LIOUVILLE CLOSED H_T-FIELDS.

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ABSTRACT. Let T be a complete, model complete o-minimal theory extending the theory of real closed ordered fields. An H_T-*field* is a model K of T equipped with a T-derivation ∂ such that the underlying ordered differential field of (K, ∂) is an H-field. We study H_T-fields and their extensions. Our main result is that if T is power bounded, then every H_T-field K has either exactly one or exactly two minimal Liouville closed H_T-field extensions up to K-isomorphism. The assumption of power boundedness can be relaxed to allow for certain exponential cases, such as $T = \mathrm{Th}(\mathbb{R}_{\mathrm{an,exp}})$.

CONTENTS

INTRODUCTION

In [1], Aschenbrenner and van den Dries introduced the class of H-fields. This class consists of Hardy fields containing $\mathbb{R}$ as well as various ordered differential fields of formal power series, most notably the field $\mathbb{T}$ of *logarithmic-exponential transseries*. In [3], Aschenbrenner, van den Dries, and van der Hoeven showed that the complete theory of $\mathbb{T}$ is the model companion of the theory of H-fields with small derivation.

In this article, we consider H-fields equipped with additional o-minimal structure. Let T be a complete, model complete o-minimal theory which extends the theory RCF of real closed ordered fields, and let K be a model of T. Let ∂ be a T-derivation on K as defined in [13], and let $C := \ker(\partial)$ be the constant field of K (we recall the definition of a T-*derivation* in Subsection 1.4 below). We say that (K, ∂) is an H_T-**field** if the following two conditions hold:

(H1) $\partial f > 0$ for all $f \in K$ with $f > C$, and
(H2) $\mathcal{O} = C + o$, where $\mathcal{O}$ is the convex hull of C in K and o is the unique maximal ideal of $\mathcal{O}$.

If, moreover, $\partial o \subseteq o$, then (K, ∂) is said to have *small derivation*. Axioms (H1) and (H2) are the axioms for H-fields, given in [1], so an H_T-field is just a model of T equipped with a T-derivation making it an H-field. In fact, the class of H_{RCF}-fields coincides with the class of real closed H-fields.

Our long-term goal is to show that the theory of H_T-fields has a model companion (under the assumption that T is itself model complete). To this end, we examine various extensions of H_T-fields, often under the assumption that T is *power bounded*. Our investigation is based on the study of H-fields and their extensions, conducted in [3]. The main results in this article are on the existence and uniqueness of T-*Liouville closures*. Before giving an overview of the results, let us discuss two motivating examples: $\mathcal{R}$-Hardy fields and expansions of the field $\mathbb{T}$.

$\mathcal{R}$-Hardy fields. Perhaps the most natural examples of H_T-fields are $\mathcal{R}$-Hardy fields, introduced in [9]. Let $\mathcal{R}$ be an o-minimal expansion of the real field $\mathbb{R}$ in an appropriate language $\mathcal{L}_{\mathcal{R}}$ and let $T_{\mathcal{R}}$ be the complete $\mathcal{L}_{\mathcal{R}}$-theory of $\mathcal{R}$. By adding a function symbol to $\mathcal{L}_{\mathcal{R}}$ for each definable function, we may arrange that $T_{\mathcal{R}}$ has quantifier elimination and a universal axiomatization. For convenience, we will also assume that $\mathcal{L}_{\mathcal{R}}$ contains a constant symbol for each $r \in \mathbb{R}$, so any model of $T_{\mathcal{R}}$ contains $\mathcal{R}$ as an elementary substructure.

Recall that a **Hardy field** is an ordered differential field of germs at $+\infty$ of unary real-valued functions, where the germ of a function $f \colon \mathbb{R} \to \mathbb{R}$ at $+\infty$ is the equivalence class

$$[f] := \big\{ g : f|_{(a,+\infty)} = g|_{(a,+\infty)} \text{ for some } a \in \mathbb{R} \big\}.$$

An **$\mathcal{R}$-Hardy field** is a Hardy field $\mathcal{H}$ which is closed under all function symbols in $\mathcal{L}_{\mathcal{R}}$. That is, $\mathcal{H}$ is an $\mathcal{R}$-Hardy field if for every n-ary function symbol F in $\mathcal{L}_{\mathcal{R}}$ and all germs $[f_1], \ldots, [f_n] \in \mathcal{H}$, the germ $\big[F(f_1, \ldots, f_n)\big]$ is in $\mathcal{H}$, where $F(f_1, \ldots, f_n)$ is the composite function $x \mapsto F\big(f_1(x), \ldots, f_n(x)\big)$. We view constant symbols as nullary functions, so we may identify $\mathbb{R}$ with a subfield of $\mathcal{H}$ by identifying $r \in \mathbb{R}$ with the germ of the constant function $x \mapsto r$.

If $\mathcal{H}$ is an $\mathcal{R}$-Hardy field, then we view $\mathcal{H}$ as an $\mathcal{L}_{\mathcal{R}}$-structure as follows:

- If F is an n-ary function symbol in $\mathcal{L}_{\mathcal{R}}$ and $[f_1], \ldots, [f_n] \in \mathcal{H}$, then

$$F\big([f_1], \ldots, [f_n]\big) := \big[F(f_1, \ldots, f_n)\big].$$

- If R is an n-ary predicate in $\mathcal{L}_{\mathcal{R}}$ and $[f_1], \ldots, [f_n] \in \mathcal{H}$, then

$$\mathcal{H} \models R\big([f_1], \ldots, [f_n]\big) :\Longleftrightarrow \mathcal{R} \models R\big(f_1(x), \ldots, f_n(x)\big) \text{ for all sufficiently large } x.$$

By [9, Lemma 5.8], $\mathcal{R}$ is an elementary $\mathcal{L}_{\mathcal{R}}$-substructure of $\mathcal{H}$. As a consequence of our assumption on $T_{\mathcal{R}}$, each $\mathcal{L}_{\mathcal{R}}(\emptyset)$-definable function F is given piecewise by terms, so the identity

$$F\big([f_1], \ldots, [f_n]\big) := \big[F(f_1, \ldots, f_n)\big]$$

holds for arbitrary $\mathcal{L}_{\mathcal{R}}(\emptyset)$-definable functions F, not just for function symbols in $\mathcal{L}_{\mathcal{R}}$.

Let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field. As a Hardy field, $\mathcal{H}$ admits a derivation $\partial \colon \mathcal{H} \to \mathcal{H}$ given by $\partial[f] := [f']$. Using the chain rule from elementary calculus, it is easy to see that this is even a $T_{\mathcal{R}}$-derivation. We claim that with this derivation, $\mathcal{H}$ is an $H_{T_{\mathcal{R}}}$-field. Note that the constant field $C = \ker(\partial)$ of $\mathcal{H}$ is equal to $\mathbb{R}$, so (H2) follows from Dedekind completeness of the reals. For $[f] \in \mathcal{H}$, we have

$$[f] > \mathbb{R} \implies \lim_{x \to \infty} f(x) = \infty \implies [f'] = \partial[f] > 0,$$

so (H1) holds as well.

Transseries. Let $\mathbb{R}_{\mathrm{an}}$ be the expansion of the real field by restricted analytic functions, and let $\mathbb{R}_{\mathrm{an,exp}}$ be the further expansion of $\mathbb{R}_{\mathrm{an}}$ by the unrestricted exponential function. Let $\mathbb{R}_{\mathrm{re}}$ be the expansion of the real field by only the restricted sine, cosine, and exponential functions (collectively, *restricted elementary functions*). Let T_{re}, T_{an}, and $T_{\mathrm{an,exp}}$ be the elementary theories of $\mathbb{R}_{\mathrm{re}}$, $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{an,exp}}$ respectively. By [10, Corollary 2.8], the field $\mathbb{T}$ admits a canonical expansion to a model of $T_{\mathrm{an,exp}}$, which we denote by $\mathbb{T}_{\mathrm{an,exp}}$. Let $\mathbb{T}_{\mathrm{an}}$ and $\mathbb{T}_{\mathrm{re}}$ denote the corresponding reducts of $\mathbb{T}_{\mathrm{an,exp}}$.

We claim that $\mathbb{T}_{\mathrm{an,exp}}$, with its natural derivation, is an $H_{\mathrm{an,exp}}$-field, where we write "$H_{\mathrm{an,exp}}$-field" instead of "$H_{T_{\mathrm{an,exp}}}$-field" for easier reading. It is well-known that the ordered differential field $\mathbb{T}$ is an H-field; the

axiom (H1) is verified in [11, Proposition 4.3] and the axiom (H2) follows easily since the constant field of $\mathbb{T}$ is isomorphic to $\mathbb{R}$, which is Dedekind complete. Moreover, the derivation on $\mathbb{T}_{\mathrm{an,exp}}$ is a $T_{\mathrm{an,exp}}$-derivation, since it is compatible with all restricted analytic functions and the exponential function by [11, Corollaries 3.3 and 3.4]; see [13, Lemma 2.9] for why this is sufficient. It follows that $\mathbb{T}_{\mathrm{an}}$ is an H_{an}-field, and that $\mathbb{T}_{\mathrm{re}}$ is an H_{re}-field.

We conjecture that $\mathbb{T}_{\mathrm{an,exp}}$ and $\mathbb{T}_{\mathrm{an}}$ are both model complete. We also conjecture that the theory of $\mathbb{T}_{\mathrm{an,exp}}$ is the model companion of the theory of $H_{\mathrm{an,exp}}$-fields with small derivation, and that the theory of $\mathbb{T}_{\mathrm{an}}$ is the model companion of the theory of H_{an}-fields with small derivation. These conjectures are, of course, inspired by the corresponding results for $\mathbb{T}$ from [3]. In [16, Chapter 8], we show that $\mathbb{T}_{\mathrm{re}}$ is model complete, and that its theory is the model companion of the theory of H_{re}-fields with small derivation. The proof of this result relies heavily on machinery from [3], as well as basic facts about restricted elementary functions. The methods used to investigate $\mathbb{T}_{\mathrm{re}}$ are almost surely too case-specific to handle the richer expansions $\mathbb{T}_{\mathrm{an}}$ and $\mathbb{T}_{\mathrm{an,exp}}$.

Outline and overview of results. In this article, we fix a complete, model complete o-minimal theory T which extends the theory RCF of real closed ordered fields in some appropriate language $\mathcal{L} \supseteq \{0, 1, +, -, \cdot, <\}$. The class of H_T-fields is axiomatized in the language $\mathcal{L}^{\mathcal{O},\partial} := \mathcal{L} \cup \{\mathcal{O}, \partial\}$, where $\mathcal{O}$ is a unary predicate interpreted as in (H2). By [13, Lemma 2.3], the constant field C of an H_T-field K is an underlying elementary $\mathcal{L}$-substructure of K. It follows that the valuation ring $\mathcal{O}$ in (H2) is T-convex, in the sense of van den Dries and Lewenberg [8]. In studying H_T-fields, valuation theory plays a key role. Analogs of many classical results about valued fields have been proven in the o-minimal setting under the assumption of power boundedness (a generalization of polynomial boundedness introduced in [18]). Accordingly, we will assume at various times that T is *power bounded*. Valuation-theoretic background is given in Section 2, along with some new technical results on T-convex valuation rings in the power bounded setting.

Expansions of models of T by a T-convex valuation ring and a continuous T-derivation were first studied in [17]. There, the focus was on immediate extensions. Our first result on H_T-fields is a corollary of the main result in [17].

Corollary 4.7. *Suppose that T is power bounded. Then every H_T-field has a spherically complete immediate H_T-field extension.*

In addition to studying H_T-fields, we consider the broader class of *pre-H_T-fields*, which arise as substructures of H_T-fields. Most of the study of extensions of H_T-fields and pre-H_T-fields is carried out in Section 4. Here is one such extension result from that section, which shows that every pre-H_T-field has an "H_T-field hull."

Theorem 4.14. *Suppose that T is power bounded and let K be a pre-H_T-field. Then K has an H_T-field extension $H_T(K)$ such that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$.*

The extension theory developed in Section 4 is applied in Section 5 to study *T-Liouville closures*. An H_T-field K is said to be *Liouville closed* if every element in K has an integral and an exponential integral in K, that is, if for all $y \in K$, there is $f \in K$ and $g \in K^\times$ with $\partial f = \partial g/g = y$. A *$T$-Liouville closure* of an H_T-field K is a Liouville closed H_T-field extension of K which is built from K by adjoining integrals and exponential integrals (a precise definition is given in Section 5).

In [1], Aschenbrenner and van den Dries proved that every H-field has at least one and at most two Liouville closures up to isomorphism. They proved that *grounded H-fields* have exactly one Liouville closure and that certain types of ungrounded H-fields have exactly two. Gehret later showed that the precise dividing line for ungrounded H-fields is the property of λ-*freeness* [15]. The terms "grounded" and "λ-free" are defined in Section 3. In Section 5, we show that when T is power bounded, the number of T-Liouville closures of an H_T-field can likewise be characterized in terms of being grounded or λ-free:

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Theorem 5.16. *Suppose that T is power bounded and let K be an H_T-field. If K is grounded or if K is ungrounded and λ-free, then K has exactly one T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. If K is ungrounded and not λ-free, then K has exactly two T-Liouville closures up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. For any Liouville closed H_T-field extension M of K, there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of some T-Liouville closure of K into M.*

The assumption of power boundedness excludes some important o-minimal theories, such as $T_{\mathrm{an,exp}}$. Fortunately, many of our results can be extended to o-minimal theories which are "controlled" by a power bounded base theory in the same way that $T_{\mathrm{an,exp}}$ is "controlled" by its reduct T_{an}, as shown in [9]. In formalizing precisely what we mean, we use the framework from Foster's thesis [14]. Suppose that T is power bounded, that T defines a restricted exponential function, and that the field of exponents of T is cofinal in the prime model of T. Foster axiomatizes an extension T^{e} of T in the language $\mathcal{L}_{\log} \coloneqq \mathcal{L} \cup \{\log\}$ whose models are expansions of models of T by an unrestricted exponential function (which is compatible with the restricted exponential function and the power functions of T). This extension is also complete, model complete, and o-minimal. We give an overview of Foster's results, as well as some historical context, in Subsection 1.3.

In Section 6 we show that many of the extension results proven in Section 4 under the assumption of power boundedness also go through for $H_{T^{\mathrm{e}}}$-fields and pre-$H_{T^{\mathrm{e}}}$-fields. Instead of working with (pre)-$H_{T^{\mathrm{e}}}$-fields directly, we work with a broader class of $\mathcal{L}_{\log}^{\mathcal{O},\partial}$-structures, called *logarithmic (pre)-H_T-fields*, where the logarithm is not assumed to be surjective. In Section 7, we show that Theorem 5.16 generalizes to the class of logarithmic H_T-fields:

Theorem 7.11. *Let K be a logarithmic H_T-field. If K is λ-free, then K has exactly one logarithmic T-Liouville closure up to $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphism. Otherwise, K has exactly two logarithmic T-Liouville closures up to $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphism. For any Liouville closed logarithmic H_T-field extension M of K, there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding of some logarithmic T-Liouville closure of K into M.*

In the statement of Theorem 7.11, a *logarithmic T-Liouville closure* of K is just a logarithmic H_T-field extension of K which is also a T-Liouville closure of K. The "grounded" case in Theorem 5.16 cannot occur for logarithmic H_T-fields; see Corollary 6.3.

In [1], Aschenbrenner and van den Dries showed that any H-field embedding of a Hardy field $\mathcal{H}$ into $\mathbb{T}$ extends to the smallest Liouville closed Hardy field extension of $\mathcal{H}$. As an application of Theorem 7.11, we prove an analog of this embedding theorem for $\mathbb{R}_{\mathrm{an,exp}}$-Hardy fields.

Theorem 7.12. *Let $\mathcal{H}$ be an $\mathbb{R}_{\mathrm{an,exp}}$-Hardy field. Then any $H_{\mathrm{an,exp}}$-field embedding $\mathcal{H} \to \mathbb{T}_{\mathrm{an,exp}}$ extends to an $H_{\mathrm{an,exp}}$-field embedding $\mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H}) \to \mathbb{T}_{\mathrm{an,exp}}$, where $\mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H})$ is the minimal Liouville closed $\mathbb{R}_{\mathrm{an,exp}}$-Hardy field extension of $\mathcal{H}$.*

Our final result is that every pre-H_T-field has a pre-H_T-field extension which satisfies the "order 1 intermediate value property." An analog of this theorem was shown for pre-H-fields in [2] and for $\mathcal{R}$-Hardy fields in [7]. Unlike many of the other results, this requires no power boundedness assumptions on T.

Theorem 8.1. *Let K be a pre-H_T-field. Then K has a pre-H_T-field extension M with the following property: for any $\mathcal{L}(M)$-definable continuous function $F\colon M \to M$ and any $b_1, b_2 \in M$ with*

$$b_1' \ < \ F(b_1), \qquad b_2' \ > \ F(b_2),$$

there is $a \in M$ between b_1 and b_2 with $a' = F(a)$.

In Remark 8.4, we discuss whether the pre-H_T-field extension M above can be taken to be an H_T-field.

1. Preliminaries

1.1. **Notation and conventions.** In this article, k, m, and n always denote elements of $\mathbb{N} = \{0, 1, 2, \ldots\}$. By "ordered set" we mean "totally ordered set." Let S be an ordered set, let $a \in S$, and let $A \subseteq S$. We put

$$S^{>a} \;\coloneqq\; \{s \in S : s > a\};$$

similarly for $S^{\geq a}$, $S^{<a}$, $S^{\leq a}$, and $S^{\neq a}$. We write "$a > A$" (respectively "$a < A$") if a is greater (less) than each $s \in A$, and we let $A^{\downarrow}$ denote the **downward closure of** A. A **cut** in S is a downward closed subset of S, and if A is a cut in S, then an element y in an ordered set extending S is said to **realize the cut** A if $A < y < S \setminus A$. Note that the empty set is a cut which is realized by any element $y < S$. Likewise, S itself is a cut realized by any element $y > S$. If Γ is an ordered abelian group, then we set $\Gamma^{>} \coloneqq \Gamma^{>0}$, and we define $\Gamma^{\geq}$, $\Gamma^{<}$, $\Gamma^{\leq}$, and $\Gamma^{\neq}$ analogously. If R is a ring, then $R^{\times}$ denotes the multiplicative group of units in R. A **well-indexed sequence** is a sequence (a_{ρ}) whose terms are indexed by ordinals ρ less than some infinite limit ordinal ν.

We always use K, L, and M for models of T (or expansions thereof). Let $A \subseteq K$ and let $D \subseteq K^n$. We say that D is $\mathcal{L}(A)$**-definable** if

$$D \;=\; \varphi(K) \;\coloneqq\; \big\{ y \in K^n : K \models \varphi(y) \big\}$$

for some $\mathcal{L}(A)$-formula φ. A function $F \colon D \to K$ is said to be $\mathcal{L}(A)$-definable if its graph is a definable subset of K^{n+1}. Note that the domain of an $\mathcal{L}(A)$-definable function is $\mathcal{L}(A)$-definable.

For $A \subseteq K$, let $\mathrm{dcl}_{\mathcal{L}}(A)$ denote the $\mathcal{L}$-definable closure of A (in K, implicitly, but this doesn't change if we pass to elementary extensions of K). If $b \in \mathrm{dcl}_{\mathcal{L}}(A)$, then $b = F(a)$ for some $\mathcal{L}(\emptyset)$-definable function F and some tuple a from A. It is well-known that $(K, \mathrm{dcl}_{\mathcal{L}})$ is a pregeometry. A set $B \subseteq K$ is said to be $\mathcal{L}(A)$**-independent** if $b \notin \mathrm{dcl}_{\mathcal{L}}\big(A \cup (B \setminus \{b\})\big)$ for all $b \in B$. A tuple $a = (a_i)_{i \in I}$ is said to be $\mathcal{L}(A)$-independent if its set of components $\{a_i : i \in I\}$ is $\mathcal{L}(A)$-independent and no components are repeated. Since T has definable Skolem functions, any definably closed subset $A \subseteq K$ is an elementary $\mathcal{L}$-substructure of K. Together with our assumption that T is complete, this guarantees that T has a prime model, which can be uniquely identified with $\mathrm{dcl}_{\mathcal{L}}(\emptyset)$ in any model of T.

Let M be a T-extension of K, that is, a model of T which contains K as an $\mathcal{L}$-substructure. Given an $\mathcal{L}(K)$-definable set $D \subseteq K^n$, we let D^M denote the subset of M^n defined by the same $\mathcal{L}(K)$-formula as D. We sometimes refer to D^M as the **natural extension of** D **to** M. Since T is assumed to be model complete, this natural extension does not depend on the choice of defining formula. If $F \colon D \to K$ is an $\mathcal{L}(K)$-definable function, then let F^M be the $\mathcal{L}(K)$-definable function whose graph is the natural extension of the graph of F. Then the domain of F^M is D^M, and we often drop the superscript and just write $F \colon D^M \to M^m$.

For $B \subseteq M$, let $K\langle B \rangle$ denote the $\mathcal{L}$-substructure of M with underlying set $\mathrm{dcl}_{\mathcal{L}}(K \cup B)$. If $B = \{b_1, \ldots, b_n\}$, we write $K\langle b_1, \ldots, b_n \rangle$ instead of $K\langle B \rangle$. Note that $K\langle B \rangle$ is an elementary $\mathcal{L}$-substructure of M. If B is $\mathcal{L}(K)$-independent and $M = K\langle B \rangle$, then B is called a **basis for** M **over** K. The **rank of** M **over** K, denoted $\mathrm{rk}_{\mathcal{L}}(M|K)$, is the cardinality of a basis for M over K (this doesn't depend on the choice of basis). We say that M is a **simple extension** of K if $\mathrm{rk}_{\mathcal{L}}(M|K) = 1$. If M is a simple extension of K, then $M = K\langle b \rangle$ for some $b \in M \setminus K$.

Let $\mathcal{L}^* \supseteq \mathcal{L}$, let T^* be an $\mathcal{L}^*$-theory extending T, and let $K \models T^*$. We use the same conventions for $\mathcal{L}^*$-definability as we do for $\mathcal{L}$-definability. A T^***-extension of** K is a model $M \models T^*$ which contains K as an $\mathcal{L}^*$-substructure. If M is an *elementary* T^*-extension of K and $D \subseteq K^n$ is $\mathcal{L}^*(K)$-definable, then let D^M denote the subset of M^n defined by the same formula as D.

1.2. Power functions and power boundedness.

Here we mention some basic facts about power functions and exponentials. Proofs can be found in [18] or [14, Section 2.2.4]. A **power function on** K is an $\mathcal{L}(K)$-definable endomorphism of the multiplicative group $K^{>}$. Each power function F is $\mathcal{C}^1$ on $K^{>}$ and uniquely determined by $F'(1)$. Set

$$\Lambda := \big\{F'(1) : F \text{ is a power function on } K\big\}.$$

Then Λ is a subfield of K, and it is called the **field of exponents of** K. For $a \in K^{>}$ and a power function F, we suggestively write $F(a)$ as a^λ, where $\lambda = F'(1)$. A straightforward computation tells us that the derivative of the power function $x \mapsto x^\lambda$ at a is $\lambda a^{\lambda-1}$.

We say that K is **power bounded** if for each $\mathcal{L}(K)$-definable function $F \colon K^{>} \to K^{>}$, there is λ in the field of exponents of K with $|F(x)| < x^\lambda$ for all sufficiently large positive x. Note that K is **polynomially bounded** (any unary $\mathcal{L}(K)$-definable function is eventually bounded by x^n for some n) if and only if K is power bounded with archimedean field of exponents.

An **exponential function on** K is an ordered group isomorphism from the additive group K to the multiplicative group $K^{>}$. Any exponential function on K grows more quickly than every power function on K. By [18], either K is power bounded or K defines an exponential function. Any definable exponential function on K is everywhere differentiable, and if K defines an exponential function, then it is fairly easy to see that there is a unique $\mathcal{L}(\emptyset)$-definable exponential function which is equal to its own derivative. Thus, defining an exponential function is a property of the theory T (we say that T **defines an exponential**), and so power boundedness is a property of the theory T as well (we say that T **is power bounded**). If T is power bounded, then each power function on K is $\mathcal{L}(\emptyset)$-definable, so we refer to the field of exponents Λ as the **field of exponents of** T, as Λ does not depend on K.

1.3. Exponentials and logarithms.

Most of the results in Sections 3, 4, and 5 are proved under the assumption that T is power bounded. In Sections 6 and 7, we show that many of these results can be extended to theories which define an exponential, so long as these theories are essentially controlled by a power bounded reduct. The prototypical example of this is the theory $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$. Let T_{rexp} be the (polynomially bounded) theory of the real field expanded by the restriction of the exponential function to the interval $[-1, 1]$. Then T_{rexp} is model complete by Wilkie [23]. In [19], Ressayre demonstrated that $T_{\exp}$ can be axiomatized by extending T_{rexp} by the following axioms:

(E1) exp is an exponential function, as defined above,
(E2) exp agrees with its restricted counterpart on the interval $[-1, 1]$, and
(E3) exp grows sufficiently fast, that is, if $x \geqslant n^2$ then $\exp x \geqslant x^n$ for each $n > 0$.

Ressayre also showed that if T_{rexp} eliminates quantifiers in some language $\mathcal{L}^*$, then $T_{\exp}$ eliminates quantifiers in the language $\mathcal{L}^* \cup \{\exp, \log\}$.

Ressayre's method was expounded on in [9], where the authors took T_{an} as the polynomially bounded base theory and showed that $T_{\mathrm{an,exp}}$ can be axiomatized by extending T_{an} by the axioms (E1), (E2), and (E3) above. They used this to show that $T_{\mathrm{an,exp}}$ has quantifier elimination and a universal axiomatization in the language $\mathcal{L}_{\mathrm{an}} \cup \{\exp, \log\}$, where $\mathcal{L}_{\mathrm{an}}$ extends the language of ordered rings by function symbols for all restricted analytic functions, inversion away from zero, and n^{th} roots for all $n > 1$.

In [12], van den Dries and Speissegger further generalized this approach. Let $\mathcal{R}$ be a polynomially bounded expansion of the real field which defines the restriction of the exponential function to $[-1, 1]$ and let $\mathcal{L}_{\mathcal{R}}$ be a language in which $T_{\mathcal{R}}$, the theory of $\mathcal{R}$, has quantifier elimination and a universal axiomatization. Then the expansion of $\mathcal{R}$ by the unrestricted exponential function is o-minimal and completely axiomatized by the theory $T_{\mathcal{R}}^{\mathrm{e}}$, which expands $T_{\mathcal{R}}$ by axioms (E1), (E2), and (E3) above along with an axiom scheme stating that $\exp(rx) = (\exp x)^r$ whenever the power function $x \mapsto x^r$ is definable in $\mathcal{R}$ [12, Theorem B]. As with $T_{\mathrm{an,exp}}$, the theory $T_{\mathcal{R}}^{\mathrm{e}}$ has quantifier elimination and a universal axiomatization in the language $\mathcal{L}_{\mathcal{R}} \cup \{\exp, \log\}$.

