MODEL COMPLETENESS FOR THE DIFFERENTIAL FIELD OF TRANSSERIES WITH EXPONENTIATION

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Abstract. Let $\mathbb{T}$ be the differential field of logarithmic-exponential transseries. We show that the expansion of $\mathbb{T}$ by its natural exponential function is model complete and locally o-minimal. We give an axiomatization of the theory of this expansion that is effective relative to the theory of the real exponential field. We adapt our results to show that the expansion of $\mathbb{T}$ by this exponential function and by its natural restricted sine and restricted cosine functions is also model complete and locally o-minimal.

Introduction

The differential field $\mathbb{T}$ of logarithmic-exponential transseries is a real closed ordered field extension of $\mathbb{R}$ which was introduced independently by Écalle [16] in his solution to the Dulac conjecture and by Dahn and Göring [10] in their work on Tarski's problem on real exponentiation. This structure $\mathbb{T}$ serves as a sort of universal domain for studying the asymptotic behavior of non-oscillating solutions to differential equations over the reals. In [3], Aschenbrenner, van den Dries, and van der Hoeven show that the elementary theory of $\mathbb{T}$ as an ordered valued differential field is model complete.

There is a natural exponential exp on the field $\mathbb{T}$ that makes $\mathbb{T}$ an elementary extension of the real exponential field $\mathbb{R}_{\exp}$ [14]. Let $\mathbb{T}_{\exp}$ be the expansion of the *differential* field $\mathbb{T}$ by exp. Then $\mathbb{T}_{\exp}$ is a proper expansion of $\mathbb{T}$ in terms of definability, since exp is not definable in $\mathbb{T}$ by [3, 16.6.7]. In the introduction of [3], it was asked whether the model completeness result for $\mathbb{T}$ can be extended to $\mathbb{T}_{\exp}$. In this paper, we provide a positive answer to this question.

Theorem A (Corollary 5.6)**.** *The elementary theory of $\mathbb{T}_{\exp}$ is model complete in the language of ordered valued differential exponential fields.*

The elementary theory of $\mathbb{T}$ is axiomatized by one of two completions of T^{nl}—the theory of ω-free newtonian Liouville closed H-fields [3]. Introduced by van den Dries and Aschenbrenner [1], H-fields form a class of ordered valued differential fields that includes $\mathbb{T}$, all differential subfields of $\mathbb{T}$ containing $\mathbb{R}$, and any Hardy field containing $\mathbb{R}$. The theory T^{nl} is the model companion of the theory of H-fields and the study of H-fields and their extensions is a key part of the proof that $\mathbb{T}$ is model complete.

In studying $\mathbb{T}_{\exp}$ (or valued exponential fields more generally), it is convenient to work with the logarithm function instead of the exponential function. One has a lot of control over the value group of valued field extensions which are only closed under taking logarithms, but not that much control over those extensions which are also closed under taking exponentials; see Section 4 of [13] for evidence supporting these heuristics. Accordingly, we study *logarithmic H-fields*, a class of H-fields which are equipped with a (not necessarily surjective) logarithm. We call a logarithmic H-field with a surjective logarithm an *exponential H-field* and we show that every logarithmic H-field has an exponential H-field extension with the same constant field. We further show that every exponential H-field can be extended to an ω-free newtonian exponential H-field with the same constant field. Every ω-free newtonian exponential H-field is automatically Liouville closed; see Remark 3.7.

Let $T_{\log}$ extend the theory of logarithmic H-fields by axioms asserting that the constant field C equipped with the restricted logarithm $\log|_{C^>}$ is elementarily equivalent to $\mathbb{R}_{\exp}$. Let $T^{\mathrm{nt}}_{\exp}$ extend the theory of ω-free newtonian exponential H-fields by the same constant field axioms. Using Wilkie's celebrated theorem that $\mathbb{R}_{\exp}$ is model complete [31], we show the following:

Theorem B (Theorem 5.4)**.** *$T^{\mathrm{nt}}_{\exp}$ is model complete and it is the model companion of $T_{\log}$.*

Since $\mathbb{T}_{\exp} \models T^{\mathrm{nt}}_{\exp}$, the first theorem is an immediate corollary of the second theorem. Our proof closely follows the proof that T^{nl} is the model companion of the theory of H-fields. Many of the tools used for that result go through in our case with little change, but extending the results in Sections 16.1 and 16.6 of [3] to our setting requires us to prove a nontrivial result about differential field extensions. Using the model completeness result above, we are able to describe the completions of $T^{\mathrm{nt}}_{\exp}$. An H-field is said to have *small derivation* if the derivative of every infinitesimal element is infinitesimal and *large derivation* otherwise.

Theorem C (Theorem 5.5). *$T^{\mathrm{nt}}_{\exp}$ has two completions: $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$, whose models are the models of $T^{\mathrm{nt}}_{\exp}$ with small derivation, and $T^{\mathrm{nt}}_{\exp,\mathrm{lg}}$, whose models have large derivation.*

This mirrors the situation with T^{nl}, whose two completions are $T^{\mathrm{nl}}_{\mathrm{sm}}$ (small derivation) and $T^{\mathrm{nl}}_{\mathrm{lg}}$ (large derivation). Since $\mathbb{T}_{\exp}$ has small derivation, $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ completely axiomatizes the theory of $\mathbb{T}_{\exp}$. The only part of this axiomatization that is not known to be effective is the part that states that the constant field is elementarily equivalent to $\mathbb{R}_{\exp}$; see [25]. Thus, the theory of $\mathbb{T}_{\exp}$ is decidable relative to the theory of $\mathbb{R}_{\exp}$.

The theory $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ has other models of interest: any maximal Hardy field is a model of $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$, as is Conway's field of surreal numbers equipped with the Kruskal-Gonshor exponential and the Berarducci-Mantova derivation. In fact, any model of $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ with constant field $\mathbb{R}$ admits an elementary differential exponential field embedding into the surreals. This answers the first question posed in [6].

We are also able to use our model completeness result to show that unary definable sets in models of $T^{\mathrm{nt}}_{\exp}$ are topologically tame.

Theorem D (Theorem 5.13). *If $K \models T^{\mathrm{nt}}_{\exp}$, then for each $y \in K$ and each definable $X \subseteq K$, there is an open interval (a, b) around y such that $X \cap (a, b)$ is a finite union of points and intervals.*

In the literature, this property is called *local o-minimality*; see [30]. In the future, we hope to better understand the definable sets of arbitrary arity. We also hope to show that $T^{\mathrm{nt}}_{\exp}$ is combinatorially tame (NIP or even distal). This likely requires us to first prove a quantifier elimination result in some reasonable extended language. At the end of this paper, we give some ideas about what this language might be.

Given a real analytic function $f\colon U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^n$ is an open neighborhood of the compact box $[-1,1]^n$, we associate to f its restriction $f|_{[-1,1]^n}$. The expansion of the real field by all of these *restricted analytic functions* and the unrestricted exponential function, denoted $\mathbb{R}_{\mathrm{an},\exp}$, is model complete [15]. There is a natural expansion of the exponential field $\mathbb{T}$ by restricted analytic functions using Taylor series, and this expansion is an elementary extension of $\mathbb{R}_{\mathrm{an},\exp}$; see [14] for details. Ultimately, we would like to prove a model completeness result for $\mathbb{T}_{\mathrm{an},\exp}$: the expansion of the *differential* field $\mathbb{T}$ by these restricted analytic functions and the unrestricted exponential. As a step in this direction, we modify our model completeness proof to study the expansion of $\mathbb{T}_{\exp}$ by restricted sine and cosine functions. We call this expansion $\mathbb{T}_{\mathrm{rt},\exp}$, where the subscript rt stands for "restricted trigonometric functions".

Theorem E (Corollary 6.6). *The elementary theory of $\mathbb{T}_{\mathrm{rt},\exp}$ is model complete.*

This theorem uses that the corresponding expansion $\mathbb{R}_{\mathrm{rt},\exp}$ of the real field by restricted sine and cosine functions and the unrestricted exponential function is model complete, a theorem of van den Dries and Miller [15]. We use our result to show that $\mathbb{T}_{\mathrm{rt},\exp}$ is also locally o-minimal.

Outline. Section 1 is devoted to preliminaries on valued fields, differential fields, and especially H-fields. In Section 2, we prove a somewhat technical result about differential field extensions. Logarithmic and exponential H-fields are introduced in Section 3 and we study logarithmic H-field extensions and logarithmic H-field embeddings in Section 4. We use these results in Section 5 to prove our main theorems about $T^{\mathrm{nt}}_{\exp}$ and in Section 6, we study restricted trigonometric functions. In Section 7, we provide an alternative axiomatization for $T^{\mathrm{nt}}_{\exp}$, indicate a language in which $T^{\mathrm{nt}}_{\exp}$ might eliminate quantifiers, and briefly discuss the issue of proving model completeness for $\mathbb{T}_{\mathrm{an},\exp}$.

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1. Preliminaries

We draw heavily from [3] and for brevity, we cite the results that we use from there only by their number. For example, we say [3, 3.5.19] instead of [3, Corollary 3.5.19]. The book [3] is almost entirely self-contained, so some of the results we cite (like [3, 3.5.19]) were not originally proven there. We use the same notational conventions as [3], but we will repeat what is needed in this paper.

We let m, n and r range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. All fields are assumed to be of characteristic zero and if K is a field, then we set $K^\times := K \setminus \{0\}$. By "ordered set" we mean "totally ordered set". If S is an ordered set and $a \in S$, then we set $S^{>a} := \{s \in S : s > a\}$; similarly for $S^{\geqslant a}$, $S^{<a}$, $S^{\leqslant a}$, and $S^{\neq a}$. If S is an ordered abelian group, then we let $S^{>} := S^{>0}$ and we define $S^{\geqslant}$, $S^{<}$, $S^{\leqslant}$, and $S^{\neq}$ analogously. For $A \subseteq S$, we set

$$A^{\downarrow} \ := \ \{s \in S : s < a \text{ for some } a \in A\}$$

and we say that A **is downward closed in** S if $A = A^{\downarrow}$.

Ordered fields and ordered exponential fields. Let K be an ordered field. We let $[-1,1]_K$ denote the closed interval $\{x \in K : -1 \leqslant x \leqslant 1\}$. We let K^{rc} denote the real closure of K. If L is a real closed field extension of K, then there is a unique ordered field embedding $K^{\mathrm{rc}} \to L$ over K.

An **exponential** on K is an ordered group isomorphism $\exp\colon K \to K^{>}$. We define an **ordered exponential field** to be an ordered field K equipped with an exponential exp. In other literature, exponentials are sometimes not required to be surjective onto $K^{>}$, but it is convenient for us to impose this condition. We let $\log\colon K^{>} \to K$ denote the inverse of exp. If K and L are ordered exponential fields, then an ordered field embedding $\imath\colon K \to L$ is said to be an **ordered exponential field embedding** if $\imath(\exp a) = \exp \imath(a)$ for all $a \in K$.

One important ordered exponential field is the real exponential field $\mathbb{R}_{\exp}$. Let $\mathrm{Th}(\mathbb{R}_{\exp})$ be the first-order theory of $\mathbb{R}_{\exp}$ in the natural language of ordered exponential fields. This theory is model complete and o-minimal by Wilkie's theorem [31].

Valued fields. Let K be a valued field with valuation $v\colon K^\times \to \Gamma$. Then the value group of K is the ordered (additively written) abelian group Γ. We let $\mathcal{O} := \big\{a \in K : va \geqslant 0\big\}$ denote the valuation ring of K, we let $\smallo := \big\{a \in K : va > 0\big\}$ denote the unique maximal ideal of $\mathcal{O}$, and we let $\mathrm{res}(K) := \mathcal{O}/\smallo$ denote the residue field of K. Our standing assumption that all fields are of characteristic zero applies to $\mathrm{res}(K)$ as well, so all valued fields are of equicharacteristic zero. For $a, b \in K$ we set

$$a \asymp b \ :\Longleftrightarrow \ va = vb, \qquad a \preccurlyeq b \ :\Longleftrightarrow \ va \geqslant vb, \qquad a \prec b \ :\Longleftrightarrow \ va > vb, \qquad a \sim b \ :\Longleftrightarrow \ a - b \prec a.$$

Note that $a \sim b$ if and only if $a, b \neq 0$ and $a/b \in 1 + \smallo$. We let K^{h} denote the henselization of K. There is a unique valued field embedding of K^{h} over K into any henselian valued field extending K. If L is also a valued field, then we denote its value group, valuation ring, and maximal ideal by Γ_L, $\mathcal{O}_L$, and $\smallo_L$ respectively. Let L be a valued field extension of K. We identify Γ with a subgroup of Γ_L and we identify $\mathrm{res}(K)$ with a subfield of $\mathrm{res}(L) = \mathcal{O}_L/\smallo_L$ in the natural way. The Zariski-Abhyankar Inequality states that the transcendence degree of L over K is at least the transcendence degree of $\mathrm{res}(L)$ over $\mathrm{res}(K)$ plus the $\mathbb{Q}$-linear dimension of $\mathbb{Q}\Gamma_L$ over $\mathbb{Q}\Gamma$, where $\mathbb{Q}\Gamma$ is the divisible hull of Γ. We say that L is an **immediate extension of** K if $\Gamma_L = \Gamma$ and $\mathrm{res}(L) = \mathrm{res}(K)$.

Differential fields. Let K be a differential field with derivation $\partial\colon K \to K$. For $a \in K$ we often write a' instead of $\partial(a)$ and a'' instead of $\partial^2(a)$. More generally, we write $a^{(n)}$ for $\partial^n(a)$. We set $a^\dagger := a'/a$ if $a \neq 0$ and we call $a^\dagger$ the **logarithmic derivative** of a. We define the iterates of the logarithmic derivative as follows:

$$a^{\langle 0\rangle} \ := \ a, \qquad a^{\langle n+1\rangle} \ := \ \begin{cases} \big(a^{\langle n\rangle}\big)^\dagger & \text{if } a^{\langle n\rangle} \text{ is defined and nonzero} \\ \text{undefined} & \text{if } a^{\langle n\rangle} \text{ is undefined or zero.} \end{cases}$$

We let $C := \ker(\partial)$ denote the constant field of K and if L is also a differential field, then we denote its constant field by C_L. Given $\phi \in K^\times$, we let K^ϕ be the differential field with underlying field K and derivation $\phi^{-1}\partial$. We call K^ϕ the **compositional conjugate of** K **by** ϕ. Note that $C_{K^\phi} = C$.

We set $K' := \{a' : a \in K\}$ and we set $(K^\times)^\dagger := \{a^\dagger : a \in K^\times\}$. Note that $K^\dagger$ is a subgroup of K since $a^\dagger + b^\dagger = (ab)^\dagger$ for $a, b \in K^\times$. Note also that K' is a C-vector subspace of K. We say that $a \in K$ can be

integrated if $a \in K'$ and we call an element $b \in K$ with $b' = a$ an **integral** of a. If b_0 and b_1 are integrals of a, then $b_0 - b_1 \in C$. We say that $a \in K$ can be **exponentially integrated** if $a \in (K^\times)^\dagger$ and we call an element $b \in K^\times$ with $b^\dagger = a$ an **exponential integral** of a. If b_0 and b_1 are exponential integrals of a, then $b_0/b_1 \in C^\times$. In this paper, we will make use of Ostrowski's Theorem about the algebraic independence of integrals [26].

Theorem (Ostrowski)**.** *Let L be a differential field extension of K with $C_L = C$. Let $a_1, \dots, a_n \in L$ with $a_1', \dots, a_n' \in K$. Then either $a_1, \dots, a_n$ are algebraically independent over K or $a_1, \dots, a_n$ are C-linearly dependent over K.*

If y is an element of a differential field extension of K, then we let $K\langle y\rangle := K(y, y', y'', \dots)$ denote the differential field extension of K generated by y. We say that y is d**-transcendental over** K if the sequence $y, y', y'', \dots$ is algebraically independent over K (in the field-theoretic sense) and we say that y is d**-algebraic over** K otherwise. If y is d-algebraic over K, then $K\langle y\rangle$ has finite transcendence degree over K.

Asymptotic fields. Let K be a valued differential field (that is, a valued field of equicharacteristic zero equipped with a derivation). Let Γ, $\mathcal{O}$, and $\scriptstyle\mathcal{O}$ be the value group, valuation ring, and maximal ideal of K respectively. If $a' \in \scriptstyle\mathcal{O}$ for all $a \in \scriptstyle\mathcal{O}$, then we say that K has **small derivation**. We say that K has **large derivation** if K does not have small derivation.

