath^*$ is initial, so $\jmath$ is initial as well by Lemma 6.4. □

Corollary 6.1. *There is an initial* log*-embedding $\jmath : \Gamma \to \mathbf{No}$ which extends $\imath$ if and only if K is molecular over Δ.*

Proof. First, assume that K is molecular over Δ and let $(\Delta_\alpha)_{\alpha\leqslant\beta}$ and $(y_\alpha)_{\alpha<\beta}$ witness this, $\beta < \mathbf{On}$. Fix $\alpha < \beta$ and assume that we have an initial log-embedding $\imath_\alpha : \Delta_\alpha \to \mathbf{No}$. This holds when $\alpha = 0$, since $\Delta_0 = \Delta$ so we may take $\imath_0 = \imath$. Fix $y^* \in \mathbf{No}$ which is the simplest element realizing the same cut over $\tilde{\imath}_\alpha(K_{\Delta_\alpha})$ that y_α realizes over K_{Δ_α}. By Proposition 6.1, $\imath_\alpha$ extends uniquely to an initial log-embedding $\imath_{\alpha+1} : \Delta_{\alpha+1} \to \mathbf{No}$ which sends y_α to y^*. Taking unions at limit stages, we can construct an initial log-embedding $\jmath = \imath_\beta$ of $\Delta_\beta = \Gamma$ into $\mathbf{No}$.

Now let $\jmath : \Gamma \to \mathbf{No}$ be an initial log-embedding extending $\imath$. We may identify K with an initial exponential subfield of $\mathbf{No}$ via $\tilde{\jmath}$. Assume that for some $\beta < \mathbf{On}$ we have a continuous chain $(\Delta_\alpha)_{\alpha\leqslant\beta}$ of initial log-exp-subspaces of Γ, and a corresponding family of elements $(y_\alpha)_{\alpha<\beta}$ from K such that the following holds:

(a) $\Delta_0 = \Delta$.
(b) y_α is Δ_α-atomic and $\Delta_{\alpha+1} = \Delta\langle y_\alpha\rangle^E$ for each $\alpha < \beta$.

If $\Delta_\beta = \Gamma$ we are done. Suppose that $\Delta_\beta \subsetneq \Gamma$. Take $\gamma \in \Gamma^{>0}$ which is the simplest element in $\mathbf{No}$ realizing a cut over Δ_β and set $y_\beta := \omega^\gamma$. Then y_β is the simplest element in K realizing a cut over K_{Δ_β}, so it is Δ_β-atomic by Lemma 6.9. Then the identity map on $\Delta_\beta\langle y_\beta\rangle^E$ is a log-embedding, and so by Proposition 6.1, it is initial. We set $\Delta_{\beta+1} := \Delta_\beta\langle y_\beta\rangle^E$. Continuing in this way and taking unions at limit stages, we eventually exhaust all of Γ. □

7. Initial embeddings of ordered exponential fields

We now turn to the first of our main results.

Theorem 7.1. *Let A be an ordered exponential field whose universe is a set or a proper class. Then A is isomorphic to an initial exponential subfield of* $\mathbf{No}$ *if and only if A is isomorphic to an ordered exponential field K, where:*

(i) *K is a truncation closed, cross sectional subfield of a Hahn field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$;*
(ii) $\exp_K x = e^x$ *for all $x \in \mathbb{R}_K$;*
(iii) $\exp_K x = \sum_{n=0}^{\infty} x^n/n!$ *for all infinitesimal $x \in K$;*
(iv) $\exp_K x > x^n$ *for each positive infinite $x \in K$ and each n;*
(v) $\log_K(t^\Gamma) = K_{\mathrm{PI}}$*;*
(vi) *There is $\beta \leqslant \mathbf{On}$, a continuous chain of* $\log$*-subspaces $(\Gamma_\alpha)_{\alpha<\beta}$ of Γ with $\Gamma = \bigcup_{\alpha<\beta} \Gamma_\alpha$, and a continuous chain $(K_\alpha)_{\alpha<\beta}$ of ordered exponential subfields of K with $K = \bigcup_{\alpha<\beta} K_\alpha$ such that for each $\alpha < \beta$:*
   (a) Γ_α and K_α are sets;
   (b) $\mathbb{R}_K \subseteq K_0$ and K_α is a truncation closed, cross sectional subfield of $\mathbb{R}((t^{\Gamma_\alpha}))$;
   (c) K_0 is molecular and $K_{\alpha+1}$ is molecular over Γ_α.

Proof. Let $(K_\alpha)_{\alpha<\beta}$ and $(\Gamma_\alpha)_{\alpha<\beta}$ be as in the statement of the theorem. Note that $\log_{K_\alpha}(t_\alpha^\Gamma) = (K_\alpha)_{\mathrm{PI}}$ for each α. Applying Corollary 6.1 to K_0 and $\{0\}$ in place of K and Δ, we get an initial log-embedding $\imath_0 : \Gamma_0 \to \mathbf{No}$. Now fix α with $\alpha + 1 < \beta$ and suppose that we have an initial log-embedding $\imath_\alpha : \Gamma_\alpha \to \mathbf{No}$. By Lemma 6.4, we may extend $\imath_\alpha$ to an initial log-embedding $\Gamma_\alpha^E \to \mathbf{No}$, where Γ_α^E is computed inside of $K_{\alpha+1}$. Now we apply Corollary 6.1 to $K_{\alpha+1}$ and Γ_α^E in place of K and Δ to further extend to an initial log-embedding $\Gamma_{\alpha+1} \to \mathbf{No}$.

For the converse, note that any initial exponential subfield of $\mathbf{No}$ satisfies (i)–(v), so it remains to show (vi). Let K be an initial exponential subfield of $\mathbf{No}$ and let Γ be the value group of K. If K is a set, then we let $\beta = 1$ and $K_0 = K$. Applying Corollary 6.1 with $\Delta = \{0\}$, we see that K is molecular. If K is a proper class, then we let $\beta = \mathbf{On}$, we let $K_0 := \mathbb{R}_K$, and for each $\alpha > 0$, we let

$$K_\alpha \ := \ K \cap \mathbf{No}(\omega_\alpha).$$

Then $(K_\alpha)_{\alpha<\mathbf{On}}$ is a continuous chain of exponential subfields of K with union K, and K_α is a truncation closed, cross sectional subfield of $\mathbb{R}((\omega^{\Gamma_\alpha}))$ for each α, where $\Gamma_\alpha \ := \ \{\gamma \in \Gamma : \omega^\gamma \in K_\alpha\}$. Since K_0 is trivially molecular, we need only show that $K_{\alpha+1}$ is molecular over Γ_α for each α. It is equivalent to show that $K_{\alpha+1}$ is molecular over Γ_α^E for each α, where Γ_α^E is computed in $K_{\alpha+1}$. This follows from Corollary 6.1, with $K_{\alpha+1}$ and Γ_α^E in place of K and Δ. □

The last condition of Theorem 7.1 is somewhat lengthy. However, the just provided proof of the theorem allows us to replace it with a substantially simpler condition in the case that A is a set.