Note that the axiom expressing compatibility with the power function $x \mapsto x^r$ is really only necessary when r is irrational; if $r \in \mathbb{Q}$, then compatibility follows from (E1).

The most general application of this method is in Foster's thesis [14], where $T_{\mathcal{R}}$ above is replaced by any complete power bounded o-minimal theory, subject to some natural constraints. We will use Foster's framework in Sections 6 and 7, and we spend the remainder of this subsection describing his hypotheses and results.

Definition 1.1. A **restricted exponential on** K is a strictly increasing map $e \colon K \to K$ which is zero outside of $[-1, 1]$, which satisfies the identity

$$e(x + y) \;=\; e(x)\,e(y)$$

for $x, y \in [-1, 1]$ with $|x + y| \leqslant 1$, and which is differentiable at 0 with derivative 1.

As is the case with total exponential functions, any $\mathcal{L}(K)$-definable restricted exponential on K is actually $\mathcal{L}(\emptyset)$-definable, continuous on the interval $[-1, 1]$, differentiable on the interval $(-1, 1)$, and equal to its own derivative on that interval. Thus, we say that T **defines a restricted exponential** to mean that any model of T admits an $\mathcal{L}(\emptyset)$-definable restricted exponential. For the remainder of this subsection, we assume that T defines a restricted exponential e. We also assume T is power bounded and that Λ, the field of exponents of T, is cofinal in the prime model of T.

Lemma 1.2. *Let* $y \in [-1, 1]$. *Then* $y + 1 \leqslant e(y)$, *with equality if and only if* $y = 0$.

Proof. Since $e(0) = 1$, we may assume that y is nonzero. The o-minimal mean value theorem gives some u between y and 0 with

$$e(y) - 1 \;=\; e(y) - e(0) \;=\; e(u)y.$$

Treating the cases $y > 0$ and $y < 0$ separately, we see that $e(u)y > y$, so $e(y) > y + 1$. $\qquad\square$

For $y \in K$ with $e(-1) \leqslant y \leqslant e(1)$, let $\ln(y)$ be the unique element of $[-1, 1]$ with $e(\ln y) = y$. Then $\ln$ is $\mathcal{L}(\emptyset)$-definable and continuous where defined. A straightforward computation gives that $\ln$ is differentiable at y strictly between $e(-1)$ and $e(1)$ with derivative y^{-1}.

Definition 1.3. A **logarithm on** K is a function $\log \colon K^{>} \to K$ which satisfies the following axioms for all $a \in K^{>}$ and all $\lambda \in \Lambda$:

    (L1) $\log$ is an ordered group embedding of the multiplicative group $K^{>}$ into the additive group K;
    (L2) if $e(-1) \leqslant a \leqslant e(1)$, then $\log a = \ln a$;
    (L3) if $\lambda > 1$ and $a \geqslant \lambda^2$, then $a \geqslant \lambda \log a$;
    (L4) $\log(a^{\lambda}) = \lambda \log a$.

Let $\log$ be a logarithm on K. We set $\mathcal{L}_{\log} \coloneqq \mathcal{L} \cup \{\log\}$, and we view K as an $\mathcal{L}_{\log}$-structure, where $\log$ is interpreted to be identically zero on $K^{\leqslant}$. Let $\exp$ denote the compositional inverse of $\log$, where it is defined. Combining (L1) and (L2), we see that $\log$ is differentiable at any $y > 0$ with derivative y^{-1}. Thus, $\exp$ is also differentiable where defined, and equal to its own derivative.

The axioms (L1)–(L4) are analogs of Foster's axioms (A1)–(A5) [14, Section 6.5], though they are presented in terms of logarithms rather than exponentials. Axiom (L1) is an analog of Ressayre's axiom (E1), axiom (L2) corresponds to Ressayre's axiom (E2), and axiom (L3) is a version of (E3) which works for arbitrary (possibly non-archimedean) fields of exponents. Axiom (L4) is an analog of van den Dries and Speissegger's additional axiom.

We prove here a couple of lemmas for later use.

Lemma 1.4. *Let* $\log$ *be a logarithm on* K, *let* $f, g \in K$ *with* $|f - g| \leqslant 1$, *and suppose that* $g \in \log(K^{>})$. *Then* $f \in \log(K^{>})$.

Proof. Let $h \coloneqq \exp(g)\,e(f - g) \in K^{>}$. Then $f = \log h$. $\qquad\square$

The inequality in axiom (L3) is not strict (a useful formulation for verifying that it holds in certain situations). However, the strict version of (L3) follows:

Lemma 1.5. *Let* log *be a logarithm on K, let $\lambda \in \Lambda$ with $\lambda > 1$, and let $a \in K$ with $a > \lambda^2$. Then $a > \lambda \log a$.*

Proof. Let $\rho \in \Lambda$ with $1 < \rho^2 \leqslant \mathrm{e}(1)$. For example, we may take $\rho = 4/3$, since $\mathrm{e}(1) > 2$ by Lemma 1.2. If $a \geqslant \rho^2 \lambda^2$, then $a \geqslant \rho \lambda \log a > \lambda \log a$ by (L3), so we may assume that $\lambda^2 < a < \rho^2 \lambda^2$. Take $u \in K$ with $\lambda^2 u = a$, so $1 < u < \mathrm{e}(1)$. We have

$$\lambda \log a \;=\; \lambda \log \lambda^2 + \lambda \ln u \;\leqslant\; \lambda^2 + \lambda^2 \ln u,$$

where the equality uses (L1) and (L2) and the inequality uses (L3) and our assumption that $\lambda > 1$. Lemma 1.2 gives $u - 1 = \mathrm{e}(\ln u) - 1 > \ln u$, so $\lambda \log a < \lambda^2 + \lambda^2(u-1) = \lambda^2 u = a$. $\qquad\square$

Let T^{e} be the $\mathcal{L}_{\log}$-theory which extends the axioms (L1)–(L4) by an additional axiom which states that log is surjective. Foster's main results on the theory T^{e} are as follows:

Fact 1.6 ([14], Theorems 6.5.2 and 6.6.9)**.**

- *(1) T^{e} is complete, model complete, and o-minimal.*
- *(2) The prime model of T admits a unique expansion to a model of T^{e}.*
- *(3) If T has quantifier elimination and a universal axiomatization, then so does T^{e} in the language $\mathcal{L}_{\log} \cup \{\exp\}$, where $\exp$ is interpreted as the compositional inverse of log.*

Corollary 1.7. *Any $\mathcal{L}_{\log}(\emptyset)$-definable function in any model of T^{e} is given piecewise by a composition of $\mathcal{L}(\emptyset)$-definable functions,* log, *and* exp.

Proof. Let $\mathcal{L}^*$ be the extension of $\mathcal{L}$ by function symbols for each $\mathcal{L}(\emptyset)$-definable function. Then T has quantifier elimination and a universal axiomatization in the language $\mathcal{L}^*$, since T has definable Skolem functions. Thus, T^{e} has quantifier elimination and a universal axiomatization in the language $\mathcal{L}^*_{\log} \cup \{\exp\}$ by Fact 1.6 (3) above. It follows that any $\mathcal{L}^*_{\log}(\emptyset)$-definable function in any model of T^{e} is given piecewise by terms in the language $\mathcal{L}^*_{\log} \cup \{\exp\}$. These terms are exactly compositions of $\mathcal{L}(\emptyset)$-definable functions, log, and exp. $\qquad\square$

1.4. *T-derivations.* Let $\partial \colon K \to K$ be a map. For $a \in K$, we use a' or ∂a in place of $\partial(a)$, and we use $a^{(r)}$ in place of $\partial^r(a)$. If $a \neq 0$, then we set $a^{\dagger} := a'/a$. Given a set $A \subseteq K$, we set $\partial A := \{a' : a \in A\}$. Given a tuple $b = (b_1, \ldots, b_n) \in K^n$, we use ∂b or b' to denote the tuple $(b'_1, \ldots, b'_n)$.

Let $F \colon U \to K$ be an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with $U \subseteq K^n$ open. Let ∇F denote the gradient

$$\nabla F \;:=\; \left(\frac{\partial F}{\partial Y_1}, \ldots, \frac{\partial F}{\partial Y_n} \right),$$

viewed as an $\mathcal{L}(K)$-definable map from U to K^n. If $n = 1$, then we write F' instead of ∇F. We say that ∂ is **compatible with** F if

$$F(u)' \;=\; \nabla F(u) \cdot u'$$

for each $u \in U$, where $a \cdot b$ denotes the dot product $\sum_{i=1}^n a_i b_i$ for $a, b \in K^n$. We say that ∂ is a **T-derivation on** K if ∂ is compatible with every $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with open domain.

T-derivations were introduced in [13]. Compatibility with the functions $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ gives that any T-derivation on K is a *derivation* on K, that is, a map satisfying the identities $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for $a, b \in K$. For the remainder of this subsection, we assume that ∂ is a T-derivation on K, and we let $C := \ker(\partial)$ denote the constant field of K. By [13, Lemma 2.3], the constant field C is the underlying set of an elementary $\mathcal{L}$-substructure of K. We recall two facts from [13] for later use:

Fact 1.8 ([13], Lemma 2.12)**.** *Let $U \subseteq K^n$ be open and let $F\colon U \to K$ be an $\mathcal{L}(K)$-definable $\mathcal{C}^1$-function. Then there is a (necessarily unique) $\mathcal{L}(K)$-definable function $F^{[\partial]}\colon U \to K$ such that*

$$F(u)' \;=\; F^{[\partial]}(u) + \nabla F(u) \cdot u'$$

for all $u \in U$.

Fact 1.9 ([13], Lemma 2.13)**.** *Let M be a T-extension of K, let A be a $\mathrm{dcl}_{\mathcal{L}}$-basis for M over K, and let $s\colon A \to M$ be a map. Then there is a unique extension of ∂ to a T-derivation on M such that $a' = s(a)$ for all $a \in A$.*

Let us also note for future use that if $x \mapsto x^\lambda\colon K^> \to K$ is an $\mathcal{L}(\emptyset)$-definable power function on K, then $(y^\lambda)' = \lambda y^{\lambda-1} y'$ for all $y \in K^>$.

Let $s \in K$. An element a with $a' = s$ is called an **integral of** s and a nonzero element b with $b^\dagger = s$ is called an **exponential integral of** s. If $s \neq 0$, then an element f with $f' = s^\dagger$ is called a d-**logarithm of** s. In differential algebra, "d-logarithms" are often just called "logarithms" (this is the case in [3]). We include the "d" since there may be actual logarithms present. Suppose that T defines an exponential function with compositional inverse log. Then for $s > 0$, the d-logarithms of s are exactly the elements $\log(s) + c$ for some $c \in C$.

We say that K is **closed under integration** if every element of K has an integral in K, and we say that K is **closed under exponential integration** if every element of K has an exponential integral in $K^\times$. We say that K is **Liouville closed** if K is closed under both integration and exponential integration. Finally, we say that K is **closed under taking** d-**logarithms** if every element of $K^\times$ has a d-logarithm in K. Note that if K is closed under integration, then K is closed under taking d-logarithms. Of course, if T defines an exponential function, then K is closed under taking d-logarithms.

2. T-convex valuation rings

Following [8], a subset $\mathcal{O} \subseteq K$ is said to be a T-**convex valuation ring of** K if $\mathcal{O}$ is nonempty and convex and if $F(\mathcal{O}) \subseteq \mathcal{O}$ for all $\mathcal{L}(\emptyset)$-definable continuous functions $F\colon K \to K$. Let $\mathcal{L}^{\mathcal{O}} := \mathcal{L} \cup \{\mathcal{O}\}$ be the extension of $\mathcal{L}$ by a unary predicate $\mathcal{O}$ and let $T^{\mathcal{O}}$ be the $\mathcal{L}^{\mathcal{O}}$-theory which extends T by axioms asserting that $\mathcal{O}$ is a T-convex valuation ring. For the rest of this section, let $K = (K, \mathcal{O}) \models T^{\mathcal{O}}$. Unlike in [8], we allow $\mathcal{O} = K$, in which case K is said to be **trivially valued**. The theory $T^{\mathcal{O}}$ is *weakly o-minimal*—every $\mathcal{L}^{\mathcal{O}}(K)$-definable subset of K is a finite union of convex subsets of K [8, Corollary 3.14].

Any T-convex valuation ring is a valuation ring, so $\mathcal{O}$ has a unique maximal ideal, which we denote by o. We let Γ denote the value group of K, and we denote the (surjective) Krull valuation by $v\colon K^\times \to \Gamma$. If T is power bounded with field of exponents Λ, then $\Lambda \subseteq \mathcal{O}$ since each $\lambda \in \Lambda$ is $\mathcal{L}(\emptyset)$-definable, and the value group Γ naturally admits the structure of an ordered Λ-vector space by

$$\lambda v(a) \;:=\; v(a^\lambda)$$

for $a \in K^>$ (this does not depend on the choice of a). We set $\Gamma_\infty := \Gamma \cup \{\infty\}$ where $\infty > \Gamma$, and we extend v to all of K by setting $v(0) := \infty$. For $a, b \in K$, set

$$a \preccurlyeq b \;:\Longleftrightarrow\; va \geqslant vb, \qquad a \prec b \;:\Longleftrightarrow\; va > vb, \qquad a \asymp b \;:\Longleftrightarrow\; va = vb, \qquad a \sim b \;:\Longleftrightarrow\; v(a-b) > va.$$

Note that $\sim$ is an equivalence relation on $K^\times$, and that if $a \sim b$, then $a \asymp b$ and a is positive if and only if b is.

Let $\mathrm{res}(K) := \mathcal{O}/\mathit{o}$ denote the **residue field of** K, and for $a \in \mathcal{O}$, let $\bar{a}$ denote the image of a under the residue map $\mathcal{O} \to \mathrm{res}\,K$. Under this map, $\mathrm{res}\,K$ admits a natural expansion to a T-model; see [8, Remark 2.16] for details. A **lift of** $\mathrm{res}\,K$ is an elementary $\mathcal{L}$-substructure $\boldsymbol{k}$ of K contained in $\mathcal{O}$ such that the map $a \mapsto \bar{a}\colon \boldsymbol{k} \to \mathrm{res}\,K$ is an $\mathcal{L}$-isomorphism. By [8, Theorem 2.12], one can always find a lift of $\mathrm{res}\,K$. For $D \subseteq K$, set $\overline{D} := \{\bar{a} : a \in D \cap \mathcal{O}\} \subseteq \mathrm{res}(K)$.

Let M be a $T^{\mathcal{O}}$-extension of K with T-convex valuation ring $\mathcal{O}_M$ and maximal ideal $\mathfrak{o}_M$. We view Γ as a subgroup of Γ_M and $\operatorname{res} K$ as an $\mathcal{L}$-substructure of $\operatorname{res} M$ in the obvious way. Let v and $x \mapsto \bar{x}$ denote their extensions to functions $M^\times \to \Gamma_M$ and $\mathcal{O}_M \to \operatorname{res} M$. By [8, Corollary 3.13], the extension of $T^{\mathcal{O}}$ by an axiom stating that $\mathcal{O}$ is a *proper* subring of K is complete and model complete, so if $\mathcal{O} \neq K$, then M is an elementary $T^{\mathcal{O}}$-extension of K. If $\mathcal{O}_M = M$, then $\mathcal{O} = K$ so M is again an elementary $T^{\mathcal{O}}$-extension of K.

Fact 2.1 ([8], Section 3). *Let $K\langle a \rangle$ be a simple T-extension of K. There are at most two T-convex valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ of $K\langle a \rangle$ which make $K\langle a \rangle$ a $T^{\mathcal{O}}$-extension of K:*

$$\mathcal{O}_1 \;:=\; \big\{y \in K\langle a \rangle : |y| < u \text{ for some } u \in \mathcal{O}\big\}, \qquad \mathcal{O}_2 \;:=\; \big\{y \in K\langle a \rangle : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

If there is $b \in K\langle a \rangle$ which realizes the cut $\mathcal{O}^\downarrow$, then b is contained in $\mathcal{O}_2$ but not $\mathcal{O}_1$, so $\mathcal{O}_1 \subsetneq \mathcal{O}_2$. If there is no such b, then $\mathcal{O}_1 = \mathcal{O}_2$.

2.1. **The Wilkie inequality.** In this subsection, we assume that T is power bounded with field of exponents Λ. The following fact is an analog of the Abhyankar-Zariski inequality, and it is referred to in the literature as the **Wilkie inequality**.

Fact 2.2 ([6], Section 5). *Let M be a $T^{\mathcal{O}}$-extension of K and suppose $\operatorname{rk}_{\mathcal{L}}(M|K)$ is finite. Then*

$$\operatorname{rk}_{\mathcal{L}}(M|K) \geqslant \operatorname{rk}_{\mathcal{L}}(\operatorname{res} M \,|\, \operatorname{res} K) + \dim_\Lambda(\Gamma_M/\Gamma).$$

We most frequently use the Wilkie inequality when M is a simple extension of K. Here is a consequence of the Wilkie inequality:

Lemma 2.3. *Let S be a cut in Γ. Then there is a simple $T^{\mathcal{O}}$-extension $K\langle f \rangle$ of K where $f > 0$ and where vf realizes the cut S. This extension is unique up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism and is completely described as follows: f realizes the cut*

$$\{y \in K : y \leqslant 0 \text{ or } vy > S\}$$

and $\mathcal{O}_{K\langle f \rangle}$ is the convex hull of $\mathcal{O}$ in $K\langle f \rangle$.

Proof. Let $K\langle f \rangle$ be a simple extension of K where f realizes the cut

$$\{y \in K : y \leqslant 0 \text{ or } vy > S\},$$

and let $\mathcal{O}_{K\langle f \rangle}$ be the convex hull of $\mathcal{O}$ in $K\langle f \rangle$. Then $K\langle f \rangle$ with this T-convex valuation ring is indeed a $T^{\mathcal{O}}$-extension of K by Fact 2.1, and vf clearly realizes the cut S. It remains to show uniqueness. Let $\mathcal{O}^*$ be another T-convex valuation ring of $K\langle f \rangle$ with $\mathcal{O}^* \cap K = \mathcal{O}$. If $\mathcal{O}^* \neq \mathcal{O}_{K\langle f \rangle}$, then by Fact 2.1, there is $g \in \mathcal{O}^*$ with $g > \mathcal{O}$. Then the residue field of $K\langle f \rangle$ with respect to $\mathcal{O}^*$ is strictly bigger than $\operatorname{res} K$, as it contains the image of g. By the Wilkie inequality, the value group of $K\langle f \rangle$ with respect to $\mathcal{O}^*$ is equal to Γ, so the valuation of f with respect to $\mathcal{O}^*$ cannot realize the cut S. $\qquad\square$

In Proposition 2.6 below, we use the Wilkie inequality to bound the derivative of a unary $\mathcal{L}(K)$-definable function. This proposition will be used a number of times in Section 4. First, we need two lemmas.

Lemma 2.4. *Let $M = K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \succ 1$ and $va \notin \Gamma$. Let $F \colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \preccurlyeq 1$. Then $F'(a) \prec a^{-1}$.*

Proof. By replacing a with $-a$ if need be, we may assume that $a > 0$. The Wilkie inequality gives $\operatorname{res} M = \operatorname{res} K$, so we may take $u \in \mathcal{O}^\times$ with $F(a) - u \prec 1$. Replacing F with $F - u$, we may assume that $F(a) \prec 1$. Note that this does not change F'.

We first handle the case that $\mathcal{O} = K$, so $a > K$ and $|F(a)| < K^>$. Phrased in terms of limits, we have

$$\lim_{x \to \infty} |F(x)| \;=\; 0,$$

and we want to show that

$$\lim_{x \to \infty} x|F'(x)| \;=\; 0.$$

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Let $\varepsilon, g \in K^>$ be given. We need to find $d > g$ with $d|F'(d)| < \varepsilon$. By increasing g, we may assume that $|F(g)| < \varepsilon/4$ and that $|F|$ is decreasing and $\mathcal{C}^1$ on a neighborhood of $[g, +\infty)$. The o-minimal mean value theorem gives $d \in (g, 2g)$ with

$$|F'(d)| \;=\; \left| \frac{F(2g) - F(g)}{g} \right| \;\leqslant\; \frac{2|F(g)|}{g}.$$

Since $d < 2g$ and $|F(g)| < \varepsilon/4$ we have

$$d|F'(d)| \;\leqslant\; d\frac{2|F(g)|}{g} \;<\; 4|F(g)| \;<\; \varepsilon.$$

Now suppose $\mathcal{O} \neq K$, so M is an elementary $T^{\mathcal{O}}$-extension of K. We need to show that a does not belong to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set $\left\{ y \in M^> : F'(y) \succcurlyeq y^{-1} \right\}$. We do this by showing that for *any* $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq K^>$ with $a \in A^M$, there is $y \in A$ with $F'(y) \prec y^{-1}$. Let A be such a set. Since $T^{\mathcal{O}}$ is weakly o-minimal, we may assume that A is open and convex. By shrinking A, we arrange that F is $\mathcal{C}^1$ on A and that $F(y) \prec 1$ for all $y \in A$. Since $va \notin \Gamma$ and Γ is densely ordered, the set A contains elements y_1, y_2 with $y_1 \prec y_2$. The o-minimal mean value theorem gives

$$F'(y) \;=\; \frac{F(y_2) - F(y_1)}{y_2 - y_1}.$$

for some $y \in A$ between y_1 and y_2. Since $F(y_2) - F(y_1) \prec 1$ and $y_2 - y_1 \asymp y_2 \succcurlyeq y$, we have $F'(y) \prec y^{-1}$, as desired. $\qquad\square$

Lemma 2.5. *Let $M = K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \asymp 1$ and $\bar{a} \notin \operatorname{res} K$. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \preccurlyeq 1$. Then $F'(a) \preccurlyeq 1$.*

Proof. This is trivial if $\mathcal{O} = K$, so we may assume that K is nontrivially valued. Let $\boldsymbol{k} \subseteq \mathcal{O}^\times$ be a lift of $\operatorname{res} K$, so $\boldsymbol{k}\langle a \rangle$ is a lift of $\operatorname{res} M$ by [8, Lemma 5.1]. Take an $\mathcal{L}(\boldsymbol{k})$-definable function $G\colon K \to K$ with $|F(a)| < G(a)$. The Wilkie inequality gives that $\Gamma_M = \Gamma$, so since $\Gamma^<$ has no largest element, it suffices to show that $|F'(a)| < d$ for each $d \in K^>$ with $d \succ 1$. Let d be given and let I be an arbitrary subinterval of $K^>$ with $a \in I^M$. It suffices to find some $y \in I$ with $|F'(y)| < d$. By shrinking I, we arrange that F is $\mathcal{C}^1$ on I and that $|F(y)| < G(y)$ for all $y \in I$. As $\bar{a} \in \bar{I}^{\operatorname{res} M}$, we see that $\bar{I}$ must be infinite, so $I \cap \boldsymbol{k}$ is infinite. Take $y_1, y_2 \in I \cap \boldsymbol{k}$ with $y_1 < y_2$, so $y_2 - y_1 \asymp 1$. Note that $G(y_i) \in \boldsymbol{k}$, so $|F(y_i)| < G(y_i) \prec d$ for $i = 1, 2$. The o-minimal mean value theorem gives

$$F'(y) \;=\; \frac{F(y_2) - F(y_1)}{y_2 - y_1} \;\prec\; d$$

for some $y \in I$ between y_1 and y_2. In particular, $|F'(y)| < d$. $\qquad\square$

Proposition 2.6. *Let $M = K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \not\sim f$ for all $f \in K$ and let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function. Then $F'(a) \preccurlyeq a^{-1}F(a)$.*

Proof. First, suppose $a \succ 1$ and $va \notin \Gamma$. The Wilkie inequality gives $\Gamma_M = \Gamma \oplus \Lambda va$, so take $d \in K^>$ and $\lambda \in \Lambda$ with $F(a) \asymp da^\lambda$. Then $d^{-1}a^{-\lambda}F(a) \asymp 1$ and, applying Lemma 2.4 to the function $y \mapsto d^{-1}y^{-\lambda}F(y)$, we get

$$d^{-1}a^{-\lambda}F'(a) - \lambda d^{-1}a^{-\lambda-1}F(a) \;\prec\; a^{-1}.$$

Since $\lambda \preccurlyeq 1$ and $d^{-1}a^{-\lambda}F(a) \asymp 1$, we have $-\lambda d^{-1}a^{-\lambda-1}F(a) \preccurlyeq a^{-1}$. It follows that $d^{-1}a^{-\lambda}F'(a) \preccurlyeq a^{-1}$, so

$$F'(a) \;\preccurlyeq\; a^{-1}da^\lambda \;\asymp\; a^{-1}F(a).$$

Now, suppose $a \prec 1$ and $va \notin \Gamma$. Let $G\colon K \to K$ be the function given by

$$G(y) \;=\; \begin{cases} F(y^{-1}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

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Then $F(a) = G(a^{-1})$. By applying the previous case to G and $a^{-1} \succ 1$, we get

$$F'(a) = -G'(a^{-1})a^{-2} \preccurlyeq aG(a^{-1})a^{-2} = a^{-1}F(a).$$

Finally, suppose $va \in \Gamma$ and take $b \in K$ with $b \asymp a$, so $b^{-1}a \asymp 1$. Note that $\overline{b^{-1}a} \notin \operatorname{res} K$, for otherwise we would have $a \sim bu$ for some $u \in \mathcal{O}^\times$, contradicting our assumption on a. The Wilkie inequality gives $\Gamma_M = \Gamma$, so take $d \in K^>$ with $F(a) \asymp d$. Applying Lemma 2.5 with $b^{-1}a$ in place of a and with the function $y \mapsto d^{-1}F(by)$ in place of F, we see that

$$d^{-1}bF'(a) \preccurlyeq 1,$$

so $F'(a) \preccurlyeq b^{-1}d \asymp a^{-1}F(a)$. $\qquad\square$

Note that our standing assumption of power boundedness is necessary for Proposition 2.6, as the proposition clearly fails when a is infinite and F is an exponential function with $F' = F$. Our assumption that $a \not\sim f$ for all $f \in K$ is also necessary. To see this, suppose $a \sim f \in K$ and let $F(Y) = Y - f$. Then $F(a) \prec a$ so $a^{-1}F(a) \prec 1$, but $F'(a) = 1$. Here is an application of Proposition 2.6 for use in the proof of Lemma 4.12.

