T-CONVEX T-DIFFERENTIAL FIELDS AND THEIR IMMEDIATE EXTENSIONS

ELLIOT KAPLAN

Abstract. Let T be a polynomially bounded o-minimal theory extending the theory of real closed ordered fields. Let K be a model of T equipped with a T-convex valuation ring and a T-derivation. If this derivation is continuous with respect to the valuation topology, then we call K a *T-convex T-differential field*. We show that every T-convex T-differential field has an immediate strict T-convex T-differential field extension which is spherically complete. In some important cases, the assumption of polynomial boundedness can be relaxed to power boundedness.

Introduction

In this article, T is a complete, model complete o-minimal theory which extends the theory RCF of real closed ordered fields in some appropriate language $\mathcal{L} \supseteq \{0, 1, +, -, \cdot, <\}$. Let $K \models T$, let $\mathcal{O}$ be a T-convex valuation ring of K, as defined in [7], and let ∂ be a T-derivation on K, as defined in [11]. If ∂ is continuous with respect to the valuation topology induced by $\mathcal{O}$, then we call $K = (K, \mathcal{O}, \partial)$ a **T-convex T-differential field**. Motivating examples include $\mathcal{R}$-Hardy fields, defined in [8], and the expansion of the field of logarithmic-exponential transseries by restricted analytic functions, considered in [9]; see Examples 3.3 and 3.4 below.

Let K be a T-convex T-differential field. By [2, Lemma 4.4.7], continuity of the derivation is equivalent to the existence of $\phi \in K^\times$ with $\partial\smallo \subseteq \phi\smallo$, where $\smallo$ is the unique maximal ideal of $\mathcal{O}$. Following [3], we say that a T-convex T-differential field extension M of K is **strict** if

$$\partial\smallo \subseteq \phi\smallo \implies \partial_M\smallo_M \subseteq \phi\smallo_M, \qquad \partial\mathcal{O} \subseteq \phi\smallo \implies \partial_M\mathcal{O}_M \subseteq \phi\smallo_M$$

for each $\phi \in K^\times$. In this article, we show the following:

Theorem A (Corollary 7.4). *If T is polynomially bounded, then K has an immediate strict T-convex T-differential field extension which is spherically complete.*

When $T = \mathrm{RCF}$, Theorem A says that every real closed valued field equipped with a continuous derivation has a spherically complete immediate strict extension, a result previously established in [3]. A case unique to this article is when $T = T_{\mathrm{an}} := \mathrm{Th}(\mathbb{R}_{\mathrm{an}})$, the theory of the expansion of the real field by all restricted analytic functions. The theory T_{an} is model complete, o-minimal, and polynomially bounded [4, 12]. We give an example of a spherically complete T_{an}-convex T_{an}-differential field below (Example 3.5). Valued fields equipped with both analytic structure and an operator have been studied before by Rideau [19]. Our setting, however, is quite different from Rideau's.

In some important cases, the assumption of polynomial boundedness can be relaxed to power boundedness (a generalization of polynomial boundedness introduced in [17]).

Theorem B (Corollaries 6.4 and 6.5). *Suppose T is power bounded.*

(i) If $\partial\smallo \subseteq \smallo$ and the induced derivation on the residue field of K is nontrivial, then K has a spherically complete immediate T-convex T-differential field extension with small derivation.

(ii) If K is asymptotic (in the sense of [2]), then K has a spherically complete immediate asymptotic T-convex T-differential field extension.

Theorems A and B both follow from something a bit more general. Before stating this, we recall some definitions from [3]. Let $v\colon K^\times \to \Gamma$ be the (surjective) Krull valuation corresponding to $\mathcal{O}$ and set

$$\Gamma(\partial) := \big\{v(\phi) : \phi \in K^\times \text{ and } \partial\smallo \subseteq \phi\smallo\big\}, \qquad S(\partial) := \big\{\gamma \in \Gamma : \Gamma(\partial) + \gamma = \Gamma(\partial)\big\}.$$

Theorem C (Theorem 7.3)**.** *Suppose that T is power bounded with field of exponents Λ and that $S(\partial)$ is a Λ-subspace of Γ. Then K has an immediate strict T-convex T-differential field extension which is spherically complete.*

The condition that $S(\partial)$ is a Λ-subspace of Γ is satisfied when Λ is archimedean (that is, when T is polynomially bounded). It is also satisfied when $S(\partial) = \{0\}$, a situation which arises when K is asymptotic or when $\partial\smallo \subseteq \smallo$ and the induced derivation on the residue field of K is nontrivial. We are unsure whether $S(\partial)$ is always a Λ-subspace of Γ, or whether this assumption is necessary.

The assumption of power boundedness, on the other hand, we know to be necessary. If T is not power bounded and $\mathcal{O} \neq K$, then K has no T-convex extension which is spherically complete by Miller's dichotomy [17] and a negative result of Kuhlmann, Kuhlmann, and Shelah [16], established independently by van der Hoeven [14, Proposition 2.2]. See Remark 1.12 below for details.

In [3], Aschenbrenner, van den Dries, and van der Hoeven proved that every equicharacteristic zero valued field with a continuous derivation has an immediate strict valued differential field extension which is spherically complete. Our proof of Theorem C is similar in structure to the proof of this result. As in [3], we first handle the case $S(\partial) = \{0\}$, and then we reduce to this case using a coarsening argument. Our definition of the set $Z(K, \ell)$ is also quite similar to that in [3]. However, since we work with $\mathcal{L}(K)$-definable functions instead of differential polynomials over K, many of the key tools from [3] are not available to us. For example, the property of *having positive Newton degree*, which plays a starring role in [3], is here replaced with the property of *eventual smallness*.

Outline. In Section 1, we review the basics of T-convex valuation rings and power boundedness. We verify that a number of well-known facts and constructions in valuation theory (such as the existence and uniqueness of spherically complete immediate extensions) hold in the power bounded T-convex setting. Section 2 begins with a review of T-derivations. This is followed by a discussion of how definable functions can be additively, multiplicatively, and compositionally conjugated. This section ends with a subsection on *thin sets*, a notion which may also be of use when studying T-derivations in other settings.

In Section 3, we prove some basic lemmas about T-convex T-differential fields. In Section 4, we assign a valuation to definable functions in implicit form, which acts as an analog of the Gaussian valuation for polynomials. We use this to study the behavior of these functions. We also introduce the aforementioned property of eventual smallness.

In Section 5, we introduce the sets $Z(K, \ell)$. These sets tell us when a cut in K can be filled by an element a satisfying the identity $a^{(r)} = F(a, a', \ldots, a^{(r-1)})$ for some $\mathcal{L}(K)$-definable function F. In Section 6 we consider the special case $S(\partial) = \{0\}$, and in Section 7, we prove Theorem C.

Acknowledgements. This article contains results from my PhD thesis [15]. I would like to thank Allen Gehret and Erik Walsberg for helpful conversations. I'd especially like to thank Nigel Pynn-Coates and Lou van den Dries for providing feedback on earlier drafts of this article.

Notation and conventions. In this article, we always use k, m, n, p, q, and r to denote elements of $\mathbb{N} = \{0, 1, 2, \ldots\}$. By "ordered set" we mean "totally ordered set." Let S be an ordered set, let $a \in S$, and let $A \subseteq S$. We let

$$S^{>a} \ := \ \{s \in S : s > a\};$$

similarly for $S^{\geqslant a}$, $S^{<a}$, $S^{\leqslant a}$, and $S^{\neq a}$. We write "$a > A$" (respectively "$a < A$") if a is greater (less) than each $s \in A$. For $b \in S^{>a}$, we put

$$[a, b]_A \ := \ \{s \in A : a \leqslant s \leqslant b\}.$$

If $A = S$, we drop the subscript and write $[a, b]$ instead. We say that A is *downward closed* if $s \in A$ whenever $s \in S$ is less than some $a \in A$, and we let

$$A^{\downarrow} \ := \ \bigcup_{a \in A} S^{\leqslant a}$$

denote the **downward closure of** A, so A is downward closed if and only if $A = A^{\downarrow}$. A **cut** in S is just a downward closed subset of S. If A is a cut in S and y is an element in an ordered set extending S, then we say that y *realizes the cut* A if

$$A \ < \ y \ < \ S \setminus A.$$

If Γ is an ordered abelian group, then we let $\Gamma^{>} := \Gamma^{>0}$ and we define $\Gamma^{\geqslant}$, $\Gamma^{<}$, $\Gamma^{\leqslant}$, and $\Gamma^{\neq}$ analogously. If R is a ring, then we let $R^{\times}$ denote the multiplicative group of units in R. We let $\mathrm{Mat}_{m,n}(R)$ be the collection of $m \times n$ matrices with entries in R, so $Ab \in R^m$ for $A \in \mathrm{Mat}_{m,n}(R)$ and $b \in R^n$. We identify $\mathrm{Mat}_{m,n}(R)$ with $R^{m\times n}$ in the usual way. If $m = n$, we just write $\mathrm{Mat}_n(R)$.

We always use K, L, and M for models of T (or expansions thereof). We regard K^0 as the one-point space $\{0\}$, and we identify each nullary map $F\colon K^0 \to K^n$ with its value $F(0) \in K^n$.

Let $A \subseteq K$ and let $D \subseteq K^n$. We say that D is **$\mathcal{L}(A)$-definable** if

$$D \ = \ \varphi(K) \ := \ \big\{ y \in K^n : K \models \varphi(y) \big\}$$

for some $\mathcal{L}(A)$-formula $\varphi(y)$. If $D \subseteq K^n$ is $\mathcal{L}(A)$-definable, then we let $\dim_{\mathcal{L}}(D) \in \{-\infty, 0, \ldots, n\}$ denote the o-minimal dimension of D. Let $k \leqslant n$. We denote the projection of D onto the first k coordinates by $\pi_k(D)$ and for $y \in K^k$, we set $D_y := \big\{ z \in K^{n-k} : (y, z) \in D \big\}$. Let $F\colon D \to K^m$ be a map. We let $\mathrm{Gr}(F) \subseteq K^{n+m}$ denote the graph of F, and we say that F is $\mathcal{L}(A)$-definable if $\mathrm{Gr}(F)$ is. Note that the domain of an $\mathcal{L}(A)$-definable map is $\mathcal{L}(A)$-definable. If F is $\mathcal{C}^1$ and D is open, then we let $\mathbf{J}_F$ denote the Jacobian matrix

$$\mathbf{J}_F \ := \ \left(\frac{\partial F_i}{\partial Y_j} \right)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n},$$

viewed as an $\mathcal{L}(K)$-definable map from U to $\mathrm{Mat}_{m,n}(K)$. If $m = n = 1$, then we write F' instead of $\mathbf{J}_F$.

For $A \subseteq K$, we let $\mathrm{dcl}_{\mathcal{L}}(A)$ be the $\mathcal{L}$-definable closure of A (in K, implicitly, but this doesn't change if we pass to elementary extensions of K). If $b \in \mathrm{dcl}_{\mathcal{L}}(A)$, then $b = F(a)$ for some $\mathcal{L}(\emptyset)$-definable function F and some tuple a from A. It is well-known that $(K, \mathrm{dcl}_{\mathcal{L}})$ is a pregeometry. A set $B \subseteq K$ is said to be **$\mathcal{L}(A)$-independent** if $b \notin \mathrm{dcl}_{\mathcal{L}}\big(A \cup (B \setminus \{b\})\big)$ for all $b \in B$. A tuple $a = (a_i)_{i \in I}$ is said to be $\mathcal{L}(A)$-independent if its set of components $\{a_i : i \in I\}$ is $\mathcal{L}(A)$-independent and no components are repeated.

Let M be a T-extension of K, that is, a model of T which contains K as an $\mathcal{L}$-substructure. Given an $\mathcal{L}(K)$-definable set $D \subseteq K^n$, we let D^M denote the subset of M^n defined by the same $\mathcal{L}(K)$-formula as D. We sometimes refer to D^M as the **natural extension of D to M**. Since T is assumed to be model complete, this natural extension does not depend on the choice of defining formula. If $F\colon D \to K^m$ is an $\mathcal{L}(K)$-definable map, then we let $F^M\colon D^M \to M^m$ be the $\mathcal{L}(K)$-definable map with graph $\mathrm{Gr}(F^M) = \mathrm{Gr}(F)^M$. We often drop the superscript for definable maps and just write $F\colon D^M \to M^m$.

Let $A \subseteq M$. We let $K\langle A\rangle$ denote the $\mathcal{L}$-substructure of M with underlying set $\mathrm{dcl}_{\mathcal{L}}(K \cup A)$. If $A = \{a_1, \ldots, a_n\}$, we write $K\langle a_1, \ldots, a_n\rangle$ instead of $K\langle A\rangle$. Since T has definable Skolem functions, $K\langle A\rangle$ is an elementary $\mathcal{L}$-substructure of M. If A is $\mathcal{L}(K)$-independent and $M = K\langle A\rangle$, then A is called a **basis for M over K**. The **rank of M over K**, denoted $\mathrm{rk}_{\mathcal{L}}(M|K)$, is the cardinality of a basis for M over K (this doesn't depend on the choice of basis). We say that M is a **simple extension** of K if $\mathrm{rk}_{\mathcal{L}}(M|K) = 1$. Then $M = K\langle a\rangle$ for some $a \in M \setminus K$.

Let $\mathcal{L}^* \supseteq \mathcal{L}$, let T^* be an $\mathcal{L}^*$-theory extending T, and let $K \models T^*$. We use the same conventions for $\mathcal{L}^*$-definability as we do for $\mathcal{L}$-definability. A **T^*-extension of** K is a model $M \models T^*$ which contains K as an $\mathcal{L}^*$-substructure. If M is an *elementary* T^*-extension of K and $D \subseteq K^n$ is $\mathcal{L}^*(K)$-definable, then we let D^M denote the subset of M^n defined by the same formula as D.

1. T-convex valuation rings

The fundamentals of valuation theory on o-minimal fields were established by van den Dries and Lewenberg in [7]. In this section, we set up valuation theoretic notation, recall some important results, and establish lemmas for later use.

Following [7], we say that a nonempty convex set $\mathcal{O} \subseteq K$ is a **T-convex valuation ring of** K if $F(\mathcal{O}) \subseteq \mathcal{O}$ for all $\mathcal{L}(\emptyset)$-definable continuous functions $F\colon K \to K$. Let $\mathcal{L}^{\mathcal{O}} := \mathcal{L} \cup \{\mathcal{O}\}$ be the extension of $\mathcal{L}$ by a unary predicate $\mathcal{O}$ and let $T^{\mathcal{O}}$ be the $\mathcal{L}^{\mathcal{O}}$-theory which extends T by axioms asserting that $\mathcal{O}$ is a T-convex valuation ring. For the rest of this section, let $K = (K, \mathcal{O}) \models T^{\mathcal{O}}$ (so K is now an $\mathcal{L}^{\mathcal{O}}$-structure). Unlike in [7], we allow $\mathcal{O} = K$, in which case K is said to be **trivially valued**. The following is an easy consequence of the o-minimal monotonicity theorem:

Fact 1.1. *The convex hull of an elementary $\mathcal{L}$-substructure of K is a T-convex valuation ring of K.*

We let $\scriptstyle\mathcal{O}$ denote the unique maximal ideal of $\mathcal{O}$ and for $a, b \in K$, we set

$$\begin{aligned} &a \preccurlyeq b \ :\Longleftrightarrow\ a \in b\mathcal{O}, && a \prec b \ :\Longleftrightarrow\ b \neq 0 \text{ and } a \in b\smallo, && a \asymp b \ :\Longleftrightarrow\ a \preccurlyeq b \text{ and } b \preccurlyeq a \\ &a \succcurlyeq b \ :\Longleftrightarrow\ b \preccurlyeq a, && a \succ b \ :\Longleftrightarrow\ b \prec a, && a \sim b \ :\Longleftrightarrow\ a - b \prec a. \end{aligned}$$

Note that if $a \sim b$, then a, b are nonzero and that $\sim$ is an equivalence relation on $K^{\times}$. Moreover, if $a \sim b$, then $a \asymp b$ and a is positive if and only if b is. We let $v\colon K^{\times} \to \Gamma$ be the (surjective) Krull valuation corresponding to $\mathcal{O}$, so $\ker(v) = \mathcal{O}^{\times}$. The **value group** Γ (or Γ_K if K is not clear from context) is written additively and ordered as follows:

$$va \geqslant 0 \iff a \in \mathcal{O}, \qquad va > 0 \iff a \in \smallo.$$

We set $\Gamma_{\infty} := \Gamma \cup \{\infty\}$ where $\infty > \Gamma$, and we extend v to all of K by setting $v(0) := \infty$. For $a, b \in K$, we have

$$v(a+b) \ \geqslant\ \min\{va, vb\}, \qquad v(ab) \ =\ va + vb.$$

Subsets of K of the form

$$\big\{y \in K : v(y-a) > \gamma\big\}, \qquad \big\{y \in K : v(y-a) \geqslant \gamma\big\}$$

for $a \in K$ and $\gamma \in \Gamma$ are called **open v-balls** and **closed v-balls**, respectively. The open v-balls form a basis for a field topology on K, called the **valuation topology on** K. Both open and closed v-balls are clopen with respect to this topology. If $\mathcal{O} \neq K$, then the valuation topology on K agrees with the order topology. If $\mathcal{O} = K$, then the valuation topology is the discrete topology.