Corollary 7.1. *Let A be an ordered exponential field whose universe is a set. Then A is isomorphic to an initial exponential subfield of* $\mathbf{No}$ *if and only if A is isomorphic to an ordered exponential field K, where:*

(i) *K is a truncation closed, cross sectional subfield of a Hahn field $\mathbb{R}((t^\Gamma))$;*
(ii) $\exp_K x = e^x$ *for all $x \in \mathbb{R}_K$;*
(iii) $\exp_K x = \sum_{n=0}^{\infty} x^n/n!$ *for all infinitesimal $x \in K$;*
(iv) $\exp_K x > x^n$ *for each positive infinite $x \in K$ and each n;*
(v) $\log_K(t^\Gamma) = K_{\mathrm{PI}}$*;*
(vi) *K is molecular.*

Remark 7.1. *The proof of Theorem 7.1 gives us additional information which will be useful later: If K is an exponential ordered field satisfying conditions (i)-(vi) in the statement of Theorem 7.1, then there is an initial* $\log$*-embedding $\imath : \Gamma \to \mathbf{No}$, so $\tilde{\imath}$ is an initial ordered exponential field embedding. The same is true in Corollary 7.1.*

Remark 7.2. *One may ask why the simpler criterion in Corollary 7.1 does not also work for proper classes or, more precisely, why we only define "molecularity" for sets. If one allows K and Γ to be proper classes in Definition 6.3, then we run into the following problem: Let $(\Delta_\alpha)_{\alpha\leqslant\beta}$ and $(y_\alpha)_{\alpha\leqslant\beta}$ be as in the proof of*

Corollary 6.1, let $\alpha < \beta$, and suppose that we have an initial log-*embedding* $\imath_\alpha : \Delta_\alpha \to \mathbf{No}$*. The next step in the proof is to fix y^* in* **No** *which realizes the same cut over $\tilde{\imath}_\alpha(K_{\Delta_\alpha})$ that y_α realizes over K_{Δ_α}. If we allow Γ to be a proper class, then Δ_α may be a proper class as well. Then K_{Δ_α} is also a proper class and we are not guaranteed the existence of an element y^* as desired.*

Ressayre ([31]; also see [6]) showed if A is a *real closed exponential field* with residue class field $\mathbb{R}_A$ and value group Γ, then A is isomorphic to an exponential ordered field K satisfying conditions (i), (ii), (iv) and (v) of the above theorem (conditions (ii) and (iv) follow from Ressayre's axioms for real closed exponential fields). It is an open question whether in general such embeddings for models of the theory $T(\mathbb{R}, e^x)$ of real numbers with exponentiation can be found that also satisfy conditions (iii) and (vi). As such, contrary to what is stated in [19], the following remains an

Open Question. *Is every model of $T(\mathbb{R}, e^x)$ isomorphic to an initial exponential subfield of* **No***?*

However, while this question remains open, in the following section it is shown there are distinguished classes of models of $T(\mathbb{R}, e^x)$ having additional structure that are isomorphic to initial exponential subfields of **No**. We remark that if Ressayre's embedding theorem can be amended to also satisfy (iii), then the methods in the next section can likely be used to show that it satisfies (vi) as well.

8. Exponential fields which define convergent Weierstrass systems

Let $I = [-1, 1]$. Given n, an open neighborhood $U \supseteq I^n$, and a real analytic function $f : U \to \mathbb{R}$, we define a corresponding **restricted analytic function** $\bar{f} : U \to \mathbb{R}$ by

$$\bar{f}(x) \ := \ \begin{cases} f(x) & \text{if } x \in I^n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}_r^\omega$ denote the family of all restricted analytic functions (of any arity) and let $\mathcal{F} \subseteq \mathcal{C}_r^\omega$.

Definition 8.1.

(i) Let $\mathcal{L}_\mathcal{F}$ be the language $(+, \cdot, -, 0, 1, <, \bar{f} \in \mathcal{F})$ and let $\mathbb{R}_\mathcal{F}$ be the natural expansion of $\mathbb{R}$ to an $\mathcal{L}_\mathcal{F}$-structure. Let $T_\mathcal{F}$ be the complete $\mathcal{L}_\mathcal{F}$-theory of $\mathbb{R}_\mathcal{F}$.
(ii) Let $\mathcal{L}_{\mathcal{F},\exp}$ be the language $\mathcal{L}_\mathcal{F} \cup \{\exp\}$ and let $\mathbb{R}_{\mathcal{F},\exp}$ be the expansion of $\mathbb{R}_\mathcal{F}$ by the total exponential function. Let $T_{\mathcal{F},\exp}$ be the complete $\mathcal{L}_{\mathcal{F},\exp}$-theory of $\mathbb{R}_{\mathcal{F},\exp}$.

As a consequence of Proposition 1.6, we have the following:

Fact 1. **No** *admits a natural expansion to a model of $T_\mathcal{F}$. In this expansion, the interpretation of any restricted analytic function agrees with its Taylor series expansion.*

Lemma 8.1. *Let Γ be a divisible ordered abelian group (whose universe is a set or a proper class). Then the Hahn field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ admits a natural expansion to a model of $T_\mathcal{F}$. In this expansion, the interpretation of any restricted analytic function agrees with its Taylor series expansion.*

Proof. If Γ is a set, this this is a result of van den Dries, Macintyre and Marker [10]. If Γ is a proper class, then we have $\Gamma = \bigcup_{\alpha<\mathbf{On}} \Gamma_\alpha$ where $(\Gamma_\alpha)_{\alpha<\mathbf{On}}$ is an increasing family of divisible ordered abelian subgroups of Γ whose universes are sets. Then $\mathbb{R}((t^\Gamma))_{\mathbf{On}} = \bigcup_{\alpha<\mathbf{On}} \mathbb{R}((t^{\Gamma_\alpha}))$. Moreover, since $\big(\mathbb{R}((t^{\Gamma_\alpha}))\big)_{\alpha<\beta<\mathbf{On}}$ is a continuous chain of set-models of an $\forall\exists$-theory $T_\mathcal{F}$, $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ is a model of $T_\mathcal{F}$ as well [13, page 41, (i)]. $\square$

Let $\mathcal{F}^{\mathrm{df}} \subseteq \mathcal{C}_r^\omega$ be the collection of all restricted analytic functions which are 0-definable in the structure $\mathbb{R}_\mathcal{F}$. Let $K \models T_\mathcal{F}$ and let $A \subseteq K$. We say that A is **$\mathcal{F}$-closed** if A is a real closed subfield of K which is closed under all functions $\bar{f} \in \mathcal{F}^{\mathrm{df}}$. We let the **$\mathcal{F}$-closure** of A be the smallest $\mathcal{F}$-closed subfield of K containing A. Given an $\mathcal{F}$-closed subfield $A \subseteq K$ and an element $y \in K$, we let $A\langle y\rangle$ be the $\mathcal{F}$-closure of $A \cup \{y\}$. Note that $\mathcal{F}^{\mathrm{df}}$ is closed under taking partial derivatives, so the following is a consequence of [30, Lemma 3.5] and [11, Lemma 3.3].

Lemma 8.2. *Let Γ be a divisible ordered abelian group whose universe is a set and let A be a truncation closed subset of $\mathbb{R}((t^\Gamma))$. Then the $\mathcal{F}$-closure of A is truncation closed.*

We say that $T_{\mathcal{F}}$ **defines a convergent Weierstrass system** if the family

$$\{f(a+x) : \bar{f} \in \mathcal{F}^{\mathrm{df}} \text{ and } a \in I^n\}$$

forms a convergent Weierstrass system, as defined in [8]. Our main result in this section is the following:

Theorem 8.1. *Suppose that $T_{\mathcal{F}}$ defines a convergent Weierstrass system and that the restriction $\overline{\exp}$ is in $\mathcal{F}^{\mathrm{df}}$. Then any model $K \models T_{\mathcal{F},\exp}$ admits an initial $\mathcal{L}_{\mathcal{F},\exp}$-elementary embedding into* $\mathbf{No}$.

In preparation for the proof of Theorem 8.1, we devote the next subsection to proving a simpler result: if $T_{\mathcal{F}}$ defines a convergent Weierstrass system then any model $K \models T_{\mathcal{F}}$ admits an initial $\mathcal{L}_{\mathcal{F}}$-elementary embedding into $\mathbf{No}$.

8.1. **Initial embeddings of models of $T_{\mathcal{F}}$.** In this subsection, we assume that $T_{\mathcal{F}}$ defines a convergent Weierstrass system and we fix a model $K \models T_{\mathcal{F}}$ whose universe K is a set or a proper class. As any convergent Weierstrass system contains all of the constant functions, we may naturally identify $\mathbb{R}$ with a subfield of K.