We say that K is **asymptotic** if $f \prec g \iff f' \prec g'$ for all nonzero $f, g \in \scriptstyle\mathcal{O}$. Note that if K is asymptotic, then the constant field C of K is contained in $\mathcal{O}$. Thus, any asymptotic field with nontrivial valuation must also have nontrivial derivation. On the other hand, any valued differential field with trivial valuation is automatically asymptotic. Let K be asymptotic. For $f \in K^\times$ with $f \not\asymp 1$, the values $v(f')$ and $v(f^\dagger)$ only depend on vf, so for $\gamma = vf$, we set

$$\gamma^\dagger \;:=\; v(f^\dagger), \qquad \gamma' \;:=\; v(f') \;=\; \gamma + \gamma^\dagger.$$

This gives us a map

$$\psi\colon \Gamma^{\neq} \to \Gamma, \qquad \psi(\gamma) = \gamma^\dagger$$

and, following Rosenlicht [28], we call the pair (Γ, ψ) the **asymptotic couple of** K. We have the following important subsets of Γ:

$$(\Gamma^<)' \;:=\; \{\gamma' : \gamma \in \Gamma^<\}, \qquad (\Gamma^>)' \;:=\; \{\gamma' : \gamma \in \Gamma^>\}, \qquad \Psi \;:=\; \{\gamma^\dagger : \gamma \in \Gamma^{\neq}\}.$$

It is always the case that $(\Gamma^<)' < (\Gamma^>)'$ and that $\Psi < (\Gamma^>)'$. If there is $\beta \in \Gamma$ with $\Psi < \beta < (\Gamma^>)'$, then we call β a **gap in** K. There is at most one such β, and if Ψ has a largest element, then there is no such β. If K has trivial valuation, then the three important subsets above are empty and 0 is a gap in K. We say that K **is grounded** if Ψ has a largest element and we say that K **is ungrounded** otherwise. Finally, we say that K **has asymptotic integration** if $\Gamma = (\Gamma^<)' \cup (\Gamma^>)'$.

We say that K is H**-asymptotic** if K is asymptotic and $f \preccurlyeq g \implies f^\dagger \succcurlyeq g^\dagger$ for all nonzero $f, g \in \scriptstyle\mathcal{O}$. We have an important trichotomy for the structure of H-asymptotic fields:

Fact 1.1 ([3], 9.2.16)**.** *If K be an H-asymptotic field, then exactly one of the following is true:*

(1) K has asymptotic integration;
(2) K has a gap;
(3) K is grounded.

Let K be an ungrounded H-asymptotic field with asymptotic couple (Γ, ψ). Then $\Psi \subseteq (\Gamma^{<})'$, so we have a **contraction map** $\chi \colon \Gamma^{<} \to \Gamma^{<}$ where $\chi(\alpha)$ is the unique element in $\Gamma^{<}$ with $\chi(\alpha)' = \alpha^{\dagger}$. We say that $s \in K$ is a **gap creator over** K if vf is a gap in $K(f)$ for some f in an H-asymptotic field extension of K with $f^{\dagger} = s$. In the lemma below, we summarize some facts about gap creators from Section 11.5 in [3].

Lemma 1.2. *Let K be an H-asymptotic field with asymptotic integration and divisible value group. Then K has an immediate H-asymptotic field extension with a gap creator. If $s \in K$ is a gap creator over K, then vf is a gap in $K(f)$ for any nonzero f in an H-field extension of K with $f^{\dagger} = s$.*

Proof. By [3, 11.4.10], K has a spherically complete immediate H-field extension. By [3, 11.5.14], this extension has a gap creator. The second part of the lemma is the remark after [3, 11.5.14]. $\square$

*H***-fields.** Let K be an ordered differential field with constant field C. We say that K is an ***H*-field** if

(H1) $f' > 0$ for all $f \in K$ with $f > C$;
(H2) $\mathcal{O} = C + \scriptstyle{\mathcal{O}}$, where $\mathcal{O}$ is the convex hull of C in K and $\scriptstyle{\mathcal{O}}$ is the unique maximal ideal of $\mathcal{O}$.

Let K be an H-field. For $\phi \in K^{>}$, the compositional conjugate K^{ϕ} of K is also an H-field. Any H-field is a valued field with valuation ring $\mathcal{O}$ as defined in (H2), and we view H-fields as ordered valued differential fields. Any ordered field with trivial derivation is an H-field, and every H-field with nontrivial derivation has a nontrivial valuation ring.

For the remainder of this section, let K be an H-field with valuation ring $\mathcal{O}$, value group Γ, and constant field C. By (H2), the projection map $\mathcal{O} \to \operatorname{res}(K)$ maps C isomorphically onto $\operatorname{res}(K)$. Consequently, an H-field extension L of K is an immediate extension of K if and only if $\Gamma_L = \Gamma$ and $C_L = C$. As a valued differential field, K is H-asymptotic, so K has an asymptotic couple and the trichotomy in Fact 1.1 applies to K. Here is a useful lemma about adjoining integrals to henselian H-fields.

Lemma 1.3. *Let K be henselian and let $s \in K \setminus K'$. Suppose that either*

(1) K has asymptotic integration, or
(2) $vs \in (\Gamma^{>})'$.

Then there is an immediate H-field extension $K(a)$ of K with $a \not\asymp 1$ and $a' = s$. If L is also an H-field extension of K and $b \in L$ satisfies $b \not\asymp 1$ and $b' = s$, then there is a unique H-field embedding $K(a) \to L$ over K that sends a to b.

Proof. By [3, 10.2.5], the set $\{v(s - y') : y \in K\}$ has no largest element. Thus $K(a)$ is an immediate H-field extension of K by either [3, 10.2.4] or [3, 10.2.6]. The universal property of $K(a)$ follows from the universal properties in [3, 10.2.4] and [3, 10.2.6]. $\square$

A major result in [3] is that the theory of H-fields has a model companion, namely the theory T^{nl} of ω-free newtonian Liouville closed H-fields. We say that K is **Liouville closed** if K is real closed and every $a \in K$ can be integrated and exponentially integrated in K. The axioms "ω-free" and "newtonian" are more technical and we will not define these axioms precisely, but we will list some facts about these axioms that will be useful later in this paper.

The axiom of **ω-freeness** is a rather subtle axiom that, among other things, rules out the existence of gap creators. If K is ω-free, then K is ungrounded by definition and K has no gap creator. In particular, K has no gap, so by Fact 1.1 we see that every ω-free H-field has asymptotic integration. The property of ω-freeness is quite robust; it passes to d-algebraic H-field extensions and it is inherited by certain H-subfields:

Fact 1.4 ([3], Section 11.7 and 13.6.1)**.** *If K is ω-free and L is a* d*-algebraic H-field extension of K, then L is ω-free. If E is an ungrounded H-subfield of K and $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$, then E is ω-free.*

In connection with gap creation, let us mention another consequence of ω-freeness:

Lemma 1.5. *Let K be ω-free and let f be an element in an H-field extension of K such that vf is a gap in $K\langle f\rangle$. Then $K\langle f^\dagger\rangle$ is an immediate extension of K.*

Proof. This follows from [3, 11.4.7, 11.5.6, and 11.5.9]. For readers who are familiar with λ-sequences from [3], we provide a proof. If vf is a gap in $K\langle f\rangle$, then $-f^\dagger$ is a pseudolimit of a λ-sequence in K by [3, 11.5.6 and 11.5.9], but since K is ω-free, any λ-sequence in K is divergent and of d-transcendental type over K [3, 13.6.3], so using [3, 11.4.7], we get that $K\langle f^\dagger\rangle$ is an immediate extension of K. $\square$

The axiom of newtonianity is also rather subtle. It is, however, quite a strong axiom, especially when coupled with ω-freeness:

Fact 1.6. *If K is an ω-free newtonian H-field, then K is **asymptotically** d-**algebraically maximal**, that is, K has no proper immediate* d*-algebraic H-field extension.*

Fact 1.6 was shown under the assumption that K also have divisible value group [3, 14.0.2], but this divisibility assumption can be removed; see [27]). By Lemma 1.3 and Fact 1.6, every element in an ω-free newtonian H-field has an integral. Fact 1.6, combined with [3, 2.4.2] and facts from Section 11.4 in [3], gives us the following:

Fact 1.7. *Let K be ω-free and newtonian and let $K\langle y\rangle$ be an immediate H-field extension of K. Let L be also an H-field extension of K and let $z \in L$ realize the same cut as y over K. Then there is an H-field embedding $K\langle y\rangle \to L$ over K that sends y to z.*

If K is ω-free, then a **newtonization of** K is by definition an immediate newtonian H-field extension of K that embeds over K into any ω-free newtonian H-field extension of K.

Fact 1.8. *If K is ω-free, then K has a newtonization K^{nt} which is* d*-algebraic over K. Any two newtonizations of K are isomorphic over K.*

Fact 1.8 was shown under the assumption that K also have divisible value group [3, 14.3.12 and 14.5.4] but again, this divisibility assumption can be removed by [27]. Both ω-freeness and newtonianity are preserved under compositional conjugation by elements of $K^>$.

If K is ω-free, then a **Newton-Liouville closure of** K is by definition a newtonian liouville closed H-field extension of K that embeds over K into any newtonian Liouville closed H-field extension of K.

Fact 1.9 ([3], 14.5.10)**.** *If K is ω-free, then K has a Newton-Liouville closure K^{nl}. Any such K^{nl} is* d*-algebraic over K and the constant field of K_{nl} is a real closure of K.*

We have a sort of converse to Fact 1.9:

Lemma 1.10. *Let K be ω-free and let L is a newtonian Liouville closed H-field extension of K. If L is* d*-algebraic over K and if C_L is a real closure of C, then L is a Newton-Liouville closure of K.*

Proof. Let K^{nl} be a Newton-Liouville closure of K. Then there is an embedding $K^{\mathrm{nl}} \to L$ over K and C_L is contained in the image of this embedding. By [3, 16.2.1], this embedding is surjective. $\square$

Finally, we note that Newton-Liouville closures are unique and minimal.

Fact 1.11 ([3], 16.2.2)**.** *Let K be ω-free. Any two Newton-Liouville closures of K are isomorphic over K. If K^{nl} is a Newton-Liouville closure of K, then the only newtonian Liouville subfield of K^{nl} containing K is K^{nl} itself.*

2. Linear independence of ILD-sequences.

Let K be a differential field with constant field C and let y be an element in a differential field extension of K that is d-transcendental over K.

Definition 2.1. *An **ILD-sequence for** y **over** K is a sequence (y_n) in $K\langle y\rangle$ where $y_0 = y$ and where $y_{n+1}/y_n^\dagger \in K^\times$ for each n.*

In the above definition, ILD stands for "iterated logarithmic derivative." The simplest ILD-sequence is the sequence $\left(y^{\langle n\rangle}\right)$. In this section, we will prove the following proposition.

Proposition 2.2. *If (y_n) is an ILD-sequence for y over K, then $(y_n^\dagger)$ is C-linearly independent over $K\langle y\rangle'$.*

The proof of this proposition is inspired by ideas of Srinivasan [29]. This proposition will come in handy in a couple of places: in Subsections 4.4 and 4.5, we will encounter situations where we need to show that the logarithms of certain ILD-sequences for y over K are algebraically independent over K. Using Ostrowski's Theorem, we can reduce this problem to showing that these logarithms are C-linearly independent over K, and then by taking derivatives, we can solve this problem by applying the above proposition. Towards proving Proposition 2.2, we begin with a weaker version:

Lemma 2.3. $y^\dagger \notin K\langle y\rangle'$.

Proof. Set $L := K\langle y'\rangle$, so $K\langle y\rangle = L(y)$ and $L(y)$ is isomorphic to the field of rational functions $L(Y)$. Suppose towards a contradiction that $y^\dagger \in L(y)'$, so there are coprime polynomials $P, Q \in L[Y]^{\neq}$ with

$$\frac{y'}{y} \;=\; y^\dagger \;=\; \left(\frac{P(y)}{Q(y)}\right)' \;=\; \frac{P(y)'Q(y) - P(y)Q(y)'}{Q(y)^2}.$$

Multiplying by y and $Q(y)^2$ gives

$$Q(y)^2 y' \;=\; y\big(P(y)'Q(y) - P(y)Q(y)'\big)$$

and so y divides $Q(y)$ in the ring $L[y]$. Let $k > 0$ be maximal such that y^k divides $Q(y)$ in $L[y]$. Then y^{2k} divides $Q(y)^2$ and so y^{2k-1} divides $P(y)'Q(y) - P(y)Q(y)'$ in $L[y]$. In particular, y^k divides $P(y)'Q(y) - P(y)Q(y)'$ since $2k - 1 \geqslant k$. Since y^k divides $Q(y)$ but not $P(y)$ we see that y^k must divide $Q(y)'$ in $L[y]$. Take $R \in L[Y]$ with $Q(y) = y^k R(y)$. We have $Q(y)' = ky^{k-1}y'R(y) + y^k R(y)'$ so y must divide $R(y)$ in $L[y]$. Then y^{k+1} divides $Q(y)$ in $L[y]$, contradicting that we chose k to be maximal. ☐

Remark 2.4. *In Lemma 2.3, we can relax the assumption that y is* d*-transcendental over K; in the proof we only use that y is transcendental over $K\langle y'\rangle$. However, in this paper we only apply this lemma to* d*-transcendental elements.*

For the remainder of this section, let (y_n) be an ILD-sequence for y over K and for each n, take $d_n \in K^\times$ with $y_n^\dagger = d_n y_{n+1}$, so $y_n' = d_n y_n y_{n+1}$. For each n we set $K_n := K(y_n, y_{n+1}, \ldots) = K\langle y_n\rangle$. By [3, 4.1.5], each element in $K\langle y\rangle \setminus K$ is d-transcendental over K. In particular, each y_n is d-transcendental over K and $C_{K\langle y\rangle} = K$.

Lemma 2.5. *The sequence (y_n) is algebraically independent over K.*

Proof. An easy induction on n gives that $y^{(n)} \in K[y_0, \ldots, y_n]$ for each n. In particular,

$$K(y, \ldots, y^{(n)}) \;\subseteq\; K(y_0, \ldots, y_n).$$

Thus, $K(y_0, \ldots, y_n)$ has transcendence degree $n+1$ over K for each n, since y is assumed to be d-transcendental over K. This shows that the sequence (y_n) is algebraically independent over K. ☐

It follows from Lemma 2.5 that the ring $K_{n+1}[y_n]$ is isomorphic to the polynomial ring $K_{n+1}[Y]$ for each n.

Lemma 2.6. *Let $n, k \geqslant 0$ and $a_0, \ldots, a_k \in K_{n+1}$. If $a_0 + \cdots + a_k y_n^k \in K_{n+1}[y_n]'$, then $a_i y_n^i \in (K_{n+1} y_n^i)'$ for each $i \in \{0, \ldots, k\}$.*

Proof. Take $m \geqslant k$ and $P = P_0 + \cdots + P_m Y^m \in K_{n+1}[Y]$ with

$$\sum_{i=0}^{k} a_i y_n^i \;=\; P(y_n)' \;=\; \sum_{j=0}^{m} P_j' y_n^j + P_j (y_n^j)'.$$

Since $(y_n^j)' = j d_n y_{n+1} y_n^j$ for each j, we have

$$\sum_{i=0}^{k} a_i y_n^i \;=\; \sum_{j=0}^{m} (P_j' + P_j j d_n y_{n+1}) y_n^j,$$

so $a_i y_n^i = (P_i' + P_i i d_n y_{n+1}) y_n^i = (P_i y_n^i)'$ for each $i \in \{0, \ldots, k\}$. ☐

Lemma 2.7. *For each n we have $K_0' \cap K_{n+1}[y_n] = K_{n+1}[y_n]'$.*

Proof. We need the following two claims:

Claim 1. *For each n we have $K_n' \cap K_{n+1}[y_n] \subseteq K_{n+1}[y_n]'$.*

Proof of Claim 1. Since y_n is transcendental over K_{n+1}, the ring $K_{n+1}[y_n]$ is isomorphic to the polynomial ring $K_{n+1}[Y]$. Let $P, Q \in K_{n+1}[Y]^{\neq}$ be coprime polynomials with Q monic such that

$$\left(\frac{P(y_n)}{Q(y_n)}\right)' \;=\; \frac{P(y_n)'Q(y_n) - P(y_n)Q(y_n)'}{Q(y_n)^2} \;\in\; K_{n+1}[y_n].$$

We will show that $Q = 1$. Note that $Q(y_n)^2$ must divide $P(y_n)'Q(y_n) - P(y_n)Q(y_n)'$ in the ring $K_{n+1}[y_n]$. Since $Q(y_n)$ divides $P(y_n)'Q(y_n)$ in $K_{n+1}[y_n]$ and since no factor of $Q(y_n)$ divides $P(y_n)$, we see that $Q(y_n)$ must divide $Q(y_n)'$ in $K_{n+1}[y_n]$. Let $k = \deg Q$ and take $Q_0, \ldots, Q_{k-1}, \in K_{n+1}$ with $Q(Y) = Y^k + Q_{k-1}Y^{k-1} + \cdots + Q_0$. Then

$$Q(y_n)' \;=\; kd_n y_{n+1} y_n^k + \sum_{i=0}^{k-1} \big(Q_i' + Q_i i d_n y_{n+1}\big) y_n^i,$$

so $kd_n y_{n+1} Q(y_n) = Q(y_n)'$. For $i < k$, we have

$$kd_n y_{n+1} Q_i \;=\; Q_i' + Q_i i d_n y_{n+1}.$$

We claim that $Q_i = 0$ for each $i < k$. Suppose towards a contradiction that there is $i < k$ with $Q_i \neq 0$. Then $Q_i^\dagger = (k-i)d_n y_{n+1} = (k-i)y_n^\dagger$ and so $y_n^{k-i} = cQ_i$ for some $c \in C_{K_n}^\times = C^\times$. Since $cQ_i \in K_{n+1}$ and since y_n is transcendental over K_{n+1} by Lemma 2.5, this is a contradiction. Thus $Q_0, \ldots, Q_{k-1}$ are all 0 and $Q(y_n) = y_n^k$.

