HILBERT POLYNOMIALS FOR FINITARY MATROIDS

ANTONGIULIO FORNASIERO AND ELLIOT KAPLAN

Abstract. We consider a tuple $\Phi = (\phi_1, \ldots, \phi_m)$ of commuting maps on a finitary matroid X. We show that if Φ satisfies certain conditions, then for any finite set $A \subseteq X$, the rank of $\{\phi_1^{r_1} \cdots \phi_m^{r_m}(a) : a \in A \text{ and } r_1 + \cdots + r_m = t\}$ is eventually a polynomial in t (we also give a multivariate version of the polynomial). This allows us easily recover Khovanskii's theorem on the growth of sumsets, the existence of the classical Hilbert polynomial, and the existence of the Kolchin polynomial. We also prove some new Kolchin polynomial results for differential exponential fields and derivations on o-minimal fields.

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Introduction

Let $(X, \mathrm{c}\ell)$ be a finitary matroid, let $m \in \mathbb{N}^{>0}$, and let $\Phi \coloneqq (\phi_1, \ldots, \phi_m)$ be a finite tuple of commuting maps $X \to X$. The tuple Φ is said to be a **triangular system** if

$$a \in \mathrm{c}\ell(B) \implies \phi_j a \in \mathrm{c}\ell\big(\{\phi_i b : b \in B \text{ and } i \leq j\}\big)$$

for all $j \in \{1, \ldots, m\}$, all $a \in X$, and all $B \subseteq X$. For $\bar{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, we let $|\bar{r}| = r_1 + \cdots + r_m$, and we let $\phi^{\bar{r}} : X \to X$ be the composite map $\phi^{\bar{r}} = \phi_1^{r_1} \cdots \phi_m^{r_m}$. For $t \in \mathbb{N}$ and $A \subseteq X$, set

$$\Phi^{(t)}(A) \coloneqq \big\{\phi^{\bar{r}}(a) : a \in A \text{ and } |\bar{r}| = t\big\}.$$

In this paper, we prove the following:

Theorem. *Suppose that Φ is a triangular system, let $A \subseteq X$ be finite, and let $B \subseteq X$ be arbitrary. Then there is a polynomial $P \in \mathbb{Q}[Y]$ of degree at most $m - 1$ such that*

$$\mathrm{rk}\big(\Phi^{(t)}(A)\big|\Phi^{(t)}(B)\big) = P(t)$$

for all sufficiently large $t \in \mathbb{N}$, where rk *is the rank function associated to $(X, \mathrm{c}\ell)$.*

Using this theorem in various settings, we can easily deduce classical results such as Khovanskii's theorem on the growth of sumsets $A + tB$ in commutative semigroups, the existence of the Hilbert polynomial for finitely generated graded $K[\bar{x}]$-modules, and the existence of the Kolchin polynomial for partial differential fields $(K, \delta_1, \ldots, \delta_m)$.

The above theorem is a special case of a more general multivariate version. Fix

$$0 = m_1 < m_2 < \cdots < m_k < m_{k+1} = m,$$

2020 *Mathematics Subject Classification.* Primary 05B35. Secondary 03C64, 05E40, 12H05, 12H10, 13D40.

and set $d_i := m_{i+1} - m_i$ for $i \in \{1, \ldots, k\}$. For each i, we set $\Phi_i := (\phi_{m_i+1}, \phi_{m_i+2}, \ldots, \phi_{m_i+d_i})$, and we call the tuple $(\Phi_1, \ldots, \Phi_k)$ a **partition of** Φ. For a tuple $\bar{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, we set

$$\|\bar{r}\| \ := \ (r_{m_1+1} + \cdots + r_{m_1+d_1}, \ldots, r_{m_k+1} + \cdots + r_{m_k+d_k}) \ \in \ \mathbb{N}^k,$$

and for $\bar{s} \in \mathbb{N}^k$ and $A \subseteq X$, we set $\Phi^{(\bar{s})}(A) := \big\{\phi^{\bar{r}}(a) : a \in A \text{ and } \|\bar{r}\| = \bar{s}\big\}$.

Theorem A. *Suppose that each Φ_i is a triangular system, let $A \subseteq X$ be finite, and let $B \subseteq X$ be arbitrary. Then there is a polynomial $P \in \mathbb{Q}[Y_1, \ldots, Y_k]$ of degree at most $d_i - 1$ in each variable Y_i such that*

$$\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big) \ = \ P(\bar{s})$$

for $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ with $\min\{s_1, \ldots, s_k\}$ sufficiently large.

We call the polynomial P in Theorem A the **Hilbert polynomial of A over B with respect to the partition** $(\Phi_1, \ldots, \Phi_k)$, and we denote it by $P^{\Phi}_{A|B}$. The existence of the Hilbert polynomial can be used to recover Nathanson's generalization of Khovanskii's sumset theorem [17] and Levin's multivariate generalizations of the Kolchin polynomial [14]. We also give some novel generalizations of the Kolchin polynomial to difference-differential exponential fields and to derivations on o-minimal fields. In the proof of Theorem A, we use Proposition 1.4, which describes the generating function associated to a decreasing function, and might be of independent interest.

In addition to the Hilbert polynomial, we define a closure operator $\mathrm{c}\ell^{\Phi}$, called the **Φ-closure on X with respect to the partition** $(\Phi_1, \ldots, \Phi_k)$, as follows:

$$a \in \mathrm{c}\ell^{\Phi}(B) \ :\Longleftrightarrow \ \mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\big) < \big|\{\bar{r} \in \mathbb{N}^m : \|\bar{r}\| = \bar{s}\}\big| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

Our second theorem relates this closure operator to the Hilbert polynomial.

Theorem B. *Suppose that each Φ_i is a triangular system. Then $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid and for any $A, B \subseteq X$ with A finite, we have*

$$P^{\Phi}_{A|B}(Y_1, \ldots, Y_k) \ = \ \frac{\mathrm{rk}^{\Phi}(A|B)}{(d_1 - 1)! \cdots (d_k - 1)!} Y_1^{d_1 - 1} \cdots Y_k^{d_k - 1} + \textit{lower degree terms},$$

where rk^{Φ} is the rank function associated to $(X, \mathrm{c}\ell^{\Phi})$.

After some preliminaries in Section 1, we prove Theorems A and B in Section 2. In Section 3, we collect some consequences of Theorem A in the case that the maps ϕ_i are all endomorphisms. In Section 4, we turn to the case that the maps ϕ_i are *quasi-endomorphisms*. Theorem A doesn't apply directly in this case (see Remark 4.2), but we establish an appropriate analog in Theorem C below. Some applications of Theorem C for difference-differential fields (as well as difference-differential exponential fields and differential o-minimal fields) are considered in Section 5.

Acknowledgements. The first author was partially supported by GNSAGA–INdAM. The second author was located at the Fields Institute for Research in Mathematical Sciences and was supported by the National Science Foundation under Award No. 2103240. The authors would like to thank Piotr Kowalski and George Shakan for helpful conversations.

1. Preliminaries

1.1. **Notation and conventions.** Throughout, $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$. Let $d \in \mathbb{N}^{>0}$, and let $\bar{r} = (r_1, \ldots, r_d)$ and $\bar{s} = (s_1, \ldots, s_d)$ range over $\mathbb{N}^d$. We write $\min(\bar{r})$ to mean $\min\{r_1, \ldots, r_d\}$. Let $\preccurlyeq$ denote the partial order on $\mathbb{N}^d$ given by

$$\bar{r} \preccurlyeq \bar{s} \ :\Longleftrightarrow \ r_i \leq s_i \text{ for each } i = 1, \ldots, d,$$

and let $<_{\mathrm{lex}}$ denote the lexicographic order on $\mathbb{N}^d$ with emphasis on the last coordinate, so

$$\bar{r} <_{\mathrm{lex}} \bar{s} \ :\Longleftrightarrow \ \text{there is } i \in \{1, \ldots, d\} \text{ such that } r_i < s_i \text{ and } r_j = s_j \text{ for } i < j \leq d.$$

Let $\bar{0}_d$ be the tuple $(0, \ldots, 0)$ consisting of d zeros, so $\bar{0}_d$ is both the minimal element of $\mathbb{N}^d$ with respect to both of the orders $\preccurlyeq$ and $\leq_{\mathrm{lex}}$. For $i \in \{1, \ldots, d\}$, let $\hat{e}_{i,d} := (0, \ldots, 1, \ldots, 0)$ be the tuple which consists of a 1 in the ith spot and zeros everywhere else.

1.2. **Finitary matroids and triangular systems.** Recall that a **finitary matroid** consists of a set X, together with a map $\mathrm{c}\ell\colon \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies the following conditions:

(1) Monotonicity: if $A \subseteq B \subseteq X$, then $A \subseteq \mathrm{c}\ell(A) \subseteq \mathrm{c}\ell(B)$;
(2) Idempotence: $\mathrm{c}\ell\big(\mathrm{c}\ell(A)\big) = \mathrm{c}\ell(A)$ for $A \subseteq X$;
(3) Finite character: if $A \subseteq X$ and $a \in \mathrm{c}\ell(A)$, then $a \in \mathrm{c}\ell(A_0)$ for some finite subset $A_0 \subseteq A$;
(4) Steinitz exchange: For $a, b \in X$ and $A \subseteq X$, if $a \in \mathrm{c}\ell\big(A \cup \{b\}\big) \setminus \mathrm{c}\ell(A)$, then $b \in \mathrm{c}\ell\big(A \cup \{a\}\big)$.