The following result, [8, Theorem 1.7], is key:

Proposition 8.1. *$\mathbb{R}_{\mathcal{F}}$ admits quantifier elimination in the language $\mathcal{L}_{\mathcal{F}^{\mathrm{df}}} \cup \{(-)^{-1}\}$, where $(-)^{-1}$ is interpreted as multiplicative inversion away from zero.*

Corollary 8.1. *Suppose that $T_{\mathcal{F}}$ defines a convergent Weierstrass system, let $K \models T_{\mathcal{F}}$, and suppose that A is an $\mathcal{F}$-closed subset of K. Then A is an $\mathcal{L}_{\mathcal{F}}$-elementary substructure of K. Moreover, for any $y \in K \setminus A$, the complete $\mathcal{L}_{\mathcal{F}}$-type of y over A is determined by the cut of y in A.*

Proof. By the same argument as in [10], we have that a subset $A \subseteq K$ is an elementary substructure of K if and only if A is $\mathcal{F}$-closed, so the $\mathcal{F}$-closure of A is the same as the $\mathcal{L}_{\mathcal{F}}$-definable closure of A. Since $T_{\mathcal{F}}$ is o-minimal, the complete $\mathcal{L}_{\mathcal{F}}$-type of y over A is determined by the cut of y in A. $\square$

The following fact is an immediate consequence of [10, Corollary 3.7]:

Fact 2. *Suppose that $T_{\mathcal{F}}$ defines a convergent Weierstrass system, let $K \models T_{\mathcal{F}}$ and let A be an $\mathcal{F}$-closed subset of K. Let $\mathfrak{M}$ be a cross section for A and let $\mathfrak{m} \in K$ be such that $\mathfrak{m} \not\asymp a$ for all $a \in A$. Then $\mathfrak{M} \times \mathfrak{m}^{\mathbb{Q}}$ is a cross section for $A\langle \mathfrak{m} \rangle$.*

Proposition 8.2. *Suppose that $T_{\mathcal{F}}$ defines a convergent Weierstrass system and let $K \models T_{\mathcal{F}}$. Then K admits a truncation closed, cross sectional $\mathcal{L}_{\mathcal{F}}$-elementary embedding into a Hahn field $\mathbb{R}((t^{\Gamma}))_{\mathbf{On}}$.*

Proof. We first consider the case that K is a set. Fix a cross section $\mathfrak{M} \subseteq K^{>0}$ and let Γ^* be an $|\mathfrak{M}|^+$-saturated divisible ordered abelian group, written additively. Let us assume that we have an embedding $\imath : A \to \mathbb{R}((t^{\Gamma}))$ where

(i) Γ is a divisible subgroup of Γ^* and $\imath(\mathfrak{M} \cap A) = t^{\Gamma}$, and
(ii) A is an $\mathcal{F}$-closed subset of K, $\imath(A)$ is truncation closed, and $\imath$ is $\mathcal{L}_{\mathcal{F}}$-elementary.

Such an embedding exists: since $T_{\mathcal{F}}$ defines a convergent Weierstrass system, we have $\mathbb{R} \subseteq K$ and so the identification of $\mathbb{R}$ with $\{rt^0 : r \in \mathbb{R}\}$ is such an embedding (where $\Gamma = \{0\}$). If $A = K$ then we are done, so we assume that $A \neq K$ and show any such embedding $\imath$ can be extended to a new embedding $\imath^*$ with the same properties. We consider two possibilities:

Case 1: Suppose that $D_A(y) \notin \imath(A)$ for some $y \in K \setminus A$. Then $D_A(y)$ realizes the image under $\imath$ of the cut of y over A. By o-minimality, $\imath$ extends to an $\mathcal{L}_{\mathcal{F}}$-elementary embedding

$$\imath^* : A\langle y \rangle \to \mathbb{R}((t^{\Gamma})),$$

which sends y to $D_A(y)$. As all truncations of $D_A(y)$ are in $\imath(A)$, the set $\imath(A) \cup \{D_A(y)\}$ is truncation closed. Thus, $\imath^*(A\langle y \rangle) = \imath^*(A)\langle D_A(y) \rangle$ is truncation closed as well by Lemma 8.2. Since $\imath^*(A\langle y \rangle) \subseteq \mathbb{R}((t^{\Gamma}))$ we have that for each each element $x \in A\langle y \rangle$ there is $\mathfrak{m} \in \mathfrak{M} \cap A$ with $x \asymp \mathfrak{m}$. Therefore, $\imath^*\big(\mathfrak{M} \cap A\langle y \rangle\big) = \imath(\mathfrak{M} \cap A) = t^{\Gamma}$.

Case 2: Suppose that $D_A(y) \in A$ for all $y \in K \setminus A$. This means that for each $y \in K \setminus A$, we have

$$y - \imath^{-1}\big(D_A(y)\big) \not\asymp \mathfrak{m}$$

for each $\mathfrak{m} \in \mathfrak{M} \cap A$ (otherwise, $D_A(y)$ would not be maximal). Thus $\mathfrak{M} \cap A \neq \mathfrak{M}$, so we take $\mathfrak{m} \in \mathfrak{M} \setminus A$. Using saturation, we take $\gamma \in \Gamma^* \setminus \Gamma$ such that t^{γ} realizes the same cut over $\imath(\mathfrak{M} \cap A) = t^{\Gamma}$ that $\mathfrak{m}$ realizes over $\mathfrak{M} \cap A$. One easily verifies that t^γ actually realizes the same cut over $\imath(A)$ that $\mathfrak{m}$ realizes over A, and so, again by o-minimality, $\imath$ extends to a truncation closed $\mathcal{L}_{\mathcal{F}}$-elementary embedding

$$\imath^* : A\langle \mathfrak{m}\rangle \to \mathbb{R}((t^{\Gamma^*})),$$

which sends $\mathfrak{m}$ to t^γ. Using Fact 2, we see that $\tilde{\imath}(A\langle \mathfrak{m}\rangle) \subseteq \mathbb{R}((t^{\Gamma \oplus \mathbb{Q}\gamma}))$, that $\mathfrak{M} \cap A\langle \mathfrak{m}\rangle = (\mathfrak{M} \cap A) \times \mathfrak{m}^{\mathbb{Q}}$, and that $\imath\big(\mathfrak{M} \cap A\langle \mathfrak{m}\rangle\big) = t^{\Gamma \oplus \mathbb{Q}\gamma}$.

Now suppose that K is a proper class. We may write K as an increasing union $\bigcup_{\alpha < \mathbf{On}} K_\alpha$ where K_α is a continuous chain of models of $T_{\mathcal{F}}$. Using the previous part of this proof, we can arrange that for each α, we have an embedding $\imath_\alpha : K_\alpha \to \mathbb{R}((t^{\Gamma_\alpha}))$ satisfying (i) and (ii) above such that Γ_α extends Γ_β and $\imath_\alpha$ extends $\imath_\beta$ for $\beta < \alpha < \mathbf{On}$. Set $\Gamma := \bigcup_{\alpha<\mathbf{On}} \Gamma_\alpha$ and set $\imath := \bigcup_{\alpha<\mathbf{On}} \imath_\alpha$. Then $\imath : K \to \mathbb{R}((t^\Gamma))_{\mathbf{On}}$ is truncation closed, cross sectional, and $\mathcal{L}_{\mathcal{F}}$-elementary. □

Corollary 8.2. *Suppose that $T_{\mathcal{F}}$ defines a convergent Weierstrass system and let $K \models T_{\mathcal{F}}$. Then K admits an initial $\mathcal{L}_{\mathcal{F}}$-elementary embedding into* $\mathbf{No}$.