We now claim that k must be 0, so $Q = 1$. We have

$$\left(\frac{P(y_n)}{Q(y_n)}\right)' \;=\; \left(\frac{P(y_n)}{y_n^k}\right)' \;=\; \frac{P(y_n)'y_n^k - P(y_n)ky_n^{k-1}y_n'}{y_n^{2k}} \;=\; \frac{P(y_n)' - P(y_n)kd_n y_{n+1}}{y_n^k}.$$

Let $m = \deg P$ and take $P_0, \ldots, P_m \in K_{n+1}$, $P_m \neq 0$ with $P(Y) = P_m Y^m + P_{m-1}Y^{m-1} + \cdots + P_0$. Then

$$P(y_n)' - P(y_n)kd_n y_{n+1} \;=\; \sum_{i=0}^{m} \big(P_i' + P_i i d_n y_{n+1} - P_i k d_n y_{n+1}\big) y_n^i.$$

Suppose towards a contradiction that $k > 0$. Then y_n^k must divide $P(y_n)' - P(y_n)kd_n y_{n+1}$ in $K_{n+1}[y_n]$, so the $i = 0$ term $P_0' - P_0 k d_n y_{n+1}$ in the above sum must be 0. If $P_0 = 0$, then y_n divides $P(y_n)$ in $K_{n+1}[y_n]$, contradicting the assumption that P and Q are coprime. However if $P_0 \neq 0$, then

$$P_0^\dagger \;=\; kd_n y_{n+1} \;=\; ky_n^\dagger,$$

so $y_n^k = cP_0 \in K_{n+1}$ for some $c \in C^\times$ contradicting that y_n is transcendental over K_{n+1}. Thus $k = 0$, so $Q = 1$ and

$$\left(\frac{P(y_n)}{Q(y_n)}\right)' \;=\; P(y_n)' \;\in\; K_{n+1}[y_n]'. \qquad \square$$

Claim 2. *For each n we have $K_0' \cap K_n \subseteq K_n'$.*

Proof of Claim 2. We will show by induction on $m \leqslant n$ that $K_0' \cap K_n \subseteq K_m'$. This is clear for $m = 0$. Suppose this holds for a given $m < n$. Then

$$K_0' \cap K_n \;\subseteq\; K_m' \cap K_n \;\subseteq\; \big(K_m' \cap K_{m+1}[y_m]\big) \cap K_{m+1} \;\subseteq\; K_{m+1}[y_m]' \cap K_{m+1},$$

where the last containment follows from Claim 1. By Lemma 2.6 with $k = 0$ we have

$$K_{m+1}[y_m]' \cap K_{m+1} \;\subseteq\; K_{m+1}'.$$

Therefore, $K_0' \cap K_n \subseteq K_{m+1}'$ as required. $\square$

Now we turn to the statement of the lemma. We have

$$K_0' \cap K_{n+1}[y_n] \;=\; (K_0' \cap K_n) \cap K_{n+1}[y_n] \;\subseteq\; K_n' \cap K_{n+1}[y_n] \;\subseteq\; K_{n+1}[y_n]'$$

by Claims 2 and 1. The other containment $K_{n+1}[y_n]' \subseteq K_0' \cap K_{n+1}[y_n]$ is clear. □

We are now ready to prove the proposition.

Proof of Proposition 2.2. We need to show that the sequence $(y_n^\dagger)$ is C-linearly independent over $K\langle y\rangle' = K_0'$. Suppose this is not the case and take $r > 0$, indices $n_1 < \cdots < n_r$, and nonzero constants $c_1, \ldots, c_r \in C^\times$ such that $c_1 y_{n_1}^\dagger + \cdots + c_r y_{n_r}^\dagger \in K_0'$. Since $c_i y_{n_i}^\dagger = c_i d_{n_i} y_{n_i+1}$ for each $i \in \{1, \ldots, r\}$, we also have

$$c_1 y_{n_1}^\dagger + \cdots + c_r y_{n_r}^\dagger \;=\; c_1 d_{n_1} y_{n_1+1} + \cdots + c_r d_{n_r} y_{n_r+1} \;\in\; K_{n_1+2}[y_{n_1+1}].$$

By Lemma 2.7 with $n_1 + 1$ in place of n, we have

$$c_1 d_{n_1} y_{n_1+1} + \cdots + c_r d_{n_r} y_{n_r+1} \;\in\; K_{n_1+2}[y_{n_1+1}]'.$$

By Lemma 2.6 with $k = 1$, $n = n_1 + 1$, $a_0 = \sum_{i=2}^r c_i d_{n_i} y_{n_i+1}$, and $a_1 = c_1 d_{n_1}$, we have

$$c_1 y_{n_1}^\dagger \;=\; c_1 d_{n_1} y_{n_1+1} \;\in\; (K_{n_1+2} y_{n_1+1})' \;\subseteq\; K_{n_1+1}'.$$

In particular, $y_{n_1}^\dagger \in K_{n_1}' = K\langle y_{n_1}\rangle'$. However, Lemma 2.3 with y_{n_1} in place of y gives us that $y_{n_1}^\dagger \notin K\langle y_{n_1}\rangle'$, a contradiction. □

3. Logarithmic and exponential H-fields

We will be working only with H-fields (possibly with additional structure) for the remainder of this paper, so we make the following convention.

Convention. *For the remainder of this paper, K is an H-field with valuation ring $\mathcal{O}$, maximal ideal $\smallo$, derivation ∂, constant field C, and asymptotic couple (Γ, ψ).*

In this section, we look at H-fields that are equipped with a logarithm. Logarithms on H-fields were previously studied in [2].

Definition 3.1. *A **logarithm on** K is a map $\log\colon K^> \to K$ such that*

- *(L1) $\log$ embeds the multiplicative group $K^>$ into the additive group of K;*
- *(L2) $\log(1 + \smallo) \subseteq \smallo$;*
- *(L3) $(\log f)' = f^\dagger$ for all $f \in K^>$.*

We use exp to denote the inverse of log where it is defined. Let log be a logarithm on K. Then

$$(K^\times)^\dagger \;=\; (K^>)^\dagger \;=\; (\log K^>)' \;\subseteq\; K',$$

so K is ungrounded. Condition (L3) tells us that the trace of the logarithm on $\Gamma^<$ is given by the contraction map: for $a \in K^>$ with $a \succ 1$ we have $v(\log a) = \chi(va)$. By [3, 9.2.18] we have $\alpha < n\chi(\alpha) < 0$ for each $\alpha \in \Gamma^<$ and each $n > 0$. This shows that any logarithm on K is sufficiently "slow-growing":

Lemma 3.2. *If $\log$ is a logarithm on K, then $a \succ (\log a)^n \succ 1$ for each $a \in K^>$ with $a \succ 1$ and each $n > 0$.*

Here is another consequence of (L3).

Lemma 3.3. *If $\log$ is a logarithm on K, then $\log(K^>) \cap C = \log(C^>)$.*

Proof. If $c \in C^>$, then $(\log c)' = c^\dagger = 0$ so $\log(C^>) \subseteq \log(K^>) \cap C$. For the other containment, let $a \in K^>$ and suppose $\log a \in C$. Then $a^\dagger = (\log a)' = 0$ so $a \in C^>$. □

Though we only require that a logarithm on K is a group embedding, any logarithm on K is actually an *ordered* group embedding provided that its restriction to $C^>$ is.

Lemma 3.4. *Let* $\log$ *be a logarithm on* K *and suppose* $\log c > 0$ *for each* $c \in C^{>1}$. *Then* $\log f > 0$ *for each* $f \in K^{>1}$.

Proof. Let $f > 1$. First, consider the case that $f \succ 1$. Then $\log f \succ 1$ and $(\log f)' = f^\dagger > 0$, so $\log f$ is positive by the H-field axiom (H1). Next consider the case that $f \asymp 1$ but $f \not\sim 1$. Then $f = c(1+\varepsilon)$ for some $c \in C^{>1}$ and some $\varepsilon \in \mathcal{O}$ so

$$\log f \ = \ \log c + \log(1+\varepsilon).$$

We have $\log c > 0$ by our assumption on $\log|_{C^>}$ and condition (L2) in Definition 3.1 gives us that $\log(1+\varepsilon) \prec 1$, so $\log f \sim \log c > 0$. Finally, consider the case that $f \sim 1$. Then $f = 1+\varepsilon$ for some $\varepsilon \in \mathcal{O}^>$, so

$$(\log f)' \ = \ f^\dagger \ \sim \ \varepsilon' \ < \ 0.$$

Since $\log f \in \mathcal{O}$ axiom (H1) again gives $\log f > 0$. $\square$

Definition 3.5. *A* ***logarithmic*** H***-field*** *is a henselian* H*-field* K *equipped with a logarithm* $\log$ *on* K *such that the inverse* $\exp$ *of* $\log$ *is defined on all of* C *and such that* C *equipped with* $\exp|_C$ *is a real closed ordered exponential field.*

Let K be a logarithmic H-field with logarithm log. Then $\log|_{C^>}$ is order-preserving by our definition of ordered exponential fields, so log is order-preserving by Lemma 3.4. For all $\phi \in K^>$, the compositional conjugate K^ϕ of K is also a logarithmic H-field with the same logarithm as K.

Definition 3.6. *An* ***exponential*** H***-field*** *is a logarithmic* H*-field* K *with* $\log(K^>) = K$.

Let K be an exponential H-field. Then Γ is divisible since

$$v\exp\Big(\frac{\log a}{n}\Big) \ = \ \frac{va}{n}$$

for each $n > 0$. Then K is real closed, since C is assumed to be real closed and K is assumed to be henselian; see [3, 3.5.19]. If an element $a \in K$ can be integrated, then it can be exponentially integrated since $b' = (\exp b)^\dagger$ for all $b \in K$. To summarize:

Remark 3.7. *If* K *is an exponential* H*-field, then* K *is real closed. If in addition* $K' = K$*, then* K *is Liouville closed. In particular, every* ω*-free newtonian exponential* H*-field is Liouville closed.*

Logarithmic H-field embeddings are defined in the obvious way:

Definition 3.8. *Let* K *and* L *be logarithmic* H*-fields and let* $\imath\colon K \to L$ *be an* H*-field embedding. We say that* $\imath$ ***preserves logarithms*** *if* $\imath(\log f) = \log \imath(f)$ *for all* $f \in K^>$. *A* ***logarithmic*** H***-field embedding*** *is an* H*-field embedding that preserves logarithms. The notions of a logarithmic* H*-field extension and a logarithmic* H*-subfield are defined analogously.*

3.1. **The extension** K^ℓ**.** Recall our convention that K is an H-field. In this subsection, we show that K has an H-field extension K^ℓ that admits a definable logarithm on $1+\mathcal{O}$, and we use this extension to develop some tools that will be used in the next section.

Lemma 3.9. K *has an immediate* H*-field extension* K^ℓ *such that*

(1) K^ℓ *is henselian;*
(2) $(1+\mathcal{O}_{K^\ell})^\dagger \subseteq \mathcal{O}'_{K^\ell}$*;*
(3) K^ℓ *embeds uniquely over* K *into any henselian* H*-field extension* L *of* K *with* $(1+\mathcal{O}_L)^\dagger \subseteq \mathcal{O}'_L$.

Proof. We define an *ℓ-tower* on K to be an increasing chain of henselian H-fields $(K_\mu)_{\mu\leqslant\nu}$ such that

i. $K_0 = K^{\mathrm{h}}$;
ii. $K_\mu = \bigcup_{\lambda<\mu} K_\lambda$ for each limit ordinal μ;
iii. $K_{\mu+1} = K_\mu(y_\mu)^{\mathrm{h}}$ where $y'_\mu = (1+\varepsilon_\mu)^\dagger$ for some $\varepsilon_\mu \in \mathcal{O}_{K_\mu}$ with $(1+\varepsilon_\mu)^\dagger \notin \mathcal{O}'_{K_\mu}$.

We claim that if $(K_\mu)_{\mu\leqslant\nu}$ is an ℓ-tower on K, then K_μ is an immediate extension of K for each $\mu \leqslant \nu$. Of course, K_0 is an immediate extension of K and if K_λ is an immediate extension of K for each λ below some limit $\mu \leqslant \nu$, then K_μ is also an immediate extension of K. Suppose K_μ is an immediate extension of K for some given $\mu < \nu$ and let $\varepsilon_\mu \in \mathcal{O}_{K_\mu}$ be as in the definition of $K_{\mu+1}$. Then $v(1+\varepsilon_\mu)^\dagger = v(\varepsilon_\mu)' \in (\Gamma^>_{K_\mu})'$, so the hypothesis of Lemma 1.3 is satisfied and $K_{\mu+1}$ is an immediate extension of K_μ.

An easy induction on μ shows that each K_μ is also a Liouville extension of K. That is, C_{K_μ} is algebraic over C and each $a \in K_\mu$ is contained in a field extension $K(t_1, \ldots, t_n) \subseteq K_\mu$ where for $i \in \{1, \ldots, n\}$, either

$$t_i \in K(t_1, \ldots, t_{i-1})^{\mathrm{rc}}, \quad \text{or} \quad t_i' \in K(t_1, \ldots, t_{i-1}), \quad \text{or} \quad t_i \neq 0 \text{ and } t_i^\dagger \in K(t_1, \ldots, t_{i-1}).$$

In particular, $|K_\mu| = |K|$ for each μ by [3, 10.6.8], so maximal ℓ-towers exist. Let $(K_\mu)_{\mu \leqslant \nu}$ be a maximal ℓ-tower on K. Then K_ν is a henselian immediate extension of K and $(1 + \mathcal{O}_{K_\nu})^\dagger \subseteq \mathcal{O}'_{K_\nu}$.

Now let L be a henselian H-field extension of K with $(1 + \mathcal{O}_L)^\dagger \subseteq \mathcal{O}'_L$. We claim that there is a unique H-field embedding $\imath_\mu \colon K_\mu \to L$ over K for each $\mu \leqslant \nu$. This holds for K_0 by the universal property of the henselization and if it holds for K_λ for each λ below some limit $\mu \leqslant \nu$, then it holds for K_μ. Suppose this holds for K_μ for some given $\mu < \nu$ and let $\varepsilon_\mu \in \mathcal{O}_{K_\mu}$ be as in the definition of $K_{\mu+1}$. Let $a \in L$ with $a \prec 1$ and $a' = \imath_\mu(1 + \varepsilon_\mu)^\dagger$. The universal property in Lemma 1.3 and universal property of the henselization give a unique H-field embedding $\imath_{\mu+1} \colon K_{\mu+1} \to L$ over K_μ that sends y_μ to a. Since a is the unique integral of $\imath_\mu(1 + \varepsilon_\mu)^\dagger$ in L with nonzero valuation, the uniqueness of $\imath_{\mu+1}$ does not depend on the condition that $\imath_{\mu+1}(y_\mu) = a$. Then K_ν has all of the desired properties and we may take $K^\ell := K_\nu$. □

If $K = K^\ell$, then we define a function $\ln \colon 1 + \mathcal{O} \to \mathcal{O}$ by letting $\ln a$ be the unique element of $\mathcal{O}$ satisfying

$$(\ln a)' \ = \ a^\dagger$$

for each $a \in 1 + \mathcal{O}$. Given $\varepsilon \in \mathcal{O}^{\neq}$, we have $\ln(1 + \varepsilon)' \sim \varepsilon'$, so $\varepsilon > 0$ if and only if $\ln(1 + \varepsilon) > 0$. From this it is straightforward to check that ln is an ordered group embedding. If $K = K^\ell$ and M is an H-subfield of K, then we identify M^ℓ with its unique image in K. The extension K^ℓ is related to logarithms on K in the following way:

Lemma 3.10. *Let K be a henselian H-field. If* log *is a logarithm on K, then $K = K^\ell$ and $\log a = \ln a$ for all $a \in 1 + \mathcal{O}$.*

Proof. If log is a logarithm on K, then $\log(1 + \mathcal{O}) \subseteq \mathcal{O}$ by (L2), so

$$(1 + \mathcal{O})^\dagger \ = \ \log(1 + \mathcal{O})' \ \subseteq \ \mathcal{O}'$$

and $K = K^\ell$. For $a \in 1 + \mathcal{O}$, we have $(\log a)' = a^\dagger = (\ln a)'$, so $\log a - \ln a \in C$. Since $\log a$ and $\ln a$ are both in $\mathcal{O}$, they must be equal. □

In particular, if K is a logarithmic H-field, then $K = K^\ell$, since K is henselian. The map ln gives us a criterion for checking whether a given H-field embedding preserves logarithms.