Corollary 2.7. *Suppose $\mathcal{O} = K$, let $b \in K^n$ be an $\mathcal{L}(\emptyset)$-independent tuple, and let $K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a \prec 1$. Let $G \colon K^{1+n} \to K$ be an $\mathcal{L}(\emptyset)$-definable function with $G(a,b) \prec 1$ and let $d = (d_0, \ldots, d_n) \in K^{1+n}$. Then $\nabla G(a,b) \cdot d \prec a^{-1}$.*

Proof. Viewing G as a function of the variables $Y_0, \ldots, Y_n$, we have

$$\nabla G(a,b) \cdot d = \frac{\partial G}{\partial Y_0}(a,b)d_0 + \frac{\partial G}{\partial Y_1}(a,b)d_1 + \cdots + \frac{\partial G}{\partial Y_n}(a,b)d_n.$$

Since $d_i \in K = \mathcal{O}$ for each $i = 0, \ldots n$, we have $d_i \preccurlyeq 1$ for each i, so it suffices to show that $\frac{\partial G}{\partial Y_i} \prec a^{-1}$ for each i. For $i = 0$, we apply Proposition 2.6 to the function $y \mapsto G(y, b)$ to get $\frac{\partial G}{\partial Y_0}(a,b) \preccurlyeq a^{-1}G(a,b) \prec a^{-1}$. For $i > 0$, we will again use Proposition 2.6, but doing so requires a bit of an argument. By symmetry, it suffices to show that $\frac{\partial G}{\partial Y_1}(a,b) \prec a^{-1}$. Let $E := \operatorname{dcl}(b_2, \ldots, b_n)$ and view E as an elementary $\mathcal{L}^{\mathcal{O}}$-substructure of K with $\mathcal{O}_E = \mathcal{O} \cap E = E$. Then $b_1 \notin E$, so $b_1 \not\sim f$ for any $f \in E$, since E is trivially valued. Viewing $E\langle a \rangle$ as a $T^{\mathcal{O}}$-extension of E with $va \notin \Gamma_E = \{0\}$, the Wilkie inequality gives $\operatorname{res} E\langle a \rangle = \operatorname{res} E$, so $b_1 \not\sim f$ for any $f \in E\langle a \rangle$. Thus, we may apply Proposition 2.6 with $E\langle a \rangle$ in place of K, with b_1 in place of a, and with the function $y \mapsto G(a, y, b_2, \ldots, b_n)$ in place of F to get $\frac{\partial G}{\partial Y_1}(a,b) \preccurlyeq b_1^{-1}G(a,b)$. Since $b_1^{-1} \in K$ and $G(a,b) \prec 1$, this gives $\frac{\partial G}{\partial Y_1}(a,b) \prec 1 \prec a^{-1}$. $\qquad\square$

2.2. Immediate extensions and pseudocauchy sequences.

In this subsection, let M be a $T^{\mathcal{O}}$-extension of K. If $\Gamma_M = \Gamma$ and $\operatorname{res} M = \operatorname{res} K$, then M is said to be an **immediate extension of K**. If M is an immediate extension of K, then M is an elementary $T^{\mathcal{O}}$-extension of K. Note that M is an immediate extension of K if and only if for all $a \in M^\times$ there is $b \in K^\times$ with $a \sim b$. The following fact from [17], is useful for studying how $\mathcal{L}^{\mathcal{O}}(K)$-definable functions behave in immediate extensions.

Fact 2.8 ([17], Corollary 1.6)**.** *Suppose M is an immediate extension of K, let $F \colon A \to K$ be an $\mathcal{L}^{\mathcal{O}}(K)$-definable function, and let $a \in A^M$. Then there is an $\mathcal{L}(K)$-definable cell $D \subseteq A$ with $a \in D^M$ such that either $F(y) = 0$ for all $y \in D^M$ or $F(y) \sim F(a)$ for all $y \in D^M$.*

If $K\langle a \rangle$ is a simple immediate $T^{\mathcal{O}}$-extension of K, then $v(a - K) := \{v(a - y) : y \in K\}$ is a downward closed subset of Γ without a greatest element. For each $b \in K\langle a \rangle$, the set $v(b - K)$ can be expressed as a translate of $v(a - K)$:

Lemma 2.9. *Let $K\langle a \rangle$ be a simple immediate $T^{\mathcal{O}}$-extension of K and let $b \in K\langle a \rangle \setminus K$. Then*

$$v(b - K) = \gamma + v(a - K)$$

for some $\gamma \in \Gamma$.

Proof. Let $F \colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) = b$. Take an open interval $I \subseteq K$ with $a \in I^{K\langle a \rangle}$ such that F is $\mathcal{C}^1$ on I. Since $b \notin K$, we have $F'(a) \neq 0$, so we may use Fact 2.8 to shrink I and

arrange that $F'(y) \sim F'(a)$ for all $y \in I^{K\langle a \rangle}$. Set $\gamma := vF'(a) \in \Gamma$ and let $u \in I$. By the o-minimal mean value theorem, we have

$$F(a) - F(u) \ = \ F'(c)(a - u)$$

for some $c \in K\langle a \rangle$ between a and u. Then $vF'(c) = \gamma$ since $c \in I^{K\langle a \rangle}$, so

$$v\big(b - F(u)\big) \ = \ v\big(F(a) - F(u)\big) \ = \ \gamma + v(a - u).$$

The set $\big\{v(a - u) : u \in I\big\}$ is cofinal in $v(a - K)$ and, since $\big\{F(u) : u \in I\big\}$ contains an interval around b, the set $\big\{v\big(b - F(u)\big) : u \in I\big\}$ is cofinal in $v(b - K)$. This gives $v(b - K) = \gamma + v(a - K)$, as desired. $\qquad\square$

Recall that a **pseudocauchy sequence (pc-sequence) in** K is a well-indexed sequence (a_ρ) in K such that

$$a_\tau - a_\sigma \ \prec \ a_\sigma - a_\rho$$

for all $\tau > \sigma > \rho$ greater than some index ρ_0. Let (a_ρ) be a pc-sequence. An element a in a $T^{\mathcal{O}}$-extension of K is said to be a **pseudolimit of** (a_ρ) if for some index ρ_0, we have

$$a - a_\sigma \ \prec \ a - a_\rho$$

for all $\sigma > \rho > \rho_0$. In this case, we say that (a_ρ) **pseudoconverges to** a, and we write $a_\rho \rightsquigarrow a$. The pc-sequence (a_ρ) is said to be **divergent** if it has no pseudolimit in K. Suppose (a_ρ) is divergent with pseudolimit a in some $T^{\mathcal{O}}$-extension of K. Given $y \in K$, we have $a - a_\rho \prec a - y$ for all sufficiently large ρ; otherwise, we would have $a_\rho \rightsquigarrow y$.

Under the assumption of power boundedness, pc-sequences are related to immediate extensions as follows:

Lemma 2.10. *Suppose that T is power bounded and let $K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K. The following are equivalent:*

    (1) $K\langle a \rangle$ is an immediate extension of K;
    (2) $v(a - K)$ has no largest element;
    (3) there is a divergent pc-sequence in K which pseudoconverges to a.

Proof. The equivalence of (1) and (2) follows from results in Tyne's thesis [22]; see [17, Lemma 1.10] for a full proof (this uses our assumption of power boundedness). Assume (2) holds and let (a_ρ) be a well-indexed sequence in K such that $v(a - a_\rho)$ is strictly increasing and cofinal in $v(a - K)$. One easily verifies that (a_ρ) is a divergent pc-sequence in K which pseudoconverges to a. Now, assume (3) holds and let (a_ρ) be a pc-sequence witnessing this. Then for $y \in K$, we may take ρ with $a - a_\rho \prec a - y$, proving (2). $\qquad\square$

As a corollary, we get that any divergent pc-sequence in K has a pseudolimit in an immediate extension of K.

Corollary 2.11. *Let T be power bounded and let (a_ρ) be a divergent pc-sequence in K. Then there is an immediate $T^{\mathcal{O}}$-extension $K\langle a \rangle$ of K with $a_\rho \rightsquigarrow a$. If b is an element of a $T^{\mathcal{O}}$-extension M of K with $a_\rho \rightsquigarrow b$, then there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle a \rangle \to M$ sending a to b.*

Proof. Let $K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $a_\rho \rightsquigarrow a$ (such an extension exists by compactness). Lemma 2.10 gives that $K\langle a \rangle$ is an immediate extension of K. Let b be an element of a $T^{\mathcal{O}}$-extension M of K with $a_\rho \rightsquigarrow b$. We claim that a and b realize the same cut in K. Let $y \in K$ and take a_ρ with $a - a_\rho \prec a - y$ and $b - a_\rho \prec b - y$. Then

$$a - y \ \sim \ (a - y) - (a - a_\rho) \ = \ a_\rho - y \ = \ (b - y) - (b - a_\rho) \ \sim \ b - y,$$

so $a < y$ if and only if $b < y$. This gives us a unique $\mathcal{L}(K)$-embedding $\iota \colon K\langle a \rangle \to M$ sending a to b. To get that ι is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding, let $F \colon K \to K$ be $\mathcal{L}(K)$-definable. We need to show that $F(a) \in \mathcal{O}_{K\langle a \rangle}$ if and only if $F(b) \in \mathcal{O}_M$. We assume that $F(a) \neq 0$, and we will show that $F(a) \sim F(b)$. Using Fact 2.8, take an interval $I \subseteq K$ with $a \in I^{K\langle a \rangle}$ such that $F(y) \sim F(a)$ for all $y \in I^{K\langle a \rangle}$. Since K has a proper immediate

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extension, it is not trivially valued, so M is an *elementary $T^{\mathcal{O}}$-extension* of K. Thus, $F(b) \sim F(y) \sim F(a)$, since $b \in I^M$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. H_T-ASYMPTOTIC FIELDS

In this section, we introduce the class of H_T-asymptotic fields. These fields are o-minimal analogs of the H-asymptotic fields from [3, Chapter 9]. We will collect a few facts from [3] for later use, and then we will discuss the immediate extensions of H_T-asymptotic fields. As we will see in Section 4, all (pre)-H_T-fields are H_T-asymptotic. Before proceeding, let us fix some notation. Set $\mathcal{L}^{\mathcal{O},\partial} := \mathcal{L} \cup \{\mathcal{O}, \partial\}$, and let $T^{\mathcal{O},\partial}$ be the $\mathcal{L}^{\mathcal{O},\partial}$-theory which extends $T^{\mathcal{O}}$ by axioms stating that ∂ is a T-derivation. We also make the following standing assumption:

Assumption 3.1. *For the remainder of this article, let $K = (K, \mathcal{O}, \partial) \models T^{\mathcal{O},\partial}$. We will continue to use the notation introduced in Subsection 1.4 and Section 2. In particular, we write $\mathfrak{o}$ for the maximal ideal of $\mathcal{O}$, Γ for the value group of $(K, \mathcal{O})$, and C for the constant field of (K, ∂).*

Definition 3.2. K is an H_T**-asymptotic field** if for all $g \in K$ with $g \succ 1$, we have

(HA1) $g^\dagger > 0$,
(HA2) $g^\dagger \succ f'$ for all $f \in \mathfrak{o}$, and
(HA3) $g^\dagger \succcurlyeq f'$ for all $f \in \mathcal{O}^\times$.

The definition above differs slightly from the definition of an H-asymptotic field given in [3], though we claim that every H_T-asymptotic field is H-asymptotic. Indeed, (HA2) and (HA3) along with [3, Proposition 9.1.3] imply that every H_T-asymptotic field K is *asymptotic*, that is, $f \prec g \iff f' \prec g'$ for all $f, g \in K^\times$ with $f, g \prec 1$. To see that each H_T-asymptotic field K is H-*asymptotic* in the sense of [3], let $f, g \in K^\times$ with $f \prec g \prec 1$. We need to show that $f^\dagger \succcurlyeq g^\dagger$. Applying condition (HA1) to g^{-1} and g/f, we have

$$g^\dagger = -(g^{-1})^\dagger < 0, \qquad g^\dagger - f^\dagger = (g/f)^\dagger > 0,$$

so $f^\dagger < g^\dagger < 0$. In particular, $f^\dagger \succcurlyeq g^\dagger$, as desired. Conversely, if K is asymptotic, then K satisfies (HA2) and (HA3) by [3, Proposition 9.1.3]. However, the H-asymptotic fields in [3] are not necessarily ordered, and even convexly ordered H-asymptotic fields need not satisfy (HA1).

For the remainder of this section, we assume that K is an H_T-asymptotic field. Note that if $\mathcal{O} \neq K$, then the derivation on K is nontrivial by (HA1). Indeed, (HA1) ensures that the constant field of any H_T-asymptotic field is contained in the valuation ring.

Fact 3.3 ([3], Corollary 9.1.4)**.** *Let $f, g \in K$ with $g \not\asymp 1$. If $f \prec g$, then $f' \prec g'$. If $f \preccurlyeq g$, then $f \sim g \iff f' \sim g'$.*

Let $f \in K^\times$ with $f \not\asymp 1$. As K is asymptotic, the values $v(f')$ and $v(f^\dagger)$ only depend on vf, so for $\gamma = vf$, we set

$$\gamma^\dagger := v(f^\dagger), \qquad \gamma' := v(f') = \gamma + \gamma^\dagger.$$

This gives us a map

$$\psi \colon \Gamma^{\neq} \to \Gamma, \qquad \psi(\gamma) := \gamma^\dagger.$$

Following Rosenlicht [20], we call the pair (Γ, ψ) the **asymptotic couple of** K. We have the following important subsets of Γ:

$$(\Gamma^<)' := \{\gamma' : \gamma \in \Gamma^<\}, \qquad\qquad (\Gamma^>)' := \{\gamma' : \gamma \in \Gamma^>\},$$
$$(\Gamma^{\neq})' := (\Gamma^<)' \cup (\Gamma^>)', \qquad\qquad \Psi := \psi(\Gamma^{\neq}) = \{\gamma^\dagger : \gamma \in \Gamma^{\neq}\}.$$

It is always the case that $(\Gamma^<)' < (\Gamma^>)'$ and that $\Psi < (\Gamma^>)'$. If there is $\beta \in \Gamma$ with $\Psi < \beta < (\Gamma^>)'$, then we call β a **gap in** K. There is at most one such β, and if Ψ has a largest element, then there is no such β. If K has trivial valuation, then the four subsets above are empty and 0 is a gap in K. We say that K **is**

grounded if Ψ has a largest element, and we say that K **is ungrounded** otherwise. Finally, we say that K **has asymptotic integration** if $\Gamma = (\Gamma^{\neq})'$. If β is a gap in K or if $\beta = \max \Psi$, then $\Gamma \setminus (\Gamma^{\neq})' = \{\beta\}$. In [3], the authors prove the following trichotomy for H-asymptotic fields. Of course, this trichotomy holds for H_T-asymptotic fields as well.

Fact 3.4 ([3], Corollary 9.2.16). *Exactly one of the following holds:*

(1) K has asymptotic integration;
(2) K has a gap;
(3) K is grounded.

It follows that K is ungrounded if and only if $\Psi \subseteq (\Gamma^{<})'$. This allows us to define the **contraction map** $\chi : \Gamma^{\neq} \to \Gamma^{<}$ for ungrounded K as follows: let $\gamma \in \Gamma^{\neq}$, so $\gamma^{\dagger} \in \Psi \subseteq (\Gamma^{<})'$. Let $\chi(\gamma)$ be the unique element of $\Gamma^{<}$ with $\chi(\gamma)' = \gamma^{\dagger}$. We extend χ to a map $\Gamma \to \Gamma^{\leqslant}$ by setting $\chi(0) := 0$.

3.1. **Spherically complete immediate extensions of H_T-asymptotic fields.** In this subsection, we will use the results in [17] to prove the following theorem:

Theorem 3.5. *Suppose that T is power bounded. Then K has an immediate H_T-asymptotic field extension which is spherically complete.*

The analog of Theorem 3.5 for H-asymptotic fields was established in [4]. Theorem 3.5 is a fairly immediate consequence of [17, Theorem 6.3], but we will provide some additional detail. It is also worth remarking that in [17], there is a standing assumption that $\mathcal{O} \neq K$. Of course, if $\mathcal{O} = K$, then Theorem 3.5 holds vacuously since K itself is spherically complete. Let us begin with a test for whether an immediate extension of K is H_T-asymptotic:

Lemma 3.6. *Let M be an immediate $T^{\mathcal{O},\partial}$-extension of K. If $f' \prec g^{\dagger}$ for all $f \in \mathfrak{o}_M$ and all $g \in K$ with $g \succ 1$, then M is an H_T-asymptotic field.*

Proof. Let $h \in M$ with $h \succ 1$ and take $g \in K$ with $h \sim g$. For $\varepsilon \in \mathfrak{o}_M$ with $h = g(1+\varepsilon)$, we have

$$h^{\dagger} - g^{\dagger} \ = \ (h/g)^{\dagger} \ = \ (1+\varepsilon)^{\dagger} \ = \ \frac{\varepsilon'}{1+\varepsilon} \ \sim \ \varepsilon'.$$

By assumption, $\varepsilon' \prec g^{\dagger}$, so $h^{\dagger} \sim g^{\dagger}$. As K is H_T-asymptotic, we have $g^{\dagger} > 0$, so $h^{\dagger} > 0$ as well. Additionally, we have $h^{\dagger} \sim g^{\dagger} \succ f'$ for all $f \in \mathfrak{o}_M$, by assumption. Now let $f \in \mathcal{O}_M^{\times}$. Take $u \in K$ and $\delta \in \mathfrak{o}_M$ with $f = u + \delta$, so $f' = u' + \delta'$. We have $\delta' \prec g^{\dagger}$ by assumption and $u' \preccurlyeq g^{\dagger}$, since K is H_T-asymptotic and $u, g \in K$, so $f' \preccurlyeq g^{\dagger} \sim h^{\dagger}$. $\qquad\square$

The main objects of study in [17] are T-*convex T-differential fields*: models of $T^{\mathcal{O},\partial}$ in which the T-derivation is continuous with respect to the valuation topology. Thus, the next step for us is to verify that this continuity assumption holds.

Lemma 3.7. *The derivation on K is continuous with respect to the valuation topology.*

Proof. If $\mathcal{O} = K$, then the valuation topology on K is the discrete topology, and the derivation on K is trivially continuous, so we may assume that $\mathcal{O} \neq K$. Take $g \in K$ with $g \succ 1$. Then $g^{\dagger} \succ f'$ for all $f \in \mathfrak{o}$ by (HA2), so $\partial\mathfrak{o} \subseteq g^{\dagger}\mathfrak{o}$. By [3, Lemma 4.4.7], continuity of the derivation is equivalent to the existence of $\phi \in K^{\times}$ with $\partial\mathfrak{o} \subseteq \phi\mathfrak{o}$, so ∂ is indeed continuous. $\qquad\square$

Central in [17] are the following subsets of Γ, which were first introduced in [4]:

$$\Gamma(\partial) \ := \ \big\{v\phi : \phi \in K^{\times} \text{ and } \partial\mathfrak{o} \subseteq \phi\mathfrak{o}\big\}, \qquad S(\partial) \ := \ \big\{\gamma \in \Gamma : \Gamma(\partial) + \gamma = \Gamma(\partial)\big\}.$$

Using that K is H_T-asymptotic, these subsets can be described explicitly:

Lemma 3.8. $\Gamma(\partial) = \Gamma \setminus (\Gamma^{>})'$ *and* $S(\partial) = \{0\}$.

Proof. For $\phi \in K^\times$, we have

$$v\phi \notin (\Gamma^>)' \iff v\phi < (\Gamma^>)' \iff f' \prec \phi \text{ for all } f \in o \iff v\phi \in \Gamma(\partial),$$

proving the first equality. For the second, note that $S(\partial)$ is a convex subgroup of Γ, so it suffices to show that $S(\partial) \cap \Gamma^> = \emptyset$. Let $\gamma \in \Gamma^>$, so

$$\gamma^\dagger \in \Psi \subseteq \Gamma \setminus (\Gamma^>)', \qquad \gamma^\dagger + \gamma = \gamma' \in (\Gamma^>)'.$$

Since $\Gamma(\partial) = \Gamma \setminus (\Gamma^>)'$, this tells us that $\gamma^\dagger \in \Gamma(\partial)$, but $\gamma^\dagger + \gamma \notin \Gamma(\partial)$, so $\gamma \notin S(\partial)$. $\qquad \square$

Following [4], we say that a $T^{\mathcal{O},\partial}$-extension M of K is a **strict extension of** K if

$$\partial o \subseteq \phi o \implies \partial_M o_M \subseteq \phi o_M, \qquad \partial \mathcal{O} \subseteq \phi o \implies \partial_M \mathcal{O}_M \subseteq \phi o_M$$

for each $\phi \in K^\times$. If M is an immediate strict extension of K, then ∂_M is automatically continuous. By [4, Lemma 1.5], if M is an *immediate $T^{\mathcal{O},\partial}$-extension* of K, then M is a strict extension of K if and only if $\partial_M o_M \subseteq \phi o_M$ for each $\phi \in K^\times$ with $v\phi \in \Gamma(\partial)$.

Corollary 3.9. *Let M be an immediate $T^{\mathcal{O},\partial}$-extension of K. Then M is a strict extension of K if and only if M is an H_T-asymptotic field.*

Proof. Suppose that M is a strict extension of K and let $f \in o_M$. As $\Psi \subseteq \Gamma(\partial)$ by Lemma 3.8, we have $f' \prec \phi$ for all $\phi \in K^\times$ with $v\phi \in \Psi$. Thus, M is an H_T-asymptotic field by Lemma 3.6. Conversely, suppose that M is H_T-asymptotic, let $\phi \in K^\times$ with $v\phi \in \Gamma(\partial)$, and let $f \in o_M$. By [4, Lemma 1.5], we need only show that $f' \prec \phi$. Take $g \in o$ with $f \sim g$. Then $f' \sim g' \prec \phi$ by Fact 3.3. $\qquad \square$

Proof of Theorem 3.5. Since $S(\partial) = \{0\}$, there is an immediate strict T-convex T-differential field extension M of K which is spherically complete by [17, Theorem 6.3] (this uses that T is power bounded). By Corollary 3.9, M is an H_T-asymptotic field. $\qquad \square$

3.2. **Simple immediate extensions of H_T-asymptotic fields.** In this subsection, we turn our focus to certain types of simple immediate extensions. We assume throughout this subsection that T is power bounded.

Proposition 3.10. *Let $G \colon K \to K$ be an $\mathcal{L}(K)$-definable function, let (a_ρ) be a divergent pc-sequence in K, and suppose $a_\rho' - G(a_\rho) \rightsquigarrow 0$. Then K has an immediate H_T-asymptotic field extension $K\langle a \rangle$ with $a_\rho \rightsquigarrow a$ and $a' = G(a)$. If b is a pseudolimit of (a_ρ) in an H_T-asymptotic field extension M of K with $b' = G(b)$, then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b.*

Proof. Let a be a pseudolimit of (a_ρ) in an immediate $T^{\mathcal{O}}$-extension of K; such an extension exists by Corollary 2.11. Using Fact 1.9, extend the derivation on K to a T-derivation on $K\langle a \rangle$ with $a' = G(a)$. We claim that $K\langle a \rangle$ is H_T-asymptotic. By Lemma 3.6, it suffices to show that $f' \prec g^\dagger$ for all $f \in M$ and all $g \in K$ with $f \prec 1 \prec g$. Let $F \colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \prec 1$, let $g \in K$ with $g \succ 1$, and suppose towards contradiction that $F(a)' \succcurlyeq g^\dagger$. Then $F(a) \notin K$, so $F'(a) \neq 0$. Take an open interval $I \subseteq K$ with $a \in I^{K\langle a \rangle}$ such that F is $\mathcal{C}^1$ on I, and let $H \colon I \to K$ be the $\mathcal{L}(K)$-definable function $H(Y) = F^{[\partial]}(Y) + F'(Y)G(Y)$ where $F^{[\partial]}$ is as defined in Fact 1.8. Then

$$H(a) = F^{[\partial]}(a) + F'(a)G(a) = F(a)' \succcurlyeq g^\dagger.$$

Using Fact 2.8, we may shrink I to arrange that

$$F(y) \sim F(a) \prec 1, \qquad H(y) \sim H(a) \succcurlyeq g^\dagger, \qquad F'(y) \sim F'(a)$$

for all $y \in I^{K\langle a \rangle}$. Let ρ be a sufficiently large index, so $a_\rho \in I$. Since $F(a_\rho) \prec 1$ and K is H_T-asymptotic, we have $F(a_\rho)' \prec g^\dagger \preccurlyeq H(a_\rho)$. Thus,

$$H(a_\rho) \sim H(a_\rho) - F(a_\rho)' = \big(F^{[\partial]}(a_\rho) + F'(a_\rho)G(a_\rho)\big) - \big(F^{[\partial]}(a_\rho) + F'(a_\rho)a_\rho'\big) = F'(a_\rho)\big(G(a_\rho) - a_\rho'\big).$$

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Since $H(a_\rho) \sim H(a)$ and $F'(a_\rho) \sim F'(a) \neq 0$, we have

$$G(a_\rho) - a'_\rho \ \sim \ \frac{H(a)}{F'(a)}.$$

Since this holds for all sufficiently large ρ, we have $G(a_\rho) - a'_\rho \sim G(a_\sigma) - a'_\sigma$ for σ, ρ sufficiently large, contradicting our assumption that $a'_\rho - G(a_\rho) \rightsquigarrow 0$. Thus, $K\langle a \rangle$ is an H_T-asymptotic field, as claimed. The embedding property follows from Fact 1.9 and Corollary 2.11. $\qquad\square$

Corollary 3.11. *Let $s \in K$ with $vs \in (\Gamma^>)'$ and $s \notin \partial\mathcal{o}$, and suppose $v(s - \partial\mathcal{o})$ has no largest element. Then K has an immediate H_T-asymptotic field extension $K\langle a \rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial\mathcal{o}_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$.*

Proof. Let (a_ρ) be a well-indexed sequence in $\mathcal{o}$ such that $v(s - a'_\rho)$ is strictly increasing as a function of ρ and cofinal in $v(s - \partial\mathcal{o})$. The proof of [3, Lemma 10.2.4] gives that (a_ρ) is a divergent pc-sequence in K. We apply Proposition 3.10 where G is the constant function s to get an immediate H_T-asymptotic field extension $K\langle a \rangle$ of K with $a_\rho \rightsquigarrow a$ and $a' = s$. Let M be an H_T-asymptotic field extension of K and let $b \in \mathcal{o}_M$ with $b' = s$. Then for $\rho < \sigma$, we have

$$(b - a_\rho)' \ = \ s - a'_\rho \ \sim \ (a_\sigma - a_\rho)'.$$

Since $b - a_\rho, a_\sigma - a_\rho \prec 1$, Fact 3.3 gives us that $b - a_\rho \sim a_\sigma - a_\rho$, so $a_\rho \rightsquigarrow b$. Proposition 3.10 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath \colon K\langle a \rangle \to M$ sending a to b. For uniqueness, let $\jmath \colon K\langle a \rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a) - b \in C_M$ since $\jmath(a)' = s = b'$. Since $\jmath(a), b \prec 1$ and $C_M^\times \subseteq \mathcal{O}_M^\times$, we must have $\jmath(a) = b$. This shows that $\jmath = \imath$. $\qquad\square$

Corollary 3.12. *Let $s \in K$ with $v(s - \partial K) < (\Gamma^>)'$ and suppose $v(s - \partial K)$ has no largest element. Then K has an immediate H_T-asymptotic field extension $K\langle a \rangle$ with $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b.*

Proof. Let (a_ρ) be a well-indexed sequence in K such that $v(s - a'_\rho)$ is strictly increasing as a function of ρ and cofinal in $v(s - \partial K)$ and such that $s - a'_\rho \prec s$ for each ρ. The proof of [3, Lemma 10.2.6] gives that (a_ρ) is a divergent pc-sequence in K. We apply Proposition 3.10 where G is the constant function s to get an immediate H_T-asymptotic field extension $K\langle a \rangle$ of K with $a_\rho \rightsquigarrow a$ and $a' = s$. Let M be an H_T-asymptotic field extension of K and let $b \in M$ with $b' = s$. For $\rho < \sigma$, we have

$$(b - a_\rho)' \ = \ s - a'_\rho \ \sim \ (a_\sigma - a_\rho)',$$

so $v(b - a_\rho)' \in (\Gamma^<)'$ and $b - a_\rho \succ 1$. Fact 3.3 gives $b - a_\rho \sim a_\sigma - a_\rho$, so $a_\rho \rightsquigarrow b$ and Proposition 3.10 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath \colon K\langle a \rangle \to M$ sending a to b. $\qquad\square$

3.3. λ-freeness and ω-freeness. In this subsection, let K be an ungrounded H_T-asymptotic field with $\mathcal{O} \neq K$. A **logarithmic sequence in K** is a well-indexed sequence (ℓ_ρ) in K such that:

(1) $\ell_\rho \succ \ell_\sigma \succ 1$ for all $\sigma > \rho$ and $(v\ell_\rho)$ is cofinal in $\Gamma^<$;
(2) $v\ell_{\rho+1} = \chi(v\ell_\rho)$ for all ρ.