Proof. By Proposition 8.2, we may identify K with a truncation closed, cross sectional $\mathcal{L}_{\mathcal{F}}$-elementary substructure of a Hahn field $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$. As Γ is divisible, we may fix an initial ordered group embedding $\imath : \Gamma \to \mathbf{No}$ by Proposition 1.1. Then the induced ordered field embedding

$$\sum_{\alpha<\beta} r_\alpha t^{\gamma_\alpha} \mapsto \sum_{\alpha<\beta} r_\alpha \omega^{\imath(\gamma_\alpha)} : K \to \mathbf{No}$$

is initial by Proposition 4.1. Since each restricted analytic function in $\mathcal{F}^{\mathrm{df}}$ agrees with its Taylor series expansion in both $\mathbb{R}((t^\Gamma))_{\mathbf{On}}$ and in $\mathbf{No}$, the embedding above is an $\mathcal{L}_{\mathcal{F}^{\mathrm{df}}}$-embedding. By Proposition 8.1, the image of this embedding is $\mathcal{L}_{\mathcal{F}}$-elementary. □

8.2. **Convergent Weierstrass systems with an entire exponential function.** In this subsection, we assume that $T_{\mathcal{F}}$ defines a convergent Weierstrass system and that $\overline{\exp}$ is in $\mathcal{F}^{\mathrm{df}}$. We fix a model $K \models T_{\mathcal{F},\exp}$ whose universe is a set or a proper class.

The method used in [10] gives the following:

Proposition 8.3. *$\mathbb{R}_{\mathcal{F},\exp}$ admits quantifier elimination in the language $\mathcal{L}_{\mathcal{F}^{\mathrm{df}},\exp} \cup \{\log\}$.*

Definition 8.2. Let A be an $\mathcal{F}$-closed subset of K. Let Δ be an $\mathbb{R}$-vector subspace of A. Suppose that we have a truncation closed $\mathcal{L}_{\mathcal{F}}$-elementary embedding $\imath : A \to \mathbb{R}((t^\Delta))_{\mathbf{On}}$ such that

(i) $\imath(\Delta) \subseteq \mathbb{R}((t^\Delta))_{\mathrm{PI}}$,
(ii) $\exp_K(\Delta) \subseteq A$, and
(iii) $\imath(\exp_K \delta) = t^\delta$ for each $\delta \in \Delta$.

We call $(A, \Delta, \imath)$ a **development triple**. We say that a development triple $(A, \Delta, \imath)$ is an exp**-development triple** if $\imath(\Delta) = \imath(A) \cap \mathbb{R}((t^\Delta))_{\mathrm{PI}}$.

The definition of a development triple is taken from [6]. Our development triples are slightly different than the ones in [6], as we insist that our substructures A be $\mathcal{F}$-closed. If $(A, \Delta, \imath)$ is a development triple, then the stipulation that $\imath$ be $\mathcal{L}_{\mathcal{F}}$-elementary ensures that $\exp_K x \in A$ with

$$\imath\big(\exp_K x\big) \;=\; \sum_{n=0}^{\infty} \imath(x)^n/n!$$

and that $\log_K(1+x) \in A$ for all infinitesimal $x \in A$. The fact that $K \models T_{\mathcal{F},\exp}$ also ensures that $\exp_K r = e^r$ for each $r \in \mathbb{R}$ and that $\log_K r = \ln r$ for each $r \in \mathbb{R}^{>0}$. Condition (iii) ensures that $\imath$ is cross sectional.

Lemma 8.3. *If $(A, \Delta, \imath)$ is a development triple, then A is closed under $\log_K$. If $(A, \Delta, \imath)$ is an* exp*-development triple then, A is closed under $\exp_K$.*

Proof. For each $a \in A^{>0}$, we may write $\imath(a) = rt^\delta\big(1 + \imath(x)\big)$ for some $\delta \in \Delta$, some $r \in \mathbb{R}^{>0}$, and some infinitesimal x. Then $a = r(\exp_K \delta)(1 + x)$ and so

$$\log_K a \;=\; \ln r + \delta + \log_K(1 + x) \in A.$$

If $(A, \Delta, \imath)$ is an exp-development triple, then we may write each $a \in A$ as $a = a_P + r + a_I$ where $\imath(a_P) \in \mathbb{R}((t^\Delta))_{\mathrm{PI}}$, where $r \in \mathbb{R}$, and where a_I is infinitesimal. Then

$$\exp_K a \ = \ \exp_K(a_P)e^r(\exp_K a_I).$$

Since $a_P \in \Delta$, we have that $\exp_K a \in A$. □

Definition 8.3. Let $(A, \Delta, \imath)$ be an exp-development triple. We set $A^* := \imath(A) \subseteq \mathbb{R}((t^\Delta))$ and we define $\exp_{A^*} : A^* \to A^*$ by setting

$$\exp_{A^*} a \ := \ \imath\big(\exp_K \imath^{-1}(a)\big)$$

for each $a \in A^*$.

Then A^* is truncation closed and cross sectional and

$$\log_{A^*}(t^\Delta) \ = \ \imath(\Delta) \ = \ A^*_{\mathrm{PI}}.$$

Let $(A, \Delta, \imath) \subseteq (B, \Gamma, \imath)$ be exp-development triples. Then $A^* \subseteq B^* \subseteq \mathbb{R}((t^\Gamma))$ and Δ is an exp-log-subspace of Γ. If $D_A(x) \in A^* = \imath(A)$ for each $x \in B$, then we have that $B^*_\Delta = A^*$. Conversely, for any exp-log-subspace $\Delta \subseteq \Gamma$, we have an exp-development triple $\big(A^\Delta, \Delta, \jmath\restriction_{A^\Delta}\big)$ given by setting

$$A^\Delta \ := \ \jmath^{-1}(B^*_\Delta).$$

Lemma 8.4. *Any development triple $(A, \Delta, \imath)$ can be extended to a development triple $(B, \Delta, \jmath)$ such that $D_B(x) \in \jmath(B)$ for each $x \in K$.*

Proof. Suppose that there is an $x \in K$ with $D_A(x) \notin \imath(A)$. Then $D_A(x)$ realizes the same cut over $\imath(A)$ that x realizes over A. By o-minimality, we extend $\imath$ to a truncation closed $\mathcal{L}_{\mathcal{F}}$-elementary embedding

$$\imath^* : A\langle x\rangle \to \mathbb{R}((t^\Delta))$$

which sends x to $D_A(x)$. It is straightforward to check that $\big(A\langle x\rangle, \Delta, \imath^*\big)$ is still a development triple. Continuing in this manner, we construct a development triple $(B, \Delta, \jmath)$ extending $(A, \Delta, \imath)$ where $D_B(x) \in \jmath(B)$ for each $x \in K$. □

Lemma 8.5. *If K is a set, then any development triple $(A, \Delta, \imath)$ can be extended to an* exp-*development triple $(B, \Gamma, \jmath)$. Moreover, Γ can be taken to be minimal in the following sense: if $(B', \Gamma', \jmath')$ is another* exp-*development triple extending $(A, \Delta, \imath)$, then $\Gamma \subseteq \Gamma'$.*

Proof. By Lemma 8.4, we may extend A and assume that $D_A(x) \in \imath(A)$ for all $x \in K$. Suppose that there is a $\gamma \in A \setminus \Delta$ with $\imath(\gamma) \in \mathbb{R}((t^\Delta))_{\mathrm{PI}}$. We claim that $\exp_K \gamma \not\asymp \exp_K \delta$ for all $\delta \in \Delta$. Suppose not and take $u \in K$ with $u \asymp 1$ such that $\exp_K \gamma = u \exp_K \delta$. Then $\gamma - \delta = \log_K u \preccurlyeq 1$, contradicting that $\imath(\gamma)$ and $\imath(\delta)$ are purely infinite. Since every element of A is asymptotic to some element of $\exp_K(\Delta)$, this tells us that $\exp_K \gamma \notin A$ and that the cut realized by $\exp_K \gamma$ in A is completely determined by the cut realized by γ in Δ. Let Δ^* be the ordered $\mathbb{R}$-vector space $\Delta + \mathbb{R}\gamma \subseteq A$. We extend $\imath$ to an embedding

$$\imath^* : A\langle \exp_K \gamma\rangle \to \mathbb{R}((t^{\Delta^*}))$$

by sending $\exp_K \gamma$ to t^γ. Again, it is straightforward to check that $\big(A\langle \exp_K \gamma\rangle, \Delta^*, \imath^*\big)$ is a development triple. Continuing in this manner, we construct an exp-development triple $(B, \Delta, \jmath)$ extending $(A, \Delta, \imath)$.