Lemma 3.11. *Let K and L be logarithmic H-fields and let $\imath \colon K \to L$ be an H-field embedding. Let $f, g \in K^>$ with $f \sim g$. If $\imath(\log g) = \log \imath(g)$, then $\imath(\log f) = \log \imath(f)$.*

Proof. Take $\varepsilon \in \mathcal{O}$ such that $f = g(1 + \varepsilon)$. Then

$$\log f \ = \ \log g + \log(1 + \varepsilon) \ = \ \log g + \ln(1 + \varepsilon)$$

by Lemma 3.10. We have $\imath(\log g) = \log \imath(g)$ by assumption, so it remains to show that $\imath\big(\ln(1+\varepsilon)\big) = \ln \imath(1+\varepsilon)$. To see this, note that $\imath\big(\ln(1 + \varepsilon)\big) \in \mathcal{O}_L$ and that

$$\imath\big(\ln(1 + \varepsilon)\big)' \ = \ \imath\big((1 + \varepsilon)^\dagger\big) \ = \ \imath(1 + \varepsilon)^\dagger.$$

Since $\ln \imath(1 + \varepsilon)$ is the unique element of $\mathcal{O}_L$ with derivative $\imath(1 + \varepsilon)^\dagger$, we have $\imath\big(\ln(1 + \varepsilon)\big) = \ln \imath(1 + \varepsilon)$. □

The extension K^ℓ is useful both for constructing logarithmic H-field extensions and for checking whether H-subfields are actually logarithmic H-subfields. We detail these methods below.

Corollary 3.12. *Let K be a logarithmic H-field and let E be an H-subfield of K with $C_E = C$. Assume that for each $f \in E^>$ there is $g \in E^>$ with $f \asymp g$ and $\log g \in E$. Then E^ℓ is a logarithmic H-subfield of K.*

Proof. Since E^ℓ is an immediate extension of E, the conditions on E also hold for E^ℓ so we assume without loss of generality that $E = E^\ell$. We need to show that $\log(E^>) \subseteq E$. Let $f \in E^>$ and take $g \in E^>$ with $f \asymp g$ and $\log g \in E$. Then $f = cg(1 + \varepsilon)$ for some $c \in C_E^> = C^>$ and some $\varepsilon \in \mathcal{O}_E$ so

$$\log f \ = \ \log c + \log g + \log(1 + \varepsilon) \ = \ \log c + \log g + \ln(1 + \varepsilon).$$

Our assumption gives that $\log c$ and $\log g$ are in E and, since $E = E^\ell$, we have $\ln(1 + \varepsilon) \in E$ as well. □

Lemma 3.13. *Let K be a logarithmic H-field and let M be an H-field extension of K with $C_M = C$. Let $(a_i)_{i\in I}$ be a family of elements in $M^>$ with $a_i \not\asymp 1$ for each i such that*

$$\Gamma_M \;=\; \Gamma \oplus \bigoplus_{i\in I} \mathbb{Z} v a_i$$

and let $(b_i)_{i\in I}$ be a family of elements in M such that $b_i' = a_i^\dagger$ for each $i \in I$. Then there is a unique logarithm $\log$ on M^ℓ extending the logarithm on K such that $\log a_i = b_i$ for each $i \in I$. With this logarithm, M^ℓ is a logarithmic H-field extension of K. If L is also a logarithmic H-field extension of K and $\imath\colon M \to L$ is an H-field embedding over K, then the unique H-field embedding $M^\ell \to L$ extending $\imath$ preserves logarithms if and only if $\imath(b_i) = \log \imath(a_i)$ for each $i \in I$.

Proof. Since M^ℓ is an immediate extension of M, the conditions on M also hold for M^ℓ, so we assume without loss of generality that $M = M^\ell$. Let $f \in M^>$. Our assumption on Γ_M and C_M gives

$$f \;=\; g(1+\varepsilon)\prod_{i\in I} a_i^{k_i}$$

for some $g \in K^>$, some $\varepsilon \in \smallo_M$, and some family $(k_i)_{i\in I}$ of integers where only finitely many k_i are nonzero. We set

$$\log f \;:=\; \log g + \ln(1+\varepsilon) + \sum_{i\in I} k_i b_i.$$

It is routine to show that this does not depend on the choice of g. Note that

$$(\log f)' \;=\; (\log g)' + \ln(1+\varepsilon)' + \sum_{i\in I} k_i b_i' \;=\; g^\dagger + (1+\varepsilon)^\dagger + \sum_{i\in I} k_i a_i^\dagger \;=\; f^\dagger.$$

Using that $\log|_{K^>}$ and $\ln$ are group embeddings, it is straightforward to show that $\log$ is a group embedding. Then $\log$ is indeed a logarithm on M since $\log(1+\smallo) = \ln(1+\smallo) \subseteq \smallo$, and M equipped with $\log$ is a logarithmic H-field since $M = M^\ell$ and $C_M = C$. For uniqueness, let $\log^*$ is an arbitrary logarithm on M. Then $\log^*(1+\varepsilon) = \ln(1+\varepsilon) = \log(1+\varepsilon)$ by Lemma 3.10, so if $\log^*$ extends the logarithm on K and $\log^* a_i = b_i$, then $\log^* f = \log f$.

Now let L be a logarithmic H-field extension of K and let $\imath\colon M \to L$ be an H-field embedding over K. We continue to assume that $M = M^\ell$ and we assume that $\imath(b_i) = \log \imath(a_i)$ for each $i \in I$. We need to show that $\imath(\log f) = \log \imath(f)$ where f is as above. Using the fact that $f \sim g\prod_{i\in I} a_i^{k_i}$ and Lemma 3.11, we may assume that $f = g\prod_{i\in I} a_i^{k_i}$. Since

$$\log\Big(g\prod_{i\in I} a_i^{k_i}\Big) \;=\; \log g + \sum_{i\in I} k_i \log(a_i)$$

and since $g \in K$, this further reduces to showing that $\imath(\log a_i) = \log \imath(a_i)$ for each $i \in I$. This holds by our assumption, since $\log a_i = b_i$ for each i. The other implication, that $\imath(b_i) = \log \imath(a_i)$ if $\imath$ preserves logarithms, is clear. □

The conditions on C_M and Γ_M in the above lemma are always satisfied when M is an immediate H-field extension of K:

Corollary 3.14. *Let K be a logarithmic H-field and let M be an immediate H-field extension of K. Then there is a unique logarithm $\log$ on M^ℓ extending the logarithm on K. With this logarithm, M^ℓ is a logarithmic H-field extension of K and if L is also a logarithmic H-field extension of K and $\imath\colon M \to L$ is an H-field embedding over K, then the unique embedding $M^\ell \to L$ extending $\imath$ preserves logarithms.*

4. Extensions of logarithmic H-fields

In this section, we prove a variety of extension and embedding results about logarithmic H-fields for use in Section 5. For the remainder of this section, K is a logarithmic H-field with logarithm log.

4.1. d**-algebraic extensions.** In this subsection we deal with various d-algebraic logarithmic H-field extensions of K. We will show that any ω-free logarithmic H-field has a minimal ω-free newtonian exponential H-field extension. We begin with the newtonization.

Corollary 4.1. *Let K be ω-free. Then there is a unique logarithm on K^{nt} extending the logarithm on K. If L is an ω-free newtonian logarithmic H-field extension of K, then there is a logarithmic H-field embedding $K^{\mathrm{nt}} \to L$ over K.*

Proof. We have $(K^{\mathrm{nt}})^{\ell} = K^{\mathrm{nt}}$ since K^{nt} is asymptotically d-algebraically maximal by Fact 1.6. Since K^{nt} is an immediate extension of K, the logarithm on K extends uniquely to a logarithm on K^{nt} by Corollary 3.14. For L as in the statement of the Corollary, there is an H-field embedding $K^{\mathrm{nt}} \to L$ over K by Fact 1.8, and this embedding preserves logarithms by Corollary 3.14. □

Next, we deal with constant field extensions.

Lemma 4.2. *Let M be a logarithmic H-field extension of K. Then $K(C_M)^{\ell}$ is a logarithmic H-subfield of M with value group Γ. Let L be also a logarithmic H-field extension of K and let $\imath\colon C_M \to C_L$ be an ordered exponential field embedding over C. Then $\imath$ extends uniquely to a logarithmic H-field embedding $K(C_M)^{\ell} \to L$ over K.*

Proof. The value group of $K(C_M)^{\ell}$ is the same as the value group of $K(C_M)$, which is the same as the value group of K by [3, 10.5.15]. Thus, for each $f \in K(C_M)^{\ell}$ with $f > 0$ there is $g \in K^{>0}$ with $f \asymp g$. Since K is a logarithmic H-field, Corollary 3.12 with M and $K(C_M)$ in place of K and E gives us that $K(C_M)^{\ell}$ is a logarithmic H-subfield of M. Now let L and $\imath$ be as in the statement of the lemma. By [3, 10.5.15 and 10.5.16] there is a unique H-field embedding $K(C_M) \to L$ over K that extends $\imath$. This in turn extends uniquely to an embedding $\jmath\colon K(C_M)^{\ell} \to L$. It remains to show that $\jmath$ preserves logarithms. Let $f \in K(C_M)^{\ell}$ with $f > 0$. Then there is $c \in C_M^{>}$ and $g \in K^{>}$ with $f \sim cg$. By Lemma 3.11 it suffices to show that $\jmath\big(\log(cg)\big) = \log \jmath(cg)$. This follows from our assumption on $\imath$:

$$\jmath\big(\log(cg)\big) \;=\; \jmath(\log c + \log g) \;=\; \imath(\log c) + \log g \;=\; \log \imath(c) + \log g \;=\; \log \jmath(cg). \qquad \square$$

Now we move on to real closures.

Lemma 4.3. *Set $K^{\mathrm{rc},\ell} := (K^{\mathrm{rc}})^{\ell}$. Then $K^{\mathrm{rc},\ell}$ is a real closed H-field with constant field C and there is a unique logarithm on $K^{\mathrm{rc},\ell}$ extending the logarithm on K. If L is also a real closed logarithmic H-field extension of K, then there is a unique logarithmic H-field embedding $\imath\colon K^{\mathrm{rc},\ell} \to L$ over K.*

Proof. Set $M := K^{\mathrm{rc},\ell}$. Then M is an immediate extension of K^{rc}, so $\Gamma_M = \mathbb{Q}\Gamma$ and $C_M = C^{\mathrm{rc}} = C$ since C is assumed to be real closed. Thus, M is real closed since M is henselian; see [3, 3.5.19]. For each $f \in M^{>}$ there is $g \in K^{>}$ and $q \in \mathbb{Q}$ with $f \sim g^q$. Take $\varepsilon \in \mathcal{O}_M$ with $f = g^q(1+\varepsilon)$. We set

$$\log f \;:=\; q \log g + \ln(1+\varepsilon).$$

It is routine to check that log is indeed a logarithm on M extending the logarithm on K. Any logarithm $\log^*$ on M which extends the logarithm on K must satisfy $\log^*(g^q) = q\log g \in K$ and $\log^*(1+\varepsilon) = \ln(1+\varepsilon)$, so this extension is unique. Now let L be a real closed logarithmic H-field extending K. Then by the universal property of the real closure there is a unique H-field embedding $K^{\mathrm{rc}} \to L$ over K and this in turn extends uniquely to an embedding $\imath\colon M \to L$. To see that $\imath$ preserves logarithms, let $g \in K^{>}$ and $q \in \mathbb{Q}$ and note that

$$\imath(\log g^q) \;=\; q \log g \;=\; \log \imath(g^q).$$

We are done in light of Lemma 3.11. □

Finally, we deal with adding exponentials.

Lemma 4.4. *Let $a \in K \setminus \log(K^{>})$ with $a \prec 1$. Then K has an immediate H-field extension $K(f)$ with $f \sim 1$ and $f^{\dagger} = a'$. There is a unique logarithm on $K(f)^{\ell}$ extending the logarithm on K, and this logarithm satisfies $\log f = a$. If L is also a logarithmic H-field extension of K and $a \in \log(L^{>})$, then there is a unique logarithmic H-field embedding $\imath\colon K(f)^{\ell} \to L$ over K.*

Proof. We claim that $a' \notin (K^\times)^\dagger$. Suppose not and let $b \in K^\times$ with $a' = b^\dagger$. Then $b \asymp 1$ since $va' \in (\Gamma^>)'$, so by replacing b with cb for some $c \in C^\times$, we may assume that $b \sim 1$. Lemma 3.10 gives us that $a = \ln b = \log b$, contradicting our assumption that $a \notin \log(K^>)$.

With this claim out of the way, we may apply [3, 10.5.18] with a' in place of s to get an immediate H-field extension $K(f)$ of K where $f \sim 1$ and $f^\dagger = a'$. By Corollary 3.14 there is a unique logarithm log on $K(f)^\ell$. We have $\log f = \ln f = a$, since $a' = f^\dagger$. Now let L be a logarithmic H-field extension of K with $a \in \log(L^>)$. Since $(\exp a)^\dagger = a'$, [3, 10.5.18] gives us a unique H-field embedding $K(f) \to L$ that sends f to $\exp a$. This in turn extends uniquely to an H-field embedding $\imath\colon K(f)^\ell \to L$, and $\imath$ preserves logarithms by Corollary 3.14. Since any *logarithmic* H-field embedding $K(f)^\ell \to L$ must send f to $\exp a$, the map $\imath$ is unique as claimed. □

Lemma 4.5. *Let K be real closed with asymptotic integration. Let $a \in K$ and suppose that $a - \log b \succ 1$ for all $b \in K^>$. Then K has an H-field extension $K(f)$ with constant field C where $f^\dagger = a'$. There is a logarithm on $K(f)^\ell$ that extends the logarithm on K and is uniquely determined by the condition $\log f = a$. If L is also a logarithmic H-field extension of K and $a \in \log(L^>)$, then there is a unique logarithmic H-field embedding $\imath\colon K(f)^\ell \to L$ over K.*

Proof. We may assume that $a < 0$. Then $a' < 0$ as well since $a \succ 1$. We claim that $v(a' - b^\dagger) \in \Psi^\downarrow$ for all $b \in K^\times$. If not, then there is $b \in K^\times$ with either $a' = b^\dagger$ or $v(a' - b^\dagger) \in (\Gamma^>)'$ since K has asymptotic integration. By replacing b with $-b$ if necessary, we may assume that $b > 0$, so either $a' = (\log b)'$ or $v(a' - \log b)' \in (\Gamma^>)'$. In either case we have $a - \log b \preccurlyeq 1$, a contradiction.

With this claim out of the way, we may apply [3, 10.5.20] with a' in place of s to get an H-field $K(f)$ extending K where

$$f^\dagger \ = \ a', \qquad f \ > \ C, \qquad \Gamma_{K(f)} \ = \ \Gamma \oplus \mathbb{Z}vf, \qquad C_{K(f)} \ = \ C.$$

By Lemma 3.13 there is a unique logarithm on $K(f)^\ell$ with $\log f = a$. Now let L be a logarithmic H-field extension of K with $a \in \log(L^>)$. Then $(\exp a)^\dagger = a'$ so by [3, 10.5.20], there is a unique H-field embedding $K(f) \to L$ that sends f to $\exp a$. This in turn extends uniquely to an H-field embedding $\imath\colon K(f)^\ell \to L$. Since $a = \log \imath(f)$, Lemma 3.13 gives that $\imath$ preserves logarithms. Since any logarithmic H-field embedding $K(f)^\ell \to L$ must send f to $\exp a$, the map $\imath$ is unique as claimed. □

We now show that each ω-free logarithmic H-field has a minimal exponential H-field extension.

Corollary 4.6. *If K is ω-free, then K has an ω-free exponential H-field extension K^e with constant field C and with the following property: for any exponential H-field L extending K, there is a unique logarithmic H-field embedding $K^e \to L$ over K. Moreover, K^e is* d*-algebraic over K and the only exponential H-subfield of K^e containing K is K^e itself.*

Proof. We define an *e-tower* on K to be increasing chain of ω-free logarithmic H-fields $(K_\mu)_{\mu \leqslant \nu}$ such that

i. $K_0 = K$;
ii. $K_\mu = \bigcup_{\lambda<\mu} K_\lambda$ for each limit ordinal μ;
iii. If K_μ is not real closed, then $K_{\mu+1} = K_\mu^{\mathrm{rc},\ell}$ equipped with the logarithm in Lemma 4.3;
iv. If K_μ is real closed, then $K_{\mu+1} = K_\mu(f_\mu)^\ell$ where $\log f_\mu \notin K_\mu$ and either $\log f_\mu \prec 1$ (in which case $K_{\mu+1}$ is as in Lemma 4.4) or $\log f_\mu - \log b \succ 1$ for all $b \in K_\mu^>$ (in which case $K_{\mu+1}$ is as in Lemma 4.5).

Since the extensions in Lemmas 4.3, 4.4, and 4.5 are all d-algebraic, they all preserve ω-freeness by Fact 1.4. This ensures that the assumptions in Lemma 4.5 are met whenever this lemma is applied, since ω-free H-fields have asymptotic integration. Induction on μ gives that each K_μ is a Liouville extension of K, so $|K_\mu| = |K|$ and $C_{K_\mu} = C$ for each μ; see the proof of Lemma 3.9 for the definition of a Liouville extension. Thus, maximal e-towers exist, and we let $(K_\mu)_{\mu\leqslant\nu}$ be a maximal e-tower on K.

To see that K_ν is an exponential H-field, let $a \in K_\nu$. If $a - \log b \succ 1$ for all $b \in K_\nu^>$, then $(K_\mu)_{\mu\leqslant\nu}$ can be extended by Lemma 4.5, so we take $b \in K^>$ with $a - \log b \preccurlyeq 1$. If $a - \log b \asymp 1$, then there is $c \in C^>$ with $a - \log b \sim \log c \in C^\times$ since $\log(C^>) = C$. We have

$$a - \log(cb) \ = \ a - \log b - \log c \ \prec \ 1,$$

so we may arrange that $a - \log b \prec 1$ by replacing b with cb. We may take $f \in K_\mu^>$ with $a - \log b = \log f$, since otherwise $(K_\mu)_{\mu\leqslant\nu}$ can be extended using Lemma 4.4. Then $\log(fb) = a$, so $a \in \log(K_\mu^>)$.