The **residue field of** K is the quotient $\operatorname{res}(K) = \mathcal{O}/\smallo$. For $a \in \mathcal{O}$, we let $\bar{a} := a + \smallo$ denote the image of a under the quotient map $\mathcal{O} \to \operatorname{res}(K)$. Under this map, $\operatorname{res}(K)$ admits a natural expansion to a T-model; see [7, Remark 2.16] for details. A **lift of** $\operatorname{res}(K)$ is an elementary $\mathcal{L}$-substructure $\boldsymbol{k}$ of K contained in $\mathcal{O}$ such that the map $a \mapsto \bar{a}\colon \boldsymbol{k} \to \operatorname{res}(K)$ is an $\mathcal{L}$-isomorphism. By [7, Theorem 2.12], we can always find a lift of $\operatorname{res}(K)$. For $a = (a_1, \ldots, a_n) \in \mathcal{O}^n$, we let $\bar{a} := (\bar{a}_1, \ldots, \bar{a}_n) \in \operatorname{res}(K)^n$. For $D \subseteq K^n$, we let

$$\overline{D} \ :=\ \{\bar{a} : a \in D \cap \mathcal{O}^n\} \ \subseteq\ \operatorname{res}(K)^n.$$

Fact 1.2 ([5], 1.10)**.** *If $D \subseteq K^n$ is $\mathcal{L}(K)$-definable, then $\overline{D}$ is $\mathcal{L}(\operatorname{res} K)$-definable and*

$$\dim_{\mathcal{L}} \overline{D} \ \leqslant\ \dim_{\mathcal{L}} D.$$

Let M be a $T^{\mathcal{O}}$-extension of K with T-convex valuation ring $\mathcal{O}_M$ and maximal ideal $\smallo_M$. We view Γ as a subgroup of Γ_M and $\operatorname{res}(K)$ as an $\mathcal{L}$-substructure of $\operatorname{res}(M)$ in the obvious way. We let v and $x \mapsto \bar{x}$ denote their extensions to functions $M^{\times} \to \Gamma_M$ and $\mathcal{O}_M \to \operatorname{res}(M)$. If $\mathcal{O} \neq K$, then $\mathcal{O}_M \neq M$ and M is an elementary $T^{\mathcal{O}}$-extension of K by [7, Corollary 3.13]. If $\mathcal{O}_M = M$, then $\mathcal{O} = K$ so M is again an elementary $T^{\mathcal{O}}$-extension of K. The following Lemma on simple residue field extensions will be used in Section 7:

Lemma 1.3. *Let $M = K\langle a \rangle$ be a simple $T^{\mathcal{O}}$-extension of K with $\Gamma_M = \Gamma$, $a \asymp 1$, and $\bar{a} \notin \operatorname{res}(K)$. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function. Then $F'(a) \preccurlyeq F(a)$.*

Proof. Let $\boldsymbol{k} \subseteq \mathcal{O}^{\times}$ be a lift of $\operatorname{res}(K)$, so $\boldsymbol{k}\langle a \rangle$ is a lift of $\operatorname{res}(M)$ by [7, Lemma 5.1]. Using that $\Gamma_M = \Gamma$, take $b \in K^{>}$ with $F(a) \asymp b$. We need to show that $F'(a) \preccurlyeq b$. Since $\Gamma^{<}$ has no largest element, it suffices to show that $b^{-1}|F'(a)| < d$ for each $d \in K^{>}$ with $d \succ 1$. Let such an element d be given. By $\mathcal{L}$-elementarity, it is enough to show that for any subinterval $I \subseteq K^{>}$ with $a \in I^M$, there is $y \in I$ with $b^{-1}|F'(y)| < d$. Let I be such an interval and take an $\mathcal{L}(\boldsymbol{k})$-definable function $G\colon K \to K$ with $b^{-1}|F(a)| < G(a)$. By shrinking I, we arrange that F is $\mathcal{C}^1$ on I and that $b^{-1}|F(y)| < G(y)$ for all $y \in I$. As $\bar{a} \in \bar{I}^{\operatorname{res}(M)}$, we see that $\bar{I}$ must be infinite, so $I \cap \boldsymbol{k}$ is infinite. Take $y_1, y_2 \in I \cap \boldsymbol{k}$ with $y_1 < y_2$, so $y_2 - y_1 \asymp 1$. Note that $G(y_i) \in \boldsymbol{k}$, so $b^{-1}|F(y_i)| < G(y_i) \prec d$ for $i = 1, 2$. By the o-minimal mean value theorem, we have

$$b^{-1}F'(y) \ =\ \frac{b^{-1}F(y_2) - b^{-1}F(y_1)}{y_2 - y_1} \ \prec\ d$$

for some $y \in I$ between y_1 and y_2. In particular, $b^{-1}|F'(y)| < d$. $\square$

1.1. **Immediate extensions.** In this subsection, let M be a $T^{\mathcal{O}}$-extension of K. If $\Gamma_M = \Gamma$ and $\operatorname{res}(M) = \operatorname{res}(K)$, then M is said to be an **immediate extension of** K. If M is an immediate extension of K, then M is an elementary $T^{\mathcal{O}}$-extension of K. Note that M is an immediate extension of K if and only if for all $a \in M^\times$ there is $b \in K^\times$ with $a \sim b$. The next lemma shows that in an immediate extension of K, we can approximate $\mathcal{L}^{\mathcal{O}}(K)$-definable sets by $\mathcal{L}(K)$-definable sets.

Lemma 1.4. *Suppose M is an immediate extension of K, let $A \subseteq K^n$ be an $\mathcal{L}^{\mathcal{O}}(K)$-definable set, and let $a \in A^M$. Then there is an $\mathcal{L}(K)$-definable cell $D \subseteq A$ with $a \in D^M$.*

Proof. It suffices to find an $\mathcal{L}(K)$-definable set $B \subseteq A$ with $a \in B^M$, for then we can replace B with a subcell D in a cell decomposition of B. If K is trivially valued, then A is already $\mathcal{L}(K)$-definable, so we may assume $\mathcal{O} \neq K$. Since T has definable Skolem functions, we may arrange that T has quantifier elimination and a universal axiomatization by extending $\mathcal{L}$ by function symbols for all $\mathcal{L}(\emptyset)$-definable functions. Let T^* be the extension of $T^{\mathcal{O}}$ by an axiom asserting that $\mathcal{O} \neq K$. Then T^* eliminates quantifiers by [7], so

$$A \ = \ \bigcup_{i \leqslant m} \bigcap_{j \leqslant k} A_{i,j}$$

where either $A_{i,j}$ is $\mathcal{L}(K)$-definable or

$$A_{i,j} \ = \ \big\{y \in K^n : F(y) \in \mathcal{O}\big\} \quad \text{or} \quad A_{i,j} \ = \ \big\{y \in K^n : F(y) \notin \mathcal{O}\big\}$$

for some $\mathcal{L}(K)$-definable function $F\colon K^n \to K$. For each $i \leqslant m$ and each $j \leqslant k$, we take an $\mathcal{L}(K)$-definable set $B_{i,j} \subseteq A_{i,j}$ such that if $a \in A_{i,j}^M$, then $a \in B_{i,j}^M$. We do this as follows.

(i) If $A_{i,j}$ is $\mathcal{L}(K)$-definable, then we set $B_{i,j} := A_{i,j}$.

(ii) Suppose $A_{i,j} = \big\{y \in K^n : F(y) \in \mathcal{O}\big\}$ for some $\mathcal{L}(K)$-definable F. If $F(a) \notin \mathcal{O}_M$, then we set $B_{i,j} := \emptyset$. If $F(a) \in \mathcal{O}_M$, then since $\operatorname{res}(M) = \operatorname{res}(K)$, we may take $u \in K^>$ with $u \asymp 1$ and $|F(a)| < u$. We set
$$B_{i,j} \ := \ \big\{y \in K^n : |F(y)| < u\big\}.$$

(iii) Suppose $A_{i,j} = \big\{y \in K^n : F(y) \notin \mathcal{O}\big\}$ for some $\mathcal{L}(K)$-definable F. If $F(a) \in \mathcal{O}_M$, then we set $B_{i,j} := \emptyset$. If $F(a) \notin \mathcal{O}_M$, then since $\Gamma_M = \Gamma$, we may take $d \in K^>$ with $d \succ 1$ and $|F(a)| > d$. We set
$$B_{i,j} \ := \ \big\{y \in K^n : |F(y)| > d\big\}.$$

Now set
$$B \ := \ \bigcup_{i \leqslant m} \bigcap_{j \leqslant k} B_{i,j}.$$
$\square$

Corollary 1.5. *Suppose M is an immediate extension of K, let $F\colon A \to K$ be an $\mathcal{L}^{\mathcal{O}}(K)$-definable function, and let $a \in A^M$. Then there is an $\mathcal{L}(K)$-definable cell $D \subseteq A$ with $a \in D^M$ such that either $F(y) = 0$ for all $y \in D^M$ or $F(y) \sim F(a)$ for all $y \in D^M$.*

Proof. If $F(a) = 0$, then apply Lemma 1.4 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set $\big\{y \in A : F(y) = 0\big\}$. If $F(a) \neq 0$, then take $b \in K^\times$ with $F(a) \sim b$ and apply Lemma 1.4 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set $\big\{y \in A : F(y) \sim b\big\}$. $\square$

Let ℓ be an element in an immediate $T^{\mathcal{O}}$-extension of K. Then the set

$$v(\ell - K) \ := \ \big\{v(\ell - y) : y \in K\big\}$$

is contained in Γ and has no largest element. To see this, let $y \in K$ be given and take $b \in K$ with $\ell - y \sim b$. Then $v(\ell - y - b) > v(\ell - y) = v(b) \in \Gamma$. These values $v(\ell - y)$ completely determine the extension $K\langle \ell \rangle$ up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism:

Corollary 1.6. *Let $K\langle \ell \rangle$ be a simple immediate $T^{\mathcal{O}}$-extension of K and let $a \in M$ with $v(a - y) = v(\ell - y) \in \Gamma$ for each $y \in K$. Then there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \ell \rangle \to M$ sending ℓ to a.*

Proof. First, we will show that a and ℓ realize the same cut in K. Let $y \in K$ with $y < \ell$ and take $f \in K^>$ with $\ell - y \sim f$. Then $\ell - y - f \prec f$, so $a - y - f \asymp \ell - y - f \prec f$. Thus $a - y \sim f > 0$. Likewise, if $y \in K$ and $y > \ell$, then $y > a$. This gives us a unique $\mathcal{L}(K)$-embedding $\iota\colon K\langle \ell \rangle \to M$ sending ℓ to a. To get that ι is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding, let $F\colon K \to K$ be $\mathcal{L}(K)$-definable. We need to show that $F(\ell) \in \mathcal{O}_{K\langle \ell \rangle}$ if and only if $F(a) \in \mathcal{O}_M$. We assume that $F(\ell) \neq 0$, and we will show that $F(\ell) \sim F(a)$. Using Corollary 1.5, take an interval $I \subseteq K$ with $\ell \in I^{K\langle \ell \rangle}$ such that $F(y) \sim F(\ell)$ for all $y \in I^{K\langle \ell \rangle}$. Since K has a proper immediate extension, it is not trivially valued, so M is an *elementary* $T^{\mathcal{O}}$-extension of K. Thus, $F(a) \sim F(y) \sim F(\ell)$, since $a \in I^M$. $\square$

Let $\mathcal{B}$ be a collection of closed v-balls in K. Then $\mathcal{B}$ is said to be **nested** if $B_1 \cap B_2 \neq \emptyset$ for any $B_1, B_2 \in \mathcal{B}$. If $\mathcal{B}$ is nested, then it is totally ordered by inclusion. If every nested collection of closed v-balls in K has nonempty intersection in K, then K is said to be **spherically complete**. If M is an elementary $T^{\mathcal{O}}$-extension of K, then we let $\mathcal{B}^M$ denote the collection $\{B^M : B \in \mathcal{B}\}$. Below we list some facts about spherical completeness. These are all standard facts from valuation theory, but we include brief proofs.

Lemma 1.7.

(1) *Let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection. Then K has a simple elementary $T^{\mathcal{O}}$-extension $K\langle a \rangle$ with $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$.*

(2) *Let $\mathcal{B}$ and a be as in (1). Then the set $v(a - K)$ has no largest element and for $y \in K$, the value $v(a - y)$ does not depend on the choice of a, just on the assumption $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$.*

(3) *Suppose M is an immediate extension of K and let $a \in M \setminus K$. Then the collection of all closed v-balls B in K with $a \in B^M$ is nested and has empty intersection in K.*

(4) *If K is spherically complete, then K has no proper immediate $T^{\mathcal{O}}$-extensions.*

Proof. For (1), the assumption that $\mathcal{B}$ is nested gives that $\bigcap \mathcal{B}_0$ is nonempty for any finite subcollection $\mathcal{B}_0 \subseteq \mathcal{B}$. By model theoretic compactness, we can find a in an elementary $T^{\mathcal{O}}$-extension of K with $a \in \bigcap \mathcal{B}^{K\langle a \rangle}$. For (2), let $y \in K$, take a v-ball $B \in \mathcal{B}$ which doesn't contain y, and let $b \in B$. Then $v(a - b) > v(a - y)$, so $v(a - K)$ has no largest element. We also have $v(a - y) = v(b - y)$, and $v(b - y)$ clearly does not depend on a. For (3), let B_1, B_2 be closed v-balls with $a \in B_1^M \cap B_2^M$. As M is an elementary $T^{\mathcal{O}}$-extension of K and B_1, B_2 are $\mathcal{L}^{\mathcal{O}}(K)$-definable, there is $b \in K$ with $b \in B_1 \cap B_2$. This shows that the collection of all closed v-balls B in K with $a \in B^M$ is nested. To see that this collection has empty intersection in K, let $b \in K$ be given and take $d \in K$ with $v(a - d) > v(a - b)$. Let

$$\gamma \ := \ v(a - d), \qquad B \ = \ \{y \in K : v(y - d) \geqslant \gamma\}.$$

Then $a \in B^M$ but $b \notin B$. Finally, (4) follows immediately from (3). $\square$

1.2. **Power bounded theories.** A **power function on** K is an $\mathcal{L}(K)$-definable endomorphism of the multiplicative group $K^>$. Each power function F is $\mathcal{C}^1$ on $K^>$ and uniquely determined by $F'(1)$, and we set

$$\Lambda \ := \ \{F'(1) : F \text{ is a power function on } K\}.$$

Then Λ is a subfield of K, and it is called the **field of exponents of** K. For $a \in K^>$ and a power function F, we suggestively write $F(a)$ as a^λ where $\lambda = F'(1)$. We say that K is **power bounded** if for each $\mathcal{L}(K)$-definable function $F\colon K \to K$, there is λ in the field of exponents of K with $|F(x)| < x^\lambda$ for all sufficiently large positive x.

An **exponential function on** K is an ordered group isomorphism from the additive group K to the multiplicative group $K^>$. Any exponential function on K grows more quickly than every power function on K. By [17], either K is power bounded or K defines an exponential function. It follows that being power bounded is a property of the theory T (we say that T *is power bounded*). If T is power bounded, then each power function on K is $\mathcal{L}(\emptyset)$-definable, so we refer to the field of exponents Λ as the *field of exponents of* T, as Λ does not depend on K. If T is power bounded with archimedean field of exponents, then T is said to be **polynomially bounded**.

If T is power bounded with field of exponents Λ, then the value group Γ naturally admits the structure of an ordered Λ-vector space by

$$\lambda v(a) \ := \ v(a^\lambda)$$

for $a \in K^{>}$ (this does not depend on the choice of a). If T is power bounded, then we have a better understanding of the immediate $T^{\mathcal{O}}$-extensions of K. This is due to the following result of Tyne:

Fact 1.8 ([22], 12.10 and 13.4)**.** *Let T be power bounded and let $K\langle\ell\rangle$ be a simple $T^{\mathcal{O}}$-extension of K.*
(1) If $\Gamma_{K\langle\ell\rangle} \neq \Gamma$, then there is $a \in K$ with $v(\ell - a) \notin \Gamma$.
(2) If $\operatorname{res} K\langle\ell\rangle \neq \operatorname{res} K$, then there are $a, b \in K$ with $b(\ell - a) \preccurlyeq 1$ and $\overline{b(\ell - a)} \notin \operatorname{res} K$.

Item (1) above is often referred to as the **valuation property**. Item (2) is called the **residue property**. These properties allow us to characterize the simple immediate $T^{\mathcal{O}}$-extensions of K as follows:

Lemma 1.9. *Let T be power bounded and let $K\langle\ell\rangle$ be a simple $T^{\mathcal{O}}$-extension of K. The following are equivalent:*
(1) $v(\ell - K)$ has no largest element;
(2) for each $a \in K$ there is $d \in K$ with $\ell - a \sim d$;
(3) $K\langle\ell\rangle$ is an immediate extension of K.

Proof. Assume (1) holds, let $a \in K$, and take $b \in K$ with $\ell - b \prec \ell - a$. Then $\ell - a \sim \ell - a - (\ell - b) = b - a \in K$. Now, assume (2) holds. Then $v(\ell - a) \in \Gamma$ for each $a \in K$, so $\Gamma_{K\langle\ell\rangle} = \Gamma$ by the valuation property. Let $a, b \in K$ with $b(\ell - a) \preccurlyeq 1$ and take $d \in K$ with $\ell - a \sim d$. If $b = 0$, then $\overline{b(\ell - a)} = 0 \in \operatorname{res}(K)$ and if $b \neq 0$, then $b(\ell - a) \sim bd$ so $\overline{b(\ell - a)} = \overline{bd} \in \operatorname{res}(K)$. Thus, $\operatorname{res} K\langle\ell\rangle = \operatorname{res} K$ by the residue property. Finally, suppose (3) holds, let $a \in K$, and take $d \in K$ with $\ell - a \sim d$. Then $v(\ell - a - d) > v(\ell - a)$, so $v(\ell - K)$ has no largest element. $\square$

We can use this equivalence together with Lemma 1.7 to show that if T is power bounded, then any nested collection of closed v-balls has nonempty intersection in an *immediate* $T^{\mathcal{O}}$-extension of K.