More on finitary matroids can be found in [19], where they are called *independence spaces*. Finitary matroids often appear in model theory, where they are called *pregeometries*; see [21, Appendix C.1]. For the remainder of this paper, we fix a finitary matroid $(X, \mathrm{c}\ell)$, a positive natural number $m \in \mathbb{N}^{>0}$, and a finite tuple $\Phi \coloneqq (\phi_1, \ldots, \phi_m)$ of commuting maps $X \to X$. We will always use a, b to denote elements of X and A, B to denote subsets of X. We will often abuse notation and write things like "$a \in \mathrm{c}\ell(ABb)$" to mean "$a \in \mathrm{c}\ell\big(A \cup B \cup \{b\}\big)$."

For $A \subseteq X$, we let $\mathrm{c}\ell_A$ denote the following closure operator:

$$a \in \mathrm{c}\ell_A(B) \;:\Longleftrightarrow\; a \in \mathrm{c}\ell(AB).$$

Then $(X, \mathrm{c}\ell_A)$ is also a finitary matroid, called the **localization of** $(X, \mathrm{c}\ell)$ **at** A. We say that B is $\mathrm{c}\ell$-**independent over** A if $b \notin \mathrm{c}\ell_A\big(B \setminus \{b\}\big)$ for all $b \in B$. A **basis for** B **over** A is a subset $B_0 \subseteq B$ which is $\mathrm{c}\ell$-independent over A such that $B \subseteq \mathrm{c}\ell_A(B_0)$. Steinitz exchange ensures that any two bases for B over A have the same cardinality, called the **rank of** B **over** A and denoted $\mathrm{rk}(B|A)$. We just write $\mathrm{rk}(B)$ for $\mathrm{rk}(B|\emptyset)$, and we use rk_A for the rank corresponding to the localization $(X, \mathrm{c}\ell_A)$, so $\mathrm{rk}(B|A) = \mathrm{rk}_A(B)$.

Let Θ be the free commutative monoid on Φ, so Θ consists of all operators $\phi^{\bar{r}} \coloneqq \phi_1^{r_1} \cdots \phi_m^{r_m}$ for $\bar{r} \in \mathbb{N}^m$. Note that $\phi^{\bar{0}_m}$ is the identity map on X (and also the identity element of Θ) and that $\phi^{\hat{e}_{i,m}} = \phi_i$ for $i = 1, \ldots, m$. For $\theta \in \Theta$, we let $\theta(A) \coloneqq \{\theta a : a \in A\}$, and for $\Theta_0 \subseteq \Theta$, we let

$$\Theta_0(A) \;\coloneqq\; \bigcup_{\theta \in \Theta} \theta(A).$$

Recall from the introduction that Φ is a **triangular system** for $(X, \mathrm{c}\ell)$ if

$$a \in \mathrm{c}\ell(B) \implies \phi_i a \in \mathrm{c}\ell\big(\phi_1(B) \cdots \phi_i(B)\big)$$

for each $i \in \{1, \ldots, m\}$. If Φ is a triangular system for $(X, \mathrm{c}\ell)$ and $A \subseteq X$ is closed under each map ϕ_i, then one can easily verify that Φ is a triangular system for the localization $(X, \mathrm{c}\ell_A)$. The following lemma on triangular systems will be used in the proof of the main theorem.

Lemma 1.1. *Φ is a triangular system if and only if for each $i \in \{1, \ldots, m\}$, we have*

$$\mathrm{rk}\big(\phi_i(A)\big|\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(B)\big) \;\leq\; \mathrm{rk}(A|B)$$

Proof. Suppose that Φ is a triangular system. Let A_0 be a $\mathrm{c}\ell$-basis for A over B, so $A \subseteq \mathrm{c}\ell(A_0 B)$. Since Φ is a triangular system, we have

$$\phi_i(A) \;\subseteq\; \mathrm{c}\ell\big(\phi_1(A_0B) \cdots \phi_i(A_0B)\big) \;\subseteq\; \mathrm{c}\ell\big(\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(A_0B)\big).$$

This gives

$$\mathrm{rk}\big(\phi_i(A)\big|\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(B)\big) \;\leq\; \big|\phi_i(A_0)\big| \;\leq\; |A_0| \;=\; \mathrm{rk}(A|B).$$

For the converse, let $a \in \mathrm{c}\ell(B)$. Then,

$$\mathrm{rk}\big(\phi_i(a)\big|\phi_1(aB) \cdots \phi_{i-1}(aB)\phi_i(B)\big) \;\leq\; \mathrm{rk}(a|B) \;=\; 0.$$

By induction on $i = 1, \ldots, m$, we conclude that $\phi_i(a) \in \mathrm{c}\ell\big(\phi_1(B) \cdots \phi_{i-1}(B)\phi_i(B)\big)$. $\square$

1.3. **Generating functions.** Let $k \in \mathbb{N}^{>0}$ and let $\bar{Y} = (Y_1, \ldots, Y_k)$ be a tuple of variables. Given $\bar{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$, we write $\bar{Y}^{\bar{r}}$ for the monomial $Y_1^{r_1} \cdots Y_k^{r_k}$. A polynomial $P \in \mathbb{Q}[\bar{Y}]$ is said to have **degree at most** $\bar{r}$ if P has degree at most r_i in each variable Y_i, that is, if

$$P(\bar{Y}) \;=\; a\bar{Y}^{\bar{r}} + \text{lower degree terms}$$

for some $a \in \mathbb{Q}$.

Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be a function. The **generating function of** f is the multivariate power series

$$G_f(\bar{Y}) \;=\; \sum_{\bar{r} \in \mathbb{N}^k} f(\bar{r})\bar{Y}^{\bar{r}} \;\in\; \mathbb{Z}[\![\bar{Y}]\!].$$

The following is well-known; see [6, Lemma 2.1] for a short proof.

Fact 1.2. *Suppose that G_f is a rational function with denominator $(1-Y_1)^{d_1} \cdots (1-Y_k)^{d_k}$. Then there is $P \in \mathbb{Q}[\bar{Y}]$ and of degree at most $(d_1 - 1, \ldots, d_k - 1)$ such that $f(\bar{s}) = P(\bar{s})$ whenever* $\min(\bar{s})$ *is sufficiently large.*

Remark 1.3. Let G_f and P be as in Fact 1.2 and let R be the numerator of G_f, so

$$G_f(\bar{Y}) \;=\; \frac{R(\bar{Y})}{(1-Y_1)^{d_1} \cdots (1-Y_k)^{d_k}}.$$

From the proof given in [6], we see that

$$P(\bar{Y}) \;=\; \frac{R(1, \ldots, 1)}{(d_1-1)! \cdots (d_k-1)!} Y_1^{d_1-1} \cdots Y_k^{d_k-1} + \text{lower degree terms.}$$

The proof also tells us that if R has degree at most $\bar{r}$, then $f(\bar{s}) = P(\bar{s})$ for all $\bar{s} \in \mathbb{N}^k$ with $\bar{s} \not\preccurlyeq \bar{r}$.

The function f is said to be **decreasing** if $f(\bar{r}) \leq f(\bar{s})$ whenever $\bar{r} \succcurlyeq \bar{s}$. Suppose that f is decreasing. For $n \in \mathbb{N}$, set

$$S_n(f) \;\coloneqq\; \{\bar{s} \in \mathbb{N}^k : f(\bar{s}) \leq n\},$$

so each $S_n(f)$ is a $\preccurlyeq$-upward closed subset of $\mathbb{N}^k$ and $S_n(f) = \mathbb{N}^k$ for $n \geq f(\bar{0}_k)$. Let $M_n(f)$ be the set of $\preccurlyeq$-minimal elements of $S_n(f)$, so each $M_n(f)$ is finite by Dickson's lemma. Set $M(f) \coloneqq \bigcup_{n \in \mathbb{N}} M_n(f)$, and let $\bar{m}(f)$ be the $\preccurlyeq$-least upper bound of $M(f)$.