Logarithmic sequences can be constructed by transfinite recursion; see [3, Section 11.5]. Note that if M is an H_T-asymptotic field extension of K with $\Gamma^<$ cofinal in $\Gamma_M^<$, then any logarithmic sequence in K is a logarithmic sequence in M.

A **λ-sequence in K** is a sequence (λ_ρ) where $\lambda_\rho = -\ell_\rho^{\dagger\dagger}$ for some logarithmic sequence (ℓ_ρ) in K. By [3, Proposition 11.5.3], any two λ-sequences in K are equivalent as pc-sequences (they have the same pseudolimits in every extension of K). We say that K is **λ-free** if no λ-sequence in K has a pseudolimit in K.

An **ω-sequence in** K is a sequence (ω_ρ) where $\omega_\rho = -(2\lambda'_\rho + \lambda^2_\rho)$ for some λ-sequence (λ_ρ) in K. We say that K is **ω-free** if no ω-sequence in K has a pseudolimit in K. If $\lambda_\rho \rightsquigarrow \lambda \in K$, then the corresponding ω-sequence (ω_ρ) has pseudolimit $-(2\lambda' + \lambda^2) \in K$, so ω-freeness implies λ-freeness. The property of ω-freeness plays a much larger role than λ-freeness in [3], but in this article, λ-freeness is the more central concept. Even so, ω-freeness makes an appearance in Corollary 4.18 and Proposition 4.21 below, with an eye towards future work.

Lemma 3.13. *If K is an increasing union of λ-free H_T-asymptotic fields, then K is λ-free. If K is an increasing union of ω-free H_T-asymptotic fields, then K is ω-free.*

Proof. By [3, Corollary 11.6.1], K is λ-free if and only if for all $s \in K$, there is $g \in K$ with $g \succ 1$ and $s - g^{\dagger\dagger} \succcurlyeq g^\dagger$. By [3, Corollary 11.7.8], K is ω-free if and only if for all $f \in K$, there is $g \in K$ with $g \succ 1$ and

$$f - 2(g^{\dagger\dagger})' + (g^{\dagger\dagger})^2 \succcurlyeq (g^\dagger)^2.$$

Both of these equivalent conditions are preserved by increasing unions. $\qquad\square$

For the remainder of this section, let (ℓ_ρ) be a logarithmic sequence in K with corresponding λ-sequence (λ_ρ). Nothing here will depend on the specific choice of (ℓ_ρ). The following facts about λ-sequences and λ-freeness are from [3]:

Fact 3.14 ([3], Lemma 11.5.2 and Corollary 11.6.1)**.** *If K is λ-free, then K has asymptotic integration. If K is λ-free and λ is a pseudolimit of (λ_ρ) in an H_T-asymptotic field extension of K, then $v(\lambda - K) = \Psi^\downarrow$.*

Fact 3.15 ([3], Lemma 11.5.13)**.** *Suppose K has asymptotic integration and let $\lambda \in K$ be a pseudolimit of (λ_ρ). Then $v\big(\lambda + (K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$.*

For us, the importance of λ-freeness comes from its relation to gaps:

Lemma 3.16. *Suppose that T is power bounded with field of exponents Λ and that K has asymptotic integration. Let $s \in K$ and let $M = K\langle f \rangle$ be an H_T-asymptotic field extension of K with $f \neq 0$ and $f^\dagger = s$. Then vf is a gap in M if and only $\lambda_\rho \rightsquigarrow -s$.*

Proof. One direction is by [3, Lemma 11.5.12]: if vf is a gap in M, then $\lambda_\rho \rightsquigarrow -s$. For the other direction, suppose $\lambda_\rho \rightsquigarrow -s$. We first note that $vf \notin \Gamma$. Indeed, suppose towards contradiction that $f \asymp y$ for some $y \in K^\times$ and take $u \in M$ with $f = uy$. Then

$$v(s - y^\dagger) \;=\; v(f^\dagger - y^\dagger) \;=\; v(u^\dagger) \;=\; v(u') \;>\; \Psi,$$

contradicting that $v\big(s - (K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$ by Fact 3.15.

Now we claim that $\Psi_M \subseteq \Psi^\downarrow$. We have established that $vf \notin \Gamma$, so the Wilkie inequality gives $\Gamma_M = \Gamma \oplus \Lambda vf$. Thus, we fix $\gamma \in \Gamma$ and $\lambda \in \Lambda$ with $\lambda \neq 0$, and we need to show that $\psi(\gamma + \lambda vf) \in \Psi^\downarrow$. Take $y \in K^>$ with $vy = \gamma$ and set $z := y^{-1/\lambda}$, so $y^\dagger = -\lambda z^\dagger$. Since $v\lambda = 0$, we have

$$\psi(\gamma + \lambda vf) \;=\; v(yf^\lambda)^\dagger \;=\; v(y^\dagger + \lambda s) \;=\; v(\lambda s - \lambda z^\dagger) \;=\; v(s - z^\dagger),$$

so $\psi(\gamma + \lambda vf) \in \Psi^\downarrow$ by Fact 3.15, proving the claim.

Finally, suppose toward contradiction that vf is not a gap in M. Then Ψ_M, being a cofinal subset of $\Psi^\downarrow$, has no maximum and so $vf \in (\Gamma_M^{\neq})'$. Take $\beta \in \Gamma_M^{\neq}$ with $\beta' = vf$ and take $y \in K$ with $\beta^\dagger < vy \in \Psi$. Our assumption that $\lambda_\rho \rightsquigarrow -s$ along with [3, Lemma 11.5.6 (iii)] gives $s - y^\dagger \prec y$, so

$$\psi(vf - vy) \;=\; v(f/y)^\dagger \;=\; v(s - y^\dagger) \;>\; vy,$$

contradicting [3, Lemma 9.2.2] with $\alpha = vf$ and $\gamma = vy$. $\qquad\square$

In [15], Gehret defines a property—the yardstick property—which allows us to check whether λ-freeness is preserved in various extensions. Let S be a nonempty convex subset of Γ without a largest element.

(1) We say that S has the **yardstick property** if there is $\beta \in S$ such that $\gamma - \chi(\gamma) \in S$ for all $\gamma \in S^{>\beta}$.

(2) We say that S is **jammed** if for every nontrivial convex subgroup $\{0\} \neq \Delta \subseteq \Gamma$, there is $\beta \in S$ such that $\gamma - \beta \in \Delta$ for all $\gamma \in S^{>\beta}$.

Note that if S is jammed, then so is $\gamma + S$ for any $\gamma \in \Gamma$. Both being jammed and having the yardstick property only depend on the downward closure of S: the set S is jammed if and only if $S^{\downarrow}$ is, and S has the yardstick property if and only if $S^{\downarrow}$ does. These properties are incompatible, except in the following case:

Fact 3.17 ([15], Lemma 3.17). *Let S be a nonempty convex subset of Γ without a largest element which has the yardstick property. Then S is jammed if and only if $S^{\downarrow} = \Gamma^{<}$.*

The importance of the yardstick property stems from the observation that if K is λ-free and $K\langle a \rangle$ is an immediate extension of K which is *not* λ-free, then the set $v(a - K) \subseteq \Gamma$ is jammed. Thus, we can use Fact 3.17 above to show that immediate extensions $K\langle a \rangle$ where $v(a - K)$ has the yardstick property are λ-free. A precise argument is given in the lemma below. This lemma is an analog of [15, Proposition 6.19] with virtually the same proof; only minor modifications and substitutions are required.

Lemma 3.18. *Let $K\langle a \rangle$ be a simple immediate H_T-asymptotic field extension of K. Suppose that K is λ-free and that $v(a - K) \subseteq \Gamma$ has the yardstick property. Then $K\langle a \rangle$ is λ-free.*

Proof. Suppose toward contradiction that $K\langle a \rangle$ is not λ-free and take $\lambda \in K\langle a \rangle$ with $\lambda_{\rho} \rightsquigarrow \lambda$. Since $\lambda \notin K$, Lemma 2.9 gives $\gamma \in \Gamma$ with $v(\lambda - K) = \gamma + v(a - K)$. Fact 3.14 gives $v(\lambda - K) = \Psi^{\downarrow}$, so $v(\lambda - K)$ is jammed by [15, Lemma 3.11]. Thus, $v(a - K)$ is jammed as well, so $v(a - K) = \Gamma^{<}$ by Fact 3.17. In particular, $v(a - K)$ has a supremum in Γ, so $v(\lambda - K) = \Psi^{\downarrow}$ also has a supremum in Γ. By Fact 3.4, we conclude that K does not have asymptotic integration, contradicting Fact 3.14. $\qquad\square$

4. H_T-FIELDS AND PRE-H_T-FIELDS

In this section, we introduce the main objects of study. Recall our standing assumption that $K = (K, \mathcal{O}, \partial)$ is a model of $T^{\mathcal{O},\partial}$.

Definition 4.1. K is a **pre-H_T-field** if for all $g \in K$ with $g \succ 1$, we have

(PH1) $g^{\dagger} > 0$, and
(PH2) $g^{\dagger} \succ f'$ for all $f \in \mathcal{O}$.

Every pre-H_T-field is H_T-asymptotic, and if K is H_T-asymptotic and $g^{\dagger} \succ f'$ for all $f, g \in K$ with $g \succ f \asymp 1$, then K is a pre-H_T-field. As with H_T-asymptotic fields, every pre-H_T-field is a pre-H-field, as defined in [3]. In the case of pre-H_T-fields, the converse also holds: If K is a pre-H-field, then K is a pre-H_T-field. To see this, use [3, Lemma 10.1.1] and note that (PH1) is equivalent to the condition that $g' > 0$ for all $g \in K$ with $g > \mathcal{O}$. If K is an $\mathcal{L}^{\mathcal{O},\partial}$-substructure of a pre-$H_T$-field, then K is itself a pre-H_T-field.

Lemma 4.2. *Suppose that K is a pre-H_T-field, let $a \in K^{\times}$, and let $b \in K$ be a d-logarithm of a.*

    (i) *If $a \not\asymp 1$, then $b \succ 1$ and $vb = \chi(va)$.*
    (ii) *If $a \asymp 1$, then $b \preccurlyeq 1$.*

Proof. If $a \succ 1$, then since $b' = a^{\dagger}$, we must have $b \succ 1$ by (PH2). It follows immediately that $vb = \chi(va)$. This holds more generally for $a \not\asymp 1$, since $-b' = (a^{-1})^{\dagger}$. On the other hand, if $a \asymp 1$, then

$$v(b') \;=\; v(a^{\dagger}) \;=\; v(a') \;>\; (\Gamma^{<})',$$

so $b \preccurlyeq 1$. $\qquad\square$

Corollary 4.3. *Suppose that K is a pre-H_T-field. If every element in $K^{>}$ has a d-logarithm in K, then K is ungrounded. In particular, if T defines an exponential function, then K is ungrounded.*

Proof. Suppose that every element in $K^{>}$ has a d-logarithm in K, let $\gamma \in \Psi$, and take $a \in K^{>}$ with $a \not\asymp 1$ and $va^{\dagger} = \gamma$. Let b be a d-logarithm of a, so $b \succ 1$ by Lemma 4.2 and $\gamma = vb' \in (\Gamma^{<})'$. Thus, $\Psi \subseteq (\Gamma^{<})'$. $\qquad\square$

Recall from the introduction that K is an H_T-**field** if

(H1) $f' > 0$ for all $f \in K$ with $f > \mathcal{O}$, and

(H2) $\mathcal{O} = C + o$.

Note that if K is an H_T-field, then C is a lift of $\operatorname{res} K$.

Lemma 4.4. *The following are equivalent:*

(1) K *is a pre-H_T-field and* $\mathcal{O} = C + o$;

(2) K *is an H_T-asymptotic field and* $\mathcal{O} = C + o$;

(3) K *is an H_T-field.*

Proof. It is immediate that (1) implies (2). Suppose (2) holds and let $f \in K$ with $f > \mathcal{O}$. Then $f \succ 1$ so $f^\dagger > 0$ by (HA1). As $f > 0$, this gives $f' > 0$, so (H1) is satisfied. Of course (H2) is satisfied by assumption, so (3) holds. To see that (3) implies (1), we assume that K is an H_T-field, and we will verify that (PH1) and (PH2) hold. For (PH1), let $f \in K$ with $f \succ 1$. Then $|f| > \mathcal{O}$, so $|f|' > 0$. Since $f^\dagger = |f|^\dagger$, this gives $f^\dagger > 0$. Now for (PH2), let $f, g \in K$ with $g \succ 1$ and $f \preccurlyeq 1$. We need to show that $g^\dagger \succ f'$. This is shown in [3, Lemma 10.5.1], but we repeat the proof here. First, by replacing g with $-g$ if need be, we may assume that $g > 0$. As $\mathcal{O} = C + o$, we may subtract a constant from f to arrange that $f \prec 1$. Let $c \in C^>$, so $0 < c + f, c - f \asymp 1$. This gives $g(c+f), g(c-f) > \mathcal{O}$, so $g'(c+f) + gf', g'(c-f) - gf' > 0$ by (H1), yielding

$$g'(c - f) \; > \; gf' \; > \; -g'(c + f).$$

Dividing by g gives

$$g^\dagger(c - f) \; > \; f' \; > \; -g^\dagger(c + f).$$

As $f \prec 1$ and $c \in C^>$ can be taken to be arbitrarily small, we see that $f' \prec g^\dagger$ as desired. $\square$

Corollary 4.5. *Let K be an H_T-field and let M be an H_T-asymptotic field extension of K with $\operatorname{res} M = \operatorname{res} K$. Then M is an H_T-field with $C_M = C$.*

Proof. We have $C \subseteq C_M$ and by (HA1), we have $C_M \subseteq \mathcal{O}_M$. As C is a lift of $\operatorname{res} K = \operatorname{res} M$, it is maximal among the elementary $\mathcal{L}$-substructures of M contained in $\mathcal{O}_M$, so $C = C_M$ and $\mathcal{O}_M = C + o_M$; see [8, Remark 2.11 and Theorem 2.12]. We conclude that M is an H_T-field by Lemma 4.4. $\square$

Lemma 4.6. *Let K be a pre-H_T-field and let M be an immediate H_T-asymptotic field extension of K. Then M is a pre-H_T-field. If K is an H_T-field, then M is as well.*

Proof. Let $f, g \in M$ with $g \succ f \asymp 1$. We need to show that $g^\dagger \succ f'$. Using that $\Gamma_M = \Gamma$, take $a \in K$ with $g \asymp a$, so $g^\dagger \asymp a^\dagger$. Using that $\operatorname{res} M = \operatorname{res} K$, take $b \in K$ with $f - b \prec 1$, so $(f - b)' \prec a^\dagger$, as M is H_T-asymptotic. As K is a pre-H_T-field, we also have $b' \prec a^\dagger$, so

$$f' \; = \; (f - b)' + b' \; \prec \; a^\dagger \; \asymp \; g^\dagger.$$

If K is an H_T-field, then Corollary 4.5 gives that M is an H_T-field as well. $\square$

Using Lemma 4.6, we have the following consequence of Theorem 3.5:

Corollary 4.7. *Suppose that T is power bounded. Then every pre-H_T-field has a spherically complete immediate pre-H_T-field extension and every H_T-field has a spherically complete immediate H_T-field extension.*

Assumption 4.8. *For the remainder of this section, we assume that T is power bounded with field of exponents Λ and that K is a pre-H_T-field.*

4.1. **Adjoining integrals.** To begin this subsection, let us use Corollaries 3.11 and 3.12 to say something about immediate extensions of K by integrals.

Corollary 4.9. *Let $s \in K$ with $vs \in (\Gamma^>)'$ and $s \notin \partial o$. Then K has an immediate pre-H_T-field extension $K\langle a \rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial o_M$, there is a unique $\mathcal{L}^{\mathcal{O}, \partial}(K)$-embedding $K\langle a \rangle \to M$. If K is ungrounded and λ-free, then so is $K\langle a \rangle$.*

Proof. Let $S := v(s - \partial o) \subseteq (\Gamma^>)'$. We claim that S has no largest element. Let $y \in o$ and take $b \in o$ with $s - y' \asymp b'$. Take $u \in \mathcal{O}^\times$ with $s - y' = ub'$. Then (PH2) gives $u' \prec b^\dagger$, so $u'b \prec b'$, and

$$s - (y + ub)' \;=\; s - y' - ub' - u'b \;=\; -u'b \;\prec\; b' \;\asymp\; s - y'.$$

Thus, S has no largest element as claimed, and Corollary 3.11 gives an immediate H_T-asymptotic field extension $K\langle a \rangle$ of K with $a \prec 1$, $a' = s$, and the desired embedding property. By Lemma 4.6, $K\langle a \rangle$ is itself a pre-H_T-field. By [15, Lemma 8.5], the set $v(a - K)$ has the yardstick property, so if K is ungrounded and λ-free, then $K\langle a \rangle$ is as well by Lemma 3.18. $\qquad\square$

Corollary 4.10. *Let $s \in K$ with $v(s - \partial K) \subseteq (\Gamma^<)'$. Then K has an immediate pre-H_T-field extension $K\langle a \rangle$ with $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b. If K is ungrounded and λ-free, then so is $K\langle a \rangle$.*

Proof. Let $S := v(s - \partial K) \subseteq (\Gamma^<)'$. Again, we claim that S has no largest element. Let $y \in K$ and take $b \succ 1$ and $u \asymp 1$ with $s - y' = ub'$. As in the proof of Corollary 4.9, we see that $s - (y + ub)' \prec s - y'$, as desired. Corollary 3.12 gives an immediate H_T-asymptotic field extension $K\langle a \rangle$ of K with $a' = s$ and the desired embedding property. By Lemma 4.6, $K\langle a \rangle$ is itself a pre-H_T-field. By [15, Lemma 9.6], the set $v(a - K)$ has the yardstick property, so if K is ungrounded and λ-free, then $K\langle a \rangle$ is as well by Lemma 3.18. $\qquad\square$

Now we turn to the case that K has a gap. First, we give a useful test for whether a simple extension of K is a pre-H_T-field.

Lemma 4.11. *Let K be a pre-H_T-field and let $M = K\langle a \rangle$ be a $T^{\mathcal{O},\partial}$-extension of K with $va \notin \Gamma$. Suppose that for all $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$, we have*

(i) $(ga^\lambda)^\dagger > 0$,
(ii) $(ga^\lambda)^\dagger \succ f'$ *for all $f \in K$ with $f \preccurlyeq 1$,*
(iii) $(ga^\lambda)^\dagger \succ F(a)'$ *for all $\mathcal{L}(K)$-definable functions $F: K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.*

Then M is a pre-H_T-field. If K is an H_T-field, then so is M.

Proof. Let $h \in M$ with $h \succ 1$ and take $g \in K^\times$ and $\lambda \in \Lambda$ with $h \asymp ga^\lambda$. By the Wilkie inequality, we have $\operatorname{res} M = \operatorname{res} K$, so by multiplying g with an element in $\mathcal{O}^\times$, we may even assume that $h \sim ga^\lambda$. Take $\varepsilon \in o_M$ with $h = ga^\lambda(1 + \varepsilon)$. Then

$$h^\dagger - (ga^\lambda)^\dagger \;=\; (1 + \varepsilon)^\dagger \;=\; \frac{\varepsilon'}{1 + \varepsilon} \;\sim\; \varepsilon'.$$

We have $\varepsilon' \prec (ga^\lambda)^\dagger$ by (ii) and (iii), so $h^\dagger \sim (ga^\lambda)^\dagger$. In particular, $h^\dagger > 0$ by (i), so (PH1) holds. For (PH2), let $f \in \mathcal{O}_M$. We need to show that $h^\dagger \succ f'$. Since $h^\dagger \sim (ga^\lambda)^\dagger$, it suffices to show that $(ga^\lambda)^\dagger \succ f'$. This follows from (ii) if $f \in K$, so we may assume $f \notin K$. As $\operatorname{res} M = \operatorname{res} K$, we may take $u \in \mathcal{O}$ with $f - u \prec 1$. Take an $\mathcal{L}(K)$-definable function $F: K \to K$ with $F(a) = f - u$. Then $f' = u' + F(a)' \prec (ga^\lambda)^\dagger$ by (ii) and (iii). Finally, if K is an H_T-field, then M is as well by Corollary 4.5, since $\operatorname{res} M = \operatorname{res} K$. $\qquad\square$

The next lemma shows that if K has a gap, then this gap has an integral in some pre-H_T-field extension of K.

Lemma 4.12. *Let $s \in K$ and suppose vs is a gap in K. Then K has a pre-H_T-field extension $K\langle a \rangle$ with $a \prec 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K with $s \in \partial o_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$. The pre-H_T-field $K\langle a \rangle$ is grounded with*

$$\operatorname{res} K\langle a \rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a \rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a \rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Proof. By replacing s with $-s$ if need be, we arrange that $s < 0$. Let $K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and $0 < va < \Gamma^>$. The Wilkie inequality gives $\Gamma_{K\langle a \rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a \rangle = \operatorname{res} K$. Using Fact 1.9, we equip $K\langle a \rangle$ with the unique T-derivation that extends the derivation on K and satisfies $a' = s$. We need to show that $K\langle a \rangle$ is a pre-H_T-field extension of K. Let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 4.11, it suffices to verify the following:

    (i) $(ga^\lambda)^\dagger > 0$;

    (ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;

    (iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

Since $v(ga^\lambda) = vg + \lambda va$ is assumed to be negative and since $0 < va < \Gamma^>$, it must be that either $g \succ 1$ or $g \asymp 1$ and $\lambda < 0$. If $g \succ 1$, then $vg^\dagger \in (\Gamma^<)'$ and $va^\dagger = vs - va > (\Gamma^<)'$, so

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda a^\dagger \;\sim\; g^\dagger \;>\; 0.$$

On the other hand, if $g \asymp 1$ and $\lambda > 0$, then $g^\dagger \asymp g' \preccurlyeq s \prec s/a = a^\dagger$. This gives

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda a^\dagger \;\sim\; \lambda a^\dagger,$$

since $\lambda \asymp 1$. Since $a^\dagger < 0$, we have $\lambda a^\dagger > 0$. This takes care of (i) and also tells us that $(ga^\lambda)^\dagger \succcurlyeq a^\dagger$. This can be used to quickly take care of (ii): if $f \in \mathcal{O}$, then $f' \preccurlyeq s \prec a^\dagger \preccurlyeq (ga^\lambda)^\dagger$.

    Now we turn to (iii). Let $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. We need to show that $F(a)' \prec a^\dagger = s/a$. We consider two cases. First, suppose $\mathcal{O} = K$. Take an $\mathcal{L}(\emptyset)$-definable function $G\colon K^{1+n} \to K$ and an $\mathcal{L}(\emptyset)$-independent tuple $b = (b_1, \ldots, b_n) \in K^n$ with $F(a) = G(a,b)$. Then

$$F(a)' \;=\; G(a,b)' \;=\; \nabla G(a,b) \cdot (s, b_1', \ldots, b_n'),$$

so by applying Corollary 2.7 with $(s, b_1', \ldots, b_n') \in K^{1+n}$ in place of d, we get $F(a)' \prec a^{-1}$. Since $s \asymp 1$, this gives $F(a)' \prec s/a$, as desired. Now suppose $\mathcal{O} \neq K$. We need to show that $F^{[\partial]}(a) + F'(a)s \prec s/a$. Proposition 2.6 gives $F'(a) \preccurlyeq a^{-1}F(a) \prec a^{-1}$, so $F'(a)s \prec s/a$ and it remains to show that $F^{[\partial]}(a) \prec s/a$. Since $K\langle a\rangle$ is an elementary $T^{\mathcal{O}}$-extension of K, it suffices to show that for each $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq K$ with $a \in A^{K\langle a\rangle}$, there is $y \in A$ with $F^{[\partial]}(y) \prec s/y$. Let A be such a set and, by shrinking A if need be, assume that F is $\mathcal{C}^1$ on A and that $y, F(y) \prec 1$ for all $y \in A$. Since $F'(a) \prec a^{-1}$, we can use $\mathcal{L}^{\mathcal{O}}$-elementarity to take $y \in A$ with $F'(y) \prec y^{-1}$. Multiplying by y' gives $F'(y)y' \prec y^\dagger$ for this y. Since $F(y) \prec 1$ and K is a pre-H_T-field, we have $F(y)' \prec y^\dagger$. Thus,

$$F^{[\partial]}(y) \;=\; F(y)' - F'(y)y' \;\prec\; y^\dagger.$$

Since $y \prec 1$ and vs is a gap in K, we have $y' \prec s$, so $F^{[\partial]}(y) \prec y^\dagger \prec s/y$, as desired.