Corollary 1.10. *Let T be power bounded and let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection. Then there is a simple immediate $T^{\mathcal{O}}$-extension $K\langle\ell\rangle$ of K with $\ell \in \bigcap \mathcal{B}^{K\langle\ell\rangle}$. Given $a \in \bigcap \mathcal{B}^{M}$, there is a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a.*

Proof. Using (1) of Lemma 1.7, let $K\langle\ell\rangle$ be a simple elementary $T^{\mathcal{O}}$-extension of K with $\ell \in \bigcap \mathcal{B}^{K\langle\ell\rangle}$. By (2) of Lemma 1.7, the set $v(\ell - K)$ has no largest element, so $K\langle\ell\rangle$ is an immediate extension of K by Lemma 1.9. For $a \in \bigcap \mathcal{B}^{M}$, we have $v(\ell - y) = v(a - y)$ for all $y \in K$ by (2) of Lemma 1.7, so Corollary 1.6 gives us a unique $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a. $\square$

Corollary 1.11. *Suppose T is power bounded. Then K has a spherically complete immediate $T^{\mathcal{O}}$-extension which is unique up to $\mathcal{L}^{\mathcal{O}}(K)$-isomorphism.*

Proof. If K is not itself spherically complete, then K has a proper immediate $T^{\mathcal{O}}$-extension by Corollary 1.10. It follows by Zorn's lemma that K has a spherically complete immediate $T^{\mathcal{O}}$-extension. For uniqueness, let L and M be two spherically complete immediate $T^{\mathcal{O}}$-extensions of K. We first show that there is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $L \to M$. For this, we assume $K \neq L$, and we let $\ell \in L \setminus K$. Let $\mathcal{B}$ be the collection of all closed v-balls B in K with $\ell \in B^{L}$. This collection is nested and has empty intersection in K by (3) of Lemma 1.7. Let $a \in \bigcap \mathcal{B}^{M}$ and, again using Corollary 1.10, take an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle\ell\rangle \to M$ sending ℓ to a. Continuing in this manner, we construct an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $L \to M$, and we identify L with an $\mathcal{L}^{\mathcal{O}}$-substructure of M via this embedding. Then M is an immediate extension of L, so $L = M$ by (4) of Lemma 1.7, as L is spherically complete. $\square$

The assumption of power boundedness in Corollary 1.11 is necessary.

Remark 1.12. *If T is not power bounded and $\mathcal{O} \neq K$, then K has no spherically complete $T^{\mathcal{O}}$-extension. To see this, we use Miller's dichotomy and a theorem of Kuhlmann, Kuhlmann, and Shelah. Suppose toward contradiction that K is itself spherically complete, $\mathcal{O} \neq K$, and T is not power bounded. By* [17]*, K admits an $\mathcal{L}(\emptyset)$-definable exponential function* exp*. Let $\boldsymbol{k} \subseteq \mathcal{O}$ be a lift of* $\operatorname{res}(K)$*, so $\exp|_{\boldsymbol{k}}$ is an exponential function on $\boldsymbol{k}$. Using* [2, Lemma 3.3.32]*, we take a subgroup $\mathfrak{M} \subseteq K^{>}$ such that $v|_{\mathfrak{M}} \colon \mathfrak{M} \to \Gamma$ is a group isomorphism. Using* [2, Corollary 3.3.42]*, we get an ordered valued field isomorphism from K to the ordered Hahn field $\boldsymbol{k}[[\mathfrak{M}]]$ which is the identity on $\mathfrak{M}$ and $\boldsymbol{k}$. The exponential on K induces an exponential on $\boldsymbol{k}[[\mathfrak{M}]]$ which restricts to the exponential $\exp|_{\boldsymbol{k}}$ on $\boldsymbol{k}$, contradicting the main theorem in* [16].

1.3. **Coarsening and specialization.** In this subsection, we set up notation and prove some basic lemmas about coarsening and specialization. For the remainder of this subsection, we assume that T is power bounded with field of exponents Λ, and we let Δ be a nontrivial convex Λ-subspace of Γ. We set $\dot{\Gamma} := \Gamma/\Delta$, and we let $\dot{v}\colon K^\times \to \dot{\Gamma}$ be the map $a \mapsto va + \Delta \in \dot{\Gamma}$. Then $\dot{v}$ is a Krull valuation on K with valuation ring and maximal ideal

$$\dot{\mathcal{O}} := \{y \in K : vy \geqslant \delta \text{ for some } \delta \in \Delta\}, \qquad \dot{\smallo} := \{y \in K : vy > \Delta\}.$$

Lemma 1.13. *The valuation ring $\dot{\mathcal{O}}$ is a T-convex valuation ring of K.*

Proof. Let $F\colon K \to K$ be a continuous $\mathcal{L}(\emptyset)$-definable function and let $a \in \dot{\mathcal{O}}$. We need to show that $F(a) \in \dot{\mathcal{O}}$. If $va \geqslant 0$, then $a \in \mathcal{O}$, so $F(a) \in \mathcal{O} \subseteq \dot{\mathcal{O}}$. Suppose $va < 0$, so $va \in \Delta^<$. Take $\lambda \in \Lambda$ and $b \in K^>$ such that $|F(y)| < |y|^\lambda$ for all $y \in K$ with $|y| > b$. As F is $\mathcal{L}(\emptyset)$-definable, we may assume $\{b\}$ is $\mathcal{L}(\emptyset)$-definable as well, so $vb = 0$ and $|a| > b$. Then $|F(a)| < |a|^\lambda$, so $vF(a) \geqslant \lambda va \in \Delta$. □

We let K_Δ denote the $\mathcal{L}^{\mathcal{O}}$-structure $(K, \dot{\mathcal{O}})$, so $K_\Delta \models T^{\mathcal{O}}$ by the above lemma. We refer to K_Δ as the **Δ-coarsening of** K. The residue field $\operatorname{res}(K_\Delta) = \dot{\mathcal{O}}/\dot{\smallo}$ is itself a $T^{\mathcal{O}}$-model with valuation $v\colon \operatorname{res}(K_\Delta) \to \Delta$ given by

$$v(a + \dot{\smallo}) := va$$

for $a \in \dot{\mathcal{O}} \setminus \dot{\smallo}$ and with T-convex valuation ring $\mathcal{O}_{\operatorname{res}(K_\Delta)} = \{a + \dot{\smallo} : a \in \mathcal{O}\}$. We refer to $\operatorname{res}(K_\Delta)$ with this valuation ring as the **Δ-specialization of** K. Note that $\mathcal{O}_{\operatorname{res}(K_\Delta)}/\smallo_{\operatorname{res}(K_\Delta)}$ is naturally $\mathcal{L}$-isomorphic to $\operatorname{res}(K)$.

Fact 1.14 ([2], 3.4.6)**.** *K is spherically complete if and only if K_Δ and $\operatorname{res}(K_\Delta)$ are both spherically complete.*

Let M be a $T^{\mathcal{O}}$-extension of K_Δ with $\Gamma_M = \dot{\Gamma}$. Let $\mathcal{O}_{\operatorname{res}(M)}$ be a T-convex valuation ring of $\operatorname{res}(M)$ and suppose that the expansion of $\operatorname{res}(M)$ by $\mathcal{O}_{\operatorname{res}(M)}$ is a $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_\Delta)$ with $\Gamma_{\operatorname{res}(M)} = \Delta$. Let $\mathcal{O}^*_M \subseteq M$ be the convex subring

$$\mathcal{O}^*_M := \{a \in M : a \in \mathcal{O}_M \text{ and } \bar{a} \in \mathcal{O}_{\operatorname{res}(M)}\}.$$

Lemma 1.15. *The convex subring $\mathcal{O}^*_M$ is a T-convex valuation ring of M and $\mathcal{O}^*_M \cap K = \mathcal{O}$.*

Proof. Let $a \in \mathcal{O}^*_M$ and let $F\colon M \to M$ be an $\mathcal{L}(\emptyset)$-definable continuous function. Since $\bar{a} \in \mathcal{O}_{\operatorname{res}(M)}$ and $\mathcal{O}_{\operatorname{res}(M)}$ is T-convex, we have $\overline{F(a)} = F(\bar{a}) \in \mathcal{O}_{\operatorname{res}(M)}$ by [7, Lemma 1.13]. Thus, $F(a) \in \mathcal{O}^*_M$, so $\mathcal{O}^*_M$ is T-convex. The equality $\mathcal{O}^*_M \cap K = \mathcal{O}$ follows from the equivalence

$$y \in \mathcal{O} \iff y \in \dot{\mathcal{O}} \text{ and } y + \dot{\smallo} \in \mathcal{O}_{\operatorname{res}(K_\Delta)}$$

for $y \in K$. □

Let M^* be the $T^{\mathcal{O}}$-model with underlying T-model M and T-convex valuation ring $\mathcal{O}_{M^*} = \mathcal{O}^*_M$, as defined above. Then M^* is a $T^{\mathcal{O}}$-extension of K, and we have $M^*_\Delta = M$ (as $T^{\mathcal{O}}$-models).

Lemma 1.16. *$\Gamma_{M^*} = \Gamma$ and $\operatorname{res}(M^*)$ is naturally $\mathcal{L}$-isomorphic to $\mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$.*

Proof. As M^* is a $T^{\mathcal{O}}$-extension of K, we have $\Gamma \subseteq \Gamma_{M^*}$. For the other inclusion, let $a \in (M^*)^\times$. We need to find $b \in K^\times$ with $ab^{-1} \in \mathcal{O}^\times_{M^*}$. First, take $f \in K^\times_\Delta$ with $af^{-1} \in \mathcal{O}^\times_M$ and set $u := \overline{af^{-1}} \in \operatorname{res}(M)^\times$. Next, take $g \in \dot{\mathcal{O}}^\times$ with $u\bar{g}^{-1} \in \mathcal{O}^\times_{\operatorname{res}(M)}$. Then for $b := fg$, we have $ab^{-1} \in \mathcal{O}^\times_{M^*}$, as desired. As for the residue field, note that $\{\bar{a} : a \in \mathcal{O}_{M^*}\} = \mathcal{O}_{\operatorname{res}(M)}$, so we have a surjection

$$a \mapsto \bar{a} + \smallo_{\operatorname{res}(M)} \colon \mathcal{O}_{M^*} \to \mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$$

with kernel

$$\{a \in \mathcal{O}_{M^*} : \bar{a} \in \smallo_{\operatorname{res}(M)}\} = \smallo_{M^*}.$$

This induces a natural isomorphism $\operatorname{res}(M^*) \to \mathcal{O}_{\operatorname{res}(M)}/\smallo_{\operatorname{res}(M)}$, as required. □

By the above lemma, we see that M^* is an immediate extension of K if and only if $\operatorname{res}(M)$ is an immediate extension of $\operatorname{res}(K_\Delta)$. We summarize the discussion above with a diagram:

$$\begin{tikzcd}
K_\Delta \arrow[r, "\mathcal{L}^{\mathcal{O}}"'] \arrow[d] & M = M^*_\Delta \arrow[d] \\
\operatorname{res}(K_\Delta) \arrow[r, "\mathcal{L}^{\mathcal{O}}"'] \arrow[d] & \operatorname{res}(M) \arrow[d] \\
\operatorname{res}(K) \arrow[r, "\mathcal{L}"'] & \operatorname{res}(M^*)
\end{tikzcd}$$

Horizontal arrows are all embeddings in the indicated language. Downward arrows are projections and are only defined on the T-convex valuation ring of their source. Every square commutes.

2. T-derivations

In this section, we fix a map $\partial\colon K \to K$. For $a \in K$, we use a' or ∂a in place of $\partial(a)$, and we use $a^{(r)}$ in place of $\partial^r(a)$. If $a \neq 0$, then we set $a^\dagger := a'/a$. We define the **jets of** a as follows:

$$\mathfrak{J}^r_\partial(a) \;:=\; (a, a', \ldots, a^{(r)}), \qquad \mathfrak{J}^\infty_\partial(a) := (a^{(n)})_{n\in\mathbb{N}}.$$

Occasionally, we omit the parentheses and write $\mathfrak{J}^r_\partial a$ or $\mathfrak{J}^\infty_\partial a$. To make some statements cleaner, we let $\mathfrak{J}^{-1}_\partial(a) := 0 \in K^0$. Given a set $A \subseteq K$, we let

$$\partial A \;:=\; \{a' : a \in A\}, \qquad \mathfrak{J}^r_\partial(A) \;:=\; \big\{\mathfrak{J}^r_\partial(a) : a \in A\big\}.$$

Given a tuple $b = (b_1, \ldots, b_n) \in K^n$, we use ∂b or b' to denote the tuple $(b'_1, \ldots, b'_n)$.

Given an $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function $F\colon U \to K$ with $U \subseteq K^n$ open, we say that ∂ is **compatible with** F if

$$F(u)' \;=\; \mathbf{J}_F(u)u'$$

for each $u \in U$. We say that ∂ is a **T-derivation on** K if ∂ is compatible with every $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with open domain.

The basic properties of T-derivations were systematically studied in [11]. Compatibility with the functions $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ gives that any T-derivation on K is a derivation on K. Let $\mathcal{L}^\partial := \mathcal{L} \cup \{\partial\}$, and let T^∂ be the $\mathcal{L}^\partial$-theory which extends T by axioms asserting that ∂ is a T-derivation. For the rest of this section, let $K = (K, \partial) \models T^\partial$. We let $C := \ker(\partial)$ denote the constant field of K. Lemma 2.3 in [11] gives that C is the underlying set of an elementary $\mathcal{L}$-substructure of K. We recall three facts from [11] for later use:

Fact 2.1 ([11], Lemma 2.12)**.** *Let $U \subseteq K^n$ be open and let $F\colon U \to K$ be an $\mathcal{L}(K)$-definable $\mathcal{C}^1$-function. Then there is a (necessarily unique) $\mathcal{L}(K)$-definable function $F^{[\partial]}\colon U \to K$ such that*

$$F(u)' \;=\; F^{[\partial]}(u) + \mathbf{J}_F(u)u'$$

for all $u \in U$.

Fact 2.2 ([11], Lemma 2.13)**.** *Let M be a T-extension of K, let A be a $\operatorname{dcl}_{\mathcal{L}}$-basis for M over K, and let $s\colon A \to M$ be a map. There is a unique extension of ∂ to a T-derivation on M such that $a' = s(a)$ for all $a \in A$.*

Fact 2.3 ([11], Lemma 4.2)**.** *Let M be a T-extension of K with $\operatorname{rk}_{\mathcal{L}}(M|K) = n$ and let $A \subseteq M^{n+1}$ be an $\mathcal{L}(K)$-definable set with $\dim_{\mathcal{L}} \pi_n(A) = n$. Then there is $b \in M$ and an extension of ∂ to a T-derivation on M such that $\mathfrak{J}^n_\partial(b) \in A$.*

2.1. **Affine conjugation.** In this subsection, fix $r \geqslant 0$ and let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function. For $k = 0, \dots, r$, we identify each variable Y_k with the k^{th} coordinate function $K^{1+r} \to K$. We let $Y = (Y_0, \dots, Y_r)$, so $Y\colon K^{1+r} \to K^{1+r}$ is the identity map.

Definition 2.4. *F is said to be in **implicit form** if*

$$F \ = \ \mathfrak{m}_F\big(Y_r - I_F(Y_0, \dots, Y_{r-1})\big)$$

for some $\mathfrak{m}_F \in K^\times$ and some $\mathcal{L}(K)$-definable function $I_F\colon K^r \to K$.

If F is in implicit form, then $F(a, b) = 0$ if and only if $b = I_F(a)$ for $a \in K^r$ and $b \in K$. Thus, I_F is an *implicit function* for F. This is the source of the name "implicit form" and the notation I_F. By our convention for nullary functions, the unary functions in implicit form are exactly the functions of the form $\mathfrak{m}(Y_0 - d)$ where $\mathfrak{m} \in K^\times$ and $d \in K$. Often, we omit the variables $Y_0, \dots, Y_{r-1}$ and just write $F = \mathfrak{m}_F(Y_r - I_F)$ for F in implicit form.

We may associate to F the unary $\mathcal{L}^\partial(K)$-definable function $y \mapsto F(\mathfrak{J}^r_\partial y)$. For $k \leqslant r$ and $y \in K$, we have $Y_k(\mathfrak{J}^r_\partial y) = y^{(k)}$. As is the case with differential polynomials, these functions $y \mapsto F(\mathfrak{J}^r_\partial y)$ can be additively and multiplicatively conjugated (an operation we call *affine conjugation*). They can also be compositionally conjugated, as we will see in Subsection 2.2.