Proposition 1.4. *If f is decreasing, then G_f is a rational function with numerator of degree at most $\bar{m}(f)$ and denominator $(1-Y_1) \cdots (1-Y_k)$.*

Proof. Suppose that f is decreasing, set $H \coloneqq (1-Y_1) \cdots (1-Y_k)G_f$, and let $\bar{m}(f) = (m_1, \ldots, m_k)$. We need to show for each $i \in \{1, \ldots, k\}$ and each $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ that if $s_i > m_i$, then the coefficient of $\bar{Y}^{\bar{s}}$ in H is zero. By symmetry, it suffices to consider the case $i = k$. Take power series $H_0, H_1, H_2, \ldots$ in the variables $(Y_1, \ldots, Y_{k-1})$ such that

$$H \;=\; H_0 + H_1 Y_k + H_2 Y_k^2 + \cdots.$$

We fix $t > m_k$, and we will show that $H_t = 0$. Distributing $(1 - Y_k)$ through the series G_f, we see that

$$H_t(Y_1, \ldots, Y_{k-1}) \;=\; (1-Y_1) \cdots (1-Y_{k-1}) \sum_{\bar{r} \in \mathbb{N}^{k-1}} \big(f(\bar{r}, t) - f(\bar{r}, t-1)\big) Y_1^{r_1} \cdots Y_{k-1}^{r_{k-1}},$$

so it suffices to show that $f(\bar{r}, t) = f(\bar{r}, t-1)$ for each $\bar{r} \in \mathbb{N}^{k-1}$. Let $\bar{r}$ be given and let $n = f(\bar{r}, t)$. Take $\bar{s} \in M_n(f)$ with $\bar{s} \preccurlyeq (\bar{r}, t)$. Since $s_k \leq m_k < t$, we see that $(\bar{r}, t-1) \succcurlyeq \bar{s}$ as well, so $f(\bar{r}, t-1) \leq n$. Since f is decreasing, we conclude that $f(\bar{r}, t-1) = n = f(\bar{r}, t)$. □

Remark 1.5. We have another proof of Proposition 1.4, using that the partial order on decreasing functions $\mathbb{N}^k \to \mathbb{N}$ given by $f \leq g :\Longleftrightarrow f(\bar{s}) \leq g(\bar{s})$ for all $\bar{s} \in \mathbb{N}^k$ is well-founded. The proof is as follows: let $f\colon \mathbb{N}^k \to \mathbb{N}$ be decreasing and assume that Proposition 1.4 holds for all decreasing functions $\mathbb{N}^k \to \mathbb{N}$ less than f, as well as all decreasing functions $\mathbb{N}^{k-1} \to \mathbb{N}$ (both base cases hold trivially). Let $g\colon \mathbb{N}^k \to \mathbb{N}$ be the function

$(\bar{r},t)\mapsto f(\bar{r},t+1)$, so $g\le f$, and let $h\colon \mathbb{N}^{k-1}\to\mathbb{N}$ be the function $\bar{r}\mapsto f(\bar{r},0)$. First, consider the case that $g=f$. Then $f(\bar{r},t)=f(\bar{r},0)=h(\bar{r})$ for all $\bar{r}\in\mathbb{N}^{k-1}$ and all $t\in\mathbb{N}$ and so $\bar{m}(f)=\big(\bar{m}(h),0\big)$. We have

$$G_f(\bar{Y}) \;=\; \sum_{t\in\mathbb{N}}\sum_{\bar{r}\in\mathbb{N}^{k-1}} f(\bar{r},0)\bar{Y}^{(\bar{r},t)} \;=\; (1-Y_k)^{-1}\sum_{\bar{r}\in\mathbb{N}^{k-1}} h(\bar{r})Y_1^{r_1}\cdots Y_{k-1}^{r_{k-1}} \;=\; \frac{G_h(Y_1,\ldots,Y_{k-1})}{(1-Y_k)},$$

so Proposition 1.4 holds for f by our induction hypothesis. Now, consider the case that $g<f$. In this case, $\bar{m}(f)$ is the $\preccurlyeq$-least upper bound of $\big(\bar{m}(h),0\big)$ and $\bar{m}(g)+\hat{e}_{k,k}$. We have

$$G_f(\bar{Y}) \;=\; \sum_{\bar{r}\in\mathbb{N}^{k-1}} f(\bar{r},0)\bar{Y}^{(\bar{r},0)}+\sum_{t\in\mathbb{N}}\sum_{\bar{r}\in\mathbb{N}^{k}} f(\bar{r},t+1)\bar{Y}^{(\bar{r},t+1)} \;=\; G_h(Y_1,\ldots,Y_{k-1})+Y_kG_g(\bar{Y}),$$

and we again conclude that Proposition 1.4 holds for f by our induction hypothesis.

2. The Hilbert polynomial and the Φ-closure operator

Let now $(\Phi_1,\ldots,\Phi_k)$ be the partition of Φ given in the introduction. We assume that each part of the partition Φ_i is a triangular system, and we let $A,B\subseteq X$ with A finite. In this section, we prove Theorems A and B.

2.1. **The Hilbert polynomial.** For use in the proof of Theorem A, we define the following objects: for $\bar{u}\in\mathbb{N}^m$, set

$$\Theta_{\bar{u}} \;:=\; \big\{\phi^{\bar{r}}:\|\bar{r}\|=\|\bar{u}\| \text{ and } \bar{r}<_{\mathrm{lex}}\bar{u}\big\},\qquad f^{\Phi}_{A|B}(\bar{u}) \;:=\; \mathrm{rk}\big(\phi^{\bar{u}}(A)\big|\Theta_{\bar{u}}(A)\Phi^{(\|u\|)}(B)\big).$$

Then $f^{\Phi}_{A|B}$ is bounded above by $|A|$, and $\sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{A|B}(\bar{u})=\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)$ for each $\bar{s}\in\mathbb{N}^k$.

Lemma 2.1. *The function $f^{\Phi}_{A|B}$ is decreasing.*

Proof. Let $\bar{u}'\succcurlyeq\bar{u}$ be given. We may assume that

$$\bar{u}' \;=\; \bar{u}+\hat{e}_{m_i+d,m}$$

for some $i\in\{1,\ldots,k\}$ and some $d\in\{1,\ldots,d_i\}$. Since $\Phi_i=(\phi_{m_i+1},\ldots,\phi_{m_i+d_i})$ is a triangular system, Lemma 1.1 tells us that

$$\mathrm{rk}\big(\phi_{m_i+d}\phi^{\bar{u}}(A)\big|\big\{\phi_{m_i+j}\phi^{\bar{u}}(A):0<j<d\big\}\cup\big\{\phi_{m_i+j}\big(\Theta_{\bar{u}}(A)\Phi^{(\|\bar{u}\|)}(B)\big):0<j\le d\big\}\big) \;\le\; f^{\Phi}_{A|B}(\bar{u}).$$

For $j\in\{1,\ldots,d-1\}$, we have $\|\hat{e}_{m_i+j,m}\|=\|\hat{e}_{m_i+d,m}\|$ and $\hat{e}_{m_i+j,m}<_{\mathrm{lex}}\hat{e}_{m_i+d,m}$, so

$$\|\hat{e}_{m_i+j,m}+\bar{u}\| \;=\; \|\hat{e}_{m_i+d,m}+\bar{u}\| \;=\; \|\bar{u}'\|,\qquad \hat{e}_{m_i+j,m}+\bar{u} \;<_{\mathrm{lex}}\; \hat{e}_{m_i+d,m}+\bar{u} \;=\; \bar{u}'.$$

It follows that $\phi_{m_i+j}\phi^{\bar{u}}\in\Theta_{\bar{u}'}$ for $j\in\{1,\ldots,d-1\}$. Likewise, for $\phi^{\bar{r}}\in\Theta_{\bar{u}}$ and $j\in\{1,\ldots,d\}$, we have

$$\|\hat{e}_{m_i+j,m}+\bar{r}\| \;=\; \|\hat{e}_{m_i+j,m}+\bar{u}\| \;=\|\bar{u}'\|,\qquad \hat{e}_{m_i+j,m}+\bar{r} \;<_{\mathrm{lex}}\; \hat{e}_{m_i+j,m}+\bar{u} \;\le_{\mathrm{lex}}\; \bar{u}',$$

so $\phi_{m_i+j}\Theta_{\bar{u}}\subseteq\Theta_{\bar{u}'}$. Finally, we have $\phi_{m_i+j}(\Phi^{(\|\bar{u}\|)})\subseteq\Phi^{(\|\bar{u}+\hat{e}_{m_i+j,m}\|)}=\Phi^{(\|\bar{u}'\|)}$ for $j\in\{1,\ldots,d\}$, so

$$f^{\Phi}_{A|B}(\bar{u}') \;\le\; \mathrm{rk}\big(\phi_{m_i+d}\phi^{\bar{u}}(A)\big|\big\{\phi_{m_i+j}\phi^{\bar{u}}(A):0<j<d\big\}\cup\big\{\phi_{m_i+j}\big(\Theta_{\bar{u}}(A)\Phi^{(\|\bar{u}\|)}(B)\big):0<j\le d\big\}\big). \qquad\square$$

We are now prepared to prove Theorem A:

Proof of Theorem A. Let $G^{\Phi}_{A|B}$ be the generating function of the function $\bar{s}\mapsto\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)$. We have

$$G^{\Phi}_{A|B}(\bar{Y}) \;=\; \sum_{\bar{s}\in\mathbb{N}^k}\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)\bar{Y}^{\bar{s}} \;=\; \sum_{\bar{s}\in\mathbb{N}^k}\sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{A|B}(\bar{u})\bar{Y}^{\bar{s}} \;=\; \sum_{\bar{u}\in\mathbb{N}^m} f^{\Phi}_{A|B}(\bar{u})\bar{Y}^{\|\bar{u}\|}.$$

The rightmost sum above is just the generating function of $f^{\Phi}_{A|B}$ with Y_1 substituted for the first d_1 variables, Y_2 substituted for the next d_2 variables, and so on. By Proposition 1.4, we have that $G^{\Phi}_{A|B}$ is a rational function with denominator $(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}$. The theorem follows from Fact 1.2. $\square$

Remark 2.2. Proposition 1.4 and the proof of Theorem A above tells us that the numerator of $G^{\Phi}_{A|B}$ has degree at most $\|\bar{m}(f^{\Phi}_{A|B})\|$. By Remark 1.3, we conclude that

$$\operatorname{rk}\bigl(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\bigr) \;=\; P^{\Phi}_{A|B}(\bar{s})$$

for $\bar{s} \in \mathbb{N}^k$ with $\bar{s} \succcurlyeq \|\bar{m}(f^{\Phi}_{A|B})\|$.