Set $K^e := K_\nu$. The proof that K^e has the desired universal property goes the same way as the proof of Lemma 3.9, where we now appeal to the universal properties in Lemmas 4.3, 4.4, and 4.5. Minimality of K^e follows from the universal property: if E is an exponential H-subfield of K^e containing K, then there is a unique logarithmic H-field embedding $K^e \to E \subseteq K^e$ over K. Since the unique embedding $K^e \to K^e$ over K is the identity map, E must be all of K^e. □

Remark 4.7. *The assumptions in Corollary 4.6 can be weakened a bit: we can instead assume that K is λ-free, instead of ω-free. See* [19] *for the definition of λ-freeness and an indication of how the proof of Corollary 4.6 should be changed to prove this stronger result.*

If K is ω-free, then a **Newton-exponential closure of** K is by definition a newtonian exponential H-field extension of K which embeds over K into any other newtonian exponential H-field extension of K. Alternating Corollaries 4.6 and 4.1, we see that Newton-exponential closures exist.

Proposition 4.8. *If K is ω-free, then K has a Newton-exponential closure $K^{\mathrm{nt},e}$ which is* d*-algebraic over K and which has constant field C.*

Since any other Newton-exponential closure of K embeds into $K^{\mathrm{nt},e}$ over K, we see that *any* Newton-exponential closure of K is d-algebraic over K and has constant field C. By Remark 3.7, any Newton-exponential closure is Liouville closed, so by Lemma 1.10, the underlying H-field of any Newton-exponential closure of K is a Newton-Liouville closure of the H-field K. Since any two Newton-Liouville closures of K are isomorphic over K by Fact 1.11, we see that any Newton-Liouville closure L of K admits a logarithm that makes L a Newton-exponential closure of K. Fact 1.11 also gives us uniqueness and minimality of Newton-exponential closures.

Corollary 4.9. *If K is ω-free, then any two Newton-exponential closures of K are isomorphic over K. If $K^{\mathrm{nt},e}$ is a Newton-exponential closure of K, then the only newtonian exponential H-subfield of $K^{\mathrm{nt},e}$ containing K is $K^{\mathrm{nt},e}$ itself.*

Proof. Let $K^{\mathrm{nt},e}$ and L be Newton-exponential closures of K. Then there is a logarithmic H-field embedding $K^{\mathrm{nt},e} \to L$ over K and the image of $K^{\mathrm{nt},e}$ is in particular a newtonian Liouville closed H-subfield of L. Since L is a Newton-Liouville closure of K, the image of $K^{\mathrm{nt},e}$ must equal L by Fact 1.11. Minimality also follows from Fact 1.11: any newtonian exponential H-subfield E of $K^{\mathrm{nt},e}$ is in particular a newtonian Liouville closed H-subfield of $K^{\mathrm{nt},e}$, so E must equal $K^{\mathrm{nt},e}$. □

4.2. **Constructing ω-free logarithmic H-field extensions.** In this subsection we will show that any logarithmic H-field can be extended to an ω-free logarithmic H-field.

Lemma 4.10. *K has a logarithmic H-field extension with a gap and with constant field C.*

Proof. If K has a gap, then we are done, so we may assume that K has asymptotic integration. By Lemma 4.3 there is a unique logarithm on $K^{\mathrm{rc},\ell}$ extending the logarithm on K. If $K^{\mathrm{rc},\ell}$ has a gap, then we are done. If not, then we replace K by $K^{\mathrm{rc},\ell}$ and we assume that K is real closed. Then in particular, Γ is divisible so by Lemma 1.2, K has an immediate H-field extension L with a gap creator s. By Corollary 3.14, the logarithm on K extends uniquely to a logarithm on L^ℓ. Then L^ℓ is real closed and has asymptotic integration since it is an immediate henselian extension of K. Thus we may replace K with L^ℓ and assume that $s \in K$ is a gap creator over K. If $s \notin K'$, then by Lemma 1.3, K has an immediate H-field extension $K(a)$ where $a' = s$. Again by Corollary 3.14, the logarithm on K extends uniquely to a logarithm on $K(a)^\ell$ so we may assume that there is $a \in K$ with $a' = s$. If there is $b \in K^>$ with $a - \log b \preccurlyeq 1$, then $v(s - b^\dagger) \in (\Gamma^>)'$, contradicting [3, 11.5.10]. Thus, $a - \log b \succ 1$ for all $b \in K^>$ and we may apply Lemma 4.5 to get a logarithmic H-field $K(f)^\ell$ extending K with $\log f = a$. Then $f^\dagger = a' = s$ so vf is a gap in $K(f)^\ell$ by Lemma 1.2. □

Lemma 4.11. *Let $s \in K^>$ and suppose that vs is a gap in K. Then K has an ω-free logarithmic H-field extension K_y that contains an element $y \succ 1$ with $y' = s$, has constant field C, and has the following property: If L is also a logarithmic H-field extension of K containing $z \succ 1$ with $z' = s$, then there is a unique logarithmic H-field embedding $K_y \to L$ over K that sends y to z.*

Proof. By [3, 10.5.11], K has an H-field extension $K(y)$ with $y \succ 1$, $y > 0$, and $y' = s$. This extension has the following properties:

$$C_{K(y)} \ = \ C, \qquad \Gamma_{K(y)} \ = \ \Gamma \oplus \mathbb{Z}vy, \qquad \Psi_{K(y)} \ = \ \Psi \cup \big\{\psi(vy)\big\}, \qquad \Psi \ < \ \psi(vy).$$

Set $K_0 := K(y)$. Since K_0 is grounded there is no way to define a logarithm on K_0 (or even on K_0^ℓ). Instead we use [3, 10.5.12] to build an H-field extension $K_0(y_1)$ with $y_1 \succ 1$, $y_1 > 0$, and $y_1' = y^\dagger$. We have

$$C_{K_1} \ = \ C, \qquad \Gamma_{K_1} \ = \ \Gamma_{K_0} \oplus \mathbb{Z}vy_1, \qquad \Psi_{K_1} \ = \ \Psi_{K_0} \cup \big\{\psi(vy_1)\big\}, \qquad \Psi_{K_0} \ < \ \psi(vy_1).$$

Continuing in this manner, we build an H-field $K_y = K(y_0, y_1, \ldots)^\ell$ where $y_0 = y$ and where $y_n \succ 1$, $y_n > 0$, and $y_{n+1}' = y_n^\dagger$ for each n. Then

$$C_{K_y} \ = \ C, \qquad \Gamma_{K_y} \ = \ \Gamma \oplus \bigoplus_n \mathbb{Z}vy_n, \qquad \Psi_{K_y} \ = \ \Psi \cup \big\{\psi(vy_0), \psi(vy_1), \ldots\big\}.$$

K_y is ω-free since it is the increasing union of its grounded subfields $K(y_0, \ldots, y_n)^\ell$; see [3, 11.7.15]. By Lemma 3.13 there is a unique logarithm on K_y with $\log y_n = y_{n+1}$ for each n. Let L and z be as in the statement of the lemma and for each n, let $\log_n z$ denote the nth iterated logarithm of z. Then the universal properties in [3, 10.5.11 and 10.5.12] give a unique H-field embedding $K(y_0, y_1, \ldots) \to L$ over K that sends y_n to the iterated logarithm $\log_n z$ for each n. This extends uniquely to an H-field embedding $\imath \colon K_y \to L$. Since $\log y_n = y_{n+1}$ for each n, it is straightforward using Lemma 3.13 to check that $\imath$ preserves logarithms. Note $\imath$ as a logarithmic H-field embedding is uniquely determined by its restriction to K and by the condition that $\imath(y) = z$. □

We made a choice in Lemma 4.11 to give s an infinite integral. Below, we see that we can choose to instead give s an infinitesimal integral (after replacing s with $-s$ for convenience).

Lemma 4.12. *Let $s \in K^<$ and suppose that vs is a gap in K. Then K has an ω-free logarithmic H-field extension K_y that contains an element $y \prec 1$ with $y' = s$, has constant field C, and has the following property: If L is also a logarithmic H-field extension of K containing $z \prec 1$ with $z' = s$, then there is a unique logarithmic H-field embedding $K_y \to L$ over K that sends y to z.*

Proof. By [3, 10.5.10], K has an H-field extension $K(y)$ with $y \prec 1$, $y > 0$, and $y' = s$. This extension has the following properties:

$$C_{K(y)} \ = \ C, \qquad \Gamma_{K(y)} \ = \ \Gamma \oplus \mathbb{Z}vy, \qquad \Psi_{K(y)} \ = \ \Psi \cup \big\{\psi(vy)\big\}, \qquad \Psi \ < \ \psi(vy).$$

Just as in the proof of Lemma 4.11, we build an H-field $K_y = K(y_0, y_1, \ldots)^\ell$ where $y_n \succ 1$, $y_n > 0$, and $y_{n+1}' = y_n^\dagger$ for each n but where this time, $y_0 = y^{-1}$. Since $\psi(vy^{-1}) = \psi(vy)$, we have

$$C_{K_y} \ = \ C, \qquad \Gamma_{K_y} \ = \ \Gamma \oplus \bigoplus_n \mathbb{Z}vy_n, \qquad \Psi_{K_y} \ = \ \Psi \cup \big\{\psi(vy_0), \psi(vy_1), \ldots\big\}.$$

Again, K_y is ω-free since it is the increasing union of its grounded subfields $K(y_0, \ldots, y_n)^\ell$. The proof that K_y has the desired universal property is the same as the proof of Lemma 4.11 except for the following changes: we use [3, 10.5.10] in place of [3, 10.5.11] and we send each y_n to $\log_n(z^{-1})$ instead of $\log_n z$. □

The following is immediate from Lemmas 4.10 and 4.11.

Corollary 4.13. *K has an ω-free logarithmic H-field extension with constant field C.*

Corollary 4.13 and Proposition 4.8 give us the following:

Corollary 4.14. *K has an ω-free newtonian exponential H-field extension with constant field C.*

4.3. **The cuts $\Psi^\downarrow$ and $\Gamma^<$.** In this subsection, we assume that K is ω-free and that Γ is divisible. We let $z > 0$ be an element in some H-field extension of K with $\Psi^\downarrow < vz < (\Gamma^>)'$. By [3, 13.4.10 and 13.4.12], $K\langle z\rangle$ is an H-field extension of K with

$$C_{K\langle z\rangle} \ = \ C, \qquad \Gamma_{K\langle z\rangle} \ = \ \Gamma \oplus \mathbb{Z}vz, \qquad \Psi_{K\langle z\rangle} \ = \ \Psi.$$

By [3, 9.8.6], vz is a gap in $K\langle z\rangle$ and so $K\langle z^\dagger\rangle$ is an immediate extension of K by Lemma 1.5. Moreover, z is d-transcendental over K by [3, 13.6.1].

Lemma 4.15. *K has a logarithmic H-field extension K_z that contains z and has the following property: if L is also a logarithmic H-field extension of K containing an element $z^* > 0$ with $\Psi^{\downarrow} < vz^* < (\Gamma^{>})'$, then there is a unique logarithmic H-field embedding $K_z \to L$ over K that sends z to z^*.*

Proof. Since z is d-transcendental over K, Lemma 2.3 gives that $z^\dagger \notin K\langle z\rangle'$. In particular, $z^\dagger \notin K\langle z^\dagger\rangle'$. We build K_z in three steps. First, set $M := K\langle z^\dagger\rangle^{\mathrm{h}}$. We claim that $z^\dagger \notin M'$. Suppose otherwise, so $z^\dagger = y'$ for some $y \in M$. Then y is algebraic over $K\langle z^\dagger\rangle$ and so it is in $K\langle z^\dagger\rangle$ by Ostrowski's Theorem, a contradiction. Since M is an immediate extension of K, it has asymptotic integration, so by Lemma 1.3, M has an immediate H-field extension $M(a)$ where $a' = z^\dagger$. By Corollary 3.14, the logarithm on K extends uniquely to a logarithm on $M(a)^\ell$. Set $N := M(a)^\ell$ and note that N, being an immediate extension of K, has asymptotic integration. If there is $b \in N^{>}$ with $a - \log b \preccurlyeq 1$, then $v(s - b^\dagger) \in (\Gamma^{>})'$, contradicting [3, 11.5.10]. Thus, $a - \log b \succ 1$ for all $b \in N^{>}$ and we may apply Lemma 4.5 to uniquely extend the logarithm on N to a logarithm on $N(z)^\ell$ where $\log z = a$. We let K_z be the logarithmic H-field $N(z)^\ell$ with this logarithm. Below, we include a diagram of the H-fields in this proof. All arrows are inclusions and all starred arrows marked are immediate extensions. The two unmarked arrows are not immediate extensions.

$$\begin{tikzcd}
K \arrow[r, "*"] & K\langle z^\dagger\rangle \arrow[r] \arrow[d, "*"] & K\langle z\rangle \arrow[r, "*"] & K_z \\
& M \arrow[r, "*"] & N \arrow[ur] &
\end{tikzcd}$$

Now let L and z^* be as in the statement of the lemma. By [3, 13.4.11] there is a unique H-field embedding $K\langle z\rangle \to L$ over K that sends z to z^*. This restricts to an embedding $K\langle z^\dagger\rangle \to L$ which then extends uniquely to an embedding $M \to L$. By Lemma 1.3 this in turn extends uniquely to an embedding $M(a) \to L$ that sends a to $\log z^*$ and this further extends uniquely to an embedding $N \to L$. This last embedding preserves logarithms by Corollary 3.14 and, using Lemma 4.5, we extend this embedding a unique logarithmic H-field embedding $K_z \to L$ that sends z to z^*. Since any logarithmic H-field embedding that sends z to z^* must send $a = \log z$ to $\log z^*$, we see that this embedding is uniquely determined by the condition that it sends z to z^*. $\square$

Note that K_z above has a gap, namely vz. We now let $y > 0$ be an element in some H-field extension of K with $\Gamma^{<} < vy < 0$.

Corollary 4.16. *K has an ω-free logarithmic H-field extension K_y that contains y and has the following property: if L is also a logarithmic H-field extension of K containing an element $y^* > 0$ with $\Gamma^{<} < vy^* < 0$, then there is a unique logarithmic H-field embedding $K_y \to L$ over K that sends y to y^*.*

Proof. First, note $y' > 0$ and $\Psi^{\downarrow} < vy' < (\Gamma^{>})'$. We let $K_{y'}$ be the logarithmic H-field extension of K containing y' as constructed in Lemma 4.15. We now use Lemma 4.11 with y' in place of s to construct an ω-free logarithmic H-field K_y that extends $K_{y'}$ and contains y.

Let L be also a logarithmic H-field extension of K containing an element $y^* > 0$ with $\Gamma^{<} < vy^* < 0$. Then $\Psi^{\downarrow} < v(y^*)' < (\Gamma^{>})'$ so, by Lemma 4.15, there is a unique logarithmic H-field embedding $K_{y'} \to L$ over K that sends y' to $(y^*)'$. By Lemma 4.11 this extends uniquely to a logarithmic H-field embedding $K_y \to L$ that sends y to y^*. Of course, any H-field embedding $K_y \to L$ over K that sends y to y^* must send y' to $(y^*)'$, so the embedding $K_y \to L$ is uniquely determined by the condition that y be sent to y^*. $\square$

4.4. **Extensions controlled by asymptotic couples.** In this subsection we assume that K is an ω-free newtonian exponential H-field and we let L be a logarithmic H-field extension of K with $C_L = C$. We assume that K is *maximal* in L in the sense that there is no $y \in L \setminus K$ for which $K\langle y\rangle$ is an immediate extension of K. By Section 11.4 in [3], the set

$$\{v(f-a) : a \in K\} \;\subseteq\; \Gamma_L$$

has a largest element for each $f \in L \setminus K$. We define a **best approximation** to $f \in L \setminus K$ to be an element $b \in K$ such that $v(f-b) = \max\{v(f-a) : a \in K\}$. Note that b is a best approximation to f if and only if $v(f-b) \notin \Gamma$ (this uses the fact that $C_L = C$).

Let $f \in L \setminus K$. We set $f_0 := f$, we let b_0 be a best approximation to f_0, and we set $f_1 := (f_0 - b_0)^\dagger$. Then $f_1 \notin K$ since K is Liouville closed and $C_L = C$, so we may repeat this process: let b_1 be a best approximation to f_1 and set $f_2 := (f_1 - b_1)^\dagger$. Continuing in this way we construct a sequence (f_n) of elements in $L \setminus K$ and a sequence (b_n) of elements in K such that b_n is a best approximation of f_n and such that $f_{n+1} = (f_n - b_n)^\dagger$. Now for each $n > 0$, choose an element $a_n \in K^\times$ such that $a_n^\dagger = b_n$ and such that a_n and $f_{n-1} - b_{n-1}$ have the same sign. For each n, we set $\mathfrak{m}_n := a_{n+1}^{-1}(f_n - b_n)$, so $\mathfrak{m}_n > 0$. Then

$$f_{n+1} - b_{n+1} \ = \ \left(\frac{f_n - b_n}{a_{n+1}}\right)^\dagger,$$

so $\mathfrak{m}_n^\dagger = a_{n+2}\mathfrak{m}_{n+1}$ for each n. Note that $K\langle f\rangle = K\langle \mathfrak{m}_0\rangle = K(\mathfrak{m}_0, \mathfrak{m}_1, \ldots)$. We will be using the following facts about $K\langle f\rangle$, established in [3, 16.1.2 and 16.1.3].