Let $a, d \in K$. We let $Y^\partial_{+a,\times d} = \big((Y_0)^\partial_{+a,\times d}, \dots, (Y_r)^\partial_{+a,\times d}\big)\colon K^{1+r} \to K^{1+r}$ be the map with coordinate functions

$$(Y_k)^\partial_{+a,\times d} \ := \ a^{(k)} + \sum_{i=0}^{k} \binom{k}{i} d^{(k-i)} Y_i, \qquad k = 0, \dots, r.$$

Then for $y \in K$ and $k \leqslant r$, we have

$$(Y_k)^\partial_{+a,\times d}(\mathfrak{J}^r_\partial y) \ = \ a^{(k)} + \sum_{i=0}^{k} \binom{k}{i} d^{(k-i)} y^{(i)} \ = \ (dy + a)^{(k)},$$

so $Y^\partial_{+a,\times d}(\mathfrak{J}^r_\partial y) = \mathfrak{J}^r_\partial(dy + a)$. Note that $Y^\partial_{+a,\times d}$ is a K-affine map which is bijective if $d \neq 0$ and which takes constant value $\mathfrak{J}^r_\partial(a)$ if $d = 0$. We let $F^\partial_{+a,\times d} := F \circ Y^\partial_{+a,\times d}$, so

$$F^\partial_{+a,\times d}(\mathfrak{J}^r_\partial y) \ = \ F\big(\mathfrak{J}^r_\partial(dy + a)\big)$$

for each $y \in K$. When ∂ is clear from context, we drop the superscript and just write $F_{+a,\times d}$. For notational simplicity, we let

$$F_{+a} \ := F_{+a,\times 1}, \qquad F_{\times d} \ := F_{+0,\times d}, \qquad F_{-a,\times d} \ := \ F_{+(-a),\times d}.$$

Lemma 2.5. *Suppose F is in implicit form and $d \neq 0$. Then $F_{+a,\times d}$ is also in implicit form with*

$$\mathfrak{m}_{F_{+a,\times d}} \ = \ d\mathfrak{m}_F, \qquad I_{F_{+a,\times d}} \ = \ d^{-1}\Big((I_F)_{+a,\times d} - a^{(r)} - \sum_{i=0}^{r-1} \binom{r}{i} d^{(r-i)} Y_i\Big).$$

Proof. Define $G\colon K^r \to K$ by

$$G \ := \ d^{-1}\Big((I_F)_{+a,\times d} - a^{(r)} - \sum_{i=0}^{r-1} \binom{r}{i} d^{(r-i)} Y_i\Big).$$

Then

$$\begin{aligned} F_{+a,\times d} \ &= \ \mathfrak{m}_F\big((Y_r)_{+a,\times d} - (I_F)_{+a,\times d}\big) \ = \ \mathfrak{m}_F\Big(a^{(r)} + \sum_{i=0}^{r} \binom{r}{i} d^{(r-i)} Y_i - (I_F)_{+a,\times d}\Big) \\ &= \ \mathfrak{m}_F\Big(dY_r + a^{(r)} + \sum_{i=0}^{r-1} \binom{r}{i} d^{(r-i)} Y_i - (I_F)_{+a,\times d}\Big) \ = \ d\mathfrak{m}_F(Y_r - G), \end{aligned}$$

so $\mathfrak{m}_{F_{+a,\times d}} = d\mathfrak{m}_F$ and $I_{F_{+a,\times d}} = G$. □

Given an $\mathcal{L}(K)$-definable set $A \subseteq K^{1+r}$, we set

$$A^{\partial}_{+a,\times d} \;:=\; \{u \in K^{1+r} : Y^{\partial}_{+a,\times d}(u) \in A\},$$

so $A^{\partial}_{+a,\times d}$ is $\mathcal{L}(K)$-definable and

$$\mathcal{J}^r_\partial(y) \in A^{\partial}_{+a,\times d} \iff \mathcal{J}^r_\partial(dy+a) \in A$$

for $y \in K$. If $d \neq 0$, then $Y_{+a,\times d}$ is an $\mathcal{L}(K)$-definable bijection, so $\dim_{\mathcal{L}}(A^{\partial}_{+a,\times d}) = \dim_{\mathcal{L}}(A)$. Again, we drop the superscript if ∂ is clear from context.

2.2. **Compositional conjugation.** In this subsection, $r \geqslant 0$ is again fixed and $F\colon K^{1+r} \to K$ is an $\mathcal{L}(K)$-definable function. We continue to identify $Y = (Y_0, \ldots, Y_r)$ with the identity map and each Y_k with the k^{th} coordinate function.

Let $\phi \in K^\times$. Then $\delta := \phi^{-1}\partial$ is also a T-derivation on K, since for each $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function $G\colon U \to K$ with $U \subseteq K^n$ open and for each $u \in U$, we have

$$\delta G(u) \;=\; \phi^{-1}\partial G(u) \;=\; \phi^{-1}\big(\mathbf{J}_G(u)\partial u\big) \;=\; \mathbf{J}_G(u)\delta u.$$

We let $K^\phi = (K, \delta)$ be the expansion of the $\mathcal{L}$-structure K by the T-derivation δ, and we refer to K^ϕ as the **compositional conjugate of K by ϕ**. For $\psi \in K^\times$, we have $(K^\phi)^\psi = K^{\phi\psi}$.

Let $n \geqslant 0$. Subsection 5.7 in [2] gives for each $k \leqslant n$ an element $\xi^n_k(\phi) \in \mathbb{Q}[\phi, \partial\phi, \ldots, \partial^n\phi]$ such that

$$\partial^n y \;=\; \xi^n_0(\phi)y + \xi^n_1(\phi)\delta y + \cdots + \xi^n_n(\phi)\delta^n y.$$

In [2], $\xi^n_k(\phi)$ is instead called $F^n_k(\phi)$; we use different notation here and reserve F for definable functions. The values of $\xi^n_k(\phi)$ are given by the recurrence relation:

$$\xi^n_n(\phi) \;=\; \phi^n, \qquad \xi^n_0(\phi) \;=\; 0 \text{ for } n > 0, \qquad \xi^{n+1}_k(\phi) \;=\; \partial\xi^n_k(\phi) + \phi\xi^n_{k-1}(\phi) \text{ for } 0 < k \leqslant n. \tag{2.1}$$

Let Y^ϕ_∂ be the K-linear map $K^{1+r} \to K^{1+r}$ with matrix

$$\begin{pmatrix} \xi^0_0(\phi) & 0 & 0 & \cdots & 0 \\ \xi^1_0(\phi) & \xi^1_1(\phi) & 0 & \cdots & 0 \\ \xi^2_0(\phi) & \xi^2_1(\phi) & \xi^2_2(\phi) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi^r_0(\phi) & \xi^r_1(\phi) & \xi^r_2(\phi) & \cdots & \xi^r_r(\phi) \end{pmatrix}.$$

Then Y^ϕ_∂ is bijective and $Y^\phi_\partial(\mathcal{J}^r_\delta y) = \mathcal{J}^r_\partial(y)$ for each $y \in K$. An easy induction on r using (2.1) gives that Y^1_∂ is the identity map on K^{1+r}.

Lemma 2.6. *For $\psi \in K^\times$, we have $Y^\phi_\partial \circ Y^\psi_\delta = Y^{\phi\psi}_\partial$.*

Proof. Clearly, this identity holds on a closed subset of K^{1+r}, so it is enough to show that it holds on a dense subset. Let $U \subset K^{1+r}$ be $\mathcal{L}(K)$-definable and open. Using Fact 2.3 with $\psi^{-1}\delta$ in place of ∂, we take a T^∂-extension $M = K\langle \mathcal{J}^r_{\psi^{-1}\delta} b\rangle$ with $\mathcal{J}^r_{\psi^{-1}\delta}(b) \in U^M$. We have

$$Y^\phi_\partial\big(Y^\psi_\delta(\mathcal{J}^r_{\psi^{-1}\delta} b)\big) \;=\; Y^\phi_\partial(\mathcal{J}^r_\delta b) \;=\; \mathcal{J}^r_\partial(b) \;=\; Y^{\phi\psi}_\partial(\mathcal{J}^r_{\psi^{-1}\delta} b).$$

Since $Y^\phi_\partial \circ Y^\psi_\delta$ and $Y^{\phi\psi}_\partial$ are $\mathcal{L}(K)$-definable and K is an elementary $\mathcal{L}$-substructure of M, there is a tuple $u \in U$ with

$$(Y^\phi_\partial \circ Y^\psi_\delta)(u) \;=\; Y^{\phi\psi}_\partial(u). \qquad \square$$

Lemma 2.7. *We have*

$$Y^\partial_{+a,\times d} \circ Y^\phi_\partial \;=\; Y^\phi_\partial \circ Y^\delta_{+a,\times d}.$$

Proof. As with Lemma 2.6, this identity holds on a closed subset of K^{1+r}, so it is enough to show that it holds on a dense subset. Let $U \subset K^{1+r}$ be $\mathcal{L}(K)$-definable and open and, using Fact 2.3 with δ in place of ∂, take a T^∂-extension $M = K\langle \mathcal{J}^r_\delta b\rangle$ with $\mathcal{J}^r_\delta(b) \in U^M$. We have

$$Y^\partial_{+a,\times d}\big(Y^\phi_\partial(\mathcal{J}^r_\delta b)\big) \;=\; Y^\partial_{+a,\times d}(\mathcal{J}^r_\partial b) \;=\; \mathcal{J}^r_\partial(db+a) \;=\; Y^\phi_\partial\big(\mathcal{J}^r_\delta(db+a)\big) \;=\; Y^\phi_\partial\big(Y^\delta_{+a,\times d}(\mathcal{J}^r_\delta b)\big).$$

Since $Y^{\partial}_{+a,\times d}\circ Y^{\phi}_{\partial}$ and $Y^{\phi}_{\partial}\circ Y^{\delta}_{+a,\times d}$ are $\mathcal{L}(K)$-definable and K is an elementary $\mathcal{L}$-substructure of M, there is a tuple $u\in U$ with

$$Y^{\partial}_{+a,\times d}\big(Y^{\phi}_{\partial}(u)\big) \;=\; Y^{\phi}_{\partial}\big(Y^{\delta}_{+a,\times d}(u)\big). \qquad \square$$

We set $F^{\phi}_{\partial} := F\circ Y^{\phi}_{\partial}$, so F^{ϕ}_{∂} is $\mathcal{L}(K)$-definable and

$$F^{\phi}_{\partial}(\jmath^r_{\delta}y) \;=\; F(\jmath^r_{\partial}y)$$

for all $y\in K$. When ∂ is clear from context, we drop the subscript and just write F^{ϕ}. We have $F^{1}_{\partial}=F$ and Lemma 2.6 gives $(F^{\phi}_{\partial})^{\psi}_{\delta}=F^{\phi\psi}_{\partial}$ for $\psi\in K^{\times}$. Lemma 2.7 gives

$$(F^{\partial}_{+a,\times d})^{\phi}_{\partial} \;=\; (F^{\phi}_{\partial})^{\delta}_{+a,\times d},$$

This function, which we denote by $F^{\phi}_{+a,\times d}$, satisfies the identity

$$F^{\phi}_{+a,\times d}(\jmath^r_{\delta}y) \;=\; F\big(\jmath^r_{\partial}(dy+a)\big)$$

for all $y\in K$.

Lemma 2.8. *Suppose F is in implicit form. Then F^{ϕ} is also in implicit form with*

$$\mathfrak{m}_{F^{\phi}} \;=\; \phi^r\mathfrak{m}_F, \qquad I_{F^{\phi}} \;=\; \phi^{-r}\Big(I^{\phi}_F-\sum_{i=0}^{r-1}\xi^r_i(\phi)Y_i\Big).$$

Proof. Set $G:=\phi^{-r}\big(I^{\phi}_F-\sum_{i=0}^{r-1}\xi^r_i(\phi)Y_i\big)$. Then

$$F^{\phi} \;=\; \mathfrak{m}_F(Y^{\phi}_r-I^{\phi}_F) \;=\; \mathfrak{m}_F\Big(\sum_{i=0}^{r}\xi^r_i(\phi)Y_i-I^{\phi}_F\Big) \;=\; \mathfrak{m}_F\Big(\phi^rY_r+\sum_{i=0}^{r-1}\xi^r_i(\phi)Y_i-I^{\phi}_F\Big) \;=\; \phi^r\mathfrak{m}_F(Y_r-G),$$

so $\mathfrak{m}_{F^{\phi}}=\phi^r\mathfrak{m}_F$ and $I_{F^{\phi}}=G$. $\square$

Given an $\mathcal{L}(K)$-definable set $A\subseteq K^{1+r}$, we set

$$A^{\phi}_{\partial} \;:=\; \big\{u\in K^{1+r}:Y^{\phi}_{\partial}(u)\in A\big\},$$

so A^{ϕ}_{∂} is $\mathcal{L}(K)$-definable and

$$\jmath^r_{\delta}(y)\in A^{\phi}_{\partial} \iff \jmath^r_{\partial}(y)\in A$$

for $y\in K$. As with definable functions, we drop the subscript ∂ and just write A^{ϕ} when ∂ is clear from context. Since $Y^{\phi}_{\partial}(A^{\phi})=A$ and Y^{ϕ}_{∂} is an $\mathcal{L}(K)$-definable bijection, we have $\dim_{\mathcal{L}}(A^{\phi})=\dim_{\mathcal{L}}(A)$.

2.3. **Thin sets.** A subset $Z\subseteq K$ is said to be **thin** if $\jmath^r_{\partial}(Z)\subseteq A$ for some r and some $\mathcal{L}(K)$-definable set $A\subseteq K^{1+r}$ with $\dim_{\mathcal{L}}(A)\leqslant r$. The union of any two thin sets is thin, any subset of a thin set is thin, and the singleton $\{a\}$ is thin for $a\in K$. The constant field C of K is thin, since

$$\jmath^1_{\partial}(C) \;=\; \big\{(c,0):c\in C\big\} \;\subseteq\; K\times\{0\},$$

and $K\times\{0\}$ is a 1-dimensional subset of K^2. Thus, K is thin if ∂ is trivial. Here is a strong converse:

Proposition 2.9. *If ∂ is nontrivial, then no open subinterval of K is thin.*

Before proving this proposition, we need a short lemma.

Lemma 2.10. *Let $A\subseteq K^n$ be $\mathcal{L}(K)$-definable and let $A_1,\ldots,A_n\subseteq K$ be infinite sets. If $A_1\times\cdots\times A_n\subseteq A$, then $\dim_{\mathcal{L}}(A)=n$.*

Proof. This is clear for $n=1$. Suppose it holds for a given n, let $A\subseteq K^{n+1}$ be $\mathcal{L}(K)$-definable, and let $A_1,\ldots,A_{n+1}\subseteq K$ be infinite sets with $A_1\times\cdots\times A_{n+1}\subseteq A$. By [6, Proposition 1.5], the set

$$A^* \;:=\; \big\{y\in\pi_n(A):\dim_{\mathcal{L}}(A_y)=1\big\}$$

is definable. If $y\in A_1\times\cdots\times A_n$, then $A_{n+1}\subseteq A_y$, so $y\in A^*$ by the $n=1$ case. Thus, $A_1\times\cdots\times A_n\subseteq A^*$ and $\dim_{\mathcal{L}}(A^*)=n$, by our induction hypothesis. Set

$$B \;:=\; \big\{z\in A:\pi_n(z)\in A^*\big\}.$$

Then $\dim_{\mathcal{L}}(B)=n+1$, again by [6, Proposition 1.5], so $\dim_{\mathcal{L}}(A)=n+1$ as well. $\square$

Proof of Proposition 2.9. Suppose ∂ is nontrivial, let $I \subseteq K$ be an open interval, let $A \subseteq K^{1+r}$ be $\mathcal{L}(K)$-definable, and suppose $\mathfrak{J}^r_\partial(y) \in A$ for each $y \in I$. We will show that $\dim_{\mathcal{L}}(A) = 1 + r$. By replacing I with an open subinterval, we may assume that $I = (a - d, a + d)$ for some $a \in K$ and some $d \in K^\times$. By replacing A with $A_{+a,\times d}$, we arrange that $I = (-1, 1)$. Since ∂ is nontrivial, we have $x \in K$ with $x' \neq 0$. By inverting x if need be, we arrange that $-1 < x < 1$, so

$$c_0 + c_1 x + \frac{1}{2}c_2 x^2 + \cdots + \frac{1}{r!}c_r x^r \in I$$

for any constants $c_0, \ldots, c_r \in [0, 1/3]_C$. Set $\phi := x'$. By replacing K and A with K^ϕ and A^ϕ, we arrange that $x' = 1$. Let

$$P := \begin{pmatrix} 1 & x & \frac{1}{2}x^2 & \cdots & \frac{1}{r!}x^r \\ 0 & 1 & x & \cdots & \frac{1}{(r-1)!}x^{r-1} \\ 0 & 0 & 1 & \cdots & \frac{1}{(r-2)!}x^{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{Mat}_{1+r}(K).$$

So P is invertible and for each $c = (c_0, \ldots, c_r) \in [0, 1/3]_C^{1+r}$, we have

$$Pc = \begin{pmatrix} c_0 + c_1 x + \frac{1}{2}c_2 x^2 + \cdots + \frac{1}{r!}c_r x^r \\ c_1 + c_2 x + \cdots + \frac{1}{(r-1)!}c_r x^{r-1} \\ c_2 + \cdots + \frac{1}{(r-2)!}c_r x^{r-2} \\ \vdots \\ c_r \end{pmatrix} = \mathfrak{J}^r_\partial\Big(c_0 + c_1 x + \frac{1}{2}c_2 x^2 + \cdots + \frac{1}{r!}c_r x^r\Big) \in A.$$

Thus, $[0, 1/3]_C^{1+r} \subseteq P^{-1}A := \{P^{-1}a : a \in A\}$. Lemma 2.10 (with $P^{-1}A$ in place of A and $[0, 1/3]_C$ in place of each A_i) gives

$$\dim_{\mathcal{L}}(A) = \dim_{\mathcal{L}}(P^{-1}A) = 1 + r. \qquad \square$$

3. T-convex T-differential fields

Let $\mathcal{L}^{\mathcal{O},\partial} := \mathcal{L}^{\mathcal{O}} \cup \mathcal{L}^\partial = \mathcal{L} \cup \{\mathcal{O}, \partial\}$. As defined in the introduction, a **T-convex T-differential field** is an $\mathcal{L}^{\mathcal{O},\partial}$-structure $K = (K, \mathcal{O}, \partial)$ such that

(1) $(K, \mathcal{O}) \models T^{\mathcal{O}}$;
(2) $(K, \partial) \models T^\partial$;
(3) ∂ is continuous with respect to the valuation topology.

Let $T^{\mathcal{O},\partial}$ be the $\mathcal{L}^{\mathcal{O},\partial}$-theory of T-convex T-differential fields. For the remainder of this article, let $K = (K, \mathcal{O}, \partial) \models T^{\mathcal{O},\partial}$. The following fact demonstrates how $\partial\mathcal{O}$ is controlled by $\partial\smallo$.