Remark 2.3. Let $\Psi = (\psi_1, \ldots, \psi_n)$ be another tuple of commuting maps $X \to X$, and let $(\Psi_1, \ldots, \Psi_k)$ be a partition of Ψ. Suppose that for each $i \in \{1, \ldots, k\}$, the tuple Φ_i is a subtuple of Ψ_i. One can prove the following generalization of Theorem A:

There is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ of degree at most $d_i - 1$ in each variable Y_i such that

$$\operatorname{rk}\bigl(\Phi^{(\bar{s})}(A)\big|\Psi^{(\bar{s})}(B)\bigr) \;=\; P(\bar{s})$$

for $\bar{s} \in \mathbb{N}^k$ with $\min(\bar{s})$ sufficiently large.

To prove this generalization, replace the function $f^{\Phi}_{A|B}$ in the above proof with a function $f^{\Phi,\Psi}_{A|B}$, where

$$f^{\Phi,\Psi}_{A|B}(\bar{u}) \;:=\; \operatorname{rk}\bigl(\phi^{\bar{u}}(A)\big|\Theta_{\bar{u}}(A)\Psi^{(\|u\|)}(B)\bigr)$$

for $\bar{u} \in \mathbb{N}^m$. With the obvious changes, the proof of Lemma 2.1 shows that $f^{\Phi,\Psi}_{A|B}$ is decreasing.

2.2. **The Φ-closure operator.** Recall from the introduction that the **Φ-closure** $\mathrm{c}\ell^{\Phi}$ is given by

$$a \in \mathrm{c}\ell^{\Phi}(B) \;:\Longleftrightarrow\; \operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr) < |\Phi^{(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

In this subsection, we prove Theorem B. First, we need the following lemma:

Lemma 2.4. *For $a \in X$, we have*

$$\lim_{\min(\bar{s})\to\infty} \frac{\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr)}{|\Phi^{(\bar{s})}|} \;=\; \begin{cases} 0 & \text{if } a \in \mathrm{c}\ell^{\Phi}(B) \\ 1 & \text{if } a \notin \mathrm{c}\ell^{\Phi}(B). \end{cases}$$

Proof. If $a \notin \mathrm{c}\ell^{\Phi}(B)$, then $\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr) = |\Phi^{(\bar{s})}|$ for all $\bar{s} \in \mathbb{N}^k$, so

$$\lim_{\min(\bar{s})\to\infty} \frac{\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr)}{|\Phi^{(\bar{s})}|} \;=\; 1.$$

Suppose a belongs to $\mathrm{c}\ell^{\Phi}(B)$, as witnessed by $\bar{s}_0 \in \mathbb{N}^k$. Then we have

$$\sum_{\|\bar{u}\|=\bar{s}_0} f^{\Phi}_{a|B}(\bar{u}) \;=\; \operatorname{rk}\bigl(\Phi^{(\bar{s}_0)}(a)\big|\Phi^{(\bar{s}_0)}(B)\bigr) \;<\; |\Phi^{(\bar{s}_0)}|.$$

It follows that $f^{\Phi}_{a|B}(\bar{u}_0) = 0$ for some $\bar{u}_0 \in \mathbb{N}^m$ with $\|\bar{u}_0\| = \bar{s}_0$. By Lemma 2.1, we have $f^{\Phi}_{a|B}(\bar{u}) = 0$ whenever $\bar{u} \succcurlyeq \bar{u}_0$. Let $\bar{s} \in \mathbb{N}^k$ with $\bar{s} \succcurlyeq \bar{s}_0$. Then

$$\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr) \;=\; \sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{a|B}(\bar{u}) \;=\; \sum_{\substack{\|\bar{u}\|=\bar{s}\\ \bar{u}\not\succcurlyeq\bar{u}_0}} f^{\Phi}_{a|B}(\bar{u}) \;\leq\; |\Phi^{(\bar{s})}| - |\Phi^{(\bar{s}-\bar{s}_0)}|.$$

Since $\frac{|\Phi^{(\bar{s}-\bar{s}_0)}|}{|\Phi^{(\bar{s})}|}$ approaches 1 as $\min(\bar{s})$ grows, we have

$$\lim_{\min(\bar{s})\to\infty} \frac{\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr)}{|\Phi^{(\bar{s})}|} \;=\; 0. \qquad \square$$

Proof of Theorem B. We first need to show that $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid. Monotonicity and finite character are both clear. For idempotence, let $a \in \mathrm{c}\ell^{\Phi}\bigl(\mathrm{c}\ell^{\Phi}(B)\bigr)$ and, using finite character, take elements $b_1, \ldots, b_n \in \mathrm{c}\ell^{\Phi}(B)$ with $a \in \mathrm{c}\ell^{\Phi}(b_1, \ldots, b_n)$. We have

$$\begin{aligned}\operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\bigr) \;&\leq\; \operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(b_1, \ldots, b_n)\bigr) + \operatorname{rk}\bigl(\Phi^{(\bar{s})}(b_1, \ldots, b_n)\big|\Phi^{(\bar{s})}(B)\bigr)\\ &\leq\; \operatorname{rk}\bigl(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(b_1, \ldots, b_n)\bigr) + \sum_{i=1}^{n} \operatorname{rk}\bigl(\Phi^{(\bar{s})}(b_i)\big|\Phi^{(\bar{s})}(B)\bigr).\end{aligned}$$

It follows that

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\big)}{|\Phi^{(\bar{s})}|} \;\le\; \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(b_1,\ldots,b_n)\big)+\sum_{i=1}^n \mathrm{rk}\big(\Phi^{(\bar{s})}(b_i)\big|\Phi^{(\bar{s})}(B)\big)}{|\Phi^{(\bar{s})}|}.$$

We conclude by Lemma 2.4 that $a \in \mathrm{c}\ell^{\Phi}(B)$. Finally, for exchange, suppose that $a \in \mathrm{c}\ell^{\Phi}(Bb)\setminus \mathrm{c}\ell^{\Phi}(B)$. Since $\mathrm{c}\ell^{\Phi}$ has finite character, we may assume that B is finite. Idempotence tells us that $b \notin \mathrm{c}\ell^{\Phi}(B)$, so

$$\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\big) \;=\; |\Phi^{(\bar{s})}| \;=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(b)\big|\Phi^{(\bar{s})}(B)\big)$$

for all $\bar{s}$. It follows that

$$\begin{aligned}\mathrm{rk}\big(\Phi^{(\bar{s})}(b)\big|\Phi^{(\bar{s})}(Ba)\big) \;&=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(Bab)\big)-\mathrm{rk}\big(\Phi^{(\bar{s})}(Ba)\big) \;=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(Bab)\big)-\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(B)\big)-\mathrm{rk}\big(\Phi^{(\bar{s})}(B)\big)\\ &=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(Bab)\big)-\mathrm{rk}\big(\Phi^{(\bar{s})}(b)\big|\Phi^{(\bar{s})}(B)\big)-\mathrm{rk}\big(\Phi^{(\bar{s})}(B)\big) \;=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(Bb)\big).\end{aligned}$$

Since $a \in \mathrm{c}\ell^{\Phi}(Bb)$, we see that $b \in \mathrm{c}\ell^{\Phi}(Ba)$.

As $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid, it has an associated rank function rk^{Φ}. We will now show that

$$P^{\Phi}_{A|B}(\bar{Y}) \;=\; \frac{\mathrm{rk}^{\Phi}(A|B)}{(d_1-1)!\cdots(d_k-1)!}Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \text{lower degree terms.}$$

As $\min(\bar{s})$ grows, we have

$$P^{\Phi}_{A|B}(\bar{s}) \;=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big), \qquad |\Phi^{(\bar{s})}| \;=\; \frac{s_1^{d_1-1}\cdots s_k^{d_k-1}}{(d_1-1)!\cdots(d_k-1)!}+o(s_1^{d_1-1}\cdots s_k^{d_k-1}),$$

so it suffices to show that

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)}{|\Phi^{(\bar{s})}|} \;=\; \mathrm{rk}^{\Phi}(A|B),$$

We prove this by induction on $|A|$, with the case $A=\emptyset$ holding trivially. Suppose that this holds for a given A, and let $a \in X\setminus A$. We have

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(Aa)\big|\Phi^{(\bar{s})}(B)\big)}{|\Phi^{(\bar{s})}|} \;=\; \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)+\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(AB)\big)}{|\Phi^{(\bar{s})}|},$$

and our induction hypothesis and Lemma 2.4 gives

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)+\mathrm{rk}\big(\Phi^{(\bar{s})}(a)\big|\Phi^{(\bar{s})}(AB)\big)}{|\Phi^{(\bar{s})}|} \;=\; \mathrm{rk}^{\Phi}(A|B)+\mathrm{rk}^{\Phi}(a|AB).$$

It remains to note that $\mathrm{rk}^{\Phi}(Aa|B)=\mathrm{rk}^{\Phi}(A|B)+\mathrm{rk}^{\Phi}(a|AB)$. $\square$

Remark 2.5. By Remark 1.3, the Φ-rank $\mathrm{rk}^{\Phi}(A|B)$ coincides with the numerator of $G^{\Phi}_{A|B}(\bar{Y})$, evaluated at the tuple $(1,\ldots,1)$.