Fact 4.17. *$K\langle f\rangle$ is ω-free and*

$$\Gamma_{K\langle f\rangle} \ = \ \Gamma \oplus \bigoplus_n \mathbb{Z}v(\mathfrak{m}_n).$$

By Fact 4.17 and the Zariski-Abhyankar Inequality, we see that $\mathfrak{m}_0$ is d-transcendental over K. Thus, $(\mathfrak{m}_n)$ is an ILD-sequence for $\mathfrak{m}_0$ over K, as defined in Section 2. We have the following consequence of Proposition 2.2.

Corollary 4.18. *The sequence $(\log \mathfrak{m}_n)$ is algebraically independent over $K\langle \mathfrak{m}_0\rangle$.*

Proof. Suppose towards a contradiction that $(\log \mathfrak{m}_n)$ is algebraically dependent over $K\langle \mathfrak{m}_0\rangle$. Since $(\log \mathfrak{m}_n)' = \mathfrak{m}_n^\dagger \in K\langle \mathfrak{m}_0\rangle$ for each n and since L has the same field of constants C as $K\langle \mathfrak{m}_0\rangle$, we may use Ostrowski's Theorem to deduce that the sequence $(\log \mathfrak{m}_n)$ is C-linearly dependent over $K\langle \mathfrak{m}_0\rangle$. Then the sequence $\big((\log \mathfrak{m}_n)'\big) = (\mathfrak{m}_n^\dagger)$ is C-linearly dependent over $K\langle \mathfrak{m}_0\rangle'$, contradicting Proposition 2.2. $\square$

We now build a sequence (K_n) of henselian H-subfields of L as follows:

$$K_0 \ := \ K\langle \mathfrak{m}_0\rangle^{\mathrm{h}} \ = \ K\langle f\rangle^{\mathrm{h}}, \qquad K_{n+1} \ := \ K_n(\log \mathfrak{m}_n)^{\mathrm{h}}.$$

Corollary 4.18 gives that $\log \mathfrak{m}_n \notin K_n$ for each n. We set $K_\infty := \bigcup_n K_n$ and we set $K_f := K_\infty^\ell$. By Lemma 1.3, each K_n is an immediate extension of $K\langle f\rangle$, so K_f is an immediate extension of $K\langle f\rangle$ as well. Corollary 3.12 and Fact 4.17 gives that $K\langle f\rangle$ is a logarithmic H-subfield of L. Since K_f is a d-algebraic extension of $K\langle f\rangle$, it is ω-free by Fact 1.4.

Proposition 4.19. *Let M be a logarithmic H-field extension of K and let $g \in M$ realize the same cut as f over K. Then there is a unique logarithmic H-field embedding $K_f \to M$ over K that sends f to g.*

Proof. By [3, 16.1.5] and the universal property of the henselization, there is a unique H-field embedding $\imath_0 \colon K_0 \to M$ over K that sends f to g. For each n, set $\mathfrak{n}_n := \imath_0(\mathfrak{m}_n)$. Let $n \geqslant 0$ and suppose we have H-field embeddings $\imath_m \colon K_m \to M$ for each $m \leqslant n$ such that $\imath_n|_{K_m} = \imath_m$ and $\imath_n(\log \mathfrak{m}_m) = \log \mathfrak{n}_m$ for each $m < n$. Since $\log \mathfrak{m}_n \notin K_n$, we use Lemma 1.3 and the universal property of the henselization to get a unique H-field embedding $\imath_{n+1} \colon K_{n+1} \to M$ that extends $\imath_n$ and sends $\log \mathfrak{m}_n$ to $\log \mathfrak{n}_n$. The union of these embeddings is an embedding $\imath_\infty \colon K_\infty \to M$ that sends $\log \mathfrak{m}_n$ to $\log \mathfrak{n}_n$ for each n. This extends to an H-field embedding $K_f \to M$ that preserves logarithms in light of Lemma 3.13 and Fact 4.17. Since $\imath_0$ is the unique H-field embedding $K_0 \to M$ which sends f to g and since any logarithmic H-field embedding $K_f \to M$ must send $\log \mathfrak{m}_n$ to $\log \imath_0(\mathfrak{m}_n) = \log \mathfrak{n}_n$ for each n, our embedding is unique as claimed. $\square$

4.5. **Adding elements at infinity.** In this subsection we assume that K is ω-free and that Ψ is downward closed in Γ. This subsection will not be used in the proof of model completeness, but it will be used in the proof of local o-minimality. Let L be a logarithmic H-field extension of K with $C_L = C$ and let $a \in L$ with $a > K$.

Lemma 4.20. *$K\langle a\rangle$ is ω-free and*

$$\Gamma_{K\langle a\rangle} \ = \ \Gamma \oplus \bigoplus_n \mathbb{Z}v\big(a^{\langle n\rangle}\big).$$

Proof. If we assume that K is Liouville closed, then this is just [3, 16.6.9 and 16.6.10]. However, the only consequence of being Liouville closed that is used in the proof of [3, 16.6.9 and 16.6.10] is that Ψ is downward closed. $\square$

It follows from the above lemma and the Zariski-Abhyankar Inequality that a is d-transcendental over K. Thus, $\big(a^{\langle n\rangle}\big)$ is an ILD-sequence for a over K and we have the following:

Corollary 4.21. *The sequence $\big(\log a^{\langle n\rangle}\big)$ is algebraically independent over $K\langle a\rangle$.*

Proof. Suppose not. Since $\big(\log a^{\langle n\rangle}\big)' = a^{\langle n+1\rangle} \in K\langle a\rangle$ for each n and since L has the same field of constants C as $K\langle a\rangle$, we may use Ostrowski's Theorem to deduce that the sequence $\big(\log a^{\langle n\rangle}\big)$ is C-linearly dependent over $K\langle a\rangle$. Then the sequence $\big((\log a^{\langle n\rangle})'\big) = \big((a^{\langle n\rangle})^\dagger\big)$ must be C-linearly dependent over $K\langle a\rangle'$, contradicting Proposition 2.2. □

As in the previous subsection, we build a sequence (K_n) of henselian H-subfields of L as follows:

$$K_0 \ := \ K\langle a\rangle^{\mathrm{h}}, \qquad K_{n+1} \ := \ K_n\big(\log a^{\langle n\rangle}\big)^{\mathrm{h}}.$$

Corollary 4.21 gives that $\log a^{\langle n\rangle} \notin K_n$ for each n. We set $K_\infty := \bigcup_n K_n$ and we set $K_a := K_\infty^\ell$. Then K_a is an immediate d-algebraic extension of $K\langle a\rangle$, so K_a is a logarithmic H-subfield of L by Corollary 3.12 and Lemma 4.20 and K_a is ω-free by Fact 1.4.

Proposition 4.22. *Let M be a logarithmic H-field extension of K and let $b \in M$ with $b > K$. Then there is a unique logarithmic H-field embedding $K_a \to M$ over K that sends a to b.*

Proof. This is proven in much the same way as Proposition 4.19. By the proof of [3, 16.6.10] and the universal property of the henselization, there is a unique H-field embedding $\imath\colon K_0 \to M$ over K that sends a to b. Repeated applications of Lemma 1.3 and the universal property of the henselization gives us a unique H-field embedding $K_\infty \to M$ that extends $\imath$ and sends $\log a^{\langle n\rangle}$ to $\log b^{\langle n\rangle}$ for each n. This extends uniquely to an H-field embedding $\jmath\colon K_a \to M$ which preserves logarithms in light of Lemmas 3.13 and 4.20. As with Proposition 4.19, $\jmath$ is unique as a logarithmic H-field embedding since $\imath$ is unique as an H-field embedding. □

5. Model completeness for $\mathbb{T}_{\exp}$ and applications

In this section, we axiomatize a model complete theory of logarithmic H-fields. We show that $\mathbb{T}_{\exp}$ is a model of this theory and we list some other models of this theory. Finally, we examine some consequences of our model completeness result. The main ingredient is the following proposition.

Proposition 5.1. *Let K and L be ω-free newtonian exponential H-fields and assume the underlying ordered set of L is $|K|^+$-saturated and the cofinality of $\Gamma_L^<$ is greater than $|\Gamma|$. Let E be an ω-free logarithmic H-subfield of K with $C_E = C$ and let $\imath\colon E \to L$ be a logarithmic H-field embedding. Then $\imath$ extends to a logarithmic H-field embedding $K \to L$.*

Proof. We can assume $E \neq K$, in which case it suffices to show that $\imath$ can be extended to an embedding of an ω-free logarithmic H-subfield of K that properly contains E. If E is not also an ω-free newtonian exponential H-field, then we use Proposition 4.8 to extend $\imath$ to a logarithmic H-field embedding $E^{\mathrm{nt},\mathrm{e}} \to L$. We assume for the remainder of the proof that E is an ω-free newtonian exponential H-subfield of K. In particular, Γ_E is assumed to be divisible.

Suppose $\Gamma_E^<$ is not cofinal in $\Gamma^<$, so there is $y \in K^>$ with $\Gamma_E^< < vy < 0$. By our cofinality assumption on $\Gamma_L^<$ there is $y^* \in L^>$ with $\Gamma_{\imath(E)}^< < vy^* < 0$. Using Corollary 4.16, we extend $\imath$ to a logarithmic H-field embedding of some ω-free logarithmic H-subfield $E_y \subseteq K$ containing y. We assume for the remainder of the proof that $\Gamma_E^<$ is cofinal $\Gamma^<$. Thus every differential subfield of K containing E is an ω-free H-subfield of K by Fact 1.4.

Next, suppose $E\langle y\rangle$ is an immediate extension of E for some $y \in K \setminus E$. Then $E\langle y\rangle^\ell$ is a logarithmic H-subfield of K by Corollary 3.12. The saturation assumption on L gives $z \in L$ which realizes the $\imath$-image of the cut over $\imath(E)$ that y realizes over E. By Fact 1.7 we may extend $\imath$ to an H-field embedding of $E\langle y\rangle$ into L that sends y to z. This in turn extends to an H-field embedding $E\langle y\rangle^\ell \to L$. By Corollary 3.14 this H-field embedding preserves logarithms.

Finally assume that there is no $y \in K \setminus E$ such that $E\langle y\rangle$ is an immediate extension of E. Take $f \in K \setminus E$ and, using the saturation assumption on L, take $g \in L$ which realizes the $\imath$-image of the cut over $\imath(E)$ that

f realizes over E. By Proposition 4.19 we may extend $\imath$ to a logarithmic H-field embedding of an ω-free logarithmic H-subfield of K containing f. □

Here is a useful consequence of Proposition 5.1.

Corollary 5.2. *Let K and L be as in the statement of Proposition 5.1. If both K and L have small derivation, then any ordered exponential field embedding $\imath\colon C \to C_L$ extends to a logarithmic H-field embedding $K \to L$. The same is true if both K and L have large derivation.*

Proof. We first consider the case that K and L have small derivation. Since K is ω-free and newtonian, there is $x \in K$ with $x' = 1$. Additionally, $x \succ 1$ since K has small derivation. We view C as a logarithmic H-subfield of K with trivial derivation and gap 0 and we let C_x be the ω-free logarithmic H-field extension of C containing x constructed in Lemma 4.11. We may identify C_x with a logarithmic H-subfield of K. Since L is also ω-free and newtonian with small derivation, there is $f \in L$ with $f \succ 1$ and $f' = 1$, so $\imath$ extends to a logarithmic H-field embedding $C_x \to L$ which sends x to f by Lemma 4.11. This further extends to a logarithmic H-field embedding $K \to L$ by Proposition 5.1.

Now suppose K and L have large derivation. Again, since K is ω-free and newtonian, there is $y \in K$ with $y' = -1$. Then $y \preccurlyeq 1$ since K has large derivation, so by subtracting a constant from y, we may assume that $y \prec 1$. Let C_y be the ω-free logarithmic H-field extension of C containing y constructed in Lemma 4.12, and identify C_y with a logarithmic H-subfield of K. Take $g \in L$ with $g \prec 1$ and $g' = -1$ and extend $\imath$ to a logarithmic H-field embedding $C_y \to L$ which sends y to g. This further extends to a logarithmic H-field embedding $K \to L$, again by Proposition 5.1. □

5.1. **Model completeness and completeness.** To get a model completeness result, we need to remove the assumption that $C_E = C$ in Proposition 5.1. In order to do this, we impose some additional requirements on the constant fields of K, L, and E. A logarithmic H-field K is said to have **real exponential constant field** if its constant field C equipped with $\exp|_C$ models $\mathrm{Th}(\mathbb{R}_{\exp})$.

Corollary 5.3. *Let E, K, L and $\imath$ be as in the statement of Proposition 5.1, except we drop the assumption that $C_E = C$. Assume in addition that the underlying ordered set of C_L is $|C|^+$-saturated and that E, K, and L all have real exponential constant fields. Then $\imath$ extends to a logarithmic H-field embedding $K \to L$.*

Proof. The theory of $\mathbb{R}_{\exp}$ is model complete and o-minimal by Wilkie's theorem [31]. The saturation assumption on the underlying ordered set of C_L gives us that C_L is saturated as an ordered exponential field by o-minimality. By model completeness, the ordered exponential field embedding $\imath|_{C_E}\colon C_E \to C_L$ extends to an ordered exponential field embedding $\jmath\colon C \to C_L$. By Lemma 4.2 there is a unique logarithmic H-field embedding $E(C)^\ell \to L$ that extends both $\imath$ and $\jmath$. Since $E(C)^\ell$ is d-algebraic over E, it is ω-free by Fact 1.4. Now apply Proposition 5.1 with $E(C)^\ell$ in place of E. □

Let $\mathcal{L}_{\log} := \{+, \times, 0, 1, \leqslant, \preccurlyeq, \partial, \log\}$, where $\leqslant$ and $\preccurlyeq$ are binary relation symbols and where ∂ and log are unary function symbols. We view each logarithmic H-field K as an $\mathcal{L}_{\log}$-structure in the obvious way, where log is defined to be identically zero on $K^{\leqslant}$. Let $T_{\log}$ be the $\mathcal{L}_{\log}$-theory of logarithmic H-fields with real exponential constant field and let $T^{\mathrm{nt}}_{\exp}$ be the $\mathcal{L}_{\log}$-theory of ω-free newtonian exponential H-fields with real exponential constant field. The theory $T^{\mathrm{nt}}_{\exp}$ is consistent since it has a model; see Corollary 5.6.

Theorem 5.4. *The $\mathcal{L}_{\log}$-theory $T^{\mathrm{nt}}_{\exp}$ is model complete and it is the model companion of $T_{\log}$.*

Proof. The saturation assumptions for L and C_L and the cofinality assumption for Γ_L in Corollary 5.3 all hold when L is $|K|^+$-saturated as an $\mathcal{L}_{\log}$-structure. Model completeness for $T^{\mathrm{nt}}_{\exp}$ follows from Corollary 5.3 and a standard model completeness test; see [3, B.10.4]. By Corollary 4.14, every logarithmic H-field with real exponential constant field can be extended to an ω-free newtonian exponential H-field with the same real exponential constant field. Thus, $T^{\mathrm{nt}}_{\exp}$ is the model companion of $T_{\log}$. □

We can use Theorem 5.4 and Corollary 5.2 to characterize the completions of $T^{\mathrm{nt}}_{\exp}$. Let $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ be the $\mathcal{L}_{\log}$-theory extending $T^{\mathrm{nt}}_{\exp}$ whose models have small derivation and let $T^{\mathrm{nt}}_{\exp,\mathrm{lg}}$ be the $\mathcal{L}_{\log}$-theory extending $T^{\mathrm{nt}}_{\exp}$ whose models have large derivation. Let $K \models T^{\mathrm{nt}}_{\exp}$. Then $K^\phi \models T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ for any $\phi \in K^>$ with $v\phi \in (\Gamma^<)'$ and $K^\psi \models T^{\mathrm{nt}}_{\exp,\mathrm{lg}}$ for any $\psi \in K^>$ with $v\psi \in (\Gamma^>)'$. Thus, $T^{\mathrm{nt}}_{\exp,\mathrm{sm}}$ and $T^{\mathrm{nt}}_{\exp,\mathrm{lg}}$ are both consistent since $T^{\mathrm{nt}}_{\exp}$ is consistent.