We now turn to the claim that Γ is minimal. Let $(B', \Gamma', \jmath')$ be another exp-development triple extending $(A, \Delta, \imath)$. It is enough to show that $\Delta^* \subseteq \Gamma'$ and for this, it is enough to show that $\gamma \in \Gamma'$. Since

$$\jmath'(\Gamma') \ = \ \jmath'(B') \cap \mathbb{R}((t^{\Gamma'}))_{\mathrm{PI}} \ \supseteq \ \imath(A) \cap \mathbb{R}((t^\Delta))_{\mathrm{PI}},$$

this follows from the assumption that $\imath(\gamma) \in \mathbb{R}((t^\Delta))_{\mathrm{PI}}$. □

Proposition 8.4. *Suppose that K is a set and let $(A, \Delta, \imath)$ be an* exp-*development triple with $A \subsetneq K$ and $D_A(x) \in \imath(A)$ for all $x \in K$. Then for any $y \in K \setminus A$, there is an* exp-*development triple $(B, \Gamma, \imath)$ extending $(A, \Delta, \imath)$ such that $y \in B$ and such that $D_B(x) \in \jmath(B)$ for each $x \in K$. Moreover, given a* log-*subspace $\Delta_0 \subseteq \Delta$, if A^* is molecular over Δ_0, then B^* is molecular over Δ_0.*

Proof. If B contains $|y - D_A(y)|$, then it contains y so by replacing y with $|y - D_A(y)|$, we may assume that y is positive and that $y \not\asymp \exp_K \delta$ for each $\delta \in \Delta$. By replacing y with y^{-1}, we may assume that y is infinite. We construct a sequence (y_n) of elements of K by setting $y_0 := y$ and

$$y_{n+1} \;:=\; \left|\log_K y_n - \imath^{-1}\big(D_A(\log_K y_n)\big)\right|.$$

Methods akin to those in the proof of Lemma 6.5 show that each y_n is infinite and that the sequence $(\log_K y_n)$ is $\mathbb{R}$-linearly independent over Δ. We construct a sequence (Δ_n) of additive $\mathbb{R}$-vector subspaces of K by setting $\Delta_0 := \Delta$ and $\Delta_{n+1} := \Delta_n + \mathbb{R}\log_K y_n$ and we construct a sequence of subfields (A_n) of K by setting $A_0 := A$ and setting $A_{n+1} := A_n\langle y_n\rangle$. We also construct a series of $\mathcal{L}_{\mathcal{F}}$-elementary embeddings

$$\imath_n : A_n \to \mathbb{R}((t^{\Delta_n}))$$

which are defined setting $\imath_0 := \imath$ and by letting $\imath_{n+1}$ extend $\imath_n$ by sending y_n to $t^{\log_K y_n}$. We set

$$A_\infty \;:=\; \bigcup_n A_n, \qquad \Delta_\infty \;:=\; \bigcup_n \Delta_n$$

and we let $\imath_\infty : A_\infty \to \mathbb{R}((t^{\Delta_\infty}))$ be the common extension of the maps $\imath_n$. We claim that $(A_\infty, \Delta_\infty, \imath_\infty)$ is a development triple. Note that $\imath_\infty$ is $\mathcal{L}_{\mathcal{F}}$-elementary and truncation closed. By design, we have $\exp_K(\Delta_\infty) \subseteq B$ and $\jmath(\exp_K \gamma) = t^\gamma$ for each $\gamma \in \Delta_\infty$, so A_∞, Δ_∞ and $\imath_\infty$ satisfy (ii) and (iii). As for (i), it is enough to show that $\imath_\infty(\log_K y_n)$ is purely infinite for a given n. We have

$$\imath_\infty(\log_K y_n) \;=\; \imath_\infty\big(\imath^{-1}\big(D_A(\log_K y_n)\big) \pm y_{n+1}\big) \;=\; D_A(\log_K y_n) \pm t^{\log_K y_{n+1}}.$$

Since y_{n+1} is infinite, we know that $D_A(\log_K y_n)$ is purely infinite and that $\log_K y_{n+1}$ is positive. This gives $\imath_\infty(\log_K y_n) \in \mathbb{R}((t^{\Delta_\infty}))_{\mathrm{PI}}$.

Since $(A_\infty, \Delta_\infty, \imath_\infty)$ is a development triple, we may use Lemmas 8.4 and 8.5 to extend $(A_\infty, \Delta_\infty, \imath_\infty)$ to an exp-development triple $(B, \Gamma, \imath)$ such that $D_B(x) \in \jmath(B)$ for each $x \in K$. We choose Γ to be a minimal extension of Δ_∞, in the sense of Lemma 8.5.

We now turn to molecularity. Suppose that A^* is molecular over Δ_0, as witnessed by an increasing family $(\Delta_\alpha)_{\alpha \leqslant \beta}$ of log-exp-subspaces of Δ and a sequence of elements $(y_\alpha)_{\alpha<\beta}$ from A^*. Note that $\jmath(y)$ is a Δ-atomic element since $\jmath(y_n) = t^{\log_K y_n}$ for each n. Therefore $\Delta_\infty = \Delta\langle \jmath(y)\rangle$ and we claim that $\Gamma = \Delta_\infty^E$. Suppose that $\Delta_\infty^E \subsetneq \Gamma$. Then $\big(A^{\Delta_\infty^E}, \Delta_\infty^E, \jmath\restriction_{A^{\Delta_\infty^E}}\big)$ is an exp-development triple extending $(A_\infty, \Delta_\infty, \imath_\infty)$. This contradicts the minimality of Γ. We extend $(\Delta_\alpha)_{\alpha\leqslant\beta}$ to a family $(\Delta_\alpha)_{\alpha\leqslant\beta+1}$ by setting $\Delta_{\beta+1} := \Gamma$ and we extend $(y_\alpha)_{\alpha<\beta}$ to a sequence $(y_\alpha)_{\alpha<\beta+1}$ from B^* by setting $y_\beta := \jmath(y)$. Since $\Delta_\beta = \Delta$, we see that $(\Delta_\alpha)_{\alpha\leqslant\beta+1}$ and $(y_\alpha)_{\alpha<\beta+1}$ witness that B^* is molecular over Δ_0. $\square$

Corollary 8.3. *Suppose that K is a set and let $(A, \Delta, \imath)$ be a development triple. Then $(A, \Delta, \imath)$ can be extended to an* exp*-development triple $(K, \Gamma, \jmath)$ such that K^* is molecular over Δ.*

Proof. By Lemma 8.5, we may extend $(A, \Delta, \imath)$ to an exp-development triple $(A_0, \Delta_0, \imath_0)$ where $D_{A_0}(x) \in A_0^*$ for each $x \in K$ and where $\Delta_0 = \Delta^E$, as computed in A_0^*. Let $\beta < \mathbf{On}$ and suppose that we have an increasing family of exp-development triples $\big((A_\alpha, \Delta_\alpha, \imath_\alpha)\big)_{\alpha\leqslant\beta}$ such that for each $\alpha \leqslant \beta$:

(a) $D_{A_\alpha}(x) \in A_\alpha^*$ for each $x \in K$ and
(b) A_α^* is molecular over Δ.