Remark 2.6. The Φ-closure operator is dependent on our partition, as the following example illustrates: let $(X,\mathrm{c}\ell)$ be the set $\mathbb{Z}$ with the trivial closure operator, so the rank of any subset of $\mathbb{Z}$ is its cardinality. Let $a \in \mathbb{Z}$, let ϕ be the map $x \mapsto x+1$, let $\psi=\phi$, and let $\Phi=(\phi,\psi)$. First, suppose Φ is partitioned trivially (so $k=1$). Then $\Phi^{(t)}(a)=\{a+t\}$ for any $t\in\mathbb{N}$, so $\mathrm{rk}\big(\Phi^{(t)}(a)\big)=1<|\Phi^{(t)}|$ for $t>0$ and $\mathrm{rk}^{\Phi}(a)=0$. Now, suppose Φ is given the partition (Φ_1,Φ_2), where $\Phi_1=(\phi)$ and $\Phi_2=(\psi)$. Then $\Phi^{(s_1,s_2)}(a)=\{a+s_1+s_2\}$ for $s_1,s_2\in\mathbb{N}$, so $\mathrm{rk}\big(\Phi^{(s_1,s_2)}(a)\big)=1=|\Phi^{(s_1,s_2)}|$ and $\mathrm{rk}^{\Phi}(a)=1$. As we will see, this dependence on the partition doesn't occur in the setting of Sections 4 and 5

3. Applications I: Commuting endomorphisms and some classical results

A map $\phi\colon X\to X$ is said to be an **endomorphism of** $(X,\mathrm{c}\ell)$ if

$$a \in \mathrm{c}\ell(B) \implies \phi a \in \mathrm{c}\ell(\phi B).$$

Note that if Φ is a triangular system, then ϕ_1 is necessarily an endomorphism. Also, if $\phi_1,\ldots,\phi_m$ are endomorphisms, then Φ is a triangular system, as is each part Φ_i of our partition of Φ. In this section, we give a handful of applications of Theorem A, all of which fall into this case.

3.1. **Polynomial growth of sumsets.** Let G be a commutative semigroup. In [7], Khovanskii showed that for finite subsets $A, B \subseteq G$, the size of the sumset $A + tB$ is given by a polynomial in t for t sufficiently large (here, $A + tB$ is the set of all elements $a + b_1 + \cdots + b_t$, where $a \in A$ and each $b_i \in B$). Moreover, the degree of this polynomial is less than the size of B. Using Theorem A, we can recover a generalization of Khovanskii's theorem, originally proven by Nathanson [17].

Corollary 3.1 (Nathanson)**.** *Let $A, B_1, \ldots, B_k$ be finite subsets of G. Then there is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$|A + s_1 B_1 + \cdots + s_k B_k| \;=\; P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large. Moreover, the degree of P in each variable Y_i is less than $|B_i|$.*

Proof. Let $(X, \mathrm{c}\ell)$ be the underlying set of the semigroup G with the trivial closure operator, so the rank of any subset of G is its cardinality. Let $b_1, \ldots, b_{m_1}$ be an enumeration of B_1, let $b_{m_1+1}, \ldots, b_{m_2}$ be an enumeration of B_2, and so on. For each $i \in \{1, \ldots, m\}$, let $\phi_i \colon G \to G$ be the map $x \mapsto x + b_i$, so each ϕ_i is an endomorphism. Then

$$\Phi^{(\bar{s})}(A) \;=\; A + s_1 B_1 + \cdots + s_k B_k$$

for each $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$, and it remains to invoke Theorem A. ☐

The set B in Theorem A makes no appearance in the argument above. This suggests a slight improvement:

Corollary 3.2. *Let $A, B_1, \ldots, B_k$ be finite subsets of G and let B be an arbitrary subset of G. Then there a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$\big|(A + s_1 B_1 + \cdots + s_k B_k) \setminus (B + s_1 B_1 + \cdots + s_k B_k)\big| \;=\; P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Remark 3.3. Khovanskii and Nathanson proved their theorems by constructing appropriate graded modules and using the existence of the Hilbert polynomial (or a multivariate variant thereof). Nathanson later gave a more elementary proof of Corollary 3.1 with Ruzsa [18].

3.2. **Counting elements in an ideal.** In this next application, we make use of the fact that if ϕ is an endomorphism of $(X, \mathrm{c}\ell)$ and $\phi(C) \subseteq C$ for some $C \subseteq X$, then ϕ is also an endomorphism of $(X, \mathrm{c}\ell_C)$.

Corollary 3.4. *Let I be an ideal of $\mathbb{N}^m$, that is, a $\preccurlyeq$-downward closed subset of $\mathbb{N}^m$. For each $\bar{s} \in \mathbb{N}^k$, let $H_I(\bar{s})$ be the number of elements $\bar{r} \in I$ with $\|\bar{r}\| = \bar{s}$. Then there is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$H_I(\bar{s}) \;=\; P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Proof. Let $C \coloneqq \mathbb{N}^m \setminus I$, let $(X, \mathrm{c}\ell)$ be the set $\mathbb{N}^m$ with the trivial closure operator, and let $(X, \mathrm{c}\ell_C)$ be the localization of $(X, \mathrm{c}\ell)$ at C. Then $\mathrm{rk}_C(Y) = |Y \setminus C| = |Y \cap I|$ for any subset $Y \subseteq \mathbb{N}^m$. For each $i \in \{1, \ldots, m\}$, let $\phi_i \colon \mathbb{N}^m \to \mathbb{N}^m$ be the map $\bar{r} \mapsto \bar{r} + \hat{e}_{i,m}$, so

$$\mathrm{rk}_C\big(\Phi^{(\bar{s})}(\bar{0}_m)\big) \;=\; \big|\{\bar{r} \in \mathbb{N}^m : \|\bar{r}\| = \bar{s}\} \cap I\big| \;=\; H_I(\bar{s}),$$

Since I is $\preccurlyeq$-downward closed, we have $\phi_i(C) \subseteq C$ for each $i \in \{1, \ldots, m\}$. Thus, $\phi_1, \ldots, \phi_m$ are commuting endomorphisms of $(X, \mathrm{c}\ell_C)$, and the corollary follows from Theorem A. ☐

The case $k = 1$ of the above corollary was posed as an "elementary problem" by Stanley [20]. In Corollary 4.1 below, we prove a variant where one instead counts the number of elements $\bar{r} \in I$ with $\|\bar{r}\| \preccurlyeq \bar{s}$. This variant was first considered in [11].

3.3. **The Hilbert polynomial.** Let K be a field and let $R \coloneqq K[x_1, \ldots, x_m]$, where $x_1, \ldots, x_m$ are indeterminants. Using our partition $(\Phi_1, \ldots, \Phi_k)$, we associate to R an $\mathbb{N}^k$-grading $R = \bigoplus_{\bar{s} \in \mathbb{N}^k} R_{\bar{s}}$ as follows: the graded part $R_{\bar{s}}$ is the K-vector space generated by monomials $x_1^{r_1} \cdots x_m^{r_m}$ with $\|\bar{r}\| = \bar{s}$. Let $M = \bigoplus_{\bar{s} \in \mathbb{Z}^k} M_{\bar{s}}$ be a finitely generated graded R-module. Then each graded piece $M_{\bar{s}}$ is a finite-dimensional K-vector space. The **Hilbert function of** M, denoted $H_M \colon \mathbb{Z}^k \to \mathbb{N}$, is given by $H_M(\bar{s}) \coloneqq \dim_K(M_{\bar{s}})$. The following result is classical:

Corollary 3.5. *There is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$H_M(\bar{s}) \;=\; P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Proof. Let $(X, \mathrm{c}\ell)$ be the underlying set of our R-module M with the K-linear closure operator (that is, the closure of any subset of M is its K-linear span). For each $i \in \{1, \ldots, m\}$, let $\phi_i \colon M \to M$ be the map $a \mapsto x_i \cdot a$. Since M is finitely generated, we can find an index $\bar{s}_0$ and a finite set $A \subseteq M_{\bar{s}_0}$ such that the R-submodule $\bigoplus_{\bar{s} \succcurlyeq \bar{s}_0} M_{\bar{s}}$ is generated by A. By reindexing, we may assume that $\bar{s}_0 = \bar{0}_k$. Then for each $\bar{s} \in \mathbb{N}^k$, the graded part $M_{\bar{s}}$ is generated as a K-vector space by $\Phi^{(\bar{s})}(A)$, so

$$\dim_K(M_{\bar{s}}) \;=\; \mathrm{rk}(M_{\bar{s}}) \;=\; \mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big)$$

for $\bar{s} \in \mathbb{N}^k$. The corollary follows from Theorem A. □

4. Quasi-endomorphisms and the Kolchin Polynomial

A map $\phi \colon X \to X$ is a **quasi-endomorphism of** $(X, \mathrm{c}\ell)$ if

$$a \in \mathrm{c}\ell(B) \implies \phi a \in \mathrm{c}\ell(B\phi B).$$

Quasi-endomorphisms were first considered in [4, Section 3.1]. Of course, any endomorphism is a quasi-endomorphism. Also, ϕ is a quasi-endomorphism if and only if (id, ϕ) is a triangular system, where $\mathrm{id} \colon X \to X$ is the identity map. As we will see, derivations on fields are quasi-endomorphisms with respect to the pregeometry of algebraic closure, so the study of quasi-endomorphisms will allow us to prove various Kolchin polynomial results.