Theorem 5.5. *$T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ and $T^{\mathrm{nt}}_{\mathrm{exp,lg}}$ are both the two completions of $T^{\mathrm{nt}}_{\mathrm{exp}}$.*

Proof. Let $K, L \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ and assume L is $|K|^+$-saturated. Then C_L is $|C|^+$-saturated, so there is an ordered exponential field embedding $\imath\colon C \to C_L$ since C and C_L are elementarily equivalent; see [3, B.9.5]. This in turn extends to an embedding $\jmath\colon K \to L$ by Corollary 5.2. Then $\jmath$ is elementary since $T^{\mathrm{nt}}_{\mathrm{exp}}$ is model complete, so K and L are elementarily equivalent. This shows that $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ is complete, and the same proof shows that $T^{\mathrm{nt}}_{\mathrm{exp,lg}}$ is complete. $\square$

5.2. **Transseries.** Let $\mathbb{T}_{\mathrm{exp}}$ be the natural expansion of the field of logarithmic exponential transseries to an $\mathcal{L}_{\log}$-structure. In the introduction, $\mathbb{T}_{\mathrm{exp}}$ was the expansion of $\mathbb{T}$ by the exponential function and here, it is the expansion of $\mathbb{T}$ by the logarithm function, but there is really no difference in terms of model completeness or definability.

Corollary 5.6. *The $\mathcal{L}_{\log}$-theory of $\mathbb{T}_{\mathrm{exp}}$ is model complete and completely axiomatized by $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$.*

Proof. The logarithm log on $\mathbb{T}_{\mathrm{exp}}$ extends the usual logarithm on its constant field $\mathbb{R}$ and satisfies (L1)–(L3) of Definition 3.1; see [14]. Moreover, log is surjective and by [3, 15.0.2], $\mathbb{T}$ is an ω-free newtonian H-field with small derivation, so $\mathbb{T}_{\mathrm{exp}} \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$. $\square$

The only part of this axiomatization of $\mathbb{T}_{\mathrm{exp}}$ that is not known to be effective is the condition that models of $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ have real exponential constant field. Of course, any effective axiomatization of $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$ can be used to give an effective axiomatization of $\mathbb{T}_{\mathrm{exp}}$, but for the time being, such an axiomatization seems to be out of reach; see [25]. Many important H-subfields of $\mathbb{T}$ are logarithmic H-subfields of $\mathbb{T}_{\mathrm{exp}}$. To help us find examples, we use the following criterion:

Lemma 5.7. *Let $\log$ denote the logarithm on $\mathbb{T}_{\mathrm{exp}}$ and let $K \supseteq \mathbb{R}$ be an H-subfield of $\mathbb{T}_{\mathrm{exp}}$. If K is henselian and $\log(K^>) \subseteq K$, then K expanded by $\log|_{K^>}$ is a model of $T_{\log}$. If K is ω-free, newtonian, and Liouville closed, then $\log(K^>) \subseteq K$ and K expanded by $\log|_{K^>}$ is a model of $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$.*

Proof. Since the axioms in Definition 3.1 are universal, if K is henselian and closed under log, then $K \models T_{\log}$. Now assume that K is ω-free, newtonian, and Liouville closed. We need to show that K is closed under exp and log. To see that K is closed under log, let $a \in K^>$. Since $(\log a)' = a^\dagger$ and since T^{nl} is model complete, there is $b \in K$ with $b' = a^\dagger$. Then $\log a = b + r$ for some $r \in \mathbb{R}$, so $\log a \in K$. Showing that K is closed under exp is similar. $\square$

By Lemma 5.7, the subfield $\mathbb{T}^{\mathrm{da}}$ of transseries that are d-algebraic over $\mathbb{Q}$ is an ω-free exponential H-subfield of $\mathbb{T}_{\mathrm{exp}}$, as is the subfield $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries. Thus, $\mathbb{T}^{\mathrm{da}}$ and $\mathbb{T}_{\mathrm{g}}$ are both elementary $\mathcal{L}_{\log}$-substructures of $\mathbb{T}_{\mathrm{exp}}$. The field $\mathbb{T}_{\log}$ of logarithmic transseries is an ω-free newtonian logarithmic H-subfield of $\mathbb{T}_{\mathrm{exp}}$, but $\mathbb{T}_{\log}$ is not an exponential H-subfield of $\mathbb{T}_{\mathrm{exp}}$. See [23] for more information about $\mathbb{T}_{\mathrm{g}}$ and see [20] for more information about $\mathbb{T}_{\log}$.

Using Theorem 5.5 and Corollary 5.6, we can transfer some known facts about $\mathbb{T}_{\mathrm{exp}}$ into formal consequences of $T^{\mathrm{nt}}_{\mathrm{exp}}$.

Corollary 5.8. *Let $K \models T^{\mathrm{nt}}_{\mathrm{exp}}$. Then the underlying ordered exponential field of K models $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$.*

Proof. If this corollary holds for K^ϕ where $\phi \in K^>$, then it holds for K, so by compositionally conjugating by a suitable element of $K^>$, we may assume that $K \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$. Then K is elementarily equivalent to $\mathbb{T}_{\mathrm{exp}}$, and the underlying ordered exponential field of $\mathbb{T}_{\mathrm{exp}}$ is a model of $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$ by [14]. $\square$

Let T be an o-minimal theory extending the theory of real closed ordered fields in a suitable language $\mathcal{L}$. Let $M \models T$ and let δ be a derivation on M. Following the terminology of [18], we say that δ is a **T-derivation on** M if for any open $U \subseteq M^n$, any continuously differentiable function $f\colon U \to M$ that is $\mathcal{L}$-definable without parameters, and any $u = (u_1, \ldots, u_n) \in U$, we have

$$\delta f(u) \;=\; \nabla f(u) \cdot (\delta u_1, \ldots, \delta u_n)$$

where ∇f is the gradient of f. Note that the statement "δ is a T-derivation" can be expressed by sentences in the language $\mathcal{L} \cup \{\delta\}$.

Corollary 5.9. *Let $K \models T^{\mathrm{nt}}_{\mathrm{exp}}$. Then ∂ is a $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$-derivation on K.*

Proof. If ∂ is a $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$-derivation on K, then so is $\phi^{-1}\partial$ for any $\phi \in K^{>}$. Thus, by compositionally conjugating by a suitable element of $K^{>}$, we may assume that $K \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$. Then K is elementarily equivalent to $\mathbb{T}_{\mathrm{exp}}$ and the derivation on $\mathbb{T}_{\mathrm{exp}}$ is a $\mathrm{Th}(\mathbb{R}_{\mathrm{exp}})$-derivation; see Example 2.15 in [18]. $\square$

5.3. **Surreal numbers.** The field $\mathbf{No}$ of surreal numbers is a real closed field extension of $\mathbb{R}$ introduced by Conway [9]. The surreals may be defined in a number of equivalent ways, but for our purposes, we define a surreal number to be a map $a\colon \gamma \to \{-,+\}$, where γ is an ordinal. For such a, the ordinal γ is called the **length of** a (sometimes called the *tree-rank* or *birthday* of a, depending on which definition of the surreals is being used). The collection of all surreal numbers is a proper class, and each ordinal γ is identified with the surreal number of length γ which takes constant value $+$. For each γ, we let $\mathbf{No}(\gamma)$ be the set of surreal numbers of length $< \gamma$.

The surreals admit a (surjective) logarithm, defined by Kruskal and Gonshor [21] and with this logarithm, $\mathbf{No}$ is an elementary extension of $\mathbb{R}_{\mathrm{exp}}$ [12]. More recently, Berarducci and Mantova equipped the surreals with a derivation that makes $\mathbf{No}$ an exponential H-field with small derivation and real exponential constant field $\mathbb{R}$ [7]. With this derivation, the H-field $\mathbf{No}$ is newtonian and ω-free [5], so $\mathbf{No}$ is a model of $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$.

Let κ be a regular uncountable cardinal. Then the set $\mathbf{No}(\kappa)$ is an ω-free newtonian Liouville closed H-subfield of $\mathbf{No}$ containing $\mathbb{R}$ [5, Corollary 4.6]. By adapting the proof of Lemma 5.7, we see that $\mathbf{No}(\kappa)$ is closed under exp and log, so it is an elementary $\mathcal{L}_{\log}$-substructure of $\mathbf{No}$. The next proposition, an analogue of [5, Theorem 3], shows that the surreal numbers are *universal* among models of $T_{\log}$ with small derivation and real exponential constant field $\mathbb{R}$.

Proposition 5.10. *Every set-sized logarithmic H-field with small derivation and real exponential constant field $\mathbb{R}$ admits a logarithmic H-field embedding into $\mathbf{No}$ over $\mathbb{R}$.*

Proof. Let K be a logarithmic H-field with real exponential constant field $\mathbb{R}$ and small derivation. It suffices to show that some logarithmic H-field extension of K with the same constant field $\mathbb{R}$ admits a logarithmic H-field embedding into $\mathbf{No}$ over $\mathbb{R}$. Since K has small derivation, either $0 \in (\Gamma^{<})'$ or K has gap 0. In the second case, we can use Lemma 4.11 to extend K to an ω-free logarithmic H-field with $0 \in (\Gamma^{<})'$, so by passing to an extension, we may assume that $0 \in (\Gamma^{<})'$. Then any logarithmic H-field extension of K has small derivation. By Corollary 4.14, K has an ω-free exponential H-field extension with the same constant field $\mathbb{R}$, so we may assume that $K \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$. Let $\kappa := |K|^{+}$. Then $\mathbf{No}(\kappa) \models T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ and by [5, Lemma 5.3], the underlying ordered sets of $\mathbf{No}(\kappa)$ and $\Gamma_{\mathbf{No}(\kappa)}$ are κ-saturated. By Corollary 5.2 with $\mathbf{No}(\kappa)$ in place of L, the identity map on $\mathbb{R}$ extends to a logarithmic H-field embedding $K \to \mathbf{No}(\kappa)$. $\square$

There is a natural field embedding $\imath\colon \mathbb{T} \to \mathbf{No}$ which was shown to be an elementary H-field embedding in [5]. Since $\imath$ preserves logarithms, it is even an elementary logarithmic H-field embedding. In [17], $\imath$ was shown to also be *initial*: if a is in $\imath(\mathbb{T})$, then so is $a|_{\lambda}$ for all λ less than the length of a. The following question is similar to a question asked in [5].

Question 5.11. *Does every model of $T^{\mathrm{nt}}_{\mathrm{exp,sm}}$ with real exponential constant field $\mathbb{R}$ admit an initial logarithmic H-field embedding into $\mathbf{No}$ over $\mathbb{R}$?*

5.4. **Hardy fields.** Recall that a Hardy field is a set of germs of real-valued functions at $+\infty$ that is closed under differentiation and that forms a field under addition and multiplication. Every Hardy field containing $\mathbb{R}$ (that is, containing the germs of all constant functions) is an H-field with constant field $\mathbb{R}$. Given a Hardy field $\mathcal{H}$ and a germ $f^* \in \mathcal{H}^{>}$, we define the germ $\log f^*$ as follows: take any representative function $f(x)$ for f^* that is strictly positive on an interval $(a,+\infty)$. Then $\log f(x)$ is also defined on the interval $(a,+\infty)$, and we let $\log f^*$ be the germ of the function $\log f(x)$. Note that $\log f^*$ is not necessarily in $\mathcal{H}$.

Lemma 5.12. *Let $\mathcal{H}$ be a henselian Hardy field containing $\mathbb{R}$. If $\log f^* \in \mathcal{H}$ for each $f^* \in \mathcal{H}^{>}$, then $\mathcal{H}$ equipped with the logarithm $f^* \mapsto \log f^*$ is a model of $T_{\log}$.*

Proof. The logarithm in the statement of the lemma is a logarithm on $\mathcal{H}$ as defined in Definition 3.1: properties (L1) and (L2) are basic properties of the real logarithm function and property (L3) is just the chain rule. The restriction of this logarithm to $\mathbb{R}$ is just the normal logarithm on $\mathbb{R}$, so $\mathcal{H}$ has real exponential constant field $\mathbb{R}$. Since $\mathcal{H}$ is assumed to be henselian, we are done. $\square$

Every Hardy field has small derivation so by Proposition 5.10, every Hardy field satisfying the conditions in Lemma 5.12 admits a logarithmic H-field embedding into the surreal numbers over $\mathbb{R}$.

Recently, Aschenbrenner, van den Dries and van der Hoeven finished the proof of their conjecture in [4] that all maximal Hardy fields are ω-free and newtonian. They are currently preparing a paper on this result. By [8], any Hardy field has a Hardy field extension that contains $\mathbb{R}$ and is closed under log and exp. Thus, any maximal Hardy field is a model of $T^{\mathrm{nl}}_{\exp,\mathrm{sm}}$. In particular, all maximal Hardy fields are elementarily equivalent as differential exponential fields.

5.5. **Local o-minimality.** Let $K \models T^{\mathrm{nt}}_{\exp}$. Our model completeness result can be used to show that K is *locally o-minimal* in the sense of [30].

Theorem 5.13. *For each $y \in K$ and each $X \subseteq K$ that is $\mathcal{L}_{\log}$-definable with parameters from K, there is an interval I around y such that $X \cap I$ is a finite union of points and intervals.*

Proof. We begin with the following claim.

Claim. *Let M be an elementary extension of K and let $a \in M$ with $a > K$. Let N be an $|M|^+$-saturated elementary extension of K and let $b \in N$ with $b > K$. Then there is a logarithmic H-field embedding $\imath \colon M \to N$ over K that sends a to b.*

Proof of Claim. As in Corollary 5.3, we extend the identity map $K \to N$ to a logarithmic H-field embedding $K(C_M)^\ell \to N$. Set $L := K(C_M)^\ell$. Then L is ω-free by Fact 1.4. Our assumption on K gives $K = (K^\times)^\dagger$, so Ψ is downward closed in Γ. Then Ψ_L is downward closed in Γ_L as well, since $\Gamma_L = \Gamma$. The fact that $\Gamma_L = \Gamma$ also gives that a is greater than L and that b is greater than the image of L in N, so the assumptions in Subsection 4.5 are met with L and M in place of K and L. Let L_a be the ω-free logarithmic H-field extension of L containing a constructed in that subsection. By Proposition 4.22 with L and N in place of K and M, there is a logarithmic H-field embedding $L_a \to N$ that sends a to b. Using the saturation assumption on N and Proposition 5.1, we extend this to a logarithmic H-field embedding $M \to N$. $\square$

This claim and the fact that $T^{\mathrm{nt}}_{\exp}$ is model complete gives that K is *o-minimal at infinity*, that is, for each definable $X \subseteq K$ there is $f \in K$ with $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$. Deducing this from the claim is a standard model theoretic argument: suppose towards a contradiction that both X and $K \setminus X$ contain arbitrarily large elements of K. Then we can arrange that $a \in X^M$ and $b \in N \setminus X^N$, where a, b, M, and N are as in the claim and where X^M and X^N are the natural extensions of X to M and N. Since $T^{\mathrm{nt}}_{\exp}$ is model complete, the map $\imath \colon M \to N$ constructed in the claim is elementary, so

$$b \;=\; \imath(a) \;\in\; \imath(X^M) \;\subseteq\; X^N,$$

a contradiction. The statement of the theorem follows from o-minimality at infinity by taking fractional linear transformations. $\square$

6. Restricted trigonometric functions

In this section, we examine H-fields with restricted trigonometric functions and we prove that $\mathbb{T}_{\mathrm{rt},\exp}$ is model complete.

Definition 6.1. ***Restricted trigonometric functions on*** *K are functions $\sin\colon [-1,1]_K \to K$ and $\cos\colon [-1,1]_K \to K$ such that*

- *(RT1) $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos(a+b) = \cos a \cos b - \sin a \sin b$ for all $a, b \in [-1,1]_K$ with $a + b \in [-1,1]_K$;*
- *(RT2) $(\sin a)' = (\cos a)a'$ and $(\cos a)' = -(\sin a)a'$ for all $a \in [-1,1]_K$;*
- *(RT3) $\sin(\smallo) \subseteq \smallo$ and $\cos(\smallo) \subseteq 1 + \smallo$.*

Let sin and cos be restricted trigonometric functions on K. Then sin and cos remain restricted trigonometric functions in any compositional conjugate of K. For each $c \in [-1,1]_C$ we have $(\sin c)' = (\cos c)c' = 0$ so $\sin c \in C$. Likewise, $\cos c \in C$ for all $c \in [-1,1]_C$. With (RT3), this gives us $\sin(0) = 0$ and $\cos(0) = 1$. The next lemma shows that the restrictions of sin and cos to $\smallo$ are definable in the underlying H-field of K.

Lemma 6.2. *Let* sin *and* cos *be restricted trigonometric functions on K, let $a \in \mathcal{O}^{\neq}$, and let A be the homogeneous linear differential polynomial*

$$A(Y) \;=\; -(a')^2 Y + (a')^\dagger Y' - Y''.$$

Then $\sin a$ *is the unique zero of* A *in* $a(1+\mathcal{O})$ *and* $\cos a$ *is the unique zero of* A *in* $1 + a\mathcal{O}$.

Proof. We will first show that $\sin a$ and $\cos a$ are both zeros of A. We have

$$\begin{aligned} A(\sin a) \;&=\; -(a')^2(\sin a) + (a')^\dagger(\sin a)' - (\sin a)'' \;=\; -(a')^2(\sin a) + (a')^\dagger(\cos a)a' - \big((\cos a)a'\big)' \\ &=\; -(a')^2(\sin a) + (a')^\dagger(\cos a)a' + (\sin a)(a')^2 - (\cos a)a'' \;=\; 0. \end{aligned}$$

Likewise, $A(\cos a) = 0$. Now we will show that $\sin a \in a(1+\mathcal{O})$ and that $\cos a \in 1 + a\mathcal{O}$. Since $\cos a \sim 1$, we have

$$(\sin a)' \;=\; (\cos a)a' \;\sim\; a'.$$

Then $\sin a \sim a$ since K is asymptotic and $a, \sin a \prec 1$ [3, 9.1.4]. Since $\sin a \sim a \prec 1$, we have

$$(\cos a - 1)' \;=\; -(\sin a)a' \;\prec\; a'.$$

Again, this gives $\cos a - 1 \prec a$ since K is asymptotic.