If $A_\beta = K$, then we take $\Gamma := \Delta_\beta$ and $\jmath := \imath_\beta$ and we are done. Suppose that $A_\beta \neq K$ and take $y \in K \setminus A_\beta$. By Proposition 8.4, we can extend $(A_\beta, \Delta_\beta, \imath_\beta)$ to an exp-development triple $(A_{\beta+1}, \Delta_{\beta+1}, \imath_{\beta+1})$ such that $y \in A_{\beta+1}$, such that $D_{A_{\beta+1}}(x) \in A_{\beta+1}^*$ for each $x \in K$, and such that $A_{\beta+1}^*$ is molecular over Δ. We continue this process transfinitely, taking increasing unions at limit stages, until we exhaust all of K. $\square$

Proof of Theorem 8.1. We may write K as the union of a continuous chain $(K_\alpha)_{\alpha<\beta}$ of set-models of $T_{\mathcal{F},\exp}$ for some $\beta \leqslant \mathbf{On}$. We may assume that $K_0 = \mathbb{R}$. Taking $\Gamma_0 := \{0\}$ and $\imath_0 := \mathrm{id} : K_0 \to \mathbb{R} = \mathbb{R}((t^{\Gamma_0}))$, we get an exp-development triple $(K_0, \Gamma_0, \mathrm{id})$. For each α, suppose that we have an exp-development triple $(K_\alpha, \Gamma_\alpha, \imath_\alpha)$ and use Corollary 8.3 to extend this to an exp-development triple $(K_{\alpha+1}, \Gamma_{\alpha+1}, \imath_{\alpha+1})$ such that $K_{\alpha+1}^*$ is molecular over Γ_α. If α is a limit ordinal, then we set $(K_\alpha, \Gamma_\alpha, \imath_\alpha) := \bigcup_{\sigma<\alpha}(K_\sigma, \Gamma_\sigma, \imath_\sigma)$. K is isomorphic to $K^* := \bigcup_{\alpha<\beta} K_\alpha^*$, which satisfies the conditions in Theorem 7.1, as witnessed by the continuous chain $(K_\alpha^*)_{\alpha<\beta}$. This gives an initial ordered exponential field embedding $K \to \mathbf{No}$ which is even an $\mathcal{L}_{\mathcal{F}^{\mathrm{df}},\exp}$-embedding by Remark 7.1. By Proposition 8.3, the image of this embedding is $\mathcal{L}_{\mathcal{F},\exp}$-elementary. $\square$

8.3. **Examples.** We collect below some consequences of Theorem 8.1. First, if $\mathcal{F} = \mathcal{C}^\omega_r$, then $T_\mathcal{F}$ defines a convergent Weierstrass system, so we have the following:

Corollary 8.4. *Let $\mathbb{R}_{\text{an,exp}}$ be the expansion of $\mathbb{R}$ by all restricted analytic functions and the total exponential function and let $\mathcal{L}_{\text{an,exp}}$ be the corresponding language. If $K \equiv \mathbb{R}_{\text{an,exp}}$ then K admits an initial $\mathcal{L}_{\text{an,exp}}$-elementary embedding into* **No***.*

As was noted above, Corollary 8.4 was first proven by Fornasiero [23]. By [8, 5.4], the collection of all *differentially algebraic* analytic functions which converge in a neighborhood of 0 form a convergent Weierstrass system. This provides another example:

Corollary 8.5. *Let $\mathbb{R}_{\text{da,exp}}$ be the expansion of $\mathbb{R}$ by all differentially algebraic restricted analytic functions and the total exponential function and let $\mathcal{L}_{\text{da,exp}}$ be the corresponding language. If $K \equiv \mathbb{R}_{\text{da,exp}}$ then K admits an initial $\mathcal{L}_{\text{da,exp}}$-elementary embedding into* **No***.*

By [8], if $\mathcal{F} = \left\{\overline{\exp}, \overline{\sin}, r \in \mathbb{R}\right\}$ then $T_\mathcal{F}$ defines a convergent Weierstrass system (where, as the reader will recall, $\overline{\exp}$ and $\overline{\sin}$ are the restrictions of exp and sin to the interval $[-1,1]$). The domains of exp and sin here don't matter, so long as they are closed intervals. This gives us the following:

Corollary 8.6. *Let $\mathbb{R}_{\text{trig,exp}}$ be the expansion of $\mathbb{R}$ by $\sin\restriction_{[0,2\pi]}$, a constant for each $r \in \mathbb{R}$, and the total exponential function and let $\mathcal{L}_{\text{trig,exp}}$ be the corresponding language. If $K \equiv \mathbb{R}_{\text{trig,exp}}$ then K admits an initial $\mathcal{L}_{\text{trig,exp}}$-elementary embedding into* **No***.*

The method that van den Dries uses in the case $\mathcal{F} = \left\{\overline{\exp}, \overline{\sin}, r \in \mathbb{R}\right\}$ has been generalized by Sfouli, who provides sufficient conditions on $\mathcal{F}$ under which $T_\mathcal{F}$ defines a convergent Weierstrass system [32]:

Lemma 8.6 (Sfouli)**.** *Suppose that $\mathcal{F}$ satisfies the following two properties:*

(i) If $\bar{f} : I^n \to \mathbb{R}$ is in $\mathcal{F}$, then there is $\bar{g} : I^n \to \mathbb{R}$ in $\mathcal{F}$ such that either $\bar{f} + i\bar{g}$ or $\bar{g} + i\bar{f}$ is holomorphic on the interior of $I^{2n} \subseteq \mathbb{C}^n$.

(ii) If $\bar{f} : I^n \to \mathbb{R}$ is in $\mathcal{F}$, then there is $\varepsilon \in (0,1)$ and $\bar{g} : I^n \to \mathbb{R}$ in $\mathcal{F}$ such that $\bar{f}(x) = \bar{g}(\varepsilon x)$ for all $x \in I^n$.

Then $T_\mathcal{F}$ defines a convergent Weierstrass system.

Sfouli goes on to show that the family $\mathcal{F}_{\text{har}}$ of all restricted harmonic functions $\bar{f} : I^2 \to \mathbb{R}$ satisfies these properties. While $\exp\restriction_{[-1,1]}$ is not in $\mathcal{F}_{\text{har}}$, it is in $\mathcal{F}^{\text{df}}_{\text{har}}$ since it can be obtained by evaluating the harmonic function $e^x \cos(y)$ at $y = 0$. Thus, we have the following:

Corollary 8.7. *Let $\mathbb{R}_{\text{har,exp}}$ be the expansion of $\mathbb{R}$ by all restricted harmonic functions and the total exponential function and let $\mathcal{L}_{\text{har,exp}}$ be the corresponding language. If $K \equiv \mathbb{R}_{\text{har,exp}}$ then K admits an initial $\mathcal{L}_{\text{har,exp}}$-elementary embedding into* **No***.*

9. Trigonometric fields and surcomplex exponentiation

At the *mini-workshop on surreal numbers, surreal analysis, Hahn fields and derivations* held in Oberwolfach in 2016, the following question was raised: "Let $i = \sqrt{-1}$. Is there a good way to introduce sin and cos on **No** and an exponential map on **No**$[i]$?" [3, page 3315]. In this section we provide a positive answer to this question.

Let T_{trig} be the theory of the real field expanded by $\sin\restriction_{[0,2\pi]}$ and $\cos\restriction_{[0,2\pi]}$. We call a model

$$(K, \sin\restriction_{[0,2\pi]}, \cos\restriction_{[0,2\pi]}) \models T_{\text{trig}}$$

a **trigonometric ordered field**. Let K be such a field. Then K is real closed, so there is a discrete subring $Z \subseteq K$ such that for all $a \in K$ there is a $d \in Z$ with $d \leqslant a < d + 1$. Following tradition, we call Z an **integer part** of K. Using this integer part, we may extend sine and cosine to all of K by setting

$$\sin(a + 2\pi d) \ := \ \sin a, \qquad \cos(a + 2\pi d) \ := \ \cos a$$

where $a \in [0, 2\pi]$ and where $d \in Z$. Since K may have many integer parts, the extension of sin and cos to K is not necessarily unique (indeed, if $\sin_1$ and $\sin_2$ arise from different integer parts, then they have different zero classes). However, in the case that K is an initial trigonometric subfield of **No**, then K has a unique *initial* integer part, namely $\mathbf{Oz} \cap K$. Thus, we have the following

Proposition 9.1. *If K is an initial trigonometric ordered subfield of* $\mathbf{No}$ *then K admits canonical sine and cosine functions by taking $Z = \mathbf{Oz} \cap K$ in the above definition.*

9.1. **Trigonometric-exponential fields.** By $T_{\text{trig,exp}}$ we mean the theory of the real field expanded by $\sin\restriction_{[0,2\pi]}$, the total exponential function, and a constant symbol for each real number. We call a model

$$(K, \sin\restriction_{[0,2\pi]}, \exp) \models T_{\text{trig,exp}}$$

a **trigonometric-exponential field**. Let K be such a field. Then $\cos\restriction_{[0,2\pi]}$ is 0-definable in K, so K may be naturally viewed as an expansion of a trigonometric ordered field.