For the remainder of this section, we assume that $\Phi = (\phi_1, \ldots, \phi_m)$ consists of commuting quasi-endomorphisms of $(X, \mathrm{c}\ell)$. We have the following consequence of Theorem A:

Theorem C. *Let $A \subseteq X$ be finite and let $B \subseteq X$ be arbitrary. Then there is a polynomial $Q \in \mathbb{Q}[\bar{Y}]$ of degree at most d_i in each variable Y_i such that*

$$\mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(A)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) \;=\; Q(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large, where* $\Phi^{\preccurlyeq(\bar{s})} := \big\{\phi^{\bar{r}} : \|\bar{r}\| \preccurlyeq \bar{s}\big\}$.

Proof. Let us first consider the case $k = 1$, so we need to show that $\mathrm{rk}\big(\Phi^{\leq(t)}(A)\big|\Phi^{\leq(t)}(B)\big)$ is eventually polynomial in t of degree at most m, where $\Phi^{\leq(t)} := \big\{\phi^{\bar{r}} : |\bar{r}| \leq t\big\}$. Let $\Phi_* := (\mathrm{id}, \phi_1, \ldots, \phi_m)$ and let Θ_* be the corresponding free monoid over Φ_*, so Φ_* is a triangular system. Take $Q := P^{\Phi_*}_{A|B}$, so Q has degree at most m and

$$\mathrm{rk}\big(\Phi_*^{(t)}(A)\big|\Phi_*^{(t)}(B)\big) \;=\; Q(t)$$

for sufficiently large t. For $\bar{r} \in \mathbb{N}^m$, the maps $\phi^{\bar{r}} \in \Phi^{\leq(t)}$ and $\mathrm{id}^{t-|\bar{r}|}\,\phi^{\bar{r}} \in \Phi_*^{(t)}$ act the same way on X, so we may identify $\Phi^{\leq(t)}$ with $\Phi_*^{(t)}$. This gives

$$\mathrm{rk}\big(\Phi^{\leq(t)}(A)\big|\Phi^{\leq(t)}(B)\big) \;=\; Q(t)$$

for sufficiently large t, as desired. For arbitrary k, the above proof can be adapted in the obvious way: one must add k new identity maps in order to extend each subtuple $\Phi_i = (\phi_{m_i+1}, \ldots, \phi_{m_i+d_i})$ to a triangular system $(\mathrm{id}, \phi_{m_i+1}, \ldots, \phi_{m_i+d_i})$. □

We call the polynomial Q in Theorem C the **Kolchin polynomial of** A **over** B **with respect to the partition** $(\Phi_1, \ldots, \Phi_k)$, and we denote it by $Q^{\Phi}_{A|B}$. The following result, first established by Kondratieva, Levin, Mikhalev, and Pankratiev [11], can be deduced in exactly the same way as Corollary 3.4; just use Theorem C in place of Theorem A:

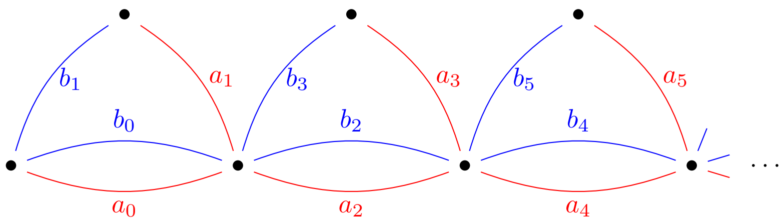
Corollary 4.1 (Theorem 2.2.5 in [11])**.** *Let I be an ideal of $\mathbb{N}^m$, and for each $\bar{s} \in \mathbb{N}^k$, let $H^*_I(\bar{s})$ be the number of elements $\bar{r} \in I$ with $\|\bar{r}\| \preccurlyeq \bar{s}$. Then there is a polynomial $Q \in \mathbb{Q}[\bar{Y}]$ such that*

$$H^*_I(\bar{s}) \;=\; Q(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

The $k = 1$ case of Corollary 4.1 was first proven by Kolchin [10, Lemma 0.16].

Remark 4.2. It is natural to ask: in our current setting where Φ is a tuple of quasi-endomorphisms, is $\mathrm{rk}\big(\Phi^{(\bar{s})}(A)\big|\Phi^{(\bar{s})}(B)\big)$ given by a polynomial for $\min(\bar{s})$ large enough? As it turns out, this can fail even with $m = k = 1$. To see this consider the following graph G:



Let V be the set of vertices and let E be the set of edges. The **graphic matroid** of G consists of the underlying set E, together with the closure operator given by

$$e \in \mathrm{c}\ell(S) \;:\Longleftrightarrow\; \text{the endpoints of } e \text{ are connected by a path in } S.$$

Let ϕ be the map sending a_i to a_{i+1} and sending b_i to b_{i+1} for each i. Then ϕ is a quasi-endomorphism of the graphic matroid, but

$$\mathrm{rk}(\phi^n a_0|\phi^n b_0) \;=\; \mathrm{rk}(a_n|b_n) \;=\; \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, the conclusion of Theorem A fails for the single map ϕ.

4.1. **Invariants of the Kolchin polynomial.** Theorem B and the proof of Theorem C tell us that $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid, where the Φ-closure operator $\mathrm{c}\ell^{\Phi}$ is here given by

$$a \in \mathrm{c}\ell^{\Phi}(B) \;:\Longleftrightarrow\; \mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(a)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) < |\Phi^{\preccurlyeq(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

In our present setting, this operator is much more robust, as it does not depend on our partition $(\Phi_1, \ldots, \Phi_k)$:

Proposition 4.3. *For $a \in X$, we have $a \in \mathrm{c}\ell^{\Phi}(B)$ if and only if $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$.*

Proof. Clearly, if $a \in \mathrm{c}\ell^{\Phi}(B)$, then $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$. Suppose that $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$, and take a finite subset $\Theta_0 \subseteq \Theta$ with $\mathrm{rk}\big(\Theta_0(a)\big|\Theta(B)\big) < |\Theta_0|$. Since $\mathrm{c}\ell$ has finite character, we can take $\bar{s} \in \mathbb{N}^k$ with $\Theta_0 \subseteq \Phi^{\preccurlyeq(\bar{s})}$ such that $\mathrm{rk}\big(\Theta_0(a)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) < |\Theta_0|$. Set $\Theta_1 := \Phi^{\preccurlyeq(\bar{s})} \setminus \Theta_0$, so

$$\mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(a)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) \;\le\; \mathrm{rk}\big(\Theta_0(a)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) + \mathrm{rk}\big(\Theta_1(a)\big|\Phi^{\preccurlyeq(\bar{s})}(B)\big) \;<\; |\Theta_0| + |\Theta_1| \;=\; |\Phi^{\preccurlyeq(\bar{s})}|.$$

We conclude that $a \in \mathrm{c}\ell^{\Phi}(B)$. ☐

When paired with Theorem B, the above proposition tells us the following:

Corollary 4.4. *The coefficient of $\bar{Y}^{\bar{d}}$ in $Q^{\Phi}_{A|B}$ times $d_1! \cdots d_k!$ equals the maximal size of a subset $A_0 \subseteq A$ such that $(\theta a)_{\theta\in\Theta, a\in A_0}$ is $\mathrm{c}\ell$-independent over $\Theta(B)$.*

Given a permutation σ of $\{1, \ldots, k\}$, we define a total order $\le_\sigma$ on $\mathbb{N}^k$ by setting

$$\bar{d} \le_\sigma \bar{e} \;:\Longleftrightarrow\; (d_{\sigma(1)}, \ldots, d_{\sigma(k)}) \le_{\mathrm{lex}} (e_{\sigma(1)}, \ldots, e_{\sigma(k)}).$$

Given a polynomial

$$P(\bar{Y}) \;=\; \sum_{\bar{e}\in\mathbb{N}^k} a_{\bar{e}} \bar{Y}^{\bar{e}} \in \mathbb{Q}[\bar{Y}],$$

we define the **dominant terms of** P to be the terms $a_{\bar{d}}\bar{Y}^{\bar{d}}$ such that $\bar{d}$ is the $\le_\sigma$-maximal element of the set $\{\bar{e} : a_{\bar{e}} \neq 0\}$ for some permutation σ of $\{1, \ldots, k\}$. One can show by induction on k that $a_{\bar{d}}\bar{Y}^{\bar{d}}$ is a dominant term for P, as witnessed by the permutation σ, if and only if

$$\lim_{s_{\sigma(1)}\to\infty}\left(\lim_{s_{\sigma(2)}\to\infty}\left(\cdots \lim_{s_{\sigma(k)}\to\infty}\left(\frac{P(\bar{s})}{a_{\bar{d}} s_1^{d_1}\cdots s_k^{d_k}}\right)\cdots\right)\right) \;=\; 1. \tag{4.1}$$