    Finally, let M be an H_T-asymptotic field extension of K and let $b \in o_M$ with $b' = s$. Then $b^\dagger = s/b$ must be negative by (HA1), so b is positive since s is negative. Moreover, vb must realize the cut $\Gamma^\leqslant$ since $vs \in (\Gamma_M^>)'$ and $vs < (\Gamma^>)'$. Lemma 2.3 gives a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $\imath\colon K\langle a\rangle \to M$ sending a to b and Fact 1.9 tells us that $\imath$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Let $\jmath\colon K\langle a\rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a) - b \in C_M$ since $\jmath(a)' = s = b'$. Since $\jmath(a), b \prec 1$, we see that $\jmath(a) = b$. This shows that $\jmath = \imath$, so $\imath$ is unique. $\qquad\square$

If we further assume that K is an H_T-field, then one can find an H_T-field extension of K with an infinite integral for a gap in K.

Lemma 4.13. *Let K be an H_T-field, let $s \in K$, and suppose vs is a gap in K. Then K has an H_T-field extension $K\langle a\rangle$ with $a \succ 1$ and $a' = s$ such that for any H_T-asymptotic field extension M of K and $b \in M$ with $b \succ 1$ and $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ sending a to b. The H_T-field $K\langle a\rangle$ is grounded with*

$$\operatorname{res} K\langle a\rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a\rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a\rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Proof. We may assume that $s > 0$. Let $K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and $\Gamma^< < va < 0$, so $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a\rangle = \operatorname{res} K$ by the Wilkie inequality. Using Fact 1.9, we equip $K\langle a\rangle$ with the unique T-derivation that extends the derivation on K and satisfies $a' = s$. To see that $K\langle a\rangle$ is an H_T-field extension of K, let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 4.11, it suffices to verify the following:

    (i) $(ga^\lambda)^\dagger > 0$;

    (ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;

(iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

Proving (i) is similar to the proof of Lemma 4.12. This time, either $g \succ 1$ or $g \asymp 1$ and $\lambda > 0$. If $g \succ 1$, then $g^\dagger \succ a^\dagger$, so $(ga^\lambda)^\dagger \sim g^\dagger > 0$. Suppose $g \asymp 1$. We need to show that $g^\dagger \prec a^\dagger$. Using that K is an H_T-field, take $c \in C$ with $g - c \in \mathfrak{o}$. Then $g^\dagger \asymp g' = (g - c)' \in \partial\mathfrak{o}$, so $v(g^\dagger) \in (\Gamma^>)'$. Since $va^\dagger < (\Gamma^>)'$, we have $(ga^\lambda)^\dagger \sim \lambda a^\dagger > 0$. This takes care of (i) and tells us that $(ga^\lambda)^\dagger \succcurlyeq a^\dagger$. For (ii), let $f \in \mathcal{O}$ and take $c \in C$ with $f - c \in \mathfrak{o}$. Then $f' = (f - c)'$, so $v(f') \in (\Gamma^>)' > va^\dagger \geqslant v(ga^\lambda)^\dagger$.

Now we turn to (iii). Let $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. As in the proof of Lemma 4.12, we need to show that $F(a)' = F^{[\partial]}(a) + F'(a)s \prec s/a$. Proposition 2.6 gives $F'(a) \preccurlyeq a^{-1}F(a) \prec a^{-1}$, so $F'(a)s \prec s/a$ and it remains to show that $F^{[\partial]}(a) \prec s/a$. We claim that $|F^{[\partial]}(a)| < s/a^2 \prec s/a$. Since $F^{[\partial]}$ is $\mathcal{L}(K)$-definable, it suffices to show that for each interval $I \subseteq K$ with $a \in I^{K\langle a \rangle}$, there is $y \in I$ with $|F^{[\partial]}(y)| < s/y^2$. Let I be such an interval and, by shrinking I if need be, assume that F is $\mathcal{C}^1$ on I and that $|F(y)| < 1$ for all $y \in I$. Since K is an H_T-field and a realizes the cut $\mathcal{O}^\downarrow$, the interval I contains a constant $c \in C^>$. Since $|F(c)| < 1$ and s is a gap in K, we have

$$F(c)' \ = \ F^{[\partial]}(c) + F'(c)c' \ = \ F^{[\partial]}(c) \ \prec \ s.$$

Since $c^2 \asymp 1$, we have $c^2 F^{[\partial]}(c) \prec s$, which yields $|F^{[\partial]}(c)| < s/c^2$, as desired.

Finally, let M be an H_T-asymptotic field extension of K and let $b \in M$ with $b \succ 1$ and $b' = s$. Then $b^\dagger = s/b$ must be positive, so b is positive since s is positive. Since $vs \in (\Gamma^<_M)'$ and $vs > (\Gamma^<)'$, we see that vb must realize the cut $\Gamma^<$. Lemma 2.3 gives a unique $\mathcal{L}^\mathcal{O}(K)$-embedding $\imath\colon K\langle a \rangle \to M$ sending a to b, and this is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Fact 1.9. $\qquad\square$

4.2. **The H_T-field hull.** We now show that the pre-H_T-field K has a minimal H_T-field extension. We say that $\beta \in \Gamma$ is a **fake gap in K** if β is a gap in K and $\beta = v(b')$ for some $b \in K$. Then necessarily $b \asymp 1$, for otherwise $\beta \in (\Gamma^{\neq})'$. Likewise, $b \not\sim c$ for any $c \in C$, for otherwise $b' = (b - c)'$ and $\beta \in (\Gamma^>)'$. Thus, no H_T-field has a fake gap. Of course, if K is grounded or has asymptotic integration, then K does not have a fake gap. Suppose K does not have a fake gap and let M be an immediate pre-H_T-field extension of K. We claim that M does not have a fake gap. Let $b \in M$ with $b \asymp 1$ and take $a \in K$ with $b - a \prec 1$. Then $v(b - a)' \in (\Gamma^>)'$. As K has no fake gap, we also have $v(a') \in (\Gamma^>)'$, so

$$v(b') \ = \ v\big((b - a)' + a'\big) \ \geqslant \ \min\big\{v(b - a)', v(a')\big\} \in (\Gamma^>)'.$$

Theorem 4.14. *K has an H_T-field extension $H_T(K)$ such that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$. For $L := H_T(K)$, we have*

$$L \ = \ K\langle C_L \rangle, \qquad \operatorname{res} L = \operatorname{res} K.$$

Proof. We first construct a pre-H_T-field extension K_0 of K which does not have a fake gap as follows: if K does not have a fake gap, then let $K_0 := K$. Suppose that K has a fake gap $\beta = v(b')$, and apply Lemma 4.12 with $s = b'$ to get a pre-H_T-field extension $K\langle a \rangle$ of K with $a \prec 1$ and $a' = b'$. Then $K\langle a \rangle$ does not have a fake gap as it is grounded, and we set $K_0 := K\langle a \rangle$. We claim that $K_0 = K\langle C_{K_0} \rangle$, that $\operatorname{res} K_0 = \operatorname{res} K$, and that for any H_T-field extension M of K, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_0 \to M$. This is all trivial if $K_0 = K$, so we assume that $K_0 \neq K$ and we let a, b be as above. Then $K_0 = K\langle b - a \rangle$ and $b - a \in C_{K_0}$, so $K_0 = K\langle C_{K_0} \rangle$, and Lemma 4.12 gives $\operatorname{res} K_0 = \operatorname{res} K$. For the embedding property, take $c \in C_M$ with $b \sim c$. Then $b' = (b - c)' \in \partial\mathfrak{o}_M$, so Lemma 4.12 gives a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_0 \to M$.

Suppose K_0 is not an H_T-field, so there is $b \in \mathcal{O}_{K_0}$ with $b \notin C_{K_0} + \mathfrak{o}_{K_0}$. Then $b' \notin \partial\mathfrak{o}_{K_0}$, for otherwise we would have $b - \varepsilon \in C_{K_0}$ for some $\varepsilon \in \mathfrak{o}_{K_0}$. Since $v(b')$ is not a fake gap, we have $v(b') \in (\Gamma^>_{K_0})'$. Corollary 4.9 gives an immediate pre-H_T-field extension $K^* := K_0\langle a \rangle$ of K_0 with $a \prec 1$ and $a' = b'$. Given an H_T-field extension M of K_0, take $c \in C_M$ with $b \sim c$. Then $b' = (b - c)' \in \partial\mathfrak{o}_M$, so Corollary 4.9 gives a unique $\mathcal{L}^{\mathcal{O},\partial}(K_0)$-embedding $K^* \to M$. Note that $K^* = K_0\langle b - a \rangle$ and $b - a \in C_{K^*}$, so $K^* = K_0\langle C_{K^*} \rangle = K\langle C_{K^*} \rangle$. As K^* is an immediate extension of K_0, there is no fake gap in K^*. Note that for any $u \in \mathcal{O}_{K^*}$ with $\bar{u} = \bar{b} \in \operatorname{res} K^* = \operatorname{res} K$, we have $u \in C_{K^*} + \mathfrak{o}_{K^*}$. Thus, by transfinitely iterating this process (with the

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number of iterations bounded by the cardinality of $\operatorname{res} K$), we build an immediate pre-H_T-field extension L of K_0 with $L = K\langle C_L \rangle$ and $\mathcal{O}_L = C_L + o_L$ such that for any H_T-field extension M of K_0, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K_0)$-embedding $L \to M$. Then L is an H_T-field by Lemma 4.4 and $\operatorname{res} L = \operatorname{res} K_0 = \operatorname{res} K$. Using also the embedding property for K_0 over K, we see that there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of L into any H_T-field extension M of K, so we may take $H_T(K)$ to be this H_T-field extension L. $\qquad\square$

The universal property in Theorem 4.14 determines $H_T(K)$ uniquely up to unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. We call $H_T(K)$ the H_T-**field hull** of K. If K does not have a fake gap, then $H_T(K)$ is an immediate extension of K; in particular, $\Gamma_{H_T(K)} = \Gamma$. If β is a fake gap in K, then $\Gamma_{H_T(K)} = \Gamma \oplus \Lambda va$ for $a \in H_T(K)$ with $0 < va < \Gamma^>$ and $v(a') = \beta$. The following consequence of Theorem 4.14 is not used anywhere, but it may be worth noting.

Corollary 4.15. *The following are equivalent:*

> *(1) every spherically complete immediate H_T-asymptotic field extension of K is an H_T-field;*
> *(2) K has a spherically complete immediate H_T-field extension;*
> *(3) K does not have a fake gap.*

Proof. By Theorem 3.5, we know that K has a spherically complete immediate H_T-asymptotic field extension, so (1) implies (2). Suppose (2) holds and let M be a spherically complete immediate H_T-field extension of K. By the universal property of the H_T-field hull, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $H_T(K) \to M$. Then $H_T(K)$ is an immediate extension of K, so $\Gamma_{H_T(K)} = \Gamma$ and K does not have a fake gap by the remarks preceding this corollary. Finally, suppose (3) holds and let M be a spherically complete immediate H_T-asymptotic field extension of K. Then M is a pre-H_T-field by Lemma 4.6, and the remarks before Theorem 4.14 tell us that M does not have a fake gap. Thus, $H_T(M)$ is an immediate extension of M, so $M = H_T(M)$, as M has no proper immediate extensions. $\qquad\square$

4.3. ω-free extensions of grounded pre-H_T-fields. In this subsection, we show that each grounded pre-H_T-field K has a canonical ungrounded ω-free extension, denoted K_ω. First, we show how to extend a grounded pre-H_T-field by an integral for the maximum of the set Ψ.

Lemma 4.16. *Let $s \in K$ and suppose $vs = \max \Psi$. Then K has a pre-H_T-field extension $K\langle a \rangle$ with $a' = s$ such that for any pre-H_T-field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b. The pre-H_T-field $K\langle a \rangle$ is grounded with*

$$\operatorname{res} K\langle a \rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a \rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a \rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Proof. Let $L = H_T(K)$, so L is an immediate extension of K and $vs = \max \Psi_L$, since K is grounded. Then L is an H_T-field, and a proof identical to the proof of Lemma 4.13 tells us that L has a grounded H_T-field extension $L\langle a \rangle$ with $a \succ 1$ and $a' = s$, where

$$\operatorname{res} L\langle a \rangle \;=\; \operatorname{res} L, \qquad \Gamma_{L\langle a \rangle} \;=\; \Gamma_L \oplus \Lambda va, \qquad \Psi_{L\langle a \rangle} \;=\; \Psi_L \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi_L.$$

Then $K\langle a \rangle$, being an $\mathcal{L}^{\mathcal{O},\partial}$-substructure of the H_T-field $L\langle a \rangle$, is a pre-H_T-field. Since L is an immediate extension of K, we have $\operatorname{res} L = \operatorname{res} K$, $\Gamma_L = \Gamma$, and $\Psi_L = \Psi$, so

$$\operatorname{res} K\langle a \rangle \;=\; \operatorname{res} K, \qquad \Gamma_{K\langle a \rangle} \;=\; \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a \rangle} \;=\; \Psi \cup \{va^\dagger\}, \qquad va^\dagger \;>\; \Psi.$$

Since $a \succ 1$, we get that va realizes the cut $\Gamma^<$ and $a^\dagger = s/a$ is positive. Let M be a pre-H_T-field extension of K and let $b \in M$ with $b' = s$. Then $b \succ 1$ by (PH2), so vb must realize the cut $\Gamma^<$ and $b^\dagger = s/b$ must be positive. Lemma 2.3 gives a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $\imath\colon K\langle a \rangle \to M$ sending a to b, and this is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Fact 1.9. $\qquad\square$

Remark 4.17. Our passage through the H_T-field hull of K is admittedly circuitous, but we don't have a better argument that $K\langle a \rangle$ is a pre-H_T-field than the one given above.

One may wonder whether the assumption that K is an H_T-field can be relaxed in Lemma 4.13. That is, if vs is a gap in the pre-H_T-field K with $s \in K$, then does K have a pre-H_T-field extension $K\langle a \rangle$ with $a \succ 1$ and $a' = s$? Arguing as in the proof of Lemma 4.16, we see that this is true so long as vs is not a fake gap in K. If vs is a fake gap in K, then s cannot have an infinite integral, so there is no such extension.

The version of Lemma 4.16 in the author's thesis [16, Lemma 7.34] contains a slight error. There, it was claimed that $K\langle a \rangle$ embeds into any H_T-asymptotic field extension of K. It is only true that $K\langle a \rangle$ embeds into any pre-H_T-field extension of K.

We now turn to the construction of K_{ω}.

Corollary 4.18. *Let K be a grounded pre-H_T-field. Then K has an ungrounded ω-free (hence, λ-free) pre-H_T-field extension K_{ω} with $\operatorname{res} K_{\omega} = \operatorname{res} K$ which embeds over K into any pre-H_T-field extension of K which is closed under taking d-logarithms.*

Proof. Let $s \in K$ with $vs^\dagger = \max \Psi$. Using Lemma 4.16, take a pre-H_T-field extension $K\langle a \rangle$ where $a' = s^\dagger$. We have

$$\operatorname{res} K\langle a \rangle \ = \ \operatorname{res} K, \qquad \Gamma_{K\langle a \rangle} \ = \ \Gamma \oplus \Lambda va, \qquad \Psi_{K\langle a \rangle} \ = \ \Psi \cup \{va^\dagger\}, \qquad va^\dagger \ > \ \Psi.$$

Repeating this process, we construct for each n a pre-H_T-field extension K_n of K with

$$K_0 \ = \ K, \qquad K_{n+1} \ = \ K_n\langle a_n \rangle, \qquad a_0 \ = \ a, \qquad a'_{n+1} \ = \ a_n^\dagger.$$

Set $K_{\omega} := \bigcup_n K_n$. Then $\operatorname{res} K_{\omega} = \operatorname{res} K$ and

$$\Gamma_{K_{\omega}} \ = \ \Gamma \oplus \bigoplus_n \Lambda va_n, \qquad \Psi_{K_{\omega}} \ = \ \Psi \cup \{va_0^\dagger, va_1^\dagger, \dots\}, \qquad \Psi \ < \ va_0^\dagger \ < \ va_1^\dagger \ < \ \cdots.$$

Moreover, K_{ω} is ω-free by [3, Corollary 11.7.15], since K_{ω} is ungrounded and each K_n is grounded. Let M be a pre-H_T-field extension of K which is closed under taking d-logarithms. Then there are elements $b_0, b_1, \dots \in M$ with $b_0' = s^\dagger$ and $b_{n+1}' = b_n^\dagger$ for each n. Repeated use of the embedding property in Lemma 4.16 allows us construct an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_{\omega} \to M$ which sends a_n to b_n for each n. $\qquad \square$

4.4. Adjoining exponential integrals.

Lemma 4.19. *Let $s \in K$ with $vs \in (\Gamma^{>})'$ and suppose that $s \neq y^\dagger$ for all $y \in K^{\times}$. Then K has an immediate pre-H_T-field extension $K\langle a \rangle$ with $a \sim 1$ and $a^\dagger = s$ such that for any H_T-asymptotic field extension M of K with $s \in (1 + \mathcal{O}_M)^\dagger$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$. If K is ungrounded and λ-free, then so is $K\langle a \rangle$.*

Proof. Let $S := v\big(s - (1 + \mathcal{O})^\dagger\big) \subseteq (\Gamma^{>})'$. By the proof of [3, Lemma 10.4.3], S has no largest element. Let (a_ρ) be a well-indexed sequence in $1 + \mathcal{O}$ such that $v(s - a_\rho^\dagger)$ is strictly increasing in S as a function of ρ. Then (a_ρ) is a divergent pc-sequence in K, again by the proof of [3, Lemma 10.4.3]. We apply Proposition 3.10 with $G(Y) = sY$ to get an immediate pre-H_T-field extension $K\langle a \rangle$ of K with $a_\rho \rightsquigarrow a$ and $a' = sa$. Note that $a \sim 1$, since each $a_\rho \sim 1$. Let M be an H_T-asymptotic field extension of K and let $b \in M$ with $b \sim 1$ and $b^\dagger = s$. Then $a_\rho^\dagger \rightsquigarrow b^\dagger$, and so $a_\rho \rightsquigarrow b$ by the proof of [3, Lemma 10.4.3]. Proposition 3.10 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath \colon K\langle a \rangle \to M$ that sends a to b. For uniqueness, let $\jmath \colon K\langle a \rangle \to M$ be an arbitrary $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\jmath(a)/b \in C_M^{\times}$ since $\jmath(a)^\dagger = s = b^\dagger$. Since $\jmath(a) \sim 1 \sim b$, we see that $\jmath(a) = b$, so $\jmath = \imath$. By [15, Lemma 7.6], the set $v(a - K)$ has the yardstick property, so if K is ungrounded and λ-free, then $K\langle a \rangle$ is as well by Lemma 3.18. $\qquad \square$

Lemma 4.20. *Let $s \in K$ with $v\big(s - (K^{\times})^\dagger\big) \subseteq \Psi^{\downarrow}$. Then K has a pre-H_T-field extension $K\langle a \rangle$ with $a > 0$ and $a^\dagger = s$ such that for any pre-H_T-field extension M of K and $b \in M^{>}$ with $b^\dagger = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b. Moreover, the extension $K\langle a \rangle$ has the following properties:*

    (1) $va \notin \Gamma$ and $\Gamma_{K\langle a \rangle} = \Gamma \oplus \Lambda va$;
    (2) $\operatorname{res} K\langle a \rangle = \operatorname{res} K$;

(3) Ψ is cofinal in $\Psi_{K\langle a\rangle}$;

(4) a gap in K remains a gap in $K\langle a\rangle$;

(5) if K is ungrounded and λ-free, then so is $K\langle a\rangle$.

Proof. Suppose that b is an element in a pre-H_T-field extension M of K with $b > 0$ and $b^\dagger = s$. Then $vb \notin \Gamma$; otherwise there is $f \in K$ and $u \in \mathcal{O}_M^\times$ with $b/f = u$, so $s - f^\dagger = u^\dagger \asymp u'$ and $v(s - f^\dagger) > \Psi$, a contradiction. Let $y \in K^\times$ with $y \prec b$. Then $y/b \prec 1$ so $y^\dagger < b^\dagger = s$. Likewise if $y \in K^\times$ with $y \succ b$, then $y^\dagger > s$. Thus, vb realizes the cut

$$S := \{vy : y^\dagger > s\} \subseteq \Gamma.$$

Let $K\langle a\rangle$ be a simple $T^{\mathcal{O}}$-extension of K where $a > 0$ and va realizes the cut S. The Wilkie inequality gives $\Gamma_{K\langle a\rangle} = \Gamma \oplus \Lambda va$ and $\operatorname{res} K\langle a\rangle = \operatorname{res} K$. Using Fact 1.9, we equip $K\langle a\rangle$ with the unique T-derivation which extends the derivation on K and satisfies $a^\dagger = s$. If we can show that $K\langle a\rangle$ is a pre-H_T-field, then the embedding property of $K\langle a\rangle$ follows from Fact 1.9, Lemma 2.3 and the discussion above.

To see that $K\langle a\rangle$ is a pre-H_T-field, let $g \in K^\times$ and $\lambda \in \Lambda$ with $ga^\lambda \succ 1$. By Lemma 4.11, it suffices to verify the following:

(i) $(ga^\lambda)^\dagger > 0$;

(ii) $(ga^\lambda)^\dagger \succ f'$ for all $f \in K$ with $f \preccurlyeq 1$;

(iii) $(ga^\lambda)^\dagger \succ F(a)'$ for all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$.

First we deal with (i). If $\lambda = 0$, then $g \succ 1$ and $(ga^\lambda)^\dagger = g^\dagger > 0$. If $\lambda > 0$, then since $ga^\lambda \succ 1$ we have $a \succ g^{-\lambda^{-1}}$, so $s > (g^{-\lambda^{-1}})^\dagger = -\lambda^{-1}g^\dagger$. This gives $\lambda^{-1}g^\dagger + s > 0$, so

$$(ga^\lambda)^\dagger \;=\; g^\dagger + \lambda s \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;>\; 0.$$

On the other hand, if $\lambda < 0$, then $a \prec g^{-\lambda^{-1}}$, so $\lambda^{-1}g^\dagger + s < 0$ and again,

$$(ga^\lambda)^\dagger \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;>\; 0.$$

Note that since

$$(ga^\lambda)^\dagger \;=\; \lambda(\lambda^{-1}g^\dagger + s) \;=\; \lambda\big(s - (g^{-\lambda^{-1}})^\dagger\big) \;\asymp\; s - (g^{-\lambda^{-1}})^\dagger,$$

we have $v(ga^\lambda)^\dagger \in \Psi^\downarrow$. Thus, for (ii) it suffices to note that $h^\dagger \succ f'$ for all $f, h \in K$, since K is a pre-H_T-field. Likewise, for (iii), it suffices to show that $h^\dagger \succ F(a)'$ for all $h \in K$ with $h \succ 1$ and all $\mathcal{L}(K)$-definable functions $F\colon K \to K$ with $F(a) \prec 1$ and $F(a) \notin K$. Suppose toward contradiction that there are F, h for which this does not hold, so $F(a) \prec 1$ but

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)as \;\succcurlyeq\; h^\dagger.$$

By replacing F with $-F$ if necessary, we may assume that $F(a)' > 0$. Since $\operatorname{res} K\langle a\rangle = \operatorname{res} K$, we have $F^{[\partial]}(a) + F'(a)as > uh^\dagger > 0$ for some $u \in K$ with $u \asymp 1$. Take an interval $I \subseteq K^>$ with $a \in I^{K\langle a\rangle}$ such that

$$|F(y)| \;<\; 1, \qquad F^{[\partial]}(y) + F'(y)ys \;>\; uh^\dagger$$

for all $y \in I$. For $y \in I$, we have $F(y) \prec 1$, so $F(y)' = F^{[\partial]}(y) + F'(y)y' \prec h^\dagger$ since K is a pre-H_T-field. This gives

$$\big(F^{[\partial]}(y) + F'(y)ys\big) - \big(F^{[\partial]}(y) + F'(y)y'\big) \;=\; F'(y)y(s - y^\dagger) \;>\; \tfrac{1}{2}uh^\dagger \;>\; 0.$$

In particular, the function $F'(y)y(s - y^\dagger)$ has constant sign on I. By shrinking I, we may assume that $F'(y)$ and y have constant sign on I, so $s - y^\dagger$ has constant sign on I as well. This is a contradiction: if $y \in I$ is greater than a, then $y \succ a$ and $y^\dagger > s$ and if $y \in I$ is less than a, then $y \prec a$ and $y^\dagger < s$.

Now that we know that $K\langle a\rangle$ is a pre-H_T-field extension of K with the required embedding property, all that remains is to check that $K\langle a\rangle$ satisfies properties (1)–(5). We have already verified properties (1) and (2). For (3), let $h \in K\langle a\rangle$ with $h \succ 1$. Then $h \asymp ga^\lambda$ for some $g \in K$ and some $\lambda \in \Lambda$ by (1), so $v(h^\dagger) = v(ga^\lambda)^\dagger$, since $K\langle a\rangle$ is H_T-asymptotic. We have already shown that $v(ga^\lambda)^\dagger \in \Psi^\downarrow$, so $v(h^\dagger) \in \Psi^\downarrow$ as well. As for (4), let β be a gap in K. Then $\beta > \Psi$, so $\beta > \Psi_{K\langle a\rangle}$ since Ψ is cofinal in $\Psi_{K\langle a\rangle}$. Suppose

toward contradiction that β is not a gap in $K\langle a\rangle$, so $\beta = \alpha'$ for some $\alpha \in \Gamma^{>}_{K\langle a\rangle}$. The universal property in Lemma 4.12 gives $\max \Psi_{K\langle a\rangle} = \alpha^\dagger > \Psi$, contradicting that Ψ is cofinal in $\Psi_{K\langle a\rangle}$.

Finally, suppose K is ungrounded and let (ℓ_ρ) be a logarithmic sequence in K with corresponding λ-sequence (λ_ρ). Then $K\langle a\rangle$ is ungrounded since Ψ is cofinal in $\Psi_{K\langle a\rangle}$. It follows that $\Gamma^{<}$ is cofinal in $\Gamma^{<}_{K\langle a\rangle}$. To see this, let $f \in K\langle a\rangle$ with $f \succ 1$ and suppose toward contradiction that $f \prec g$ for all $g \in K$ with $g \succ 1$. Then $f^\dagger \preccurlyeq g^\dagger$ for such g, so $v(f^\dagger) > \Psi$ since K is ungrounded, contradicting (3). Therefore, (ℓ_ρ) remains a logarithmic sequence in $K\langle a\rangle$ and (λ_ρ) remains a λ-sequence in $K\langle a\rangle$. Suppose toward contradiction that K is λ-free and that $\lambda_\rho \rightsquigarrow \lambda \in K\langle a\rangle$. Then $\lambda \notin K$, so $a \in K\langle\lambda\rangle$. This is a contradiction, as $K\langle\lambda\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 2.10 and $va \notin \Gamma$. This proves (5). □

We can use Lemma 4.20 along with Lemma 3.16 and Corollary 4.7 to prove the following extension result for pre-H_T-fields.