Since K is real closed, $K[i]$ is algebraically closed (where i is a square root of -1). Let

$$S \;:=\; \big\{a + bi : a \in K, b \in [0, 2\pi)_K\big\} \subseteq K[i].$$

Then S admits a natural group structure given by addition of the real parts and addition modulo 2π of the imaginary parts. More precisely:

$$(a + bi) + (c + di) \;=\; \begin{cases} (a + c) + (b + d)i & \text{if } b + d < 2\pi \\ (a + c) + (b + d - 2\pi)i & \text{if } b + d \geqslant 2\pi. \end{cases}$$

The class S, as well as its group structure is 0-definable in K, where we identify $K[i]$ with K^2 via the usual correspondence $a + bi \mapsto (a, b)$. The multiplication on $K[i]$ is also 0-definable in K. We define a map

$$E : S \to K[i]^\times, \qquad E(x + iy) \;=\; \exp x(\sin y + i \cos y).$$

Then E is also 0-definable in K, and so the $\mathcal{L}_{\text{trig,exp}}$-sentence "$E$ is a group isomorphism" is a consequence of $T_{\text{trig,exp}}$, since it is true in $\mathbb{R}$.

We now fix an integer part $Z \subseteq K$ and extend sin and cos to all of K, as is done above. We define a map:

$$a + ib \mapsto \exp a(\cos b + i \sin b) : K[i] \to K[i]^\times.$$

Note that this map extends the map $\exp : K \to K^\times$, so we denote this map by exp as well. Using the fact that E is a group isomorphism and that sine and cosine are periodic with period 2π, we have the following:

Proposition 9.2. *The map* $\exp : K[i] \to K[i]^\times$ *is a surjective group homomorphism with kernel $2\pi i Z$.*

Since the extension of exp depends on the extensions of sine and cosine, it depends on the choice of the integer part Z. However, we have the following corollary to Proposition 9.1:

Corollary 9.1. *If K is an initial trigonometric-exponential subfield of* $\mathbf{No}$*, then $K[i]$ admits a canonical exponential function by taking $Z = \mathbf{Oz} \cap K$ in the above definition.*

By Corollary 8.6 any trigonometric-exponential field K admits an initial embedding into $\mathbf{No}$. However, this initial embedding may not be unique, so there is no way to equip $K[i]$ with a canonical exponential function in general.

9.2. **Surcomplex exponentiation.** For each ordinal α, let

$$\mathbf{No}(\alpha) \;:=\; \big\{x \in \mathbf{No} : \rho_{\mathbf{No}}(x) < \alpha\big\}.$$

By Proposition 1.6, $\mathbf{No}$ is a trigonometric-exponential field. Moreover, by [9, Corollary 5.5], $\mathbf{No}(\alpha)$ is a trigonometric-exponential subfield of $\mathbf{No}$ whenever α is an *epsilon number* (that is, whenever $\omega^\alpha = \alpha$). Furthermore, $\mathbf{No}(\alpha)$ is initial for each α. Thus, in virtue of Corollary 9.1, we have the following:

Theorem 9.1. *The surcomplex numbers* $\mathbf{No}[i]$ *admits a canonical exponential function with kernel* $2\pi i\mathbf{Oz}$*. Moreover,* $\mathbf{No}(\alpha)[i]$ *admits a canonical exponential function with kernel* $2\pi i\big(\mathbf{Oz} \cap \mathbf{No}(\alpha)\big)$ *for each epsilon number α.*

10. Initial embeddings of some additional trigonometric-exponential fields

Berarducci and Mantova [4] introduced the exponential ordered field $\mathbb{R}\langle\langle\omega\rangle\rangle$ of *omega-series*. It is the smallest exponential subfield of $\mathbf{No}$ containing $\mathbb{R}$ and ω that is closed under exp, log and taking infinite sums. They further isolated the exponential subfields $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}((\omega))^{EL}$ of $\mathbf{No}$ that are isomorphic to the exponential ordered fields of *LE-series* [10, 11, 2] and *EL-series* [26, 27, 28], respectively. The system of *LE*-series in turn is isomorphic to the exponential ordered field $\mathbb{T}$ of *transseries*. In this section, we prove that $\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}((\omega))^{EL}$, which are models of $T(\mathbb{R}_{\text{an}}, e^x)$, and hence, models of $T_{\text{trig,exp}}$, are initial. The methods employed for the proofs are different from the methods used in §7 and §8, and only depend on material from the preliminary sections.

For a subclass $X \subseteq \mathbf{No}$, we let X^{rc} be the smallest real closed subfield of $\mathbf{No}$ containing X. We let X^{Σ} be the collection of all sums of all summable sequences of elements in X written in normal form. X is said to be **closed under sums** if $X = X^{\Sigma}$.

For the rest of this section, let K be an initial ordered subfield of $\mathbf{No}$.

Lemma 10.1. *K^{Σ} is an initial ordered subfield of $\mathbf{No}$. If K is also real closed, then so is K^{Σ}.*

Proof. Using Neumann's Lemma, e.g. [1, pages 260-261], we see that K^{Σ} is indeed an ordered field. Let Γ be the value group of K. By [15, Theorem 18], Γ is an initial subgroup of $\mathbf{No}$. Since K is cross sectional, we see that $K^{\Sigma} = \mathbb{R}_K((\omega^{\Gamma}))_{\mathbf{On}}$, so again by [15, Theorem 18], we see that K^{Σ} is an initial field of $\mathbf{No}$. If, in addition, K is real closed, then $\mathbb{R}_K$ is real closed and Γ is divisible. Thus, $K^{\Sigma} = \mathbb{R}_K((\omega^{\Gamma}))_{\mathbf{On}}$ is also real closed. $\square$

Lemma 10.2. *If K is real closed and X is a subset of $\mathbf{No}$ each of whose members is the simplest element of $\mathbf{No}$ that realizes a cut in K, then $(K \cup X)^{\text{rc}}$ is initial.*

Proof. This readily follows by iterating the result for the case where X is a singleton established by the first author in [18, pages 8, 37-38, Theorem 7]. $\square$

Lemma 10.3. *If K is an initial real closed subfield of $\mathbf{No}$ then $\big(K \cup \exp(K)\big)^{\text{rc}}$ is initial.*

Proof. We prove this by induction on the tree-rank of $x \in K$. If $\rho_{\mathbf{No}}(x) = 0$, then $x = 0$ and $K \cup \{\exp(0)\} = K$ is initial by assumption. Fix $x \in K$ with $\rho_{\mathbf{No}}(x) = \alpha > 0$ and suppose that

$$K_\alpha \;:=\; \big(K \cup \{\exp y : y \in K,\ \rho_{\mathbf{No}}(y) < \alpha\}\big)^{\text{rc}}$$

is initial. Then since $[x - x^L]_n$, $[x - x^R]_{2n+1}$, $\frac{1}{[x^R - x]_n}$, $\frac{1}{[x^L - x]_{2n+1}}$, $\exp x^L$, and $\exp x^R$ are all in K_α, we have that $\exp x$ is the simplest element in $\mathbf{No}$ realizing a cut in K_α. By Lemma 10.2,

$$K_{\alpha+1} \;=\; \big(K_\alpha \cup \{\exp x : x \in K,\ \rho_{\mathbf{No}}(x) = \alpha\}\big)^{\text{rc}}$$

is initial. Taking the union of the $K_{\alpha+1}$ over all $\alpha \in \{\rho_{\mathbf{No}}(x) : x \in K\}$, we see that $\big(K \cup \exp(K)\big)^{\text{rc}}$ is initial. $\square$