In the case that $B = \Theta(B)$, the dominant terms of the Kolchin polynomial for A over B only depend on $c\ell\big(\Theta(A)B\big)$:

Proposition 4.5. *Let A, A' be finite subsets of X, let $B \subseteq X$ with $\Theta(B) = B$, and suppose that $c\ell\big(\Theta(A)B\big) = c\ell\big(\Theta(A')B\big)$. Then $Q^{\Phi}_{A|B}$ and $Q^{\Phi}_{A'|B}$ have the same dominant terms.*

Proof. Take $\bar{s}_0$ with
$$A' \;\subseteq\; c\ell\big(\Phi^{\preccurlyeq(\bar{s}_0)}(A)B\big), \qquad A \;\subseteq\; c\ell\big(\Phi^{\preccurlyeq(\bar{s}_0)}(A')B\big).$$
Since each ϕ_i is a quasi-endomorphism, a routine induction on $|\bar{s}|$ gives
$$\Phi^{\preccurlyeq(\bar{s})}(A') \;\subseteq\; c\ell\big(\Phi^{\preccurlyeq(\bar{s}_0+\bar{s})}(A)B\big), \qquad \Phi^{\preccurlyeq(\bar{s})}(A) \;\subseteq\; c\ell\big(\Phi^{\preccurlyeq(\bar{s}_0+\bar{s})}(A')B\big),$$
for all $\bar{s} \in \mathbb{N}^k$. It follows that
$$\mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(A')\big|B\big) \;\leq\; \mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s}_0+\bar{s})}(A)\big|B\big), \qquad \mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(A)\big|B\big) \;\leq\; \mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s}_0+\bar{s})}(A')\big|B\big)$$
for each $\bar{s}$. Taking $\min(\bar{s})$ to be sufficiently large, we get
$$Q^{\Phi}_{A'|B}(\bar{s}) \;\leq\; Q^{\Phi}_{A|B}(\bar{s}_0+\bar{s}), \qquad Q^{\Phi}_{A|B}(\bar{s}) \;\leq\; Q^{\Phi}_{A'|B}(\bar{s}_0+\bar{s}).$$
The proposition follows easily, using that the limits (4.1) agree for $Q^{\Phi}_{A|B}$ and $Q^{\Phi}_{A'|B}$. □

5. Applications II: Difference-differential fields and expansions thereof

5.1. **Kolchin polynomials for difference-differential fields.** Let K be a field of characteristic zero. For $a \in K$ and $B \subseteq K$, we write $a \in \mathrm{acl}(B)$ to mean that a is algebraic over (the field generated by) B. Then (K, acl) is a finitary matroid. For a subfield $F \subseteq K$, we let trdeg_F denote the transcendence degree over F. For any subset $A \subseteq K$, we have $\mathrm{rk}(A|F) = \mathrm{trdeg}_F F(A)$, where $F(A)$ is the subfield of K generated by F and A.

Clearly, any field endomorphism $\sigma\colon K \to K$ is an endomorphism of (K, acl). A **derivation** on K is a map $\delta\colon K \to K$ which satisfies the identities
$$\delta(a+b) \;=\; \delta a + \delta b, \qquad \delta(ab) \;=\; a\delta b + b\delta a$$
for all $a, b \in K$.

Lemma 5.1. *Let δ be a derivation on K. Then δ is a quasi-endomorphism of (K, acl).*

Proof. Let $a \in K$ and $B \subseteq K$ with $a \in \mathrm{acl}(B)$. Then there is a tuple $\bar{b} = (b_1, \ldots, b_n)$ from B and a polynomial $P \in \mathbb{Z}[\bar{X}, Y]$ such that $P(\bar{b}, a) = 0$ but $\frac{\partial P}{\partial Y}(\bar{b}, a) \neq 0$. We have
$$\delta P(\bar{b}, a) \;=\; \frac{\partial P}{\partial X_1}(\bar{b}, a)\delta b_1 + \cdots + \frac{\partial P}{\partial X_n}(\bar{b}, a)\delta b_n + \frac{\partial P}{\partial Y}(\bar{b}, a)\delta a.$$
Let $\delta\bar{b}$ denote the tuple $(\delta b_1, \ldots, \delta b_n)$, and let $Q \in \mathbb{Z}[\bar{b}, \delta\bar{b}, a][Y]$ be the polynomial
$$Q(Y) \;\coloneqq\; \frac{\partial P}{\partial X_1}(\bar{b}, a)\delta b_1 + \cdots + \frac{\partial P}{\partial X_n}(\bar{b}, a)\delta b_n + \frac{\partial P}{\partial Y}(\bar{b}, a)Y.$$
Then $Q(\delta a) = \delta P(\bar{b}, a) = 0$, but $\frac{\partial Q}{\partial Y}(\delta a) = \frac{\partial P}{\partial Y}(\bar{b}, a) \neq 0$, so $\delta a \in \mathrm{acl}(\bar{b}, \delta\bar{b}, a)$. Since $a \in \mathrm{acl}(\bar{b})$, this gives $\delta a \in \mathrm{acl}(\bar{b}, \delta\bar{b}) \subseteq \mathrm{acl}(B\delta B)$, as desired. □

Suppose now that $\Phi = (\phi_1, \ldots, \phi_m)$ is a collection of commuting maps $K \to K$, each of which is either a derivation or a field endomorphism (hence, a quasi-endomorphism of (K, acl) by Lemma 5.1). Then $(K, \phi_1, \ldots, \phi_m)$ is called a **difference-differential field** (or a **d-field** for short). Let $\bar{a}$ be a tuple from K and let F be a d-subfield of K, that is, a subfield of K which is closed under each ϕ_i. Note that
$$\mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big|\Phi^{\preccurlyeq(\bar{s})}(F)\big) \;=\; \mathrm{rk}\big(\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big|F\big) \;=\; \mathrm{trdeg}_F F\big(\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big).$$
The following corollary is a direct application of Theorem C, Corollary 4.4, and Proposition 4.5.

Corollary 5.2. *Let $Q^{\Phi}_{\bar{a}|F}$ be the Kolchin polynomial of $\bar{a}$ over F with respect to the finitary matroid (K, acl). Then*
$$\mathrm{trdeg}_F F\big(\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big) \;=\; Q^{\Phi}_{\bar{a}|F}(\bar{s})$$
whenever $\min(\bar{s})$ is sufficiently large. Moreover:

(i) The dominant terms of $Q^{\Phi}_{\bar{a}|F}$ *only depend on the* d*-field extension* $F\big(\Theta(\bar{a})\big)$*, that is, if* $F\big(\Theta(\bar{a})\big) = F\big(\Theta(\bar{b})\big)$ *for* $\bar{b} \in F\big(\Theta(\bar{a})\big)$*, then* $Q^{\Phi}_{\bar{a}|F}$ *and* $Q^{\Phi}_{\bar{b}|F}$ *have the same dominant terms.*

(ii) The coefficient of $\bar{Y}^{\bar{d}}$ *times* $d_1! \cdots d_k!$ *is equal to the maximal size of a subset* $B \subseteq F\big(\Theta(\bar{a})\big)$ *such that the tuple* $(\theta b)_{\theta \in \Theta, b \in B}$ *is algebraically independent over* F*.*

The case when $k = 1$ and each ϕ_i is a derivation was first shown by Kolchin [9]. The multivariate differential case (each ϕ_i is a derivation but k is arbitrary) was first proven by Levin [12]. The case where each ϕ_i is an endomorphism is also due to Levin [13], as is the most general case to date: where each ϕ_i may be either a derivation or an endomorphism, but each part of the partition Φ_i must consist of only derivations or endomorphisms; see [14]. Our Corollary 5.2 is slightly more general than the result in [14], since the parts of the partition can consist of both derivations and endomorphisms.

5.2. **Kolchin polynomials for difference-differential exponential fields.** Let K be a field of characteristic zero. An **exponential on** K is a group homomorphism $E\colon A(K) \to K^{\times}$, where $A(K)$ is a divisible subgroup of the additive group of K. If E is an exponential on K, then the pair (K, E) is called an **exponential field**. A subfield F of K is an **exponential subfield** if $E(a) \in F$ for all $a \in A(F) := A(K) \cap F$. The fields $\mathbb{R}$ and $\mathbb{C}$ with their usual exponential functions are exponential fields (where the domain of the exponential is the entire field). If p is a prime greater than 2, then the function $E_p\colon \mathbb{Z}_p \to \mathbb{Q}_p^{\times}$ given by

$$x \mapsto \lim_{n \to \infty} \sum_{i=0}^{n} (px)^i / i!$$

is an exponential on $\mathbb{Q}_p$; see [16]. We may also define an exponential $E_2\colon \mathbb{Z}_2 \to \mathbb{Q}_2^{\times}$ by replacing p in the above definition by 4.