It remains to show uniqueness. Since A is an order two homogeneous linear differential polynomial, the set of zeros of A in K is a C-linear subspace of K of dimension at most 2; see [3, 4.1.14]. Moreover, $\sin a$ and $\cos a$ are C-linearly independent since $c \sin a \in \mathcal{O}$ for all $c \in C$, so the set $\{\sin a, \cos a\}$ forms a basis for this subspace. Let

$$b \;=\; c_1 \sin a + c_2 \cos a$$

be an arbitrary zero of A in K, where $c_1, c_2 \in C$. If $b \sim a$, then c_2 must be 0, since otherwise $b \sim c_2 \asymp 1$. This gives $b = c_1 \sin a \sim c_1 a$, so c_1 must be 1 and $b = \sin a$. If $b - 1 \prec a$, then c_2 must be 1, since otherwise $b - 1 \sim c_2 - 1 \asymp 1$. This gives

$$b - 1 \;=\; c_1 \sin a + \cos a - 1 \;\in\; c_1 \sin a + a\mathcal{O}$$

and so c_1 must be 0 and $b = \cos a$, since otherwise $b \sim c_1 \sin a \asymp a$. □

Let $\mathcal{L}_{\mathrm{rt,log}} := \mathcal{L}_{\log} \cup \{\sin, \cos\}$, where sin and cos are unary function symbols. Let

$$\hat{\mathcal{L}} \;:=\; \mathcal{L}_{\mathrm{rt,log}} \setminus \{\preccurlyeq, \partial\} \;=\; \{+, \times, 0, 1, \leqslant, \log, \sin, \cos\}.$$

Let $\mathbb{R}_{\mathrm{rt,exp}}$ be the natural expansion of $\mathbb{R}_{\exp}$ by restricted trigonometric functions and let $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$ be the $\hat{\mathcal{L}}$-theory of $\mathbb{R}_{\mathrm{rt,exp}}$. Then $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$ is model complete and o-minimal by van den Dries and Miller [15]. Let $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ be the $\mathcal{L}_{\mathrm{rt,log}}$-theory asserting that for $K \models T^{\mathrm{nt}}_{\mathrm{rt,exp}}$:

(1) K is an ω-free newtonian exponential H-field;
(2) sin and cos are identically zero off of $[-1,1]_K$ and the restrictions of sin and cos to $[-1,1]_K$ are restricted trigonometric functions on K;
(3) The expansion of C by $\exp|_C$, $\sin|_{[-1,1]_C}$ and $\cos|_{[-1,1]_C}$ models $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$.

We will see that $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ is has a model in Corollary 6.6. If $K \models T^{\mathrm{nt}}_{\mathrm{rt,exp}}$, then we view C as an $\hat{\mathcal{L}}$-structure in the natural way. Lemma 6.2 gives us a criterion for when a logarithmic H-field embedding is actually an $\mathcal{L}_{\mathrm{rt,log}}$-embedding.

Corollary 6.3. *Let $K, L \models T^{\mathrm{nt}}_{\mathrm{rt,exp}}$, let $\imath\colon K \to L$ be a logarithmic H-field embedding, and suppose $\imath|_C\colon C \to C_L$ is an $\hat{\mathcal{L}}$-embedding. Then $\imath$ is an $\mathcal{L}_{\mathrm{rt,log}}$-embedding.*

Proof. We need to show that $\imath(\sin f) = \sin \imath(f)$ and $\imath(\cos f) = \cos \imath(f)$ for all $f \in [-1,1]_K$. This holds if $f \in C$, so let $f \in [-1,1]_K$ and suppose $f \notin C$. Then there is a unique $c \in [-1,1]_C$ and a unique $a \in \mathcal{O}^{\neq}$ with $f = c + a$. We have

$$\imath(\sin f) \;=\; \imath(\sin c \cos a + \cos c \sin a) \;=\; \imath(\sin c)\imath(\cos a) + \imath(\cos c)\imath(\sin a)$$

by (RT1). Likewise, $\imath(\cos f) = \imath(\cos c)\imath(\cos a) - \imath(\sin c)\imath(\sin a)$. By our assumption on $\imath$, it is enough to show that $\imath(\sin a) = \sin \imath(a)$ and that $\imath(\cos a) = \cos \imath(a)$. Let A be the homogeneous linear differential polynomial over K from Lemma 6.2 and let $\imath A$ be the image of A under $\imath$, that is,

$$\imath A(Y) \;=\; -\imath(a')^2 Y + \imath(a')^\dagger Y' - Y''.$$

By Lemma 6.2 we know that $\sin a$ is a zero of A and that $\sin a \sim a$, so $\imath(\sin a)$ is a zero of $\imath A$ and $\imath(\sin a) \sim \imath(a)$. Then $\imath(\sin a) = \sin \imath(a)$, since $\sin \imath(a)$ is the unique zero of $\imath A$ in $\imath(a)(1+\mathcal{O}_L)$. Likewise, $\imath(\cos a) = \cos \imath(a)$. $\square$

Theorem 6.4. *$T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ is model complete.*

Proof. Let K, L, and E be models of $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ where E is an $\mathcal{L}_{\mathrm{rt,log}}$-substructure of K and where L is $|K|^+$-saturated as an $\mathcal{L}_{\mathrm{rt,log}}$-structure. Let $\imath\colon E \to L$ be an $\mathcal{L}_{\mathrm{rt,log}}$-embedding. To show that $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ is model complete, it is enough to show that $\imath$ extends to an $\mathcal{L}_{\mathrm{rt,log}}$-embedding $K \to L$; see [3, B.10.4]. We may view $\imath|_{C_E}\colon C_E \to C_L$ as an $\hat{\mathcal{L}}$-embedding and, using that $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$ is model-complete, we may extend $\imath|_{C_E}$ to an $\hat{\mathcal{L}}$-embedding $\jmath\colon C \to C_L$. By Lemma 4.2 there is a unique logarithmic H-field embedding $E(C)^{\ell} \to L$ that extends both $\imath$ and $\jmath$. Since $E(C)^{\ell}$ is d-algebraic over E, it is $\boldsymbol{\omega}$-free by Fact 1.4, so by Proposition 5.1 with $E(C)^{\ell}$ in place of E, we have a logarithmic H-field embedding $K \to L$ which extends both $\imath$ and $\jmath$. This is even an $\mathcal{L}_{\mathrm{rt,log}}$-embedding by Corollary 6.3. $\square$

Again, we can characterize the completions of $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$.

Theorem 6.5. *$T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ has two completions: $T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$, whose models are the models of $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ with small derivation, and $T^{\mathrm{nt}}_{\mathrm{rt,exp,lg}}$, whose models have large derivation.*

Proof. Consistency of $T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$ and $T^{\mathrm{nt}}_{\mathrm{rt,exp,lg}}$ follows from consistency of $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ and the remarks before Theorem 5.5. For completeness, let $K, L \models T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$ and assume L is $|K|^+$-saturated. Then C and C_L are elementarily equivalent as $\hat{\mathcal{L}}$-structures and C_L is $|C|^+$-saturated, so there is an $\hat{\mathcal{L}}$-embedding $\imath\colon C \to C_L$. In particular, $\imath$ is a logarithmic H-field embedding, so it extends to a logarithmic H-field embedding $\jmath\colon K \to L$ by Corollary 5.2. It follows from Corollary 6.3 and Theorem 6.4 that $\jmath$ is an elementary $\mathcal{L}_{\mathrm{rt,log}}$-embedding, so K and L are elementarily equivalent. This shows that $T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$ is complete, and the same proof shows that $T^{\mathrm{nt}}_{\mathrm{rt,exp,lg}}$ is complete. $\square$

Let $\mathbb{T}_{\mathrm{rt,exp}}$ be the natural expansion of $\mathbb{T}_{\mathrm{exp}}$ to an $\mathcal{L}_{\mathrm{rt,log}}$-structure; see [14] for details. One can easily check that the restricted trigonometric functions in this expansion satisfy (RT1)–(RT3), so we have the following:

Corollary 6.6. *The $\mathcal{L}_{\mathrm{rt,log}}$-theory of $\mathbb{T}_{\mathrm{rt,exp}}$ model complete and completely axiomatized by $T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$.*

Again, $T^{\mathrm{nt}}_{\mathrm{rt,exp,sm}}$ is effective relative to $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$. We can use Theorem 6.5 and Corollary 6.6 to deduce the following analogues of Corollaries 5.8 and 5.9:

Corollary 6.7. *If $K \models T^{\mathrm{nt}}_{\mathrm{rt,exp}}$, then the $\hat{\mathcal{L}}$-reduct of K models $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$ and ∂ is a $\mathrm{Th}(\mathbb{R}_{\mathrm{rt,exp}})$-derivation on K.*

Proof. Both of these properties are invariant under compositional composition and hold for $\mathbb{T}_{\mathrm{rt,exp}}$; see [14] and [18]. $\square$

We can also amend our proof of Theorem 5.13 to show that any model of $T^{\mathrm{nt}}_{\mathrm{rt,exp}}$ is locally o-minimal.

Corollary 6.8. *Let $K \models T^{\mathrm{nt}}_{\mathrm{rt,exp}}$. For each $y \in K$ and each $X \subseteq K$ that is $\mathcal{L}_{\mathrm{rt,log}}$-definable with parameters from K, there is an interval I around y such that $X \cap I$ is a finite union of points and intervals.*

Proof. Let M be an elementary extension of K and let $a \in M$ with $a > K$. Let N be an $|M|^+$-saturated elementary extension of K and let $b \in N$ with $b > K$. As in the proof of Theorem 5.13, it suffices to show that there is an $\mathcal{L}_{\mathrm{rt,log}}$-embedding $\imath\colon M \to N$ over K that sends a to b. Following the proof of Theorem 6.4, we extend the identity map $K \to N$ to an $\mathcal{L}_{\mathrm{rt,log}}$-embedding $\imath\colon K(C_M)^{\ell} \to N$. By Theorem 5.13, $\imath$ extends to a logarithmic H-field embedding $\jmath\colon M \to N$ that sends a to b. Then $\jmath$ is even an $\mathcal{L}_{\mathrm{rt,log}}$-embedding by Corollary 6.3. $\square$

7. Final Remarks and Future Directions

An alternative axiomatization. Let K be an H-field. We say that K has **the Intermediate Value Property** (IVP) if for all $r \geqslant 0$, all $P \in K[Y_0, \ldots, Y_r]$ and all $f < g \in K$ with

$$P(f, f', \ldots, f^{(r)}) \ < \ 0 \ < \ P(g, g', \ldots, g^{(r)}),$$

there is $y \in K$ with $f < y < g$ and $P(y, y', \ldots, y^{(r)}) = 0$. In [4], it was announced that the theory of $\boldsymbol{\omega}$-free newtonian Liouville closed H-fields has an alternative axiomatization:

Fact 7.1. *A Liouville closed H-field is ω-free and newtonian if and only if it has IVP.*

This alternative axiomatization relies heavily on the fact that the field $\mathbb{T}_{\mathrm{g}}$ of grid-based transseries has IVP [22]. Now let K be an exponential H-field. If K is ω-free and newtonian, then K is Liouville closed by Remark 3.7, so K has IVP by Fact 7.1. Now assume K has nontrivial derivation and IVP. Let $a \in K$ and take $f \in K^{>}$ with $f \succ 1$ and $f' \succ a$ (finding such an f uses that K have nontrivial derivation). Then $f' - a \sim f' > 0$ and $(-f)' - a < 0$, so by IVP, there is $y \in K$ with $|y| < f$ and $y' = a$. This shows that $K' = K$, so K is Liouville closed by Remark 3.7 and K is ω-free and newtonian by Fact 7.1. We summarize below:

Remark 7.2. *An exponential H-field with nontrivial derivation is ω-free and newtonian if and only if it has IVP. In particular, the models of $T^{\mathrm{nt}}_{\exp}$ are exactly the exponential H-fields with nontrivial derivation, real exponential constant field, and IVP.*

One can make an analogous definition for *differential exponential polynomials*: say that an exponential H-field K has exp-IVP if for all $r \geqslant 0$, all $P \in K[Y_0, \ldots, Y_{2r}]$ and all $f < g \in K$ with

$$P\big(f, \ldots, f^{(r)}, \exp f, \ldots, \exp f^{(r)}\big) \ < \ 0 \ < \ P\big(g, \ldots, g^{(r)}, \exp g, \ldots, \exp g^{(r)}\big),$$

there is $y \in K$ with $f < y < g$ and $P\big(y, \ldots, y^{(r)}, \exp y, \ldots, \exp y^{(r)}\big) = 0$.

Question 7.3. *Does every model of $T^{\mathrm{nt}}_{\exp}$ have* exp-*IVP?*

Quantifier elimination. It seems reasonable to believe that the theory $T^{\mathrm{nt}}_{\exp}$ is combinatorially tame and that the definable sets in any model of this theory are geometrically tame. For instance, $T^{\mathrm{nt}}_{\exp}$ is likely NIP or even distal and the constant field of any model of $T^{\mathrm{nt}}_{\exp}$ is probably stably embedded as a model of $\mathrm{Th}(\mathbb{R}_{\exp})$. The appropriate analogues of all of these properties hold for T^{nl}, but in order to prove that $T^{\mathrm{nt}}_{\exp}$ enjoys these properties, it is invaluable to have a quantifier elimination result at hand. As in the case of T^{nl}, quantifier elimination will almost surely have to involve some additions to our language. Let $K \models T^{\mathrm{nt}}_{\exp}$ and let $\mathcal{L}^{\Lambda,\Omega,\mathrm{df}}_{\log}$ be the extension of $\mathcal{L}_{\log}$ by the following symbols:

(1) A unary predicate Λ, to be interpreted in K as the set $\{-y^{\dagger\dagger} : y \in K,\ y \succ 1\}$;
(2) A unary predicate Ω, to be interpreted in K as the set
$$\{a \in K : 4y'' + ay = 0 \text{ for some } y \in K^{\times}\};$$
(3) A function symbol $\widetilde{f}$ for each function $f\colon \mathbb{R}^n \to \mathbb{R}$ that is definable without parameters in $\mathbb{R}_{\exp}$. Since $K \models \mathrm{Th}(\mathbb{R}_{\exp})$ by Corollary 5.8, each of these function symbols has a natural interpretation as a function $K^n \to K$.

Question 7.4. *Does $T^{\mathrm{nt}}_{\exp}$ eliminate quantifiers in the language $\mathcal{L}^{\Lambda,\Omega,\mathrm{df}}_{\log}$?*

The additions of Ω and Λ are necessary to prove quantifier elimination for the theory T^{nl}, so they will almost certainly be required to prove quantifier elimination for $T^{\mathrm{nt}}_{\exp}$. It is not currently known whether $\mathrm{Th}(\mathbb{R}_{\exp})$ admits quantifier elimination in a "nice" language, but it has been known for some time that $\mathrm{Th}(\mathbb{R}_{\exp})$ does *not* eliminate quantifiers in the natural language of ordered exponential fields [11]. Since $\mathrm{Th}(\mathbb{R}_{\exp})$ is o-minimal, this theory has definable Skolem functions, so we do know that it admits quantifier elimination after adding a function symbol for each function that is definable without parameters. A similar approach should work with $T^{\mathrm{nt}}_{\mathrm{rt},\exp}$. We note that it may be possible to get quantifier elimination by only interpreting the function symbols in (3) as functions from $C^n \to C$ (and as identically zero off of C^n).

Model completeness for $\mathbb{T}_{\mathrm{an},\exp}$. We do not expect that our proof that $\mathbb{T}_{\mathrm{rt},\exp}$ is model complete can be generalized to show that $\mathbb{T}_{\mathrm{an},\exp}$—the expansion of $\mathbb{T}_{\exp}$ by *all* restricted analytic functions—is model complete. Indeed, our model completeness result for $\mathbb{T}_{\mathrm{rt},\exp}$ relies heavily on the fact that the restrictions of sin and cos to $\mathcal{O}$ are definable in the underlying H-field of any model of $T^{\mathrm{nt}}_{\mathrm{rt},\exp}$. Our proof that $\mathbb{T}_{\exp}$ is model complete suggests a model completeness result for $\mathbb{T}_{\mathrm{an}}$ can be "upgraded" to a model completeness result for $\mathbb{T}_{\mathrm{an},\exp}$ by "adding" a logarithm function to each step. The proof that $\mathbb{R}_{\mathrm{an},\exp}$ eliminates quantifiers in [13] further substantiates the philosophy that one should start with restricted analytic functions and add the logarithms later. First steps towards a model completeness result for $\mathbb{T}_{\mathrm{an}}$ are considered under the umbrella of *H_T-fields* in [24], but a full proof of model completeness for $\mathbb{T}_{\mathrm{an}}$ will likely take quite a bit of work.

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