Lemma 10.4. *Suppose that K is an initial real closed subfield of $\mathbf{No}$, that K contains $\mathbb{R}$, and that K is closed under sums. Then $\big(K \cup \log(K^{>0})\big)^{\text{rc}}$ is initial.*

Proof. Let Γ be the value group of K. Then Γ is an initial divisible subgroup of $\mathbf{No}$. We first claim that $\log(K^{>0}) \subseteq \big(K \cup \log(\omega^{\Gamma})\big)^{\text{rc}}$. For $x \in K^{>0}$, we may write $x = r\omega^{\gamma}(1+\varepsilon)$ for some $r \in \mathbb{R}^{>0}$, some $\gamma \in \Gamma$, and some $\varepsilon \in K$ with $\varepsilon \prec 1$. We have $\ln(r) \in \mathbb{R} \subseteq K$ and, since K is closed under sums, we have

$$\log(\varepsilon) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\varepsilon^k}{k} \in K.$$

Thus, $\log(x) \in K + \log(\omega^{\Gamma}) \subseteq \big(K \cup \log(\omega^{\Gamma})\big)^{\text{rc}}$.

We will now show that $\big(K \cup \log(\omega^{\Gamma})\big)^{\text{rc}}$ is initial by induction on the simplicity of $\gamma \in \Gamma$. If $\rho_{\mathbf{No}}(\gamma) = 0$, then $\omega^{\gamma} = 1$ and so $K \cup \{\log(1)\} = K$ is initial by assumption. Fix $\gamma \in \Gamma$ with $\rho_{\mathbf{No}}(\gamma) = \alpha > 0$ and suppose that

$$K_\alpha \;:=\; \Big(K \cup \big\{\log(\omega^{\delta}) : \delta \in \Gamma,\ \rho_{\mathbf{No}}(\delta) < \alpha\big\}\Big)^{\text{rc}}$$

is initial. Using that K is cross sectional and that Γ is divisible, we see that ω^{γ^L}, ω^{γ^R}, $\omega^{\frac{\gamma^R-\gamma}{n}}$, and $\omega^{\frac{\gamma-\gamma^L}{n}}$ are all in $K \subseteq K_\alpha$. Since $\log(\omega^{\gamma^L})$, $\log(\omega^{\gamma^R})$ are also in K_α, we have $\log(\omega^\gamma)$ is the simplest element in $\mathbf{No}$ realizing a cut in K_α. By Lemma 10.2,

$$K_{\alpha+1} \;=\; \Big(K_\alpha \cup \big\{\log(\omega^\gamma) : \gamma \in \Gamma,\ \rho_{\mathbf{No}}(\gamma) = \alpha\big\}\Big)^{\mathrm{rc}}$$

is initial. By taking the union of the $K_{\alpha+1}$ over $\alpha \in \{\rho_{\mathbf{No}}(\gamma) : \gamma \in \Gamma\}$, we deduce that $\big(K \cup \log(\omega^\Gamma)\big)^{\mathrm{rc}}$ is initial. □

The following definitions are due to Berarducci and Mantova [4]:

Definition 10.1.

(i) $\mathbb{R}\langle\langle\omega\rangle\rangle$ is the smallest subfield of $\mathbf{No}$ containing $\mathbb{R}(\omega)$ and closed under exp, log, and sums.

(ii) $\mathbb{R}((\omega))^{LE}$ is the union $\bigcup_n X_n$ where $X_0 = \mathbb{R}(\omega)$ and

$$X_{n+1} = \big(X_n \cup \exp(X_n) \cup \log(X_n^{>0})\big)^{\Sigma}.$$

(iii) $\mathbb{R}((\omega))^{EL}$ is the union $\bigcup_n Y_n$ where $Y_0 = \mathbb{R}\big(\omega, \log(\omega), \log_2(\omega), \dots\big)$ and

$$Y_{n+1} = \big(Y_n \cup \exp(Y_n) \cup \log(Y_n^{>0})\big)^{\Sigma}.$$

Theorem 10.1. *The fields $\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$, and $\mathbb{R}((\omega))^{EL}$ are all initial.*

Proof. Lemmas 10.1-10.4 show that $\mathbb{R}\langle\langle\omega\rangle\rangle$ is initial. Since $\mathbb{R}((\omega))^{LE}$ is real closed, it is also equal to the union $\bigcup_n K_n$ where

$$K_0 = \big(\mathbb{R} \cup \{\omega\}\big)^{\mathrm{rc},\Sigma}, \quad K_{n+1} = \big(K_n \cup \exp(K_n) \cup \log(K_n^{>0})\big)^{\mathrm{rc},\Sigma}.$$

Lemmas 10.1-10.4 likewise show that this is an initial subfield of $\mathbf{No}$. As for $\mathbb{R}((\omega))^{EL}$, we first build a subfield L_0 by setting

$$L_{0,0} := \big(\mathbb{R} \cup \{\omega\}\big)^{\mathrm{rc},\Sigma}, \quad L_{0,m+1} := \big(L_m \cup \log(L_m^{>0})\big)^{\mathrm{rc},\Sigma}, \quad L_0 := \Big(\bigcup_m L_{0,m+1}\Big)^{\mathrm{rc},\Sigma}.$$

Note that $\mathbb{R}\big(\omega, \log(\omega), \log_2(\omega), \dots\big) \subseteq L_0 \subseteq \mathbb{R}((\omega))^{EL}$. Now, we repeat the same process above: that is, we set

$$L_{n+1} = \big(L_n \cup \exp(L_n) \cup \log(L_n^{>0})\big)^{\mathrm{rc},\Sigma}$$

and observe that $\mathbb{R}((\omega))^{EL} = \bigcup_n L_n$. □

Corollary 10.1. *$\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$, and $\mathbb{R}((\omega))^{EL}$ are all models of $T_{\mathrm{trig,exp}}$. Thus, by Corollary 9.1, these fields all admit a canonical exponential function on their algebraic closures.*

Proof. $\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$, and $\mathbb{R}((\omega))^{EL}$ are all closed under exponentiation by definition. Additionally, $\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$, and $\mathbb{R}((\omega))^{EL}$ are all increasing unions of Hahn fields (this is by definition for $\mathbb{R}((\omega))^{LE}$ and $\mathbb{R}((\omega))^{EL}$, and this is the case for $\mathbb{R}\langle\langle\omega\rangle\rangle$ by [4, Remark 4.24 and Corollary 4.28]). Since each restricted analytic function on $\mathbf{No}$ agrees with its Taylor series expansion, this gives that $\mathbb{R}\langle\langle\omega\rangle\rangle$, $\mathbb{R}((\omega))^{LE}$, and $\mathbb{R}((\omega))^{EL}$ are all closed under restricted analytic functions and so these three fields are elementary $\mathcal{L}_{\mathrm{an,exp}}$-substructures of $\mathbf{No}$ by Proposition 8.3. In particular, they are all models of $T_{\mathrm{trig,exp}}$. □

By [4, Theorem 4.11], $\mathbb{R}((\omega))^{LE}$ is the image of the canonical embedding $\imath : \mathbb{T} \to \mathbf{No}$ which sends x to ω (see [2] for an explicit definition of $\imath$). Thus, we have the following:

Corollary 10.2. *The image of the canonical embedding $\imath : \mathbb{T} \to \mathbf{No}$ is initial.*

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