Let (K, E) be an exponential field. An **E-term** is a partial function given by arbitrary compositions of E and polynomials over $\mathbb{Z}$. Model theoretically speaking, an E-term is a term in the language $(+, \cdot, -, 0, 1, E)$; to avoid partially defined functions, one may take E to be identically zero away from $A(K)$. Let $B \subseteq K$. A tuple $\bar{a} = (a_1, \ldots, a_n)$ is said to be a **regular solution to a Khovanskii system over** B if there is a tuple $\bar{b} = (b_1, \ldots, b_m)$ from B and E-terms $t_1, \ldots, t_n$ in $n + m$ variables such that

$$t_1(\bar{a}, \bar{b}) \;=\; \cdots \;=\; t_n(\bar{a}, \bar{b}) \;=\; 0, \qquad \det \begin{pmatrix} \frac{\partial t_1}{\partial X_1}(\bar{a}, \bar{b}) & \cdots & \frac{\partial t_1}{\partial X_n}(\bar{a}, \bar{b}) \\ \vdots & \ddots & \vdots \\ \frac{\partial t_n}{\partial X_1}(\bar{a}, \bar{b}) & \cdots & \frac{\partial t_n}{\partial X_n}(\bar{a}, \bar{b}) \end{pmatrix} \;\neq\; 0.$$

The **exponential-algebraic closure of** B, written $\operatorname{ecl}(B)$, consists of all components of any regular solution to a Khovanskii system over B. The exponential-algebraic closure was first defined by Macintyre [15], and Kirby later showed that (K, ecl) is always a finitary matroid [8, Theorem 1.1]. If F is an exponential subfield of K and A is a subset of K, then we let $F(A)^E$ denote the exponential subfield of K generated by F and A, and we define the **exponential transcendence degree of** $F(A)^E$ **over** F, written $\operatorname{etrdeg}_F F(A)^E$, to be the rank $\operatorname{rk}\big(F(A)^E \big| F\big) = \operatorname{rk}(A|F)$ given by the matroid (K, ecl).

An **exponential endomorphism of** K is a field endomorphism $\sigma\colon K \to K$ such that $\sigma E(a) = E(\sigma a)$ for all $a \in A(K)$. An **exponential derivation on** K is a derivation $\delta\colon K \to K$ which satisfies the identity $\delta E(a) = E(a)\delta a$ for all $a \in K$.

Lemma 5.3. *Any exponential endomorphism of* K *is an endomorphism of* (K, ecl)*. Any exponential derivation on* K *is a quasi-endomorphism of* (K, ecl)*.*

Proof. Let σ be an exponential endomorphism of K, let δ be an exponential derivation on K, and let $B \subseteq K$. If $a \in K$ is a component of a regular solution to a Khovanskii system over B, then $\sigma(a)$ is a component to a regular solution to a Khovanskii system over $\sigma(B)$, namely, the same Khovanskii system but with the parameters from B replaced with the corresponding parameters from $\sigma(B)$. Thus, σ is an endomorphism of (K, ecl).

To see that δ is a quasi-endomorphism of (K, ecl), we use [8, Theorem 1.1], which states that $a \in K$ belongs to $\operatorname{ecl}(B)$ if and only if every exponential derivation on K which vanishes on B also vanishes at a. Suppose $a \in \operatorname{ecl}(B)$, and let ε be an exponential derivation on K which vanishes on $B \cup \delta(B)$. We need to show that $\varepsilon\delta a = 0$. Consider the map $\varepsilon\delta - \delta\varepsilon \colon K \to K$, where $(\varepsilon\delta - \delta\varepsilon)(y) = \varepsilon\delta y - \delta\varepsilon y$. It is routine to show that $\varepsilon\delta - \delta\varepsilon$ is an exponential derivation. For $b \in B$, we have $\varepsilon\delta b - \delta\varepsilon b = 0$, since ε vanishes on $B \cup \delta(B)$. Thus, $\varepsilon\delta - \delta\varepsilon$ vanishes on B, so it also vanishes at a. Since ε also vanishes at a, we see that

$$0 \;=\; \varepsilon\delta a - \delta\varepsilon a \;=\; \varepsilon\delta a. \qquad \square$$

Suppose now that $\Phi = (\phi_1, \ldots, \phi_m)$ is a collection of commuting maps $K \to K$, each of which is either an exponential derivation or an exponential endomorphism. The structure $(K, E, \phi_1, \ldots, \phi_m)$ is called a **difference-differential exponential field** (or a d**-exponential field** for short). Let $\bar{a}$ be a tuple from K and let F be a d-exponential subfield of K, that is, an exponential subfield of K which is closed under each ϕ_i. Theorem C, Corollary 4.4, and Proposition 4.5 give us the following:

Corollary 5.4. *Let $Q^{\Phi}_{\bar{a}|F}$ be the Kolchin polynomial of $\bar{a}$ over F with respect to the finitary matroid (K, ecl). Then*

$$\operatorname{etrdeg}_F F\big(\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big)^E \;=\; Q^{\Phi}_{\bar{a}|F}(\bar{s})$$

whenever $\min(\bar{s})$ is sufficiently large. Moreover:

(1) The dominant terms of $Q^{\Phi}_{\bar{a}|F}$ only depend on the d*-exponential field extension $F\big(\Theta(\bar{a})\big)^E$.*

(2) The coefficient of $\bar{Y}^{\bar{d}}$ times $d_1! \cdots d_k!$ is equal to the maximal size of a subset $B \subseteq F\big(\Theta(\bar{a})\big)^E$ such that the tuple $(\theta b)_{\theta \in \Theta, b \in B}$ is exponential-algebraically independent over F.

Remark 5.5. With the obvious changes, Corollary 5.4 may be applied to j-fields. These fields, introduced in [3], are equipped with partially defined functions which behave like the modular j-function. The relevant closure operator in this setting is the $j$cl-closure, defined in [3], and the tuple Φ should consist of commuting j-field endomorphisms and j-derivations, also defined in [3]. See [1] for more on the j-closure operator. Thanks to Vincenzo Mantova for bringing this to our attention.

5.3. **Derivations on o-minimal fields.** Let T be an o-minimal theory extending the theory of real closed ordered fields, and let K be a model of T; see [2] for definitions and background. The **definable closure operator on** K, denoted dcl, is given by

$$a \in \mathrm{dcl}(B) \;:\Longleftrightarrow\; a = f(\bar{b}) \text{ for some tuple } \bar{b} \text{ from } B \text{ and some } \emptyset\text{-definable function } f.$$

It is well-known that (K, dcl) is a finitary matroid, and we denote the corresponding rank function by rk_T. Given an elementary substructure F of K and a set $A \subseteq K$, we let $F\langle A\rangle$ denote the definable closure of $F \cup A$ in K. Then $F\langle A\rangle$ is also a model of T, and $\mathrm{rk}_T\big(F\langle A\rangle\big|F\big) = \mathrm{rk}_T(A|F)$.

A T**-derivation on** K is a map $\delta \colon K \to K$ such that for each tuple $\bar{a} = (a_1, \ldots, a_n) \in K^n$ and each $\emptyset$-definable function f which is $\mathcal{C}^1$ in a neighborhood of $\bar{a}$, we have

$$\delta f(\bar{a}) \;=\; \frac{\partial f}{\partial Y_1}(\bar{a})\delta a_1 + \cdots + \frac{\partial f}{\partial Y_n}(\bar{a})\delta a_n. \tag{5.1}$$

The study of T-derivations was initiated by the authors in [4] and was expanded on by the second author [5]. The following fact can be proven along the lines of Lemma 5.1, using (5.1) above:

Fact 5.6. *Any T-derivation on K is a quasi-endomorphism of (K, dcl).*

Now suppose that $\Phi = (\phi_1, \ldots, \phi_m)$ is a collection of commuting T-derivations on K. Then $(K, \phi_1, \ldots, \phi_m)$ is called a T**-differential field**. Let $\bar{a}$ be a tuple from K and let F be a T-differential subfield of K, that is, an elementary substructure of K which is closed under each ϕ_i.

Corollary 5.7. *Let $Q^{\Phi}_{\bar{a}|F}$ be the Kolchin polynomial of $\bar{a}$ over F with respect to the finitary matroid (K, dcl). Then*

$$\mathrm{rk}_T\big(F\big\langle\Phi^{\preccurlyeq(\bar{s})}(\bar{a})\big\rangle\big|F\big) \;=\; Q^{\Phi}_{\bar{a}|F}(\bar{s})$$

whenever $\min(\bar{s})$ is sufficiently large. Moreover:

(1) The dominant terms of $Q^{\Phi}_{\bar{a}|F}$ only depend on the T-differential field extension $F\big\langle\Theta(\bar{a})\big\rangle$.

(2) The coefficient of $\bar{Y}^{\bar{d}}$ times $d_1! \cdots d_k!$ is equal to the maximal size of a subset $B \subseteq F\big\langle\Theta(\bar{a})\big\rangle$ such that the tuple $(\theta b)_{\theta \in \Theta, b \in B}$ is dcl*-independent over F.*

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