Proposition 4.21. *K has an ungrounded ω-free H_T-field extension.*

Proof. By passing to the H_T-field hull of K, we may assume that K is an H_T-field. First, we will show that every H_T-field with asymptotic integration has an H_T-field extension with a gap. Then, we will show that every H_T-field with a gap has a grounded H_T-field extension. Finally, we will show that every grounded H_T-field has an ungrounded ω-free H_T-field extension.

For the first part, suppose K has asymptotic integration. As having asymptotic integration is a property of the asymptotic couple of K, every immediate extension of K also has asymptotic integration. By applying Corollary 4.7 we can pass to a spherically complete extension of K, so we may assume that K is spherically complete. Let (λ_ρ) be a λ-sequence in K, so (λ_ρ) has a pseudolimit $\lambda \in K$ by spherical completeness. The set $v\big(\lambda + (K^\times)^\dagger\big)$ is a cofinal subset of $\Psi^\downarrow$ by Fact 3.15, so Lemma 4.20 gives an H_T-field extension $K\langle a\rangle$ of K with $a^\dagger = -\lambda$. This extension has a gap, namely va, by Lemma 3.16.

Now, assume that K has a gap $\beta \in \Gamma$. Take $s \in K$ with $vs = \beta$ and use Lemma 4.12 to get a grounded H_T-field extension $K\langle a\rangle$ of K with $a' = s$. Finally, if K is grounded, apply Corollary 4.18 to get an ungrounded ω-free H_T-field extension K_ω of K. □

4.5. **Constant field extensions.** Recall that the constant field C is naturally a model of T. In the following proposition, we show that if K is an H_T-field, then we may take extensions of K by constants corresponding to T-extensions of C.

Proposition 4.22. *Let K be an H_T-field and let E be a T-extension of C. Then there is an H_T-field extension L of K where C_L is $\mathcal{L}(C)$-isomorphic to E such that for any H_T-field extension M of K and any $\mathcal{L}(C)$-embedding $\imath\colon C_L \to C_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$ extending $\imath$.*

Proof. It suffices to consider the case $E = C\langle f\rangle$ where $f \notin C$. Let $L = K\langle a\rangle$ be a simple T-extension of K where a realizes the cut

$$(C^{<f} + o)^\downarrow \;=\; \{y \in K : y < \mathcal{O}\} \cup \{c + \varepsilon : c \in C^{<f} \text{ and } \varepsilon \in o\}.$$

We expand L to an $\mathcal{L}^{\mathcal{O}}$-structure by letting

$$\mathcal{O}_L \;:=\; \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

This expansion of L is a $T^{\mathcal{O}}$-extension of K by Fact 2.1. Note that $a \in \mathcal{O}_L$ and $\bar{a} \notin \operatorname{res} K$, so the Wilkie inequality gives $\Gamma_L = \Gamma$. Using Fact 1.9, we extend ∂ uniquely to a T-derivation on L with $a' = 0$.

We claim that L is an H_T-field extension of K. We may assume that $\mathcal{O} \neq K$, since otherwise, K and L both have trivial valuation and derivation, so L is trivially an H_T-field extension of K. To see that L satisfies (H1), let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) > \mathcal{O}_L$. We need to show

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)a' \;=\; F^{[\partial]}(a) \;>\; 0.$$

As $F^{[\partial]}$ is $\mathcal{L}(K)$-definable, it suffices to show that for any subinterval $I \subseteq K$ with $a \in I^L$, there is $y \in I$ with $F^{[\partial]}(y) > 0$. Let I be such a subinterval. Using that $\Gamma_L = \Gamma$ and that $\Gamma^{>}$ has no least element, take $d \in K$

with $F(a) > d > \mathcal{O}_L$. By shrinking I, we arrange that $F(y) > d$ for all $y \in I$. Since $\bar{a} \in \bar{I}^{\operatorname{res} L}$, we see that $\bar{I}$ must be infinite. Thus, $I \cap C$ is infinite, so take $c \in I \cap C$. As $c \in C$, we have $F(c)' = F^{[\partial]}(c)$. As $c \in I$, we have $F(c) > d > \mathcal{O}$, so $F(c)' = F^{[\partial]}(c) > 0$, as desired. Now, let us show that (H2) holds. By (H1), we have $C_L \subseteq \mathcal{O}_L$. Clearly, C_L contains $C\langle a \rangle$. Since $C\langle a \rangle$ is a lift of res L, it is maximal among the elementary $\mathcal{L}$-substructures of L contained in $\mathcal{O}_L$, so $C\langle a \rangle = C_L$ and $\mathcal{O}_L = C_L + \mathfrak{o}_L$; see [8, Remark 2.11 and Theorem 2.12]. This completes the proof that L is an H_T-field. It also tells us that C_L is $\mathcal{L}(C)$-isomorphic to E.

Given an H_T-field extension M of K and an $\mathcal{L}(C)$-embedding $\imath \colon C_L \to M$, there is at most one possible $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\jmath \colon L \to M$ which extends $\imath$, namely the one which sends a to $\imath(a)$. Let us show that this is actually an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. By assumption, a and $\imath(a)$ realize the same cut over C. Since $a - c \notin \mathfrak{o}_L$ and $\imath(a) - c \notin \mathfrak{o}_M$ for all $c \in C$, this assumption gives that a and $\imath(a)$ realize the same cut over $\mathcal{O}$. As $a \in \mathcal{O}_L$ and $\imath(a) \in \mathcal{O}_M$, we see that a and $\imath(a)$ realize the same cut over K, so $\jmath$ is an $\mathcal{L}(K)$-embedding. Fact 1.9 ensures that $\jmath$ is an $\mathcal{L}^{\partial}(K)$-embedding. To see that $\jmath$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, let $f \in K\langle a \rangle$. If $f \in \mathcal{O}_L$, then $|f| < c$ for some $c \in C_L$, so $|\jmath(f)| < \imath(c) \in C_M$, which gives $\jmath(f) \in \mathcal{O}_M$. Conversely, if $f \notin \mathcal{O}_L$, then $|f| > d$ for some $d \in K$ with $d > \mathcal{O}$, so $|\jmath(f)| > d$, which gives $\jmath(f) \notin \mathcal{O}_M$. $\qquad\square$

5. Liouville closed H_T-fields

In this section, we assume that K is an H_T-field. For now, we drop the assumption that T is power bounded (though we will re-introduce this assumption at the beginning of Subsection 5.1). Recall that K is *Liouville closed* if for each $y \in K$, there is $f \in K$ and $g \in K^\times$ with $f' = g^\dagger = y$.

Definition 5.1. A **T-Liouville extension** of K is an H_T-field extension L of K where

(1) $C_L = C$, and
(2) each $a \in L$ is contained in an H_T-subfield $K\langle t_1, \ldots, t_n \rangle \subseteq L$ where for $i = 1, \ldots, n$, either $t_i' \in K\langle t_1, \ldots, t_{i-1} \rangle$ or $t_i \neq 0$ and $t_i^\dagger \in K\langle t_1, \ldots, t_{i-1} \rangle$.

Below we list some easily verified facts about T-Liouville extensions of K.

Fact 5.2.

(1) If L is a T-Liouville extension of K and M is a T-Liouville extension of L, then M is a T-Liouville extension of K.
(2) If M is a T-Liouville extension of K and L is an H_T-field extension of K contained in M, then M is a T-Liouville extension of L.
(3) If $(L_i)_{i \in I}$ is an increasing chain of T-Liouville extensions of K, then the union $\bigcup_{i \in I} L_i$ is a T-Liouville extension of K.
(4) Every T-Liouville extension of K has the same cardinality as K.

Lemma 5.3. *Suppose K is Liouville closed. Then K has no proper T-Liouville extensions.*

Proof. Let L be a T-Liouville extension of K, and let $a \in L$. We will show that $a \in K$. By definition, a is contained in an H_T-subfield $K\langle t_1, \ldots, t_n \rangle \subseteq L$ where for $i = 1, \ldots, n$, either $t_i' \in K\langle t_1, \ldots, t_{i-1} \rangle$ or $t_i^\dagger \in K\langle t_1, \ldots, t_{i-1} \rangle$. We show by induction that $K\langle t_1, \ldots, t_i \rangle = K$ for each $i \leqslant n$. Fix $i \leqslant n$ and suppose that $K\langle t_1, \ldots, t_{i-1} \rangle = K$. If $t_i' \in K$, then since K is Liouville closed, there is $f \in K$ with $f' = t_i'$. Then $t_i = f + c$ for some $c \in C_L = C$, so $t_i \in K$ as well. Likewise, if $t_i^\dagger \in K$, then there is $g \in K^\times$ with $g^\dagger = t_i^\dagger$, so $t_i = cg$ for some $c \in C_L^\times = C^\times$, again giving $t_i \in K$. $\qquad\square$

Definition 5.4. A **T-Liouville closure** of K is a T-Liouville extension of K which is Liouville closed.

Corollary 5.5. *Let L be a Liouville closed H_T-field extension of K. If L is a T-Liouville closure of K, then L has no proper Liouville closed H_T-subfields which contain K. If L has no proper Liouville closed H_T-subfields which contain K and $C_L = C$, then L is a T-Liouville closure of K.*

Proof. Suppose that L is a T-Liouville closure of K and let M be a Liouville closed H_T-subfield of L containing K. Then L is a T-Liouville extension of M by Fact 5.2, so $M = L$ by Lemma 5.3. Now suppose that $C_L = C$ and that L is not a T-Liouville closure of K. We will find a proper Liouville closed subfield of L containing K. Let M be a maximal T-Liouville extension of K contained in L (we know that M exists by Fact 5.2 and Zorn's lemma). We claim that M is Liouville closed. Let $s \in M$ and take $a \in L$ and $b \in L^\times$ with $a' = b^\dagger = s$. Since $C_{M\langle a \rangle}, C_{M\langle b \rangle} \subseteq C_L = C$, we see that $M\langle a \rangle$ and $M\langle b \rangle$ are both T-Liouville extensions of M, and therefore equal to M by maximality and Fact 5.2. Thus, every element of M has an integral in M and an exponential integral in $M^\times$. $\qquad\square$

5.1. T-Liouville towers.

Assumption 5.6. *For the remainder of this section, we assume that T is power bounded.*

Definition 5.7. A **T-Liouville tower on K** is a strictly increasing chain $(K_\mu)_{\mu \leqslant \nu}$ of H_T-fields such that:

(1) $K_0 = K$;
(2) if $\mu \leqslant \nu$ is an infinite limit ordinal, then $K_\mu = \bigcup_{\eta < \mu} K_\eta$;
(3) if $\mu < \nu$, then $K_{\mu+1} = K_\mu \langle a_\mu \rangle$ with $a_\mu \notin K_\mu$ and one of the following holds:
    (a) $a'_\mu = s_\mu \in K_\mu$ with $a_\mu \prec 1$ and vs_μ is a gap in K_μ;
    (b) $a'_\mu = s_\mu \in K_\mu$ with $a_\mu \succ 1$ and vs_μ is a gap in K_μ;
    (c) $a'_\mu = s_\mu \in K_\mu$ with $vs_\mu = \max \Psi_{K_\mu}$;
    (d) $a'_\mu = s_\mu \in K_\mu$ with $a_\mu \prec 1$, $vs_\mu \in (\Gamma^>_{K_\mu})'$, and $s_\mu \notin \partial \mathcal{O}_{K_\mu}$;
    (e) $a'_\mu = s_\mu \in K_\mu$ with $v(s_\mu - \partial K_\mu) \subseteq (\Gamma^<_{K_\mu})'$;
    (f) $a^\dagger_\mu = s_\mu \in K_\mu$ with $a_\mu \sim 1$, $vs_\mu \in (\Gamma^>_{K_\mu})'$, and $s_\mu \neq y^\dagger$ for all $y \in K^\times_\mu$;
    (g) $a^\dagger_\mu = s_\mu \in K_\mu$ with $a_\mu > 0$ and $v\big(s_\mu - (K^\times_\mu)^\dagger\big) \subseteq \Psi^\downarrow_{K_\mu}$.

The H_T-field K_ν is called the **top of the tower** $(K_\mu)_{\mu \leqslant \nu}$.

Let $(K_\mu)_{\mu \leqslant \nu}$ be a T-Liouville tower on K. Note that (a), (b), (c), (f), and (g) correspond to Lemmas 4.12, 4.13, 4.16, 4.19, and 4.20 and that (d) and (e) correspond to Corollaries 4.9 and 4.10, respectively. In each of these extensions, we have $\operatorname{res} K_{\mu+1} = \operatorname{res} K_\mu$, so $C_{K_{\mu+1}} = C_{K_\mu}$ by Corollary 4.5. Thus, $K_{\mu+1}$ is a T-Liouville extension of K_μ for each $\mu \leqslant \nu$. Using also Fact 5.2, we see that each K_μ is a T-Liouville extension of K. If $(K_\mu)_{\mu \leqslant \nu}$ cannot be extended to a larger T-Liouville tower $(K_\mu)_{\mu \leqslant \nu+1}$ on K, then $(K_\mu)_{\mu \leqslant \nu}$ is said to be *maximal*. Since each T-Liouville extension of K has the same cardinality as K, maximal T-Liouville towers on K exist by Zorn's lemma.

Lemma 5.8. *Let L be the top of a maximal T-Liouville tower on K. Then L is Liouville closed and, therefore, L is a T-Liouville closure of K.*

Proof. By (a) and (b), L does not have a gap, and by (c), L is not grounded, so L has asymptotic integration by Fact 3.4. Let $s \in L$. We will show that s has an integral in L and an exponential integral in $L^\times$. We have $v(s - \partial L) \nsubseteq (\Gamma^<_L)'$ by (e), so there is $y \in L$ with $v(s - y') > (\Gamma^<_L)'$. Since L has asymptotic integration, we have $v(s - y') \in (\Gamma^>_L)'$ so by (d), there is $f \in \mathcal{O}_L$ with $f' = s - y'$. Then $s = (f + y)'$. Likewise, by (g) there is $b \in L^\times$ with $v(s - b^\dagger) > \Psi^\downarrow_L$. Asymptotic integration gives $v(s - b^\dagger) \in (\Gamma^>_L)'$, and we may take $g \in L^\times$ with $g \sim 1$ and $g^\dagger = s - b^\dagger$ by (f). Then $s = (bg)^\dagger$. $\qquad\square$

Lemma 5.8 gives the existence of T-Liouville closures under our standing assumption that T is power bounded. The rest of the section is focused on uniqueness.

Lemma 5.9. *Let L be a Liouville closed H_T-field extension of K and let $(K_\mu)_{\mu \leqslant \nu}$ be a T-Liouville tower on K. Suppose that $(K_\mu)_{\mu \leqslant \nu}$ is a tower in L, that is, each K_μ is an H_T-subfield of L. Suppose also that $(K_\mu)_{\mu \leqslant \nu}$ cannot be extended to a T-Liouville tower $(K_\mu)_{\mu \leqslant \nu+1}$ in L. Then $(K_\mu)_{\mu \leqslant \nu}$ is a maximal T-Liouville tower on K.*

Proof. By Fact 5.2 and Lemma 5.3, it suffices to show that K_ν is Liouville closed. If vs is a gap in K_ν for some $s \in K_\nu$, then L contains an element a with $a' = s$. By subtracting a constant from a, we may assume that $a \not\asymp 1$. By Lemma 4.12 (if $a \prec 1$) or Lemma 4.13 (if $a \succ 1$), we see that $K_\nu\langle a \rangle \subseteq L$ is a T-Liouville extension of K_ν, contradicting the maximality of $(K_\mu)_{\mu \leqslant \nu}$ in L. Thus, K_ν has no gap and likewise, Lemma 4.16 shows that K_ν is ungrounded, so K_ν has asymptotic integration by Fact 3.4.

Fix $s \in K_\nu$. If $v(s - \partial K_\nu) \subseteq (\Gamma^<_{K_\nu})'$, then $K\langle f \rangle$ is a T-Liouville extension of K_ν contained in L for any $f \in L$ with $f' = s$ by Corollary 4.10, contradicting the maximality of $(K_\mu)_{\mu \leqslant \nu}$ in L. Therefore, we may take $y \in K_\nu$ with $v(s - y') > (\Gamma^<_{K_\nu})'$. As K_ν has asymptotic integration, we have $v(s - y') \in (\Gamma^>_{K_\nu})'$. If $s - y' \notin \partial o_{K_\nu}$, then $K\langle g \rangle$ is a T-Liouville extension of K_ν contained in L for any $g \in o_L$ with $g' = s - y'$ by Corollary 4.9, again contradicting the maximality of $(K_\mu)_{\mu \leqslant \nu}$ in L. Thus, $s - y' \in \partial o_{K_\nu}$, so $s \in \partial K_\nu$. A similar argument, using Lemmas 4.19 and 4.20 shows that s has an exponential integral in $K_\nu^\times$. $\square$

Lemmas 5.8 and 5.9 can be used to remove the assumption "$C_L = C$" from Corollary 5.5 under our current assumption of power boundedness.

Corollary 5.10. *Let L be a Liouville closed H_T-field extension of K. Then L is a T-Liouville closure of K if and only if L has no proper Liouville closed H_T-subfields which contain K.*

Proof. One implication holds by Corollary 5.5. For the other, suppose that L is not a T-Liouville closure of K and let $(K_\mu)_{\mu \leqslant \nu}$ be a maximal T-Liouville tower on K in L. Then K_ν is a T-Liouville closure of K by Lemmas 5.8 and 5.9. In particular, K_ν is a proper Liouville closed H_T-subfield of L containing K. $\square$

5.2. **λ-freeness and the uniqueness of T-Liouville closures.** Whether K has a unique T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism is closely tied to the existence of gaps, which is in turn related to λ-freeness.

Lemma 5.11. *Let $(K_\mu)_{\mu \leqslant \nu}$ be a T-Liouville tower on K and suppose K_μ does not have a gap for all $\mu < \nu$. Then K_ν embeds over K into any Liouville closed H_T-field extension of K.*

Proof. Let M be a Liouville closed H_T-field extension of K. We will construct an increasing chain of $\mathcal{L}^{\mathcal{O},\partial}(K)$-embeddings $(\imath_\mu : K_\mu \to M)_{\mu \leqslant \nu}$. Let $\imath_0 : K_0 \to M$ be the identity on K, and take increasing unions at limits. For successors, fix $\mu < \nu$ and let $\imath_\mu : K_\mu \to M$ be an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Since K_μ has no gap, $K_{\mu+1}$ is an extension of type (c), (d), (e), (f), or (g). The embedding properties in Lemmas 4.16, 4.19 and 4.20 and Corollaries 4.9 and 4.10 give an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath_{\mu+1} : K_{\mu+1} \to M$ extending $\imath_\mu$. $\square$

Proposition 5.12. *Suppose K is ungrounded and λ-free. Then K has a T-Liouville closure L which embeds over K into any Liouville closed H_T-field extension of K. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to L.*

Proof. Let $(K_\mu)_{\mu \leqslant \nu}$ be a maximal T-Liouville tower on K. We will prove by induction on $\mu \leqslant \nu$ that each K_μ is ungrounded and λ-free. This holds when $\mu = 0$ by assumption and if $\mu \leqslant \nu$ is an infinite limit ordinal, then this follows from Lemma 3.13. Let $\mu < \nu$ and suppose that K_μ is ungrounded and λ-free. Then K_μ has no gap, so $K_{\mu+1}$ must be an extension of type (d), (e), (f), or (g). Then $K_{\mu+1}$ is ungrounded and λ-free by Lemmas 4.19 and 4.20 and Corollaries 4.9 and 4.10. Set $L := K_\nu$, so L is a T-Liouville closure of K, and let M be a Liouville closed H_T-field extension of K. By Lemma 5.11, there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\imath : L \to M$. Moreover, if M is a T-Liouville closure of K, then $\imath(L)$ is a Liouville closed H_T-subfield of M containing K, so $\imath(L) = M$ by Corollary 5.5. Thus, L is unique up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. $\square$

Proposition 5.13. *Suppose K is grounded. Then K has a T-Liouville closure L which embeds over K into any Liouville closed H_T-field extension of K. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to L.*

Proof. Let K_ω be as in Corollary 4.18. Then K_ω is the union of an increasing chain of T-Liouville extensions of K, so K_ω is a T-Liouville extension of K by Fact 5.2. Moreover, K_ω is ungrounded and λ-free, so K_ω has a T-Liouville closure L which embeds over K_ω into any Liouville closed H_T-field extension of K_ω by

Proposition 5.12. Then L is a T-Liouville closure of K as well, since K_ω is a T-Liouville extension of K. Let M be a Liouville closed H_T-field extension of K. Then M is closed under taking d-logarithms, so Corollary 4.18 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_\omega \to M$ which further extends to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$. As in Proposition 5.12, uniqueness follows from this embedding property and Corollary 5.5. $\qquad\square$

5.3. **Gaps and the nonuniqueness of T-Liouville closures.** If K has a gap vs, then we have a choice to make. Either we can adjoin an integral a of s with $a \prec 1$, as is done in Lemma 4.12, or we can adjoin an integral b of s with $b \succ 1$, as in Lemma 4.13. This "fork in the road" prevents K from having a unique T-Liouville closure, but as we will see below, this is really the only obstruction to uniqueness.

Proposition 5.14. *Let $\beta \in \Gamma$ be a gap in K. Then K has T-Liouville closures L_1 and L_2 with $\beta \in (\Gamma_{L_1}^>)'$ and $\beta \in (\Gamma_{L_2}^<)'$. Let M be a Liouville closed H_T-field extension of K. If $\beta \in (\Gamma_M^>)'$, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. Likewise, if $\beta \in (\Gamma_M^<)'$, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

Proof. Let $s \in K$ with $vs = \beta$. Let $K_1 := K\langle a\rangle$ be the H_T-field extension of K given in Lemma 4.12, so $a \prec 1$ and $a' = s$, and let $K_2 := K\langle b\rangle$ be the H_T-field extension of K given in Lemma 4.13, so $b \succ 1$ and $b' = s$. Then K_1 is grounded, so it has a T-Liouville closure L_1 which embeds over K_1 into any Liouville closed H_T-field extension of K_1 by Proposition 5.13. Likewise, K_2 has a T-Liouville closure L_2 which embeds over K_2 into any Liouville closed H_T-field extension of K_2. Now let M be a Liouville closed H_T-field extension of K. If $\beta \in (\Gamma_M^>)'$, then the embedding property in Lemma 4.12 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K_1 \to M$, which in turn extends to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. If $\beta \in (\Gamma_M^<)'$, then using the embedding property in Lemma 4.13 instead, we get an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. If M is a T-Liouville closure of K, then M is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2 by Corollary 5.5 since M contains the $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic image of either L_1 or L_2 as a Liouville closed H_T-subfield. $\qquad\square$

We can use Lemma 3.16 to show that H_T-fields with asymptotic integration which are not λ-free also have two distinct T-Liouville closures.

Proposition 5.15. *Suppose that K has asymptotic integration and is not λ-free. Then K has T-Liouville closures L_1 and L_2 which are not $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic. If M is a Liouville closed H_T-field extension of K, then there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of either L_1 or L_2 into M. Any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

Proof. Let (λ_ρ) be a λ-sequence in K with pseudolimit $\lambda \in K$, so $v(\lambda + (K^\times)^\dagger)$ is a cofinal subset of $\Psi^\downarrow$ by Fact 3.15. Lemma 4.20 gives an H_T-field extension $K\langle a\rangle$ of K with $a > 0$ and $a^\dagger = -\lambda$. By Lemma 3.16, va is a gap in $K\langle a\rangle$. By Proposition 5.14, $K\langle a\rangle$ has T-Liouville closures L_1 and L_2 with $va \in (\Gamma_{L_1}^>)'$ and $va \in (\Gamma_{L_2}^<)'$, one of which embeds over $K\langle a\rangle$ into any Liouville closed H_T-field extension of $K\langle a\rangle$. We claim that there is no $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to L_2$; in particular, L_1 and L_2 are nonisomorphic over K. To see this, take $b_1 \in L_1$ and $b_2 \in L_2$ with $b_1 \prec 1$, $b_2 \succ 1$, and $b_1' = b_2' = a$. Then $(b_1')^\dagger = (b_2')^\dagger = a^\dagger = -\lambda$. Suppose toward contradiction that $\iota\colon L_1 \to L_2$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Then $\iota(b_1')^\dagger = (b_2')^\dagger$ so $\iota(b_1) = c_1 b_2 + c_2$ for some $c_1, c_2 \in C_{L_2}$ with $c_1 \neq 0$. Since $b_2 \succ 1$, this gives $\iota(b_1) \succ 1$, contradicting that $b_1 \prec 1$.

Let M be a Liouville closed H_T-field extension of K. Lemma 4.20 gives an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a\rangle \to M$ and Proposition 5.14 allows us to extend this embedding to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of either L_1 or L_2 into M. As in Proposition 5.14, we may use Corollary 5.5 to see that any T-Liouville closure of K is $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2. $\qquad\square$

Putting together the above propositions, we can now precisely state our main theorem on the existence and uniqueness of T-Liouville closures.

Theorem 5.16. *If K is grounded or if K is ungrounded and λ-free, then K has exactly one T-Liouville closure up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. If K is ungrounded and not λ-free, then K has exactly two T-Liouville*

closures up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism. For any Liouville closed H_T-field extension M of K, there is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding of some T-Liouville closure of K into M.

6. Logarithmic H_T-fields and logarithmic pre-H_T-fields

In this section, we assume that T is power bounded with field of exponents Λ. We further assume that T defines a restricted exponential function e and that Λ is cofinal in the prime model of T. We let ln denote the compositional inverse of e.

Recall from Subsection 1.3 the language $\mathcal{L}_{\log} = \mathcal{L} \cup \{\log\}$ and the $\mathcal{L}_{\log}$-theory T^{e}, which extends T by axioms stating that log is a surjective logarithm. In this section and the next, we study pre-$H_{T^{\mathrm{e}}}$-fields, $H_{T^{\mathrm{e}}}$-fields, and their extensions. From a valuation-theoretic perspective, it is inconvenient to work with pre-$H_{T^{\mathrm{e}}}$-fields directly, so we instead work with a broader class of $\mathcal{L}_{\log}^{\mathcal{O},\partial}$-structures, called *logarithmic pre-H_T-fields*, where the logarithm is not assumed to be surjective.

Definition 6.1. Let K be a pre-H_T-field and let log be a logarithm on K. We say that K is a **logarithmic pre-H_T-field** if

(LH1) $\log(a)' = a^{\dagger}$ for $a \in K^{>}$, and
(LH2) $\mathcal{O} \subseteq \log(K^{>})$.

We say that K is a **logarithmic H_T-field** if K is a logarithmic pre-H_T-field which is also an H_T-field.