Fact 3.1 ([2], 4.4.2)**.** *If K has small derivation, then $\partial\mathcal{O} \subseteq \mathcal{O}$. Consequently, $\partial\smallo \subseteq \phi\smallo \Longrightarrow \partial\mathcal{O} \subseteq \phi\mathcal{O}$ for each $\phi \in K^\times$.*

We say that $K \models T^{\mathcal{O},\partial}$ has **small derivation** if $\partial\smallo \subseteq \smallo$. Suppose K has small derivation. By the above fact, ∂ induces a map $\bar{a} \mapsto \overline{\partial a}\colon \mathrm{res}(K) \to \mathrm{res}(K)$. We denote this map also by ∂, and we call it the **induced derivation on** $\mathrm{res}(K)$. This induced derivation is even a T-derivation. To see this, let F be an n-ary $\mathcal{L}(\emptyset)$-definable $\mathcal{C}^1$-function with open domain and let φ be the $\mathcal{L}(\emptyset)$-formula defining the domain of F. Set

$$U := \varphi(K) \subseteq K^n, \qquad V := \varphi(\mathrm{res}\, K) \subseteq \mathrm{res}(K)^n,$$

so $V \subseteq \overline{U}$. Let F denote both its interpretation as a function $U \to K$ and its interpretation as a function $V \to \mathrm{res}(K)$ and let $u \in U$ with $\bar{u} \in V$. By [7, Lemma 1.13], we have $\overline{F(u)} = F(\bar{u})$, so

$$\partial F(\bar{u}) = \partial\overline{F(u)} = \overline{\partial F(u)} = \overline{\mathbf{J}_F(u)\partial u} = \mathbf{J}_F(\bar{u})\partial\bar{u}.$$

Accordingly, we view $\mathrm{res}(K)$ as a model of T^∂. Note that the induced derivation on $\mathrm{res}(K)$ is trivial if and only if $\partial\mathcal{O} \subseteq \smallo$.

Let Δ be a nontrivial convex Λ-subspace of Γ. Recall the $T^{\mathcal{O}}$-models K_Δ and $\mathrm{res}(K_\Delta)$ associated to K and Δ from Subsection 1.3. We may view K_Δ as a $T^{\mathcal{O},\partial}$-model with the same derivation as K; the valuation topology induced by $\dot{\mathcal{O}}$ is either discrete or the same as the topology induced by $\mathcal{O}$, so ∂ is still continuous in K_Δ. Suppose K has small derivation. Then K_Δ does as well by [2, Corollary 4.4.4], so we may consider the induced derivation on $\operatorname{res}(K_\Delta)$. This induced derivation is a T-derivation by the remarks above, and it is also small, hence continuous. Thus, we may view $\operatorname{res}(K_\Delta)$ as a $T^{\mathcal{O},\partial}$-model as well.

3.1. **Examples.** As mentioned in the introduction, the following example fits comfortably within the framework of [3]:

Example 3.2 (Real closed valued differential fields)**.** Let $(R, \mathcal{O})$ be a real closed valued field. Then $\mathcal{O}$ is RCF-convex by [7, Proposition 4.2]. Let ∂ be a derivation on R which is continuous with respect to the valuation topology. By [11, Proposition 2.8], ∂ is an RCF-derivation, so $(R, \mathcal{O}, \partial) \models \mathrm{RCF}^{\mathcal{O},\partial}$. Conversely, every model of $\mathrm{RCF}^{\mathcal{O},\partial}$ is a real closed valued field with a continuous derivation.

Hardy fields, which are deeply connected to o-minimality, provide a rich source new of examples:

Example 3.3 ($\mathcal{R}$-Hardy fields)**.** Let $\mathcal{R}$ be an arbitrary o-minimal expansion of the real field in a language $\mathcal{L}_\mathcal{R}$. By extending $\mathcal{L}_\mathcal{R}$, we may assume that $T_\mathcal{R} := \mathrm{Th}(\mathcal{R})$ has quantifier elimination and a universal axiomatization. Following [8], we define an **$\mathcal{R}$-Hardy field** to be a Hardy field $\mathcal{H}$ which is closed under all function symbols in $\mathcal{L}_\mathcal{R}$. That is, $\mathcal{H}$ is an ordered field of germs at $+\infty$ of unary functions $f\colon \mathbb{R} \to \mathbb{R}$ such that:

(1) If the germ of f belongs to $\mathcal{H}$, then so does the germ of f';
(2) If F is an n-ary function symbol in $\mathcal{L}_\mathcal{R}$ and the germs of $f_1, \ldots, f_n$ belong to $\mathcal{H}$, then the germ of the composite function $x \mapsto F\big(f_1(x), \ldots, f_n(x)\big)$ belongs to $\mathcal{H}$.

Let $\mathcal{H}$ be an $\mathcal{R}$-Hardy field. Then $\mathcal{H}$ admits a natural expansion to a model of $T_\mathcal{R}$ [8, Lemma 5.8]. Let ∂ denote the natural derivation on $\mathcal{H}$, which sends the germ of f to the germ of f', and let $\mathcal{O}$ be the set of germs which are bounded $n \in \mathbb{N}$. Then $\mathcal{O}$ is the convex hull of the prime model of $T_\mathcal{R}$, so $\mathcal{O}$ is $T_\mathcal{R}$-convex by Fact 1.1. The chain rule from elementary calculus tells us that ∂ is a $T_\mathcal{R}$-derivation. As the derivation on $\mathcal{H}$ is small, it is continuous, so $\mathcal{H} \models T_\mathcal{R}^{\mathcal{O},\partial}$.

The expansion of the field of logarithmic-exponential transseries by restricted analytic functions serves as another motivating example:

Example 3.4 (Transseries)**.** Let $\mathbb{T}$ be the ordered valued differential field of logarithmic-exponential transseries with convex valuation ring $\mathcal{O} = \{f \in \mathbb{T} : |f| < n \text{ for some } n\}$ and derivation ∂; see Appendix A of [2] for a detailed definition. By [9, Corollary 2.8], $\mathbb{T}$ admits a canonical expansion to a model of T_{an}, which we denote by $\mathbb{T}_{\mathrm{an}}$. Then $\mathcal{O}$, being the convex hull of the T_{an}-model $\mathbb{R}_{\mathrm{an}}$, is T_{an}-convex by Fact 1.1. Moreover, ∂ is a T_{an}-derivation since by [10, Corollary 3.3], it is compatible with all restricted analytic functions; see [11, Lemma 2.9] for why this is sufficient. As ∂ is small by [10, Proposition 4.1], it is continuous, so $\mathbb{T}_{\mathrm{an}} \models T_{\mathrm{an}}^{\mathcal{O},\partial}$.

Examples 3.3 and 3.4 above are both *pre-H-fields*, as introduced in [1]. Models of $T^{\mathcal{O},\partial}$ which are also pre-H-fields are studied in more depth in [15, Chapter 7]. For an example which is not a pre-H-field, consider the following:

Example 3.5 (Ordered Hahn differential fields)**.** Let $\boldsymbol{k} = (\boldsymbol{k}, \partial_{\boldsymbol{k}}) \models T_{\mathrm{an}}^{\partial}$ and let $\mathfrak{M}$ be a divisible ordered abelian group, written multiplicatively. Consider the ordered Hahn field $\boldsymbol{k}[[\mathfrak{M}]]$. We identify $\boldsymbol{k}$ with a subfield of $\boldsymbol{k}[[\mathfrak{M}]]$ via the embedding $a \mapsto a \cdot 1_{\mathfrak{M}}$. We expand $\boldsymbol{k}[[\mathfrak{M}]]$ to a model of T_{an} as follows: for $y = (y_1, \ldots, y_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ with $|y_1|, \ldots, |y_n| \leqslant 1$, we take tuples $a = (a_1, \ldots, a_n) \in \boldsymbol{k}^n$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \boldsymbol{k}[[\mathfrak{M}]]^n$ with

$$|a_k| \;\leqslant\; 1, \qquad |\varepsilon_k| \;<\; \boldsymbol{k}^{>}, \qquad y_k \;=\; a_k + \varepsilon_k$$

for $k = 1, \ldots, n$. For an n-ary restricted analytic function F, we set

$$F(y) \;=\; F(a + \varepsilon) \;:=\; \sum_{\boldsymbol{i} \in \mathbb{N}^n} \frac{F^{(\boldsymbol{i})}(a)}{\boldsymbol{i}!} \varepsilon^{\boldsymbol{i}},$$

where

$$F^{(\boldsymbol{i})} \;=\; \frac{\partial^{i_1 + \cdots + i_n} F}{\partial^{i_1} Y_1 \cdots \partial^{i_n} Y_n}, \qquad \boldsymbol{i}! \;=\; i_1! \cdots i_n!, \qquad \varepsilon^{\boldsymbol{i}} \;=\; \varepsilon_1^{i_1} \cdots \varepsilon_n^{i_n}.$$

We denote this expansion by $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$. It is fairly routine to verify that $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ is indeed a model of T_{an}; this was done when $\boldsymbol{k} = \mathbb{R}$ in [8], and the reader may consult [15, Proposition 2.13] for full details.

Now we extend $\partial_{\boldsymbol{k}}$ to a derivation ∂ on $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ by setting

$$\partial\Big(\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m}\Big) \ := \ \sum_{\mathfrak{m}} \partial_{\boldsymbol{k}}(f_{\mathfrak{m}})\mathfrak{m}$$

for $\sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m} \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$. Using that ∂ is strongly additive, one can verify that $\partial F(y) = \mathbf{J}_F(y)\partial y$ for each n-ary restricted analytic function F and each $y = (y_1, \ldots, y_n) \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}^n$ with $|y_1|, \ldots, |y_n| < 1$; see [15, Proposition 3.14] for full details. It follows by [11, Lemma 2.9] that ∂ is a T_{an}-derivation. The constant field of $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ is the subfield $C_{\boldsymbol{k}}[[\mathfrak{M}]]$, where $C_{\boldsymbol{k}}$ is the constant field of $\boldsymbol{k}$.

Let $\mathcal{O}$ be the convex hull of $\boldsymbol{k}$ in $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$, so $\mathcal{O}$ is T_{an}-convex by Fact 1.1. With respect to $\mathcal{O}$, the derivation ∂ is **monotone**, that is, $\partial f \preccurlyeq f$ for all $f \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$. In particular, ∂ is small, so $\big(\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}, \mathcal{O}, \partial\big) \models T_{\mathrm{an}}^{\mathcal{O},\partial}$. This model is spherically complete and has *many constants*: for each $f \in \boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$, there is a constant c with $f \asymp c$.

In [20], Scanlon proved an Ax-Kochen-Eršov (AKE) result for Hahn differential fields $\boldsymbol{k}[[\mathfrak{M}]]$ where $\boldsymbol{k}$ is an (unordered) differential field of characteristic zero and where the derivation on $\boldsymbol{k}$ is extended to $\boldsymbol{k}[[\mathfrak{M}]]$ as in the example above. As a consequence, he showed that if $\mathfrak{M}$ is divisible and $\boldsymbol{k}$ is differentially closed, then the valued differential field $\boldsymbol{k}[[\mathfrak{M}]]$ is model complete. Scanlon's AKE result can also be used to show that if $\mathfrak{M}$ is divisible and $\boldsymbol{k}$ is a closed ordered differential field (as introduced by Singer [21]), then the *ordered* valued differential field $\boldsymbol{k}[[\mathfrak{M}]]$ is model complete. In [11], it is shown that $T_{\mathrm{an}}^{\partial}$ has a model completion. This raises the question: if $\boldsymbol{k}$ is a model of this model completion and $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}} \models T_{\mathrm{an}}^{\mathcal{O},\partial}$ is as above, then is $\boldsymbol{k}[[\mathfrak{M}]]_{\mathrm{an}}$ model complete? As with the uniqueness questions discussed at the end of this article, answering this question likely requires some analog of *differential henselianity.* For a generalization of Scanlon's AKE result, see [13].

3.2. **Strict extensions.** Let M be a $T^{\mathcal{O},\partial}$-extension of K. Recall from the introduction that M is a **strict** extension of K if

$$\partial\smallo \subseteq \phi\smallo \implies \partial_M\smallo_M \subseteq \phi\smallo_M \qquad \partial\mathcal{O} \subseteq \phi\smallo \implies \partial_M\mathcal{O}_M \subseteq \phi\smallo_M$$

for each $\phi \in K^{\times}$.

We are interested in when K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension. If $\mathcal{O} = K$, then K is itself spherically complete. Moreover, if ∂ is trivial and T is power bounded, then K has a spherically complete immediate $T^{\mathcal{O}}$-extension M by Corollary 1.11. Viewed as a $T^{\mathcal{O},\partial}$-model with trivial derivation, M is a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension of K. If T is not power bounded and $\mathcal{O} \neq K$, then K has no spherically complete $T^{\mathcal{O}}$-extension at all by Remark 1.12. Accordingly, we make the following assumption:

Assumption 3.6. *For the remainder of this article, T is power bounded with field of exponents Λ and the derivation and valuation on K are nontrivial.*

A consequence of this assumption which we will use freely is that any $T^{\mathcal{O}}$-extension of K is elementary. Below we list some basic but important facts about strict extensions.

1. M is a strict $T^{\mathcal{O},\partial}$-extension of K if and only if M^{ϕ} is a strict $T^{\mathcal{O},\partial}$-extension of K^{ϕ} for $\phi \in K^{\times}$.
2. If M is a $T^{\mathcal{O},\partial}$-extension of K and M is contained in a strict $T^{\mathcal{O},\partial}$-extension of K, then M is itself a strict extension of K.
3. If M is a strict $T^{\mathcal{O},\partial}$-extension of L and L is a strict $T^{\mathcal{O},\partial}$-extension of K, then M is a strict extension of K.
4. If M is an elementary $T^{\mathcal{O},\partial}$-extension of K, then M is a strict extension of K.

We have a useful test for determining whether an immediate $T^{\mathcal{O},\partial}$-extension is strict:

Fact 3.7 ([3], 1.5)**.** *Suppose that M is an immediate $T^{\mathcal{O}}$-extension of K and let ∂_M be a T-derivation on M which extends ∂. If*

$$\partial\smallo \subseteq \phi\smallo \implies \partial_M\smallo_M \subseteq \phi\smallo_M$$

for each $\phi \in K^{\times}$, then M is a strict $T^{\mathcal{O},\partial}$-extension of K.

Recall from the introduction the sets

$$\Gamma(\partial) \;:=\; \{v\phi : \partial\smallo \subseteq \phi\smallo\}, \qquad S(\partial) \;:=\; \big\{\gamma \in \Gamma : \Gamma(\partial) + \gamma = \Gamma(\partial)\big\}.$$

We sometimes write $\Gamma_K(\partial)$ and $S_K(\partial)$ if K is not clear from context. Note that $\Gamma(\partial) < v(\partial\smallo)$ is a downward closed subset of Γ and that $S(\partial)$, the *stabilizer* of $\Gamma(\partial)$, is a convex subgroup of Γ. For $\phi \in K^\times$, we have

$$\Gamma(\phi^{-1}\partial) \;=\; \Gamma(\partial) - v\phi, \qquad S(\phi^{-1}\partial) \;=\; S(\partial),$$

so $S(\partial)$ is invariant under compositional conjugation. If M is a strict $T^{\mathcal{O},\partial}$-extension of K with $\Gamma_M = \Gamma$, then $\Gamma_M(\partial) = \Gamma(\partial)$ and $S_M(\partial) = S(\partial)$. Using that K is real closed, we can show that $\Gamma(\partial)$ is closed in Γ (with respect to the order topology).

Lemma 3.8. *Let $\alpha \in \Gamma$. If $\alpha - \varepsilon \in \Gamma(\partial)$ for each $\varepsilon \in \Gamma^>$, then $\alpha \in \Gamma(\partial)$.*

Proof. Suppose $\alpha \notin \Gamma(\partial)$. Take $a \in \smallo$ with $a > 0$ and $v(a') \leqslant \alpha$. Then

$$v\big((a^{1/2})'\big) \;=\; v(a') - \frac{1}{2}va \;\leqslant\; \alpha - \frac{1}{2}va,$$

so $\alpha - \frac{1}{2}va \notin \Gamma(\partial)$. □

4. Behavior of definable functions

In this section, let $F\colon K^{1+r} \to K$ be an $\mathcal{L}(K)$-definable function in implicit form. We set $vF := v(\mathfrak{m}_F) \in \Gamma$, and we call vF the **valuation of** F. Lemmas 2.5 and 2.8 give

$$vF_{+a,\times d} \;=\; vF + vd, \qquad vF^\phi \;=\; vF + rv\phi$$

for $a \in K$ and $d, \phi \in K^\times$. The valuation of F acts as a sort of crude replacement for the Gaussian valuation associated to differential polynomials, used frequently throughout [2].

4.1. **Bounding $vF(\mathcal{J}^r_\partial u)$ from above.** In this subsection, we use vF to find points $u \in K$ where $F(\mathcal{J}^r_\partial u)$ is "not too small."

Lemma 4.1. *Suppose that K has small derivation and that the induced derivation on $\operatorname{res}(K)$ is nontrivial. Then*

$$vF(\mathcal{J}^r_\partial u) \;\leqslant\; vF$$

for some $u \in \mathcal{O}^\times$.