Clearly, every pre-$H_{T^{\mathrm{e}}}$-field is a logarithmic pre-H_T-field and, likewise, every $H_{T^{\mathrm{e}}}$-field is a logarithmic H_T-field. After proving some basic results, we will show in Proposition 6.6 below that a logarithmic (pre)-H_T-field is a (pre)-$H_{T^{\mathrm{e}}}$-field if and only if the logarithm is surjective.

Assumption 6.2. *For the remainder of this section, let $K = (K, \log, \mathcal{O}, \partial)$ be a logarithmic pre-H_T-field.*

For $a \in K^{>}$, axiom (LH1) tells us that $\log a$ is a d-logarithm of a. Thus, we have the following consequence of Corollary 4.3:

Corollary 6.3. *Any logarithmic pre-H_T-field is ungrounded.*

We now investigate the induced logarithms on the residue field and the constant field of K.

Lemma 6.4. *The logarithm on K induces a well-defined logarithm on $\operatorname{res} K$ (also denoted by $\log$). With this induced logarithm, $\operatorname{res} K$ is a model of T^{e}.*

Proof. Let $a \in \mathcal{O}$ with $\bar{a} > 0$. Then $a \asymp 1$, so Lemma 4.2 gives $\log a \in \mathcal{O}$. Since $\mathcal{O}$ is T-convex and ln is $\mathcal{L}(\emptyset)$-definable and continuous at 1, we have $\ln(1 + \mathfrak{o}) \subseteq \mathfrak{o}$ by [8, Lemma 1.13]. Since $a + \mathfrak{o} = a(1 + \mathfrak{o})$, axioms (L1) and (L2) give

$$\log(a + \mathfrak{o}) \ = \ \log a + \log(1 + \mathfrak{o}) \ = \ \log a + \ln(1 + \mathfrak{o}) \ \subseteq \ \log a + \mathfrak{o}.$$

Thus, log induces a well-defined map on $\operatorname{res} K$. It is routine to verify that this induced map is a logarithm on $\operatorname{res} K$, and it remains to show that this induced logarithm is surjective. To see this, let $a \in \mathcal{O}$ and, using (LH2), take $b \in K^{>}$ with $\log b = a$. Since $\log b \in \mathcal{O}$ is a d-logarithm of b, Lemma 4.2 tells us that $b \asymp 1$. Then $\bar{b} > 0$ and $\log \bar{b} = \bar{a}$. $\qquad\square$

Lemma 6.5. *The restriction of the logarithm on K to C is a logarithm on C. With this restricted logarithm, C is a model of T^{e}.*

Proof. For $c \in C^{>}$, we have $\log(c)' = c^{\dagger} = 0$, so $\log c \in C$ as well. Thus, $\log(C^{>}) \subseteq C$, and it follows immediately that $\log |_C$ is a logarithm on C. To see that $\log(C^{>}) = C$, let $a \in C$ and, using that $C \subseteq \mathcal{O} \subseteq \log(K^{>})$, take $f \in K^{>}$ with $\log f = a$. Then $f^{\dagger} = a' = 0$, so $f \in C^{>}$. $\qquad\square$

The natural $\mathcal{L}$-embedding $C \to \operatorname{res} K$ is even an $\mathcal{L}_{\log}$-embedding, where $\operatorname{res} K$ and C are equipped with the logarithms from Lemmas 6.4 and 6.5, respectively. Now, we can say more about the relationship between logarithmic pre-H_T-fields and pre-H_{T^e}-fields.

Proposition 6.6. *The logarithmic pre-H_T-field K is a pre-H_{T^e}-field if and only if $\log(K^>) = K$. Moreover, if K is a logarithmic H_T-field, then K is an H_{T^e}-field if and only if $\log(K^>) = K$.*

Proof. One direction is immediate. For the other, suppose that $\log(K^>) = K$. Then the $\mathcal{L}_{\log}$-reduct of K is a model of T^e. By Lemma 6.4, the residue field $\operatorname{res} K$ with the induced logarithm models T^e as well. It follows that $\mathcal{O}$ is T^e-convex, and it remains to show that ∂ is a T^e-derivation. By Corollary 1.7, any $\mathcal{L}_{\log}(\emptyset)$-definable function on K is given piecewise by a composition of $\mathcal{L}(\emptyset)$-definable functions, log, and exp. By [13, Lemma 2.6], ∂ is compatible with a composition of functions so long as it is compatible with each constituent function. Since ∂ is compatible with log and with all $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-functions by assumption, it remains to show that ∂ is compatible with exp. For $u \in K$, we have $u = \log(\exp u)$, so taking derivatives gives

$$u' \ = \ \log(\exp u)' \ = \ (\exp u)^{-1} \exp(u)'.$$

Thus, $\exp(u)' = \exp(u)u'$, as desired. $\qquad\square$

Next, we provide a short test for checking whether an $\mathcal{L}^{\mathcal{O},\partial}$-embedding of logarithmic pre-H_T-fields is also an $\mathcal{L}_{\log}^{\mathcal{O},\partial}$-embedding, along with a longer proposition on extending the logarithm on K to certain pre-H_T-field extensions of K.

Lemma 6.7. *Let M be a logarithmic pre-H_T-field and let $\imath \colon K \to M$ be an $\mathcal{L}^{\mathcal{O},\partial}$-embedding. Let $f, g \in K^>$ with $f \sim g$. If $\imath(\log g) = \log_M \imath(g)$, then $\imath(\log f) = \log_M \imath(f)$.*

Proof. Take $\varepsilon \in o$ such that $f = g(1 + \varepsilon)$. Then

$$\log f \ = \ \log g + \log(1 + \varepsilon) \ = \ \log g + \ln(1 + \varepsilon).$$

As $\imath$ is an $\mathcal{L}$-embedding, we have $\imath\big(\ln(1 + \varepsilon)\big) = \ln \imath(1 + \varepsilon)$. We have $\imath(\log g) = \log_M \imath(g)$ by assumption, so $\imath(\log f) = \log_M \imath(f)$. $\qquad\square$

Proposition 6.8. *Let L be a pre-H_T-field extension of K with $\operatorname{res} L = \operatorname{res} K$. Let $(a_i)_{i \in I}$ be a family of elements in $L^>$ with $a_i \not\asymp 1$ for each i such that*

$$\Gamma_L \ = \ \Gamma \oplus \bigoplus_{i \in I} \Lambda v a_i,$$

and let $(b_i)_{i \in I}$ be a family of elements in L such that $b_i' = a_i^\dagger$ for each $i \in I$. Then there is a unique logarithm on L extending the logarithm on K such that $\log_L a_i = b_i$ for each $i \in I$. With this logarithm, L is a logarithmic pre-H_T-field extension of K. If M is also a logarithmic pre-H_T-field extension of K and if $\imath \colon L \to M$ is an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, then $\imath$ is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding if and only if $\imath(b_i) = \log_M \imath(a_i)$ for each $i \in I$.

Proof. Let $f \in L^>$. Our assumption on Γ_L and $\operatorname{res} L$ gives

$$f \ = \ g(1 + \varepsilon) \prod_{i \in I} a_i^{\lambda_i}$$

for some $g \in K^>$, some $\varepsilon \in o_L$, and some family $(\lambda_i)_{i \in I}$ of exponents in Λ where only finitely many λ_i are nonzero. Set

$$\log_L f \ := \ \log g + \ln(1 + \varepsilon) + \sum_{i \in I} \lambda_i b_i.$$

It is routine to show that this does not depend on the choice of g. Before we show that $\log_L$ is a logarithm on L, we first note that

$$(\log_L f)' \ = \ (\log g)' + \ln(1 + \varepsilon)' + \sum_{i \in I} \lambda_i b_i' \ = \ g^\dagger + (1 + \varepsilon)^\dagger + \sum_{i \in I} \lambda_i a_i^\dagger \ = \ f^\dagger,$$

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so (LH1) holds. We now turn to verifying (L1)–(L4). A straightforward computation gives $\log_L(f_1 f_2) = \log_L f_1 + \log_L f_2$ for all $f_1, f_2 \in L^>$, so $\log_L$ is a group homomorphism. To see that $\log_L$ is even an ordered group embedding, we assume that $f > 1$, and we need to show that $\log_L f > 0$. If $f \succ 1$, then $(\log_L f)' = f^\dagger > 0$ by (PH1). Since $\log_L f$ is a d-logarithm of f, Lemma 4.2 gives $v(\log_L f) = \chi(vf) < 0$. Thus, $(\log_L f)^\dagger > 0$ as well, so $\log_L f = (\log_L f)'/(\log_L f)^\dagger > 0$. Now, assume that $f \asymp 1$, so each $\lambda_i = 0$ and $f = g(1 + \varepsilon)$. If $f \not\sim 1$, then $\bar{g} = \bar{f} > 1$, so $\log \bar{g} > 0$ by Lemma 6.4. Thus, $\log g > \mathcal{o}_L$, and since $\ln(1 + \varepsilon) \in \mathcal{o}_L$, we have

$$\log_L f \;=\; \log g + \ln(1 + \varepsilon) \;\sim\; \log g \;>\; 0.$$

If $f \sim 1$, then we may assume that $g = 1$, so $\varepsilon > 0$ and $\log_L f = \ln(1 + \varepsilon) > 0$.

For (L2), we assume that $\mathrm{e}(-1) \leqslant f \leqslant \mathrm{e}(1)$, and we need to show that $\log_L f = \ln f$. Our assumption on f gives that each $\lambda_i = 0$ and $\mathrm{e}(-1) \leqslant g \leqslant \mathrm{e}(1)$. Then

$$\log_L f \;=\; \log g + \ln(1 + \varepsilon) \;=\; \ln g + \ln(1 + \varepsilon) \;=\; \ln f.$$

For (L3), let $\lambda \in \Lambda$ with $\lambda > 1$, and assume that $f \geqslant \lambda^2$. We need to show that $f \geqslant \lambda \log_L f$. If $f \succ 1$, then $v(\log_L f) = \chi(vf) > vf$ by [3, Lemma 9.2.18], so $f \succ \log_L f$. In particular, $f \geqslant \lambda \log_L f$. Thus, we may assume that $f \asymp 1$, so each $\lambda_i = 0$ and $f = g(1 + \varepsilon)$, where $g \asymp 1$. Lemma 1.2 gives $\varepsilon = \mathrm{e}(\ln(1 + \varepsilon)) - 1 \geqslant \ln(1 + \varepsilon)$, so

$$\lambda \log_L f \;=\; \lambda \log g + \lambda \ln(1 + \varepsilon) \;\leqslant\; \lambda \log g + \lambda \varepsilon. \tag{6.1}$$

If $f \sim \lambda^2$, then we may arrange that $g = \lambda^2$ and $\varepsilon \geqslant 0$. Thus, $\lambda \log g = \lambda \log \lambda^2 \leqslant \lambda^2$ and $\lambda \varepsilon \leqslant \lambda^2 \varepsilon$. Combined with (6.1), this gives $\lambda \log_L f \leqslant \lambda^2 + \lambda^2 \varepsilon = \lambda^2(1 + \varepsilon) = f$, as desired. If $f \not\sim \lambda^2$, then $\bar{g} = \bar{f} > \lambda^2$, so Lemmas 1.5 and 6.4 give $\bar{g} > \lambda \log \bar{g}$. Thus, $g + (g - \lambda)\varepsilon > \lambda \log g$ since $(g - \lambda)\varepsilon \prec 1$. Again, (6.1) gives

$$\lambda \log_L f \;<\; g + (g - \lambda)\varepsilon + \lambda \varepsilon \;=\; g(1 + \varepsilon) \;=\; f.$$

For (L4) let $\rho \in \Lambda$. Then $f^\rho = g^\rho (1 + \varepsilon)^\rho \prod_{i \in I} a_i^{\rho \lambda_i}$, so

$$\log_L f^\rho \;=\; \log g^\rho + \ln\big((1 + \varepsilon)^\rho\big) + \sum_{i \in I} \rho \lambda_i b_i \;=\; \rho \log g + \rho \ln(1 + \varepsilon) + \rho \sum_{i \in I} \lambda_i b_i \;=\; \rho \log_L f,$$

where the equality $\ln\big((1 + \varepsilon)^\rho\big) = \rho \ln(1 + \varepsilon)$ holds by [14, Lemma 6.4.1].

Finally, for (LH2), let $a \in \mathcal{O}_L$ and, using that $\operatorname{res} L = \operatorname{res} K$, take $b \in \mathcal{O}$ with $a - b \prec 1$. Then $b \in \log(K^>) \subseteq \log(L^>)$, so $a \in \log(L^>)$ by Lemma 1.4.

Now let M be a logarithmic pre-H_T-field extension of K and let $\imath \colon L \to M$ be an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding. Clearly, if $\imath$ is an $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding, then $\imath(b_i) = \log_M \imath(a_i)$ for each $i \in I$. For the other implication, we assume that $\imath(b_i) = \log_M \imath(a_i)$ for each $i \in I$, and we need to show that $\imath(\log_L f) = \log_M \imath(f)$, where f is as above. Using Lemma 6.7 and the fact that $f \sim g \prod_{i \in I} a_i^{\lambda_i}$, we may assume that $f = g \prod_{i \in I} a_i^{\lambda_i}$. Since

$$\log_L \Big(g \prod_{i \in I} a_i^{\lambda_i} \Big) \;=\; \log_L g + \sum_{i \in I} \lambda_i \log_L(a_i)$$

and since $g \in K$, this further reduces to showing that $\imath(\log_L a_i) = \log_M \imath(a_i)$ for each $i \in I$. This holds by our assumption, since $\log_L a_i = b_i$ for each i. Uniqueness of $\log_L$ follows from this embedding property by taking $\imath \colon L \to L$ to be the identity map. $\qquad\square$

For the remainder of this article, we will just write $\log$ instead of $\log_L$ when the logarithmic pre-H_T-field L is clear from context. The conditions on $\operatorname{res} L$ and Γ_L in the above proposition are always satisfied when L is an immediate pre-H_T-field extension of K.

Corollary 6.9. *Let L be an immediate pre-H_T-field extension of K. Then there is a unique logarithm on L extending the logarithm on K, and with this logarithm, L is a logarithmic pre-H_T-field extension of K. If M is also a logarithmic pre-H_T-field extension of K, then any $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $L \to M$ is necessarily an $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding.*

6.1. **The exponential closure.** In this subsection, we prove that every logarithmic pre-H_T-field has a minimal logarithmic pre-H_T-field extension with a surjective logarithm. Below is the key Lemma.

Lemma 6.10. *Let $f \in K \setminus \log(K^>)$. Then K has a logarithmic pre-H_T-field extension $K\langle a \rangle$ with $a > 0$ and $\log a = f$ such that for any logarithmic pre-H_T-field extension M of K with $f \in \log(M^>)$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding $K\langle a \rangle \to M$. Moreover, the extension $K\langle a \rangle$ has the following properties:*

(1) $va \notin \Gamma$ and $\Gamma_{K\langle a \rangle} = \Gamma \oplus \Lambda va$;
(2) $\operatorname{res} K\langle a \rangle = \operatorname{res} K$;
(3) Ψ is cofinal in $\Psi_{K\langle a \rangle}$;
(4) a gap in K remains a gap in $K\langle a \rangle$;
(5) if K is λ-free, then so is $K\langle a \rangle$.

Proof. We claim that $v\big(f' - (K^\times)^\dagger\big) \subseteq \Psi^\downarrow$. Suppose not and take $g \in K^\times$ with $v(f' - g^\dagger) > \Psi$. By replacing g with $-g$ if necessary, we may assume that $g > 0$, so $v(f' - g^\dagger) = v(f - \log g)' > \Psi$. Since Ψ is cofinal in $(\Gamma^<)'$, we have $f - \log g \in \mathcal{O} \subseteq \log(K^>)$. Take $h \in K^>$ with $f - \log g = \log h$. Then $f = \log(gh)$, a contradiction.

With this claim out of the way, we apply Lemma 4.20 with f' in place of s to get a pre-H_T-field $K\langle a \rangle$ extending K with $a^\dagger = f'$ which has properties (1)–(5) above. By property (1) and Proposition 6.8, there is a unique logarithm on $K\langle a \rangle$ with $\log a = f$ making $K\langle a \rangle$ a logarithmic pre-H_T-field extension of K. Now let M be a logarithmic pre-H_T-field extension of K with $f \in \log(M^>)$ and set $b := \exp f \in M$. Then $b^\dagger = f'$, so Lemma 4.20 gives a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle a \rangle \to M$ that sends a to b. By the uniqueness part of Proposition 6.8, this is even an $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding. Since any $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding $K\langle a \rangle \to M$ must send a to $b = \exp f$, this embedding is unique, even without the requirement that a be sent to b. $\qquad\square$

Theorem 6.11 below follows by iterating Lemma 6.10 (we also use that an increasing union of λ-free logarithmic H_T-fields is λ-free; see Lemma 3.13).

Theorem 6.11. *K has a logarithmic pre-H_T-field extension K^e with a surjective logarithm such that for any logarithmic pre-H_T-field extension M of K with a surjective logarithm, there is a unique $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding $K^e \to M$. The extension K^e has the following properties:*

(1) $\operatorname{res} K^e = \operatorname{res} K$;
(2) Ψ is cofinal in Ψ_{K^e};
(3) a gap in K remains a gap in K^e;
(4) if K is λ-free, then so is K^e.

We refer to K^e as the **exponential closure of K**. By Proposition 6.6, the extension K^e is a pre-H_{T^e}-field. If K is a logarithmic H_T-field, then K^e is an H_{T^e}-field by (1) and Corollary 4.5. The universal property in Theorem 6.11 gives that K^e is unique up to unique $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-isomorphism. This also gives the aforementioned minimality: if M is a logarithmic pre-H_T-subfield of K^e containing K with $M = \log(M^>)$, then $M = K^e$.

6.2. **Adjoining integrals and the logarithmic H_T-field hull.** In this subsection, we prove variants of the results in Subsection 4.1. We begin with the following immediate consequences of Corollaries 4.9, 4.10, and 6.9:

Corollary 6.12. *Let $s \in K$ with $vs \in (\Gamma^>)'$ and $s \notin \partial\mathcal{O}$. Then K has an immediate logarithmic pre-H_T-field extension $K\langle a \rangle$ with $a \prec 1$ and $a' = s$ such that for any logarithmic pre-H_T-field extension M of K with $s \in \partial\mathcal{O}_M$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding $K\langle a \rangle \to M$. If K is λ-free, then so is $K\langle a \rangle$.*

Corollary 6.13. *Let $s \in K$ with $v(s - \partial K) \subseteq (\Gamma^<)'$. Then K has an immediate logarithmic pre-H_T-field extension $K\langle a \rangle$ with $a' = s$ such that for any logarithmic pre-H_T-field extension M of K and $b \in M$ with $b' = s$, there is a unique $\mathcal{L}^{\mathcal{O},\partial}_{\log}(K)$-embedding $K\langle a \rangle \to M$ sending a to b. If K is λ-free, then so is $K\langle a \rangle$.*

Now we show how to find an integral for $s \in K$ when vs is a gap. We can't use Lemma 4.12 directly, as the pre-H_T-field extension constructed in that lemma is grounded, so it does not admit a logarithm. We rectify this issue by invoking the extension in Corollary 4.18.

Lemma 6.14. *Let $s \in K$ and suppose vs is a gap in K. Then K has a logarithmic pre-H_T-field extension $K\langle a \rangle_\omega$ with $a \prec 1$ and $a' = s$ such that for any logarithmic pre-H_T-field extension M of K with $s \in \partial \mathcal{O}_M$, there is a unique $\mathcal{L}_{\log}^{\mathcal{O}, \partial}(K)$-embedding $K\langle a \rangle_\omega \to M$. The logarithmic pre-$H_T$-field $K\langle a \rangle_\omega$ is ω-free with $\operatorname{res} K\langle a \rangle_\omega = \operatorname{res} K$.*

Proof. Lemma 4.12 provides a grounded pre-H_T-field extension $K\langle a \rangle$ of K with $a \prec 1$, $a' = s$, and $\operatorname{res} K\langle a \rangle = \operatorname{res} K$. Applying Corollary 4.18 with $|a|^{-1}$ in place of s, we further extend $K\langle a \rangle$ to an ω-free pre-H_T-field $K\langle a \rangle_\omega$ with $\operatorname{res} K\langle a \rangle_\omega = \operatorname{res} K$. Lemma 4.12 and the proof of Corollary 4.18 tell us that

$$\Gamma_{K_\omega} \ = \ \Gamma \oplus \bigoplus_n \Lambda v a_n, \qquad \Psi_{K_\omega} \ = \ \Psi \cup \{v a_0^\dagger, v a_1^\dagger, \dots\}, \qquad \Psi \ < \ v a_0^\dagger \ < \ v a_1^\dagger \ < \ \cdots,$$

where $a_0 = |a|^{-1}$ and $a'_{n+1} = a_n^\dagger$ for each n. Using the pre-H_T-field axioms, one can easily check that each a_n is positive and infinite, so by Proposition 6.8, there is a unique logarithm on $K\langle a \rangle_\omega$ which extends the logarithm on K such that $\log a_n = a_{n+1}$ for each n. Let M be a logarithmic pre-H_T-field extension of K with $s \in \partial \mathcal{O}_M$. By Lemma 4.12, there is a unique $\mathcal{L}^{\mathcal{O}, \partial}(K)$-embedding $\imath \colon K\langle a \rangle \to M$. Let $b_0 := \imath(a_0) \in M$ and for each n, let $b_{n+1} := \log b_n$. By the proof of Corollary 4.18, the embedding $\imath$ extends to an $\mathcal{L}^{\mathcal{O}, \partial}(K)$-embedding $K\langle a \rangle_\omega \to M$ which sends a_n to b_n for each n. By Proposition 6.8, this is even an $\mathcal{L}_{\log}^{\mathcal{O}, \partial}$-embedding and, as an $\mathcal{L}_{\log}^{\mathcal{O}, \partial}$-embedding, it is unique. $\qquad \square$

If K is a logarithmic H_T-field with a gap, then we can instead use Lemma 4.13 and apply Corollary 4.18 with $|a|$ in place of s to show the following:

Lemma 6.15. *Let K be a logarithmic H_T-field, let $s \in K$, and suppose vs is a gap in K. Then K has a logarithmic H_T-field extension $K\langle a \rangle_\omega$ with $a \succ 1$ and $a' = s$ such that for any logarithmic pre-H_T-field extension M of K and $b \in M$ with $b \succ 1$ and $b' = s$, there is a unique $\mathcal{L}_{\log}^{\mathcal{O}, \partial}(K)$-embedding $K\langle a \rangle \to M$ sending a to b. The logarithmic H_T-field $K\langle a \rangle$ is ω-free with $\operatorname{res} K\langle a \rangle_\omega = \operatorname{res} K$.*

The following proposition has the same proof as Theorem 4.14, except we use Lemma 6.14 in place of Lemma 4.12, and we use Corollary 6.12 in place of Corollary 4.9.

Proposition 6.16. *K has a logarithmic H_T-field extension $H_T^{\log}(K)$ with $\operatorname{res} H_T^{\log}(K) = \operatorname{res} K$ such that for any logarithmic H_T-field extension M of K, there is a unique $\mathcal{L}_{\log}^{\mathcal{O}, \partial}(K)$-embedding $H_T^{\log}(K) \to M$.*

6.3. **Constant field extensions.** We end this section with a proposition on extending logarithmic H_T-fields by constants.

Proposition 6.17. *Let K be a logarithmic H_T-field and let E be a T^{e}-extension of C. Then there is a logarithmic H_T-field extension L of K where C_L is $\mathcal{L}_{\log}(C)$-isomorphic to E such that for any logarithmic H_T-field extension M of K and any $\mathcal{L}_{\log}(C)$-embedding $\imath \colon C_L \to C_M$, there is a unique $\mathcal{L}_{\log}^{\mathcal{O}, \partial}(K)$-embedding $L \to M$ extending $\imath$.*

Proof. Let L be the H_T-field extension of K given by Proposition 4.22, so C_L is $\mathcal{L}(C)$-isomorphic to E. By the proof of Proposition 4.22, we also have $\Gamma_L = \Gamma$. Using the $\mathcal{L}(C)$-isomorphism $C_L \to E$, we pull the logarithm on E back to a logarithm on C_L, and we view C_L as a model of T^{e} with this logarithm. We need to extend this logarithm on C_L to a logarithm on L. Let $f \in L^>$ and take $g \in K^>$ with $f \asymp g$. Take $c \in C_L^>$ with $f/g \sim c$, and take $\varepsilon \in \mathcal{O}_L$ with $f = cg(1 + \varepsilon)$. Set

$$\log f \ := \ \log g + \log c + \ln(1 + \varepsilon),$$

where $\log g$ is evaluated in K and $\log c$ is evaluated in C_L. It is routine to check that this assignment doesn't depend on the choice of g. Checking that this map is a logarithm is also quite straightforward; for

convenience, we may assume that $g = 1$ in the case that $f \asymp 1$. For (LH1), we have
$$\log(f)' = \log(g)' + \log(c)' + \ln(1 + \varepsilon)' = g^\dagger + c^\dagger + (1 + \varepsilon)^\dagger = f^\dagger,$$
where the second equality uses that $\log(c)' = c^\dagger = 0$ for $c \in C_L^>$. Since for each $a \in \mathcal{O}_L$, there is $d \in C_L$ with $|a - d| \leqslant 1$ and since the logarithm on C_L is surjective, (LH2) follows from Lemma 1.4.

Now let M be a logarithmic H_T-field extension of K and let $\imath \colon C_L \to C_M$ be an $\mathcal{L}_{\log}(C)$-embedding. Using the embedding property of L, we uniquely extend $\imath$ to an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $\jmath \colon L \to M$. We will show that $\jmath$ is even an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding. For $c \in C_L^>$ and $g \in K^>$, we have
$$\log \jmath(cg) = \log \imath(c) + \log g = \imath(\log c) + \log g = \jmath\big(\log(cg)\big).$$

Let $f \in L^>$. Since we can find $c \in C_L^>$ and $g \in K^>$ with $f \sim cg$, Lemma 6.7 and the above computation allow us to conclude that $\log \jmath(f) = \jmath(\log f)$. $\qquad\square$

7. LIOUVILLE CLOSED LOGARITHMIC H_T-FIELDS

In this section, we keep the same assumptions as in the previous section (T is power bounded, T defines a restricted exponential function, Λ is cofinal in the prime model of T). Let K be a logarithmic H_T-field. As in Section 5, we investigate Liouville closed extensions of K. Now that we have a logarithm present, we can relate exponential integrals to actual exponentials:

Lemma 7.1. *The following are equivalent:*
  (i) $\log(K^>) = K$ *and every element in K has an integral in K;*
  (ii) K *is Liouville closed.*

Proof. Suppose (i) holds and let $f \in K$. We need to find an exponential integral for f in $K^{\neq}$. Take $g \in K$ with $g' = f$. Then $(\exp g)^\dagger = g' = f$. Now, suppose (ii) holds and let $a \in K$. We need to show that $a \in \log(K^>)$. Take $b \in K^>$ with $b^\dagger = a'$. Then $\log(b)' = a'$ so $a - \log b \in C$. Since $C = \log(C^>)$ by Lemma 6.5, we may take $c \in C^>$ with $a - \log b = \log c$. Then $a = \log(bc)$. $\qquad\square$

Definition 7.2. A **logarithmic T-Liouville extension** of K is a logarithmic H_T-field extension L of K which is also a T-Liouville extension of K, as defined in Section 5. A **logarithmic T-Liouville closure** of K is a logarithmic T-Liouville extension of K which is Liouville closed.

Any logarithmic T-Liouville extension of K has the same cardinality as K, and an increasing union of logarithmic T-Liouville extensions is itself a logarithmic T-Liouville extension. The logarithmic H_T-field extension K^e in Theorem 6.11 is a logarithmic T-Liouville extension, as are the extensions considered in Corollaries 6.12 and 6.13 and in Lemmas 6.14 and 6.15.

Definition 7.3. A **logarithmic T-Liouville tower on** K is a strictly increasing chain $(K_\mu)_{\mu \leqslant \nu}$ of logarithmic H_T-fields such that:
  (1) $K_0 = K$;
  (2) if $\mu \leqslant \nu$ is an infinite limit ordinal, then $K_\mu = \bigcup_{\eta < \mu} K_\eta$;
  (3) if $\mu < \nu$ and $K_\mu \neq \log(K_\mu^>)$, then $K_{\mu+1} = K_\mu^e$;
  (4) if $\mu < \nu$ and $K_\mu = \log(K_\mu^>)$, then $K_{\mu+1}$ is one of the following extensions of K_μ:
      (a) $K_{\mu+1} = K_\mu \langle a_\mu \rangle_\omega$, where $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \prec 1$ and vs_μ is a gap in K_μ;
      (b) $K_{\mu+1} = K_\mu \langle a_\mu \rangle_\omega$, where $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \succ 1$ and vs_μ is a gap in K_μ;
      (c) $K_{\mu+1} = K_\mu \langle a_\mu \rangle$, where $a_\mu' = s_\mu \in K_\mu$ with $a_\mu \prec 1$, $vs_\mu \in (\Gamma_{K_\mu}^>)'$, and $s_\mu \notin \partial \mathcal{O}_{K_\mu}$;
      (d) $K_{\mu+1} = K_\mu \langle a_\mu \rangle$, where $a_\mu' = s_\mu \in K_\mu$ with $v(s_\mu - \partial K_\mu) \subseteq (\Gamma_{K_\mu}^<)'$.
The logarithmic H_T-field K_ν is called the **top of the tower** $(K_\mu)_{\mu \leqslant \nu}$.

The extensions in (a) and (b) correspond to Lemmas 6.14 and 6.15, respectively, and the logarithm in these extensions is the logarithm defined in those Lemmas. The extensions in (c) and (d) correspond to

Corollaries 6.12 and 6.13. If $(K_\mu)_{\mu \leqslant \nu}$ is a logarithmic T-Liouville tower on K, then each K_μ is a logarithmic T-Liouville extension of K. Maximal logarithmic T-Liouville towers on K (that is, logarithmic T-Liouville towers on K which cannot be properly extended) exist by Zorn's lemma, and we briefly verify below that the analogs of Lemmas 5.8 and 5.9 hold in this setting.

Lemma 7.4. *Let L be the top of a maximal logarithmic T-Liouville tower on K. Then L is Liouville closed and, therefore, L is a logarithmic T-Liouville closure of K.*

Proof. As in the proof of Lemma 5.8, maximality tells us each element in L has an integral in L. Maximality also tells us that $L = \log(L^>)$, so L is Liouville closed by Lemma 7.1. $\qquad\square$

Lemma 7.5. *Let L be a Liouville closed logarithmic H_T-field extension of K and let $(K_\mu)_{\mu \leqslant \nu}$ be a logarithmic T-Liouville tower on K. Suppose that $(K_\mu)_{\mu \leqslant \nu}$ is a tower in L which cannot be extended to a logarithmic T-Liouville tower $(K_\mu)_{\mu \leqslant \nu+1}$ in L. Then $(K_\mu)_{\mu \leqslant \nu}$ is a maximal logarithmic T-Liouville tower on K.*

Proof. By Fact 5.2 and Lemma 5.3, it suffices to show that K_ν is Liouville closed. The proof that each element in K_ν has an integral in K_ν is essentially the same as the proof of Lemma 5.9. Since L is Liouville closed, we have $L = \log(L^>)$ by Lemma 7.1, so Theorem 6.11 allows us to identify K_ν^{e} with a logarithmic H_T-subfield of L. Since K_ν^{e} is a logarithmic T-Liouville extension of K_ν, we have $K_\nu = K_\nu^{\mathrm{e}}$. Thus, K_ν is Liouville closed, again by Lemma 7.1. $\qquad\square$

We have the following analog of Corollary 5.10, where Lemmas 7.4 and 7.5 are used in place of Lemmas 5.8 and 5.9.

Corollary 7.6. *Let L be a Liouville closed logarithmic H_T-field extension of K. Then L is a logarithmic T-Liouville closure of K if and only if L has no proper Liouville closed logarithmic H_T-subfields which contain K.*

An analog of Lemma 5.11 also goes through with the obvious changes to the proof.

Lemma 7.7. *Let $(K_\mu)_{\mu \leqslant \nu}$ be a logarithmic T-Liouville tower on K and suppose K_μ does not have a gap for all $\mu < \nu$. Then K_ν admits an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding into any Liouville closed logarithmic H_T-field extension of K.*

7.1. Uniqueness and nonuniqueness of logarithmic T-Liouville closures.

Proposition 7.8. *Suppose K is λ-free. Then K has a logarithmic T-Liouville closure L which embeds over K into any Liouville closed logarithmic H_T-field extension of K. Any logarithmic T-Liouville closure of K is $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic to L.*

Proof. Let $(K_\mu)_{\mu \leqslant \nu}$ be a maximal logarithmic T-Liouville tower on K. We will prove by induction on $\mu \leqslant \nu$ that each K_μ is λ-free. This holds when $\mu = 0$ by assumption and if $\mu \leqslant \nu$ is an infinite limit ordinal, then this follows from Lemma 3.13. Let $\mu < \nu$ and suppose that K_μ is λ-free. If $K_\mu \neq \log(K_\mu^>)$, then $K_{\mu+1} = K_\mu^{\mathrm{e}}$ is λ-free by Theorem 6.11. Suppose that $K_\mu = \log(K_\mu^>)$. Since K_μ is λ-free, it has no gap, so $K_{\mu+1}$ must be an extension of type (c) or (d). Then $K_{\mu+1}$ is λ-free by Corollaries 6.12 and 6.13.

Set $L := K_\nu$, so L is a logarithmic T-Liouville closure of K. Let M be a Liouville closed logarithmic H_T-field extension of K. By Lemma 7.7, there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $\imath\colon L \to M$. Moreover, if M is a logarithmic T-Liouville closure of K, then in particular, M is a T-Liouville closure of K and $\imath(L)$ is a Liouville closed H_T-subfield of M containing K, so $\imath(L) = M$ by Corollary 5.5. Thus, L is unique up to $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphism. $\qquad\square$

If K is not λ-free, then K has two distinct logarithmic T-Liouville closures. As in the case of H_T-fields, it is helpful to handle the case that K has a gap and the case that K has asymptotic integration separately.

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Proposition 7.9. *Let $\beta \in \Gamma$ be a gap in K. Then K has logarithmic T-Liouville closures L_1 and L_2 with $\beta \in (\Gamma_{L_1}^>)'$ and $\beta \in (\Gamma_{L_2}^<)'$. Let M be a Liouville closed logarithmic H_T-field extension of K. If $\beta \in (\Gamma_M^>)'$, then there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. Likewise, if $\beta \in (\Gamma_M^<)'$, then there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. Any logarithmic T-Liouville closure of K is $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

Proof. Let $s \in K$ with $vs = \beta$. Let $K_1 := K\langle a\rangle_\omega$ be the logarithmic H_T-field extension of K given in Lemma 6.14, so $a \prec 1$ and $a' = s$, and let $K_2 := K\langle b\rangle_\omega$ be the logarithmic H_T-field extension of K given in Lemma 6.15, so $b \succ 1$ and $b' = s$. Then K_1 is λ-free, so it has a logarithmic T-Liouville closure L_1 which embeds over K_1 into any Liouville closed logarithmic H_T-field extension of K_1 by Proposition 7.8. Likewise, K_2 has a logarithmic T-Liouville closure L_2 which embeds over K_2 into any Liouville closed logarithmic H_T-field extension of K_2. Now let M be a Liouville closed logarithmic H_T-field extension of K. If $\beta \in (\Gamma_M^>)'$, then the embedding property in Lemma 6.14 gives an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $K_1 \to M$, which in turn extends to an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $L_1 \to M$. If $\beta \in (\Gamma_M^<)'$, then using the embedding property in Lemma 6.15 instead, we get an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding $L_2 \to M$. If M is a logarithmic T-Liouville closure of K, then M is $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2 by Corollary 5.5, since M contains the $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic image of either L_1 or L_2 as a Liouville closed logarithmic H_T-subfield. $\square$

Proposition 7.10. *Suppose that K has asymptotic integration and is not λ-free. Then K has logarithmic T-Liouville closures L_1 and L_2 which are not $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic. If M is a Liouville closed logarithmic H_T-field extension of K, then there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding of either L_1 or L_2 into M. Any logarithmic T-Liouville closure of K is $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2.*

Proof. Let (λ_ρ) be a λ-sequence in K with pseudolimit $\lambda \in K$. First, we consider the case that $-\lambda$ has an integral $f \in K$. If there were $a \in K^>$ with $\log a = f$, then we would have $a^\dagger = f' = -\lambda$, contradicting that $v(\lambda + (K^\times)^\dagger)$ is a cofinal subset of $\Psi^\downarrow$ by Fact 3.15. Thus, f has no exponential in $K^>$, so we use Lemma 6.10 to extend K to a logarithmic H_T-field $K\langle a\rangle$ with $\log a = f$. Then $a^\dagger = -\lambda$, so va is a gap in $K\langle a\rangle$ by Lemma 3.16. By Proposition 7.9, $K\langle a\rangle$ has logarithmic T-Liouville closures L_1 and L_2 with $va \in (\Gamma_{L_1}^>)'$ and $va \in (\Gamma_{L_2}^<)'$, one of which embeds over $K\langle a\rangle$ into any Liouville closed logarithmic H_T-field extension of $K\langle a\rangle$. Using Lemma 7.1 and the universal property of Lemma 6.10, we see that either L_1 or L_2 embeds over K into any Liouville closed logarithmic H_T-field extension of K.

Now, consider the case that $-\lambda$ has no integral in K. Since K has asymptotic integration, we may use either Corollary 6.12 or Corollary 6.13 to extend K to an immediate logarithmic H_T-field $K\langle f\rangle$ where $f' = s$. Since $K\langle f\rangle$ is an immediate extension of K, the λ-sequence (λ_ρ) remains a λ-sequence in $K\langle f\rangle$ and $K\langle f\rangle$ has asymptotic integration, so by the previous case, $K\langle f\rangle$ has logarithmic T-Liouville closures L_1 and L_2 with $v(\exp f) \in (\Gamma_{L_1}^>)'$ and $v(\exp f) \in (\Gamma_{L_2}^<)'$, one of which embeds over $K\langle f\rangle$ into any Liouville closed logarithmic H_T-field extension of $K\langle f\rangle$. The embedding properties in Corollaries 6.12 and 6.13 ensure either L_1 or L_2 embeds over K into any Liouville closed logarithmic H_T-field extension of K.

The embedding properties of L_1 and L_2, together with Corollary 5.5, ensure that any logarithmic T-Liouville closure of K is $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphic to either L_1 or L_2. The proof that L_1 and L_2 are not isomorphic to each other is exactly the same as in the proof of Proposition 5.15. $\square$

Let us combine the above propositions into one theorem.

Theorem 7.11. *If K is λ-free, then K has exactly one logarithmic T-Liouville closure up to $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphism. Otherwise, K has exactly two logarithmic T-Liouville closures up to $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-isomorphism. For any Liouville closed logarithmic H_T-field extension M of K, there is an $\mathcal{L}_{\log}^{\mathcal{O},\partial}(K)$-embedding of some logarithmic T-Liouville closure of K into M.*

7.2. **An application to $\mathbb{R}_{\mathrm{an,exp}}$-Hardy fields.** In this subsection, let $\mathcal{R}$, $\mathcal{L}_\mathcal{R}$, and $T_\mathcal{R}$ be as in the introduction, that is, $\mathcal{R}$ is an o-minimal expansion of the real field, $T_\mathcal{R}$ is the $\mathcal{L}_\mathcal{R}$-theory of $\mathcal{R}$, $\mathcal{L}_\mathcal{R}$ is assumed

to include a constant symbol for each $r \in \mathbb{R}$, and $T_{\mathcal{R}}$ is assumed to have quantifier elimination and a universal axiomatization. Let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field and let $[f]$ be a germ of a real-valued unary function at $+\infty$. Then $[f]$ is said to be **comparable to** $\mathcal{H}$ if for each $[g] \in \mathcal{H}$, either $g(x) < f(x)$ eventually, or $g(x) > f(x)$ eventually, or $g(x) = f(x)$ eventually (where *eventually* means *for all sufficiently large x*). If $[f]$ is comparable to $\mathcal{H}$, then set

$$\mathcal{H}\langle[f]\rangle \ :=\ \big\{t([f]) : t \text{ is a unary } \mathcal{L}_{\mathcal{R}}(\mathcal{H})\text{-term}\big\}.$$

If, in addition to being comparable with $\mathcal{H}$, the function f is eventually $\mathcal{C}^1$ and $[f'] \in \mathcal{H}\langle[f]\rangle$, then $\mathcal{H}\langle[f]\rangle$ is an $\mathcal{R}$-Hardy field; see [9, Lemma 5.12] and the remarks at the end of [9]. If $f' = g$ or $f = \exp(g)$ for some $[g] \in \mathcal{H}$, then $[f]$ is comparable to $\mathcal{H}$ by Boshernitzan [5, Theorem 5.3]; see also [21]. In both cases, f is eventually $\mathcal{C}^1$ and $[f'] \in \mathcal{H}\langle[f]\rangle$, so it follows that

- $\mathcal{H}\langle[\exp g]\rangle$ is an $\mathcal{R}$-Hardy field for $[g] \in \mathcal{H}$, and
- $\mathcal{H}\langle[f]\rangle$ is an $\mathcal{R}$-Hardy field if $[f'] \in \mathcal{H}$.

Since any increasing union of $\mathcal{R}$-Hardy fields is an $\mathcal{R}$-Hardy field and since Hardy fields are bounded in size, Zorn's lemma and the remarks above give us a *Liouville closed* $\mathcal{R}$-Hardy field extension of $\mathcal{H}$ where every germ has an integral and an exponential (thus, every germ also has a nonzero exponential integral). We denote by $\mathrm{Li}_{\mathcal{R}}(\mathcal{H})$ the intersection of all Liouville closed $\mathcal{R}$-Hardy field extensions of $\mathcal{H}$. Then $\mathrm{Li}_{\mathcal{R}}(\mathcal{H})$ is a $T_{\mathcal{R}}$-Liouville closure of $\mathcal{H}$ by Corollary 5.5, since the constant field of any $\mathcal{R}$-Hardy field is $\mathbb{R}$.

Here is an application when $\mathcal{R} = \mathbb{R}_{\mathrm{an,exp}}$. The appropriate language here is $\mathcal{L}_{\mathrm{an,exp}}$, which includes a function symbol for each restricted analytic function, as well as function symbols for exp and log. By [9, Corollary 4.6], $\mathbb{R}_{\mathrm{an,exp}}$ has quantifier elimination and a universal axiomatization in this language. Since each constant function is analytic, our assumptions at the beginning of the subsection hold for this expansion. For an $\mathbb{R}_{\mathrm{an,exp}}$-Hardy field $\mathcal{H}$, let us write $\mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H})$ instead of $\mathrm{Li}_{\mathbb{R}_{\mathrm{an,exp}}}(\mathcal{H})$. Recall that $\mathbb{T}_{\mathrm{an,exp}}$ is the canonical expansion of the field $\mathbb{T}$ of logarithmic-exponential transseries to an $H_{\mathrm{an,exp}}$-field. The following theorem is an analog of a theorem on Hardy fields from [1].

Theorem 7.12. *Let $\mathcal{H}$ be an $\mathbb{R}_{\mathrm{an,exp}}$-Hardy field and let $\imath \colon \mathcal{H} \to \mathbb{T}_{\mathrm{an,exp}}$ be an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding. Then $\imath$ extends to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding $\mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H}) \to \mathbb{T}_{\mathrm{an,exp}}$.*

Proof. If $\mathcal{H} = \mathbb{R}_{\mathrm{an,exp}}$, then we extend $\imath$ to an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding of

$$\mathcal{H}(\mathbb{R}_{\mathrm{an,exp}}) \ :=\ \big\{[t] : t \text{ is a unary } \mathcal{L}_{\mathrm{an,exp}}(\emptyset)\text{-term}\big\}$$

by sending $[t] \in \mathcal{H}(\mathbb{R}_{\mathrm{an,exp}})$ to $t(x) \in \mathbb{T}_{\mathrm{an,exp}}$ where $x \in \mathbb{T}_{\mathrm{an,exp}}$ is the distinguished positive infinite element with derivative $x' = 1$. One can easily verify that this is an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding. Thus, by replacing $\mathcal{H}$ by $\mathcal{H}(\mathbb{R}_{\mathrm{an,exp}})$ if need be, we may assume that $\mathcal{H}$ is a proper extension of $\mathbb{R}_{\mathrm{an,exp}}$. Let $K := \imath(\mathcal{H}) \subseteq \mathbb{T}_{\mathrm{an,exp}}$ and let $(K_\mu)_{\mu \leqslant \nu}$ be a maximal logarithmic T_{an}-Liouville tower on K in $\mathbb{T}_{\mathrm{an,exp}}$. Lemma 7.5 tells us that K_ν is a logarithmic T_{an}-Liouville closure of K. Moreover, none of the H_T-fields K_μ have a gap by [1, Lemma 6.6], so Lemma 7.7 gives an $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding $\jmath \colon K_\nu \to \mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H})$ which extends $\imath^{-1}$. Since $\mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H})$ is a logarithmic T_{an}-Liouville closure of $\mathcal{H}$, we have $\jmath(K_\nu) = \mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H})$ by Corollary 5.5. Thus, we may take $\jmath^{-1} \colon \mathrm{Li}_{\mathrm{an,exp}}(\mathcal{H}) \to \mathbb{T}_{\mathrm{an,exp}}$ to be our $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an,exp}}$-embedding. $\qquad\square$

Remark 7.13. Using T_{rexp}, the theory of the reals with the restricted exponential function, in place of T_{an} in the proof above, one can also show that any $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{exp}}$-embedding of an $\mathbb{R}_{\mathrm{exp}}$-Hardy field $\mathcal{H} \supseteq \mathbb{R}$ into $\mathbb{T}_{\mathrm{exp}}$ can be extended to $\mathrm{Li}_{\mathrm{exp}}(\mathcal{H})$. Here, $\mathbb{T}_{\mathrm{exp}}$ is the expansion of $\mathbb{T}$ by just the exponential function.

Using Lemmas 5.9 and 5.11 in place of Lemmas 7.5 and 7.7 in the proof above, one can show that any $\mathcal{L}^{\mathcal{O},\partial}_{\mathrm{an}}$-embedding of an $\mathbb{R}_{\mathrm{an}}$-Hardy field $\mathcal{H}$ into $\mathbb{T}_{\mathrm{an}}$ can be extended to $\mathrm{Li}_{\mathrm{an}}(\mathcal{H})$. The same holds for any polynomially bounded reduct $\mathcal{R}$ of $\mathbb{R}_{\mathrm{an,exp}}$, so long as $\mathcal{R}$ satisfies the assumptions at the beginning of this subsection.

8. The order 1 intermediate value property

In this section, let K be a pre-H_T-field. We do not assume that T is power bounded. Our goal is to prove the following extension result:

Theorem 8.1. K has a pre-H_T-field extension M with the following property: for every $\mathcal{L}(M)$-definable continuous function $F\colon M \to M$ and every $b_1, b_2 \in M$ with

$$b_1' \; < \; F(b_1), \qquad b_2' \; > \; F(b_2),$$

there is $a \in M$ between b_1 and b_2 with $a' = F(a)$.

Before proving this theorem, we need a lemma about extensions of pre-H_T-fields:

Lemma 8.2. Suppose K is ungrounded, let M be a $T^{\mathcal{O},\partial}$-extension of K, and suppose that

$$\mathcal{O}_M \; = \; \big\{y \in M : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\},$$

and that $g' > 0$ for all $g \in M$ with $g > \mathcal{O}_M$. Then M is a pre-H_T-field extension of K.

Proof. Let $g \in M$ with $g \succ 1$. Then $|g|' > 0$ by assumption, so $g^\dagger = |g|^\dagger > 0$, proving (PH1). For (PH2), let $f \in \mathcal{O}_M$. We need to show that $f' \prec g^\dagger$. We will do this by showing that $v(f') > \Psi$ and that $vg^\dagger \in \Psi^\downarrow$. To see that $vf' > \Psi$, let $\gamma \in \Psi$ and, using that K is ungrounded, take $d \in K$ with $d > \mathcal{O}$ and $vd' > \gamma$. Then $d + f, d - f > \mathcal{O}_M$, so $d' + f', d' - f' > 0$. This gives $-d' < f' < d'$, so $vf' \geqslant vd' > \gamma$. To see that $vg^\dagger \in \Psi^\downarrow$, take $d \in K$ with $|g| > d > \mathcal{O}$. Then $|g|d^{-1/2} > d^{1/2} > \mathcal{O}$ so $\big(|g|d^{-1/2}\big)^\dagger = g^\dagger - d^\dagger/2 > 0$, which gives $g^\dagger > d^\dagger/2$. As $d^\dagger > 0$, we have $vg^\dagger \leqslant vd^\dagger \in \Psi$, so $vg^\dagger \in \Psi^\downarrow$. $\qquad\square$

Theorem 8.1 follows by iterating the following proposition; an analog of [2, Theorem 4.3]. See also [7], where van den Dries proves this for $\mathcal{R}$-Hardy fields.

Proposition 8.3. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable continuous function and let $b_1, b_2 \in K$ with

$$b_1' \; < \; F(b_1), \qquad b_2' \; > \; F(b_2).$$

Then there is a pre-H_T-field extension M of K and $a \in M$ between b_1 and b_2 with $a' = F(a)$.

Proof. If $\mathcal{O} = K$, then let $M := K\langle a\rangle$ be a simple T-extension of K where a lies between b_1 and b_2. Using Fact 1.9, extend the T-derivation on K uniquely to a T-derivation on M by so that $a' = F(a)$. Then M, with this derivation and the T-convex valuation ring $\mathcal{O}_M := M$, is the desired pre-H_T-field extension of K. Having handled this case, we assume for the remainder of the proof that $\mathcal{O} \neq K$ (so ∂ is continuous by Lemma 3.7).

Next, we arrange that K is ungrounded. If T is not power bounded, then T defines an exponential function [18], so K is necessarily ungrounded by Corollary 4.3. If T is power bounded and K is grounded, then we use Corollary 4.18 to replace K with an ungrounded pre-H_T-field extension.

Now, let us handle the case that $b_1 < b_2$. Let I be the interval (b_1, b_2) and set

$$A \; := \; \big\{y \in I : y' < F(y)\big\}.$$

Since $b_1' < F(b_1)$ and since F and ∂ are continuous, we have $y' < F(y)$ for all $y \in I$ sufficiently close to b_1. Thus, A is nonempty. Likewise, $y' > F(y)$ for all $y \in I$ sufficiently close to b_2, so A is not cofinal in I. If A has a supremum $b \in I$, then $b' = F(b)$ by continuity, and we may take $M = K$. Thus, we may assume that A has no supremum in I. Let $M := K\langle a\rangle$ be a simple T-extension of K where a realizes the cut $A^\downarrow$. Using Fact 1.9, we extend the T-derivation on K uniquely to a T-derivation on M by with $a' = F(a)$. We also equip M with the T-convex valuation ring

$$\mathcal{O}_M \; := \; \big\{y \in M : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

We claim that M is a pre-H_T-field extension of K. By Lemma 8.2, it is enough to show that $g' > 0$ for all $g \in M$ with $g > \mathcal{O}_M$. Let $G\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $G(a) > \mathcal{O}_M$. We may assume

$G(a) \notin K$. Suppose toward contradiction that

$$G(a)' \;=\; G^{[\partial]}(a) + G'(a)F(a) \;\leqslant\; 0.$$

Take $d \in K$ with $G(a) > d$ and take a subinterval $J \subseteq I$ with $a \in J^M$ such that G is $\mathcal{C}^1$ on J and such that for all $y \in J$, we have

$$G(y) \;>\; d, \qquad G^{[\partial]}(y) + G'(y)F(y) \;\leqslant\; 0.$$

Let $y \in J$. Since $G(y) > d > \mathcal{O}$ and K is a pre-H_T-field, we have $G(y)' = G^{[\partial]}(y) + G'(y)y' > 0$, so

$$\bigl(G^{[\partial]}(y) + G'(y)y'\bigr) - \bigl(G^{[\partial]}(y) + G'(y)F(y)\bigr) \;=\; G'(y)\bigl(y' - F(y)\bigr) \;>\; 0.$$

By shrinking J, we may assume that $G'(y)$ has constant sign on J, so $y' - F(y)$ has constant sign on J as well. This is a contradiction, as J contains elements of A as well as elements of $I \setminus A$. The case that $b_1 > b_2$ is virtually identical; we instead let I be the interval (b_2, b_1) and let

$$A \;:=\; \bigl\{y \in I : y' > F(y)\bigr\}. \hspace{4cm} \square$$

Remark 8.4. Note that in Proposition 8.3, the residue field of the M may be strictly larger than the residue field of K, so even if K is an H_T-field, M may only be a pre-H_T-field. Accordingly, we do not know if in Theorem 8.1, the pre-H_T-field extension M of K can be taken to be an H_T-field in general. However, if we assume that T is power bounded, then M can be taken to be an H_T-field: simply use Theorem 4.14 to replace the pre-H_T-field extension in Proposition 8.3 with its H_T-field hull. We can also take M to be an H_T-field if T is an exponential theory of the form considered in Subsection 1.3 (for example, if $T = T_{\mathrm{an,exp}}$). For this, we use Proposition 6.16 to pass to the logarithmic H_T-field hull, followed by Theorem 6.11 to pass to the exponential closure.

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