Proof. Fact 1.2 gives that $\overline{\operatorname{Gr}(I_F)} \subseteq \operatorname{res}(K)^{1+r}$ is $\mathcal{L}(\operatorname{res} K)$-definable and $\dim_{\mathcal{L}}\big(\overline{\operatorname{Gr}(I_F)}\big) \leqslant r$. Thus, the set

$$\big\{y \in \operatorname{res}(K) : \mathcal{J}^r_\partial(y) \in \overline{\operatorname{Gr}(I_F)}\big\} \cup \{0\}$$

is a thin subset of $\operatorname{res}(K)$. Proposition 2.9 applied to $\operatorname{res}(K)$ gives $u \in \mathcal{O}^\times$ with $\mathcal{J}^r_\partial(\bar{u}) = \overline{\mathcal{J}^r_\partial(u)} \notin \overline{\operatorname{Gr}(I_F)}$. Then either $I_F(\mathcal{J}^{r-1}_\partial u) \succ 1$ or $I_F(\mathcal{J}^{r-1}_\partial u) \preccurlyeq 1$ and

$$\overline{u^{(r)}} \;\neq\; \overline{I_F(\mathcal{J}^{r-1}_\partial u)}.$$

In either case, $u^{(r)} - I_F(\mathcal{J}^{r-1}_\partial u) \succcurlyeq 1$, so

$$F(\mathcal{J}^r_\partial u) \;=\; \mathfrak{m}_F\big(u^{(r)} - I_F(\mathcal{J}^{r-1}_\partial u)\big) \;\succcurlyeq\; \mathfrak{m}_F. \qquad \square$$

Lemma 4.2. *Suppose $S(\partial) = \{0\}$ and let $\beta \in \Gamma^>$. Then there is $\gamma \in \Gamma(\partial)$ and $u \in K$ with $|vu| < \beta$ such that*

$$vF(\mathcal{J}^r_\partial u) \;\leqslant\; vF + r\gamma + \beta.$$

Proof. We claim that for any $\varepsilon \in \Gamma^>$, we can find $\gamma \in \Gamma(\partial)$ and $a \in \smallo$ such that $v(a') - \gamma \leqslant \varepsilon$. Let $\varepsilon \in \Gamma^>$ be given and, using that $\varepsilon \notin S(\partial) = \{0\}$, take $\gamma \in \Gamma(\partial)$ with $\gamma + \varepsilon \notin \Gamma(\partial)$. Then there is $a \in \smallo$ with $v(a') \leqslant \gamma + \varepsilon$, as desired. This claim yields an elementary $T^{\mathcal{O},\partial}$-extension M of K with $\gamma \in \Gamma_M(\partial)$ and $a \in \smallo_M$ such that

$$v(a') - \gamma \;<\; \Gamma^>.$$

Let Δ be the convex Λ-subspace of Γ_M consisting of all $\varepsilon \in \Gamma_M$ with $|\varepsilon| < \Gamma^>$ and let $\phi \in M^\times$ with $v\phi = \gamma$. Then M^ϕ has small derivation, so M^ϕ_Δ does as well by [2, Corollary 4.4.4]. Since $v(\phi^{-1}a') < \Gamma^>$

and $v(a') > v\phi$, we have $\dot{v}(\phi^{-1}a') = 0$, so the derivation on $\operatorname{res}(M^{\phi}_{\Delta})$ is nontrivial. Applying Lemma 4.1 to M^{ϕ}_{Δ} and F^{ϕ}, we get $u \in M$ with $\dot{v}u = 0$ and

$$\dot{v}F^{\phi}(\jmath^{r}_{\phi^{-1}\partial_M}u) \;\leqslant\; \dot{v}F^{\phi}.$$

Then $|vu| < \Gamma^{>}$ and

$$vF^{\phi}(\jmath^{r}_{\phi^{-1}\partial_M}u) - vF^{\phi} \;=\; vF(\jmath^{r}_{\partial_M}u) - (vF + rv\phi) \;=\; vF(\jmath^{r}_{\partial_M}u) - vF - r\gamma \;<\; \Gamma^{>}.$$

In particular, $|vu| < \beta$ and $vF(\jmath^{r}_{\partial_M}u) < vF + r\gamma + \beta$. As M is an elementary $T^{\mathcal{O},\partial}$-extension of K, the lemma follows. □

Corollary 4.3. *Suppose $S(\partial) = \{0\}$, let $\beta \in \Gamma^{>}$, and suppose that $vF(\jmath^{r}_{\partial}a) = vF(\jmath^{r}_{\partial}b)$ for all $a, b \in K$ with $|va|, |vb| < \beta$. Then there is $\gamma \in \Gamma(\partial)$ such that $vF(\jmath^{r}_{\partial}u) \leqslant vF + r\gamma$ for all $u \in K$ with $|vu| < \beta$.*

Proof. We first handle the case $r = 0$, so $I_F \in K$. Take $a \in \mathcal{O}^{\times}$ with $a \not\sim I_F$. Then

$$F(a) \;=\; \mathfrak{m}_F(a - I_F) \;\succcurlyeq\; \mathfrak{m}_F,$$

so $vF(a) \leqslant vF$. For $u \in K$ with $|vu| < \beta$, we have $vF(u) = vF(a) \leqslant vF$, as desired. Now assume $r > 0$. Let $a \in K$ with $|va| < \beta$ and set

$$\alpha \;:=\; r^{-1}\big(vF(\jmath^{r}_{\partial}a) - vF\big).$$

For $u \in K$ with $|vu| < \beta$, we have $vF(\jmath^{r}_{\partial}u) = vF(\jmath^{r}_{\partial}a) = vF + r\alpha$, so it suffices to show that $\alpha \in \Gamma(\partial)$. Let $\varepsilon \in \Gamma^{>}$ with $r\varepsilon < \beta$. Using Lemma 4.2, take $b \in K^{\times}$ and $\gamma \in \Gamma(\partial)$ with $|vb| < r\varepsilon < \beta$ and

$$vF(\jmath^{r}_{\partial}b) \;\leqslant\; vF + r\gamma + r\varepsilon.$$

By assumption, $vF(\jmath^{r}_{\partial}a) = vF(\jmath^{r}_{\partial}b)$, so

$$\alpha - \varepsilon \;=\; r^{-1}\big(vF(\jmath^{r}_{\partial}a) - vF\big) - \varepsilon \;=\; r^{-1}\big(vF(\jmath^{r}_{\partial}b) - vF\big) - \varepsilon \;\leqslant\; \gamma \;\in\; \Gamma(\partial).$$

As ε can be taken to be arbitrarily small, we have $\alpha \in \Gamma(\partial)$ by Lemma 3.8. □

4.2. **Eventual smallness.** In this subsection, let ϕ range over $K^{\times}$, let $\ell \prec 1$ be an element in a $T^{\mathcal{O},\partial}$-extension of K, and suppose $v(\ell - K)$ has no largest element.

A property is said to hold *for all $y \in K$ sufficiently close to ℓ* if there is $\eta \in v(\ell - K)$ such that the property holds for all $y \in K$ with $v(\ell - y) > \eta$. We say that F **is small near** (K, ℓ) if $I_F(\jmath^{r-1}_{\partial}y) \prec 1$ for all $y \in K$ sufficiently close to ℓ.

Lemma 4.4. *Let $\phi_0 \in K^{\times}$ with $v\phi_0 \in \Gamma(\partial)$ and suppose $v\phi \leqslant v\phi_0$. If F^{ϕ_0} is small near (K^{ϕ_0}, ℓ), then F^{ϕ} is small near (K^{ϕ}, ℓ).*

Proof. By replacing K, F, and ϕ with K^{ϕ_0}, F^{ϕ_0}, and $\phi_0^{-1}\phi$, we may assume $\phi_0 = 1$ (so K has small derivation) and $\phi \succcurlyeq 1$. Set $\mathfrak{d} := \phi^{-1}\partial$. Suppose F is small near (K, ℓ) and let $y \in \mathcal{O}$ be close enough to ℓ so that $I_F(\jmath^{r-1}_{\partial}y) \prec 1$. We claim that $I_{F^{\phi}}(\jmath^{r-1}_{\mathfrak{d}}y) \prec 1$. By Lemma 2.8, we have

$$I_{F^{\phi}}(\jmath^{r-1}_{\mathfrak{d}}y) \;=\; \phi^{-r}\Big(I^{\phi}_{F}(\jmath^{r-1}_{\mathfrak{d}}y) - \sum_{i=0}^{r-1}\xi^{r}_{i}(\phi)\mathfrak{d}^{i}y\Big) \;=\; \phi^{-r}I_F(\jmath^{r-1}_{\partial}y) - \sum_{i=0}^{r-1}\phi^{-r}\xi^{r}_{i}(\phi)\mathfrak{d}^{i}y.$$

As $I_F(\jmath^{r-1}_{\partial}y) \prec 1$ and $\phi \succcurlyeq 1$, we have $\phi^{-r}I_F(\jmath^{r-1}_{\partial}y) \prec 1$, so it remains to show

$$\sum_{i=0}^{r-1}\phi^{-r}\xi^{r}_{i}(\phi)\mathfrak{d}^{i}y \;\prec\; 1.$$

We claim that $\phi^{\dagger} \preccurlyeq \phi$. This is clear in the case that $\phi' \preccurlyeq \phi$, for then $\phi^{\dagger} \preccurlyeq 1 \preccurlyeq \phi$ (note that if $\phi \asymp 1$, then we are in this case by Fact 3.1). On the other hand, if $\phi' \succ \phi \succ 1$, then $\phi^{\dagger} \preccurlyeq \phi$ by [2, Lemma 6.4.1]. Now [3, Lemma 2.2] gives $\phi^{-r}\xi^{r}_{i}(\phi) \preccurlyeq 1$ for each $i < r$. Since K^{ϕ} has small derivation and $y \prec 1$, we have $\phi^{-r}\xi^{r}_{i}(\phi)\mathfrak{d}^{i}y \prec 1$ for each $i < r$ as desired. □

We say that F **is eventually small near** (K,ℓ) if F^ϕ is small near (K^ϕ,ℓ) whenever $v\phi\in\Gamma(\partial)$. Unlike smallness, eventual smallness is invariant under compositional conjugation: F is eventually small near (K,ℓ) if and only if F^ϕ is eventually small near (K^ϕ,ℓ). By the above lemma, the set

$$\big\{v\phi\in\Gamma(\partial): F^\phi \text{ is small near } (K^\phi,\ell)\big\}$$

is a downward closed subset of $\Gamma(\partial)$. Thus, F is eventually small near (K,ℓ) if and only if F^ϕ is small near (K^ϕ,ℓ) for all sufficiently large $v\phi\in\Gamma(\partial)$, and F is not eventually small near (K,ℓ) if and only if F^ϕ is not small near (K^ϕ,ℓ) for all sufficiently large $v\phi\in\Gamma(\partial)$. Eventual smallness serves as a crude analog of the Newton degree in [2] and [3]; one should think of F being eventually small as analogous to F having positive Newton degree. Of course, Newton degree makes no sense for arbitrary definable functions.

Lemma 4.5. *Suppose that $S(\partial)=\{0\}$ and that F is eventually small near (K,ℓ). For each $\beta\in\Gamma^>$, we have*

$$F(\mathcal{J}^r_\partial a)\ \not\asymp\ F(\mathcal{J}^r_\partial b)$$

for some $a,b\in K$ with $va,vb>-\beta$.

Proof. Let $\beta\in\Gamma^>$ and let $a\in K$ with $|va|<\beta$. If $F(\mathcal{J}^r_\partial a)\not\asymp F(\mathcal{J}^r_\partial b)$ for some $b\in K$ with $|vb|<\beta$, then we are done, so we may assume $F(\mathcal{J}^r_\partial a)\asymp F(\mathcal{J}^r_\partial b)$ for all $b\in K$ with $|vb|<\beta$. Then Corollary 4.3 gives $\gamma\in\Gamma(\partial)$ with $vF(\mathcal{J}^r_\partial a)\leqslant vF+r\gamma$. Take ϕ with $v\phi=\gamma$. Then F^ϕ is small near (K^ϕ,ℓ), so we may take $y\in\smallo$ close enough to ℓ so that $I_{F^\phi}(\mathcal{J}^{r-1}_\delta y)\prec 1$. Since $\delta^r y\prec 1$ as well, we have

$$F(\mathcal{J}^r_\partial y)\ =\ F^\phi(\mathcal{J}^r_\delta y)\ =\ \phi^r\mathfrak{m}_F\big(\delta^r y-I_{F^\phi}(\mathcal{J}^{r-1}_\delta y)\big)\ \prec\ \phi^r\mathfrak{m}_F,$$

so $vF(\mathcal{J}^r_\partial y)>vF+r\gamma\geqslant vF(\mathcal{J}^r_\partial a)$. □

5. Vanishing

In this section, let ϕ range over $K^\times$, let ℓ be an element in a strict $T^{\mathcal{O},\partial}$-extension L of K, and suppose $v(\ell-K)$ has no largest element. Unlike in Subsection 4.2, we do not assume $\ell\prec 1$.

Definition 5.1. *Let $F\colon K^{1+r}\to K$ be an $\mathcal{L}(K)$-definable function in implicit form. We say F **vanishes at** (K,ℓ) if $F_{+a,\times d}$ is eventually small near $\big(K,d^{-1}(\ell-a)\big)$ for all $a\in K$ and $d\in K^\times$ with $\ell-a\prec d$.*

Let $Z(K,\ell)$ be the set of all $\mathcal{L}(K)$-definable functions in implicit form which vanish at (K,ℓ). For each r, we let $Z_r(K,\ell)$ be the functions in $Z(K,\ell)$ of arity $1+r$, so $Z(K,\ell)$ is equal to the disjoint union $\bigcup_r Z_r(K,\ell)$. The set $Z(K,\ell)$ serves a similar purpose to the set in [2] and [3] with the same name. Note that $Z(K,\ell)$ does not depend on ℓ, only on the $\mathcal{L}^{\mathcal{O}}$-type of ℓ over K. we will show in Proposition 5.4 below that if $Z(K,\ell)=\emptyset$ then $F(\mathcal{J}^r_\partial\ell)\neq 0$ for all $\mathcal{L}(K)$-definable functions F in implicit form.

Lemma 5.2. $Z_0(K,\ell)=\emptyset$.

Proof. Let $F\colon K\to K$ be an $\mathcal{L}(K)$-definable function in implicit form, so $I_F\in K$. Let $d\in K$ with $d\asymp\ell-I_F$ and let $a\in K$ with $\ell-a\prec d$. Then $I_F-a\asymp d$, so

$$I_{F_{+a,\times d}}\ =\ d^{-1}(I_F-a)\ \asymp\ 1$$

and $F_{+a,\times d}$ is not small near $\big(K,d^{-1}(\ell-a)\big)$. As $F^\phi_{+a,\times d}=F_{+a,\times d}$ for all ϕ, we see that $F_{+a,\times d}$ is not eventually small near $\big(K,d^{-1}(\ell-a)\big)$, so $F\notin Z(K,\ell)$. □

Lemma 5.3. *Suppose $S(\partial)=\{0\}$, let $F\in Z_r(K,\ell)$, and let $y\in K$. Then there is $z\in K$ with $\ell-z\prec\ell-y$ and $F(\mathcal{J}^r_\partial y)\not\asymp F(\mathcal{J}^r_\partial z)$.*

Proof. Let $a\in K$ and $d\in K^\times$ with $\ell-a\prec d\prec\ell-y$. Set $\beta:=vd-v(\ell-y)>0$. Since $F_{+a,\times d}$ is eventually small near $\big(K,d^{-1}(\ell-a)\big)$, Lemma 4.5 gives $b_1,b_2\in K$ with $vb_1,vb_2>-\beta$ and

$$vF_{+a,\times d}(\mathcal{J}^r_\partial b_1)\ \neq\ vF_{+a,\times d}(\mathcal{J}^r_\partial b_2).$$

Either $vF_{+a,\times d}(\mathcal{J}^r_\partial b_1)$ or $vF_{+a,\times d}(\mathcal{J}^r_\partial b_2)$ is different from $vF(\mathcal{J}^r_\partial y)$, so suppose $vF_{+a,\times d}(\mathcal{J}^r_\partial b_1)\neq vF(\mathcal{J}^r_\partial y)$ and set $z:=db_1+a$. Then $F(\mathcal{J}^r_\partial z)=F_{+a,\times d}(\mathcal{J}^r_\partial b_1)\not\asymp F(\mathcal{J}^r_\partial y)$ and, since $v(db_1)>vd-\beta=v(\ell-y)$, we have

$$v(\ell-z)\ =\ v\big((\ell-a)-db_1\big)\ \geqslant\ \min\big\{v(\ell-a),v(db_1)\big\}\ >\ v(\ell-y).$$ □

5.1. **Behavior of nonvanishing functions.** Fix r and suppose $Z_q(K,\ell)=\emptyset$ for all $q<r$. Our goal is to prove the following result:

Proposition 5.4. *Let $F\colon K^{1+r}\to K$ be an $\mathcal{L}(K)$-definable function in implicit form with $F\notin Z_r(K,\ell)$. Then $F(\mathfrak{J}^r_\partial\ell)\neq 0$ and*

$$F(\mathfrak{J}^r_\partial\ell)\ \sim\ F(\mathfrak{J}^r_\partial y)$$

for all $y\in K$ sufficiently close to ℓ.

This proposition requires a somewhat lengthy proof by induction, so we make the following hypothesis.

Induction Hypothesis (IH)**.** *We assume that for all $q<r$ and all $\mathcal{L}(K)$-definable functions $F\colon K^{1+q}\to K$ in implicit form, we have $F(\mathfrak{J}^q_\partial\ell)\neq 0$ and*

$$F(\mathfrak{J}^q_\partial\ell)\ \sim\ F(\mathfrak{J}^q_\partial y)$$

for all $y\in K$ sufficiently close to ℓ.

Lemma 5.5. *Suppose (IH) holds. Then $K\langle\mathfrak{J}^{r-1}_\partial\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K.*

Proof. Let $q<r$ be given and assume $K\langle\mathfrak{J}^{q-1}_\partial\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K (this holds vacuously when $q=0$). We will show that $K\langle\mathfrak{J}^{q}_\partial\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of $K\langle\mathfrak{J}^{q-1}_\partial\ell\rangle$, from which the lemma follows by induction. Let $G\colon K^q\to K$ be an $\mathcal{L}(K)$-definable function. For all $u\in K$ sufficiently close to ℓ, we have

$$\ell^{(q)}-G(\mathfrak{J}^{q-1}_\partial\ell)\ \sim\ y^{(q)}-G(\mathfrak{J}^{q-1}_\partial y)\ \in\ K$$

by (IH). Since G is arbitrary, we may apply Lemma 1.9 with $K\langle\mathfrak{J}^{q-1}_\partial\ell\rangle$ and $\ell^{(q)}$ in place of K and ℓ to get that $K\langle\mathfrak{J}^{q}_\partial\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of $K\langle\mathfrak{J}^{q-1}_\partial\ell\rangle$. ☐

Lemma 5.6. *Suppose (IH) holds, let $A\subseteq K^r$ be $\mathcal{L}^{\mathcal{O}}(K)$-definable, and suppose $\mathfrak{J}^{r-1}_\partial(\ell)\in A^L$. Then A has nonempty interior and $\mathfrak{J}^{r-1}_\partial(y)\in A$ for all $y\in K$ sufficiently close to ℓ.*

Proof. Using Lemmas 1.4 and 5.5, we take an $\mathcal{L}(K)$-definable cell D contained in A with $\mathfrak{J}^{r-1}_\partial(\ell)\in D^L$. Let $q<r$ be given and assume $\pi_q(D)$ is open and $\mathfrak{J}^{q-1}_\partial(y)\in\pi_q(D)$ for all $y\in K$ sufficiently close to ℓ (this holds vacuously when $q=0$). We will show that $\pi_{q+1}(D)$ is open and $\mathfrak{J}^{q}_\partial(y)\in\pi_{q+1}(D)$ for all $y\in K$ sufficiently close to ℓ, from which the lemma follows by induction. If $\pi_{q+1}(D)$ is not open, then $\pi_{q+1}(D)=\mathrm{Gr}\left(G|_{\pi_q(D)}\right)$ for some $\mathcal{L}(K)$-definable function $G\colon K^q\to K$, so $\ell^{(q)}=G(\mathfrak{J}^{q-1}_\partial\ell)$, contradicting (IH). Therefore, $\pi_{q+1}(D)$ is open. Suppose $\pi_{q+1}(D)$ is of the form

$$\big\{(a,b):a\in\pi_q(D)\text{ and }G(a)<b<H(a)\big\}$$

for some $\mathcal{L}(K)$-definable functions $G,H\colon K^q\to K$. Then (IH) gives

$$y^{(q)}-G(\mathfrak{J}^{q-1}_\partial y)\ \sim\ \ell^{(q)}-G(\mathfrak{J}^{q-1}_\partial\ell)\ >\ 0\ >\ \ell^{(q)}-H(\mathfrak{J}^{q-1}_\partial\ell)\ \sim\ y^{(q)}-H(\mathfrak{J}^{q-1}_\partial y)$$

for all $y\in K$ sufficiently close to ℓ, so $G(\mathfrak{J}^{q-1}_\partial y)<y^{(q)}<H(\mathfrak{J}^{q-1}_\partial y)$ for these y. This gives $\mathfrak{J}^{q}_\partial(y)\in\pi_{q+1}(D)$ for all $y\in K$ sufficiently close to ℓ as desired. If $\pi_{q+1}(D)$ is of the form

$$\big\{(a,b):a\in\pi_q(D)\text{ and }b>G(a)\big\}\ \text{ or }\ \big\{(a,b):a\in\pi_q(D)\text{ and }b<H(a)\big\},$$

then a simpler version of the above argument works. If $\pi_{q+1}(D)=\pi_q(D)\times K$, then the result follows immediately from the inductive assumption. ☐

Corollary 5.7. *Suppose (IH) holds and let $G\colon K^r\to K$ be an $\mathcal{L}(K)$-definable function. If $G(\mathfrak{J}^{r-1}_\partial\ell)=0$, then $G(\mathfrak{J}^{r-1}_\partial y)=0$ for all $y\in K$ sufficiently close to ℓ. If $G(\mathfrak{J}^{r-1}_\partial\ell)\neq 0$, then*

$$G(\mathfrak{J}^{r-1}_\partial\ell)\ \sim\ G(\mathfrak{J}^{r-1}_\partial y)$$

for all $y\in K$ sufficiently close to ℓ.

Proof. If $G(\jmath_\partial^{r-1}\ell)=0$, then apply Lemma 5.6 to the $\mathcal{L}(K)$-definable set

$$\{a\in K^r : G(a)=0\}.$$

If $G(\jmath_\partial^{r-1}\ell)\neq 0$, then since $K\langle\jmath_\partial^{r-1}\ell\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 5.5, we may take $g\in K^\times$ with $G(\jmath_\partial^{r-1}\ell)\sim g$. Now apply Lemma 5.6 to the $\mathcal{L}^{\mathcal{O}}(K)$-definable set

$$\{a\in K^r : G(a)\sim g\}.\qquad\square$$

We are now ready to prove Proposition 5.4.

Proof of Proposition 5.4. Suppose Proposition 5.4 holds with q in place of r for all $q<r$ (this is vacuous if $r=0$). Then (IH) holds, since we are assuming that $Z_q(K,\ell)=\emptyset$ for all $q<r$. Let $F\colon K^{1+r}\to K$ be as in the statement of the proposition. Since $F\notin Z_r(K,\ell)$, we may take $a\in K$ and $d\in K^\times$ with $\ell-a\prec d$ such that $F_{+a,\times d}$ is not eventually small near $\big(K,d^{-1}(\ell-a)\big)$. Set $e:=d^{-1}(\ell-a)\prec 1$ and take ϕ with $v\phi\in\Gamma(\partial)$ such that $F^{\phi}_{+a,\times d}$ is not small near (K^ϕ,e). Set $\delta:=\phi^{-1}\partial$ and set

$$\mathfrak{m} \;:=\; \mathfrak{m}_{F^{\phi}_{+a,\times d}},\qquad G \;:=\; \big(I_{F^{\phi}_{+a,\times d}}\big)^{\phi^{-1}}_{-d^{-1}a,\times d^{-1}}.$$

We have

$$F(\jmath_\partial^r\ell) \;=\; F^{\phi}_{+a,\times d}(\jmath_\delta^r e) \;=\; \mathfrak{m}_{F^{\phi}_{+a,\times d}}\big(\delta^r e - I_{F^{\phi}_{+a,\times d}}(\jmath_\delta^{r-1}e)\big) \;=\; \mathfrak{m}\big(\delta^r e - G(\jmath_\partial^{r-1}\ell)\big).$$

Using Corollary 5.7, we take $\eta\in v(\ell-K)$ such that for all $y\in K$ with $v(\ell-y)>\eta$, either

$$G(\jmath_\partial^{r-1}y) \;=\; G(\jmath_\partial^{r-1}\ell) \;=\; 0 \;\text{ or }\; G(\jmath_\partial^{r-1}y) \;\sim\; G(\jmath_\partial^{r-1}\ell) \;\neq\; 0.$$

Then $\eta-vd\in v(e-K)$ and, since $e\prec 1$, we may increase η and arrange $\eta-vd>0$. Since $F^{\phi}_{+a,\times d}$ is not small near (K^ϕ,e), we may take $z_0\in K$ with

$$v(e-z_0) \;>\; \eta-vd,\qquad I_{F^{\phi}_{+a,\times d}}(\jmath_\delta^{r-1}z_0) \;\succcurlyeq\; 1.$$

Then $v\big(\ell-(dz_0+a)\big)>\eta$, so we have

$$G(\jmath_\partial^{r-1}\ell) \;\sim\; G\big(\jmath_\partial^{r-1}(dz_0+a)\big) \;=\; I_{F^{\phi}_{+a,\times d}}(\jmath_\delta^{r-1}z_0) \;\succcurlyeq\; 1.$$

Since δ is small, L strictly extends K, and $e\prec 1$, we have $\delta^r e\prec 1$, so

$$F(\jmath_\partial^r\ell) \;=\; \mathfrak{m}\big(\delta^r e - G(\jmath_\partial^{r-1}\ell)\big) \;\sim\; -\mathfrak{m}G(\jmath_\partial^{r-1}\ell) \;\neq\; 0.$$

Now, let $y\in K$ with $v(\ell-y)>\eta$ and set $z:=d^{-1}(y-a)$, so

$$F(\jmath_\partial^r y) \;=\; F^{\phi}_{+a,\times d}(\jmath_\delta^r z) \;=\; \mathfrak{m}_{F^{\phi}_{+a,\times d}}\big(\delta^r z - I_{F^{\phi}_{+a,\times d}}(\jmath_\delta^{r-1}z)\big) \;=\; \mathfrak{m}\big(\delta^r z - G(\jmath_\partial^{r-1}y)\big).$$

Since $v(e-z)>\eta-vd>0$, we have $z\prec 1$, so $\delta^r z\prec 1$. Since $v(\ell-y)>\eta$, we also have $G(\jmath_\partial^{r-1}y)\sim G(\jmath_\partial^{r-1}\ell)\succcurlyeq 1$. Thus,

$$F(\jmath_\partial^r y) \;=\; \mathfrak{m}\big(\delta^r z - G(\jmath_\partial^{r-1}y)\big) \;\sim -\mathfrak{m}G(\jmath_\partial^{r-1}y) \;\sim\; -\mathfrak{m}G(\jmath_\partial^{r-1}\ell) \;\sim\; F(\jmath_\partial^r\ell).\qquad\square$$

6. Constructing immediate extensions when $S(\partial)=\{0\}$

As in the previous section, let ℓ be an element in a strict $T^{\mathcal{O},\partial}$-extension L of K and suppose $v(\ell-K)$ has no largest element.

Proposition 6.1. *Suppose $Z(K,\ell)=\emptyset$. Then $K\langle\jmath_\partial^\infty\ell\rangle$ is an immediate strict $T^{\mathcal{O},\partial}$-extension of K. Let b be an element in a strict $T^{\mathcal{O},\partial}$-extension M of K with $v(b-y)=v(\ell-y)$ for each $y\in K$. Then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle\jmath_\partial^\infty\ell\rangle\to M$ sending ℓ to b.*

Proof. By Lemma 5.5, $K\langle\jmath_\partial^\infty\ell\rangle$ is an increasing union of immediate $T^{\mathcal{O}}$-extensions of K, so it is itself an immediate $T^{\mathcal{O}}$-extension of K. It is also strict, as L is strict. As for the existence of an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle\jmath_\partial^\infty\ell\rangle\to M$, we proceed by induction. Let $r\geqslant 0$ and suppose we have an $\mathcal{L}^{\mathcal{O}}(K)$-embedding

$$\imath\colon K\langle\jmath_\partial^{r-1}\ell\rangle\to M$$

which sends the tuple $\mathcal{J}_\partial^{r-1}(\ell)$ to $\mathcal{J}_\partial^{r-1}(b)$ (this holds vacuously when $r=0$). Let $G\colon K^r \to K$ be an $\mathcal{L}(K)$-definable function. As $Z(K,b)=Z(K,\ell)=\emptyset$, Proposition 5.4 (applied to both ℓ and b) gives $\eta \in v(\ell - K)$ such that

$$\ell^{(r)} - G(\mathcal{J}_\partial^{r-1}\ell) \ \sim\ y^{(r)} - G(\mathcal{J}_\partial^{r-1}y) \ \sim\ b^{(r)} - G(\mathcal{J}_\partial^{r-1}b)$$

for $y \in K$ with $v(\ell - y) = v(b-y) > \eta$. Since G is arbitrary and $\imath\big(G(\mathcal{J}_\partial^{r-1}\ell)\big) = G(\mathcal{J}_\partial^{r-1}b)$, we may apply Corollary 1.6 with $K\langle \mathcal{J}_\partial^{r-1}\ell\rangle$, $\ell^{(r)}$, and $b^{(r)}$ in place of K, ℓ, and b to extend $\imath$ to an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \mathcal{J}_\partial^{r}\ell\rangle \to M$ sending $\ell^{(r)}$ to $b^{(r)}$. The union of these embeddings is an $\mathcal{L}^{\mathcal{O}}(K)$-embedding $K\langle \mathcal{J}_\partial^{\infty}\ell\rangle \to M$ which sends $\mathcal{J}_\partial^{\infty}(\ell)$ to $\mathcal{J}_\partial^{\infty}(b)$. This is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Fact 2.2. As an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, it is uniquely determined by the condition that ℓ be sent to b. $\square$

Proposition 6.2. *Suppose $S(\partial) = \{0\}$, let $F \in Z_{r+1}(K,\ell)$, and suppose that $Z_q(K,\ell) = \emptyset$ for all $q \leqslant r$. Then K has an immediate strict $T^{\mathcal{O},\partial}$-extension $K\langle \mathcal{J}_\partial^{r}a\rangle$ with $F(\mathcal{J}_\partial^{r+1}a) = 0$ and $v(a-y) = v(\ell-y)$ for each $y \in K$. Let b be an element in a strict $T^{\mathcal{O},\partial}$-extension M of K with $F(\mathcal{J}_\partial^{r+1}b) = 0$ and $v(b-y) = v(\ell-y)$ for each $y \in K$. Then there is a unique $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding $K\langle \mathcal{J}_\partial^{r}a\rangle \to M$ sending a to b.*

Proof. Let $(a_0,\dots,a_r)$ realize the $\mathcal{L}^{\mathcal{O}}(K)$-type of $\mathcal{J}_\partial^{r}(\ell)$ in some $T^{\mathcal{O}}$-extension of K. Then $K\langle a_0,\dots,a_r\rangle$ is an immediate $T^{\mathcal{O}}$-extension of K by Lemma 5.5. The tuple $\mathcal{J}_\partial^{r}(\ell)$ is $\mathcal{L}(K)$-independent by Proposition 5.4, so $(a_0,\dots,a_r)$ is $\mathcal{L}(K)$-independent as well. Using Fact 2.2, we extend ∂ to a T-derivation on $K\langle a_0,\dots,a_r\rangle$ with $a_i' = a_{i+1}$ for $i<r$ and $a_r' = I_F(a_0,\dots,a_r)$. Set $a := a_0$ so $K\langle a_0,\dots,a_r\rangle = K\langle \mathcal{J}_\partial^{r}a\rangle$ and $a^{(r+1)} = I_F(\mathcal{J}_\partial^{r}a)$.

We need to show that $K\langle \mathcal{J}_\partial^{r}a\rangle$ is a strict extension of K. Let $\phi \in K^\times$ with $v\phi \in \Gamma(\partial)$ and let $G\colon K^{r+1}\to K$ be an $\mathcal{L}(K)$-definable function with $G(\mathcal{J}_\partial^{r}a) \prec 1$. By Fact 3.7, it suffices to show that $G(\mathcal{J}_\partial^{r}a)' \prec \phi$. We assume that $G(\mathcal{J}_\partial^{r}a) \neq 0$, and we take an $\mathcal{L}(K)$-definable open set $U \subseteq K^{1+r}$ on which G is $\mathcal{C}^1$ and which contains $\mathcal{J}_\partial^{r}(a)$ in its natural extension. By Fact 2.1, we have

$$G(\mathcal{J}_\partial^{r}a)' \ =\ G^{[\partial]}(\mathcal{J}_\partial^{r}a) + \mathbf{J}_G(\mathcal{J}_\partial^{r}a)\big(a',a'',\dots,a^{(r)}, I_F(\mathcal{J}_\partial^{r}a)\big).$$

Let $Y = (Y_0,\dots,Y_r)$ and let $H\colon U \to K$ be the function

$$H(Y) \ :=\ G^{[\partial]}(Y) + \mathbf{J}_G(Y)\big(Y_1,\dots,Y_r, I_F(Y)\big),$$

so $H(\mathcal{J}_\partial^{r}a) = G(\mathcal{J}_\partial^{r}a)'$. Suppose toward contradiction that $H(\mathcal{J}_\partial^{r}a) \succcurlyeq \phi$. Since $\mathcal{J}_\partial^{r}(\ell)$ and $\mathcal{J}_\partial^{r}(a)$ have the same $\mathcal{L}^{\mathcal{O}}(K)$-type and $Z_r(K,\ell)=\emptyset$, Lemma 5.6 and Corollary 5.7 give $\eta \in v(\ell - K)$ with

$$\mathcal{J}_\partial^{r}(y) \ \in\ U, \qquad G(\mathcal{J}_\partial^{r}y) \ \sim\ G(\mathcal{J}_\partial^{r}a) \ \prec\ 1, \qquad H(\mathcal{J}_\partial^{r}y) \ \sim\ H(\mathcal{J}_\partial^{r}a) \ \succcurlyeq\ \phi$$

for all $y \in K$ with $v(\ell-y) > \eta$. For the remainder of this proof, we let $y \in K$ with $v(\ell - y) > \eta$. Since $G(\mathcal{J}_\partial^{r}y) \prec 1$ and $v\phi \in \Gamma(\partial)$, we have

$$G(\mathcal{J}_\partial^{r}y)' \ =\ G^{[\partial]}(\mathcal{J}_\partial^{r}y) + \mathbf{J}_G(\mathcal{J}_\partial^{r}y)(y',\dots,y^{(r)},y^{(r+1)}) \ \prec\ \phi \ \preccurlyeq\ H(\mathcal{J}_\partial^{r}y).$$

Thus

$$G(\mathcal{J}_\partial^{r}y)' - H(\mathcal{J}_\partial^{r}y) \ =\ \frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}y)\big(y^{(r+1)} - I_F(\mathcal{J}_\partial^{r}y)\big) \ \sim\ -H(\mathcal{J}_\partial^{r}y).$$

Since $H(\mathcal{J}_\partial^{r}y) \neq 0$, we have $\frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}y) \neq 0$, so

$$y^{(r+1)} - I_F(\mathcal{J}_\partial^{r}y) \ \sim\ -H(\mathcal{J}_\partial^{r}y)\left(\frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}y)\right)^{-1}.$$

We have $H(\mathcal{J}_\partial^{r}y) \sim H(\mathcal{J}_\partial^{r}a)$ and, by increasing η, we may assume $\frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}y) \sim \frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}a)$. Thus

$$F(\mathcal{J}_\partial^{r+1}y) \ =\ \mathfrak{m}_F\big(y^{(r+1)} - I_F(\mathcal{J}_\partial^{r}y)\big) \sim\ -\mathfrak{m}_F H(\mathcal{J}_\partial^{r}a)\left(\frac{\partial G}{\partial Y_r}(\mathcal{J}_\partial^{r}a)\right)^{-1}.$$

In particular, $F(\mathcal{J}_\partial^{r+1}y) \sim F(\mathcal{J}_\partial^{r+1}z)$ for all $y,z \in K$ with $v(\ell-y), v(\ell-z) > \eta$, contradicting Lemma 5.3.

Now let M and b be as in the statement of the proposition. Since $\mathcal{J}_\partial^{r}(\ell)$ and $\mathcal{J}_\partial^{r}(a)$ have the same $\mathcal{L}^{\mathcal{O}}(K)$-type and $Z_r(K,\ell)=\emptyset$, we may construct an $\mathcal{L}^{\mathcal{O}}(K)$-embedding

$$K\langle \mathcal{J}_\partial^{r}a\rangle \ \to\ M$$

which sends $\mathcal{J}_\partial^{r}(a)$ to $\mathcal{J}_\partial^{r}(b)$ as in the proof of the previous proposition. This is even an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding by Fact 2.2. As an $\mathcal{L}^{\mathcal{O},\partial}(K)$-embedding, it is uniquely determined by the condition that a be sent to b. $\square$

Theorem 6.3. *Suppose $S(\partial) = \{0\}$. Then K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension.*

Proof. We may assume that K is not itself spherically complete. It suffices to show that K has a proper immediate strict $T^{\mathcal{O},\partial}$-extension, as the property $S(\partial) = \{0\}$ is preserved by immediate strict extensions. Let $\mathcal{B}$ be a nested collection of closed v-balls in K with empty intersection in K and let ℓ be an element in an elementary $T^{\mathcal{O},\partial}$-extension of L of K with $\ell \in \bigcap \mathcal{B}^L$. Then $v(\ell - K)$ has no largest element by Lemma 1.7. If $Z(K,\ell) = \emptyset$, then $K\langle \jmath_\partial^\infty \ell\rangle$ is a proper immediate strict $T^{\mathcal{O},\partial}$-extension of K by Proposition 6.1. Suppose $Z(K,\ell) \neq \emptyset$. Lemma 5.2 gives $Z_0(K,\ell) = \emptyset$, so take r maximal such that $Z_q(K,\ell) = \emptyset$ for all $q \leqslant r$. Then Proposition 6.2 provides a proper immediate strict $T^{\mathcal{O},\partial}$-extension $K\langle \jmath_\partial^r a\rangle$ of K where a is in the natural extension of each $B \in \mathcal{B}$. □

Before moving on, let us consider a couple of cases that can be handled by Theorem 6.3. Suppose that K has small derivation and that the induced derivation on $\operatorname{res}(K)$ is nontrivial. Then $\Gamma(\partial) = \Gamma^{\leqslant}$, so $S(\partial) = \{0\}$; see [3, Corollary 1.7 and Lemma 1.15]. Given a $T^{\mathcal{O},\partial}$-extension M of K, it follows from Fact 3.1 that M is a strict extension of K if and only if M has small derivation. Thus, we have the following:

Corollary 6.4. *If K has small derivation and the induced derivation on* $\operatorname{res}(K)$ *is nontrivial, then K has a spherically complete immediate $T^{\mathcal{O},\partial}$-extension with small derivation.*

Another important case where $S(\partial) = \{0\}$ is when K is asymptotic. Recall from [2] that K is said to be **asymptotic** if for all $f, g \in K^\times$ with $f, g \prec 1$, we have $f \prec g \iff f' \prec g'$. If K is asymptotic, then $S(\partial) = \{0\}$ by [3, Lemma 1.14], so K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension. Moreover, if K is asymptotic and M is an immediate $T^{\mathcal{O},\partial}$-extension of K, then M is a strict $T^{\mathcal{O},\partial}$-extension of K if and only if M is itself asymptotic by [3, Lemmas 1.11 and 1.12]. We summarize below:

Corollary 6.5. *If K is asymptotic, then K has a spherically complete immediate asymptotic $T^{\mathcal{O},\partial}$-extension.*

7. Coarsening by $S(\partial)$

In this section, we prove our main theorem. First, we establish some results on residue field extensions.

Lemma 7.1. *Suppose K has small derivation and let $L = K\langle a\rangle$ be a simple $T^{\mathcal{O},\partial}$-extension of K with $a \asymp 1$, $\bar{a} \notin \operatorname{res}(K)$, and $a' \preccurlyeq 1$. Then L has small derivation. Moreover, if $\partial\mathcal{O} \subseteq \smallo$ and $a' \prec 1$, then $\partial_L \mathcal{O}_L \subseteq \smallo_L$.*

Proof. Let $F\colon K \to K$ be an $\mathcal{L}(K)$-definable function with $F(a) \prec 1$. We need to show that

$$F(a)' \;=\; F^{[\partial]}(a) + F'(a)a' \;\prec\; 1.$$

Since $\operatorname{res}(L) \neq \operatorname{res}(K)$ and T is power bounded, we have $\Gamma_L = \Gamma$ by [5, Corollary 5.6] (the Wilkie inequality for power bounded theories). Using Lemma 1.3, we see that $F'(a) \preccurlyeq F(a) \prec 1$, so $F'(a)a' \prec 1$ and it remains to show that $F^{[\partial]}(a) \prec 1$. By $\mathcal{L}^{\mathcal{O}}$-elementarity, it suffices to show that for any $\mathcal{L}^{\mathcal{O}}(K)$-definable set $A \subseteq \mathcal{O}$ with $a \in A^L$, there is $y \in A$ with $F^{[\partial]}(y) \prec 1$. Let A be such a set and, by shrinking A if need be, assume that F is $\mathcal{C}^1$ on A and that $F(y) \prec 1$ for all $y \in A$. Since $F'(a) \prec 1$, we can use $\mathcal{L}^{\mathcal{O}}$-elementarity to take $y \in A$ with $F'(y) \prec 1$. Since $y' \preccurlyeq 1$ by Fact 3.1, we have $F'(y)y' \prec 1$. Since $F(y)' \prec 1$ as well, this gives

$$F^{[\partial]}(y) \;=\; F(y)' - F'(y)y' \;\prec\; 1.$$

This takes care of the first part of the lemma.

For the second part, assume that $\partial\mathcal{O} \subseteq \smallo$ and that $a' \prec 1$. We need to show that $F(a)' \prec 1$ for each $\mathcal{L}(K)$-definable function $F\colon K \to K$ with $F(a) \preccurlyeq 1$. The proof is essentially the same as the proof of the first part, but now Lemma 1.3 only gives that $F'(a) \preccurlyeq 1$. We make up for this by using our assumption that $\partial\mathcal{O} \subseteq \smallo$ and that $a' \prec 1$. □

The following corollary serves as an analog of [3, Corollary 6.7].

Corollary 7.2. *Suppose $\partial\mathcal{O} \subseteq \smallo$ and let E be a T-extension of* $\operatorname{res}(K)$. *Then there is a strict $T^{\mathcal{O},\partial}$-extension L of K such that $\Gamma_L = \Gamma$, the derivation on* $\operatorname{res}(L)$ *is trivial, and* $\operatorname{res}(L)$ *is $\mathcal{L}(\operatorname{res} K)$-isomorphic to E.*

Proof. It suffices to consider the case $E = \operatorname{res}(K)\langle f\rangle$ where $f \notin \operatorname{res}(K)$. Let $L = K\langle a\rangle$ be a simple T-extension of K where a realizes the cut

$$\{y \in K : y < \mathcal{O} \text{ or } y \in \mathcal{O} \text{ and } \bar{y} < f\}.$$

We expand L to an $\mathcal{L}^{\mathcal{O}}$-structure by letting

$$\mathcal{O}_L \ := \ \big\{y \in L : |y| < d \text{ for all } d \in K \text{ with } d > \mathcal{O}\big\}.$$

This expansion of L is a $T^{\mathcal{O}}$-extension of K by [7, Main Lemma 3.6]. Note that $a \in \mathcal{O}_L$ and that $\operatorname{res}(L) = \operatorname{res}(K)\langle \bar{a}\rangle$ is $\mathcal{L}(\operatorname{res} K)$-isomorphic to E, since $\bar{a}$ and f realize the same cut in $\operatorname{res}(K)$. In particular, $\operatorname{res}(L) \neq \operatorname{res}(K)$, so $\Gamma_L = \Gamma$ by the Wilkie inequality. Using Fact 2.2, we extend ∂ uniquely to a T-derivation on L with $a' = 0$. We claim that L is a strict $T^{\mathcal{O},\partial}$-extension of K. Let $\phi \in K^{\times}$ and note that $\phi^{-1}a' = a' = 0$. If $\partial\smallo \subseteq \phi\smallo$, then Lemma 7.1 applied to K^{ϕ} and L^{ϕ} in place of K and L gives $\partial_L \smallo_L \subseteq \phi\smallo_L$. Likewise, if $\partial\mathcal{O} \subseteq \phi\smallo$, then Lemma 7.1 gives $\partial_L\mathcal{O}_L \subseteq \phi\smallo_L$. The case $\phi = 1$ gives that the derivation on $\operatorname{res}(L)$ is trivial. ☐

We are now ready to prove the main theorem:

Theorem 7.3. *Suppose $S(\partial)$ is a Λ-subspace of K. Then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

Proof. If $S(\partial) = \{0\}$, then we are done by Theorem 6.3, so we may assume that $\Delta := S(\partial) \neq \{0\}$. We arrange by compositional conjugation that K has small derivation. The assumption that Δ is a Λ-subspace of K allows us to coarsen by Δ, which we do, yielding the $T^{\mathcal{O},\partial}$-model K_Δ. The derivation on $\operatorname{res}(K_\Delta)$ is trivial by [3, Lemma 6.1] and $S_{K_\Delta}(\partial) = \{0\}$ by [3, Lemma 6.2]. Let E be a spherically complete immediate $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_\Delta)$; such an extension exists by Corollary 1.11. Using Corollary 7.2, we take a strict $T^{\mathcal{O},\partial}$-extension L of K_Δ such that $\Gamma_L = \dot{\Gamma}$, the derivation on $\operatorname{res}(L)$ is trivial, and $\operatorname{res}(L)$ is $\mathcal{L}(\operatorname{res} K_\Delta)$-isomorphic to E. Then $S_L(\partial) = \{0\}$ as well, and we apply Theorem 6.3 to L to get a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension M of L. We have $\operatorname{res}(M) = \operatorname{res}(L)$ as T-models, so $\operatorname{res}(M)$ is $\mathcal{L}(\operatorname{res} K_\Delta)$-isomorphic to E. We equip $\operatorname{res}(M)$ with a T-convex valuation ring $\mathcal{O}_{\operatorname{res}(M)}$ so that $\operatorname{res}(M)$ is $\mathcal{L}^{\mathcal{O}}(\operatorname{res} K_\Delta)$-isomorphic to E; then $\operatorname{res}(M)$ is a spherically complete immediate $T^{\mathcal{O}}$-extension of $\operatorname{res}(K_\Delta)$. Now let M^* be the $T^{\mathcal{O},\partial}$-model with underlying T^{∂}-model M and T-convex valuation ring

$$\mathcal{O}_{M^*} \ := \ \{a \in \mathcal{O}_M : \bar{a} \in \mathcal{O}_{\operatorname{res}(M)}\}.$$

Then M^* is an immediate $T^{\mathcal{O},\partial}$-extension of K with $M^*_\Delta = M$; see Subsection 1.3. By [3, Lemma 6.4], M^* is a strict $T^{\mathcal{O},\partial}$-extension of K. As $M^*_\Delta = M$ and $\operatorname{res}(M^*_\Delta) = \operatorname{res}(M)$ are both spherically complete, M^* is spherically complete by Fact 1.14. ☐

The diagram below catalogs the objects and maps involved in the proof of Theorem C.

$$\begin{tikzcd}
K_\Delta \arrow[r, "\text{strict}", "\mathcal{L}^{\mathcal{O},\partial}"'] \arrow[d] & L \arrow[r, "\text{strict}", "\mathcal{L}^{\mathcal{O},\partial}"'] \arrow[d] & M = M^*_\Delta \arrow[d] & \\
\operatorname{res}(K_\Delta) \arrow[r, "\mathcal{L}"'] \arrow[d] \arrow[rr, bend right, "\mathcal{L}^{\mathcal{O}}"'] & \operatorname{res}(L) \arrow[r, "\sim", "\mathcal{L}"'] & \operatorname{res}(M) \arrow[r, "\sim", "\mathcal{L}^{\mathcal{O}}"'] \arrow[d] & E \arrow[d] \\
\operatorname{res}(K) \arrow[rr, "\sim", "\mathcal{L}"'] & & \operatorname{res}(M^*) \arrow[r, "\sim", "\mathcal{L}"'] & \operatorname{res}(E)
\end{tikzcd}$$

As with the diagram in Subsection 1.3, each horizontal arrow is an embedding in the indicated language and every downward arrow is a partially defined projection. Isomorphisms are labeled as such and every square commutes.

If Λ is archimedean, then $S(\partial)$ is always a Λ-subspace of Γ. Thus, we have the following:

Corollary 7.4. *If T is polynomially bounded, then K has an immediate strict $T^{\mathcal{O},\partial}$-extension which is spherically complete.*

7.1. **Uniqueness.** Suppose K has a spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension M. It is natural to ask: under which circumstances is M the *unique* spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension of K up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism? Uniqueness holds if K is itself spherically complete, for then $M = K$. In particular, it holds if $\mathcal{O} = K$. If ∂ is trivial, then any immediate strict $T^{\mathcal{O},\partial}$-extension of K has trivial derivation as well, so K has a unique spherically complete immediate strict $T^{\mathcal{O},\partial}$-extension up to $\mathcal{L}^{\mathcal{O},\partial}(K)$-isomorphism by Corollary 1.11.

Suppose $T = \mathrm{RCF}$. If ∂ is small, $\mathrm{res}(K)$ is linearly surjective, and K is monotone, then M is unique up to isomorphism over K by [2, Section 7.4]. If ∂ is small, $\mathrm{res}(K)$ is linearly surjective, and K is asymptotic then again M is unique up to isomorphism over K by [18]. An example of a real closed H-field R which does not have a unique spherically complete immediate strict $\mathrm{RCF}^{\mathcal{O},\partial}$-extension up to isomorphism over R is given in [3].

When $T \neq \mathrm{RCF}$, nothing is known about uniqueness outside of the trivial cases. All the results in the case $T = \mathrm{RCF}$ depend crucially on *differential henselianity*, a differential-algebraic property which we have yet to generalize to our setting.

References

[1] M. Aschenbrenner and L. van den Dries. H-fields and their Liouville extensions. *Math. Z.*, 242(3):543–588, 2002.

[2] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. *Asymptotic Differential Algebra and Model Theory of Transseries*. Number 195 in Annals of Mathematics Studies. Princeton University Press, 2017.

[3] M. Aschenbrenner, L. van den Dries, and J. van der Hoeven. Maximal immediate extensions of valued differential fields. *Proc. Lond. Math. Soc. (3)*, 117(2):376–406, 2018.

[4] L. van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bull. Amer. Math. Soc. (N.S.)*, 15:189–193, 1986.

[5] L. van den Dries. T-convexity and tame extensions II. *J. Symbolic Logic*, 62:14–34, 1997.

[6] L. van den Dries. *Tame Topology and o-Minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1998.

[7] L. van den Dries and A. Lewenberg. T-convexity and tame extensions. *J. Symbolic Logic*, 60(1):74–102, 1995.

[8] L. van den Dries, A. Macintyre, and D. Marker. The elementary theory of restricted analytic fields with exponentiation. *Ann. Math.*, 140:183–205, 1994.

[9] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential power series. *J. Lond. Math. Soc. (2)*, 56(3):417–434, 1997.

[10] L. van den Dries, A. Macintyre, and D. Marker. Logarithmic-exponential series. *Ann. Pure Appl. Logic*, 111(1-2):61–113, 2001.

[11] A. Fornasiero and E. Kaplan. Generic derivations on o-minimal structures. *J. Math. Log.*, to appear.

[12] A. Gabrielov. Projections of semianalytic sets. *Funkcional. Anal. i Priložen.*, 2(4):18–30, 1968.

[13] T. Hakobyan. An Ax-Kochen-Ershov theorem for monotone differential-Henselian fields. *J. Symb. Log.*, 83(2):804–816, 2018.

[14] J. van der Hoeven. *Asymptotique Automatique*. PhD thesis, École Polytechnique, 1997.

[15] E. Kaplan. *Derivations on o-minimal fields*. PhD thesis, University of Illinois at Urbana-Champaign, 2021.

[16] F.-V. Kuhlmann, S. Kuhlmann, and S. Shelah. Exponentiation in power series fields. *Proc. Amer. Math. Soc.*, 125(11):3177–3183, 1997.

[17] C. Miller. A growth dichotomy for o-minimal expansions of ordered fields. In *Logic: from foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 385–399. Oxford Univ. Press, New York, 1996.

[18] N. Pynn-Coates. Differential-henselianity and maximality of asymptotic valued differential fields. *Pacific J. Math.*, 308(1):161–205, 2020.

[19] S. Rideau. Some properties of analytic difference valued fields. *J. Inst. Math. Jussieu*, 16(3):447–499, 2017.

[20] T. Scanlon. A model complete theory of valued D-fields. *J. Symbolic Logic*, 65(4):1758–1784, 2000.

[21] M. Singer. The model theory of ordered differential fields. *J. Symbolic Logic*, 43(1):82–91, 1978.

[22] J. Tyne. *T-Levels and T-Convexity*. PhD thesis, University of Illinois at Urbana-Champaign, 2003.

Email address: eakapla2@illinois.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801